Digital Image Processing HW3

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1 Question 1

We implement the mean shift as follows:
Input:

- Image X
- Kernel size k
- σ_s, σ_r as Gaussian standard deviations
- w_1, w_2, h as thresholding values

Output:

- Filtered Image Y
- C_arr, a list of Centroids, separated by threshold values

Note 1. : Each entry of C_arr corresponds to a centroid (mean) of all the points that are currently mapped to it and contains their mean (via a **rolling average**) x and y coordinates and intensity as well as the number of points in the original image that converge to it, i.e., $C\_arr[index] = [\bar{x}, \bar{y}, \bar{I}, n]$

Algorithm 1: Mean Shift

Data: $X, k\sigma_s, \sigma_r, w_1, w_2, h$
Result: C_arr, Y
$x, y \leftarrow 0, 0$;
$C\_arr \leftarrow [,,]$;
while $x, y \in X$ **do**
  $x_0, y_0, I_0, t \leftarrow x, y, X(x, y), 0$;
  while *! converged* **do**
    $[x_{t+1}, y_{t+1}, I_{t+1}] \leftarrow \sum_{i_k} [x_i, y_i, I_i] \mathcal{N}([x_i, y_i], [x_t, y_t], \sigma_s) \cdot \mathcal{N}(I_i, I_t, \sigma_r)$;
    $t \leftarrow t + 1$;
  end
  index $\leftarrow \underset{c \in C\_arr}{\arg\min} ||c - [x_k, y_k, I_k]||$;
  if $||C\_arr[index] - [x_k, y_k, I_k]|| \leq h$ **then**
    $n \leftarrow C\_arr[index][4]$;
    $C\_arr[index][1:3] \leftarrow (n/n+1) C\_arr[index][1:3] + (1/n+1)[x_k, y_k, I_k]$;
    $C\_arr[index][4] + 1$;
  else
    $C\_arr.append([x_k, y_k, I_k, 1])$;
  end
  if *segmentation* **then**
    $X(x, y) \leftarrow C\_arr[index][3]$;
  else
    if *smoothing* **then**
      $X(x, y) \leftarrow I_k$;
    else
  end
end

Where $||[x, y, I]|| = w_1(x^2 + y^2) + w_2(I^2)$ is a **norm** in both spatial and intensity coordinates that makes sure that each point is mapped to a centroid that is close to it both spatially and in terms of intensity. The following pages contain results on the images provided.

Figure 1: Image with Gaussian Noise $\sigma = 5$

Figure 2: $\sigma_s = 2$, $\sigma_r = 2$

Figure 3: $\sigma_s = 0.1$, $\sigma_r = 0.1$

Figure 4: $\sigma_s = 3$, $\sigma_r = 15$

Figure 5: Image with Gaussian Noise $\sigma = 10$

Figure 6: $\sigma_s = 2$, $\sigma_r = 2$

Figure 7: $\sigma_s = 0.1$, $\sigma_r = 0.1$

Figure 8: $\sigma_s = 3$, $\sigma_r = 15$

Figure 9: Image with Gaussian Noise $\sigma = 5$

Figure 10: $\sigma_s = 2$, $\sigma_r = 2$

Figure 11: $\sigma_s = 0.1$, $\sigma_r = 0.1$

Figure 12: $\sigma_s = 3$, $\sigma_r = 15$

Figure 13: Image with Gaussian Noise $\sigma = 10$

Figure 14: $\sigma_s = 2$, $\sigma_r = 2$

Figure 15: $\sigma_s = 0.1$, $\sigma_r = 0.1$

Figure 16: $\sigma_s = 3$, $\sigma_r = 15$

It seems as though the images are being segmented but they are smoothened because of non strict thresholds and updations. An extreme case of this with more strict thresholds and usage of $I(x_k, y_k)$ instead of I_k gives segmented images as follows: (in `imagesc`)

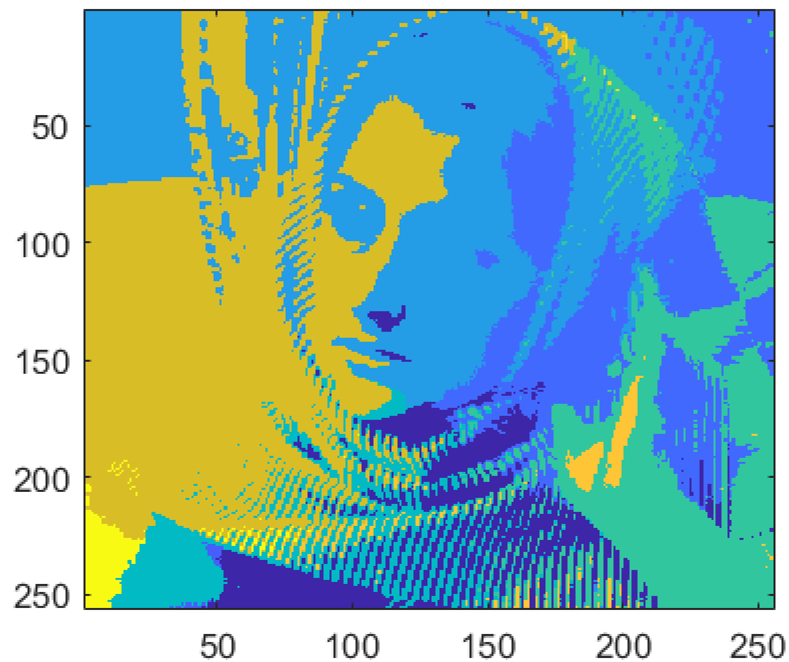

Figure 17: Segmented barbara

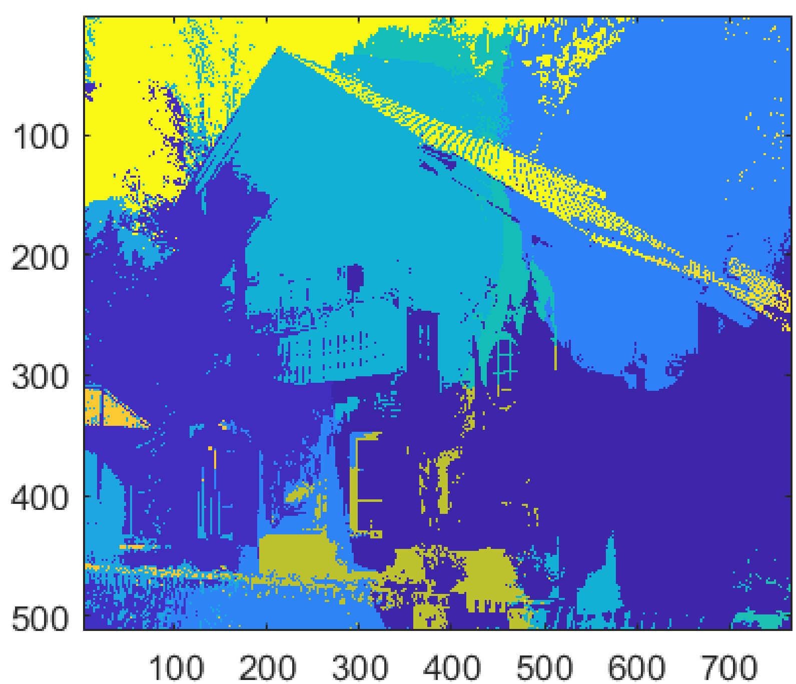

Figure 18: Segmented house

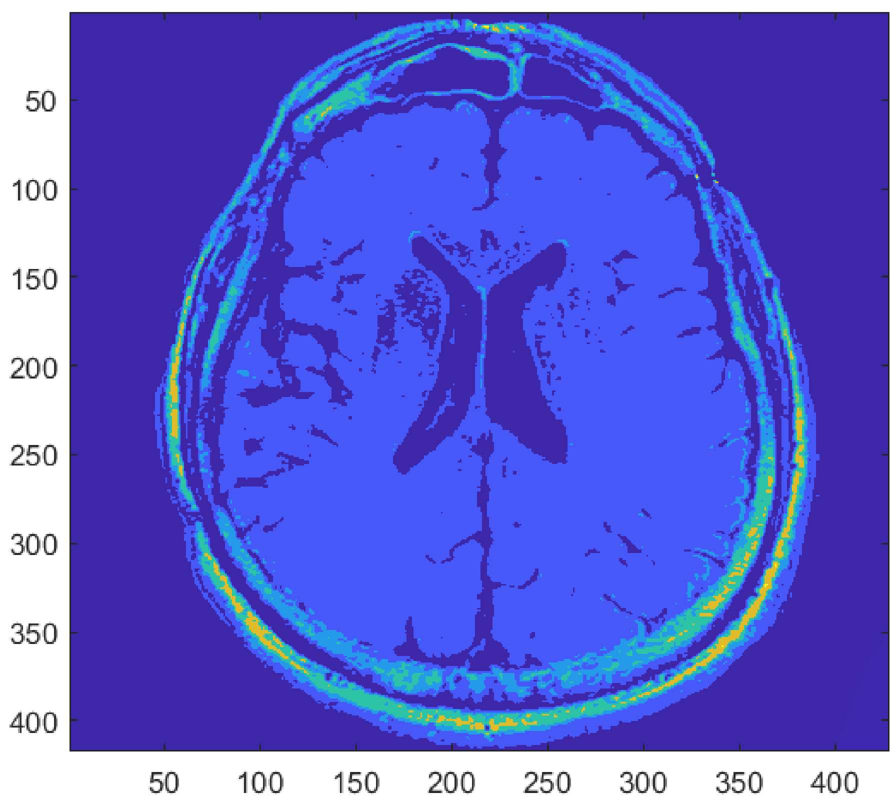

Figure 19: Segmented MRI

2 Question 2

We begin with the **Ideal low pass** filter. It has the form:

$$I(u,v)_h = \begin{cases} 0, & \text{if } u^2+v^2 < D^2 \\ 1, & \text{otherwise} \end{cases} \tag{1}$$

We use the `barbara256` image. The following are the image after adding 0 mean Gaussian with $\sigma = 5$ and its Fourier transform respectively.

Figure 20: barbara256 with noise

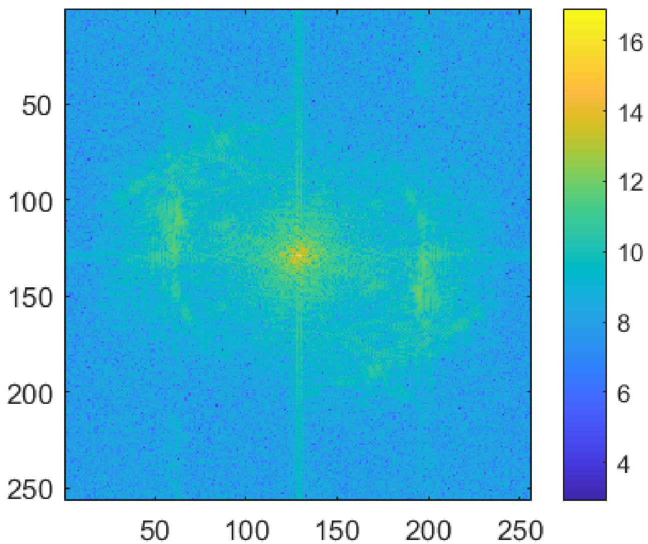

Figure 21: Fourier Transform log absolute form

The following are the transforms after the application of the ideal low pass filters with D values of 40 and 80 respectively (in log absolute format).

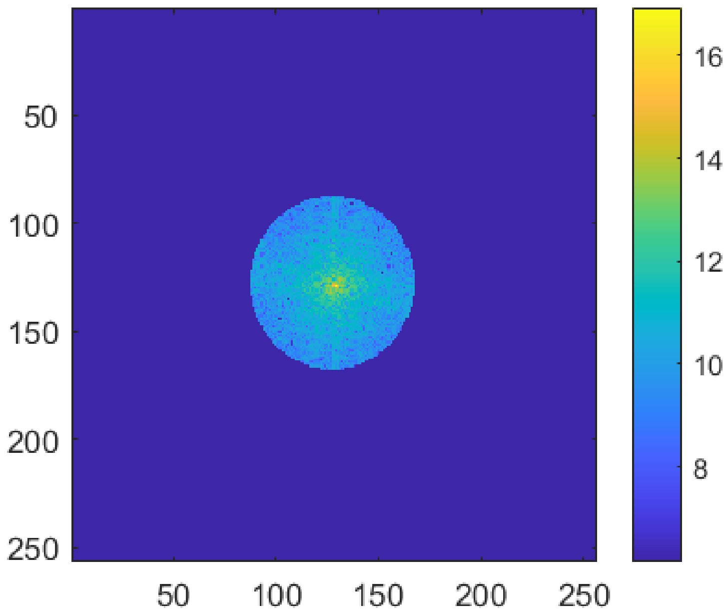

Figure 22: Ideal D=40

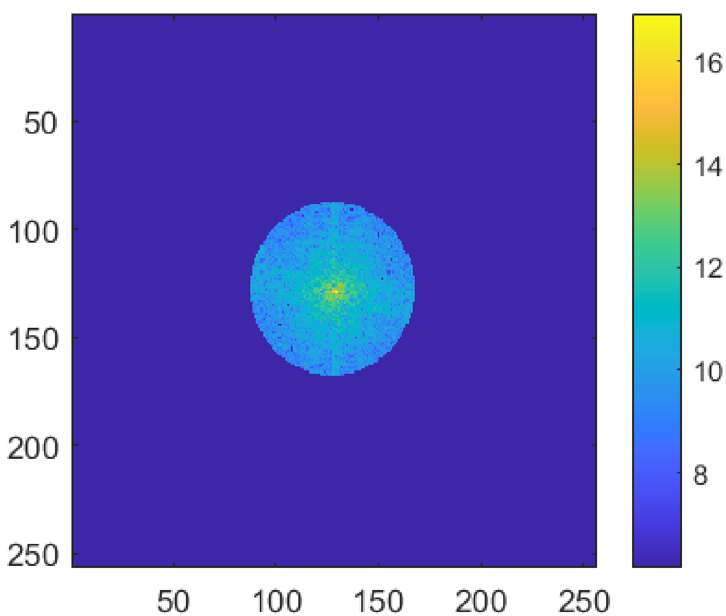

Figure 23: Ideal D=80

The following are the transformed images after applying the ideal low-pass filters.

Figure 24: Ideal D=40

Figure 25: Ideal D=80

The Gaussian low pass filter has the following form:

$$G(u,v)_\sigma = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u^2+v^2)}{2\sigma^2}} \tag{2}$$

Therefore, its corresponding spatial form is the inverse Fourier transform of G which is,

$$g(x,y) = e^{-\frac{x^2+y^2}{2\cdot(1/\sigma^2)}} \tag{3}$$

This must be convolved with our image. We build a kxk kernel for the Gaussian filter (k=71 in our case). However, due to the high values of σ, the resultant frequency domain kernel is almost constant, the spatial kernel is almost a Kronecker Delta and convolving it with the image gives back the image.

The following are the kernels' frequency and spatial domain respectively (in log absolute format) (they are almost the same for $\sigma = 40$ and 80: The following is the image after the Gaussian Low pass filter:

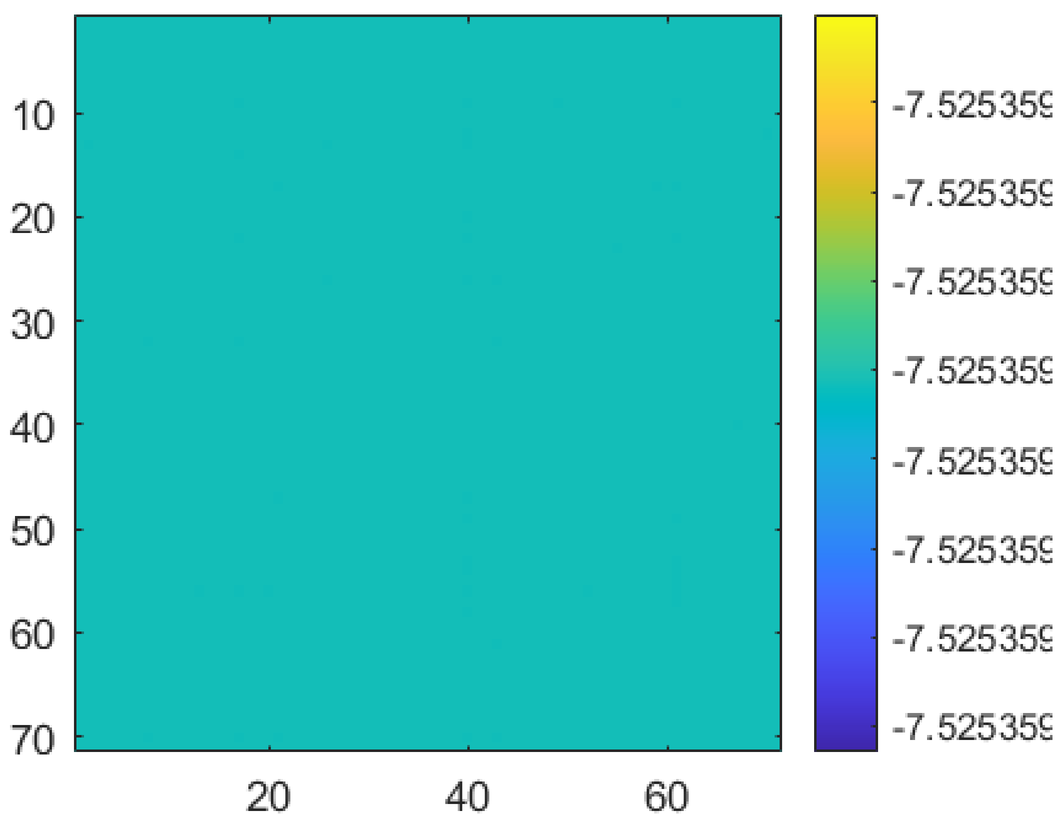

Figure 26: Gaussian Low pass Filter frequency domain

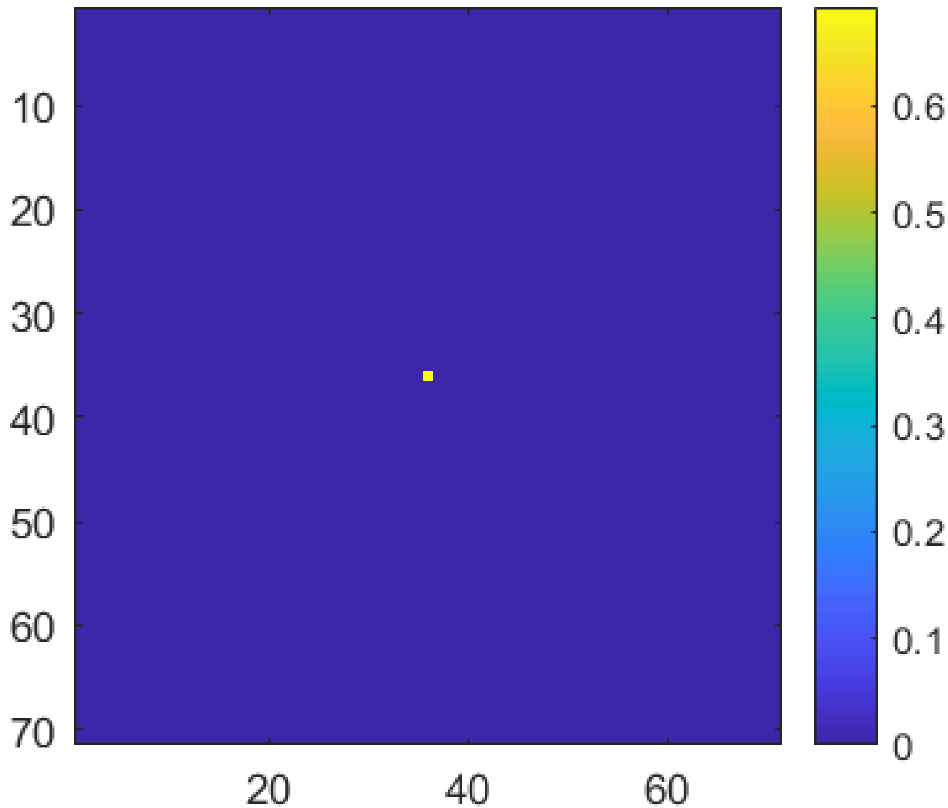

Figure 27: Gaussian Low pass Filter spatial domain

Figure 28: Filtered Image

Analysis:
The ideal low pass filters seem to have smoothened the image considerably, which is expected because all the high frequency artifats (edges, corners, etc.) are filtered out. More smoothing is seen for $D = 80$ than for $D = 40$, which is also expected because a higher D value cuts off more frequencies, giving a smoother image.

The Gaussian low pass creates a gaussian filter in the spatial domain as well. We already know that Gaussian Filters are smoothing (blurring) so it's no surprise that the low pass Gaussian filter smoothens (blurs) an image. We note that these filters aren't normalized. Further, $\sigma = 40, 80$ are very large and the filter is almost constant (if the kernel is small). We also do an analysis with a smaller standard deviation size (3) and present it below. The following are the $\sigma = 3$ standard gaussian low-pass filter in frequency domain, spatial domain and the filtered image respectively.

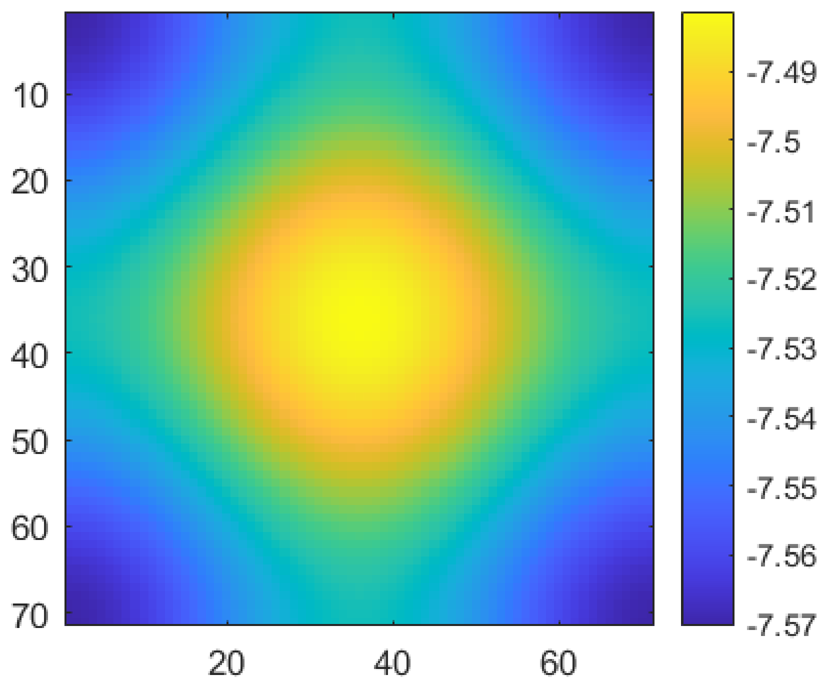

Figure 29: Gaussian Low pass Filter frequency domain

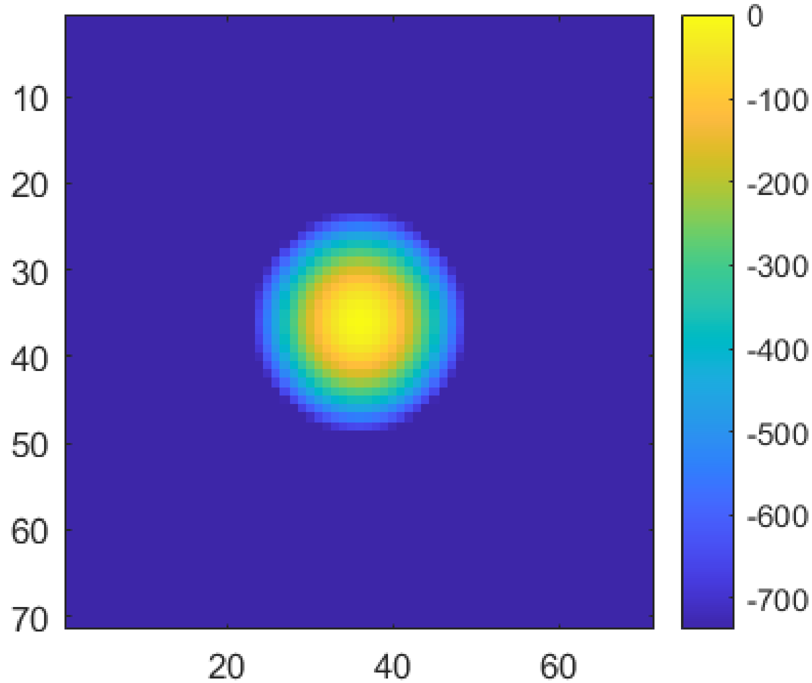

Figure 30: Gaussian Low pass Filter spatial domain

Figure 31: Filtered image for $\sigma = 3$

3 Question 3

The convolution theorem for 2D discrete Fourier transforms is as follows:

$$\mathcal{F}[f \otimes g] = F \cdot G \tag{4}$$

and

$$f \otimes g = \mathcal{F}^{-1}[F \cdot G] \tag{5}$$

The proof goes as follows:

$$\begin{aligned}
h(x,y) =& (f \otimes g)(x,y) \\
=& \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} f(m,n) \cdot g(x-m, y-n) \\
\mathcal{F}[h(x,y)](u,v) =& \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} f(m,n) \cdot g(x-m, y-n) \cdot e^{-j2\pi(ux+vy)} \\
=& \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} f(m,n) \cdot \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} g(x-m, y-n) \cdot e^{-j2\pi(ux+vy)} \\
&\text{Substituting } x-m \text{ by } x' \text{ and } y-n \text{ by } y' \text{ we get} \\
\mathcal{F}[h(x,y)](u,v) =& \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} f(m,n) \cdot \sum_{x'=-\infty}^{+\infty} \sum_{y'=-\infty}^{+\infty} g(x', y') \cdot e^{-j2\pi(u(x'+m)+v(y'+n))} \\
=& \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} f(m,n) \cdot e^{-j2\pi(um+vn)} \cdot \sum_{x'=-\infty}^{+\infty} \sum_{y'=-\infty}^{+\infty} g(x', y') \cdot e^{-j2\pi(ux'+vy')} \\
=& F(u,v) \cdot G(u,v)
\end{aligned}$$

The other side follows by taking the Fourier inverse on both sides.

4 Question 4

The Fourier transform of such an image can be done first in the y direction to get constant functions in y because the Fourier transform of a delta is constant. Then applying the Fourier transform along x gives dirac deltas along x = 0. The final transform is a delta along the line x = 0.

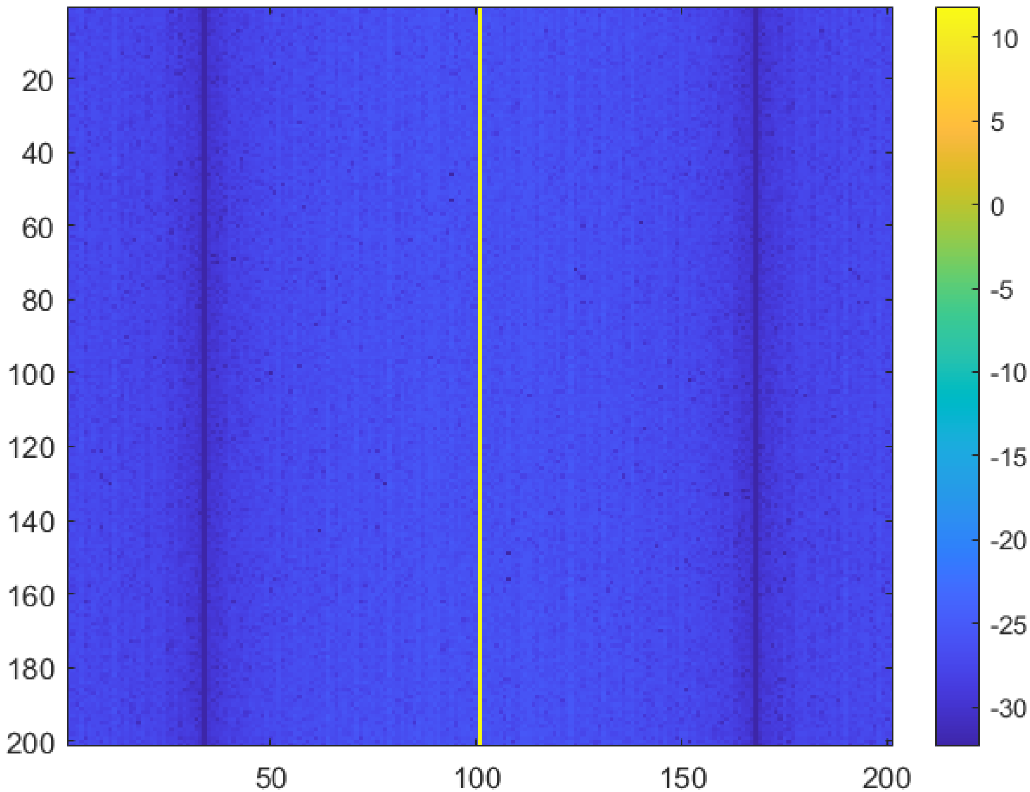

Figure 32: Fourier Transform

5 Question 5

To prove:
If $f(x, y)$ is real, then its discrete Fourier transform $F(u, v)$ satisfies $F^*(u, v) = F(-u, -v)$

Proof:

$$F(u, v) = \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} f(x, y) \cdot e^{-j2\pi(ux+vy)}$$

Taking the conjugate of both sides and using the properties,

$$(a + b)^* = a^* + b^*$$
$$(a \cdot b)^* = a^* \cdot b^*$$

$$F^*(u, v) = \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} f^*(x, y) \cdot e^{+j2\pi(ux+vy)}$$

Since $f(x, y)$is real, $f^*(x, y) = f(x, y)$

$$F^*(u, v) = \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} f(x, y) \cdot e^{-j2\pi((-u)x+(-v)y)}$$
$$=F(-u, -v)$$

To prove:
If $f(x, y)$ is real and even, then $F(u, v)$ is also real and even.
We use the fact that a function f is even **iff** $f(-x, -y) = f(x, y)$ and is real **iff** $f^*(x, y) = f(x, y)$

Then since f is real, we get from the previous equation,

$$F^*(u, v) = F(-u, -v) = \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} f(x, y) \cdot e^{+j2\pi(ux+vy)}$$

Substituting x and y by $-x$ and $-y$ respectively (note that the summation limits change in both summations so the sign change cancels out and the limits remain the same) we get,

$$F^*(u, v) = F(-u, -v) = \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} f(-x, -y) \cdot e^{-j2\pi(ux+vy)}$$

But since f is even,

$$= \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} f(x, y) \cdot e^{-j2\pi(ux+vy)}$$
$$=F(u, v)$$

i.e.,

$$F^*(u, v) = F(-u, -v) =F(u, v)$$

Hence, F is real and even.

6 Question 6

If $\mathcal{F}$ is the continuous Fourier operator, to prove that:
$\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = f(t)$

We have,

$$(\mathcal{F}(f(t)))(\mu) = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi\mu t}\, dt$$

$$(\mathcal{F}(\mathcal{F}(f(t))))(\nu) = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(t) \cdot e^{-j2\pi\mu t}\, dt\right) \cdot e^{-j2\pi\nu\mu}\, d\mu$$

Flipping integrals,

$$= \int_{-\infty}^{+\infty} f(t) \cdot \left(\int_{-\infty}^{+\infty} e^{-j2\pi\mu t} \cdot e^{-j2\pi\nu\mu}\, d\mu\right) dt$$

$$= \int_{-\infty}^{+\infty} f(t) \cdot \delta(\nu + t)\, dt$$

Using the sifting property,

$$= f(-\nu)$$

Therefore,

$$(\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t)))))(x) = (\mathcal{F}(\mathcal{F}(f(-\nu)))(x) = f(x)$$

7 Question 7

Consider the three high pass filters, the ideal, the Butterworth and the Gaussian filters in their Fourier domain. They are (respectively) as follows:

$$I(u,v)_h = \begin{cases} 0, & \text{if } u^2 + v^2 < h \\ 1, & \text{otherwise} \end{cases} \tag{6}$$

$$B(u,v)_{(D,n)} = 1 - \frac{1}{1 + (\sqrt{u^2+v^2}/D)^{2n}} \tag{7}$$

$$G(u,v)_\sigma = 1 - e^{-\frac{(u^2+v^2)}{2\sigma^2}} \tag{8}$$

Their values in the spatial domain at $x = 0$, $y = 0$ are:

$$\int\int_{-\infty}^{+\infty} F(u,v) \cdot e^{j2\pi(0u+0v)}\, du\, dv$$

$$= \int\int_{-\infty}^{+\infty} F(u,v)\, du\, dv$$

For each of the filters F.

1) For the ideal filter this is obviously ∞
2) For the Butterworth filter, this is

$$\int\int_{-\infty}^{+\infty} 1\, du\, dv - \int\int_{-\infty}^{+\infty} \frac{1}{1+(\sqrt{u^2+v^2}/D)^{2n}}\, du\, dv$$

The second integral can be converted to polar coordinates,

$$\int_0^{2\pi}\int_0^{\infty} \frac{r}{1+(r/D)^{2n}}\, dr\, d\theta$$
$$= K \cdot 2\pi \int_0^{\infty} \frac{r}{1+r^{2n}}\, dr$$

Substitute $r^2$with t,

$$= K' \cdot 2\pi \int_0^{\infty} \frac{1}{1+t^n}\, dt$$

If $n > 1$,

$$\leq K' \cdot 2\pi \int_0^1 1\, dt + K' \cdot 2\pi \int_1^{\infty} \frac{1}{1+t^2}\, dt$$
$$= K' \cdot 2\pi \left(1 + tan^{-1}(x)|_1^{\infty}\right)$$
$$= K' \cdot 2\pi(1+\pi/4)$$

Thus, the second integral is finite and the first integral is infinite, giving an unbounded value at $x = 0$ and $y = 0$ if n is greater than 1, else the analysis is as follows:

$$\int\int_{-\infty}^{+\infty} 1\, du\, dv - \int\int_{-\infty}^{+\infty} \frac{1}{1+(\sqrt{u^2+v^2}/D)^{2n}}\, du\, dv$$
$$= 2\pi \int_0^{\infty} (r - \frac{r}{1+(r/D)^2})\, dr$$
$$= \frac{2\pi}{D^2} \int_0^{\infty} \frac{r^3}{1+(r/D)^2}\, dr$$
$$= \frac{\pi}{D^2} \int_0^{\infty} \frac{t}{1+t/D^2}\, dt$$
$$\leq \frac{\pi}{D^2} \int_0^{\infty} \frac{t}{1+(t/D)^2}\, dt$$
$$= K'' ln(1+(x/D)^2)|_0^{\infty}$$
$$= \infty$$

3) For the Gaussian filter, this is

$$\int\int_{-\infty}^{+\infty} 1\, du\, dv - \int\int_{-\infty}^{+\infty} e^{-\frac{(u^2+v^2)}{2\sigma^2}}\, du\, dv$$
$$= \infty - 2\pi\sigma^2$$
$$= \infty$$

In short, the value of a function's Fourier inverse at $(0, 0)$ is its integral over all frequency space and since none of the functions are L^1, these values are unbounded.