ORF 555, Fall 2025: Homework 3

due by Friday 10/31 at 11:59pm: please upload to Gradescope

1. We re-visit the model of a producer of an exhaustible resource (oil, say), first when they have a monopoly over energy production, and second when they have competition from an inexhaustible producer (solar, say). In monopoly, the oil producer chooses an extraction rate $q \geq 0$ to draw from reserves $x(t)$ at time t, with assumed cost of extracting equal to zero. The rate will be a function of the amount left at time t, thus

$$\frac{dx}{dt} = -q(x(t)), \quad \text{as long as} \quad x(t) > 0.$$

As usual, we denote by x the initial reserves level $x(0)$. The market price is given by $1 - q(x(t))$, and the producer's objective function is

$$J(q) := \int_0^{\tau_x^q} e^{-rt} q(x(t)) \left(1 - q(x(t))\right) dt,$$

where τ_x^q is the exhaustion time from initial reserves x under the strategy q: the time at which reserves run out, $x(\tau_x^q) = 0$.

   (a) Time is measured in years. If we value \$100 received in one year's time as \$90 today, what is the (continuously compounded) discount rate r that captures lost interest and other investment opportunities from tying up capital for that year. From now on, take $r = 0.1$.

   (b) Suppose the producer acts as if they have infinite reserves and chooses constant extraction rate $q = 1/2$. However they actually begin with reserves $x(0) = 10$ units (for example, barrels). How many years will it take them to exhaust their reserves under the simple $q = 1/2$ strategy? What is the lifetime value of this well using this strategy, that is $J(q)$?

   (c) Plot the optimal strategy $q^*(x)$ (from class and in the Notes) as a function of x, as well as the previous strategy $q = 1/2$ on the same graph. What is the exhaustion time $\tau_{10}^{q^*}$ for the optimal strategy starting with reserves $x(0) = 10$?

   (d) What is the optimal value of the well, that is $v(10)$ using the formula given in the Notes.

2. Now suppose there is competition from a single renewable energy producer such as solar, whose resources are inexhaustible but have higher cost $c > 0$ per unit of energy produced. Again, $x(t)$ denotes remaining oil reserves at time t. Player 1, oil, extracts

(and delivers energy) from reserves at rate $q_1(x(t))$, and player 2, solar, produces energy at rate $q_2(x(t))$:

$$\frac{dx}{dt} = -q_1(x(t)), \quad \text{as long as} \quad x(t) > 0,$$

and we assume oil extraction policies have a *clean finish*, namely they ramp down to zero production as oil runs out: $q_1(0) = 0$. The players interact in Cournot competition, with the market energy price being $1 - q_1 - q_2$. Their value functions are, respectively:

$$u(x) = \sup_{q_1 \geq 0} \int_0^\eta e^{-rt} q_1(x(t)) \left[1 - q_1(x(t) - q_2(x(t))\right] dt,$$

$$w(x) = \sup_{q_2 \geq 0} \left\{ \int_0^\eta e^{-rt} q_2(x(t)) \left(1 - q_1(x(t) - q_2(x(t)) - c\right) dt \right.$$
$$\left. + \int_\eta^\infty e^{-rt} q_2(x(t)) \left[1 - q_2(x(t)) - c\right] dt \right\},$$

and η is the oil exhaustion time: the time at which reserves run out, $x(\eta) = 0$. As in the previous question, assume $r = 0.1$ and $x(0) = 10$, and use formulas in the Notes, but indicate which ones you are using where, and what you are calculating.

(a) For what value of c is the blockading point x_b where solar becomes competitive equal to 5? Assume this value for c from now on.

(b) What is the optimal value of the well, that is $u(10)$ using the formula given in the Notes.

(c) What is the exhaustion time η of oil reserves under this competition?

(d) Plot the optimal strategies $q_1^*(x)$ and $q_2^*(x)$ as functions of x on the same graph.

(e) Finally, plot the trajectory of the market price of energy

$$P(t) := 1 - q_1(x^*(t)) - q_2(x^*(t)),$$

as a function of time for $0 \leq t \leq \eta$. Here $x^*(t)$ means the Nash equilibrium trajectory when player 1 is using $q_1^*(x)$. What is the price for $t > \eta$? Does the price decrease when solar enters due to competition?

3. Download the natural gas and copper futures price data, which covers most of the past 4 years. Each row represents one day. Some places are blank (N/A), since there is no future with that maturity trading on that date. In the file, CLx and Cx mean maturity x months from the trading date. The goal is to categorize the shape of the forward curves (of both commodities) and how they have varied over time. Three basic shapes are: backwardation, contango and humped.

(a) Look at plots of the forward curves over time and decide if any additional categories of shape have occurred often enough to be included as a separate category. For instance: humped with peak and humped with trough. Then write a code that can classify the shape of each day's curve as being in which of your categories. Give a table of what % of time over the whole sample the curve (for each commodity) has been in each of your categories.

(b) Now plot how those percentages in each category have changed each year.

(c) Are there any structural changes when the curves flipped from one regular shape to another? Are these associated with any specific geopolitical/news events?

(d) Construct a measure of the slope of the forward curves which assigns one number to each day. Explain how this was done, and plot the daily slope over the sample.

4. Go to `pjm.com` Report the date and time you access their Current Conditions data.

   (a) What is the electricity demand in MW at that time, and what fraction is being provided by renewables?

   (b) Give a table of the percentages being supplied by the various different fuels (including renewables, the fuel mix) ordered from highest to lowest.

   (c) Describe/quantify how actuals have departed from forecasts over the previous hours for which they give data.

   (d) Where in the PJM grid are the LMPs highest and lowest? Suggest an explanation.

   (e) Repeat the above for ERCOT.