ORF 555, Fall 2025: Homework 5

due by Monday 12/8 at 11:59pm. Please upload to Gradescope and show all work, code and calculations so we can assign partial credit when grading.

PART I: Submit individual answers

1. In a one-period model of bitcoin mining, if the miner expends $\alpha \geq 0$ on computational power over a fixed time period (say one day), they have a probability

$$p = \frac{\alpha}{\alpha + \mathcal{A}}$$

of discovering a bitcoin and receiving a reward $r > 0$. That is, over the period, their gain or loss is given by $-c\alpha + rZ$, where Z is 1 with probability p, and zero with probability $1 - p$, and c is the cost of electricity per unit hash. Here $\mathcal{A} > 0$ is the given combined hashing rate of all other miners. The miner seeks to maximize their *expected profit*. In the following, the answers are in terms of $(c, r, \mathcal{A})$.

   (a) What is the optimal (best response) hashing rate $\alpha^* \geq 0$ of the miner given $\mathcal{A}$?

   (b) At what level of c (and higher) is the optimal α^* zero, so the individual miner has not enough incentive to participate because of high electricity cost? Denote this threshold level as c^*.

   (c) **Hard conceptually: use AI and/or TAs**
   Now suppose there are $M + 1$ miners facing different costs of electricity, for instance in different states or countries. The miners are competing for the next Bitcoin, their interaction being through the aggregate hashrate $\mathcal{A}$. This is framed as a mean field game, in which $\mathcal{A}$ is approximated by $M\bar{\alpha}$, where $\bar{\alpha}$ is the average hashrate under the assumption that there is an approximating continuum of miners with electricity costs distributed according to a continuous density $m(c)$. The equilibrium of the competition is as follows:

   - Given $\mathcal{A}$, a player with cost c, hashes at rate $\alpha^*(c)$, as found in part (a).
   - Then the aggregate hashrate $\mathcal{A}$ is approximated by

   $$\mathcal{A} = M\bar{\alpha}, \qquad \bar{\alpha} = \int \alpha^*(c)\, m(c)\, dc,$$

   which is a fixed point condition that substitutes for Nash equilibrium.

   Assume costs are distributed **uniformly on** $(r/2, r)$. Assume *a priori* that

   $$1 < \mathcal{A} < 2. \tag{1}$$

   i. Find $\bar{\alpha}$ and hence $\mathcal{A}$.
   ii. Validate that $\mathcal{A}$ satisfies the assumption (1) for M large enough.
   iii. Find the cutoff c^* such that for miners with costs $c \in [c^*, r)$, their optimal hash rate is zero.

iv. What fraction of miners is active, and what is the limit of this fraction as $M \to \infty$?

2. In a (different) one-period model of bitcoin mining, if a miner with capital $x > 0$ expends $\alpha \geq 0$ on computational power over a fixed time period (say one day), they have a probability

$$p = \frac{\alpha x}{\alpha x + \mathcal{A}X}$$

of discovering a bitcoin and receiving a reward $r > 0$. That is, over the period, their gain or loss is given by $-\alpha + rZ$, where Z is 1 with probability p, and zero with probability $1 - p$. Here $\mathcal{A} > 0$ is the given hashing rate of all other miners, and $X > 0$ is their combined capital. The success probability is thus dependent on a miner's capital.

   (a) What is the optimal (best response) hashing rate $\alpha^* \geq 0$ of the miner given $\mathcal{A}$?

   (b) At what level of relative capital disadvantage x/X (and lower) is the optimal α^* zero, so the individual miner has not enough incentive to participate because of being too small a player?

   (c) How does this conform to preferential attachment (increasing wealth inequality), and go against the idea of decentralized finance.

3. **Financialization Question** In a commodity market comprising market users and speculators as described in class and in the Notes, the presence of the speculators jumps the instantaneous correlation from zero (if they were not there) to

$$\rho_t = \varepsilon \frac{\theta \pi \sigma_S X_t}{A \sigma_t^{(0)}},$$

where the notation is explained in the Notes.

   (a) If the supply of the commodity A is very large, explain the financialization correlation compared with if the supply were smaller.

   (b) What values of (π, θ) among unleveraged strategies by the speculators maximizes the instantaneous correlation?

4. Options on forwards.

   (a) Let Y be a lognormal random variable (under a risk-neutral probability Q), so that $\log Y \sim \mathcal{N}(\mu, \sigma^2)$. Calculate the expectation

$$\mathbb{E}^Q\{(1 - Y)^+\},$$

   giving the answer in terms of the cumulative distribution function Φ of a standard normal random variable.

   (b) A put option with expiration date T on a forward maturing on date $T' > T$ has (cash) payoff

$$(K - F(T, T'))^+,$$

   where K is a strike price and $F(T, T')$ is the forward price on date T. For the GBM model (with convenience yield δ) for the spot price, find (with details) the distribution of $F(T, T')$ given $F(t, T')$, where $t < T$ is today. (*Hint: use directly the SDE we found for $F(t, T)$.*)

(c) Hence find the (Black's) formula for the put option in the GBM model.

(d) In the Schwartz one-factor model for the spot price, find (with details) the distribution of $F(T,T')$ given $F(t,T')$.

(e) Hence find the (Black's) formula for the put option in this Schwartz model.

5. Let (S_t) be the stock price of a firm and (X_t) a market index. A call option on S that is *benchmarked* to the index X pays $(S_T - X_T)^+$ on date T. That is, the holder is rewarded only if the firm's stock outperforms the index. (An example is stock options given to employees of the firm, and this is an example, outside of commodities of an *exchange option*). In the following, everything is in the risk-neutral world Q.

(a) Let $(W_t^{Q(1)})$ and $(W_t^{Q(2)})$ be independent Brownian motions, meaning that at each time $t > 0$, $W_t^{Q(1)}$ and $W_t^{Q(2)}$ are independent random variables. Show that

$$B_t^Q := \rho W_t^{Q(1)} + \rho' W_t^{Q(2)},$$

where $\rho \in (-1,1)$ and $\rho' := \sqrt{1-\rho^2}$, is a Brownian motion.

(b) Calculate $\mathbb{E}^Q\{W_T^{Q(1)} B_T^Q\}$.

(c) Suppose that S and X are correlated geometric Brownian motions satisfying, in the risk-neutral world

$$\begin{aligned} \frac{dS_t}{S_t} &= r\,dt + \sigma\,(\rho\,dW_t^{Q(1)} + \rho' dW_t^{Q(2)}), \\ \frac{dX_t}{X_t} &= r\,dt + \eta\,dW_t^{Q(1)}. \end{aligned}$$

The constant ρ is called the instantaneous correlation coefficient. Write down the expression for the *no arbitrage* price of the benchmarked call as an expectation of a function of $W_T^{Q(1)}$ and $W_T^{Q(2)}$.

(d) Compute this expectation to obtain a formula for the benchmarked option. (It should look like the Black-Scholes call option pricing formula with adjusted inputs: $C_{BS}(\widetilde{S}_0, \widetilde{K}, T; \widetilde{r}, \widetilde{\sigma})$.)

PART II: Submit group answers in the same groups from Homework 4.
Questions posted in separate file.