ORF 555 HW 5

(Q.1) 1-period Bitcoin mining:-

expend → $\alpha \geq 0$; prob of winning $(p) = \dfrac{\alpha}{\alpha + A}$

$$\underbrace{\text{Gain/loss}}_{\text{profit }(P_r)} = rZ - c\alpha = \begin{cases} r - c\alpha & \text{wp } p \\ -c\alpha & \text{wp } 1-p \end{cases}$$

$$E(P_r) = p(r - c\alpha) + (1-p)(-c\alpha) = rp - c\alpha$$

$$= \frac{r\alpha}{\alpha + A} - c\alpha = f(\alpha)$$

(a) $\alpha^* \Rightarrow$ $\dfrac{\partial f}{\partial \alpha} = 0\Big|_{\alpha=\alpha^*} \Rightarrow \dfrac{r(\alpha + A - \alpha)}{(\alpha + A)^2} - c = 0\Big|_{\alpha=\alpha^*}$

$$\Rightarrow \frac{rA}{(\alpha^* + A)^2} = c \Rightarrow \boxed{\alpha^* = \sqrt{\frac{rA}{c}} - A} \rightarrow \boxed{\left(\text{if } c \leq \frac{r}{A};\ \text{if } c \geq \frac{r}{A},\ \alpha^* = 0\right)}$$

$$\frac{\partial^2 f}{\partial \alpha^2}\Big|_{\alpha=\alpha^*} = \frac{-2rA}{(\alpha^* + A)^3} < 0 \Rightarrow \alpha^* = \text{maximiser of } f \ (E(P_r))$$

(b) $\alpha^* = 0 \Rightarrow \boxed{c^* = \dfrac{r}{A}}$ $\left(\sqrt{\dfrac{rA}{c}} - A \leq 0 \Rightarrow \sqrt{\dfrac{rA}{c}} \leq A \right.$

$$\left. \Rightarrow \frac{rA}{c} \leq A^2 \Rightarrow c \geq \frac{r}{A} \right)$$

(c) $M+1$ miners with different costs of electricity;
Interacting through A

$A \sim M\bar{\alpha}$: assuming distribution of electricity costs is by $m(c)$

$\rightarrow A \rightarrow c \rightarrow \alpha^*(c)$

$\rightarrow A = M\bar{\alpha}$ with $\bar{\alpha} = \int \alpha^*(c)\, m(c)\, dc$

mean α (hash) rate

$c \in U[\frac{r}{2}, r]$

$\Rightarrow m(c) = \frac{2}{r} \mathbb{1}[\frac{r}{2} \le c \le r]$

(1) $\bar{\alpha} = \int \alpha^*(c)\, m(c)\, dc$

$= \int_{r/2}^{r} \alpha^*(c) \frac{2}{r}\, dc$

$= \int_{r/2}^{r} \left(\sqrt{\frac{rA}{c}} - A\right) \frac{2}{r}\, dc$

$= \frac{2}{r}\sqrt{rA}\, 2\left[\sqrt{c}\right]_{r/2}^{r} - \frac{2A}{r}\frac{r}{2}$

$= 4\sqrt{\frac{A}{r}}\left(\sqrt{r} - \frac{\sqrt{r}}{\sqrt{2}}\right) - A$

$= 4\sqrt{A}\left(1 - \frac{1}{\sqrt{2}}\right) - A$

$= (4 - 2\sqrt{2})\sqrt{A} - A$

$A = M\bar{\alpha}$
$= M\left((4-2\sqrt{2})\sqrt{A} - A\right)$

$$\Rightarrow \sqrt{A} = M(4-2\sqrt{2}-\sqrt{A})$$

$$\Rightarrow \sqrt{A} = \frac{M(4-2\sqrt{2})}{1+M}$$

$$\Rightarrow \boxed{A = \frac{8M^2(\sqrt{2}-1)^2}{(1+M)^2}}$$

$$\Rightarrow \bar{\alpha} = \sqrt{A}(4-2\sqrt{2}-\sqrt{A})$$

$$= \frac{M(4-2\sqrt{2})}{1+M}(4-2\sqrt{2})\left(1-\frac{M}{1+M}\right)$$

$$= \frac{M \cdot 8 \cdot (\sqrt{2}-1)^2}{(1+M)^2}$$

$$= \frac{8M(\sqrt{2}-1)^2}{(1+M)^2}$$

(ii) For $M \to \infty$, $A \to 8(\sqrt{2}-1)^2$

$$= 8 \cdot 0.41^2$$

$$\approx 8 \cdot 0.16$$

$$\sim 1.28 \in (1,2) \checkmark$$

(iii) c^* is s.t.

$\alpha^*(c^*) = 0 \qquad (\& \ \alpha^*(c) = 0 \ \forall \ c \in [c^*, r))$

$\alpha^*(c^*) = \sqrt{\frac{rA}{c^*}} - A = 0 \Rightarrow c^* = \frac{r}{A}$

$\Rightarrow c^* = \dfrac{r}{\frac{8M^2(\sqrt{2}-1)^2}{(1+M)^2}} = \boxed{\dfrac{r(1+M)^2}{8M^2(\sqrt{2}-1)^2}}$

(Q.2) Now, $p = \dfrac{\alpha x}{\alpha x + AX}$ (Here, α (most probably) represents cost of elec/hash · # hashes/\$ i.e. total cost/\$)

H (profit) $= rZ - \alpha$

$$E(H) = r\,E(Z) - \alpha = rp - \alpha = \frac{r\alpha x}{\alpha x + AX} - \alpha$$

$$= r\left(1 - \frac{AX}{\alpha x + AX}\right) - \alpha$$

$$= r - \frac{rAX}{\alpha x + AX} - \alpha$$

$$= (r-\alpha) - \frac{rAX}{\alpha x + AX} = f(\alpha) \quad - (*)$$

(a) $\dfrac{\partial f}{\partial \alpha} = 0$ at $\alpha = \alpha^*$ $\Rightarrow$ $-1 + \dfrac{rAX \cdot x}{(\alpha^* x + AX)^2} = 0$

$$\Rightarrow \frac{rAXx}{(\alpha^* x + AX)^2} = 1$$

$$\Rightarrow \alpha^* x + AX = \sqrt{rAXx}$$

$$\Rightarrow \boxed{\alpha^* = \frac{\sqrt{rAXx} - AX}{x}} \rightarrow \boxed{\left(\text{if } d = \frac{x}{X} \geq \frac{A}{r};\ \text{if } d \leq \frac{A}{r},\ \alpha^* = 0\right)}$$

(b) Let $\dfrac{x}{X} = d$

$$\Rightarrow \alpha^* = \sqrt{\frac{rA}{d}} - \frac{A}{d} \leq 0 \Rightarrow \sqrt{\frac{rA}{d}} \leq \frac{A}{d} \Rightarrow \frac{rA}{d} \leq \frac{A^2}{d^2}$$

$$\Rightarrow \boxed{d \leq \frac{A}{r}}$$

(c) As Individual wealth (x) increases, firstly $\alpha^* \uparrow$ (more chance of α^* being positive now) & also due to $x \uparrow$ & $\alpha \uparrow$, (assuming A & X are constant), p(prob. reward) $\uparrow$

So, the one with more wealth has more/higher chance of getting reward (discovering bitcoin) ~~More wealth~~ due to which the wealth gap/inequality is likely to increase. This supports the idea of preferential attachment.

Hence, instead of getting decentralized (where wealth ~~distribution~~ inequality decreases & distribution is more uniform among players), here wealth inequality is increasing & so leads us to be against the idea of decentralized finance.

Q.3) ~~Financialization~~ Financialization . Q^n

$$\rho_t = \varepsilon \frac{\theta \pi \sigma_s X_t}{A \, \sigma_t^{(0)}}$$

(a) If A is very large, $\rho_t \to 0 \Rightarrow$ financialization correlation tends to 0.

In general, $\rho_t \propto \frac{1}{A}$ (Inversely proportional to A).

So, as A ↑, ρ_t ↓.

$\Rightarrow$ ~~compared to sup~~ If supply : A is small, ρ_t is large

If A is large, ρ_t is small.

$\Rightarrow$ compared to supply being less, if supply is more, ~~the~~ corr^n is less.

So, if A is large, ρ_t = small

If A is small, ρ_t = large (significant).

So, compared to A being ~~large~~ small, if A (supply) were large,

the financialization correlation (ρ_t) would decrease (be less).

why? If supply is smaller, every unit of the supply matters (i.e. marginal utility of the units is high); So, correlation would kick on. However, if the supply is large, the impact of speculators does not affect the actual trading or market of the commodity, thus decreasing ρ_t.

supply small
→ small market
→ impact of speculators high (how they trade)
→ S_t high

supply large
→ large market
→ spec. impact low
→ hence, S_t low.

(b) Note that $\theta + \pi \leq 1$ ($\because$ unleveraged)

To maximize $S_t \propto \theta\pi$, $\boxed{\theta = \pi = \frac{1}{2}}$

(Q. 4)

(a) $\log Y \in (-\infty, \infty) \Rightarrow Y \in (e^{-\infty}, e^{\infty}) = (0, \infty)$

$\log Y \sim N(\mu, \sigma^2)$ under risk-neutral prob Q

So, $E^Q\{(1-Y)^+\} = \int_0^\infty (1-y)^+ f_Y^q(y)\,dy = \int_0^1 (1-y) f_Y^q(y)\,dy + \int 0 \ldots$

$= \int_0^1 (1-y) f_Y^q(y)\,dy$

$Y = e^{N(\mu,\sigma^2)}$

$1 - Y = 1 - e^{N(\mu,\sigma^2)}$

$= \int_0^1 f_Y^q(y)\,dy - \int_0^1 y f_Y^q(y)\,dy$

$= P(Y \le 1) - \left[\left(y \int f_Y^q(y)dy\right)_0^1 - \int_0^1 \left(\int f_Y^q(y)dy\right)dy\right]$

$= P(\log Y \le 0) - \left[\left(y F_Y^q(y)\right)_0^1 - \int_0^1 F_Y^q(y)\,dy\right]$

$= P(N(\mu,\sigma^2) \le 0) - [F_Y^q(1)] + \int_0^1 F_Y^q(y)\,dy$

$\log y = t \Rightarrow \frac{dy}{y} = dt$

$\Rightarrow I = \int_{-\infty}^0 (1-e^t) f_T^q(t)\,dt = \int_{-\infty}^0 f_T^q(t)\,dt - \int_{-\infty}^0 e^t f_T^q(t)\,dt$

So, $E^Q((1-Y)^+) \cancel{=} - E^Q((Y-1)^+)$

~~$dX = \mu dt + \sigma dz$~~

$Y = \mu + \sigma Z \to N(0,1)$

$(Y_T = \ln(\frac{S_T}{X_T}) = \ln(\frac{S_0}{X_0}) + (\quad)T + \sigma B_T - \eta W_T^{Q1})$

$\sigma^2 T + \eta^2 T - 2\sigma\eta S T = (\sigma^2 + \eta^2 - 2\sigma\eta S)T$

From wikipedia, for a lognormal RV Y

$g(K) = \int_K^\infty y f_Y^q(y)\,dy = e^{\mu+\sigma^2/2}\,\phi\left(\frac{\mu - \ln K}{\sigma} + \sigma\right)$

$\Rightarrow g(1) = e^{\mu+\sigma^2/2}\,\phi\left(\frac{\mu+\sigma^2}{\sigma}\right)$

$\Rightarrow \int_0^1 y f_Y^q(y)\,dy = E(Y) - g(1) = e^{\mu+\sigma^2/2} - e^{\mu+\sigma^2/2}\,\phi\left(\frac{\mu+\sigma^2}{\sigma}\right)$

$= e^{\mu+\sigma^2/2}\,\phi\left(-\frac{\mu-\sigma^2}{\sigma}\right)$

$\Rightarrow$ Final answer $= E^Q((1-Y)^+) = \boxed{\phi\left(-\frac{\mu}{\sigma}\right) - e^{\mu+\sigma^2/2}\,\phi\left(-\frac{\mu}{\sigma} - \sigma\right)}$

How? If $X = LN(\mu, \sigma^2)$ (log normal) & $\ln X = Y \sim N(\mu, \sigma^2)$

$$\int_k^\infty x f_X(x)\,dx = E(X 1_{X \geq k}) = E(e^Y 1_{e^Y \geq k}) = E(e^Y 1_{Y \geq \ln k})$$

$$= \int_{\ln k}^\infty e^y f_Y(y)\,dy = \int_{\ln k}^\infty e^y \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(-\frac{(y^2 - 2\mu y + \mu^2)}{2\sigma^2} + y\right) dy$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(\frac{-y^2 + 2\mu y - \mu^2 + 2\sigma^2 y}{2\sigma^2}\right) dy$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(-\frac{[(y - (\mu+\sigma^2))^2 - (\mu+\sigma^2)^2 + \mu^2]}{2\sigma^2}\right) dy$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(-\frac{(y - (\mu+\sigma^2))^2}{2\sigma^2} + \frac{\cancel{\mu^2} + 2\mu\sigma^2 + \sigma^4 - \cancel{\mu^2}}{2\sigma^2}\right) dy$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \frac{1}{\sigma\sqrt{2\pi}} \int_{\ln k}^\infty \exp\left(-\frac{(y - (\mu+\sigma^2))^2}{2\sigma^2}\right) dy$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) P\left(N(\mu+\sigma^2, \sigma^2) > \ln k\right)$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) P\left(N(0,1) > \frac{\ln k - \mu - \sigma^2}{\sigma}\right)$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \left(1 - \phi\left(\frac{\ln k - \mu}{\sigma} - \sigma\right)\right)$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \phi\left(\frac{\mu - \ln k}{\sigma} + \sigma\right)$$

(Q.4)

(b) put option : expiration date T on a forward maturing at time $T' > T$ $\Rightarrow$ payoff $= (K - F(T,T'))^+$

$\hookrightarrow$ Terminal payoff.

~~So,~~ $\cancel{F(T,T') = F(t,T')\, e^{(r-\delta-\frac{\sigma^2}{2})(T-t) + \sigma W^{q}_{(T-t)}}}$

~~Bae~~ Premidterm notes $\rightarrow$ 3.5.1 (Page 30)

Eqn (36) :-

$$F(t,T') = S_t\, e^{(r-\delta)(T'-t)}$$

$$F(T,T') = S_T\, e^{(r-\delta)(T'-T)}$$

$$\Rightarrow \frac{F(T,T')}{F(t,T')} = \frac{S_T}{S_t}\, e^{(r-\delta)(t-T)}$$

$$= e^{(r-\delta-\frac{\sigma^2}{2})(T-t) + \sigma(W^q_T - W^q_t)}\, e^{(r-\delta)(t-T)}$$

$$= e^{-\frac{\sigma^2}{2}(T-t) + \sigma(W^q_T - W^q_t)}$$

$$\Rightarrow \boxed{F(T,T') = F(t,T')\, e^{-\frac{\sigma^2}{2}(T-t) + \sigma(W^q_T - W^q_t)}}$$

(c) put (p) $= e^{-r(T-t)} E^{q}((K - F(T,T'))^{+})$

$= e^{-r(T-t)} E^{Q}\left(\left(K - F(t,T') e^{-\frac{\sigma^2}{2}(T-t) + \sigma(W^{q}_{T} - W^{q}_{t})}\right)^{+}\right)$

$= e^{-r(T-t)} K E^{Q}\left(\left(1 - \frac{F(t,T')}{K} e^{-\frac{\sigma^2}{2}(T-t) + \sigma(W^{q}_{T} - W^{q}_{t})}\right)^{+}\right)$

Formula: $p = K e^{-r(T-t)} N(-d_2) - S_t N(-d_1)$ $\quad d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$

$= K e^{-r(T-t)} N(-d_2) - F(t,T') e^{-r(T'-t)} N(-d_1)$

↳ T or T'?

$= e^{-r(T-t)} N(d$

so $p_0 = e^{-rT}(K N(-d_2) - F(0,T') N(-d_1))$

where $d_1 = \frac{\ln(F(0,T')/K) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$

$\Rightarrow p_t = e^{-r(T-t)}(K N(-d_2) - F(t,T') N(-d_1))$

where $d_1 = \frac{\ln\left(\frac{F(t,T')}{K}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$

(Q.4)

(c) $\text{price} = P_t = e^{-r(T-t)} E\left((K - F(T,T'))^+\right)$

$$= e^{-r(T-t)} K\, E\left(\left(1 - \frac{F(T,T')}{K}\right)^+\right)$$

$\downarrow$ lognormal ?

$$\ln\left(\frac{F(T,T')}{K}\right) = \ln\left(\frac{F(t,T')}{K} e^{-\frac{\sigma^2}{2}(T-t) + \sigma(W^q_T - W^q_t)}\right)$$

$$= \ln\left(\frac{F(t,T')}{K}\right) - \frac{\sigma^2}{2}(T-t) + \sigma(W^q_T - W^q_t)$$

$$= N\left(\ln\left(\frac{F(t,T')}{K}\right) - \frac{\sigma^2}{2}(T-t),\ \sigma^2(T-t)\right)$$

$$= N\left(\underbrace{\ln\left(\frac{F(t,T')}{K}\right) - \frac{\sigma^2\Delta}{2}}_{\mu_c},\ \underbrace{\sigma^2\Delta}_{\sigma_c^2}\right) \quad \text{where } \Delta = T-t$$

$$\Rightarrow P_t = e^{-r\Delta} K\left(\Phi\left(-\frac{\mu_c}{\sigma_c}\right) - e^{\mu_c + \sigma_c^2/2}\, \Phi\left(-\frac{\mu_c}{\sigma_c} - \sigma_c\right)\right)$$

$$= e^{-r\Delta} K\left[\Phi\left(\frac{-\left(\ln\left(\frac{F(t,T')}{K}\right) - \frac{\sigma^2\Delta}{2}\right)}{\sigma\sqrt{\Delta}}\right) - e^{\ln\left(\frac{F(t,T')}{K}\right) - \frac{\sigma^2\Delta}{2} + \frac{\sigma^2\Delta}{2}}\, \Phi\left(\frac{-\left[\ln\left(\frac{F(t,T')}{K}\right) - \frac{\sigma^2\Delta}{2}\right] - \sigma^2\Delta}{\sigma\sqrt{\Delta}}\right)\right]$$

$$= e^{-r\Delta} K\left[\Phi(-d_2) - \frac{F(t,T')}{K}\, \Phi\left(-\frac{\ln\left(\frac{F(t,T')}{K}\right) - \frac{\sigma^2\Delta}{2}}{\sigma\sqrt{\Delta}}\right)\right]$$

$$= e^{-r\Delta} K\, \Phi(-d_2) - e^{-r\Delta} F(t,T')\, \Phi(-d_1)$$

$$= e^{-r\Delta}\left(K N(-d_2) - F(t,T') N(-d_1)\right)$$

$$= \boxed{e^{-r(T-t)}\left(K N(-d_2) - F(t,T') N(-d_1)\right)}$$

where $d_1 = \dfrac{\ln\left(\frac{F(t,T')}{K}\right) + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}$ & $d_2 = d_1 - \sigma\sqrt{T-t}$

(Q-4)(d)

So, $\ln F(t,T') = \ln S_t\, e^{-\alpha(T'-t)} + m(1-e^{-\alpha(T'-t)}) + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T'-t)})$

(Eq^n 39 in notes) $= Y_t e^{-\alpha(T'-t)} + m(1-e^{-\alpha(T'-t)}) + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T'-t)})$

$\because Y_t \sim N\left(\underbrace{Y_0 e^{-\alpha t} + m(1-e^{-\alpha t})}, \underbrace{\frac{\sigma^2}{2\alpha}(1-e^{-2\alpha(t)})}\right)$

$\ln(F(t,T')) \sim N\Big(Y_0 e^{-\alpha T'} + m(e^{-\alpha(T'-t)} - e^{-\alpha T'}) + m(1-e^{-\alpha(T'-t)}) + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T'-t)}),$
$\frac{\sigma^2}{2\alpha}(e^{-2\alpha(T'-t)} - e^{-2\alpha T'})\Big)$

~~So, $\ln F(t,T') = T_1(t,T')\ln S_t + T_2(t,T') + T_3(t,T')$~~
~~$\ln F(T,T') = T_1(T,T')\ln S_t$~~

where $T_1(a,b) = e^{-\alpha(b-a)}$
$T_2(a,b) = m(1-e^{-\alpha(b-a)})$
$T_3(a,b) = \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(b-a)})$

~~So, $\ln F(t,T') =$~~

So, $\ln F(T,T') = T_1(T,T')\ln S_T + T_2(T,T') + T_3(T,T')$
$\ln F(t,T') = T_1(t,T')\ln S_t + T_2(t,T') + T_3(t,T')$

$\Rightarrow \ln F(T,T') - \ln F(t,T') = T_1(T,T')\ln S_T - T_1(t,T')\ln S_t$
$\underbrace{\boxed{+T_2(T,T') - T_2(t,T') + T_3(T,T') - T_3(t,T')}}_{T_4}$

$= e^{-\alpha(T'-T)}\ln S_T - e^{-\alpha(T'-t)}\ln S_t + T_4$

$= e^{-\alpha T'}(e^{\alpha T}\ln S_T - e^{\alpha t}\ln S_t) + T_4$

$= e^{-\alpha T'}(e^{\alpha T} Y_T - e^{\alpha t} Y_t) + T_4$ → Eq^n 37.

$= e^{-\alpha T'}\left(e^{\alpha T}\left(Y_t e^{-\alpha(T-t)} + m(1-e^{-\alpha(T-t)}) + \sigma e^{-\alpha T}\int_t^T e^{\alpha u}\, dW_u^{Q}\right) - e^{\alpha t} Y_t\right) + T_4$

$= e^{-\alpha T'}\left(\cancel{Y_t e^{\alpha t}} + m(e^{\alpha T} - e^{\alpha t}) + \sigma\int_t^T e^{\alpha u}\, dW_u^{Q} - \cancel{e^{\alpha t} Y_t}\right) + T_4$

$= m(e^{-\alpha(T'-T)} - e^{-\alpha(T'-t)}) + T_4 + \sigma\int_t^T e^{-\alpha(T'-u)}\, dW_u^{Q}$

$$\therefore\ T_4 = f^n(T_2, T_3) = \text{deterministic},$$

$$\ln F(T,T') - \ln F(t,T')$$

$$\sim N\left(m\left(e^{-\alpha(T'-T)} - e^{-\alpha(T'-t)}\right) + T_4,\ \sigma^2 \int_t^T e^{-2\alpha(T'-u)}\,du \right)$$

$$= N\left(m\left(e^{-\alpha(T'-T)} - e^{-\alpha(T'-t)}\right) + T_4,\ \frac{\sigma^2}{2\alpha}\left(e^{-2\alpha(T'-T)} - e^{-2\alpha(T'-t)}\right) \right)$$

$$\boxed{\begin{aligned}&\Rightarrow \ln F(T,T') \\ &\sim N\Big(\ln F(t,T') + m\left(e^{-\alpha(T'-T)} - e^{-\alpha(T'-t)}\right) + T_2(T,T') - T_2(t,T') + T_3(T,T') - T_3(t,T'), \\ &\qquad \frac{\sigma^2}{2\alpha}\left(e^{-2\alpha(T'-T)} - e^{-2\alpha(T'-t)}\right) \Big)\end{aligned}}$$

(Q.4)

(e) Let $\ln F(T, T') \sim N(\mu_e, \sigma_e^2)$ $\rightarrow$ refer to part(d)

$$\Rightarrow P_t = e^{-r(T-t)} (K - F(T,T'))^+$$

$$= e^{-r(T-t)} \cdot K \left(1 - \frac{F(T,T')}{K}\right)^+$$

Now, $\ln\left(\frac{F(T, T')}{K}\right) = \ln F(T,T') - \ln K$

$$= N(\mu_e - \ln K, \sigma_e^2)$$

$\Rightarrow$ Using part(a) of Q^n 4,

$$P_t = e^{-r(T-t)} K \left(\Phi\left(\frac{-(\mu_e - \ln K)}{\sigma_e}\right) - e^{(\mu_e - \ln K) + \frac{\sigma_e^2}{2}} \Phi\left(-\frac{(\mu_e - \ln K) - \sigma_e^2}{\sigma_e}\right)\right)$$

$$\Rightarrow \boxed{P_t = e^{-r(T-t)} K \left(\Phi\left(-\frac{\mu_e - \ln K}{\sigma_e}\right) - e^{\mu_e - \ln K + \frac{\sigma_e^2}{2}} \Phi\left(-\frac{(\mu_e - \ln K)}{\sigma_e} - \sigma_e\right)\right)}$$

(Here, $\mu_e = \ln F(t,T') + m(e^{-\alpha(T'-T)} - e^{-\alpha(T'-t)}) + T_2(T,T') - T_2(t,T') + T_3(T,T') - T_3(t,T')$

& $\sigma_e^2 = \frac{\sigma^2}{2\alpha}(e^{-2\alpha(T'-T)} - e^{-2\alpha(T'-t)})$

$\Downarrow$

from part (d)

(Q-5) S_t = Stock price

X_t = market Index

$c(S) \xrightarrow[\text{on index } X]{\text{bchmkd}} pf = (S_T - X_T)^+$ on date T.

Risk Neutral world : $\mathbb{Q}$

(a) $W_t^{Q1} \perp\!\!\!\perp W_t^{Q2}$ are 2 BM.

Given : $B_t^{Q} = \rho W_t^{Q1} + \sqrt{1-\rho^2}\, W_t^{Q2}$; $\rho \in (-1,1)$

To show : B_t^Q = BM.

<u>proof</u> :- since W_t^{Q1}, W_t^{Q2} = BM,

$W_t^{Q1} \sim N(0,t)$, $W_t^{Q2} \sim N(0,t)$ (Both W_t^{Q1} & W_t^{Q2} are scalars) $\downarrow$ RV

Further since $W_t^{Q1} \perp\!\!\!\perp W_t^{Q2}$ $\forall t$,

$\text{corr}(W_t^{Q1}, W_t^{Q2}) = 0$

$\Rightarrow \text{cov}(W_t^{Q1}, W_t^{Q2}) = 0$

So, B_t^Q = linear comb^n of 2 ind^t Gaussians

= Gaussian

$\sim N(0,1)$

$\sim N(0,1)$

$\frac{1}{|\Sigma|^{1/2} 2\pi} e^{-\frac{1}{2}[x_1\ x_2]\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$

$\downarrow$

Not separable in general, but still Gaussian

$E(B_t^Q) = \rho \cdot 0 + \sqrt{1-\rho^2} \cdot 0 = 0$

$\text{var}(B_t^Q) = \rho^2 \cdot t + (1-\rho^2)t + \rho\sqrt{1-\rho^2} \cdot 0$

$= t$

$\Rightarrow B_t^Q \sim N(0,t)$

Also, we need to prove some more <u>things</u>.

(Q.) How to prove that something is a Brownian Motion (BM)?

→ ① Stationary & ind^P increments : Since W_t^{Q1}, W_t^{Q2} are BM, they have S & i increments ⇒ so would B_t

② $B_0 = 0$: $B_0 = \rho W_0^{Q1} + \sqrt{1-\rho^2}\, W_0^{Q2} = \rho * 0 + \sqrt{1-\rho^2} * 0 = 0$

BM

③ Continuity : Since W_t^{Q1} & W_t^{Q2} are continuous, so will be B_t

④ : $B_t - B_s = \rho\,(W_t^{Q1} - W_s^{Q1}) + \sqrt{1-\rho^2}\,(W_t^{Q2} - W_s^{Q2})$

$= \rho\, N(0, (t-s)) + \sqrt{1-\rho^2}\; N(0, t-s)$

$= N(0, t-s)$ ✓

To prove :-

① $B_t \sim N(0,t)$

② $B_0 = 0$

③ continuity

④ stationary & ind^P increments

⑤ $B_t - B_s = N(0, t-s)$

⇒ QED

(b) $E^{Q}(W_T^{Q1} B_T^{Q})$

$= E^{Q}\left(W_T^{Q1}\left(\rho W_T^{Q1} + \sqrt{1-\rho^2}\, W_T^{Q2}\right)\right)$

$= E^{Q}\left(\rho (W_T^{Q1})^2 + \sqrt{1-\rho^2}\, W_T^{Q1} W_T^{Q2}\right)$

$= \rho \cdot (T+0) + \sqrt{1-\rho^2}\, \underbrace{E(W_T^{Q1})}_{0} \underbrace{E(W_T^{Q2})}_{0}$

$= \underline{\underline{\rho T}}$.

(c) No arb price $= e^{-rT} E\left((S_T - X_T)^+\right)$ — ⊛

Now, $\dfrac{dX_t}{X_t} = r\,dt + \eta\, dW_t^{Q1}$

$\Rightarrow X_t = X_0 e^{(r-\eta^2/2)t + \eta W_t^{Q1}}$ — ①

IIIly, $S_t = S_0 e^{(r-\sigma^2/2)t + \sigma B_t^{Q}}$ — ②

So, putting ① & ② in ⊛,

price (P) $= e^{-rT} E\left(\left(S_0 e^{(r-\sigma^2/2)T + \sigma B_T^{Q}} - X_0 e^{(r-\eta^2/2)T + \eta W_T^{Q1}}\right)^+\right)$

$= E\left(\left(S_0 e^{-\sigma^2/2\, T + \sigma B_T^{Q}} - X_0 e^{-\eta^2/2\, T + \eta W_T^{Q1}}\right)^+\right)$

~~$= E\left(S_0 e^{-\frac{\sigma^2}{2}T}\right.$~~

$= E\left(\left(S_0 \exp\left(-\frac{\sigma^2}{2}T + \sigma(\rho W_T^{Q1} + \sqrt{1-\rho^2}\, W_T^{Q2})\right) - X_0 \exp\left(-\frac{\eta^2}{2}T + \eta W_T^{Q1}\right)\right)^+\right)$

$e^{-rT} E\left((S_T - X_T)^+\right)$

(d) $C(0) = S_0 N(d_1) - K e^{-r(0)T} N(d_2)$ where $d_1 = \dfrac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

we have:- $\quad \hookrightarrow$ · from $C = E\left(e^{-r(0)T}(S_{T} - K)^+\right)$

$$C = E\left(e^{-rT}(S_T - X_T)^+\right)$$

$$\Rightarrow C = S_0 N(d_1) - X_T e^{-rT} N(d_2)$$

OR. $C = e^{-rT} E^{Q}\left(X_T\left(\frac{S_T}{X_T} - 1\right)^+\right)$

using change of numeraire,

let $\dfrac{dQ'}{dQ} = X_T = X_0 e^{\left(r - \frac{\eta^2}{2}\right)T + \eta W_T^{Q'}}$

$$\text{So, } \frac{S_T}{X_T} = \frac{S_0}{X_0} e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T - \left(r - \frac{\eta^2}{2}\right)T - \eta W_T^{Q'}}$$

$$= \frac{S_0}{X_0} e^{\frac{(\eta^2 - \sigma^2)}{2}T + \sigma B_T - \eta W_T^{Q'}}$$

Now, using Margrabe's formula, $\to$ Internet

If $pf = (S^1_T - S^2_T)^+ \to C = e^{-q_1 T} S^1_0 N(d_1) - e^{-q_2 T} S^2_0 N(d_2)$

yield $\quad q_1 \quad q_2$

vol $\quad \sigma_1 \quad \sigma_2$

corr $\quad \rho$

where $d_1 = \dfrac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \left(q_2 - q_1 + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$; $d_2 = d_1 - \sigma\sqrt{T}$

where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$.

In our case,

$S^1_T = S_T$, $S^2_T = X_T$

yield $\quad r - r = 0 \quad r - r = 0$

vol $\quad \sigma \quad \eta$

corr $\quad \rho$

$$\Rightarrow \boxed{\begin{array}{l} C = e^{-rT}\left(S_0 N(d_1) - X_0 N(d_2)\right) \\ \text{where } d_1 = \dfrac{\ln\left(\frac{S_0}{X_0}\right) + \frac{\tilde{\sigma}^2}{2}T}{\tilde{\sigma}\sqrt{T}} ; \quad \tilde{\sigma} = \sqrt{\sigma^2 + \eta^2 - 2\sigma\eta\rho} \\ d_2 = d_1 - \tilde{\sigma}\sqrt{T} \end{array}}$$

$\tilde{S}_0 = S_0$, $\quad \tilde{\sigma} = \sqrt{\sigma^2 + \eta^2 - 2\sigma\eta\rho}$

$K = X_0$, $\quad \tilde{r} = 0$

~~So, if $E^{Q}(X_T H) = E^{Q^x}(H)$,~~

~~then $\frac{dQ^x}{dQ} = e^{-rT}\frac{X_T}{X_0} = e^{-r(T-t)}\frac{X_T}{X_t}$~~

~~Assumption: Under Q, $e^{-rt}X_t$ is a martingale~~

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t + \sigma B_t}$$

$$E(S_t) = S_0 e^{rt} \Rightarrow E(e^{-rt}S_t) = S_0$$

* Who all are mtgl?
  - → f_t
  - → $e^{-rt}S_t$

Thm:- If X_t is a martingale under Q,

then $E^{Q}(X_T H) = X_0 \, E^{Q^x}(H)$

where $\frac{dQ^x}{dQ} = \frac{X_T}{X_0}$.

So, $E^{Q}(X_T H) = \int X_T H \, dQ$ ~~$= \int X_T H \frac{dQ^x}{dQ}$~~

mtgl ⇒ $X_T \sim X_0$

$$= \int X_0 H \, dQ^x$$

$$= E^{Q^x}(X_0 H)$$

$$= X_0 \, E^{Q^x}(H)$$

In our case, $e^{-rT}X_T$ is a martingale under Q.

So, $C = e^{-rT} E^{q}\left((S_T - X_T)^+\right) = e^{-rT} E^{q}\left(X_T\left(\frac{S_T}{X_T}-1\right)^+\right)$

price of benchmarked call

$$= E^{q}\left((e^{-rT}X_T)\left(\frac{S_T}{X_T}-1\right)^+\right)$$

$$= E^{\tilde{Q}}\left((e^{-r\cdot 0}X_0)\left(\frac{S_T}{X_T}-1\right)^+\right)$$

$$= E^{\tilde{Q}}\left(X_0\left(\frac{S_T}{X_T}-1\right)^+\right)$$

$$= \boxed{X_0\, E^{\tilde{Q}}\left(\left(\frac{S_T}{X_T}-1\right)^+\right)} \quad - (*)$$

Now, when $pf = (S_T - K)^+$, $\quad \frac{dS_t}{S_t} = r\,dt + \sigma\, dW_t$

Here ~~$\ln\left(\frac{S_T}{X_T}\right) = \ln\left(\frac{S_0}{X_0}\right)$~~

So, $\frac{S_t}{X_t} = f(S_t, X_t)$

$$\Rightarrow d\left(\frac{S_t}{X_t}\right) = \frac{\partial f}{\partial S_t}dS_t + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S_t^2}(dS_t)^2 + \frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 + 2\frac{\partial^2 f}{\partial S_t \partial X_t}dS_t\, dX_t\right)$$

$$\Rightarrow d\left(\frac{S_t}{X_t}\right) = \frac{1}{X_t}\cdot dS_t - \frac{S_t}{X_t^2}dX_t + \frac{1}{2}\left(0\cdot(dS_t)^2 + \frac{2S_t}{X_t^3}(dX_t)^2 + 2\frac{(-1)}{X_t^2}dS_t\, dX_t\right)$$

$$= \frac{S_t}{X_t}\left(r\,dt + \sigma\, dB_t^{q}\right) - \frac{S_t}{X_t}\left(r\,dt + \eta\, dW_t^{q1}\right) + \frac{1}{2}\left(\frac{2S_t}{X_t}\left(\frac{dX_t}{X_t}\right)^2 - 2\frac{S_t}{X_t}\left(\frac{dS_t}{S_t}\right)\left(\frac{dX_t}{X_t}\right)\right)$$

$$= \frac{S_t}{X_t}\left(\cancel{r\,dt} + \sigma\, dB_t^{Q} - \cancel{r\,dt} - \eta\, dW_t^{q1} + \eta^2 dt - \sigma\eta\delta\, dt\right)$$

$\Rightarrow$ If $\frac{S_t}{X_t} = Z_t$,

$$\frac{dZ_t}{Z_t} = (\eta^2 - \sigma\eta\delta)\,dt + \sigma\, dB_t^{Q} - \eta\, dW_t^{q1}$$

$$= (\eta^2 - \sigma\eta\delta)\,dt + (\sigma\delta - \eta)\,W_t^{q1} + \sigma\sqrt{1-\delta^2}\,W_t^{q2}$$

$$\Rightarrow \frac{dZ_t}{Z_t} = \eta(\eta - \sigma s)\,dt + \sqrt{(\cancel{\sigma^2 s^2} + \eta^2 - 2\sigma s\eta + \sigma^2 - \cancel{\sigma^2 s^2})}\,dW_t^{new}$$

$$= \boxed{\eta(\eta - \sigma s)}\,dt + \sqrt{\sigma^2 + \eta^2 - 2\sigma\eta s}\,dW_t^{new}$$

$$= \tilde{r}\,dt + \tilde{\sigma}\,dW_t^{new} \longrightarrow$$

This leads to:-

$$Z_T = Z_0 \exp\left(\left(\tilde{r} - \frac{\tilde{\sigma}^2}{2}\right)T + \tilde{\sigma} W_T^{new}\right)$$

$$= Z_0 \exp\left(\left(\eta^2 - \sigma\eta s - \frac{1}{2}(\sigma^2 + \eta^2 - 2\sigma\eta s)\right)T + \tilde{\sigma} W_T^{new}\right)$$

$$= Z_0 \exp\left(\left(\frac{\eta^2 - \sigma^2}{2}\right)T + \tilde{\sigma} W_T^{new}\right)$$

So, we have:-

$$C = X_0\, E^{\tilde{Q}}\left((Z_T - 1)^+\right)$$

$$= X_0\left(Z_0 N(\tilde{d}_1) - 1 \cdot e^{-\tilde{r}T} N(\tilde{d}_2)\right)$$

where $\tilde{d}_1 = \dfrac{\ln(Z_0) + \left(\tilde{r} + \frac{\tilde{\sigma}^2}{2}\right)T}{\tilde{\sigma}\sqrt{T}}$, $\tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}\sqrt{T}$

$$= X_0\left(\frac{S_0}{X_0} N(\tilde{d}_1) - e^{-\tilde{r}T} N(\tilde{d}_2)\right)$$

where $\tilde{d}_1 = \dfrac{\ln\left(\frac{S_0}{X_0}\right) + \left(\tilde{r} + \frac{\tilde{\sigma}^2}{2}\right)T}{\tilde{\sigma}\sqrt{T}}$, $\tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}\sqrt{T}$

$$= \boxed{S_0 N(\tilde{d}_1) - X_0 e^{-\tilde{r}T} N(\tilde{d}_2)}$$

$$= \boxed{C_{BS}(S_0, X_0, T, \tilde{r}, \tilde{\sigma})}$$

$$= C_{BS}(\tilde{S}_0, \tilde{K}, T, \tilde{r}, \tilde{\sigma})$$

where
$\tilde{S}_0 = S_0$
$\tilde{K} = X_0$
$\tilde{r} = \eta(\eta - \sigma s)$
$\tilde{\sigma} = \sqrt{\sigma^2 + \eta^2 - 2\sigma\eta s}$