ORF 555, Fall 2025: Homework 2

due by Friday 10/3 at 11.59pm: upload to Gradescope.

Full credit is for the answer and explanation/details.

Rules for all Homeworks

- *You may (and are encouraged to) discuss homework problems with others, but the written-up solutions you turn in must be your own.*
- *You must acknowledge,* **by stating their names on your solutions**, *the people (if any) you collaborated with.*
- *All homeworks count and must be turned in on time. Extensions are granted only if requested from the Professor at least 24 hours before the deadline, and only for* **unforseeable** *circumstances.*
- *Please write-up your solutions cleanly so that it is easy for the graders to identify your answer to the questions and assign credit. For example, box or underline your final answer, but also explain how you got there.*

1. Headline October 2021 (FT): **US shale drillers cannot contain oil price rise, Pioneer boss says**. *US oil producers are not able to increase supply to tame soaring crude prices that remain "under OPEC control", according to the shale patch's biggest operator.*

   (a) In a two-player static Cournot game of oil production, player 1 OPEC has cost of production $c_1 = 0$, and player 2 US has cost of production $c_2 = 1/4$. The pricing function is linear: $P(Q) = 1 - Q$, where $Q = q_1 + q_2$. In a Nash equilibrium $(q_1^*, q_2^*) \in [0,1]^2$ for the duopoly, each player maximizes profit as a best response to the other player's equilibrium strategy:

   $$\Pi_i^* = \max_{q_i \geq 0} q_i(1 - q_i - q_j^* - c_i), \qquad i = 1, 2; j \neq i.$$

   Give the Nash equilibrium quantities (q_1^*, q_2^*) and the equilibrium market oil price P (you can quote formulas for these without deriving and plug in the appropriate numbers).

   (b) Now suppose OPEC deviates from the Nash equilibrium to increase its production by a fraction $\varepsilon > 0$, so it in fact produces

   $$q_1 = q_1^*(1 + \varepsilon).$$

What is its price impact $dP/d\varepsilon$? (This should be a number).

(c) Suppose instead that the US deviates from the Nash equilibrium to increase its production by a fraction $\delta > 0$, so it in fact produces

$$q_2 = q_2^*(1+\delta).$$

What is its price impact $dP/d\delta$? (This should be a number).

(d) How do the two price impacts compare and so validate the headline quote?

2. In the GBM model for a commodity spot price with convenience yield δ (or for a stock paying continuous dividend rate δ), we have

$$\frac{dS_t}{S_t} = (\mu - \delta)\, dt + \sigma\, dW_t, \tag{1}$$

where W is a (real world) Brownian motion. We know that for $T > t$,

$$\mathbb{E}\{S_T \mid S_t\} = S_t e^{(\mu-\delta)(T-t)}.$$

(a) Find the conditional variance $\operatorname{var}\{S_T \mid S_t\}$.

(b) Find the probability that S increases by at least $\Delta\%$ between time t and T. Does the answer depend on the value S_t, and why (or not)?

(c) In this GBM (with convenience yield) spot price model

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma\, dW_t^Q$$

in the risk neutral world Q. The forward price is given by $F(t,T) = S_t e^{(r-\delta)(T-t)}$. In the real world where the spot S follows (1) above, we found in class that

$$\frac{dF(t,T)}{F(t,T)} = (\mu - r)\, dt + \sigma\, dW_t,$$

(from which we concluded no Samuelson effect in this model). What are the dynamics of $F(t,T)$ in the risk neutral world, *i.e.* find $\cdots$ in the expression

$$\frac{dF(t,T)}{F(t,T)} = \cdots\, dt + \cdots\, dW_t^Q.$$

(d) Is $F(t,T)$ a martingale in the risk neutral world? If so, is it the time t price of a derivative security expiring at time T, and if yes, what is the payoff of that security?

(e) Let (t, T, T') be three times with $t < T < T'$. Find an expression for $F(T, T')$ in terms of $F(t, T')$ and W^Q. It should not involve S or δ. (This will be useful when we consider the price at time t of an option to enter at time $T > t$ a forward contract that matures at time $T' > T$.)

3. Fitting a model to crude oil forward data. The interest rate is fixed at $r = 5\%$. The spot price is unobservable, but we observe the current ($t = 0$) one-year forward curve (in increments of 2 months):

Forward Maturity (T)	2m	4m	6m	8m	10m	1yr
Forward price	80	75	70.5	67	64.5	63

Using the GBM one factor model for spot oil

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma\, dW_t^Q,$$

find the parameters (and S_0) which best fit the observed forward curve. Use a simple least squares fit, minimizing the sum of the squared differences between model and observed prices. What are the fitted parameters δ and σ? Plot the data and the fitted curve.

4. Suppose there is a single producer in a Cournot market where the pricing function is $P(q) = 1 - q$. The producer has per-unit cost of production $c \in [0, 1)$.

(a) What is the producer's optimal quantity q^* in the *static* scenario where they solve

$$\max_{q \geq 0} q(1 - q - c).$$

(b) In a dynamic version of the problem, the producers has reserves $x(t)$ at times $t \geq 0$ and may choose an extraction rate $q_t \geq 0$ as long as they have reserves $x(t) > 0$:

$$\frac{dx}{dt} = -q_t.$$

However there is also imposed a finite time horizon $T > 0$ beyond which they cannot sell. So, given a strategy $(q_t)_{0 \leq t \leq T}$, the revenue obtained, as a function of initial reserve $x(0) = x > 0$, is

$$v(x) = \int_0^T e^{-rt} q_t (1 - q_t - c)\, dt,$$

where future cash flows are discounted at rate $r > 0$. The producer chooses to repeatedly follow the static strategy found previously, that is $q_t = q^*$ for all $t \in [0, T]$. That is, they do not optimize but follow that strategy. As they have to stop at time T, for what values of initial reserves x are the reserves exhausted by time T?

(c) Give the value for this strategy $v(x)$ for any $x > 0$.