

# CS 215: Data Analysis and Interpretation: Assignment

Instructor: Suyash P. Awate

## Submission Instructions:

- IITB and CSE have zero tolerance to plagiarism.
- For the sake of effective learning, if you submit the solution to the assignment as a group, then each member of the group agrees to have participated fully (100%) in performing every part of every question in the assignment.
- If you submit the solution to the assignment (by yourself or as a group), then you agree that every line of code and every line in the report is your (or your group's) own, and isn't a copied/modified version of any other source online (on the internet) or offline (in electronic form or paper form or any other form).
- If you submit the solution to the assignment as a group and any member of the group is determined to have committed any (non-zero amount of) plagiarism, then the full penalty (a reduction of at least 1 letter grade, e.g., from AB to BB or even lower) will be applicable to every member of the group. The penalty will be applicable to givers and takers both.
- Submit your solution to each problem, i.e., (i) the code, (ii) the results, e.g., graphs or other data, and (iii) the report (in Adobe PDF format), for each question, through moodle. Put the code within the folder "code", the results within the folder "results", and the report within the folder "report".
- Submit all code that allows the TAs to regenerate your results, exactly as they appear in the report.
- Submit a single zip file that contains the solutions to all problems in the assignment.
- To get any possible partial credit for the code, ensure that the code is very well documented. To get partial credit for the derivations, include all derivation steps in their full details.
- To avoid non-deterministic results in each program run, and to make the results reproducible during test time, use `rng(seed)` where seed is a fixed hard-coded integer in your code.
- If the question suggests the use of some function in Matlab, then you can use a corresponding function in other coding frameworks/languages.
- Delayed submissions will be penalized 25% of the total points on each day after the deadline, i.e., submitting anytime within the first 24 hours after the deadline will incur a penalty of 25% of the total points.
- If you feel there is a typo in the question, please make suitable assumptions, consistent with those in the question, and proceed to solve the problem. Also, in that case, please let the TAs or the instructor know.
- **5 points** are reserved for submission in the proper format.

1. (25 points) Sampling within a Euclidean Plane.

- (a) (7 points) Consider an ellipse, within a 2D Euclidean plane, with center at the origin, and with major and minor axes of lengths 2 and 1 along the cardinal axes. Propose an implementable algorithm for generating random points (in 2D) distributed uniformly inside the ellipse.
- (b) (5 points) Implement the algorithm described in the previous sub-question. Use it to sample  $N = 10^7$  independent points (in 2D) and plot a 2D histogram of the sample. You may use the Matlab function `histogram2()`, e.g., with 'DisplayStyle' as 'tile'.
- (c) (8 points) Consider a triangle in a 2D Euclidean plane with vertices at  $(0, 0)$ ,  $(\pi, 0)$ ,  $(\pi/3, \exp(1))$ . Propose an implementable algorithm for generating random points (in 2D) distributed uniformly inside the triangle.
- (d) (5 points) Implement the algorithm described in the previous sub-question. Use it to sample  $N = 10^7$  independent points (in 2D) and plot a 2D histogram of the sample. You may use the Matlab function `histogram2()`, e.g., with 'DisplayStyle' as 'tile'.

2. (20 points) Multivariate Gaussian.

Read all instructions before you start.

Generate  $N$  points (with  $N$  taking the values  $10, 10^2, 10^3, 10^4, 10^5$ ) from a multivariate  $2D$  Gaussian probability density function with mean  $\mu = [1, 2]'$  and a covariance matrix  $C$  with the first row as  $[1.6250, -1.9486]$  and the second row as  $[-1.9486, 3.8750]$ .

For this generation, you are only allowed to use the `randn()` and `eig()` functions in Matlab.

For each data sample of size  $N$ , compute the maximum-likelihood (ML) estimates of the mean and the covariance matrix.

For this estimation, you are only allowed to use the `sum()` function in Matlab.

- (a) (5 points) Describe and justify your method for generating sample points from the 2D Gaussian.
- (b) (5 points) For each value of  $N$ , repeat the experiment 100 times, and plot a boxplot of the error between the true mean  $\mu$  and the ML estimate  $\hat{\mu}_N$  (which depends on  $N$ ), where the error measure is  $\|\mu - \hat{\mu}_N\|_2 / \|\mu\|_2$ . Use a logarithmic scale on the horizontal axis, i.e.,  $\log_{10} N$ .
- (c) (5 points) For each value of  $N$ , repeat the experiment 100 times, and plot a boxplot of the error between the true covariance  $C$  and the ML estimate  $\hat{C}_N$  (which depends on  $N$ ), where the error measure is  $\|C - \hat{C}_N\|_{\text{Fro}} / \|C\|_{\text{Fro}}$ . Use a logarithmic scale on the horizontal axis, i.e.,  $\log_{10} N$ .
- (d) (5 points) For each value of  $N$ , for a single data sample, within a single figure, plot the  $2D$  scatter plot of the generated data and show the principal modes of variation of the data by plotting a line starting at the empirical mean and going a distance equal to the empirical eigenvalue's square root along a direction given by the empirical eigen-vector.

3. (15 points) PCA and Hyperplane Fitting.

Read all instructions before you start.

Consider the observed set of points of the form  $(x, y) \in \mathbb{R}^2$  in the file "points2D\_Set1.mat". Assume each observation  $(x, y)$  is drawn independently from the joint probability density function  $P(X, Y)$  of random variables  $X$  and  $Y$ .

For this question, you cannot use the functions `mean()`, `cov()`, and `pca()` in Matlab.

- (5 points) How can principal component analysis (PCA) be used to best approximate a linear relationship between random variables  $X$  and  $Y$ . Describe the method clearly, using appropriate mathematical descriptions for clarity. Your description should be clear enough to lead to a programmable implementation.
- (5 points) Show a scatter plot of the points. Overlay on the scatter plot, the graph of a line showing the linear relationship between  $Y$  and  $X$ .
- (5 points) Repeat the same analysis for the set of points in "points2D\_Set2.mat". Show a scatter plot of the points. Overlay on the scatter plot, the graph of a line showing the linear relationship between  $Y$  and  $X$ . Compared to the result on the other set of points, justify the quality of the approximation resulting in this question using logical arguments.

4. (25 points) Principal Component Analysis (PCA).

Read all instructions before you start.

Download the dataset comprising images of handwritten digits in <http://yann.lecun.com/exdb/mnist>; this has been downloaded in the folder “data” and stored as “mnist.mat”. Use the entire training set of 60000 examples.

Each image is stored as a matrix ( $28 \times 28$ ) of numbers. You can visualize these images (or matrices) in Matlab using the functions `imagesc()` or `imshow()`. Use the Matlab command “axis equal” to use the same units on each axis of the image.

For the following computations, make sure to convert (cast) the integer data type to a floating-point type. For this question, you cannot use the functions `mean()`, `cov()`, and `pca()` in Matlab.

For every digit, from 0 to 9, compute:

- (i) the mean  $\mu$  (3 points),
- (ii) the covariance matrix  $C$  (5 points), and
- (ii) the principal mode of variation determined by the eigenvector  $v_1$  and the corresponding eigenvalue  $\lambda_1$  (where  $\lambda_1$  is the largest of all eigenvalues) of the covariance matrix  $C$  (7 points).

Note: Before computing the mean and covariance matrix, convert each  $28 \times 28$  pixel image matrix to a  $28^2 \times 1$  vector by concatenating its columns. To visualize the  $28^2 \times 1$  mean vector, convert it back to a matrix and then visualize it using `imagesc()`. Use the `reshape()` function to change matrices to vectors and vice versa. The covariance matrix will be of size  $28^2 \times 28^2$ .

- (5 points) For each digit, sort the  $28^2$  eigenvalues of the covariance matrix and plot them as a graph. Comment and justify what you observe. How many “principal” / significant modes of variation (i.e., number of “large” eigenvalues) do you find, for each digit ? Are the significant modes of variation equal to  $28^2$  or far less ? Why ?
- (5 points) For each digit, show the 3 images side by side: (i)  $\mu - \sqrt{\lambda_1} v_1$ , (ii)  $\mu$ , and (iii)  $\mu + \sqrt{\lambda_1} v_1$ , to show the principal mode of variation of the digits around their mean. Comment and justify what you observe. For a certain digit, say 1, what does the principal mode of variation tell you about how people write that digit ?

5. (10 points) Principal Component Analysis (PCA) for Dimensionality Reduction.

Read all instructions before you start.

Download the dataset comprising images of handwritten digits in <http://yann.lecun.com/exdb/mnist>; this has been downloaded in the folder “data” and stored as “mnist.mat”.

As of now, for each digit, each  $28 \times 28$  pixel image is represented using  $28^2$  coordinate values in the Euclidean space of dimension  $28^2$ . Suppose you decide to re-represent the images using only 84 coordinates (instead of  $28^2 = 784$ ) in a 84-dimensional basis for some 84-dimensional hyperplane within the original Euclidean space, such that the chosen 84-dimensional hyperplane maximizes the total dispersion of the original data (for the chosen digit) within the hyperplane.

- (5 points) Write a function to compute those 84 coordinates, for each of the ten digits (0–9).
- (5 points) Give an algorithm for regenerating / reconstructing the image using those 84 coordinates (and the knowledge of the designed 84-dimensional basis). For each of the ten digits (0–9), pick an image, and show the original and the reconstructed images side by side.

6. (25 points) Principal Component Analysis (PCA) for Another Image Dataset

Read all instructions before you start.

Consider the dataset provided within the folder "data\_fruit"

For this question, you cannot use the functions `mean()`, `cov()`, and `pca()` in Matlab.

Each datum is an image of size  $80 \times 80$  pixels with 3 color channels red (R), green (G), and blue (B), i.e., a  $80 \times 80 \times 3$  array. For PCA, each image should be resized to a vector of length 19200. For visualization, reshape each vector back to a RGB image of size  $80 \times 80$  pixels using the function `reshape()`, followed by a shift and rescaling of the values into the range  $[0, 1]$ , followed by displaying the matrix using the function `image()`.

- (9 points: 1 + 4 + 4) Similar to the analysis done in previous question, find the mean  $\mu$ , the covariance matrix  $C$ , and the top 4 principle **eigenvectors** of  $C$ . Display the **mean** and the **eigenvectors** as images (side by side, in the same figure); you can use the function `subplot()`. Find the top 10 **eigenvalues**, sort them, and plot their values on a graph. Use the function `eigs()` for efficient computation.
- (8 points: 4 + 4) For each fruit image in the dataset, finds its **closest** representation as a linear combination of the top 4 eigenvectors added to the mean. Use the measure of closeness as the Frobenius norm of the difference. Describe the algorithm used to produce this closest representation in mathematical terms and describe the logic behind your algorithm. Display the original fruit image and its closest representation, as images (side by side, in the same figure).
- (9 points: 3 + 3 + 3) Using all of the top 4 eigenvectors and the mean image, **sample** random images to generate new images of "fruit". Describe the underlying algorithm clearly in words and including suitable mathematical notation. Display three such images that are distinct from any image in the given the dataset, but are representative of the dataset and can be considered as that of a new / generated fruit.