CS215 Assignment3 Problem 3

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1 Question

Suppose random variable X has a uniform distribution over $[0, \theta]$, where the parameter θ is unknown. Consider a Pareto distribution prior on θ, with a scale parameter $\theta_m > 0$ and a shape parameter $\alpha > 1$, as $P(\theta) \propto (\theta_m/\theta)^\alpha$ for $\theta \geq \theta_m$ and $P(\theta) = 0$ otherwise.

1.1

Find the maximum-Likelihood estimate $\hat{\theta}^{ML}$ and the maximum-a-posteriori estimate $\hat{\theta}^{MAP}$

Solution-
Let the observations be $X_1, X_2, \cdots, X_n$ and let their maximum be X_M.
The Likelihood function L is 0 for $\theta < X_M$ and 1 otherwise.
Therefore

$$\hat{\theta}^{ML} = X_M$$

The posterior is product of L and the Pareto Distribution P. Let $X_{MAX} = max(X_M, \theta_m)$
Hence posterior distribution is

$$f(x) = \begin{cases} 0 & x < X_{MAX} \\ (\theta_m/\theta)^\alpha & x \geq X_{MAX} \end{cases}$$

Therefore

$$\hat{\theta}^{MAP} = X_{MAX}$$

1.2

Does $\hat{\theta}^{MAP}$ tend to $\hat{\theta}^{ML}$ as the sample size tends to infinity ? Is this desirable or not?

Solution-
As n tends to infinity X_m tends to θ_{true} which is $\geq \theta_m$. So X_{MAX} tends to X_m.

Hence $\hat{\theta}^{MAP}$ tend to $\hat{\theta}^{ML}$ as $n \to \infty$.

This is desirable as in this case the MLE estimate is unbiased and will give extremely precise result for large sample sizes.

1.3

Find an estimator of the mean of the posterior distribution $\hat{\theta}^{\text{PosteriorMean}}$

Solution-
We have given the Posterior distribution above, now we simply find its mean.

$$\int_{X_{MAX}}^{+\infty} \theta \left(\frac{\theta_m}{\theta}\right)^\alpha d\theta$$

This exists for $\alpha > 2$ and equals

$$\hat{\theta}^{\text{PosteriorMean}} = \frac{(\theta_m)^\alpha}{(X_{MAX})^{\alpha-2} \times (\alpha - 2)}$$

1.4

Does $\hat{\theta}^{\text{PosteriorMean}}$ tend to θ^{ML} as the sample size tends to infinity ? Is this desirable or not?

Solution-
We have seen that as sample size tends to infinity X_{MAX} tend to X_M.
Hence $\hat{\theta}^{\text{PosteriorMean}}$ tends to

$$\frac{(\theta_m)^\alpha}{(X_M)^{\alpha-2} \times (\alpha - 2)}$$

But this is not equal to $X_M = \theta^{ML}$.

Hence $\hat{\theta}^{\text{PosteriorMean}}$ does not tend to θ^{ML} as the sample size tends to infinity.

This is not desirable as at large sample sizes ML estimator gives accurate estimates.