Parth Kalkar - 05.06.2000

Theory test on Game Theory

Task Introduction:

A sequential number game with interval [1..2011], moves are (+1, +2, .., +11). The player to reach cell 2011 first wins.

Task 1 - FPG Representation:

a. The formalization for a finite position game of two players A (Duplicator) and B (Spoiler) is defined as a tuple $\{G = P_D, P_S, M_D, M_S, F_D, F_S\}$,
   where -
   P_D and P_S are disjoint finite sets of positions for Duplicator and Spoiler
   $M_D \subseteq P_D \times P_D \cup P_S$ and $M_S \subseteq P_S \times P_D \cup P_S$ are admissible moves of Duplicator and Spoiler
   $F_D \subseteq P_D \cup P_S$ and $F_S \subseteq P_D \cup P_S$ are disjoint final positions respectively (where Duplicator and Spoiler won already) [1, pg-15]
b. Using the above definition, we can represent the task's game as a tuple $\{G = P_D, P_S, M_D, M_S, F_D, F_S\}$ such that:
   $P_D = P_S = \{p_0, p_0 + 1, p_0 + 2, . . . , 2011\}$ (where p_0 is the initial position)
   $M_D = M_S = \{(p, p + n) : p \geq p_0, 0 \leq n \leq 11, p + n \leq 2011\}$
   $F_D = F_S = \{2011\}$

Strategies

A strategy for a player $X \in A, B$ is any subset $S_X \subseteq M_X$

A strategy S_X for a player $X \in A, B$ is said to be a winning strategy, if for every finite complete play where X applys S_X is winning for X. [1, pg-18]

$X = F_D \cup \{p \in P_D - F_D : \exists q((p, q) \in M_D, q \in X)\} \cup \{p \in P_S - F_S : \forall q((p, q) \in M_S, q \in X)\}$ [1, pg-21]

The winning strategy is the fixed-point of the function
$F = \lambda X. F_D \cup \{p \in P_D - F_D : \exists q((p, q) \in M_D, q \in X)\} \cup \{p \in P_S - F_S : \forall q((p, q) \in M_S, q \in X)\}$ [1, pg-22].
The first four steps of calculating the winning strategy for the duplicator give us the following:
F0 = ⊘
F1 = FD = {2011}
F2 = [2000 . . . 2011]
F3 = [1988 . . . 1999] ∪ [2000 . . . 2011]

F4 = [1975 . . . 1986] ∪ [1988 . . . 1999] ∪ [2000 . . . 2011]
A pattern can be noticed: All positions except the positions 2011−12n, n ∈ N, n ≥ 1 are in the winning strategy of the duplicator in the first four steps.

Task 2 - Duplicator's winning strategy from Position 1

We want to prove that there is no winning strategy for Duplicator when the initial position p_0 satisfies $2011 - p_0 \equiv 0[12]$ $(i.e.\ p_0 = 2011 - 12n)$ $with\ n \in N,\ n \geq 1$.

a. Base case (n = 1):
   When n = 1, we have $p_0 = 2011 - 12 = 1999$. In this initial position, the Duplicator will move to any position in [2000 . . . 2010], which will make the Spoiler able to win with one move from any of the positions. There is no way for Duplicator to win here, therefore there is no winning strategy for Duplicator with initial position $p_0 = 1999$.
b. Induction step:
   Let $n \in N,\ n \geq 1$. Assuming that there is no winning strategy for Duplicator in a game with initial position $p_0 = 2011 - 12(n - 1)$, we want to prove that there is no winning strategy from initial position $p_0 = 2011 - 12n$. The Duplicator's next move will land him in one of the positions [2011 − 12n + 1 . . . 2011 − 12n + 11] = [2011 − 12(n − 1) − 11. . . 2011 − 12(n − 1) − 1]. Then the Spoiler can just move to position 2011 − 11(n − 1) from the Duplicator position and put the Duplicator in a position with no winning strategy. This is valid for all Duplicator moves, therefore there is no winning strategy with $p_0 = 2011 - 12n$ given that there is no winning strategy with $p_0 = 2011 - 12(n - 1)$.
c. Induction conclusion:
   From the above base case and induction step, we conclude the following: Duplicator has no winning strategy in a game with initial position p_0 such that $2011 - p_0 \equiv 0[12]$

For $p_0 = 1$
We get $2011 - 1 = 2010 ! \equiv 0[12]$. Therefore, according to the previously proved lemma, **the Duplicator has winning strategy from initial position 1.**

Task 3 - Spoiler's winning strategy from Position 1

The Spoiler does not have a winning strategy in a game with initial position $p_0 = 1$. Since the Duplicator has no winning strategy in positions 2011 − 12n (excluding position 1), the Spoiler must keep the Duplicator in those positions to ensure winning.
Hence, a winning strategy for the Spoiler is to follow every move +x of the Duplicator by +(12 − x). This will make the play include all the positions 2011 − 12n till the position 2011 which ends the game with the Spoiler winning.

Task 4 - Extensive form with Payoffs

A game (with complete information) in extensive form is a directed labeled tree (infinite maybe) where the root is the initial position, nodes are positions, and the leaves are final positions. Positions specify turns, all edges are moves by participating players according to their turns and all final positions are marked (by a vector of) individual payoffs for each player [2, pg-8]

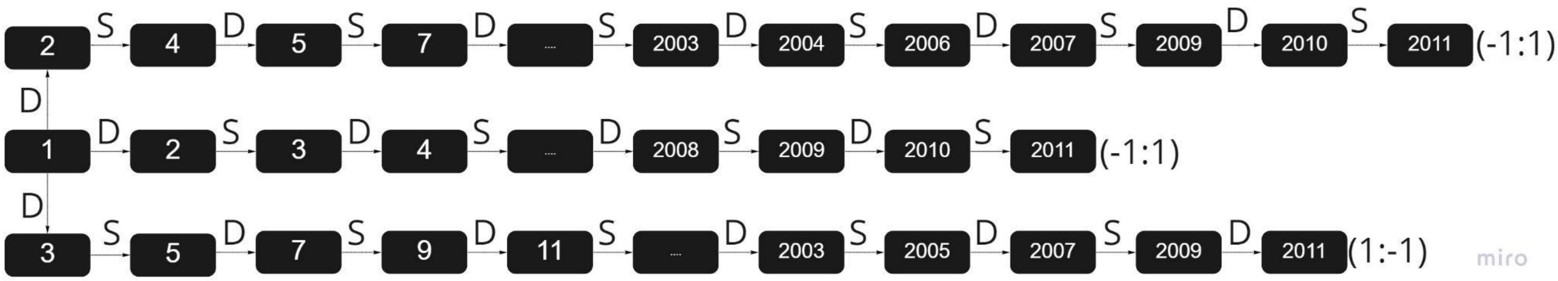

Image Url - [Link]

The strategies followed in the figure above are as follows:
1. Both Duplicator and spoiler always choose to play the move +1.
2. Duplicator always chooses to play the move +1 and Spoiler chooses to play the move +2 as long as the move is valid. Otherwise, Spoiler chooses the move +1.
3. Both Duplicator and spoiler always choose to play the move +2 as long as the move is valid. Otherwise, both choose the move +1.

Task 5 - Nash Equilibria

An acceptable play $S = (S_D, S_S)$ is acceptable for player $X \in \{D, S\}$ if $\pi_X(S) \geq \pi_X(S_X : S'_X)$ for any legal strategy S'_X that X can use. (X can't increase his payoff by changing his strategy unilaterally). [2, pg-17]
A Nash equilibrium is a play that is acceptable for all players. (i.e. no player can increase his payoff by unliterally changing his strategy).

Examples of Nash Equilibria:
1. The Spoiler follows the winning strategy, and the Duplicator moving by +1 every time. (The Spoiler is therefore moving by $+11 = 12 - 1$ every time) (Payoff $-1 : +1$). This is a Nash equilibrium because according to the definition, the Spoiler cannot increase his payoff by changing strategy (he already has the maximum payoff), and the Duplicator cannot increase his payoff because for any strategy he chooses, the Spoiler will move him back to a position with no winning strategy.
2. The Duplicator follows the winning strategy and the Spoiler chooses to move +2 when the move is legal, otherwise the move +1. Duplicator wins the game in this play and the payoff resulted is $(1, -1)$. Spoiler has no strategy that can allow them to obtain a better payoff. The same goes for Duplicator as they achieve the maximum payoff possible, therefore, the play is Nash equilibria.

Examples of Non-Nash equilibria:
1. Both players moving by +5 all the time (Duplicator wins here +1 : −1) (Spoiler can unilaterally use the better strategy of moving by +7 every time, which will make him win −1 : +1).
2. Both Duplicator and Spoiler only choose the move +1. This is not a Nash equilibria play because Duplicator could have followed the winning strategy to win the game and achieve a payoff of (+1, −1).

Task 6 - Pareto Optimal Plays

A play $S = (S_D, S_S)$ is Pareto optimal if there is no other play S' that satisfies $(\pi_D(S') \geq \pi_D(S)\ and\ \pi_S(S') \geq \pi_S(S))\ and\ (\pi_D(S') > \pi_D(S)\ or\ \pi_D(S') > \pi_D(S))$(i.e no other play would increase the payoff of one of the players without hurting the payoff of the other player). [2, pg-22]
A payoff vector (x1, y1) dominates another payoff vector (x2, y2) if all the following conditions hold: x1 ≥ x2, y1 ≥ y2 and at least one of these inequalities is strict (i.e., '>' instead of '≥').
A pair of strategies is Pareto optimal if the payoff vector for the pair is not dominated by the payoff vector for any other pair of strategies.

A game in the normal form is a zero-sum if for every play S the overall sum $\sum_{X\ is\ a\ player} \pi_X(S) = 0$ [2, pg-26].
Using the above definition, since all payoffs are of the form (a, −a) where a ∈ {1, −1}, it is a a zero-sum game. Each play in any zero-sum game of two players is Pareto optimal[3, para-2 (*Definition*)].
Thus all listed plays are Pareto optimal.

Task 7 - Fragment of the Normal Form

Two players' game in the normal form is 2-dimensional matrix where rows correspond to the strategies of the first player, columns correspond to the strategies of the second player. [2, pg-14]
In the table below, we use the following notation for strategies:

a. D_x is the strategy for the Duplicator - moves x places ahead if valid else reach the cell 2011
b. S_x is a strategy for the Spoiler - moves x places ahead if valid else reach the cell 2011

↓ Duplicator / Spoiler →	S_1	S_2
D_1	(-1 : +1)	(-1 : +1)
D_2	(-1 : +1)	(+1 : −1)
D_3	(+1 : -1)	(-1 : +1)

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References

1. Shilov, N. (2022). *Week 1* [Course notes]. Introduction to Game Theory. Retrieved from Link
2. Shilov, N. (2022). *Week 2* [Course notes]. Introduction to Game Theory. Retrieved from Link
3. Wikipedia contributors. (2022, January 15). *Zero-sum game*. Wikipedia. Retrieved February 13, 2022, from Link