# ORF 555 HW1

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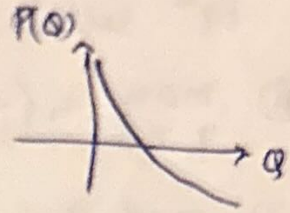
(Q.1) Given :- $P(Q) = -\log Q$ & N players




$\Rightarrow$ (a) $\lim_{Q \to 0^+} P(Q) = \lim_{Q \to 0^+} -\log Q$

$= \underline{\underline{\infty}}$.

(b) $P(Q) < 0 \Rightarrow -\log Q < 0 \Rightarrow \log Q > 0 \Rightarrow Q > 1$

$\Rightarrow$ saturation Qty $= \underline{\underline{1}}$.

(c) $N = 1$, To solve for $q$ s.t. $q(-\log q - c)$ is max$^m$

$\underbrace{\phantom{q(-\log q - c)}}_{\text{Revenue}}$

$q^* = \arg\max_{q \geq 0} q(-\log q - c)$

$\Rightarrow$ let $f(q) = q(-\log q - c) \Rightarrow q^* = \arg\max_{q \geq 0} f(q)$

$\Rightarrow \left.\frac{df}{dq}\right|_{q=q^*} = 0 \Rightarrow q^*\left(-\frac{1}{q^*}\right) + (-\log q^* - c) * 1 = 0$

$\Rightarrow +1 + \log q^* + c = 0$

$\Rightarrow \log q^* = -(c+1)$

$\Rightarrow \boxed{q^* = e^{-(c+1)}}$

To verify that $\left.\frac{d^2 f}{dq^2}\right|_{q=q^*} > 0$, $\frac{d^2 f}{dq^2} = \frac{d}{dq}(-\log q - c - 1)$

$= -\left.\frac{1}{q}\right|_{q=q^*} = -\frac{1}{q^*} < 0$

$\Rightarrow q^* = e^{-(c+1)}$ is the qty at which Revenue is maximized

$\Rightarrow P(q^*) = -\log q^* = -(-(c+1)) = \boxed{\underline{\underline{c+1}}}$

(d) $N=2$ but same cost

To find :- $q_1^*$, $q_2^*$, $P(Q^*)$

Eq$^n$ solved :-

(1) $\max\limits_{q_1 \geq 0} q_1(-\log(\overset{Q}{q_1+q_2})-c)$

(2) $\max\limits_{q_2 \geq 0} q_2(-\log(q_1+q_2)-c)$

$\frac{d(1)}{dq_1} = 0 \Rightarrow q_1\left(\frac{-1}{q_1+q_2}\right) + (-\log Q - c) = 0$

$\Rightarrow \frac{q_1}{Q} + \log Q + c = 0$ — (3)

After this,
we don't do $\frac{d(1)}{dq_2}$ $\because$ (1) represents player 1
& he/she is optimising only for $q_1$

IIIly, $\frac{d(2)}{dq_2} = 0 \Rightarrow \frac{q_2}{Q} + \log Q + c = 0$ — (4)

(3) + (4) gives :-

$\frac{Q}{Q} + 2\log Q + 2c = 0$

$\Rightarrow 2\log Q + 2c + 1 = 0$

$\Rightarrow \log Q = -c - \frac{1}{2}$

$\Rightarrow Q^* = \exp(-c-\frac{1}{2})$

(3) − (4) gives :-

$q_1 = q_2 \Rightarrow q_1^* = q_2^*$

So, $\boxed{q_1^* = q_2^* = \frac{Q^*}{2} = \frac{1}{2}\exp(-c-\frac{1}{2})}$

$P(Q^*) = -\log Q^* = \boxed{c + \frac{1}{2}}$

(e) per barrel savings $= P(Q_M^*) - P(Q_D^*)$

$= (c+1) - (c+\frac{1}{2})$

$= \frac{1}{2}$

Note that $P(Q) = -\log q$ is (assumed to be) price per barrel only

(f)

| | # barrels sold (in the optimal scenario) |
|---|---|
| Monopoly | $q^* = e^{-(c+1)}$ |
| Duopoly | $Q^* = e^{-(c+\frac{1}{2})}$ |

$\Rightarrow \boxed{Q^* > q^*}$

$\Rightarrow \boxed{Q_D > Q_M}$

(g)

| | Revenue (R) |
|---|---|
| M | $R = q^*(-\log q^* - c) = e^{-(c+1)}(\cancel{c}+1-\cancel{c}) = e^{-(c+1)}$ |
| D | $R_{Total} = Q^*(-\log Q^* - c) = e^{-(c+\frac{1}{2})}(\cancel{c}+\frac{1}{2}-\cancel{c}) = \frac{1}{2}e^{-(c+\frac{1}{2})}$ |

$R_{1player} = q_i^*(-\log Q^* - c) = \frac{1}{2} R_{TOTAL} = \frac{1}{4}e^{-(c+\frac{1}{2})}$

Now, ~~$\sqrt{e} < 2$~~ $\Rightarrow e < 4 \Rightarrow \sqrt{e} < 2$

$\Rightarrow e^{-(c+1)} > \frac{1}{2}e^{-(c+\frac{1}{2})}$

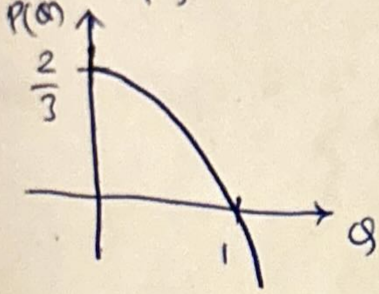
$\Rightarrow R_M > R_{D(Total)} > R_{D(1player)} = \frac{R_{D(Total)}}{2}$

(Q2) $P(Q) = \begin{cases} \frac{1}{1-\rho}(1 - Q^{1-\rho}) & ; \rho \neq 1 \\ -\log Q & ; \rho = 1 \end{cases}$ ($\rho \in \mathbb{R}$ = prudence)

$Q$ = Total Quantity

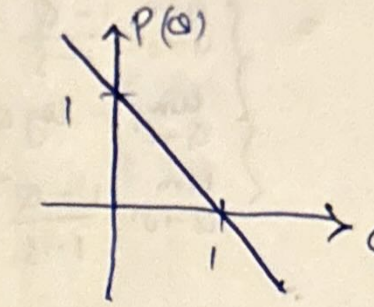
(a) (i) $\rho = -0.5$

$\Rightarrow P(Q) = \frac{1}{1.5}(1 - Q^{1.5})$

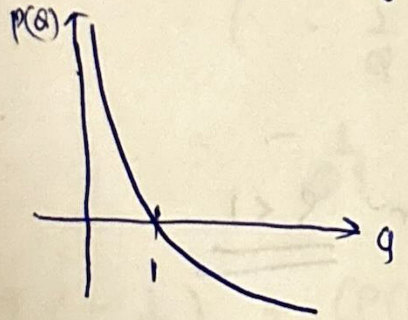



(ii) $\rho = 0$

$\Rightarrow P(Q) = 1 - Q$



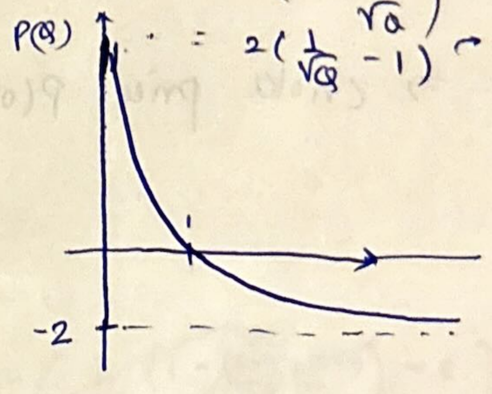

(iii) $\rho = 1 \Rightarrow P(Q) = -\log Q$




(iv) $\rho = 1.5 \Rightarrow P(Q) = \frac{1}{-0.5}(1 - Q^{-0.5})$

$= -2\left(1 - \frac{1}{\sqrt{Q}}\right)$

$= 2\left(\frac{1}{\sqrt{Q}} - 1\right)$ $f' = \frac{-1}{Q\sqrt{Q}} < 0 \quad \forall Q > 0$




(b) price $= 0 \Rightarrow P(Q) = 0$

$\Rightarrow \rho \neq 1 \Rightarrow Q^{1-\rho} = 1 \Rightarrow Q = 1$

$\rho = 1 \Rightarrow \log Q = 0 \Rightarrow Q = 1$

So, saturation demand level $Q$ (at which price becomes 0) $= \underline{\underline{1}}$

(c) $$P(Q) = \begin{cases} \frac{1}{1-\rho}\left(1 - Q^{1-\rho}\right) & ; \rho \neq 1 \\ -\log Q & ; \rho = 1 \end{cases}$$

~~for P~~

$$P(0^+) = \begin{cases} \lim\limits_{Q \to 0^+} \left(\frac{1 - Q^{1-\rho}}{1-\rho}\right) & ; \rho \neq 1 \\ \lim\limits_{Q \to 0^+} \left(-\log Q\right) & ; \rho = 1 \end{cases}$$

$$= \begin{cases} \lim\limits_{Q \to 0^+} \frac{1 - Q^{+ve}}{1-\rho} & \rho < 1 \\ \lim\limits_{Q \to 0^+} -\log Q & \rho = 1 \\ \lim\limits_{Q \to 0^+} \frac{1 - Q^{-ve}}{1-\rho} & \rho > 1 \end{cases}$$

$$= \begin{cases} \frac{1}{1-\rho} & ; \rho < 1 \\ \infty & ; \rho = 1 \\ \infty & ; \rho > 1 \end{cases}$$

$\Rightarrow$ choke price $P(0)$ is finite for $\underline{\underline{\rho < 1}}$

(d) cost (of production) $= c \quad c \in [0, P(0^+))$

To solve:- $f(q^*) = \max_{q \geq 0} q(P(q) - c)$

(or) where $q^* = \operatorname{argmax}_{q \geq 0} q(P(q) - c)$

let $\boxed{f(q) = q(P(q) - c)}$

$\Rightarrow \frac{df}{dq} = 0 \Rightarrow q(P'(q)) + (P(q) - c) \cdot 1 = 0 \quad$ — ①

Now, $P'(q) = \begin{cases} \frac{1}{1-s}\left(-(1-s) q^{-s}\right), & s \neq 1 \\ -\frac{1}{q}, & s = 1 \end{cases}$

$= \begin{cases} -\frac{1}{q^s}, & s \neq 1 \\ -\frac{1}{q}, & s = 1 \end{cases}$

$= -\frac{1}{q^s}$

$\Rightarrow q\left(-\frac{1}{q^s}\right) + (P(q) - c) = 0$

$\Rightarrow \boxed{-q^{1-s} + P(q) - c = 0} \quad$ — ②

Case (i): $s \neq 1$

$\Rightarrow -q^{1-s} + \frac{1}{1-s}(1 - q^{1-s}) - c = 0$

$\Rightarrow \frac{1}{1-s} - c = q^{1-s}\left(1 + \frac{1}{1-s}\right)$

$\frac{1-c+cs}{1-s} = q^{1-s}\left(\frac{2-s}{1-s}\right)$

$\Rightarrow \boxed{q^* = \left(\frac{1-c+cs}{2-s}\right)^{\frac{1}{1-s}}}$

$f(q^*) = q^* (P(q^*) - c)$

$= q^*\left(\frac{1}{1-s}(1 - (q^*)^{1-s}) - c\right)$

$= q^*\left(\frac{1}{1-s}\left(1 - \frac{1-c+cs}{2-s}\right) - c\right)$

$= q^*\left(\frac{2-s-1+c-cs}{(1-s)(2-s)} - c\right)$

$= q^*\left(\frac{1-s+c(1-s)}{(1-s)(2-s)} - c\right)$

$= q^*\left(\frac{1+c-2c}{(2-s)}\right)$

? $= \boxed{q^*}$

$= \boxed{\left(\frac{1-c+cs}{2-s}\right)^{\frac{1}{1-s}}}$

So, $\boxed{f(q^*) = q^* = \left(\frac{1-c+cs}{2-s}\right)^{\frac{1}{1-s}}}$

(Q.2) (d)

case (i): $s \neq 1$

$\Rightarrow -s^{1-s} + \frac{1}{1-s}(1-q^{1-s}) - c = 0$

$\Rightarrow \frac{1}{1-s} - c = q^{1-s}\left(1 + \frac{1}{1-s}\right)$

$\Rightarrow \frac{1-c+cs}{\cancel{1-s}} = q^{1-s}\left(\frac{2-s}{\cancel{1-s}}\right)$

$\Rightarrow \boxed{q^* = \left(\frac{1-c+cs}{2-s}\right)^{1/(1-s)}}$

$f(q^*) = q^*(p(q^*) - c)$

$= q^*\left(\frac{1}{1-s}(1-(q^*)^{1-s}) - c\right)$

$= q^*\left(\frac{1}{1-s}\left(1-\left(\frac{1-c+cs}{2-s}\right)\right) - c\right)$

$= q^*\left(\frac{(2-s-1+c-cs)}{(1-s)(2-s)} - c\right)$

$= q^*\left(\frac{(\cancel{1-s})(1+c)}{(\cancel{1-s})(2-s)} - c\right)$

$= q^*\left(\frac{1+c-2c+sc}{2-s}\right)$

$= q^*\left(\frac{1-c(1-s)}{2-s}\right)$

$\Rightarrow \boxed{f(q^*) = \left(\frac{1-c(1-s)}{(2-s)}\right)^{1/(1-s)} \left(\frac{1-c(1-s)}{2-s}\right)}$

Note that $\lim_{s\to1}\left(\frac{1-c(1-s)}{2-s}\right)^{1/(1-s)}\left(\frac{1-c(1-s)}{2-s}\right)$

$= \exp\left(\left(\frac{1-c(1-s)}{(2-s)} - 1\right)\frac{1}{1-s}\right)$

$= \exp\left(-\frac{(c+1)(\cancel{s-1})}{2-s}\cdot\frac{1}{\cancel{1-s}}\right)$

$= \exp(-(c+1))$

$= \exp(-c-1)$

Case (ii): $s = 1$

$\Rightarrow p(q) = -\log q$

$\Rightarrow -q^{1-1} - \log q - c = 0$

$\Rightarrow 1 + \log q + c = 0$

$\Rightarrow \log q = -c-1$

$\Rightarrow q = \exp(-c-1)$

$\Rightarrow \boxed{q^* = \exp(-c-1)}$

$f(q^*) = q^*(p(q^*) - c)$

$= q^*(-\log q^* - c)$

$= \exp(-c-1)(c+1-\cancel{c})$

$= \exp(-c-1)$

$\Rightarrow \boxed{f(q^*) = \exp(-c-1)}$

$f(q)$ = ~~Revenue~~ Profit at qty $q$

(e) • Unique NE → 2 player Cournot game
• with $\delta < 3$
• players' cost → $c_1, c_2 \in [0, P(0^+))$

P1 → $\max\limits_{q_1 \geq 0} q_1 (P(Q) - c_1)$    P2 → $\max\limits_{q_2 \geq 0} q_2 (P(Q) - c_2)$

where $Q = q_1 + q_2$

$\frac{\partial f_1}{\partial q_1} = 0 \Rightarrow q_1 P'(Q) + (P(Q) - c_1) = 0$ — ①

$\frac{\partial f_2}{\partial q_2} \Rightarrow q_2 P'(Q) + P(Q) - c_2 = 0$ — ②

① + ② gives :-

$Q P'(Q) + 2P(Q) - c = 0$ (when $c = c_1 + c_2$) — (*)

① − ② gives :-

$(q_1 - q_2) P'(Q) = c_1 - c_2$ — (**)

From (*),

case i : $\delta = 1 \Rightarrow P(Q) = -\log Q \Rightarrow$
$P'(Q) = -\frac{1}{Q}$

(*) becomes ⇒

$Q(+\frac{1}{Q}) + 2(+\log Q) + c = 0$

$\Rightarrow \log Q = \frac{-c-1}{2}$

$\boxed{Q^* = \exp(-\frac{c-1}{2})}$

To find $q_1, q_2$ using (**),

$(q_1 - q_2)(-\frac{1}{Q}) = c_1 - c_2$

$\Rightarrow (q_1 - q_2) = (c_2 - c_1) Q$

$\Rightarrow q_1^* - q_2^* = (c_2 - c_1) Q^*$

$\because q_1^* + q_2^* = Q^*$,

$q_1^* = \frac{1}{2}(c_2 - c_1 + 1) Q^*$, $q_2^* = \frac{1}{2}(-c_2 + c_1 + 1) Q^*$

$\Rightarrow \boxed{\begin{aligned} q_1^* &= \tfrac{1}{2}(c_2 - c_1 + 1)\exp(-\tfrac{c-1}{2}) \\ q_2^* &= \tfrac{1}{2}(c_1 - c_2 + 1)\exp(-\tfrac{c-1}{2}) \end{aligned}}$

case ii : $\delta \neq 1 \Rightarrow P(Q) = \frac{1}{1-\delta}(1 - Q^{1-\delta})$

$\Rightarrow P'(Q) = -\frac{1}{Q^\delta}$

⇒ (*) becomes :-

$Q(-\frac{1}{Q^\delta}) + \frac{2}{1-\delta}(1 - Q^{1-\delta}) - c = 0$

$\Rightarrow -Q^{1-\delta} - \frac{2}{1-\delta} Q^{1-\delta} = c - \frac{2}{1-\delta}$

$\Rightarrow Q^{1-\delta}\left(\frac{1-\delta+2}{1-\delta}\right) = \frac{2}{1-\delta} - c$

$\Rightarrow Q^{1-\delta}\left(\frac{3-\delta}{1-\delta}\right) = \left(\frac{2 - c + \delta c}{1-\delta}\right)$

$\Rightarrow Q^{1-\delta} = \frac{2 + c(\delta-1)}{3-\delta}$

$\delta < 3 \Rightarrow$ Nr $< 2+2c$

$\delta < 3 \Rightarrow$ Dnm $> 0$

$\Rightarrow \boxed{Q^* = \left(\frac{2 + c(\delta - 1)}{3-\delta}\right)^{\frac{1}{1-\delta}}}$

$(q_1 - q_2)(-\frac{1}{Q^\delta}) = (c_1 - c_2)$

$\Rightarrow q_1 - q_2 = (c_2 - c_1) Q^\delta$
$q_1 + q_2 = Q$

$\Rightarrow q_1 = \frac{Q + (c_2 - c_1) Q^\delta}{2}$

$q_2 = \frac{Q - (c_2 - c_1) Q^\delta}{2}$

$\boxed{\rho = 1}$

$$q_1^* = \frac{1}{2}\left(1 + (c_2 - c_1)Q^*\right)$$

$$q_2^* = \frac{1}{2}\left(1 - (c_2 - c_1)\right)Q^*$$

where $Q^* = \exp\left(-\frac{(c_1 + c_2)}{2} - 1\right)$

$\boxed{\rho \neq 1}$

$$q_1^* = \frac{Q^* + (c_2 - c_1)Q^{*\rho}}{2}$$

$$q_2^* = \frac{Q^* - (c_2 - c_1)Q^{*\rho}}{2}$$

~~where~~ $\cancel{Q^* = 2 + c(\rho - 1)}$

where $Q^* = \left(\frac{2 + (c_1 + c_2)(\rho - 1)}{3 - \rho}\right)^{\frac{1}{1-\rho}}$

(Q.3)

(b) ~~$G_1 = G_2$~~ 2 player static Cournot game ($Q \xrightarrow{\text{determines}} P$).

$c_1 = 0 \quad q_1^* \in [0,1]$ (R1)

$c_2 = 0 \quad q_2^* \in [0,1]$

$P_1 \to \max\limits_{q_1 \geq 0} q_1(1 - q_1 - q_2^*)$ → other(s) player's eq$^{lb}$ strategy

$P_2 \to \max\limits_{q_2 \geq 0} q_2(1 - q_2 - q_1^*)$

$\frac{\partial P_1}{\partial q_1} = 0 \Big|_{\text{at } q_1 = q_1^*}$

$\Rightarrow q_1(-1) + (1 - q_1 - q_2^*)(1) = 0 \Big|_{q_1 = q_1^*}$

$\Rightarrow -q_1 + 1 - q_1 - q_2^* = 0 \Big|_{q_1 = q_1^*}$

$\Rightarrow 1 - 2q_1 - q_2^* = 0 \Big|_{q_1 = q_1^*}$

$\Rightarrow 1 - 2q_1^* - q_2^* = 0 \quad - (1)$

IIIly $\frac{\partial P_2}{\partial q_2} = 0 \Big|_{q_2 = q_2^*}$

$\Rightarrow q_2(-1) + (1 - q_2 - q_1^*)(1) = 0 \Big|_{q_2 = q_2^*}$

$\Rightarrow 1 - 2q_2 - q_1^* \Big|_{q_2 = q_2^*} = 0$

$\Rightarrow 1 - 2q_2^* - q_1^* = 0 \quad - (2)$

$(1) + (2) \Rightarrow 2 = 3Q^* \Rightarrow \boxed{Q^* = q_1^* + q_2^* = \frac{2}{3}} \quad - (*)$

Also, $q_1^* = q_2^* \Rightarrow \boxed{q_1^* = q_2^* = \frac{1}{3}} \quad - (**)$

$\boxed{\text{Eq}^{lb} \text{ mkt oil price}} = 1 - Q^* = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$

(c) When profits are being maximised jointly,

$P = \max\limits_{q_1, q_2 \geq 0} q_1(1 - q_1 - q_2) + q_2(1 - q_1 - q_2)$

$\frac{\partial P}{\partial q_1} = 0 \Rightarrow q_1(-1) + (1 - q_1 - q_2)(1) + q_2(-1) = 0$

$\Rightarrow -q_1 + 1 - q_1 - q_2 - q_2 = 0$

$\Rightarrow 1 - 2q_1 - 2q_2 = 0$

$\Rightarrow 1 - 2Q = 0$

$\Rightarrow Q = \frac{1}{2}$

$\Rightarrow Q^* = \frac{1}{2}$

Total Qty produced $= \boxed{Q^* = \frac{1}{2}}$ $(< Q^*_{\text{earlier}} = \frac{2}{3})$

$\boxed{\text{Mkt oil price} = 1 - \frac{1}{2} = \frac{1}{2}}$

$>$ price earlier $= \boxed{\frac{1}{3}}$

IIIly, $\frac{\partial P}{\partial q_2} = 0 \Rightarrow q_1(-1) + q_2(-1) + (1 - q_1 - q_2)(1) = 0$

$\Rightarrow \boxed{Q^* = \frac{1}{2}}$

$\Rightarrow$ If both players produce equally,

$q_1^* = q_2^* = \frac{1}{4}$

(a) Date → April 2020 → Just after covid hit

Cournot model (of oil prices) → Quantity determines price

slashing → cutting (oil production)

So, if slashing is done (when mutually agreed upon),
Lower ↓ quantity (Q) produced ⇒ price per unit increases
overall / total

Stabilizing

⇓

price is stabilized

Intuitively, if the players decide to maximise joint revenue, they can perhaps earn more (total) revenue (than / as compared to adding ind^r revenue together by maximising ind^r profit) by slashing production. Hence, they may want to do slashing.
In the context in which this headlines came out (just after covid), players may also be interested in ~~raising the steeply fallen prices &~~ maximising their ind^r revenues which would require raising steeply fallen prices ⇒ ↓ supply ⇒ cutting prod^n.

Stabilizing oil prices → Not fluctuating much → will ↓ if more quantity (of oil) is produced → even if the players have the capacity to produce more (oil), just to avoid falling & hence fluctuating of oil prices, players won't produce more & possibly slash the production.

All this happened just after covid was announced. Covid led to ↓ in demand steeply, thus ↓ (oil) prices suddenly. So, to stabilize prices ⇒ prices were required to rise ⇒ ∵ demand won't change much, supply had to reduce ⇒ slashing would help (If this wouldn't have been done, prices would fall farther ⇒ more volatility / losses / unstability in the mkt).

(Q.4) Electricity production (single producer)

- $C = 0$
- $P(q) = 1 - q$ $\Big\} \to q^* = \frac{1}{2}$

with shortfall (trembling hand), $q \to q - \varepsilon$

So, $q^*_{us} = \frac{1+\mu-C}{2}\Big|_{C=0} = \frac{1+\mu}{2}$ (supply)

$\varepsilon \sim \text{EXP}(10) \Rightarrow f_\varepsilon(x) = 10e^{-10x}$

$\Rightarrow \mu = E(\varepsilon) = \frac{1}{10} = 0.1$

$\Rightarrow q^*_{us} = \frac{1+0.1}{2} = 0.55$

$\Rightarrow$ supply $= 0.55$

Demand

$D \sim \text{EXP}(2) \Rightarrow f_D(x) = 2e^{-2x}$

$\Rightarrow E(D) = \frac{1}{2} = 0.5$

Demand $= 0.5$

$D \perp\!\!\!\perp \varepsilon$

~~D, ε ...~~

(a) ~~$P(q^* - \varepsilon < D)$~~

~~$P(D + \varepsilon > q^*)$~~

Now, if $X \sim \text{EXP}(\lambda_1)$, $Y \sim \text{Exp}(\lambda_2)$ & $X \perp\!\!\!\perp Y$,

$$P(X+Y < \alpha) = \int_0^\alpha P(X = x, 0 < Y < \alpha - x)\,dx$$

$$= \int_0^\alpha f_X(x)\left(\int_0^{\alpha-x} f_Y(y)\,dy\right)dx$$

$$= \int_0^\alpha f_X(x)\left(\int_0^{\alpha-x} f_Y(y)\,dy\right)dx$$

$$= \int_0^\alpha f_X(x)\left(\int_0^{\alpha-x} \lambda_2 e^{-\lambda_2 y}\,dy\right)dx$$

$$= \int_0^\alpha f_X(x)\left(1 - e^{-\lambda_2(\alpha-x)}\right)dx$$

$$= \int_0^\alpha \lambda_1 e^{-\lambda_1 x} \left(1 - e^{-\lambda_2(\alpha - x)}\right) dx$$

$$= \int_0^\alpha \left(\lambda_1 e^{-\lambda_1 x} - \lambda e^{-\lambda_2 \alpha} e^{(\lambda_2 - \lambda_1)x}\right) dx$$

(Assuming $\lambda_1 \neq \lambda_2$ which is our case)

$$= \left[e^{-\lambda_1 x}\right]_\alpha^0 - \frac{\lambda_1 e^{-\lambda_2 \alpha}}{\lambda_2 - \lambda_1} \left[e^{(\lambda_2 - \lambda_1)x}\right]_0^\alpha$$

$$= 1 - e^{-\lambda_1 \alpha} - \frac{\lambda_1 e^{-\lambda_2 \alpha}}{\lambda_2 - \lambda_1}\left(e^{(\lambda_2 - \lambda_1)\alpha} - 1\right)$$

$$= 1 - e^{-\lambda_1 \alpha} + \frac{\lambda_1 e^{-\lambda_2 \alpha}}{\lambda_2 - \lambda_1}\left(1 - e^{(\lambda_2 - \lambda_1)\alpha}\right)$$

$$= 1 - e^{-\lambda_1 \alpha} + \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(e^{-\lambda_2 \alpha} - e^{-\lambda_1 \alpha}\right)$$

$$= 1 - e^{-\lambda_1 \alpha}\left(1 + \frac{\lambda_1}{\lambda_2 - \lambda_1}\right) + \left(\frac{\lambda_1}{\lambda_2 - \lambda_1}\right) e^{-\lambda_2 \alpha}$$

$$= 1 - e^{-\lambda_1 \alpha} \frac{\lambda_2}{(\lambda_2 - \lambda_1)} + \frac{\lambda_1}{(\lambda_2 - \lambda_1)} e^{-\lambda_2 \alpha}$$

$$= 1 - \left(\frac{\lambda_2 e^{-\lambda_1 \alpha} - \lambda_1 e^{-\lambda_2 \alpha}}{\lambda_2 - \lambda_1}\right)$$

$\Rightarrow F_Z(\alpha) =$ $\quad (Z = X + Y)$

$$f_Z(\alpha) = \frac{dF}{d\alpha} = +\frac{\lambda_2 e^{-\lambda_1 \alpha}}{\lambda_2 - \lambda_1}(+\lambda_1) + \frac{\lambda_1 e^{-\lambda_2 \alpha}}{\lambda_2 - \lambda_1}(-\lambda_2)$$

$$\Rightarrow \boxed{f_Z(\alpha) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 \alpha} - e^{-\lambda_2 \alpha}\right)} \quad \alpha > 0$$

In our case, $\lambda_1$ fro log) $\leftarrow \neq \lambda_2$

$D \sim Exp(2)$, $\varepsilon \sim Exp(10)$

- Let $D + \varepsilon = Z \Rightarrow$ $\boxed{f_Z(z) = \frac{2 \cdot 10}{10 - 2}\left(e^{-2z} - e^{-10z}\right);\ z > 0}$

$$\boxed{= \frac{5}{2}\left(e^{-2z} - e^{-10z}\right);\ z > 0}$$

(a) $P(q^* - \varepsilon < D) = P(q^* < D+\varepsilon)$

$= P(q^* < z)$

$= P(z > q^*)$

$= P(z > 0.5)$

$= \frac{5}{2}\left(\frac{e^{-1}}{2} - \frac{e^{-5}}{10}\right)$

$= \frac{5}{2}\left(\frac{0.3679}{2} - \frac{0.0067}{10}\right)$

$= \frac{5}{2}(0.18395 - 0.00067)$

$= \frac{5}{2}(0.18328)$

$= \frac{0.91640}{2}$

$= 0.45820$

$= 0.4582$

$\Rightarrow \boxed{\text{Prob} = 0.4582}$

In our case,

$D \sim \text{Exp}(2)$, $\varepsilon \sim \text{Exp}(10)$ with $\lambda_1$, $\lambda_2$ ( $\lambda_2 = 10$ made to be $\lambda_2$ & not 2 for $f_Z$ to be $>0$)

Let $D + \varepsilon = Z$

$$f_Z(z) = \frac{2 \times 10}{10-2}\left(e^{-2z} - e^{-10z}\right), \quad z > 0$$

$$= \frac{5}{2}\left(e^{-2z} - e^{-10z}\right)$$

(b) So, $P(q_{us}^2 - \varepsilon < D) = P(q_{us}^2 < D + \varepsilon)$

$= P(q_{us}^2 < Z)$

$= P(Z > q_{us}^2)$

$= P\left(Z > \frac{1+0.1}{2} = 0.55\right)$

$$= \int_{0.55}^{\infty} f_Z(z)\,dz$$

$$= \frac{5}{2}\left(\frac{e^{-1.1}}{2} - \frac{e^{-5.5}}{10}\right)$$

$$= \frac{5}{2}\left(\frac{0.3329}{2} - \frac{0.0041}{10}\right)$$

$$= \frac{5}{2}\left(0.16645 - 0.00041\right)$$

$$= \frac{5}{2} \cdot 0.16604$$

$$= \frac{0.83020}{2}$$

$= 0.41510$

$= 0.4151$

$\rightarrow$ $\boxed{\text{Prob} = 0.4151}$

(c) usd $\Rightarrow$ $1 - 2q^*_{usd} + u + \alpha \bar{F}(q^*_{usd}) = 0$ — (*)

Now, $F_z(z) = P(Z \le z)$

$\Rightarrow \bar{F}_z(z) = P(Z > z)$

$= \frac{5}{2}\left(\frac{e^{-2z}}{2} - \frac{e^{-10z}}{10}\right)$

$\hookrightarrow \alpha = \dfrac{2q^*_{usd} + 1 - u}{\bar{F}(q^*_{usd})}$ $\rightarrow \uparrow$ in $q^*_{usd}$ (numerator), $\rightarrow \downarrow$ in $q^*_{usd}$ (denominator),

$q^*_{usd}\uparrow \Rightarrow \frac{Nmr\uparrow}{Dnml\downarrow} \Rightarrow RHS\uparrow \Rightarrow LHS = \alpha\uparrow$

So, $\alpha\downarrow \Rightarrow q^*_{usd}\downarrow$ — (†)

$\Rightarrow$ (*) becomes :-

$$1.1 - 2q^*_{usd} + \alpha \cdot \frac{5}{2}\left(\frac{e^{-2q^*_{usd}}}{2} - \frac{e^{-10q^*_{usd}}}{10}\right) = 0$$

$\because q^*_{usd} > 0$,
if $\alpha > 0$,
Term 3 $> 0$

$\Rightarrow 2q^*_{usd} = 1.1 +$ Term 3

$= 1.1 + {>0}$

$> 1.1$

$\Rightarrow q^*_{usd} > 0.55$

$P(q^*_{usd} - \epsilon < D) = P(q^*_{usd} < \epsilon + D)$

$= P(q^*_{usd} < z)$

$= P(z > q^*_{usd})$

$= \bar{F}(q^*_{usd})$

$= \dfrac{2q^*_{usd} - 1.1}{\alpha}$

```python
import math
from matplotlib import pyplot as plt
import numpy as np
from scipy.optimize import fsolve
```

## Q4(c)

sympy

```python
# from sympy import Eq, exp, solve, symbols

# z = symbols('z')

# # setting up the complex equation z^2 + 1 = 0
# equation = Eq(z**2 + 1, 0)

# # solving the equation symbolically to find complex solutions
# solutions = solve(equation, z)

# print(solutions)

# def get_qty(alpha):
    # qvar = symbols('q')
    # eq = Eq(1.1- 2*qvar + alpha*2.5*(exp(-2*qvar)/2 - exp(-
10*qvar)/10), 0)
    # q = solve(eq, qvar)
    # return q

# alphavec = [0, 0.01, 0.05, 0.10, 0.20, 0.50, 1]
# qvec = []
# for alpha in alphavec:
#     q = get_qty(alpha)
#     print(alpha, q[0])
#     qvec.append(q[0])
# plt.plot(alphavec, qvec)
# plt.show()
```

fsolve

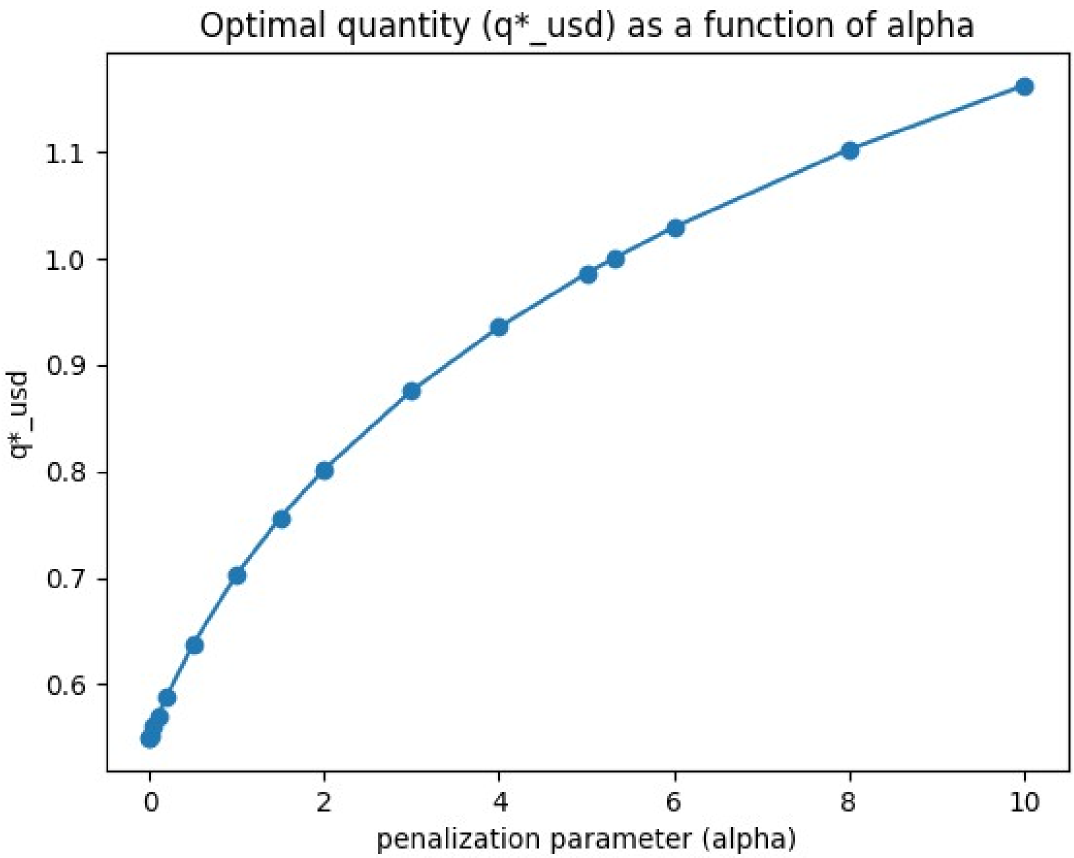
```python
def get_qty(alpha):
    return lambda q: 1.1 - 2*q + alpha*2.5*(math.exp(-2*q)/2 -
math.exp(-10*q)/10)

alphavec = [0.0001, 0.001, 0.01, 0.05, 0.10, 0.20, 0.50, 1, 1.5, 2, 3,
4, 5, 5.33, 6, 8, 10]
qvec = []
for alpha in alphavec:
    func = get_qty(alpha)
    # fsolve requires an initial guess that we put 0 here (everytime)
    q = fsolve(func, 0)
    print(alpha, q[0])
```

```python
    qvec.append(q[0])
plt.plot(alphavec, qvec)
plt.scatter(alphavec, qvec)
plt.xlabel("penalization parameter (alpha)")
plt.ylabel("q*_usd")
plt.title("Optimal quantity (q*_usd) as a function of alpha")
plt.show()
```

```
0.0001 0.5500207525052178
0.001 0.550207448340487
0.01 0.5520668580941308
0.05 0.5601697108837163
0.1 0.5699488864037865
0.2 0.5884588647330021
0.5 0.6372577593095735
1 0.7030693463334905
1.5 0.756418945622789
2 0.8015213346535381
3 0.875464466325547
4 0.9351500912986259
5 0.9854114069825046
5.33 1.0004236821903072
6 1.0289406269435826
8 1.1018991155302036
10 1.1618861037460062
```

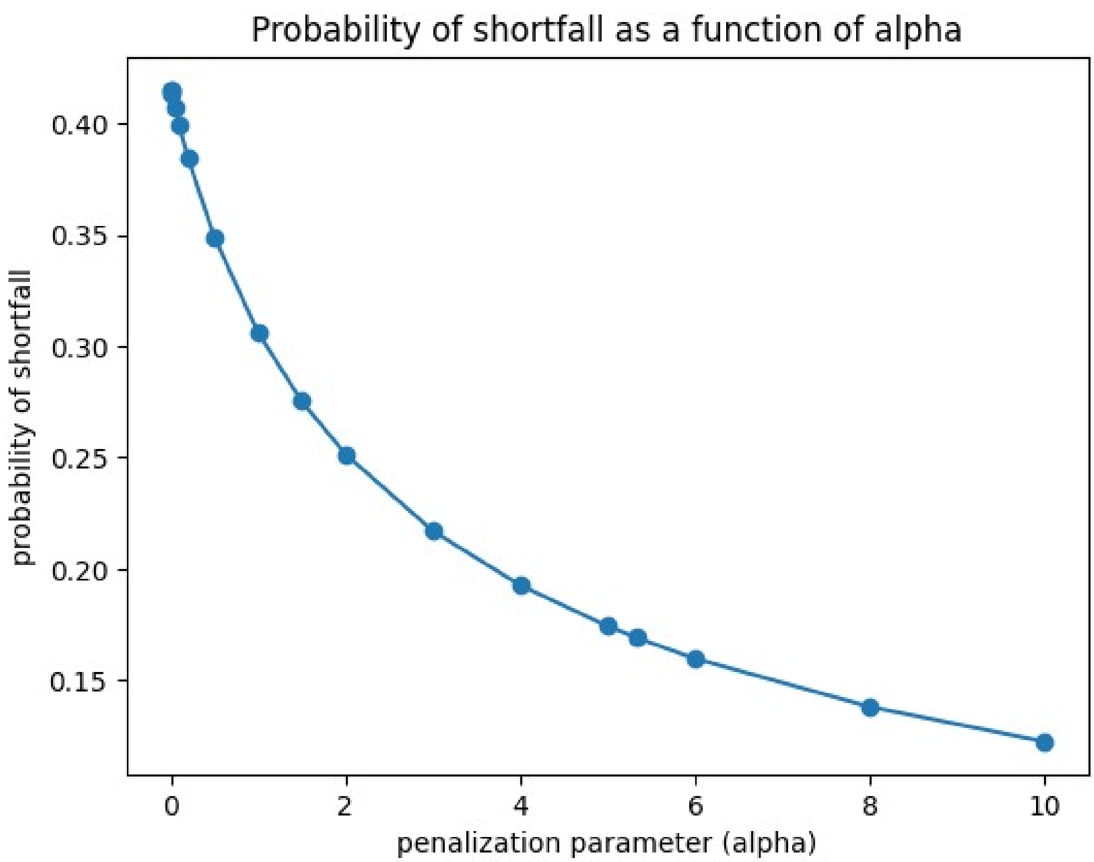
```
/var/folders/h6/k0hvphys1p5b28sdmwwfkh9r0000gn/T/
ipykernel_77742/249288800.py:2: DeprecationWarning: Conversion of an
array with ndim > 0 to a scalar is deprecated, and will error in
future. Ensure you extract a single element from your array before
performing this operation. (Deprecated NumPy 1.25.)
  return lambda q: 1.1 - 2*q + alpha*2.5*(math.exp(-2*q)/2 -
math.exp(-10*q)/10)
```




```
probvec = (2*np.array(qvec)-1.1)/np.array(alphavec)
for i in range(len(alphavec)):
    print(alphavec[i], probvec[i])
plt.plot(alphavec, probvec)
plt.scatter(alphavec, probvec)
plt.xlabel("penalization parameter (alpha)")
plt.ylabel("probability of shortfall")
plt.title("Probability of shortfall as a function of alpha")
plt.show()
```

```
0.0001 0.41505010435427536
0.001 0.4148966809738308
0.01 0.4133716188261527
0.05 0.4067884353486484
0.1 0.39897772807572895
0.2 0.38458864733002085
0.5 0.3490310372382939
1 0.3061386926669809
1.5 0.2752252608303853
2 0.2515213346535381
3 0.21697631088369795
4 0.19257504564931294
5 0.17416456279300183
```

```
5.33 0.1690145148931734
6 0.15964687564786084
8 0.13797477888255089
10 0.12237722074920124
```




## Q4 (d)

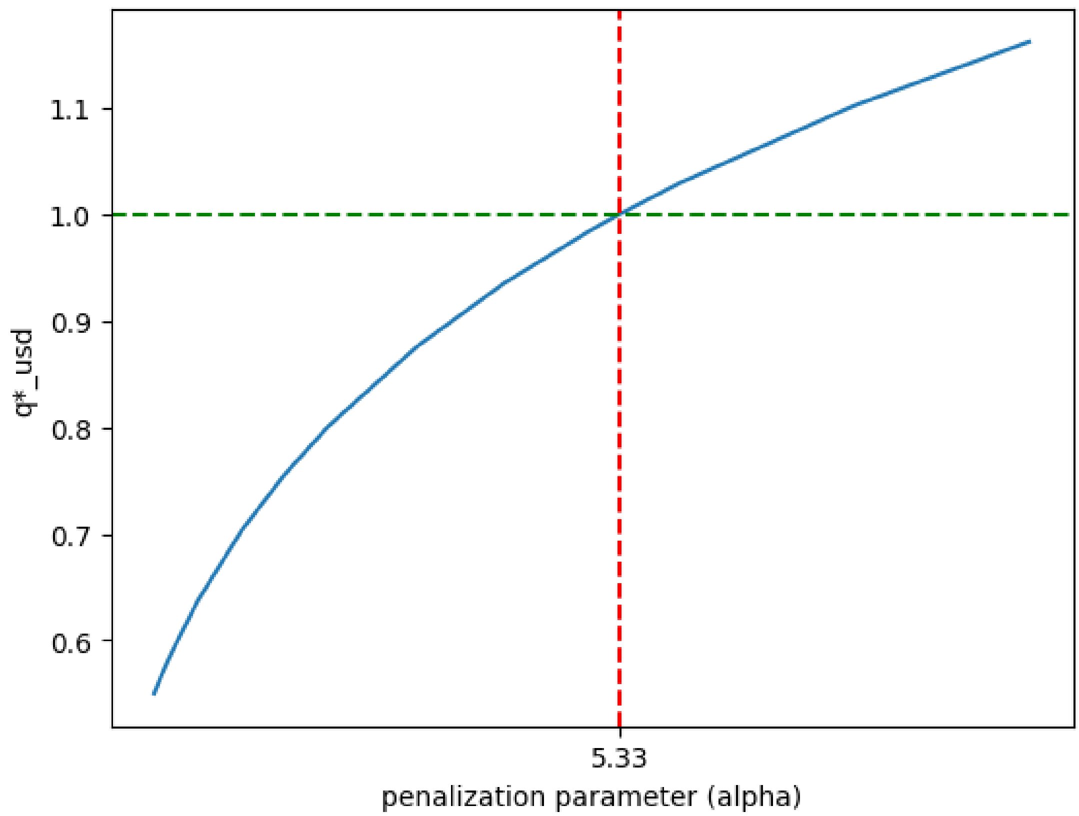
We would pick such an alpha for which:

- probability of shortfall is as low as possible (to make shortfall acceptable, we want its probability to be as low as possible since it is a negative/undesirable event)
- price>=0 (price<0 does not make sense)

So, this is how we think:

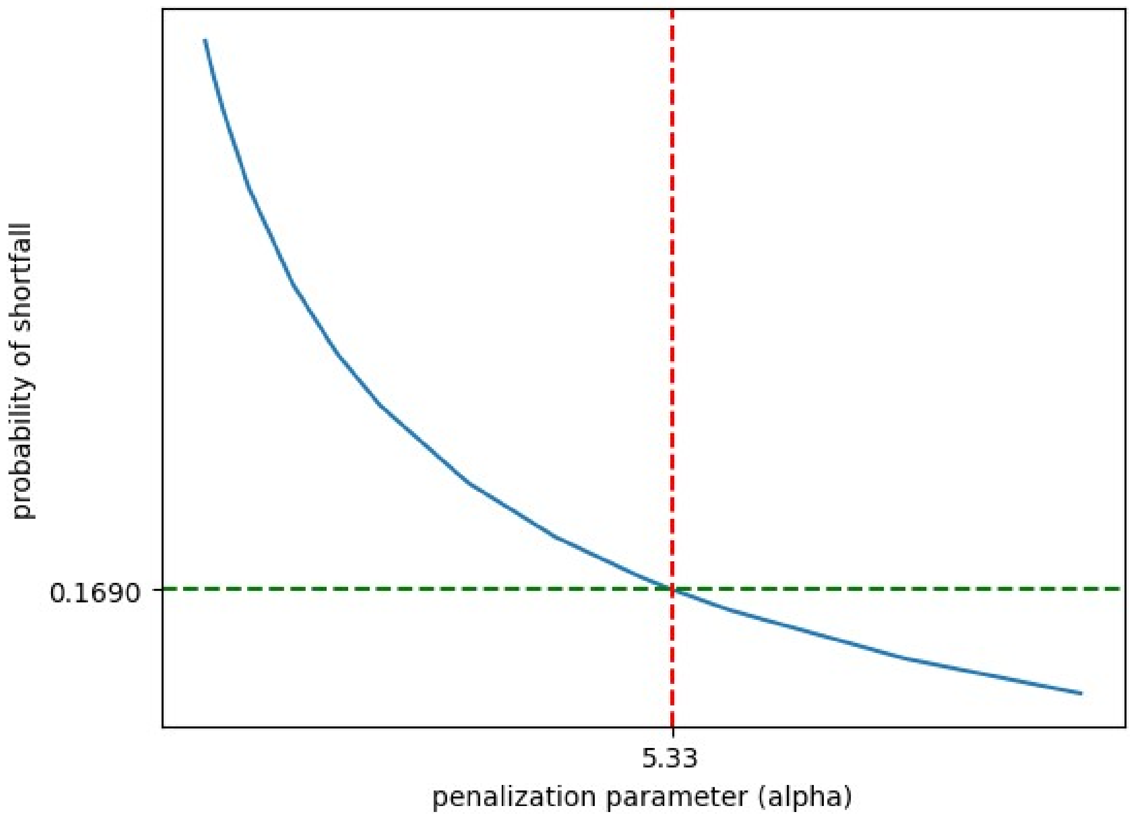
- The probability of shortfall is decreasing in alpha; so the higher the alpha, the lower the probability and the better.
- However, q*_usd is increasing in alpha. As we increase alpha, q*_usd increases.
- Since price = 1-quantity, price>=0 ---> quantity <= 1 ---> q*_usd <= 1.
- So, to minimize probability of shortfall, pick maximum possible alpha that is feasible ---> pick maximum possible quantity that is feasible ---> q*_usd = 1.

```python
plt.plot(alphavec, qvec)
plt.xlabel("penalization parameter (alpha)")
plt.ylabel("q*_usd")
plt.axhline(y = 1, color='green', linestyle='--')
plt.axvline(x = 5.33, color='red', linestyle='--')
plt.xticks([5.33], ["5.33"])
plt.show()
```




- For this q*_usd, alpha = 5.33

```python
plt.plot(alphavec, probvec)
plt.xlabel("penalization parameter (alpha)")
plt.ylabel("probability of shortfall")
plt.xticks([5.33], ["5.33"])
plt.axvline(x = 5.33, color='red', linestyle='--')
plt.axhline(y = 0.1690, color='green', linestyle='--')
plt.yticks([0.1690], ["0.1690"])
plt.show()
```




- For this alpha, probability of shortfall = 16.90%.

So, I recommend alpha = 5.33 to make the shortfall acceptable!