CS 215 Assignment 1 Problem 2

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1 Sum of two Random Variables

1.1 Empirical Calculation

SumOfPoisson function generates 10^6 random outputs of each X and Y and adds them. The value is recorded in the variable ranz. There is also a variable Empirical which stores the number of occurrences of 1 to 25 divided by 10^6 to get their probability.

1.2 Theoretical Derivation

We calculate the theoretical probability of $Z = X+Y$, where X, Y are Poisson random variables with $\lambda_X = 3$ and $\lambda_Y = 4$

$$\begin{aligned}
P(Z = n) &= \sum_{i=0}^{n} P(X = i)P(Y = n - i) \\
&= \sum_{i=0}^{n} e^{-3}\frac{3^i}{i!} \times e^{-4}\frac{4^{n-i}}{(n-i)!} \\
&= \sum_{i=0}^{n} e^{-7} \times \frac{1}{i!(n-i)!} \times 3^i 4^{n-i} \\
&= \sum_{i=0}^{n} \frac{e^{-7}}{n!} \times \frac{n!}{i!(n-i)!} \times 3^i 4^{n-i} \\
&= \frac{e^{-7}}{n!} \sum_{i=0}^{n} \binom{n}{i} 3^i 4^{n-i} \\
&= \frac{e^{-7}}{n!} 7^n
\end{aligned}$$

Hence we get the following result :

Result

If X and Y are poisson random variables with $\lambda_X = 3$ and $\lambda_Y = 4$ then $Z = X + Y$ is also a poisson random variable with $\lambda_Z = \lambda_X + \lambda_Y = 3 + 4 = 7$

Hence

$$P(Z = k) = e^{-7}\frac{7^k}{k!}$$

1.3 Comparison

We compare the values in a tabular form and see that they indeed match to a high degree of accuracy.

	Empirical	Theoretical
1	0.0064	0.0064
2	0.0225	0.0223
3	0.0525	0.0521
4	0.0915	0.0912
5	0.1277	0.1277
6	0.1494	0.1490
7	0.1490	0.1490
8	0.1300	0.1304
9	0.1009	0.1014
10	0.0711	0.0710
11	0.0450	0.0452
12	0.0264	0.0263
13	0.0139	0.0142
14	0.0070	0.0071
15	0.0033	0.0033
16	0.0014	0.0014
17	0.0006	0.0006
18	0.0002	0.0002
19	0.0001	0.0001
20	0	0
21	0	0
22	0	0
23	1.0000e-06	9.6538e-07
24	0	2.8157e-07
25	0	7.8840e-08

Figure 1: Empirical vs Theoretical

2 Poisson Thinning

2.1 Empirical Calculation

In Poisson thinning, if n bullets hit in unit interval then we do further selection on them with every hit being chosen with a probability of 0.8.
Rigorously, let Y be a Poisson random variable. Then we defined a new random variable Z such that Given $Y = n$, the probability of $Z = k$ where $k \leq n$ is $\binom{n}{k}p^k(1-p)^{n-k}$.
Formally

$$P(Z = k|Y = n) = \binom{n}{k}p^k(1-p)^{n-k}$$

To simulate this we first get a random number with $poissrnd(4)$ and then pass it as an argument to $binrand()$ along with the probability parameter 0.8.
We then note down the number of occurences of integers between 1 and 25 for comparison.

2.2 Theoretical Derivation

We first do it for general $Y = poisson(\lambda)$.

$$P(Z = k) = \sum_{n=k}^{\infty} P(Z = k|Y = n)P(Y = n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k}p^k(1-p)^{n-k}e^{-\lambda}\frac{\lambda^n}{n!}$$

$$= e^{-\lambda}\sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}\frac{\lambda^{(n-k)+k}}{n!}$$

$$= e^{-\lambda}\sum_{n=k}^{\infty} \frac{1}{k!(n-k)!}(\lambda p)^k(\lambda(1-p))^{n-k}$$

$$= e^{-\lambda}\frac{(\lambda p)^k}{k!}\sum_{n=k}^{\infty} \frac{1}{(n-k)!}(\lambda(1-p))^{n-k}$$

$$= e^{-\lambda}\frac{(\lambda p)^k}{k!}\sum_{m=0}^{\infty} \frac{1}{m!}(\lambda(1-p))^{m}$$

$$= e^{-\lambda}\frac{(\lambda p)^k}{k!} \times e^{\lambda(1-p)}$$

$$= e^{-\lambda p}\frac{(\lambda p)^k}{k!}$$

And this is just a Poisson distribution with parameter λp.
For given values this evaluates to $4 \times 0.8 = 3.2$.
Hence the theoretical distribution is simply Poisson with $\lambda = 3.2$, that is :

$$P(Z = k) = e^{-3.2}\frac{3.2^k}{k!}$$

2.3 Comparison

The empirical and theoretical values are plotted in a table for comparison.

	answer	Theoretical
1	0.1321	0.1304
2	0.2080	0.2087
3	0.2211	0.2226
4	0.1783	0.1781
5	0.1143	0.1140
6	0.0609	0.0608
7	0.0278	0.0278
8	0.0114	0.0111
9	0.0040	0.0040
10	0.0012	0.0013
11	0.0004	0.0004
12	0.0001	0.0001
13	0	0
14	0	0
15	0	0
16	0	0
17	0	0
18	0	0
19	0	0
20	0	0
21	0	0
22	0	0
23	0	6.5496e-13
24	0	8.7328e-14
25	0	1.1178e-14

Figure 2: Empirical vs Theoretical for Thinning