

AA 609 : Computational Methods in Astronomy and Space Sciences

Assignment 1

Due Date : 19/01/2024, Time: 12 noon

1. ODE Algorithms and Error Estimates

Consider the equation

$$\frac{dy}{dx} = -xy \quad (1)$$

with initial condition $y(0) = 1$.

Obtain the numerical solution of the above ODE using the following methods a) Euler Method b) Runge-Kutta 2nd Order, c) Runge-Kutta 4th Order and d) Backward Euler Method. In particular, verify that the errors (difference between numerical and exact solutions) decrease according to the expected power of the explicit algorithms only and discuss the difference between local error and global error.

2. Stellar Orbits in Kepler Potential

The equation that governs the motion of a star in r - θ plane under a Kepler potential is given by

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{a(1-e^2)} \quad (2)$$

where $u = 1/r$, a and e are constants. Solve the above equation using values $a = 5.0$ and $e = 0.8$, plot the trajectory of the star with a) Euler Method and b) Runge Kutta 2nd Order Method. Is the motion of the star what you expect i.e., a closed ellipse with eccentricity e and semi-major axis a ?. Justify the choices of initial conditions taken and any deviation from the expected orbit.

Hint: The analytic relation between radius r and θ for the above equation can be given as

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos(\theta-\theta_0)},$$

where $\theta - \theta_0$ is called the true anomaly.

3. Chandrasekhar Mass Limit

Write the equations that govern the distribution of stellar mass $M(r)$ and $P(r)$ as a function of density $\rho(r)$ following the argument of hydrostatic balance. Using the following Equation of state, express the equations in terms of variation of $M(r)$ and $\rho(r)$, where r is the spherical radius.

$$P = \frac{1}{4}(3\pi^2)^{1/3}\hbar c \left(\frac{\alpha}{m_N} \right)^{4/3} \rho^{4/3} \quad (3)$$

where, the parameter α is the number of electrons per nucleon in the star. For numerical work it is useful to re-scale the variables involved to that their actual numerical values are neither too large nor too small. We therefore introduce : $\hat{r}, \hat{\rho}$ and \hat{M} such that $r = R_0\hat{r}$, $\rho = \rho_0\hat{\rho}$, $M = M_0\hat{M}$, where,

$$\rho_0 = \frac{n_0 m_N}{\alpha}$$

with $n_0 = \frac{m_e^3 c^3}{3\pi^2 \hbar^3}$ and $M_0 = 4\pi\rho_0 R_0^3$, Using the above scaling relations, find the expression for R_0 such that the scaled equations become

$$\begin{aligned}\frac{d\hat{M}}{d\hat{r}} &= \hat{r}^2 \hat{\rho} \\ \frac{d\hat{\rho}}{d\hat{r}} &= -\frac{\hat{M}\hat{\rho}^{2/3}}{\hat{r}^2}\end{aligned}$$

Write a program to solve the above scaled equations using any algorithm of your choice. The idea is to numerically integrate these starting at $\hat{r} = 0 + \epsilon$, (where ϵ is a small number) with the initial conditions $\hat{M} = 0$ and $\hat{\rho} = \hat{\rho}_c$. The radius R of the white dwarf is the value of r at which $\rho = 0$, and $M(R)$ is the total mass at this point. Once you have the code working, calculate the total masses and radii of white dwarfs with $\hat{\rho}_c$ values ranging from about 0.1 to 10^6 . This gives a family of equilibrium configurations. By changing the stepsize h and (maybe) the algorithm used, verify that your solutions are accurate.

Hint: A useful way to display these results is as a plot of radius versus mass. Can you identify the point at which the star can no longer be supported by degenerate electrons? This limiting mass for white dwarfs (with $\alpha = 0.5$, 1 electron per 2 nucleons (1 proton + 1 neutron)) is known as the Chandrasekhar mass. Calculate its value in units of the mass of the sun.