Basic Numerical Techniques for Astrophysics and Cosmology

1 Integration

1.1 Problem 1

Describe the motion of a planet for each of the following sets of initial condition. The sun is assumed to be at the origin and its motion is not considered. The only force involved being the force of gravitation. Assuming $GM_{\odot} = 1$, where G is gravitation constant and $M_{\odot}$ is the mass of sun, one can avoid dealing with astronomical magnitudes for positions:

- $x_0 = 0.5$; $y_0 = 0.0$; $v_{0,x} = 0.0$; $v_{0,y} = 1.63$.
- $x_0 = 0.5$; $y_0 = 0.0$; $v_{0,x} = 0.0$; $v_{0,y} = 1.80$.
- $x_0 = 0.5$; $y_0 = 0.0$; $v_{0,x} = 0.0$; $v_{0,y} = 1.40$.
- $x_0 = 0.5$; $y_0 = 0.0$; $v_{0,x} = 0.50$; $v_{0,y} = 1.80$.
- $x_0 = 0.5$; $y_0 = 0.0$; $v_{0,x} = -0.50$; $v_{0,y} = 1.40$.
- $x_0 = 0.5$; $y_0 = 0.50$; $v_{0,x} = -0.50$; $v_{0,y} = 0.50$.

1.2 Problem 2

Write two functions which will perform integration following Trapezoidal and Simpson's rules respectively. Write these functions in such a way that they can take in any function as their integral.
Hints: If you are familiar with the concepts of pointer and function pointer, that will help you in writing these functions. Once you have managed to do that, evaluate the following integrals numerically:

- $\int_0^1 x dx$
- $\int_0^1 x^2 dx$
- $\int_0^1 \sin(x) dx$
- $\int_0^1 x \sin(x) dx$

using both Trapezoidal and Simpson's rules for $N = 2, 4, 8, ...1024$, where N is the number of intervals used in the integration. As all of these integrals can be evaluated analytically, estimate the relative (with respect to the analytical value) error $\mathrm{Err}(N)$ in the numerical integration as a function of N. Show plots of $\mathrm{Err}(N)$ with varying N for all the integrals. Are these results consistent with how you expect the error to scale with N?

1.3 Problem 3

Consider the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx \,. \tag{1}$$

How will you evaluate it analytically? How will you evaluate it numerically? Compare your numerical and analytical results.

1.4 Problem 4

- Compute the following integral using 2, 3 and 5 point Gaussian quadrature method:

$$\int_{-1}^{1} x^2 e^{-x^2} dx \,. \tag{2}$$

- The arc length along a curve $f(x)$ is define as:

$$s = \int \sqrt{1 + f'(x)^2} \tag{3}$$

  find the arc length from 0 to π along a curve $f(x) = sin(x)$ analytically and also by using 2, 3 and 5-point Gaussian quadrature method and determine the % error with respect to the analytical value.

1.5 Problem 5

We consider the 1D motion of a particle of mass m in a time independent potential $V(x)$. The fact that the energy E will be conserved allows us to integrate the equation of motion and obtain a solution in a closed form:

$$t - C = \sqrt{\frac{m}{2}} \int_{x_i}^{x} \frac{dx'}{\sqrt{E - V(x')}} \,. \tag{4}$$

The particle is at the position x_i at time $t = 0$ and it is located at the position x at any arbitrary time t. For simplicity, let us assume that $C = 0$.
We consider a particular case where the particle is in bound motion between two points a and b where $V(a) = E$ and $V(b) = E$ and $V(x) < E$ for $a < x < b$. The time period of the oscillation T is given by

$$T = 2\sqrt{\frac{m}{2}} \int_{x=a}^{x=b} \frac{dx'}{\sqrt{E - V(x')}} \,. \tag{5}$$

First consider a particle with $m = 1$ kg in the potential $V(x) = \alpha x^2/2$ with $\alpha = 4\,\mathrm{kg/sec}^2$. Numerically calculate the time period of oscillation and check this against the expected value. Verify that the frequency does not depend on the amplitude of oscillation.
Next consider a potential $V(x) = \exp(\alpha x^2/2) - 1$. Numerically verify that for small amplitude oscillations you recover the same results as the simple harmonic oscillator. The time period is expected to be different for large amplitude oscillations. How does the time period vary with the amplitude of oscillations? Show this graphically.