

ORF 555 : HW2

$$(3+1)^2 P = 1P$$

$$(Q-1)(a) C_1 = 0, C_2 = \frac{1}{4}$$

$$P(Q) = 1 - Q = 1 - (q_1 + q_2)$$

$$(q_1^*, q_2^*) \in [0, 1]^2$$

$$\Pi_1^* = \max_{q_1 \geq 0} q_1(1 - q_1 - q_2^* - C_1)$$

$$\Pi_2^* = \max_{q_2 \geq 0} q_2(1 - q_2 - q_1^* - C_2)$$

$$\frac{\partial \Pi_1^*}{\partial q_1} = 0 \Rightarrow 1 - q_1 - q_2^* - C_1 - q_1 \Big|_{q_1=q_1^*} = 0$$

$$\Rightarrow 1 - 2q_1 - q_2^* - C_1 \Big|_{q_1=q_1^*} = 0$$

$$= 1 - 2q_1^* - q_2^* - C_1 = 0 \quad \text{--- (1)}$$

$$\text{likewise, } \frac{\partial \Pi_2^*}{\partial q_2} = 0 \Rightarrow 1 - q_1^* - 2q_2^* - C_2 = 0 \quad \text{--- (2)}$$

$$(1) + (2) \text{ gives :- where } Q^* = q_1^* + q_2^*$$

$$2 - 3Q^* - (C_1 + C_2) = 0$$

$$\Rightarrow \boxed{Q^* = \frac{2 - (C_1 + C_2)}{3}} \quad \text{--- (4)}$$

$$(1) - (2) \text{ gives :-}$$

$$q_1^* - q_2^* + C_1 - C_2 = 0$$

$$\Rightarrow \boxed{q_1^* - q_2^* = C_2 - C_1} \quad \text{--- (4a)}$$

$$(2) \text{ & (4a) give :-}$$

$$q_1^* = \frac{2 - C_1 - C_2 + 3C_2 - 3C_1}{6} = \frac{2 + 2C_2 - 4C_1}{6}$$

$$\Rightarrow \boxed{q_1^* = \frac{1 + C_2 - 2C_1}{3}}$$

$$\text{likewise, } q_2^* = \frac{2 - C_1 - C_2 - 3C_2 + 3C_1}{6}$$

$$\Rightarrow \boxed{q_2^* = \frac{2 + 2C_1 - 4C_2}{6}}$$

$$\text{So, } (q_1^*, q_2^*) = \left(\frac{1 + C_2 - 2C_1}{3}, \frac{1 + C_1 - 2C_2}{3} \right)$$

$$(3+1)^2 P = 1P \Rightarrow P = \frac{1}{3+1} = \frac{1}{4} = \frac{1}{4}$$

Plugging in values,

$$\boxed{(q_1^*, q_2^*) = \left(\frac{5}{12}, \frac{1}{6} \right) = \frac{9b}{3b}}$$

$$P = 1 - Q^* = 1 - (q_1^* + q_2^*)$$

$$\begin{aligned} &= 1 - \left(\frac{5}{12} + \frac{1}{6} \right) \\ &= 1 - \frac{7}{12} = \frac{5}{12} \end{aligned}$$

$$= \frac{5}{12}$$

Method of solving using (b)

$$\boxed{P = \frac{5}{12}}$$

• pd satisfies 3390 ft
• wing is present at
2U of reheat & this is
wing air passes through
stirred at base like 1720 ft
at ground or from tail a pd
and therefore at speed wing
wing exit location at 3.1
2U (2) of reheat is
• stirrer work & exit air will be H =

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$$\boxed{q_2^* = \frac{1 + C_1 - 2C_2}{3}}$$

$$(b) q_1 = q_1^* (1 + \epsilon)$$

$$\Rightarrow P_n = 1 - q_1^* - q_2^*$$

$$\therefore P = 1 - q_1^* - q_2^*$$

$$\Delta P = P_n - P = q_1^* - q_1 = q_1^* - q_1^*(1 + \epsilon)$$

$$\Rightarrow \Delta P = -q_1^* \epsilon$$

$$\Rightarrow \frac{dP}{d\epsilon} = -q_1^* = -\frac{5}{12}$$

(c) By similar logic,

$$\frac{dP}{d\delta} = -q_2^* = -\frac{1}{6}$$

(d) Price impact due to deviation from N.E. qty :-

OPEC > US

\Rightarrow If OPEC deviates by ϵ , some amount
the change in price, it will be harder for US to control the change in price
 \Rightarrow US ~~they~~ will need to deviate by a lot more to bring the price back to the original one i.e. to control the price
 \Rightarrow harder for ~~US~~ US
 \Rightarrow Headline is true & hence validated.

$$\frac{1}{P} = \omega, \omega = \frac{1}{P}(1 - \epsilon)$$

$$(P - P^*) - 1 = D - 1 = \Delta P$$

$$S(1, \epsilon) \rightarrow (P^*, P)$$

$$(P - P^*) - 1 = D - 1 = \Delta P$$

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$$(P - P^*) - 1 = D - 1 = \Delta P$$

$$D = P - P^* - 1$$

$$D = P - P^* - 1$$

$$D = P - P^* - 1 = \frac{\Delta P}{P}$$

$$D = (1 + \epsilon) - 1 = \Delta P$$

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$$\frac{P^* - P + \epsilon}{P} = \frac{P^* - P + \epsilon}{P} = \frac{\Delta P}{P}$$

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$$(Q.2) \frac{dS_t}{S_t} = (\mu - \delta) dt + \sigma dW_t$$

↳ Brownian Motion

$$= W_t \sim N(0, t)$$

$$\Rightarrow S_t = S_0 \exp\left((\mu - \delta - \frac{\sigma^2}{2})t + \sigma W_t\right)$$

$$\Rightarrow S_T = S_t \exp\left((\mu - \delta - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)\right)$$

(a) Now, if $W = \exp(Z)$ where $Z \sim N(\mu, \sigma^2)$,

$$E(W) = \exp(\mu + \frac{\sigma^2}{2}) \quad | \rightarrow \phi_z(t) = E(e^{tz}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$E(W^2) = \exp(2\mu + 2\sigma^2) \quad \Rightarrow E(e^{2z}) = e^{2\mu + 2\sigma^2}$$

$$\begin{aligned} \Rightarrow \text{Var}(W) &= E(W^2) - (E(W))^2 \\ &= \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) \\ &= \exp(2\mu + \sigma^2) [\exp(\sigma^2) - 1] \end{aligned}$$

In our case, ~~given~~ $S_T = \underbrace{S_t \exp((\mu - \delta - \frac{\sigma^2}{2})(T-t))}_{C} \exp(\sigma(W_T - W_t)) \underbrace{\sim N}_{N}$

\Rightarrow Given S_t , C is known

$$\Rightarrow \frac{S_T}{C} = \exp(\sigma(W_T - W_t)) = \exp(A) \text{ where } A \sim N(0, \sigma^2(T-t))$$

$$\begin{aligned} \Rightarrow \text{Var}\left(\frac{S_T}{C}\right) &= \exp(2 \cdot 0 + \sigma^2(T-t)) (\exp(\sigma^2(T-t)) - 1) \\ &= \exp(2\sigma^2(T-t)) - \exp(\sigma^2(T-t)) \end{aligned}$$

$$\Rightarrow \text{Var}(S_T) = C^2 (\exp(2\sigma^2(T-t)) - \exp(\sigma^2(T-t)))$$

$$= S_t^2 \exp(2(\mu - \delta)(T-t) - \cancel{\sigma^2(T-t)}) \exp(\sigma^2(T-t)) (\exp(\sigma^2(T-t)) - 1)$$

$$= S_t^2 \exp(2(\mu - \delta)(T-t)) (\exp(\sigma^2(T-t)) - 1)$$

$$\Rightarrow \text{Var}(S_T | S_t) = \underline{S_t^2 \exp(2(\mu - \delta)(T-t)) (\exp(\sigma^2(T-t)) - 1)}.$$

(Q.2)

⑥ $P\left(\frac{S_T - S_t}{S_t} \geq \frac{\Delta}{100}\right)$ cannot control at Δ (why?)

$$= P\left(\frac{S_t \exp((\mu - \delta - \frac{\sigma^2}{2})(T-t) + \sigma(w_T - w_t))}{S_t} \geq \frac{\Delta}{100}\right)$$

↓
2nd P of S_t words now (but it's not important)

$$= P\left(\exp((\mu - \delta - \frac{\sigma^2}{2})(T-t) + \sigma(w_T - w_t)) \geq \frac{\Delta}{100} + 1\right)$$

$$= P\left(\cancel{\mu - \delta - \sigma} \exp(\sigma(w_T - w_t)) \geq (\frac{\Delta}{100} + 1)c\right)$$

(where $c = \exp(-(\mu - \delta - \frac{\sigma^2}{2})(T-t))$)

$$= P\left(\sigma(w_T - w_t) \geq \log\left(\frac{\Delta}{100} + 1\right)c\right)$$

$$= P\left(w_T - w_t \geq \frac{1}{\sigma} \log\left(\frac{\Delta}{100} + 1\right)c\right)$$

$$= P(N(0, T-t) \geq c_1)$$

(where $c_1 = \frac{1}{\sigma} \log\left(\frac{\Delta}{100} + 1\right)c$)

$$= P(N(0, 1) \geq \frac{c_1}{\sqrt{T-t}})$$

$$= 1 - P(N(0, 1) < \frac{c_1}{\sqrt{T-t}})$$

$$= 1 - \Phi\left(\frac{c_1}{\sqrt{T-t}}\right)$$

$$= 1 - \Phi\left(\frac{1}{\sigma\sqrt{T-t}} \log\left(\frac{\Delta}{100} + 1\right)c\right)$$

$$= 1 - \Phi\left(\frac{1}{\sigma\sqrt{T-t}} \log\left(\frac{\Delta}{100} + 1\right) \exp(-(\mu - \delta - \frac{\sigma^2}{2})(T-t))\right)$$

$$= 1 - \Phi\left(\frac{\log(\frac{\Delta}{100} + 1)}{\sigma\sqrt{T-t}} - \frac{(\mu - \delta - \frac{\sigma^2}{2})\sqrt{T-t}}{\sigma}\right)$$

The answer does NOT depend on S_t because we are talking in relative terms (change is $\Delta \frac{V}{V}$) ~~is not~~

\downarrow
required to measure. $(\text{fw} - \text{rw})_0 + (\text{f-T})(\frac{\text{S}_0}{\text{S}} - b - u) q_{\text{av}} \cancel{+ R}) q =$

If the change requested (or mentioned) was absolute, then
 S_t would have been there in the picture) $q_{\text{av}}) q =$

$$\left(\rightarrow \left(1 + \frac{\Delta}{50} \right) \leq \left((\text{fw} - \text{rw})_0 \right) q_{\text{av}} \cancel{+ b - u} \right) q =$$

$$((\text{f-T})(\frac{\text{S}_0}{\text{S}} - b - u) -) q_{\text{av}} = 0 \text{ m/s}^2$$

$$\left(\left(\rightarrow \left(1 + \frac{\Delta}{50} \right) \right) \text{fut} \leq (\text{fw} - \text{rw})_0 \right) q =$$

$$\left(\left(\rightarrow \left(1 + \frac{\Delta}{50} \right) \right) \text{fut} \frac{1}{2} \leq \text{fw} - \text{rw} \right) q =$$

$$(1) \leq (f-T, 0) H) q =$$

$$\left(\left(\rightarrow \left(1 + \frac{\Delta}{50} \right) \right) \text{fut} \frac{1}{2} = 1 \right) \text{ m/s}^2$$

$$\left(\frac{12}{\text{f-T}V} \leq (1, 0) H \right) q =$$

$$\left(\frac{12}{\text{f-T}V} > (1, 0) H \right) q - 1 =$$

$$\left(\frac{12}{\text{f-T}V} \right) \phi - 1 =$$

$$\left(\left(\rightarrow \left(1 + \frac{\Delta}{50} \right) \right) \text{fut} \frac{1}{\frac{12}{\text{f-T}V} \phi} \right) \phi - 1 =$$

$$\left(((\text{f-T})(\frac{\text{S}_0}{\text{S}} - b - u) -) q_{\text{av}} \left(1 + \frac{\Delta}{50} \right) \text{fut} \frac{1}{\frac{12}{\text{f-T}V} \phi} \right) \phi - 1 =$$

$$\left(\cancel{\text{f-T}V} \frac{(\frac{\text{S}_0}{\text{S}} - b - u)}{\phi} - \frac{\left(1 + \frac{\Delta}{50} \right) \text{fut}}{\cancel{\text{f-T}V} \phi} \right) \phi - 1 =$$

(Q.2) (c)

when real world changes to Q,
replace μ by r (in real world formula)

$$\Rightarrow \frac{df(t, T)}{f(t, T)} = (r - r) dt + \sigma dW_t^Q$$

$$= \cancel{0} dt + \sigma dW_t^Q$$

$$= \underline{0} dt + \sigma dW_t^Q$$

$$\text{So, } df_t = \frac{\partial f}{\partial S_t} dS_t + \frac{\partial f}{\partial t} dt + \frac{1}{2} \left(\frac{\partial^2 f}{\partial S_t^2} \right) (dS_t)^2$$

$$= e^{(r-\delta)(T-t)} dS_t + -(r-\delta) S_t e^{(r-\delta)(T-t)} dt$$

$$= \frac{f_t}{S_t} dS_t - (r-\delta) f_t dt$$

$$\Rightarrow \frac{df_t}{f_t} = \frac{dS_t}{S_t} - (r-\delta) dt$$

$$= \sigma dW_t^Q$$

$$\Rightarrow \boxed{\frac{df_t}{f_t} = \underline{0} dt + \underline{\sigma} dW_t^Q}$$

$$\begin{aligned}
 d(\ln F) &= \frac{\partial \ln F}{\partial F} dF + \frac{\partial \ln F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 \ln F}{\partial F^2} (dF)^2 \\
 &= \frac{1}{F} dF + 0 dt + \frac{1}{2} \left(-\frac{1}{F^2} \right) \cancel{dt} \cancel{(dF)^2} \\
 \text{current time} &= \sigma dW_t^0 + \frac{1}{2} \sigma^2 \left(\frac{dF}{F} \right)^2
 \end{aligned}$$

↑

$$\text{Assuming } T < T^1, \quad = \sigma dW_t^0 - \frac{\sigma^2}{2} dt$$

$$\Rightarrow \ln F_{T^1} - \ln F_T = \sigma (W_{T^1}^0 - W_T^0) - \frac{\sigma^2}{2} (T^1 - T)$$

Assuming $T < T < T^1$,

$$\textcircled{6} \Rightarrow \frac{F(t, T^1)}{F(t, T)} = \exp \left(-\frac{\sigma^2}{2} (T^1 - T) + \sigma (W_{T^1}^0 - W_T^0) \right)$$

(Assuming $T < T^1$)

$$\sim N(0, T^1 - T)$$

$$\Rightarrow E \cancel{F}(t, T^1) = F(t, T) \exp \left(-\frac{\sigma^2}{2} (T^1 - T) \right) \exp \left(\sigma (W_{T^1}^0 - W_T^0) \right)$$

$$\Rightarrow E(F(t, T^1) | F(t, T)) = F(t, T) \exp \left(-\frac{\sigma^2}{2} (T^1 - T) \right) \exp \left(\frac{\sigma^2 (T^1 - T)}{2} \right)$$

$$= F(t, T)$$

$$\Rightarrow E(F(t, T^1) | f(t, T)) = F(t, T)$$

Hence, $F(t, T)$ is a martingale.

Yes, it is the time t price of a derivative security expiring at time $\otimes T$.

This security is forward. So, payoff = $S(T) - F(t, T)$

Stock price at time T
 (pre)determined at time t
 forward price ↑ (of a fwd expiring
 at time T)

(Q.2)

(e) Given:- $t < T < T'$

$$\text{So, } F(t, T') = S_t \exp$$

$$F(t, T') = F(t, T) \exp\left(-\frac{\sigma^2}{2}(T' - T) + \sigma(\omega_T^q, -\omega_T^q)\right)$$

But we want $F(t, T')$ in terms of $F(T, T')$ (or reverse).
Let's write $F(t, T)$ in terms of $S(T)$.

$$\text{So, } F(t, T') = S_t e^{(r-\delta)(T'-t)}$$

$$F(T, T') = S_T e^{(r-\delta)(T'-T)}$$

$$\Rightarrow \frac{F(T, T')}{F(t, T')} = \frac{S_T}{S_t} e^{(r-\delta)(T-t)}$$

$$= e^{(r-\delta)(T-t)} \cancel{+ \frac{\sigma^2}{2}} + \cancel{\sigma(\omega_T^q, \omega_T^q)} \cancel{- (r-\delta)(T-t)}$$

$$= e^{(r-\delta-\frac{\sigma^2}{2})(T-t)} + \sigma(\omega_T^q - \omega_t^q) e^{-(r-\delta)(T-t)}$$

$$= e^{-\frac{\sigma^2}{2}(T-t)} + \sigma(\omega_T^q - \omega_t^q)$$

$$\Rightarrow F(T, T') = \underline{F(t, T')} \underline{e^{-\frac{\sigma^2}{2}(T-t)}} \underline{e^{\sigma(\omega_T^q - \omega_t^q)}}.$$

(Q.5) F(1), we know that:-

$$F(t, T) = S_t e^{(r-\delta)(T-t)} \quad (\text{atleast under the risk neutral world})$$

$$\Rightarrow F(0, T) = S_0 e^{(r-\delta)T} \quad \downarrow w^q_t$$

$$80 = S_0 e^{(r-\delta) \cdot \frac{1}{6}} \quad \text{where } r = 0.05$$

$$75 = S_0 e^{(r-\delta) \frac{2}{6}}$$

$$70.5 = S_0 e^{(r-\delta) \frac{3}{6}}$$

$$67 = S_0 e^{(r-\delta) \frac{4}{6}}$$

$$64.5 = S_0 e^{(r-\delta) \frac{5}{6}}$$

$$63 = S_0 e^{(r-\delta) \frac{6}{6}}$$

$$\cancel{e^{(r-\delta) \frac{1}{6}}} \sim \cancel{\frac{15}{16}}$$

$$(Q.4) P(q) = 1 - q$$

cost of production = c

$$(a) \Pi = \max_{q \geq 0} q(1 - q - c)$$

$$\frac{d\Pi}{dq} = 0 \Rightarrow q(-1) + (1 - q - c) \Big|_{q=q^*} = 0$$

$$\Rightarrow 1 - 2q - c = 0 \Big|_{q=q^*}$$

$$\Rightarrow q = \frac{1-c}{2} \Big|_{q=q^*}$$

$$\Rightarrow q^* = \boxed{\frac{1-c}{2}}$$

(b) Reserves $\rightarrow x(t)$, $t \geq 0$

extraction rate $\rightarrow q_t > 0$

$$\text{so, } \frac{dx}{dt} = -q_t$$

$$\Rightarrow dx = -q_t dt$$

$$\Rightarrow \int_0^t dx = \int_0^t -q_t dt$$

$$\Rightarrow x(t) - x(0) = - \int_0^t q_p dp$$

$$\Rightarrow \boxed{x(t) = x(0) - \int_0^t q_p dp} \quad \text{--- (1)}$$

finite time horizon ($T > 0$)

beyond which no selling

~~Given~~ Given $x(0) = x > 0$,

$$\text{Revenue} = \boxed{v(x_t) = \int_0^T e^{-rt} q_t (1 - q_t - q_{\text{diff}})}$$

In this part, $q_t = q^*$ $\forall t \in [0, T]$

$$\Rightarrow x(t) = x(0) - \int_0^t q^* dp$$

$$= \boxed{x(0) - q^* t} \quad \text{--- (2)}$$

For reserves to be exhausted by time T ,

$$\text{Given } x(T) = 0 \text{ & } x(t) = 0 \text{ for some } t \leq T$$

$$\Rightarrow x(0) - q^* t = 0 \text{ for } t \leq T$$

$$\Rightarrow x(0) = q^* t \leq q^* T = (\frac{1-c}{2})T$$

$$\Rightarrow \boxed{\text{for } x \in [0, (\frac{1-c}{2})T]},$$

reserves will be exhausted by time T .

$$(\frac{1-c}{2}, T)_{\min} = (x_0, T)_{\min}$$

$$(\frac{1-c}{2}, T)_{\max} =$$

$$((\frac{1-c}{2}, T)_{\min} =$$

(c) For $q_t = \frac{q}{2}$ & $\alpha = \frac{2x}{q^2} < T$

$$V(x) = \int_0^T e^{-rt} \left(\frac{1-c}{2}\right)^2 dt$$

$$= \int_0^{\alpha} e^{-rt} \left(\frac{1-c}{2}\right)^2 dt + \int_{\alpha}^T e^{-rt} (1-\alpha-c) dt$$

where $\alpha = \alpha \left(\frac{1-c}{2}\right)$ (\because after this, $x=0 \Rightarrow q_t=0$)

$$\Rightarrow V(x) = \left(\frac{1-c}{2}\right)^2 \left(1 - e^{-r\alpha}\right)$$

$$V(x) = \left(\frac{1-c}{2}\right)^2 \left(1 - e^{-\frac{2rx}{1-c}}\right)$$

This is true when $\alpha < T$,

If $\alpha \left(\frac{2x}{q^2}\right) \geq T$,

$$V(x) = \int_0^T e^{-rt} \left(\frac{1-c}{2}\right)^2 dt$$

$$= \left(\frac{1-c}{2}\right)^2 \frac{1}{r} (1 - e^{-rT})$$

$$\Rightarrow V(x) = \left(\frac{1-c}{2}\right)^2 \frac{1 - e^{-rT}}{r}$$

So, $V(x) = \left(\frac{1-c}{2}\right)^2 \frac{1 - e^{-rT_f}}{r}$

where $T_f = \min(T, \frac{2x}{(1-c)})$

$$\min(T, \alpha) = \min(T, \frac{2x}{q^2})$$

$$= \min(T, \frac{x}{(1-c)/2})$$

$$= \min(T, \frac{2x}{1-c})$$

```

from matplotlib import pyplot as plt
import numpy as np
from scipy.optimize import minimize

Ftvec = np.array([80, 75, 70.5, 67, 64.5, 63])
T = np.array([1/6, 2/6, 3/6, 4/6, 5/6, 6/6])
r = 0.05

def compute_error(pmt):
    S0, delta = pmt
    v1 = S0 * np.exp((r-delta)*T)
    sse = (Ftvec-v1)@(Ftvec-v1)
    return sse

output = minimize(compute_error, [88, 0.11])
S0_final, delta_final = output["x"]

print("S0:", S0_final)
print("delta:", delta_final, "=", np.round(delta_final*100, 2), "%")

S0: 82.97307060124456
delta: 0.3476824593023124 = 34.77 %

plt.scatter(T, Ftvec, label = "Observed values", c = "red")
plt.plot(T, S0_final * np.exp((r-delta_final)*T), label = "Fitted
values")
plt.legend()
plt.xlabel("Time (in years)")
plt.ylabel("Price")
plt.title("Forward price vs time to maturity")
plt.show()

```

Forward price vs time to maturity

