ORF 555 : HW2

(Q.1) (a) $c_1 = 0$, $c_2 = \frac{1}{4}$

$$P(Q) = 1 - Q = 1-(q_1+q_2)$$

$$(q_1^*, q_2^*) \in [0,1]^2$$

$$\Pi_1^* = \max_{q_1 \geq 0} q_1(1-q_1-q_2^*-c_1)$$

$$\Pi_2^* = \max_{q_2 \geq 0} q_2(1-q_2-q_1^*-c_2)$$

$$\frac{\partial \Pi_1^*}{\partial q_1} = 0 \Rightarrow 1-q_1-q_2^*-c_1 - q_1 \Big|_{q_1=q_1^*} = 0$$

$$\Rightarrow 1-2q_1-q_2^*-c_1 \Big|_{q_1=q_1^*} = 0$$

$$= 1-2q_1^*-q_2^*-c_1 = 0 \quad -(1)$$

|||ly, $\frac{\partial \Pi_2^*}{\partial q_2} = 0 \Rightarrow 1 - q_1^* - 2q_2^* - c_2 = 0 \quad -(2)$

(1) + (2) gives :- where $Q^* = q_1^* + q_2^*$

$$2 - 3Q^* - (c_1+c_2) = 0$$

$$\Rightarrow \boxed{Q^* = \frac{2-(c_1+c_2)}{3}} \quad -(*)$$

(1) - (2) gives :-

$$q_1^* - q_2^* + c_1 - c_2 = 0$$

$$\Rightarrow \boxed{q_1^* - q_2^* = c_2 - c_1} \quad -(**)$$

(*) & (**) give :-

$$q_1^* = \frac{2-c_1-c_2+3c_2-3c_1}{6} = \frac{2+2c_2-4c_1}{6}$$

$$\Rightarrow \boxed{q_1^* = \frac{1+c_2-2c_1}{3}}$$

|||ly. $q_2^* = \frac{2-c_1-c_2-3c_2+3c_1}{6}$ $q_2^* = \frac{2+2c_1-4c_2}{6} \Rightarrow \boxed{q_2^* = \frac{1+c_1-2c_2}{3}}$

So, $(q_1^*, q_2^*) = \left(\frac{1+c_2-2c_1}{3}, \frac{1+c_1-2c_2}{3}\right)$

Plugging in values,

$$\boxed{(q_1^*, q_2^*) = \frac{5}{12}, \frac{1}{6}}$$

$$P = 1 - Q^* = 1-(q_1^*+q_2^*)$$

$$= 1-\left(\frac{5}{12}+\frac{1}{6}\right)$$

$$= 1 - \frac{7}{12}$$

$$= \frac{5}{12}$$

$$\Rightarrow \boxed{P = \frac{5}{12}}$$

(b) $q_1 = q_1^*(1+\epsilon)$

$\Rightarrow P_n = 1 - q_1 - q_2^*$

$\because P = 1 - q_1^* - q_2^*$

$\Delta P = P_n - P = q_1^* - q_1 = q_1^* - q_1^*(1+\epsilon)$

$\Rightarrow \Delta P = -q_1^* \epsilon$

$\Rightarrow \boxed{\dfrac{dP}{d\epsilon} = -q_1^* = -\dfrac{5}{12}}$

(c) By similar logic,

$\boxed{\dfrac{dP}{d\delta} = -q_2^* = -\dfrac{1}{6}}$

(d) Price impact due to deviation from N.E. qty:-

$\boxed{\text{OPEC} > \text{US}}$

$\Rightarrow$ If OPEC deviates by some amount,

~~the change in price~~,

it ~~will be harder for US~~

to ~~control the change in~~ price

$\Rightarrow$ US ~~they~~ will need to deviate by a lot more to bring the price back to the original one i.e. to control the price

$\Rightarrow$ harder for US

$\Rightarrow$ Headline is true & hence validated.

(Q.2) $\frac{dS_t}{S_t} = (\mu-\delta)\,dt + \sigma\, dW_t$

$\hookrightarrow$ Brownian Motion

$\Rightarrow W_t \sim N(0,t)$

$\Rightarrow S_t = S_0 \exp\left(\left(\mu-\delta-\frac{\sigma^2}{2}\right)t + \sigma W_t\right)$

$\Rightarrow S_T = S_t \exp\left(\left(\mu-\delta-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right)$

(a) Now, if $W = \exp(z)$ where $z \sim N(\mu, \sigma^2)$,

$E(W) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$

$E(W^2) = \exp(2\mu + 2\sigma^2)$

$\phi_z(t) = E(e^{tz}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$\Rightarrow E(e^{2z}) = e^{2\mu + 2\sigma^2}$

$\Rightarrow E(W^2) = e^{2\mu + 2\sigma^2}$

$\Rightarrow \operatorname{var}(W) = E(W^2) - (E(W))^2$

$= \exp(2\mu+2\sigma^2) - \exp(2\mu+\sigma^2)$

$= \exp(2\mu+\sigma^2)\left[\exp(\sigma^2)-1\right]$

In our case, ~~given S_t~~ $S_T = \underbrace{S_t \exp\left(\left(\mu-\delta-\frac{\sigma^2}{2}\right)(T-t)\right)}_{C} \underbrace{\exp(\sigma(W_T-W_t))}_{N}$

$\Rightarrow$ Given S_t, C is known

$\Rightarrow \frac{S_T}{C} = \exp(\sigma(W_T - W_t)) = \exp(A)$ where $A \sim N(0, \sigma^2(T-t))$

$\Rightarrow \operatorname{var}\left(\frac{S_T}{C}\right) = \exp(2\cdot 0 + \sigma^2(T-t))(\exp(\sigma^2(T-t))-1)$

$= \exp(2\sigma^2(T-t)) - \exp(\sigma^2(T-t))$

$\Rightarrow \operatorname{Var}(S_T) = C^2\left(\exp(2\sigma^2(T-t)) - \exp(\sigma^2(T-t))\right)$

$= S_t^2 \exp\left(2(\mu-\delta)(T-t) - \cancel{\sigma^2(T-t)}\right)\exp\left(\cancel{\sigma^2(T-t)}\right)\left(\exp(\sigma^2(T-t))-1\right)$

$= S_t^2 \exp(2(\mu-\delta)(T-t))\left(\exp(\sigma^2(T-t))-1\right)$

$\Rightarrow \operatorname{var}(S_T|S_t) = S_t^2 \exp(2(\mu-\delta)(T-t))\left(\exp(\sigma^2(T-t))-1\right)$

(Q.2)

(b) $P\left(\frac{S_T - S_t}{S_t} \geq \frac{\Delta}{100}\right)$

$$= P\left(\frac{\cancel{S_t}\exp\left(\left(\mu-\delta-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right) - \cancel{S_t}}{\cancel{S_t}} \geq \frac{\Delta}{100}\right)$$

↓ Ind P of S_t

$$= P\left(\exp\left(\left(\mu-\delta-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right) \geq \frac{\Delta}{100}+1\right)$$

$$= P\left(\cancel{\mu-\delta-\sigma}\ \exp(\sigma(W_T - W_t)) \geq \left(\frac{\Delta}{100}+1\right)c\right)$$

(where $c = \exp\left(-\left(\mu-\delta-\frac{\sigma^2}{2}\right)(T-t)\right)$)

$$= P\left(\sigma(W_T - W_t) \geq \log\left(\left(\frac{\Delta}{100}+1\right)c\right)\right)$$

$$= P\left(W_T - W_t \geq \frac{1}{\sigma}\log\left(\left(\frac{\Delta}{100}+1\right)c\right)\right)$$

$$= P\left(N(0, T-t) \geq c_1\right)$$

(where $c_1 = \frac{1}{\sigma}\log\left(\left(\frac{\Delta}{100}+1\right)c\right)$)

$$= P\left(N(0,1) \geq \frac{c_1}{\sqrt{T-t}}\right)$$

$$= 1 - P\left(N(0,1) < \frac{c_1}{\sqrt{T-t}}\right)$$

$$= 1 - \Phi\left(\frac{c_1}{\sqrt{T-t}}\right)$$

$$= 1 - \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\log\left(\left(\frac{\Delta}{100}+1\right)c\right)\right)$$

$$= 1 - \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\log\left(\left(\frac{\Delta}{100}+1\right)\exp\left(-\left(\mu-\delta-\frac{\sigma^2}{2}\right)(T-t)\right)\right)\right)$$

$$= 1 - \Phi\left(\frac{\log\left(\frac{\Delta}{100}+1\right)}{\sigma\sqrt{T-t}} - \frac{\left(\mu-\delta-\frac{\sigma^2}{2}\right)\sqrt{T-t}}{\sigma}\right)$$

The answer does NOT depend on S_t because we are talking in relative terms (change is $\Delta \%$) ~~& not~~

required to measure.

If the change requsted (or mentioned) was absolute, S_t would have been there in the picture.

(Q.2) (c)

when real world changes to Q,
replace μ by r (in real world formula)

$$\Rightarrow \frac{df(t,T)}{f(t,T)} = (r-r)\,dt + \sigma\, dW_t^Q$$

$$\cancel{= \sigma\, dW_t^Q}$$

$$= \underline{0}\,dt + \sigma\, dW_t^Q$$

$$\text{So, } df_t = \frac{\partial f}{\partial S_t} dS_t + \frac{\partial f}{\partial t} dt + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial S_t^2}}_{\to 0}(dS_t)^2$$

$$= e^{(r-\delta)(T-t)} dS_t + -(r-\delta) S_t e^{(r-\delta)(T-t)} dt$$

$$= \frac{f_t}{S_t} dS_t - (r-\delta) f_t\, dt$$

$$\Rightarrow \frac{df_t}{f_t} = \frac{dS_t}{S_t} - (r-\delta)dt$$

$$= \sigma\, dW_t^Q$$

$$\Rightarrow \boxed{\frac{df_t}{f_t} = \underline{0}\,dt + \underline{\sigma}\, dW_t^Q}$$

(Q-2)(d) $d(\ln F) = \frac{\partial \ln F}{\partial F} dF + \frac{\partial \ln F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 \ln F}{\partial F^2} (dF)^2$

$= \frac{1}{F} dF + 0\,dt + \frac{1}{2}\left(-\frac{1}{F^2}\right) \cancel{\sigma^2 dt}\ (dF)^2$

$= \sigma\, dW^q_t + \frac{1}{2} \cdot -\left(\frac{dF}{F}\right)^2$

current time

$= \sigma\, dW^q_t - \frac{\sigma^2}{2} dt$

Assuming $T<T'$,

$\Rightarrow \ln F_{T'} - \ln F_T = \sigma(W^q_{T'} - W^q_T) - \frac{\sigma^2}{2}(T' - T)$

Assuming $t < T < T'$,

(b) $\Rightarrow \frac{F(t,T')}{F(t,T)} = \exp\left(-\frac{\sigma^2}{2}(T'-T) + \sigma(W^q_{T'} - W^q_T)\right)$

(~~Assuming~~ $T<T'$)

$\sim N(0, T'-T)$

$\Rightarrow$ ~~E~~ $F(t,T') = F(t,T) \exp\left(-\frac{\sigma^2}{2}(T'-T)\right) \exp\left(\sigma\left(W^q_{T'} - W^q_T\right)\right)$

$\Rightarrow E(F(t,T') \mid F(t,T)) = F(t,T) \exp\left(-\cancel{\frac{\sigma^2}{2}(T'-T)}\right) \exp\left(\cancel{\frac{\sigma^2(T'-T)}{2}}\right)$

$= F(t,T)$

$\Rightarrow E(F(t,T') \mid F(t,T)) = F(t,T)$

Hence, $F(t,T)$ is a martingale.

Yes, it is the time t price of a derivative security expiring at time T.

This security is forward. So, payoff $= S(T) - F(t,T)$

Stock price at time T

(price) determined at time t

forward price (of a fwd expiring at time T)

(Q.2) (d)

Now, ~~F(t,T) =~~

we have :- $d(\ln F) = \sigma\, dW_t^q - \frac{\sigma^2}{2} dt$

consider a forward expiring at time $T \to F(\cdot,\cdot) = F(\cdot, T)$

we look at its price at 2 different times $s < t < T$

$$\Rightarrow \ln F(t,T) - \ln F(s,T)$$

$$= \int_s^t \sigma\, dW_t^q - \int_s^t \frac{\sigma^2}{2} dt$$

$$= \sigma(W_t^q - W_s^q) - \frac{\sigma^2}{2}(t-s)$$

$$\Rightarrow F(t,T) = F(s,T) \exp\left(-\frac{\sigma^2}{2}(t-s)\right) \exp(\sigma \underbrace{(W_t^q - W_s^q)}_{\sim N(0,\, t-s)})$$

So, $E(F(t,T) \mid F(s,T)) = F(s,T) \cancel{\exp\left(-\frac{\sigma^2}{2}(t-s)\right)} \cancel{\exp\left(\frac{\sigma^2(t-s)}{2}\right)}$

$$= F(s,T)$$

$$\Rightarrow \boxed{E(F(t,T) \mid F(s,T)) = F(s,T)} \quad (s < t)$$

$\Rightarrow$ So, $F(t,T)$ is a martingale

Yes, it is the time t price of a derivative security expiring at time T.

This derivative is a forward.

So, payoff $= S(T) - F(t,T)$

$S(T)$ → stock price at time T

$F(t,T)$ → fwd price (predetermined at time t) (for a fwd expiring at time T)

(Q.2)

(e) Given:- $t < T < T'$

~~So, $F(T,T') = S_T \exp($~~

$$F(t,T') = F(t,T) \exp\left(-\frac{\sigma^2}{2}(T'-T) + \sigma(W^q_{T'} - W^q_T)\right)$$

But we want $F(t,T')$ in terms of $F(T,T')$ (or reverse).
Let's write $F(t,T)$ in terms of $S(T)$.

$$\text{So, } F(t,T') = S_t \, e^{(r-\delta)(T'-t)}$$

$$F(T,T') = S_T \, e^{(r-\delta)(T'-T)}$$

$$\Rightarrow \frac{F(T,T')}{F(t,T')} = \frac{S_T}{S_t} e^{-(r-\delta)(T-t)}$$

$$\cancel{= e^{(r-\delta-\frac{\sigma^2}{2})(T-t) + \sigma(W^q_T - W^q_t)} \, e^{-(r-\delta)(T-t)}}$$

$$\cancel{= e^{\sigma(W^q_T - W^q_t)}}$$

$$= e^{(r-\delta-\frac{\sigma^2}{2})(T-t) + \sigma(W^q_T - W^q_t)} \, e^{-(r-\delta)(T-t)}$$

$$= e^{-\frac{\sigma^2}{2}(T-t) + \sigma(W^q_T - W^q_t)}$$

$$\Rightarrow F(T,T') = \underline{F(t,T') e^{-\frac{\sigma^2}{2}(T-t)}} \, \underline{e^{\sigma(W^q_T - W^q_t)}}$$

(Q.3) F(I), we know that:-

$$F(t,T) = S_t e^{(r-\delta)(T-t)}$$ (atleast under the risk neutral world)

W^Q_t

$$\Rightarrow F(0,T) = S_0 e^{(r-\delta)T}$$

$$80 = S_0 e^{(r-\delta)\cdot\frac{1}{6}}$$

where $r = 0.05$

$$75 = S_0 e^{(r-\delta)\frac{2}{6}}$$

$\cancel{\frac{1}{4} e^{(r-\delta)\frac{1}{6}} \sim \frac{15}{16}}$

$$70.5 = S_0 e^{(r-\delta)\frac{3}{6}}$$

$$67 = S_0 e^{(r-\delta)\frac{4}{6}}$$

$$64.5 = S_0 e^{(r-\delta)\frac{5}{6}}$$

$$63 = S_0 e^{(r-\delta)\frac{6}{6}}$$

(Q.4) $P(q) = 1 - q$

cost of production $= c$

(a) $\pi = \max_{q \geq 0} q(1-q-c)$

$\frac{d\pi}{dq} = 0 \Rightarrow q(-1) + (1-q-c)\Big|_{q=q^*} = 0$

$\Rightarrow 1 - 2q - c\Big|_{q=q^*} = 0$

$\Rightarrow q = \frac{1-c}{2}\Big|_{q=q^*}$

$\Rightarrow \boxed{q^* = \frac{1-c}{2}}$

(b) Reserves $\to x(t)$, $t \geq 0$

extraction rate $\to q_t \geq 0$

So, $\frac{dx}{dt} = -q_t$

$\Rightarrow dx = -q_t \, dt$

$\Rightarrow \int_0^t dx = \int_0^t -q_t \, dt$

$\Rightarrow x(t) - x(0) = -\int_0^t q_p \, dp$

$\Rightarrow \boxed{x(t) = x(0) - \int_0^t q_p \, dp}$ — ①

finite time horizon $(T > 0)$ beyound which no selling

$\Rightarrow$ Given $x(0) = x > 0$,

Revenue $= \boxed{v(x) = \int_0^T e^{-rt} q_t (1 - q_t - c) \, dt}$

In this part, $q_t = q^* \;\; \forall t \in [0, T]$

$\Rightarrow x(t) = x(0) - \int_0^t q^* \, dp$

$= \boxed{x(0) - q^* t}$ — ②

For reserves to be exhausted by time T,

$x(T) = 0$ & $x(t) = 0$ for some $t \leq T$

$\Rightarrow x(0) - q^* t = 0$ for $t \leq T$

$\Rightarrow x(0) = q^* t \leq q^* T = \frac{(1-c)T}{2}$

$\Rightarrow$ for $\boxed{x \in \left[0, \frac{(1-c)T}{2}\right]}$,

reserves will be exhausted by time T.

(c) For $\boxed{q_t = q^*}$ & $\boxed{\alpha = \frac{x}{q^*} < T}$ $\rightarrow \left(\frac{1-c}{2}\right)$

$$V(x) = \int_0^T e^{-rt}\left(\frac{1-c}{2}\right)^2 dt$$

$$= \int_0^\alpha e^{-rt}\left(\frac{1-c}{2}\right)^2 dt + \int_\alpha^T e^{-rt}\, 0 \cdot (1-0-c)\, dt$$

where $x = \alpha\left(\frac{1-c}{2}\right)$ ($\because$ after this, $x_t = 0 \Rightarrow q_t = 0$)

$$\Rightarrow \alpha = \frac{2x}{1-c}$$

$$\Rightarrow V(x) = \left(\frac{1-c}{2}\right)^2\left(\frac{1-e^{-r\alpha}}{r}\right)$$

$$\boxed{V(x) = \left(\frac{1-c}{2}\right)^2\left(\frac{1-e^{-\frac{2rx}{1-c}}}{r}\right)}$$

This is true when $\alpha < T$,

If $\boxed{\alpha \left(= \frac{x}{q^*}\right) \geq T}$,

$$V(x) = \int_0^T e^{-rt}\left(\frac{1-c}{2}\right)^2 dt$$

$$= \left(\frac{1-c}{2}\right)^2 \cdot \frac{1}{r}\left(1-e^{-rT}\right)$$

$$\Rightarrow \boxed{V(x) = \left(\frac{1-c}{2}\right)^2 \frac{1-e^{-rT}}{r}}$$

So, $\boxed{V(x) = \left(\frac{1-c}{2}\right)^2 \frac{1-e^{-rT_f}}{r}}$

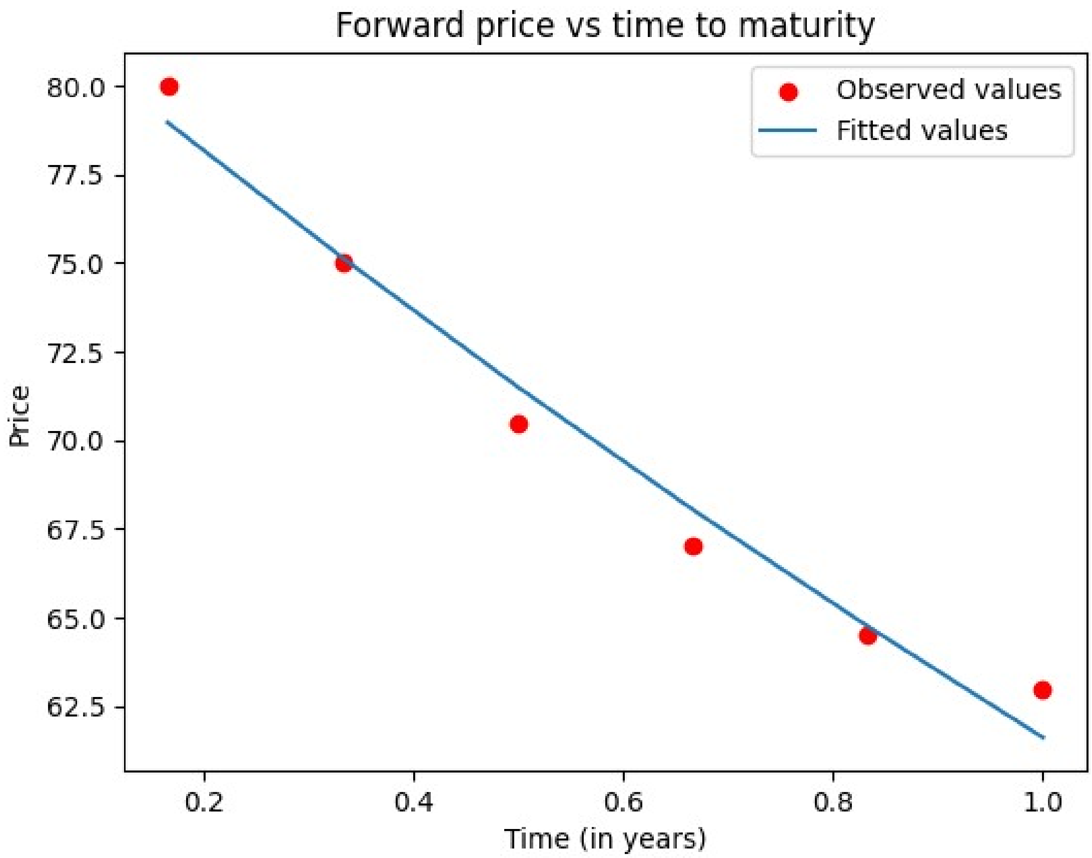
where $T_f = \min\left(T, \frac{2x}{(1-c)}\right) \Rightarrow \left(\min(T, \alpha) = \min\left(T, \frac{x}{q^*}\right)\right.$

$$= \min\left(T, \frac{x}{(1-c)/2}\right)$$

$$\left.= \min\left(T, \frac{2x}{1-c}\right)\right)$$

$\frac{x}{q^*}$

```python
from matplotlib import pyplot as plt
import numpy as np
from scipy.optimize import minimize

Ftvec = np.array([80, 75, 70.5, 67, 64.5, 63])
T = np.array([1/6, 2/6, 3/6, 4/6, 5/6, 6/6])
r = 0.05

def compute_error(pmt):
    S0, delta = pmt
    v1 = S0 * np.exp((r-delta)*T)
    sse = (Ftvec-v1)@(Ftvec-v1)
    return sse

output = minimize(compute_error, [88, 0.11])
S0_final, delta_final = output["x"]

print("S0:", S0_final)
print("delta:", delta_final, "=", np.round(delta_final*100, 2), "%")

S0: 82.97307060124456
delta: 0.3476824593023124 = 34.77 %

plt.scatter(T, Ftvec, label = "Observed values", c = "red")
plt.plot(T, S0_final * np.exp((r-delta_final)*T), label = "Fitted
values")
plt.legend()
plt.xlabel("Time (in years)")
plt.ylabel("Price")
plt.title("Forward price vs time to maturity")
plt.show()
```


Forward price vs time to maturity
Observed values
Fitted values
Price
80.0
77.5
75.0
72.5
70.0
67.5
65.0
62.5
0.2
0.4
0.6
0.8
1.0
Time (in years)