ORF 555, Fall 2025: Homework 1

due by Friday 9/19 at 5pm. Please upload to Gradescope.

1. Suppose in a Cournot oil market the pricing function is $P(Q) = -\log Q$, where Q is the total quantity brought to market by all the producers in a static game of N players.

   (a) What is the choke price for this pricing function, the price of the last barrel of oil?

   (b) What is the saturation quantity, when supply makes price go negative?

   (c) In a monopoly ($N = 1$), a single producer with marginal cost $c \geq 0$ optimizes revenue by solving
   $$\max_{q \geq 0} q(-\log q - c).$$
   Find the optimal monopolist quantity q^*, and show that the settlement price $P(q^*) = 1 + c$.

   (d) In a duopoly ($N = 2$) where both producers have the same cost c, what are the Nash equilibrium quantities q_i^* and price $P(Q^*)$, where each player solves
   $$\max_{q_i \geq 0} q_i(-\log Q - c), \quad i = 1, 2, \quad Q = q_1 + q_2.$$

   (e) What is the price per barrel saving to the consumers from the monopoly to the duopoly?

   (f) Compare how many barrels are sold in each case, which is larger between q^* and Q^*?

   (g) Compare revenue to the monopolist $q^*(P(q^*) - c)$ against revenue with an equal competitor.

2. In a Cournot oligopoly model, the family of power-type (or constant prudence) pricing functions is defined by
   $$P(Q) = \begin{cases} \frac{1}{1-\rho}\left(1 - Q^{1-\rho}\right), & \rho \neq 1 \\ -\log Q, & \rho = 1, \end{cases}$$
   where the parameter $\rho \in \mathbb{R}$ is a constant (known as prudence).

   (a) Plot the pricing function $P(Q)$ for $\rho = -0.5, 0, 1, 1.5$.

   (b) For any ρ, what is the saturation demand level Q for which the price becomes zero?

(c) For what values of ρ is the choke price $P(0)$ finite, and give its value.

(d) A monopolist with per-unit cost of production $0 \le c < P(0^+)$ in the static Cournot 'game' solves

$$\max_{q \ge 0} q\,(P(q) - c)\,.$$

For the power family of pricing functions, find the optimal solution q^* and the optimal profit

$$q^*\,(P(q^*) - c)\,.$$

(e) Find the unique Nash equilibrium for the 2-player Cournot game with these pricing functions with $\rho < 3$, where the players have costs $c_1, c_2 \in [0, P(0^+))$.

3. Headline April 2020: **Oil Nations, Prodded by Trump, Reach Deal to Slash Production**. *Oil-producing nations on Sunday agreed to the largest production cut ever negotiated, in an unprecedented coordinated effort by Russia, Saudi Arabia and the United States to stabilize oil prices and, indirectly, global financial markets.*

(a) In the context of Cournot models of oil prices, explain how and why slashing production 'stabilizes' oil prices, and what incentive the players involved may have for co-ordination.

(b) In a two-player static Cournot game of oil production, assume for simplicity both players have zero costs, and the pricing function is linear. In a Nash equilibrium $(q_1^*, q_2^*) \in [0,1]^2$ for the duopoly, each player maximizes profit as a best response to the other player's equilibrium strategy:

$$\Pi_i^* = \max_{q_i \ge 0} q_i(1 - q_i - q_j^*), \qquad i = 1, 2; j \ne i.$$

Derive the Nash equilibrium quantities (q_1^*, q_2^*) and find the equilibrium market oil price.

(c) Now suppose the players co-ordinate to maximize joint profits:

$$\max_{q_1, q_2 \ge 0} q_1(1 - q_1 - q_2) + q_2(1 - q_1 - q_2).$$

What then are the quantities they produce, and what is the market oil price? Is it higher or lower than the competitive price?

4. In a market for electricity production in which there is a single producer that can produce at marginal cost $c = 0$ (for instance solar), we have seen that in a Cournot market with pricing function $P(q) = 1 - q$, the optimal production quantity is $q^* = \frac{1}{2}$.

When there is uncertainty in production (trembling hand), so a committed quantity q actually results in a shortfall $q - \varepsilon$ delivered, we saw in Section 2.3.1 of the Notes that this alters the optimal production quantity to be $q^*_{us} = \frac{1}{2} + \frac{\mu}{2}$, where $\mu = \mathbb{E}\{\varepsilon\}$ (and the subscript us stands for uncertain supply). However, supply is expected to meet demand D, which is a random variable that is realized the next day. Assume that $\varepsilon \sim EXP(\lambda)$ with $\lambda = 10$, meaning $\mu = \frac{1}{\lambda} = 0.1$, so the average tremble in q is 0.1 units. The density function of ε is $f_\varepsilon(x) = \lambda e^{-\lambda x}$. Next, assume that $D \sim EXP(\eta)$ with $\eta = 2$, meaning $\mathbb{E}\{D\} = \frac{1}{\eta} = \frac{1}{2}$, so the expected demand is 0.5 units. The density function of D is $f_D(x) = \eta e^{-\eta x}$. Assume that D and ε are **independent** random variables.

(a) If the producer ignores its shortfall risk and demand obligation, what is $\mathbb{P}\{q^* - \varepsilon < D\}$? This should be a number using the parameter values given above.

(b) If the producer recognizes its shortfall risk, but is not penalized for not meeting demand, what is $\mathbb{P}\{q^*_{us} - \varepsilon < D\}$?

(c) When the producer is penalized for failing to meet demand, as in Section 2.4 of the Notes, the optimal quantity q^*_{usd} is solution to the equation

$$1 - 2q^*_{usd} + \mu + \alpha \bar{F}(q^*_{usd}) = 0,$$

where $\bar{F}$ is the complementary c.d.f of the random variable $Z := D + \varepsilon$, and the subscript usd stands for uncertain supply and demand. Plot q^*_{usd} as a function of the penalization parameter $\alpha \geq 0$. Also plot $\mathbb{P}\{q^*_{usd} - \varepsilon < D\}$ as a function of the penalization parameter $\alpha \geq 0$.

(d) What penalty α would you recommend to make shortfall "acceptable"?