Final written examination

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1 Task 1

1.1 Task Description

Consider a game of two players (Alice and Bob) with the following payoff matrix $\begin{bmatrix} 3 & 2 & 20 & 0 \\ 24 & 4 & 19 & 61 \end{bmatrix}$. Rows of the matrix corresponds to strategies A_1 and A_2 of Alice and columns of the matrix corresponds to strategies B_1, B_2, B_3, B_4 of Bob.

Firstly, characterize the game using terms and concepts introduced in the lecture notes. Then solve the game in mixed strategies.

1.2 Game Characterization

The game is a 2-players game in the normal form. As each cell in the payoff matrix contains only one value, which means it's equivalent to the following matrix $\begin{bmatrix} 3:-3 & 2:-2 & 20:-20 & 0:-0 \\ 24:-24 & 4:-4 & 19:-19 & 61:-61 \end{bmatrix}$ it represents a zero-sum game.

A mixed strategy of the player X is a probability distribution on the set of its pure strategies $\{X_1, X_2, .., X_n\}$.[1, pg-4]

Let G be the game in the normal form (of Alice A and Bob B), X be a player $\in \{A, B\}$, and $\{X_1, X_2, .., X_n\}$ be the set of the player X pure strategies in the game G. Any mixed strategy of the player X can be represented as a vector (a row for certainty) $(x_1, ..., x_n)$, where $1 = \Sigma_{1 \leq i \leq n} x_i$ and all $0 \leq x_1 \leq 1, ..., 0 \leq x_n \leq 1$ are probabilities that X plays strategies $X_1, ..., X_n$. Let the corresponding payoff matrix as $[G_{ij}]$. Alice has two(the number of rows) strategies A_1, A_2 and Bob has four(the number of columns) strategies B_1, B_2, B_3, B_4.

$$[G_{ij}] = \begin{bmatrix} 3 & 2 & 20 & 0 \\ 24 & 4 & 19 & 61 \end{bmatrix}$$

The value in the i-th row and j-th column express the payoff for Alice if Alice chooses the i-th strategy and Bob chooses the j-th strategy. The negative of that value is the payoff for Bob if Alice chooses the i-th strategy and Bob chooses the j-th strategy.

1.3 Solving in Mixed Strategies

Using *mixed domination*[1, pg-55] to solve the given game in mixed strategies. In any zero-sum game of two players with a matrix of $m * n$ order. If a row/column is (strictly) dominated in mixed strategies, then this row/column is non-active. Implying, strictly dominated columns or rows can be omitted and the equilibria would not be affected.[1, pg-56]

Note: All the eliminations done below were using the knowledge from [1, pg-56]

For Bob, the first column $\begin{bmatrix} 3 \\ 24 \end{bmatrix}$ is dominated by the second column $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Therefore, the strategy B_1 corresponding to the first column is non-active in the Bob's equilibrium mixed strategy. Therefore $b_1^* = 0$.

$$[G'_{ij}] = \begin{bmatrix} 2 & 20 & 0 \\ 4 & 19 & 61 \end{bmatrix}$$

Correspondingly, in G', for Bob, the second column $\begin{bmatrix} 20 \\ 19 \end{bmatrix}$ is dominated by the first column $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. therefore, the strategy B'_2 corresponding to the second column is non-active in the Bob's equilibrium mixed strategy. Therefore $b'^*_2 = 0$.

$$[G''_{ij}] = \begin{bmatrix} 2 & 1 \\ 4 & 61 \end{bmatrix}$$

For Alice, the first row $\begin{bmatrix} 2 & 1 \end{bmatrix}$ is dominated by the second row $\begin{bmatrix} 4 & 61 \end{bmatrix}$. Therefore, the strategy A_1 corresponding to the first row is non-active in the Alice's equilibrium mixed strategy. Therefore $a''^*_1 = 0$.

$$[G^3_{ij}] = \begin{bmatrix} 4 & 61 \end{bmatrix}$$

In the resultant payoff matrix G^3_{ij}, Bob's second column $\begin{bmatrix} 61 \end{bmatrix}$ is dominated by the first column $\begin{bmatrix} 4 \end{bmatrix}$. Therefore, the strategy $B_2^{(3)}$ corresponding to the second column is non-active in the Bob's equilibrium mixed strategy. Therefore $b^{(3)}{}^*_2 = 0$.

$$[G^{final}_{ij}] = \begin{bmatrix} 4 \end{bmatrix}$$

The resultant Nash equilibrium for the game G in mixed strategies is $G_{22} = (A2, B2)$ with the value of the game equal to **4**.

The solution to the game is:

$$S^* = ((0, 1), (0, 1, 0, 0))$$

2 Task 2

2.1 Task Description

Consider a game of two players (Alice and Bob) with the following payoff matrix $\begin{bmatrix} 3:24 & 2:4 \\ 20:19 & 0:61 \end{bmatrix}$. Rows of the matrix corresponds to strategies A_1 and A_2 of Alice and columns of the matrix corresponds to strategies B_1 and B_2 of Bob.

Firstly, characterize the game using terms and concepts introduced in the lecture notes. Then solve the game in mixed strategies.

2.2 Game Characterization

This game is 2-player game in the normal form between Alice and Bob. Rows A_1 and A_2 represent Alice's strategies, whereas columns B_1 and B_2 represent Bob's strategies. Matrix cells represent the payoffs when Alice and Bob play the strategies corresponding to the row and column of the cell respectively. Payoffs are denoted as $x : y$ where x is Alice's payoff and y is Bob's payoff. This game is not a zero-sum game since individual payoffs do not sum to zero for each play.

2.3 Solving in Pure Strategies

Note: This section is not strictly necessary but it may contain information that aids in the game's solution in mixed strategies.

From lecture's knowledge, A play S is called acceptable for a player X if $\pi_X(S) \geq \pi_{X:s'_X}(S)$. A Nash equilibria is any play that is acceptable for all players. Alice's acceptable plays are $(A_1, B_2), (A_2, B_1), (A_2, B_2)$ while Bob's acceptable plays are $(A_1, B_1), (A_2, B_2)$.

Hint: To solve in Pure Strategies, find all of the game's Nash Equilibria.

Looking at the Acceptable Strategies for both players, we can see that $S^* = (A_2, B_2)$ is the only Nash Equilibrium in Pure Strategies. Therefore, **the answer is:**

$$S^* = ((p, (1-p)), (q, (1-q))) = ((0, 1), (0, 1))$$

2.4 Solving in Mixed Strategies

Spectrum (or support) of a mixed strategy is the set of all pure strategies that have a positive probability in the strategy. We'll test all of the Spectrum's possibilities because it is not clear. [3, pg-74]

Considering the following cases to solve in mixed strategies:

1. Both playing pure strategies $(p, 1-p) \in \{(0, 1), (1, 0)\}$ and $(q, 1-q) \in \{(0, 1), (1, 0)\}$. This case was already discussed in the section above.
2. Both playing purely mixed strategies $((p, (1-p)), (q, (1-q)))$ where $p, q \in (0, 1)$
3. Bob is playing a purely mixed strategy $q \in (0, 1)$, but Alice is playing a pure strategy $(p, 1-p) \in \{(0, 1), (1, 0)\}$.
4. Alice is playing a purely mixed strategy $p \in (0, 1)$, but Bob is playing a pure strategy $(q, 1-q) \in \{(0, 1), (1, 0)\}$.

2.4.1 Purely Mixed Strategies

Using lemma from *Rid of inequalities in mixed equilibria: general case* [1, pg-66] to compute the expected payoffs for Alice playing her pure strategies A_1 and A_2 against the mixed strategy (b_1^*, b_2^*) of Bob.

Hint: Find only purely mixed equilibria $((p, (1-p)), (q, (1-q)))$, where $p, q \in (0, 1)$.

Finding the payoff of the player A playing their pure strategies A_1 and A_2 separately against a mixed strategy $(q, (1-q))$ of the player B:

$$\pi_A^{mix}(A2, (q, (1-q))) = 20q + (1-q) * 2 = 18q + 2$$
$$\pi_A^{mix}(A1, (q, (1-q))) = 3q + (1-q) * 2 = q + 2$$

Equalizing the payoffs of both strategies A_1 and A_2 :

$$\pi_A^{mix}(A1, (q, (1-q))) = \pi_A^{mix}(A2, (q, (1-q)))$$
$$q + 2 = 18q + 2$$
$$17q = 0 \iff q = 0$$

As per the hypothesis $q \in (0, 1) \iff q \neq 0$. Therefore, there is no purely mixed strategies.

2.4.2 Bob playing purely mixed strategy and Alice playing pure strategy

- **Alice plays the pure strategy A_1 (meaning that A_2 does not belong to the spectrum) and Bob is playing purely mixed strategy.** $((1, 0), (q, (1-q)))$, where $q \in (0, 1)$.

$$\pi_A^{mix}(A1, (q, (1-q))) = 3q + (1-q) * 2 = q + 2$$
$$\pi_B^{mix}((1, 0), B1) = 24$$
$$\pi_B^{mix}((1, 0), B2) = 4$$
$$\pi_B^{mix}((1, 0), B1)) = 24 \neq \pi_B^{mix}((1, 0), B2)) = 4$$

This strategy is rejected because it is not a Nash equilibrium.

- **Alice plays the pure strategy A_2 (meaning that A_1 does not belong to the spectrum) and Bob is playing purely mixed strategy.** $((0,1),(q,(1-q)))$, where $q \in (0,1)$.

$$\pi_A^{mix}(A2,(q,(1-q))) = 20q + (1-q) * 2 = 18q + 2$$
$$\pi_B^{mix}((0,1),B1)) = 19$$
$$\pi_B^{mix}((0,1),B2)) = 61$$
$$19 = \pi_B^{mix}((0,1),B1)) \neq \pi_B^{mix}((0,1),B2)) = 61$$

This strategy is rejected because it is not a Nash equilibrium.

2.4.3 Alice playing purely mixed strategy and Bob playing pure strategy

- **Bob plays the pure strategy B_1 (meaning that B_2 does not belong to the spectrum) and Alice is playing purely mixed strategy.** $((p,(1-p)),(1,0))$, where $p \in (0,1)$.

$$\pi_B^{mix}((p,(1-p)),B1) = 24p + 19(1-p) = 5p + 19$$
$$\pi_A^{mix}(A1,(1,0)) = 3$$
$$\pi_A^{mix}(A2,(1,0)) = 20$$
$$3 = \pi_A^{mix}(A1,(1,0)) \neq \pi_A^{mix}(A2,(1,0)) = 20$$

This strategy is rejected because it is not a Nash equilibrium.

- **Bob plays the pure strategy B_2 (meaning that B_1 does not belong to the spectrum) and Alice is playing purely mixed strategy.** $((p,(1-p)),(0,1))$, where $p \in (0,1)$.

$$\pi_B^{mix}((p,(1-p)),B2) = 4p + 61(1-p) = 61 - 57p$$
$$\pi_A^{mix}(A1,(0,1)) = 2$$
$$\pi_A^{mix}(A2,(0,1)) = 0$$
$$2 = \pi_A^{mix}(A1,(1,0)) \neq \pi_A^{mix}(A2,(1,0)) = 0$$

This strategy is rejected because it is not a Nash equilibrium.

2.5 Conclusion

The final set of Strategies to reach Nash Equilibria as follows after examining all the Possibilities and solving the game between Alice and Bob in Mixed Strategies:

Answer -

$$\{((0,1),(0,1))\}$$

References

[1] Shilov, N. (2022). Week 3 [Course notes]. Introduction to Game Theory. Retrieved from *Moodle*.

[2] Shilov, N. (2022). Week 4 [Course notes]. Introduction to Game Theory. Retrieved from *Moodle*.