Advanced Image Processing HW1

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1 Question 1

For any s-sparse vector x,

$$Ax = A_S x_S$$

where S is the set of indices with non zero value and $|S| \leq s$

Consider the matrix $B = A_S^T A_S$, where $b_{ij} =< a_i, a_j >$ and $b_{ii} = ||a_i^2|| = 1$ due to normalization of A.

By Gershgorin's Disk Theorem, every eigenvalue of the matrix $A_S^T A_S$ lies in the the union of $D(b_{ii}, r_i)$ where r_i is sum of absolute values of off-diagonal elements of the i-th row.

$$r_i = \sum_{j=1, j\neq i}^{s} < a_i, a_j > \leq \mu(s-1)$$

Since the matrix $A_S^T A_S$ is positive semidefinite, it has positive real eigenvalues and they lie in the range $[1-\mu(s-1), 1+\mu(s-1)]$

Thus for every s-sparse vector x, we can say that

$$(1-\mu(s-1))||x||^2 \leq ||Ax||^2 \leq (1+\mu(s-1))||x||^2$$

which implies $\delta_s \leq \mu(s-1)$

2 Question 2

Given $A \in \mathbb{R}^{m\times n}$ with order-s isometry constant δ_s, then:

$$\forall x, ||x||_0 \leq s,$$

$$Ax = A_{\mathcal{S}} x \text{ for some } \mathcal{S} \subseteq [n], |\mathcal{S}| = S$$

Given the isometry constant δ_s of A, we have:

$$\forall \text{ s-sparse } x, (1-\delta_s) \leq \frac{||Ax||^2}{||x||^2} \leq (1+\delta_s)$$

But for such x, $||Ax||^2 = ||A_{\mathcal{S}}x||^2 = x^T A_{\mathcal{S}}^T A_{\mathcal{S}} x$ and the minimum and maximum eigenvalues of $A_{\mathcal{S}}^T A_{\mathcal{S}}$ ($\lambda_{\mathcal{S}\min}$ and $\lambda_{\mathcal{S}\max}$ respectively) are equal to:

$$\max_x \text{ and } \min_x \frac{||A_{\mathcal{S}}x||^2}{||x||^2} \text{ respectively}$$

Therefore,

$$\lambda_{\min} = \min_{\mathcal{S},x} \frac{||A_{\mathcal{S}}x||^2}{||x||^2} = \min_x \frac{||Ax||^2}{||x||^2} \text{ and hence } (1-\delta_s) \leq \lambda_{\min}$$

Similarly,

$$\lambda_{\max} \leq (1+\delta_s)$$

Hence, $\delta_s \geq (1-\lambda_{\min})$ as well as $(\lambda_{\max}-1)$. $\therefore, \delta_s \geq \max(1-\lambda_{\min}, \lambda_{\max}-1)$. But since δ_s is the minimum such number, it is equal to $\max(1-\lambda_{\min}, \lambda_{\max}-1)$

3 Question 3

Since not all subsets were checked, the λs obtained are such that

$$\lambda_{\min} \geq \min_x \frac{||Ax||^2}{||x||^2}$$

and

$$\lambda_{\max} \leq \max_x \frac{||Ax||^2}{||x||^2}$$

Therefore, $\lambda_{\min} \geq (1-\delta_s)$ and $\lambda_{\max} \leq (1+\delta_s)$ or $\delta_s \geq (1-\lambda_{\min})$ and $\delta_s \geq (\lambda_{\max}-1)$ giving us $\delta_s \geq \max(1-\lambda_{\min}, \lambda_{\max}-1)$ or

$$\delta_s \geq \hat{\delta}_s$$

4 Question 4

(1) Even though the values of $|\theta - \theta_S||_1$ decreases as S increases, the constants C_1, C_2 are increasing functions of S itself, and thus increasing S doesn't directly translate to decreasing error bound. Further, increasing S beyond the limit $0.5(1-1/\mu)$ renders the Theorem 5 useless.

(2) Increasing measurements m decreases the value of $\mu(\Phi\Psi)$ since we have more dimensions to fit n vectors and prevent them from being correlated too much. Decreasing μ increases the value of S_{max} and thus reduces the error bound.

(3)Theorem 5 is more useful since it has a higher value of S_{max}. As per the theorem, as we can recover θ which are S-sparse with a very high accuracy, a higher bound on S is more useful.

(4)If we are setting our ϵ to below the actual noise value, we may not even obtain a solution to the P2 problem or obtain a solution that is completely useless, and the theorem doesn't help in that case. That is why it is preferred to keep ϵ such that the noise $\eta \leq \epsilon$ with a very high probability.

5 Question 5

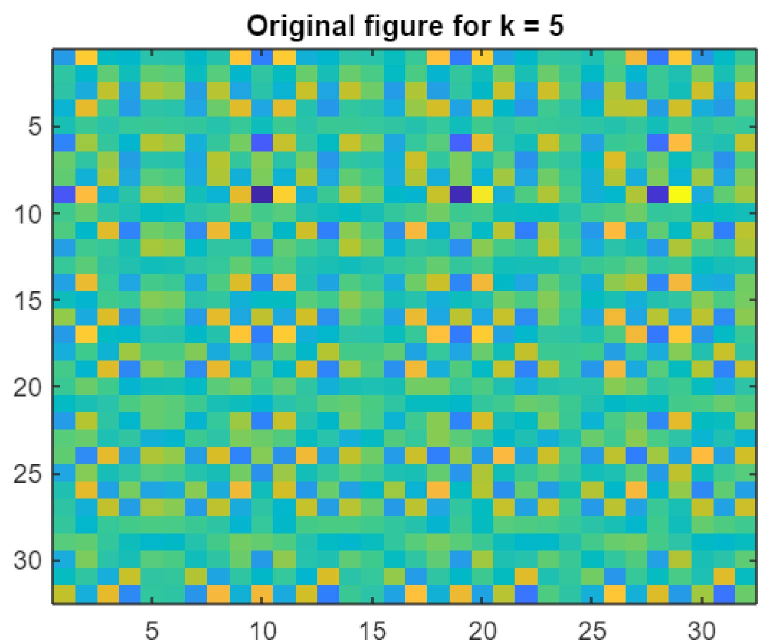
Title of the paper : Underwater Image Enhancement Using Deep Transfer Learning Based on a Color Restoration Model

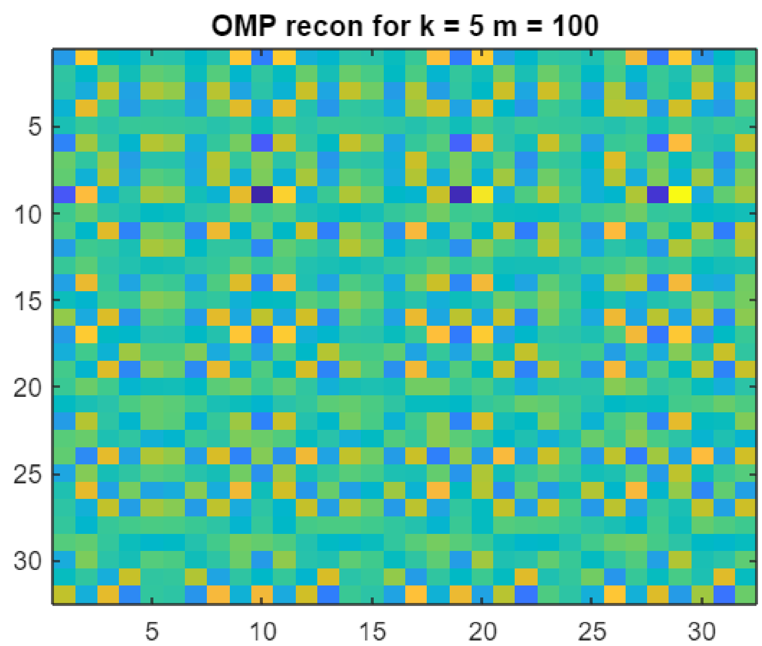
Publication : IEEE Journal of Oceanic Engineering (Volume: 48, Issue: 2, April 2023)

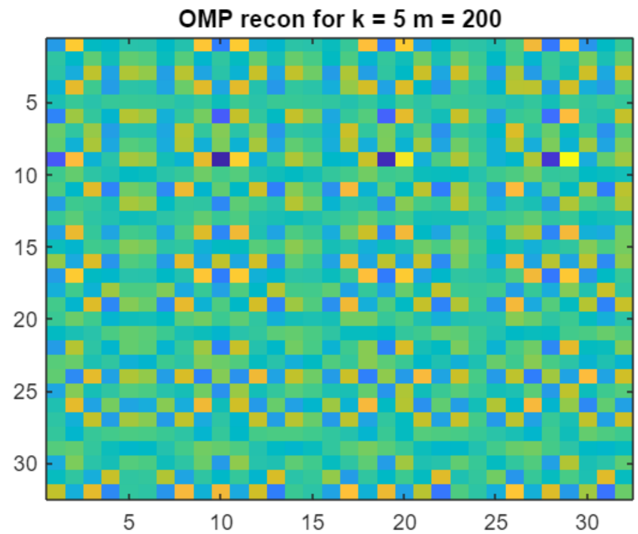
Due to the reflection and attenuation of light when propagating in water, the images captured by the visual system of an underwater vehicle in the complex underwater environment usually suffer from low visibility, blurred details, and color distortion. The paper aims to enhance and restore such underwater images using a Transfer Learning based model.

6 Question 6

The following are the results for the **OMP** algorithm.






OMP recon for k = 5 m = 200
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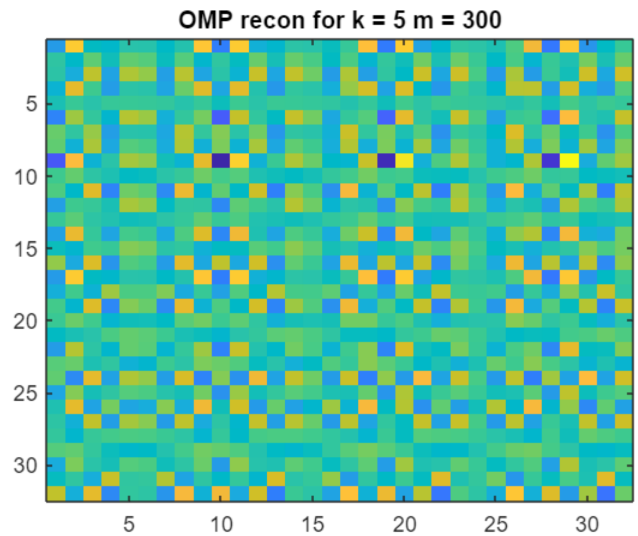
OMP recon for k = 5 m = 300
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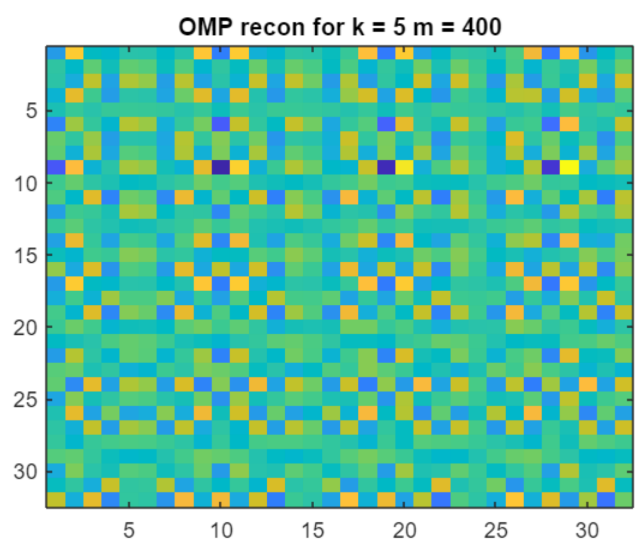
OMP recon for k = 5 m = 400
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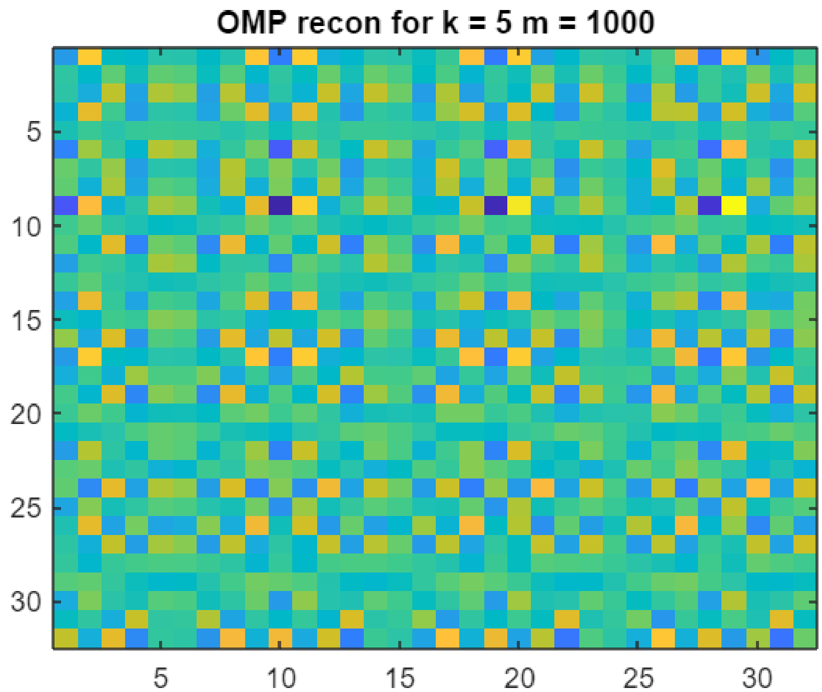
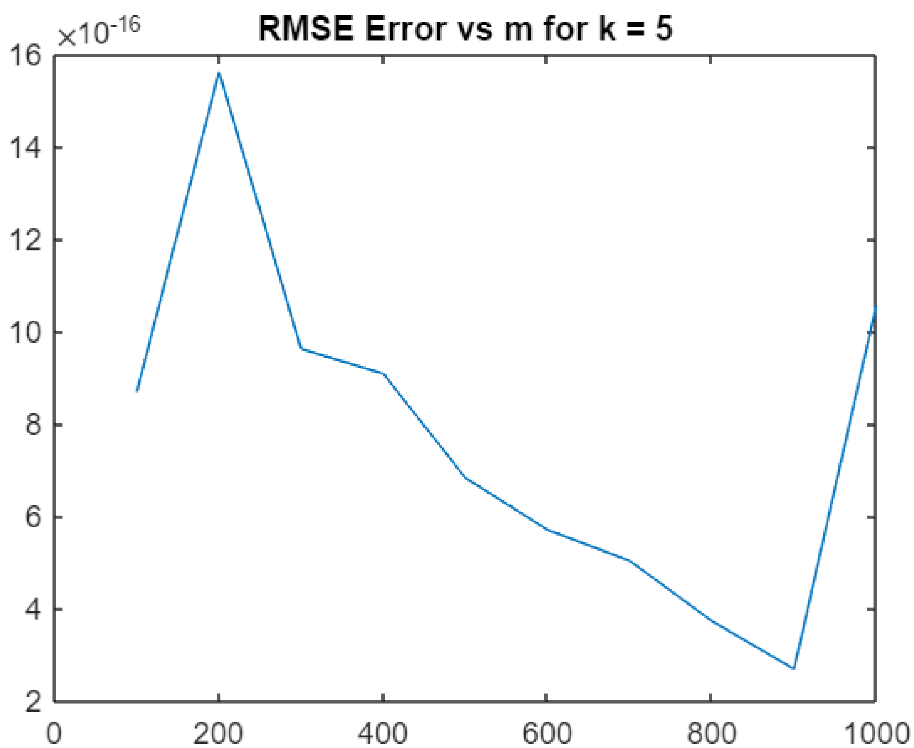
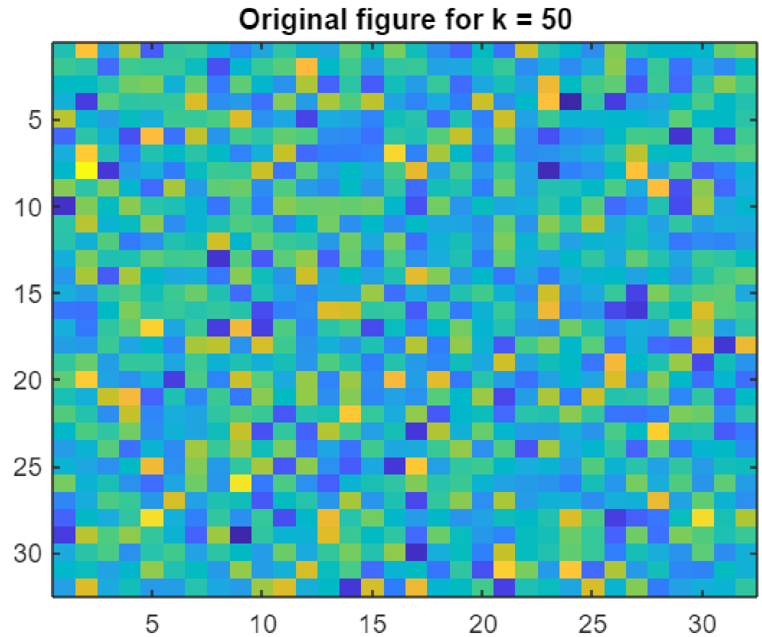

OMP recon for k = 5 m = 1000


RMSE Error vs m for k = 5



Original figure for k = 50
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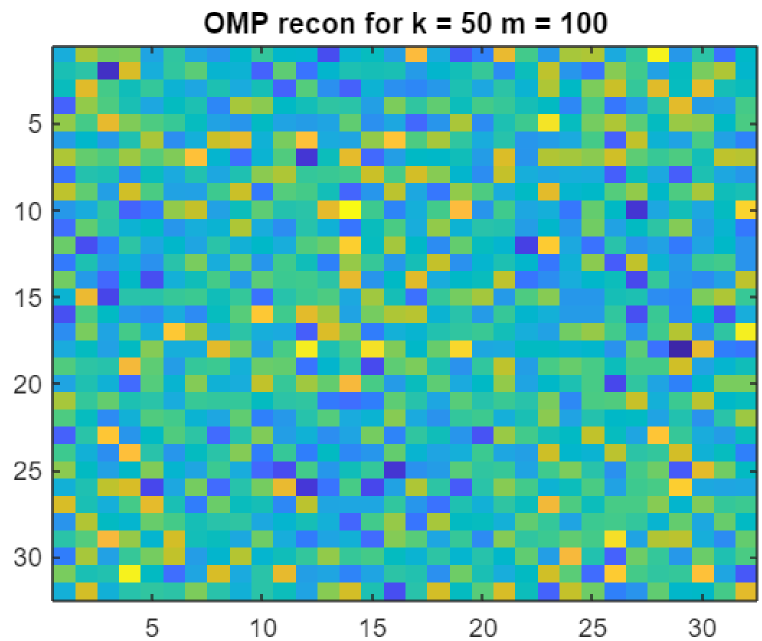

OMP recon for k = 50 m = 100
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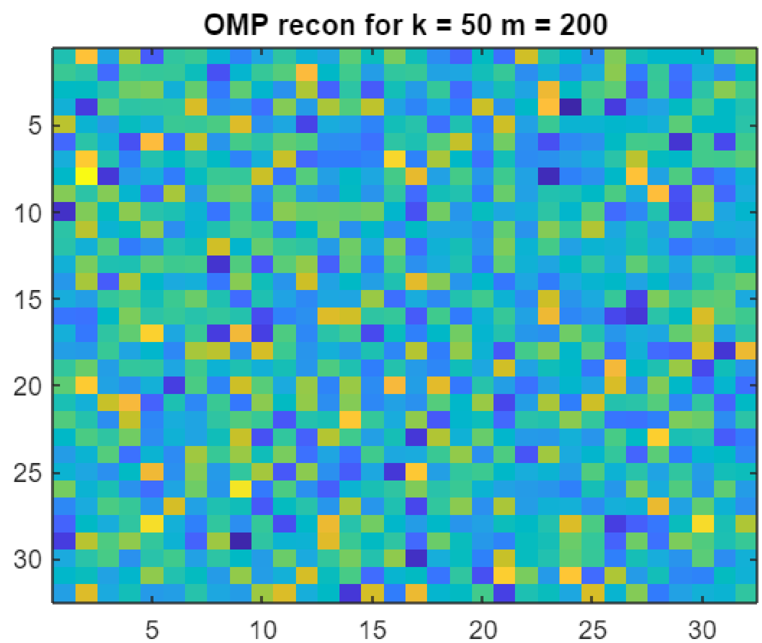
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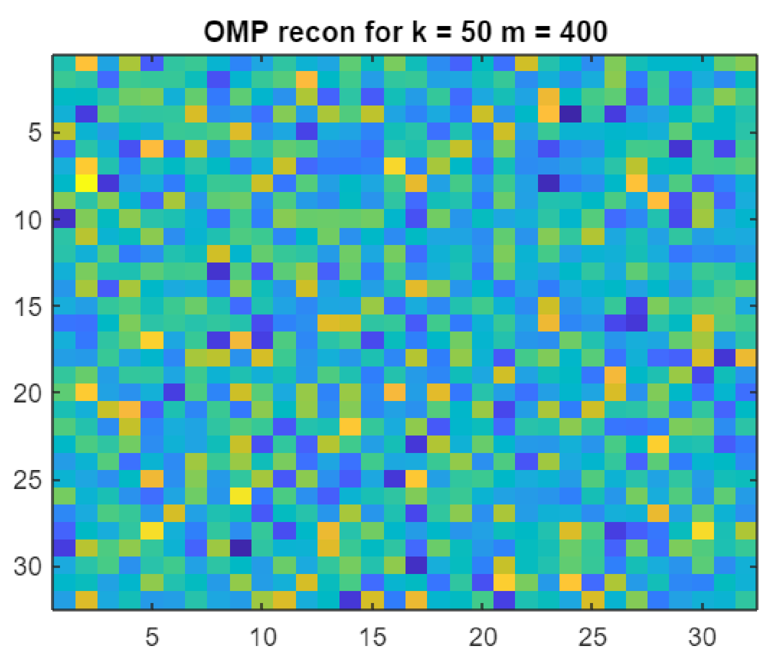
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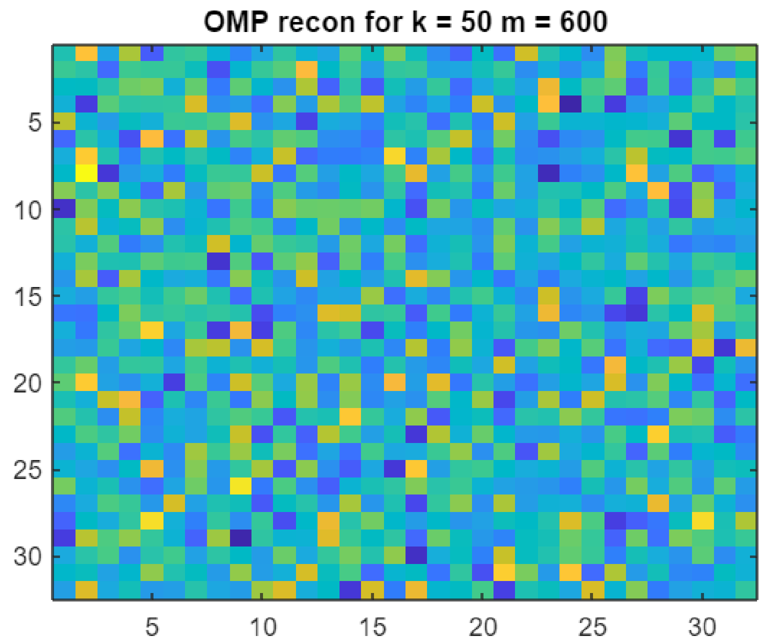
OMP recon for k = 50 m = 600
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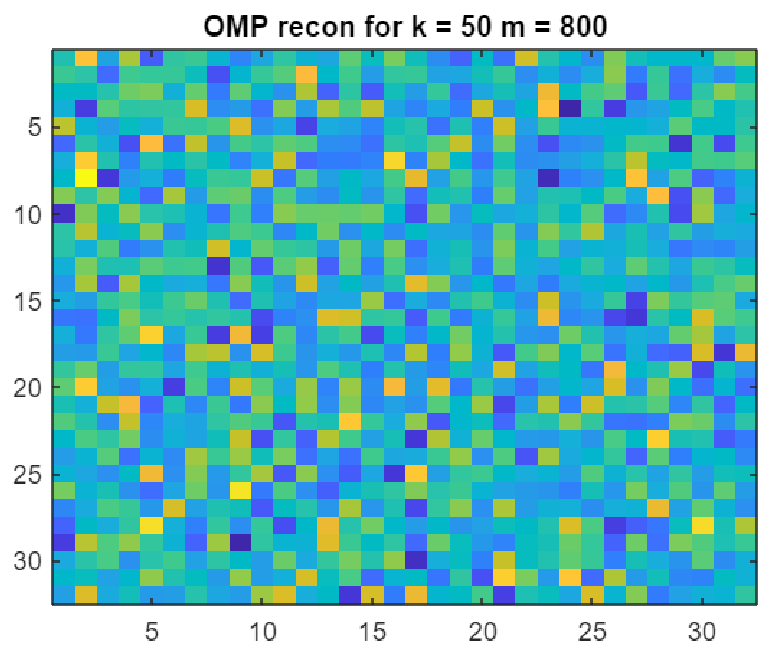
OMP recon for k = 50 m = 800
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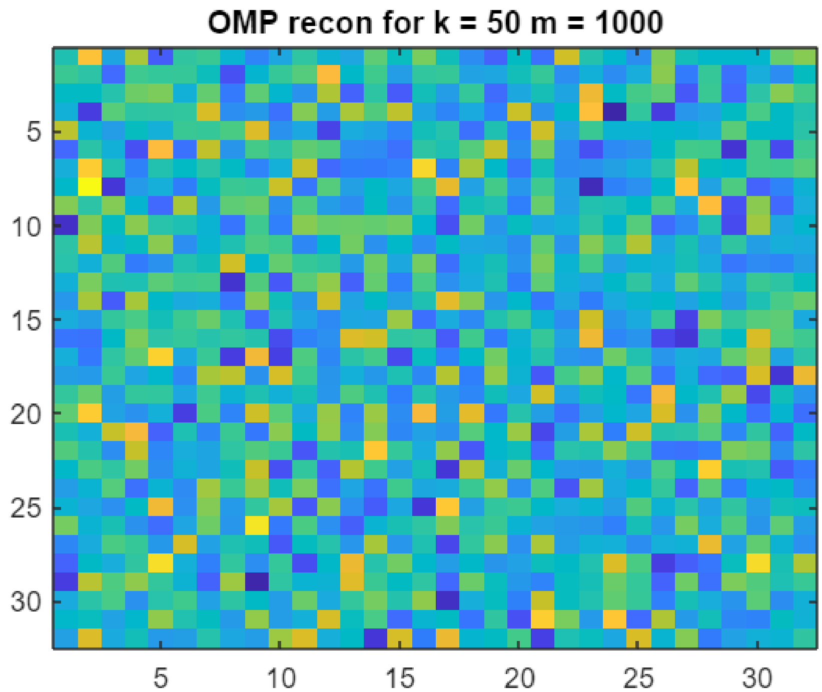
OMP recon for k = 50 m = 1000
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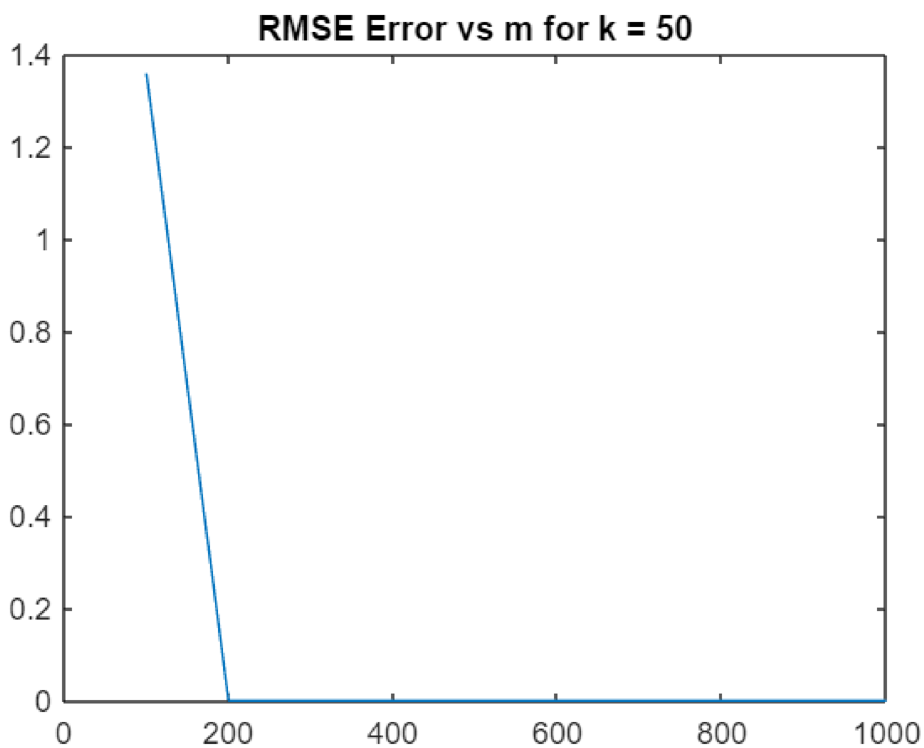
RMSE Error vs m for k = 50
1.4
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0.2
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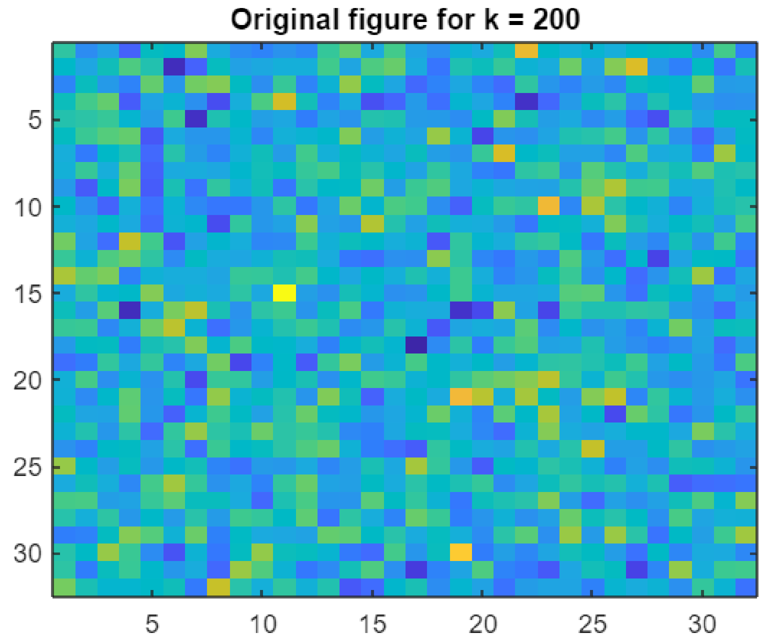
1000


Original figure for k = 200
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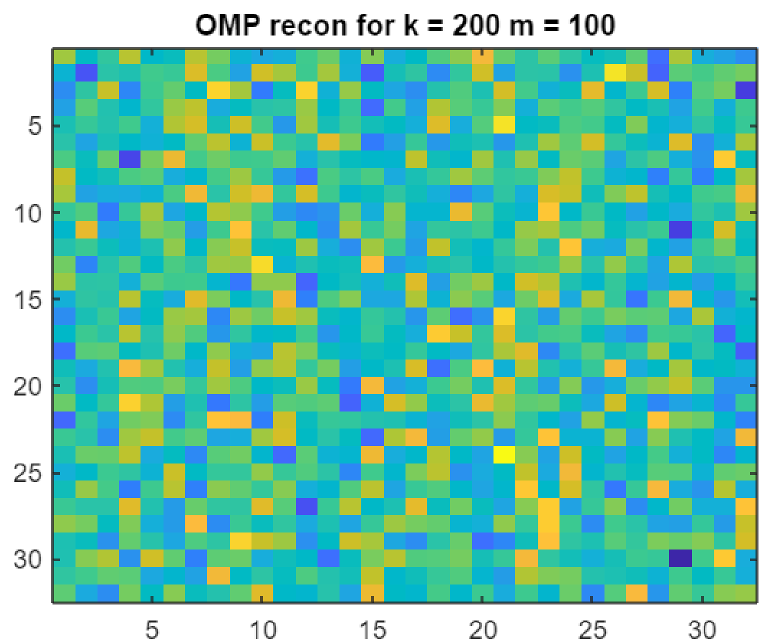
OMP recon for k = 200 m = 100
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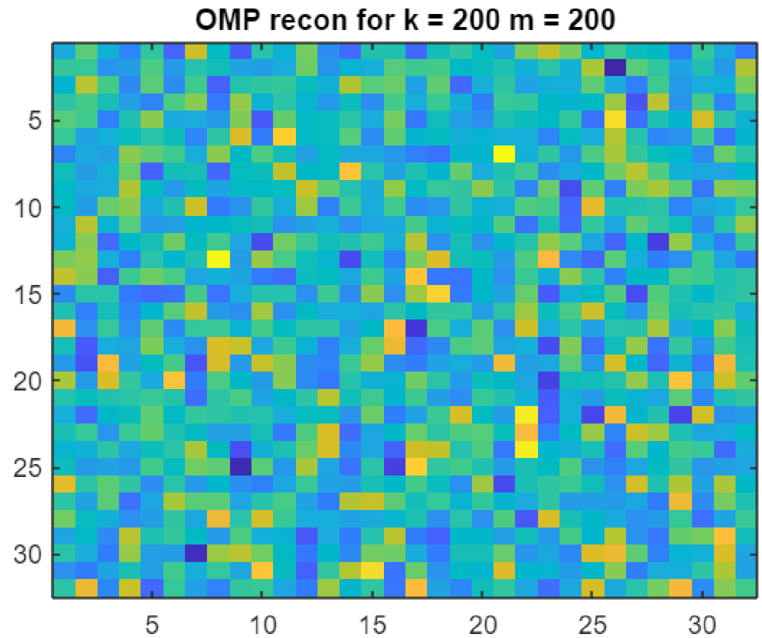
OMP recon for k = 200 m = 200
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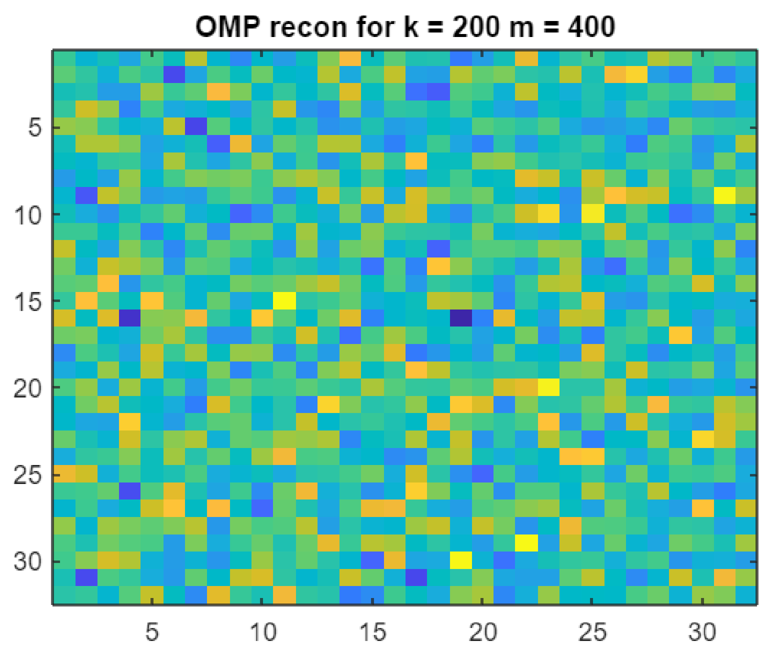
OMP recon for k = 200 m = 400
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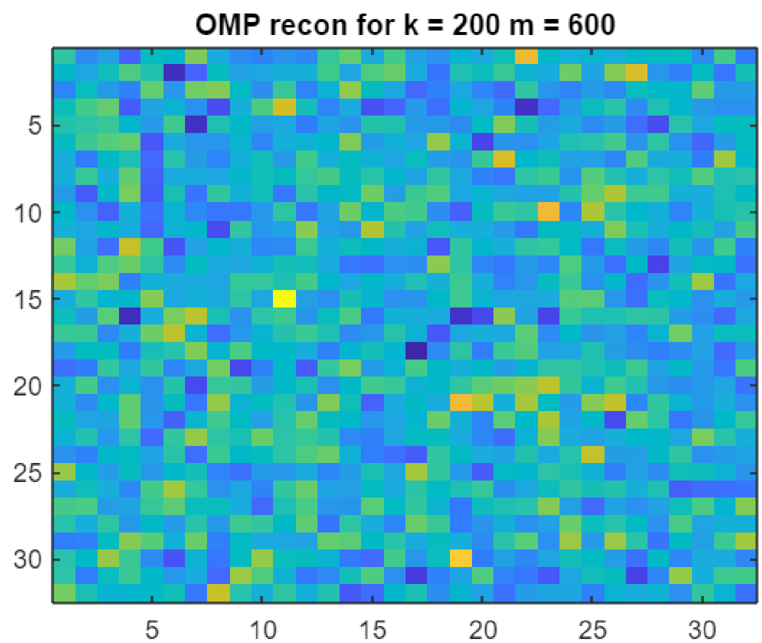
OMP recon for k = 200 m = 600
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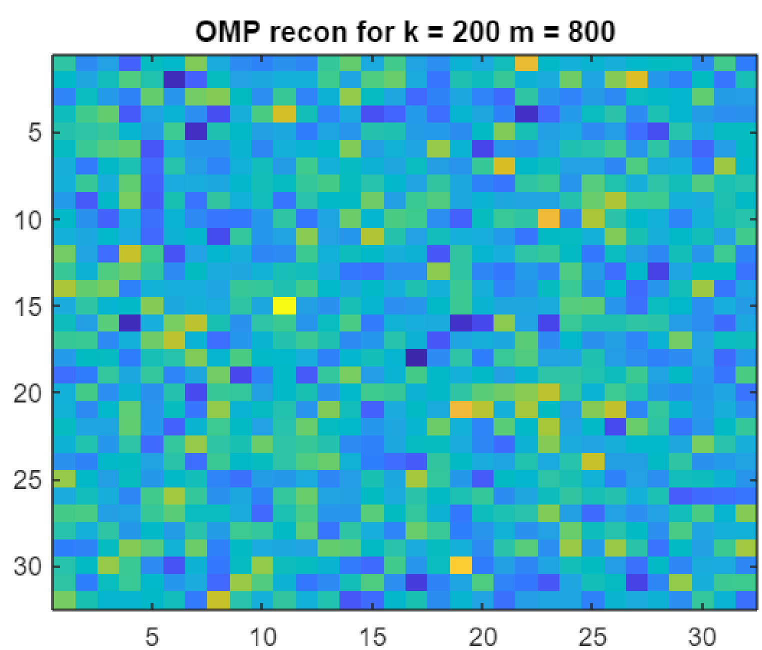
OMP recon for k = 200 m = 800
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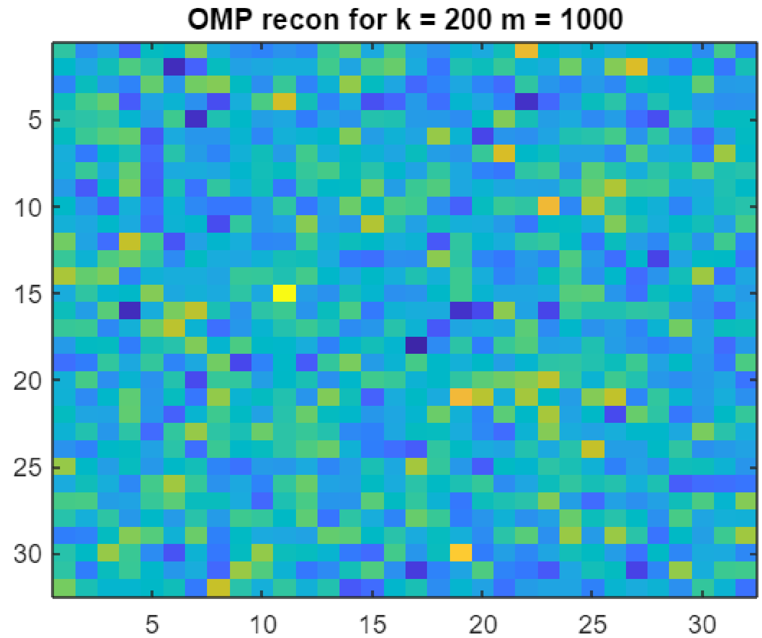
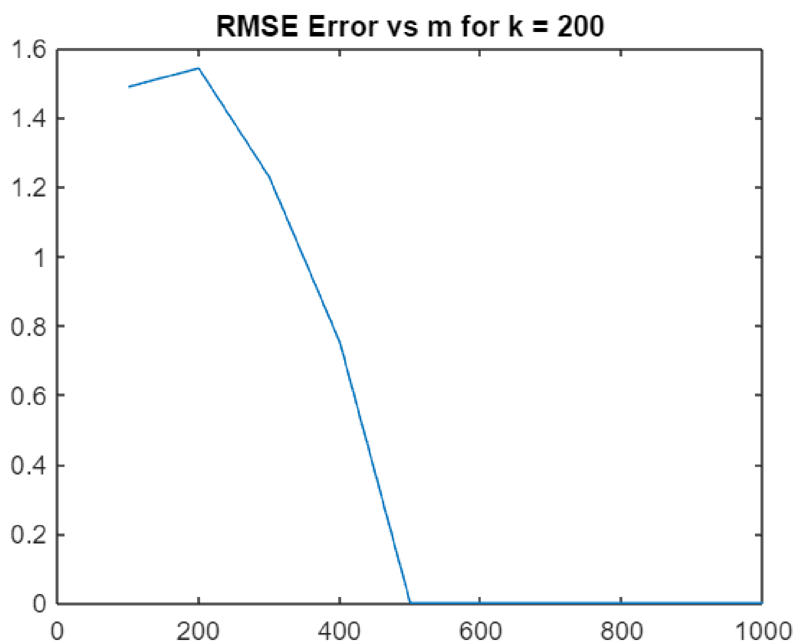
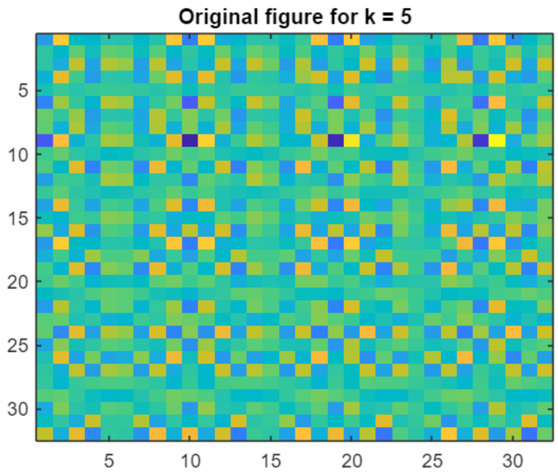


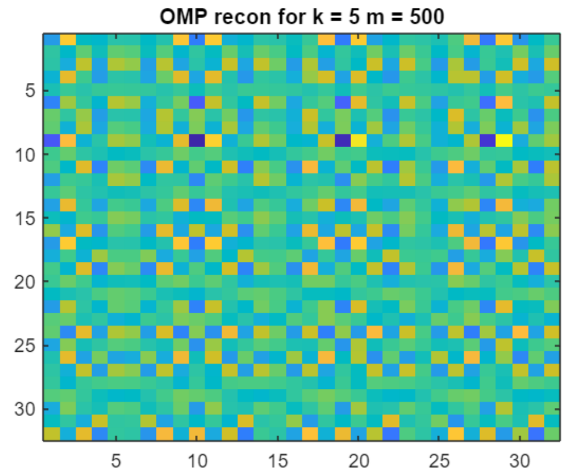


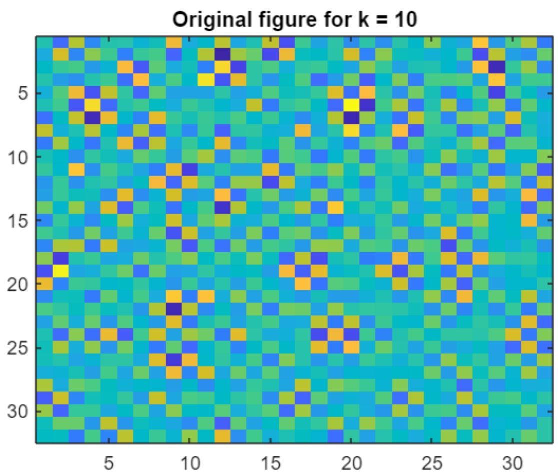
The above plots can be explained as follows:
As m varies (increases), we see an increase in accuracy of the recovered images from compressed measurements. In particular, if

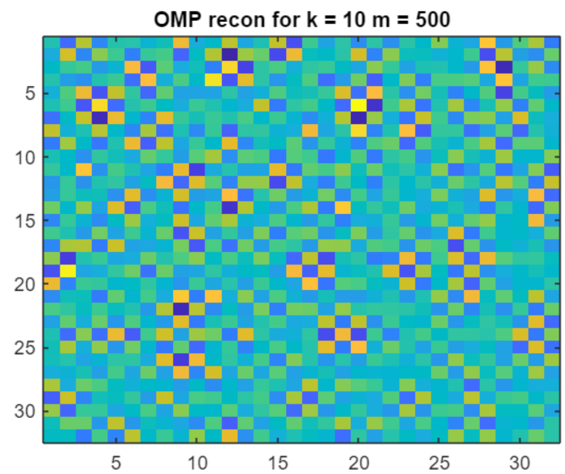
$$m \geq Cs \log n/\delta$$

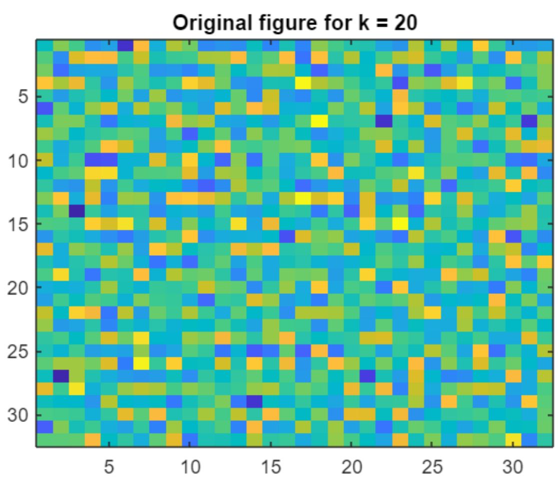
then an s sparse vector can be recovered exactly with probability $\geq 1-2\delta$. Thus a larger m allows for a smaller δ and a higher probability of accurate recovery.

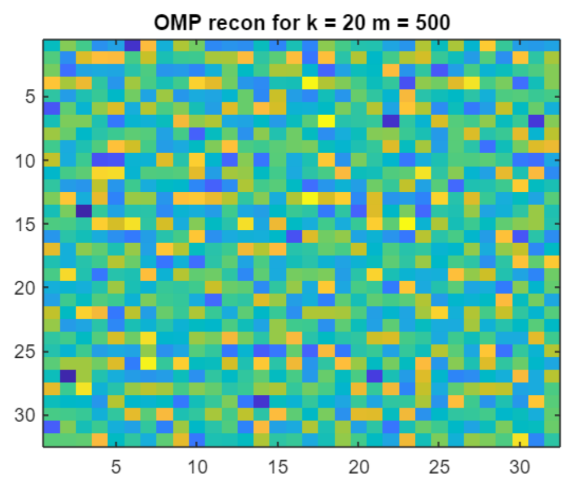

Original figure for k = 5

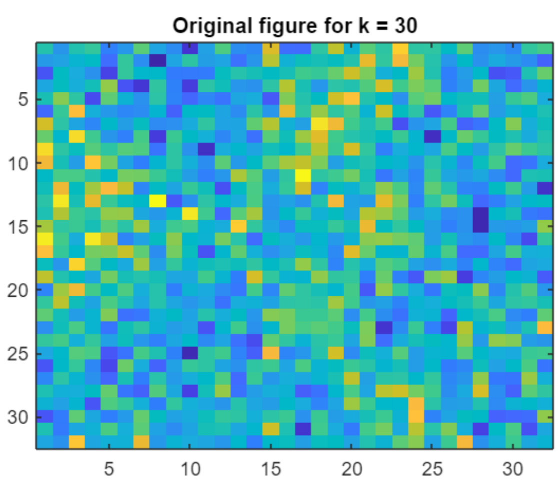

OMP recon for k = 5 m = 500

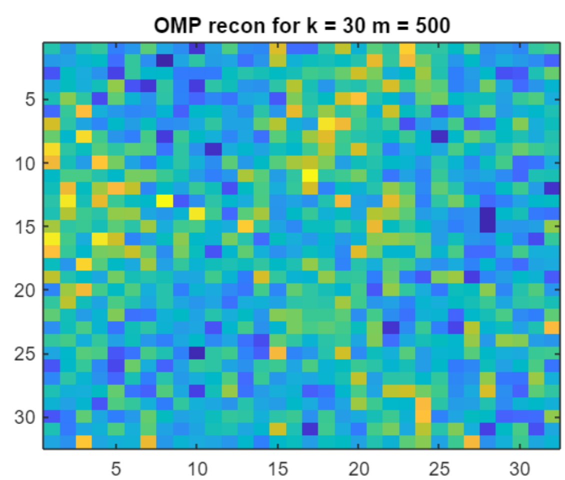

Original figure for k = 10

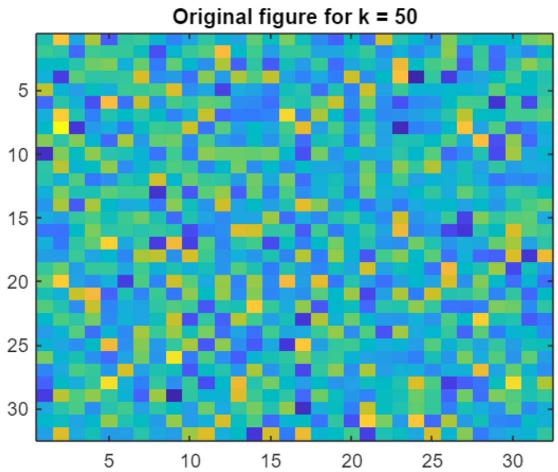

OMP recon for k = 10 m = 500

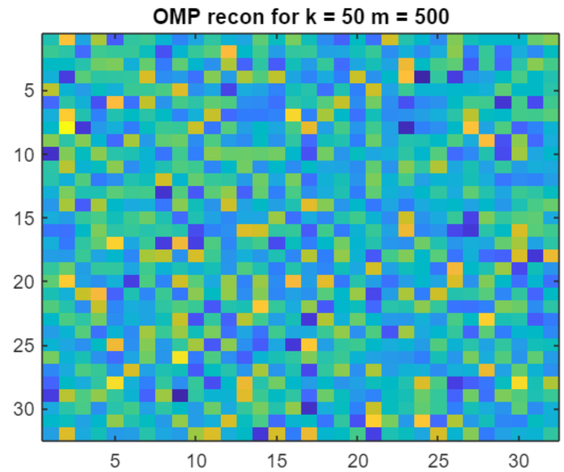

Original figure for k = 20


OMP recon for k = 20 m = 500


Original figure for k = 30


OMP recon for k = 30 m = 500


Original figure for k = 50
5 10 15 20 25 30
5 10 15 20 25 30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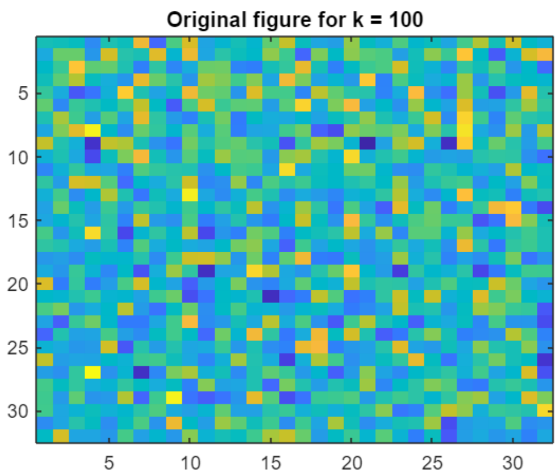
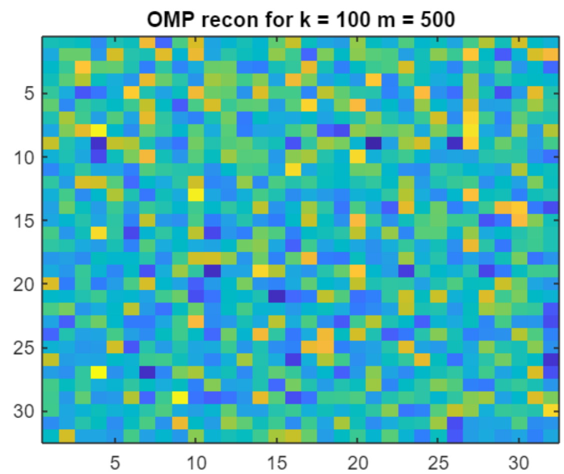
5 10 15 20 25 30


Original figure for k = 100
5 10 15 20 25 30
5 10 15 20 25 30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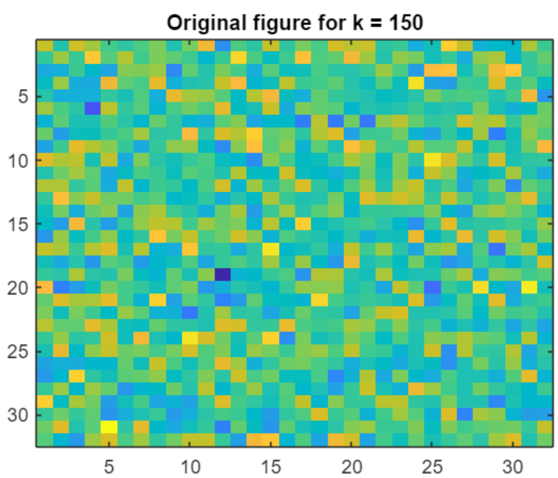
5 10 15 20 25 30
5 10 15 20 25 30


Original figure for k = 150
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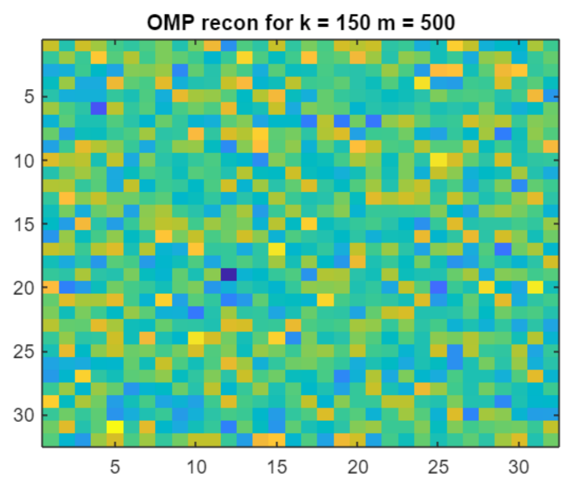
5 10 15 20 25 30


OMP recon for k = 150 m = 500
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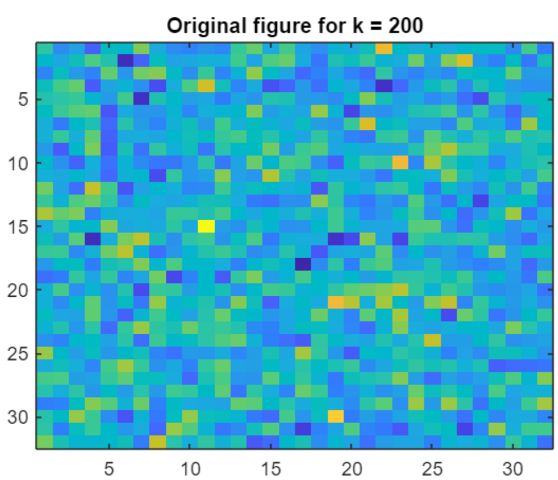
5 10 15 20 25 30


Original figure for k = 200
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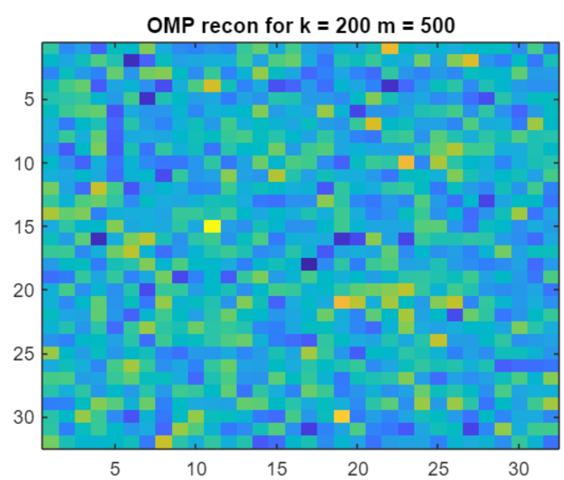
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5 10 15 20 25 30

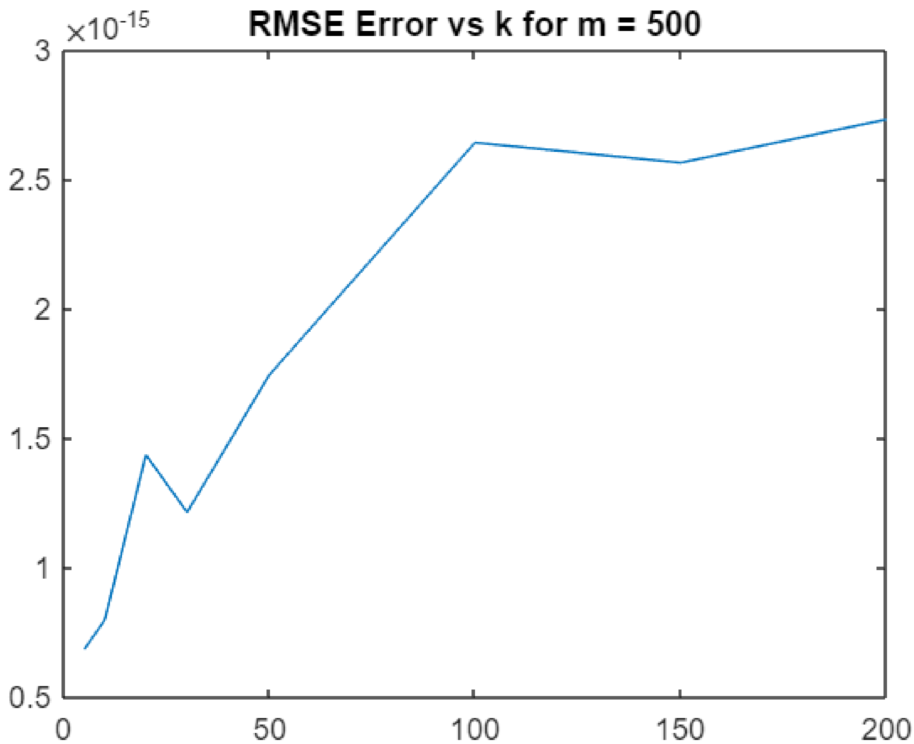
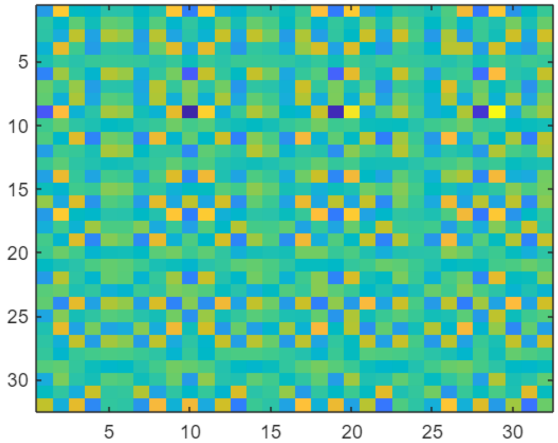
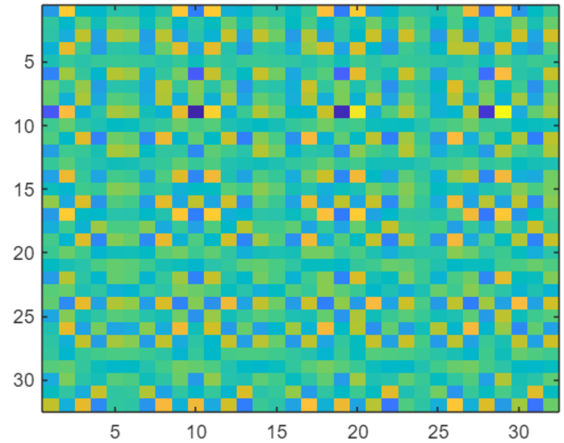


Figure 3: Caption

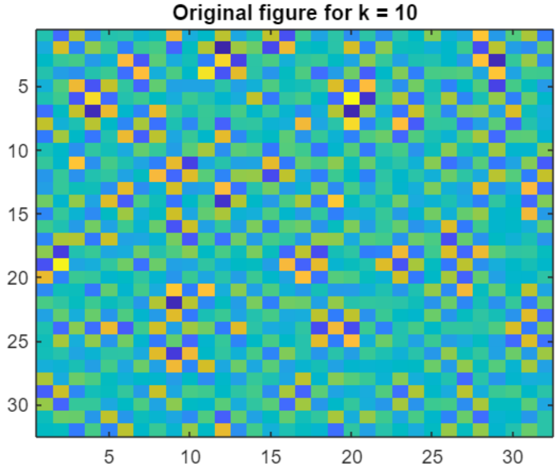
Original figure for k = 5



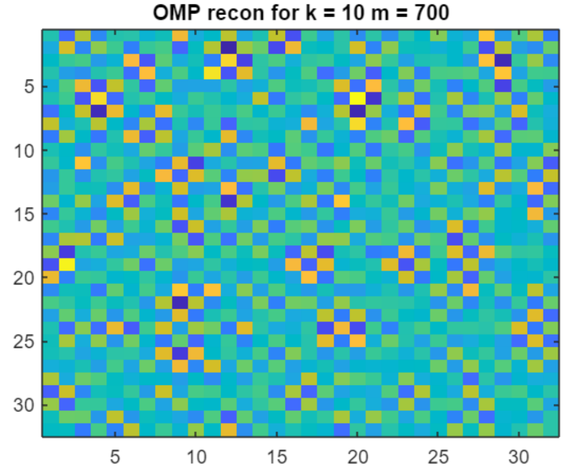
OMP recon for k = 5 m = 700



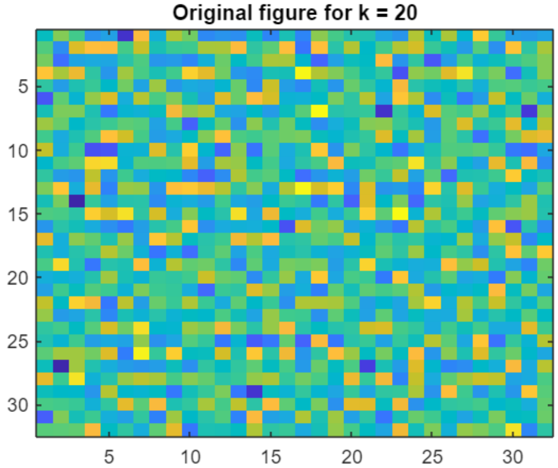
Original figure for k = 10



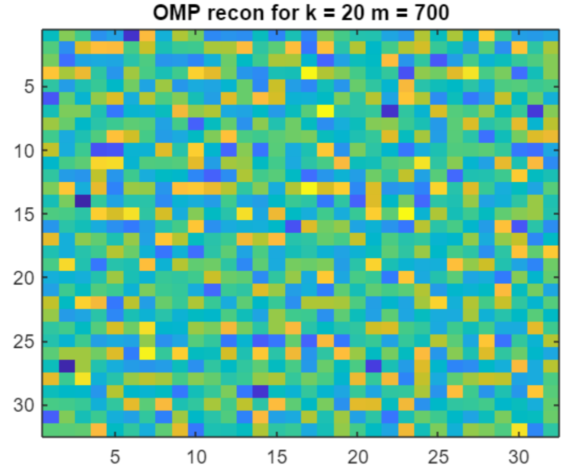
OMP recon for k = 10 m = 700



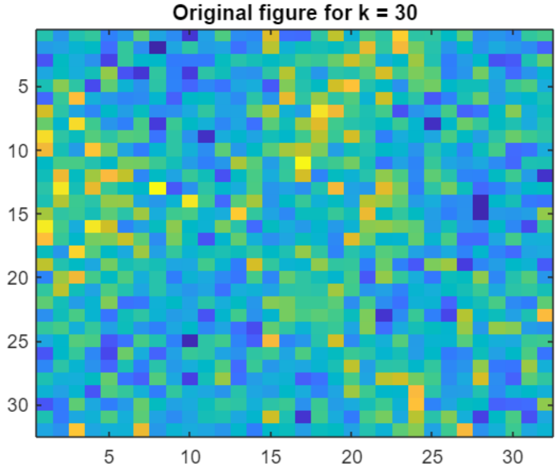
Original figure for k = 20



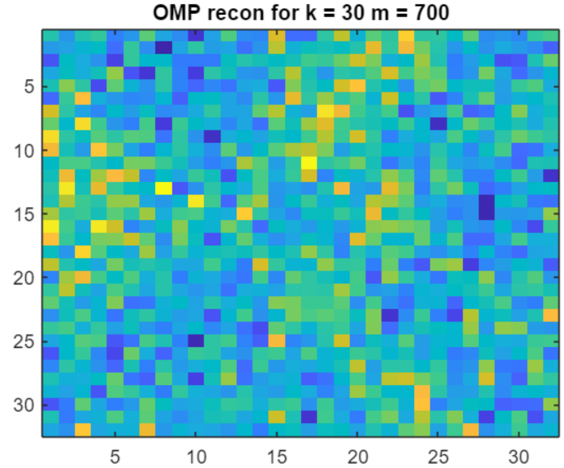
OMP recon for k = 20 m = 700

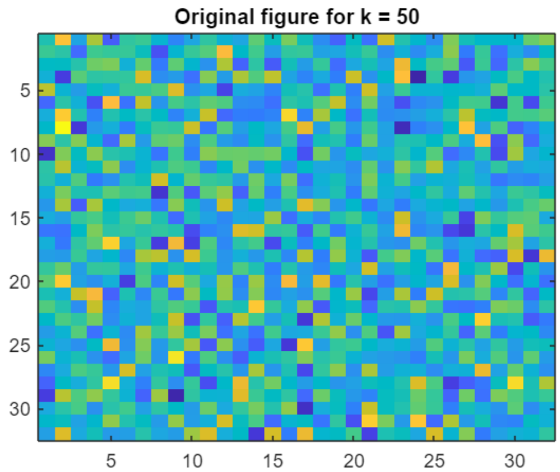


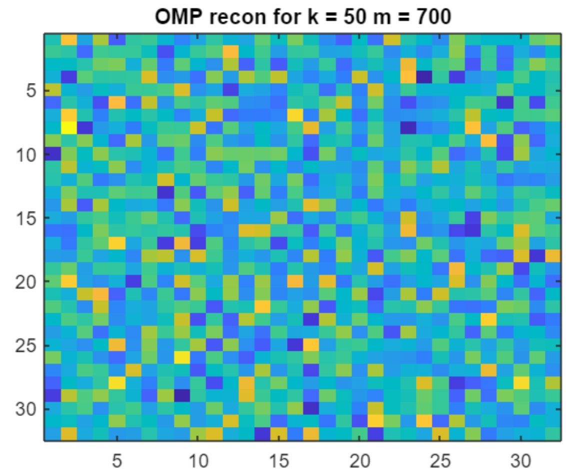
Original figure for k = 30

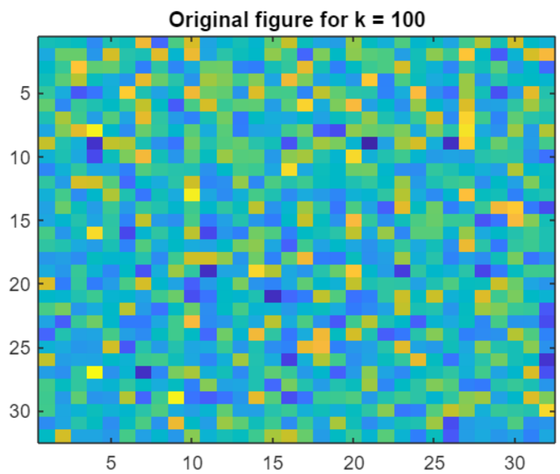


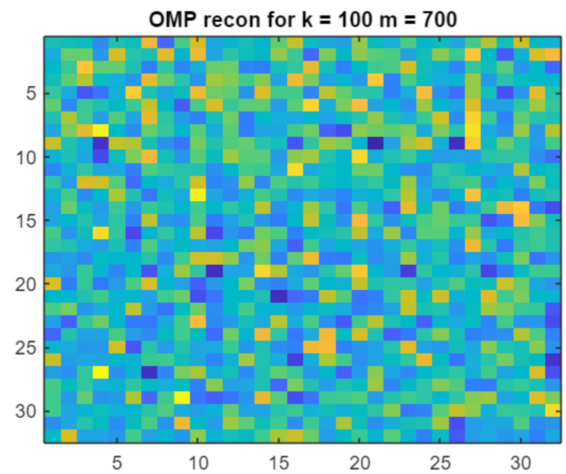
OMP recon for k = 30 m = 700

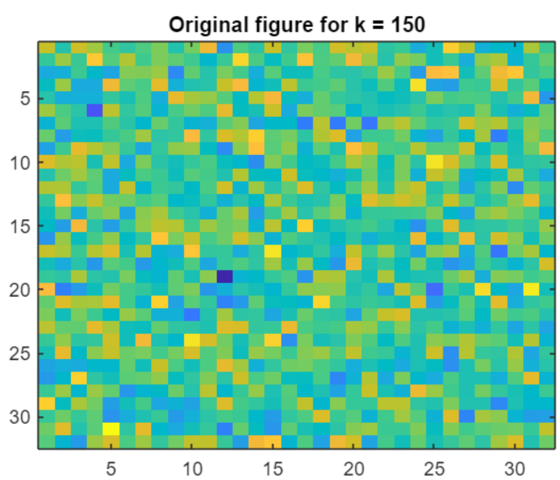


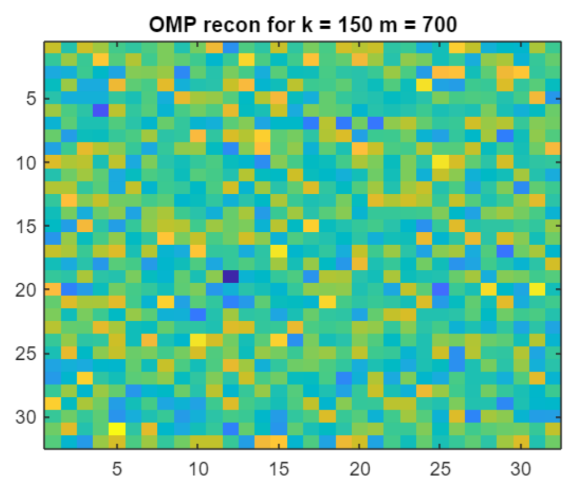

Original figure for k = 50

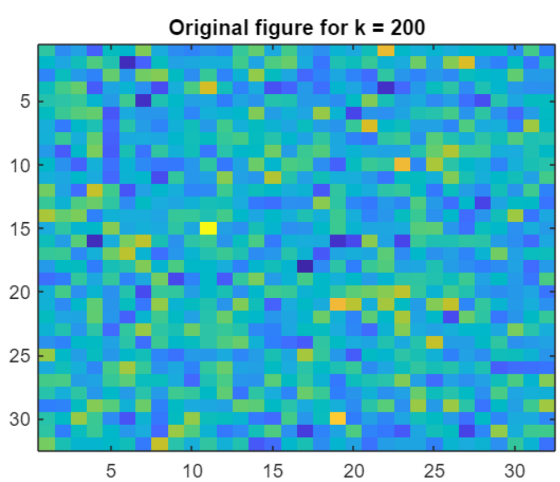

OMP recon for k = 50 m = 700

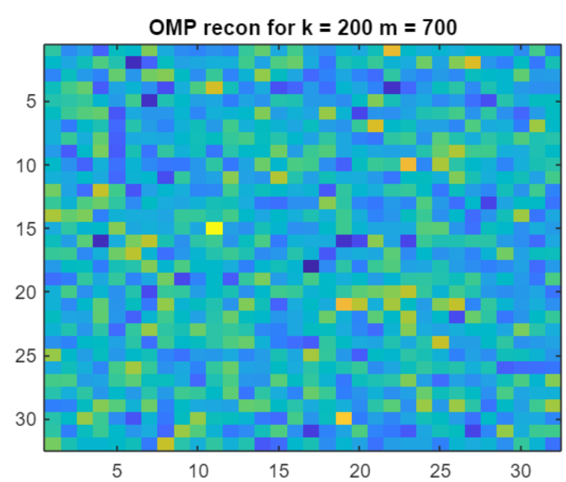

Original figure for k = 100


OMP recon for k = 100 m = 700


Original figure for k = 150


OMP recon for k = 150 m = 700


Original figure for k = 200


OMP recon for k = 200 m = 700

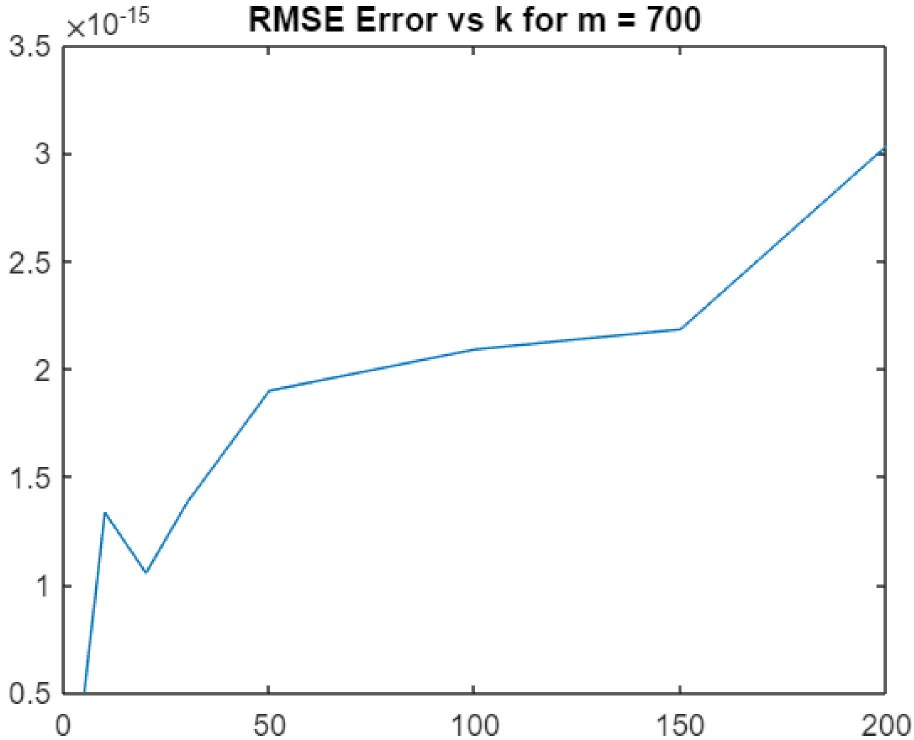
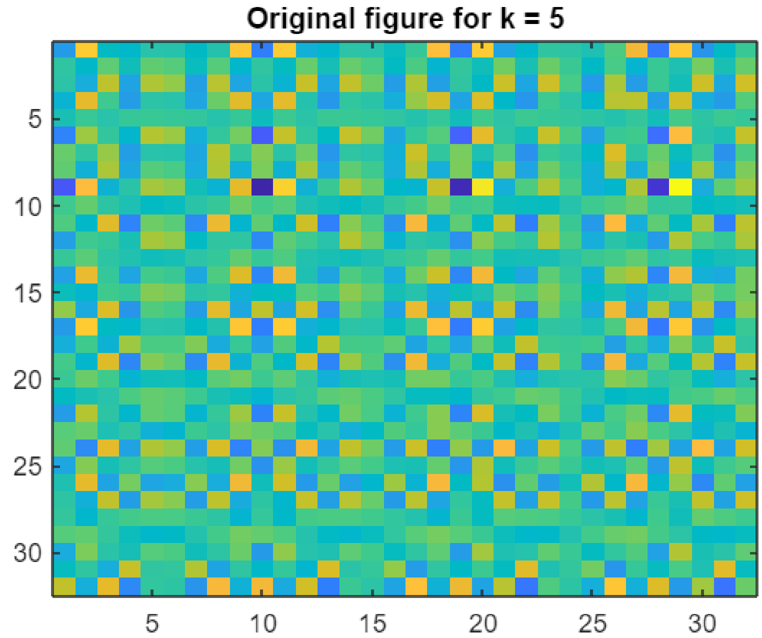
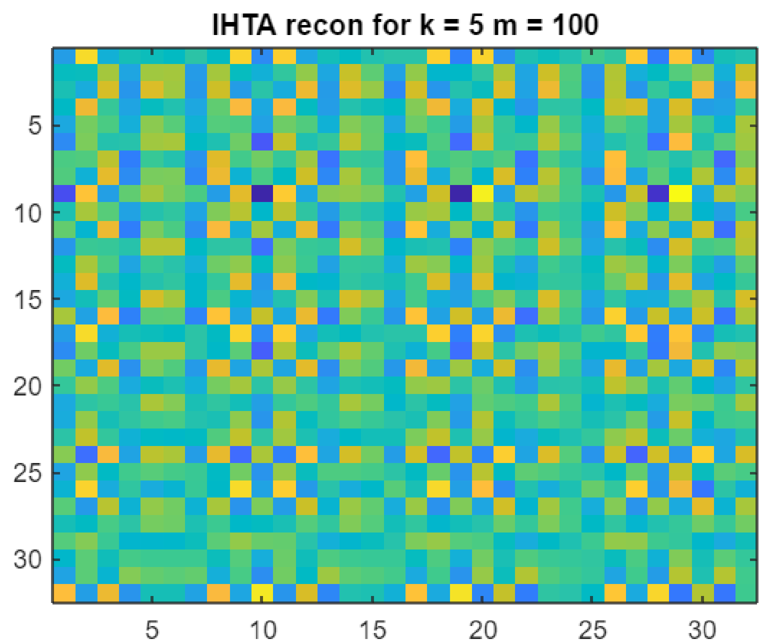


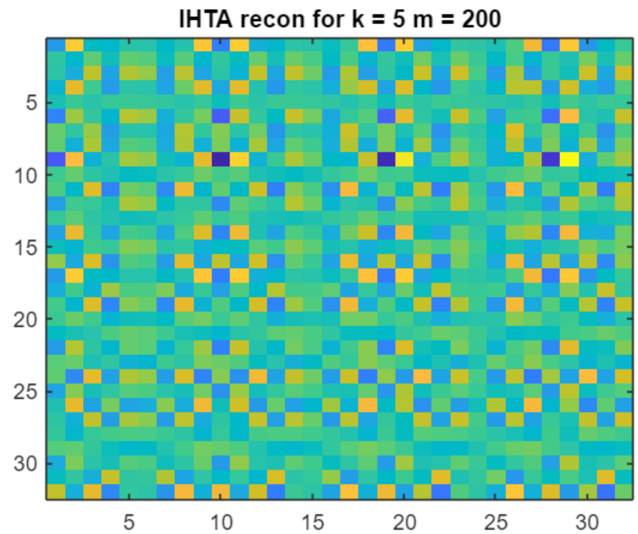
Figure 6: Caption

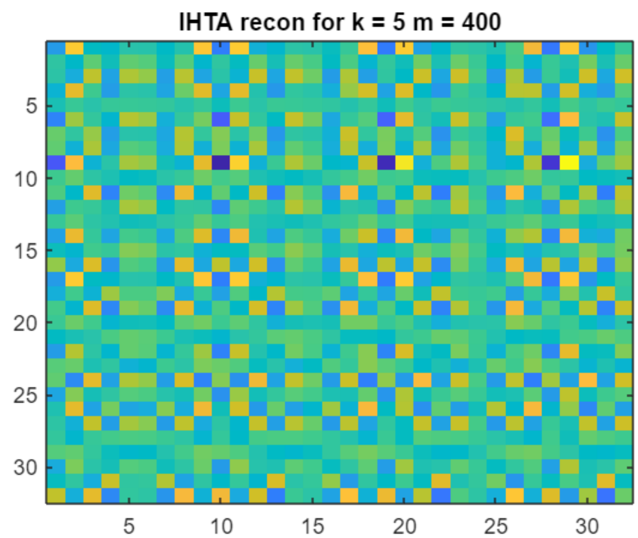
This relation between error vs k is explained easily by the fact that a higher k implies a lower sparsity. Thus sparser vectors have better reconstructions.

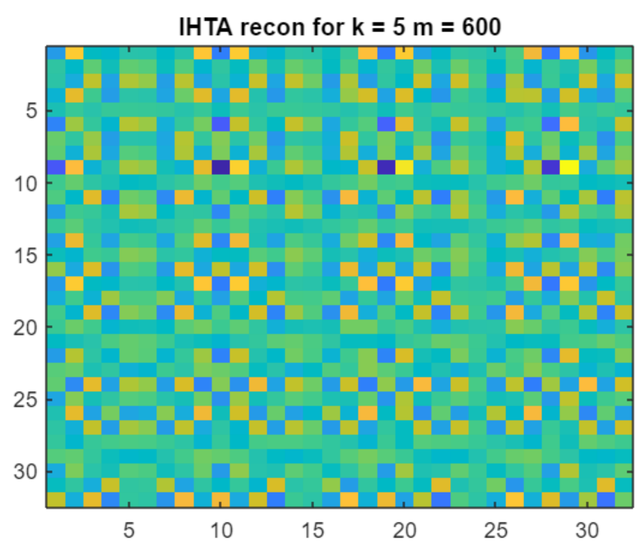
The following are the results for IHTA:

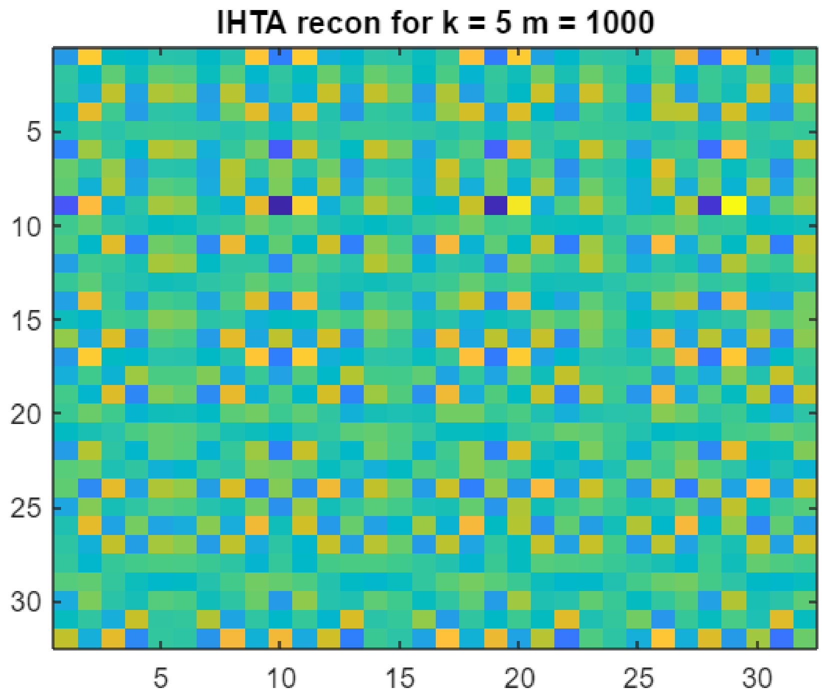


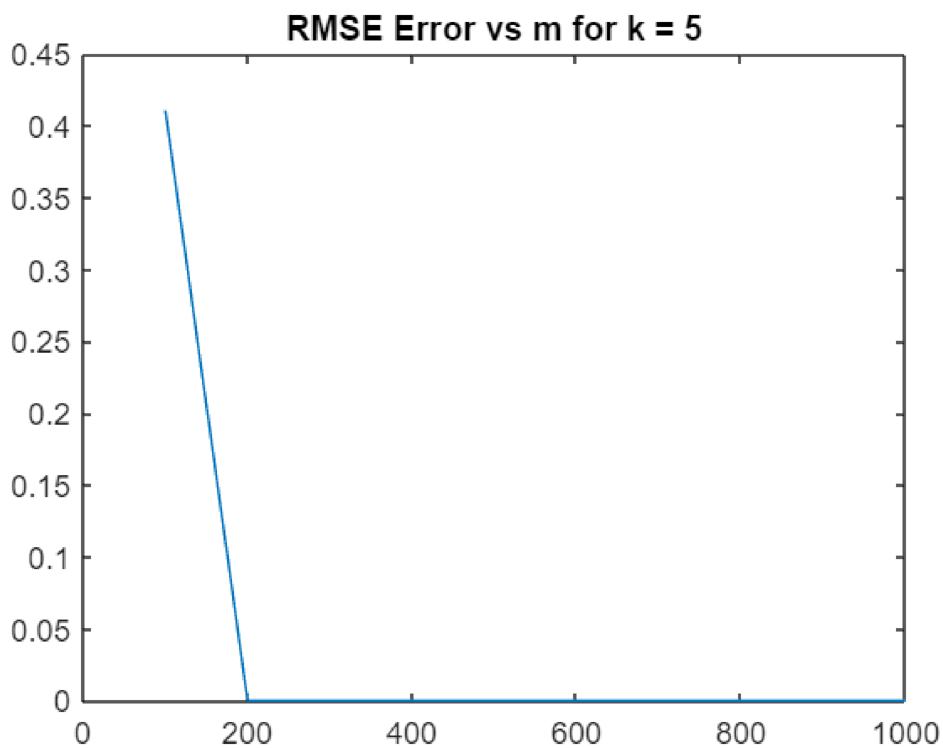


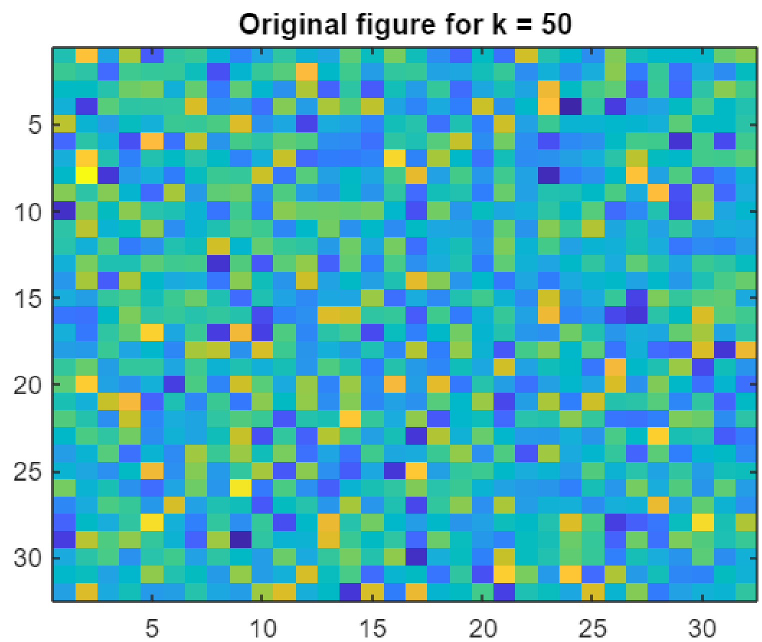

IHTA recon for k = 5 m = 200
5
10
15
20
25
30
5
10
15
20
25
30

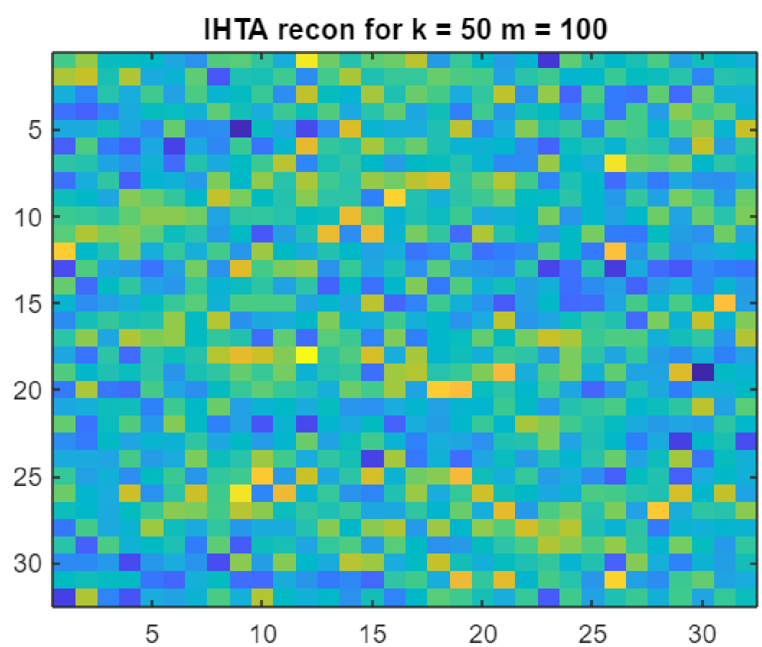

IHTA recon for k = 5 m = 400
5
10
15
20
25
30
5
10
15
20
25
30

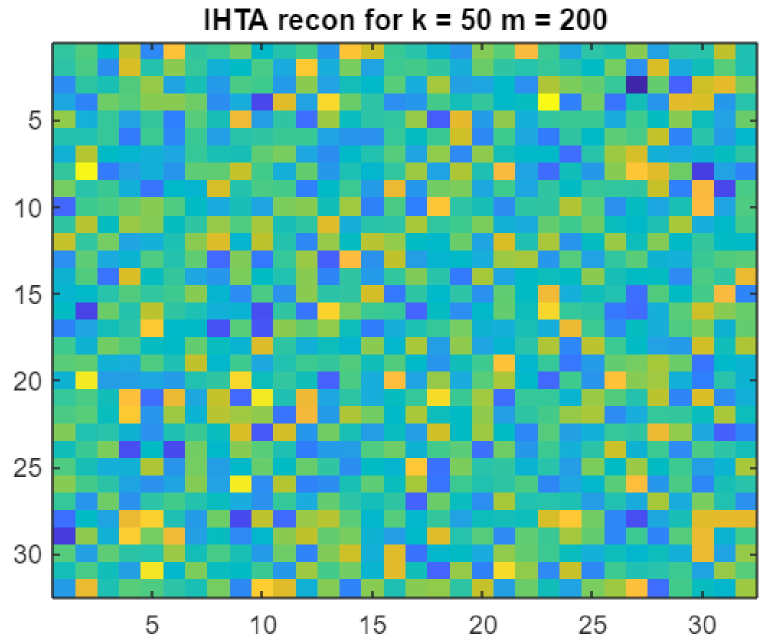

IHTA recon for k = 5 m = 600
5
10
15
20
25
30
5
10
15
20
25
30

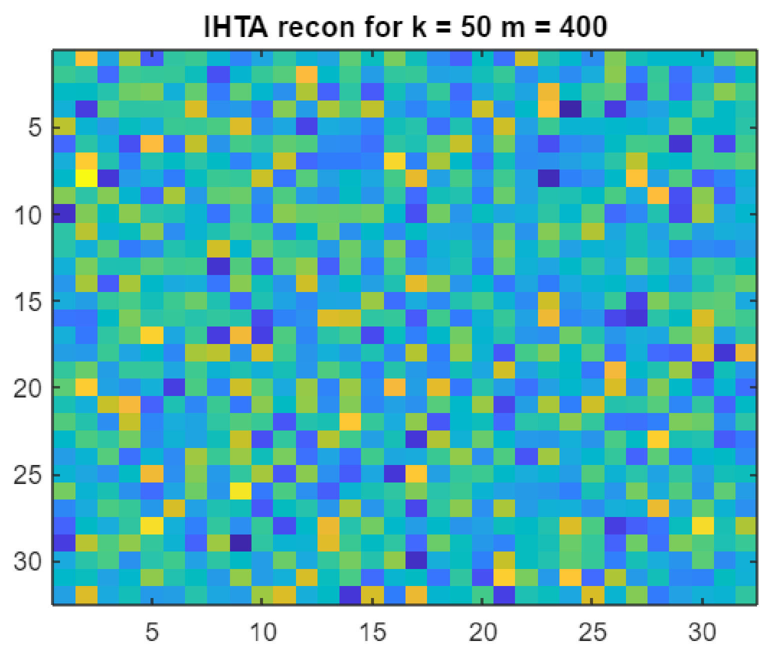

IHTA recon for k = 5 m = 1000
5
10
15
20
25
30
5
10
15
20
25
30

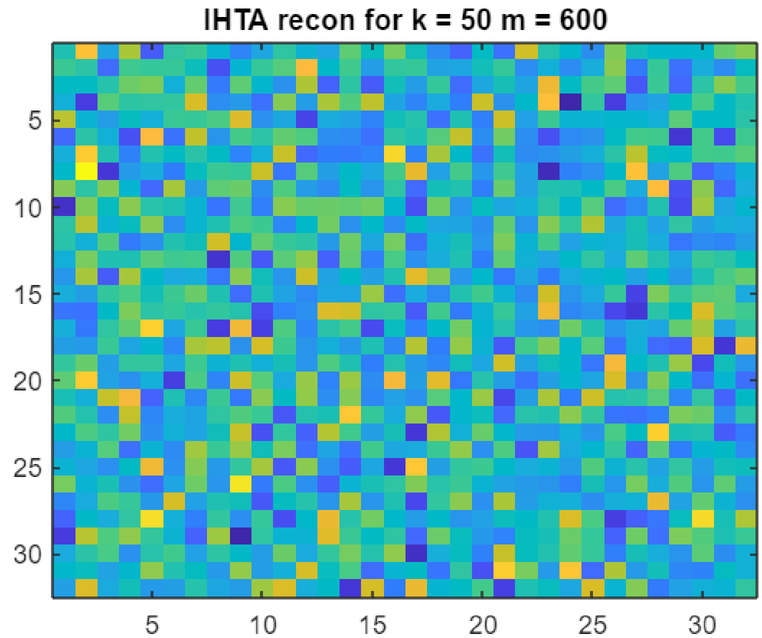

RMSE Error vs m for k = 5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
200
400
600
800
1000

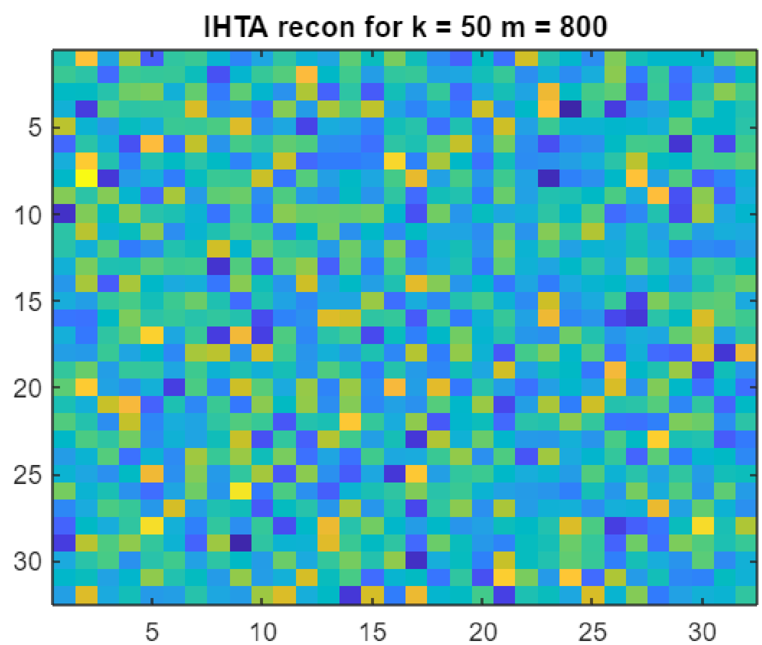

Original figure for k = 50
5
10
15
20
25
30
5
10
15
20
25
30

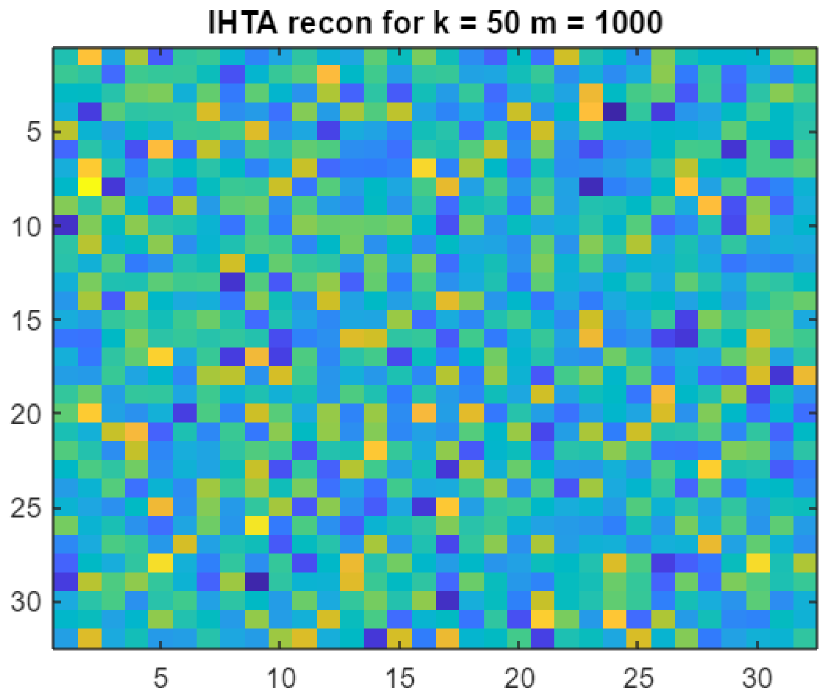

IHTA recon for k = 50 m = 100
5
10
15
20
25
30
5
10
15
20
25
30


IHTA recon for k = 50 m = 200
5
10
15
20
25
30
5
10
15
20
25
30

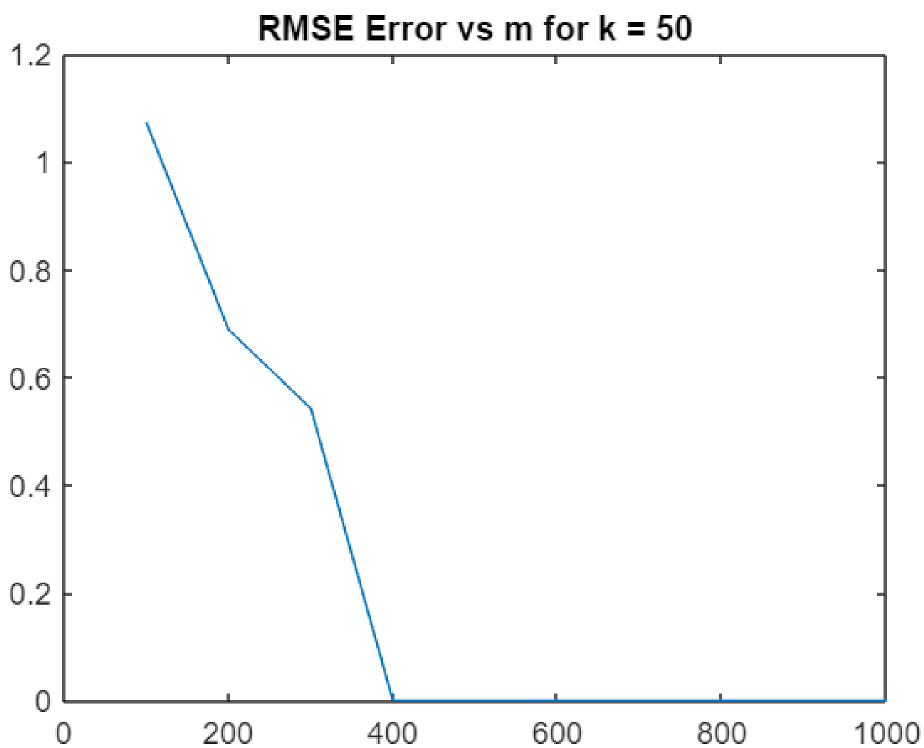

IHTA recon for k = 50 m = 400
5
10
15
20
25
30
5
10
15
20
25
30


IHTA recon for k = 50 m = 600
5
10
15
20
25
30
5
10
15
20
25
30

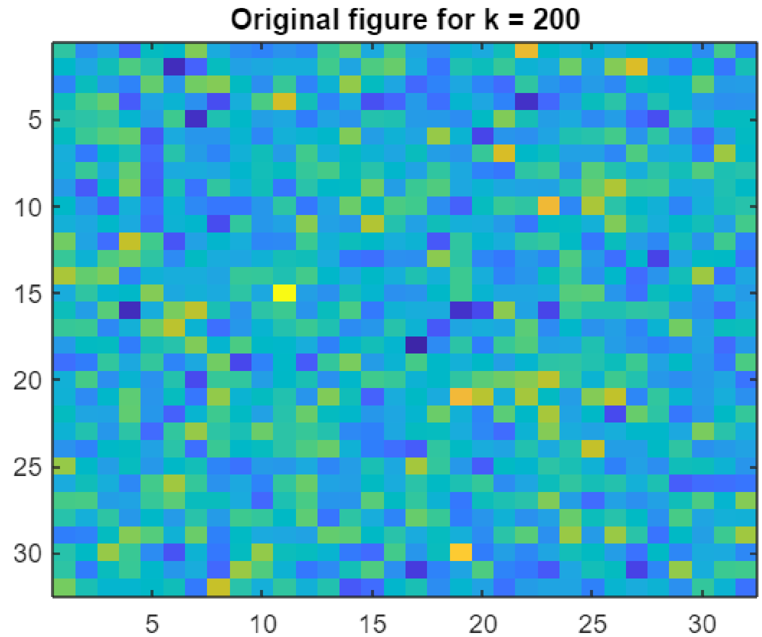

IHTA recon for k = 50 m = 800
5
10
15
20
25
30
5
10
15
20
25
30

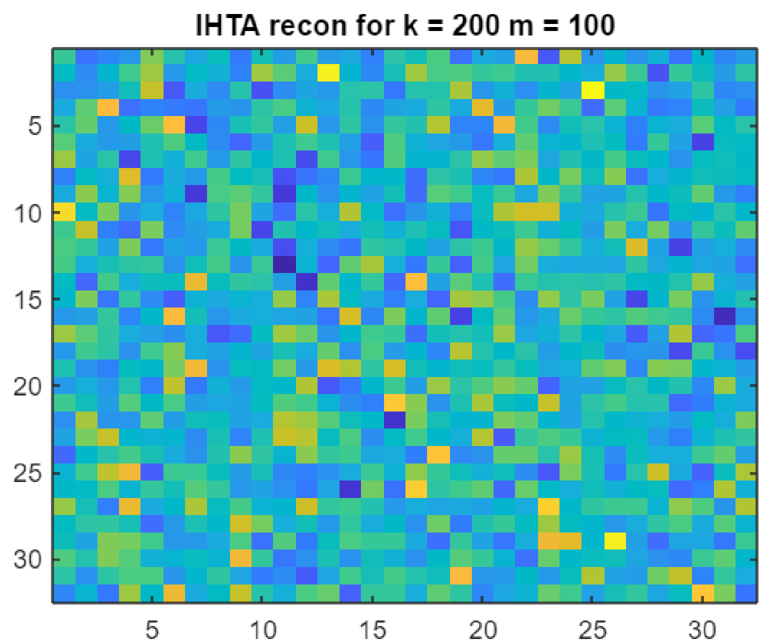


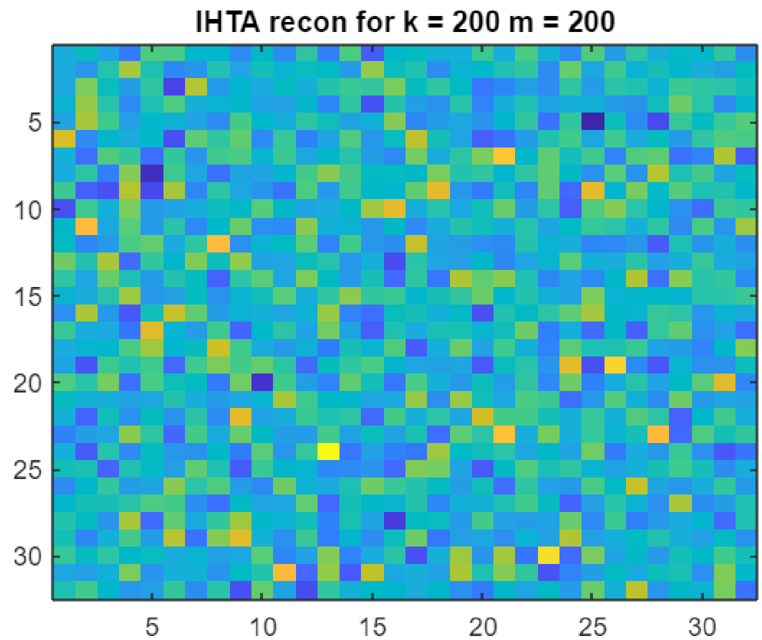
IHTA recon for k = 50 m = 1000
5
10
15
20
25
30

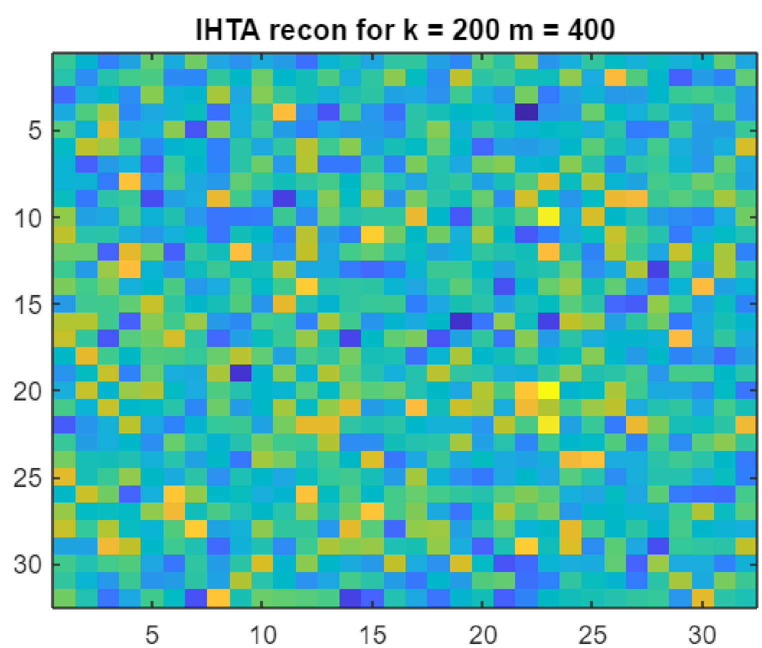


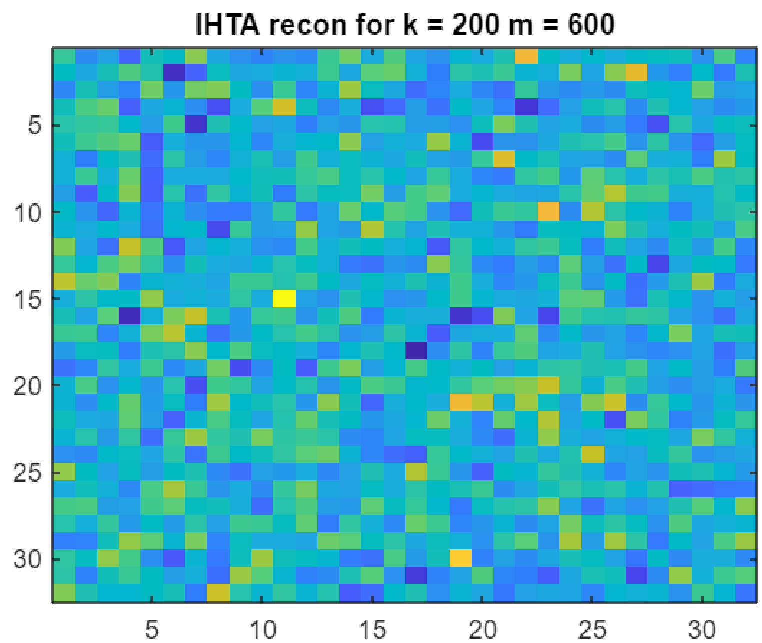
RMSE Error vs m for k = 50
1.2
1
0.8
0.6
0.4
0.2
0
200
400
600
800
1000

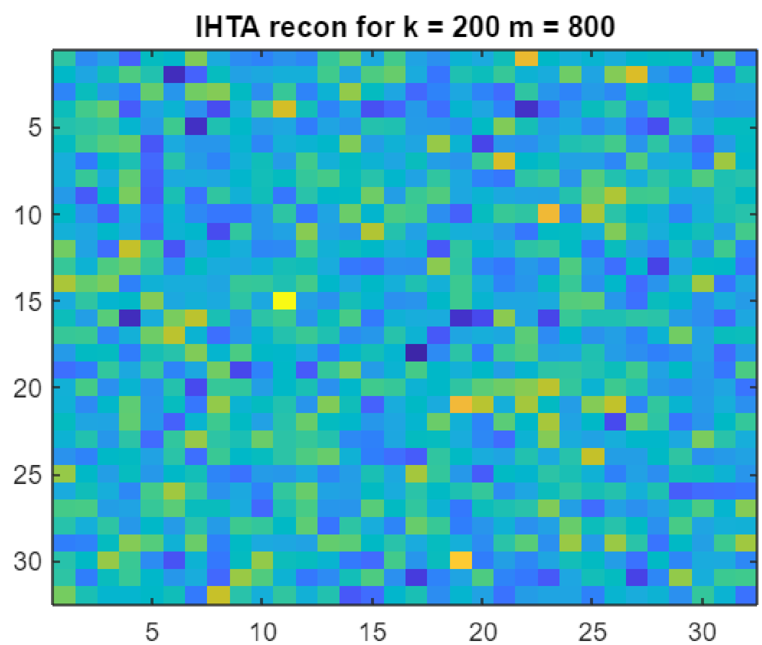

Original figure for k = 200
5
10
15
20
25
30
5
10
15
20
25
30

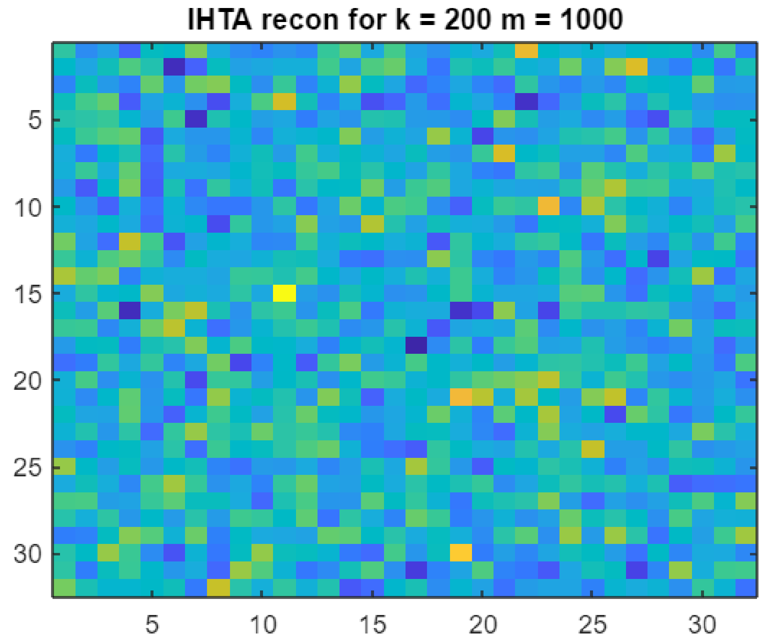

IHTA recon for k = 200 m = 100
5
10
15
20
25
30
5
10
15
20
25
30

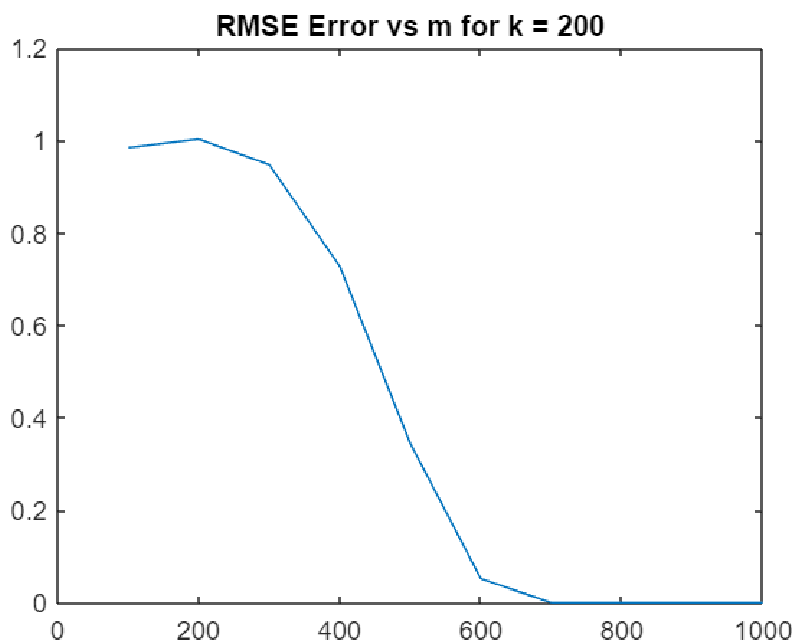

IHTA recon for k = 200 m = 200
5
10
15
20
25
30
5
10
15
20
25
30

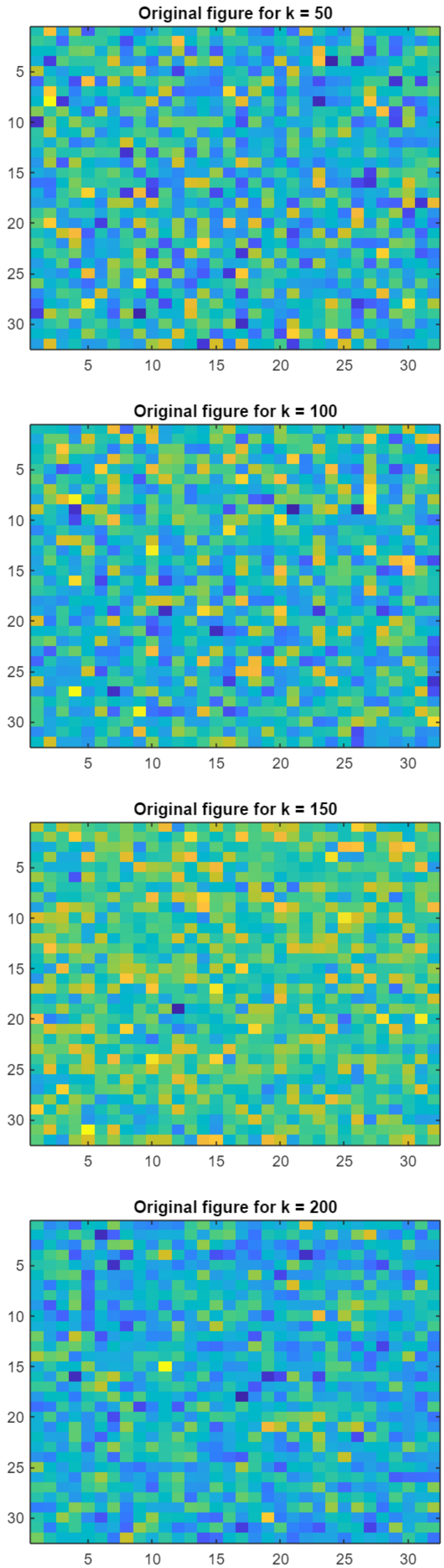

IHTA recon for k = 200 m = 400
5
10
15
20
25
30
5
10
15
20
25
30

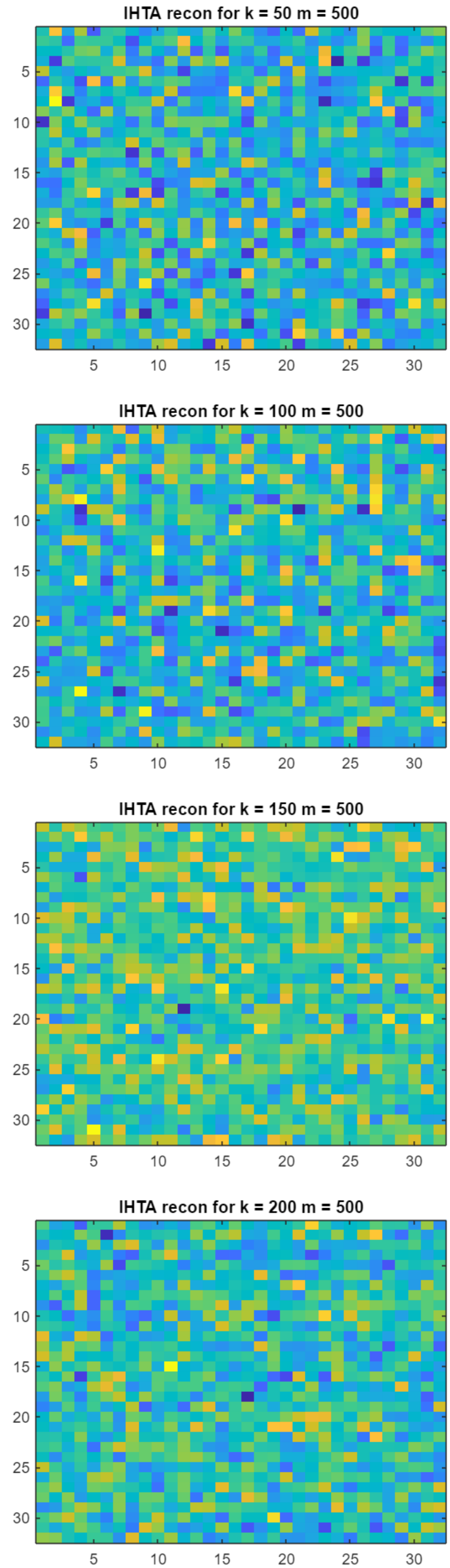

IHTA recon for k = 200 m = 600
5
10
15
20
25
30
5
10
15
20
25
30


IHTA recon for k = 200 m = 800
5
10
15
20
25
30
5
10
15
20
25
30


IHTA recon for k = 200 m = 1000
5
10
15
20
25
30
5
10
15
20
25
30


RMSE Error vs m for k = 200
1.2
1
0.8
0.6
0.4
0.2
0
0
200
400
600
800
1000


Original figure for k = 50
Original figure for k = 100
Original figure for k = 150
Original figure for k = 200


IHTA recon for k = 50 m = 500
IHTA recon for k = 100 m = 500
IHTA recon for k = 150 m = 500
IHTA recon for k = 200 m = 500

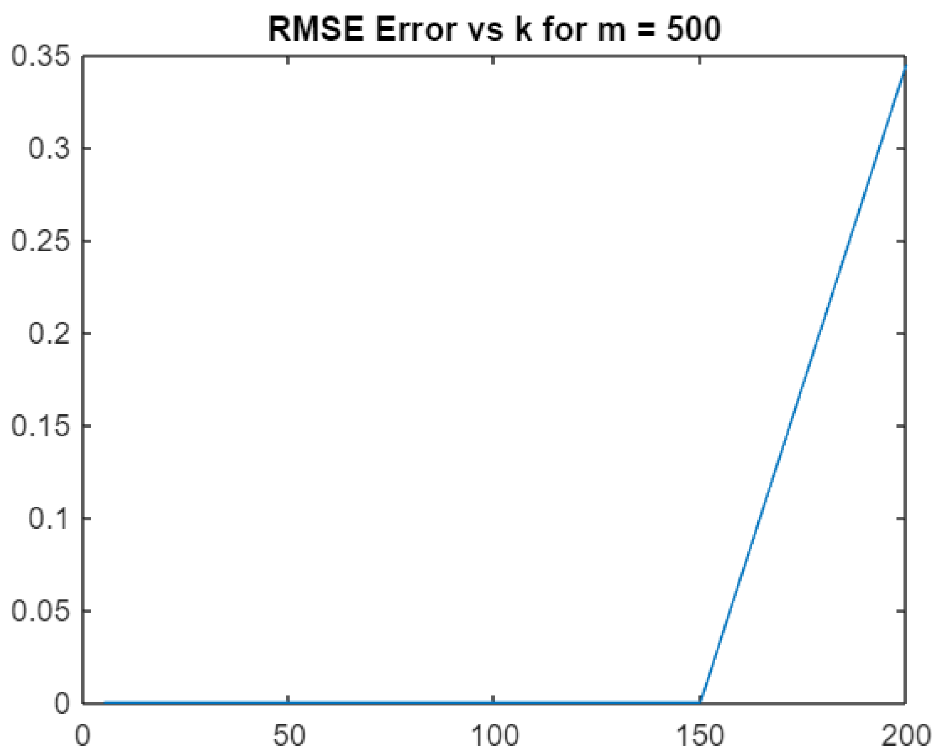
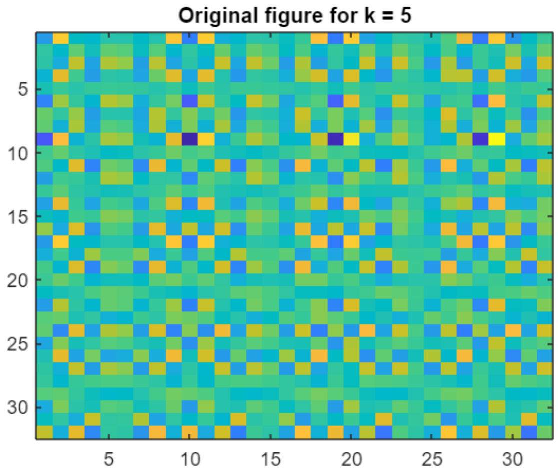
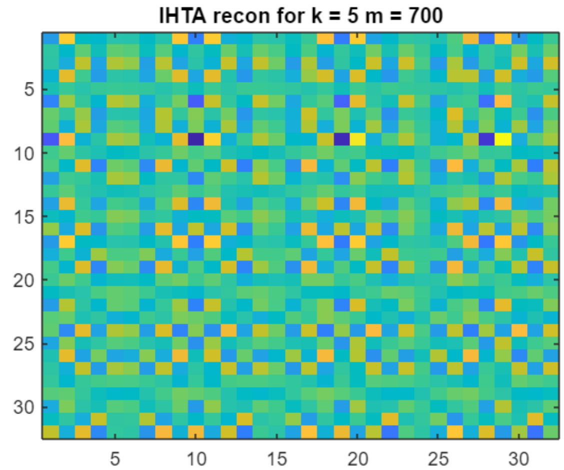
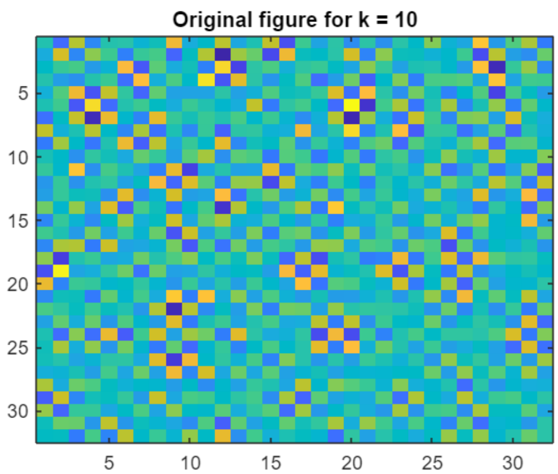
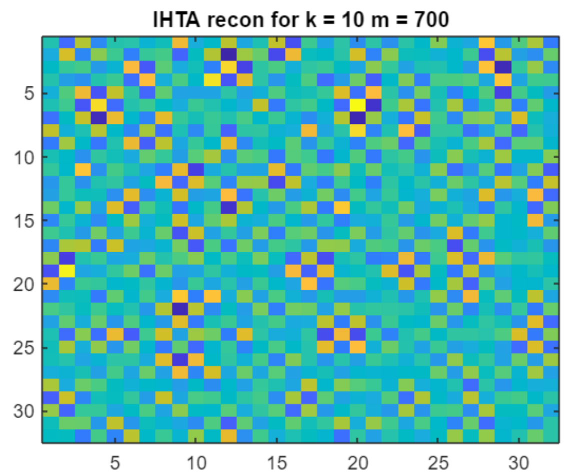


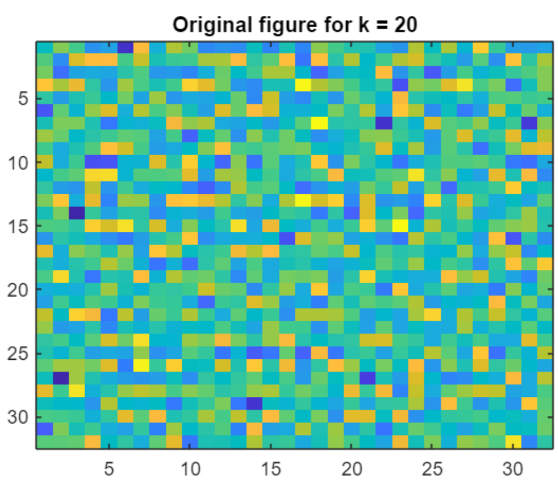
Figure 8: Caption

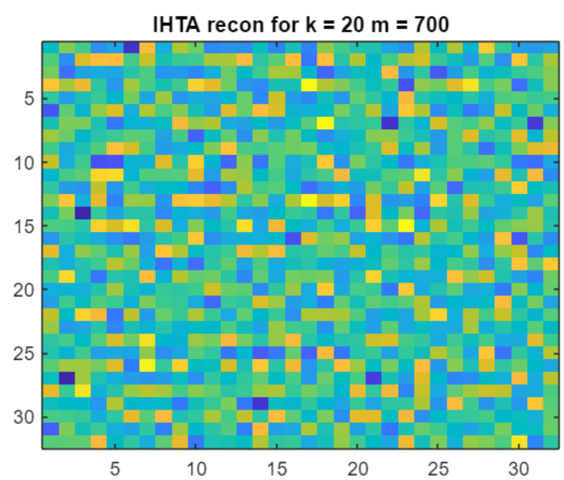

Original figure for k = 5

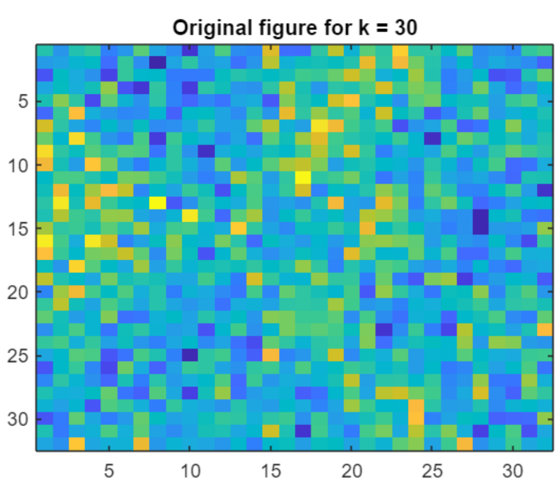

IHTA recon for k = 5 m = 700

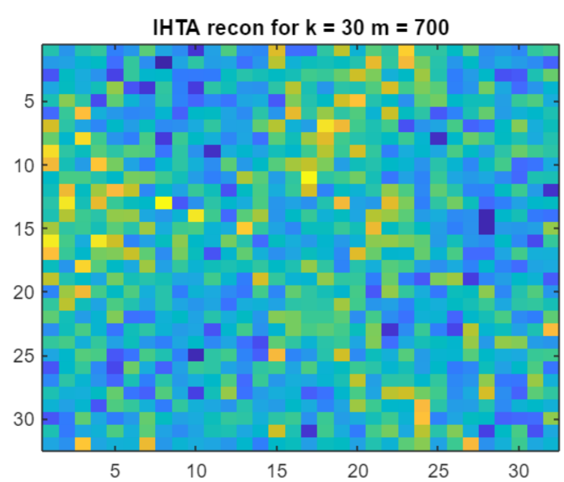

Original figure for k = 10

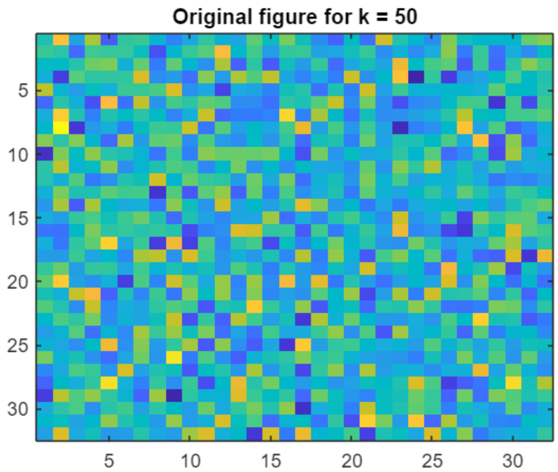

IHTA recon for k = 10 m = 700

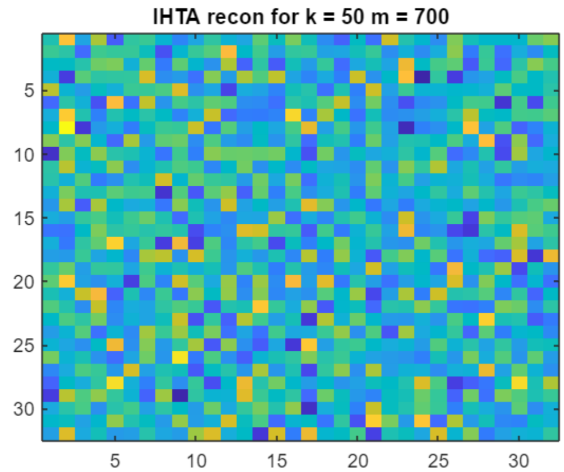

Original figure for k = 20

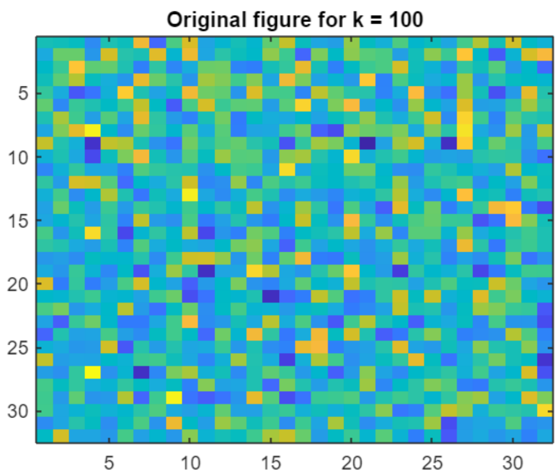

IHTA recon for k = 20 m = 700

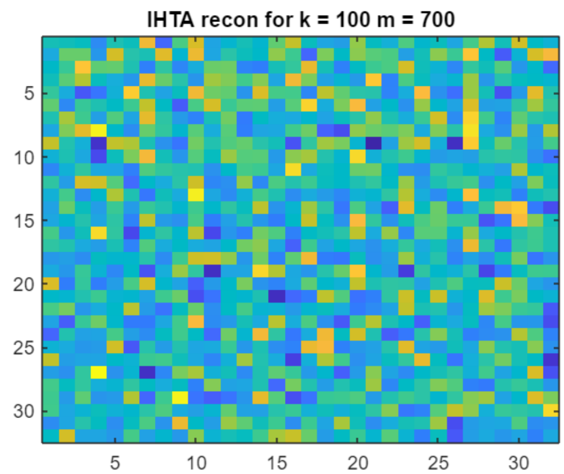

Original figure for k = 30

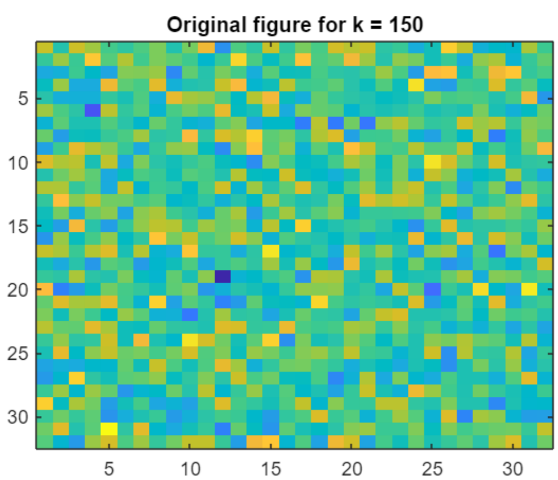

IHTA recon for k = 30 m = 700

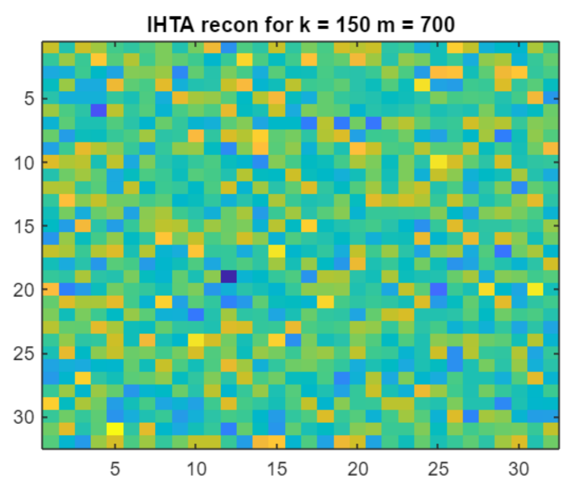

Original figure for k = 50

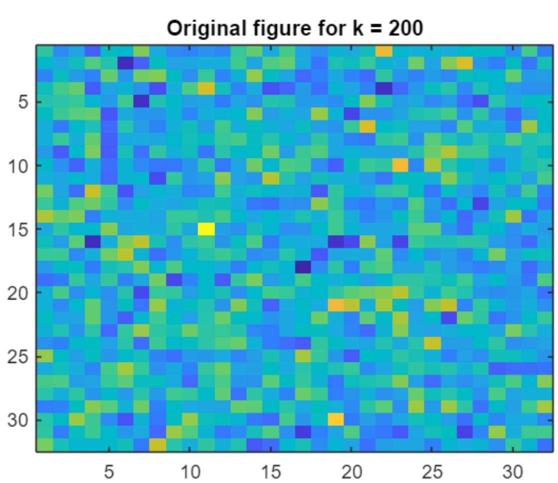

IHTA recon for k = 50 m = 700

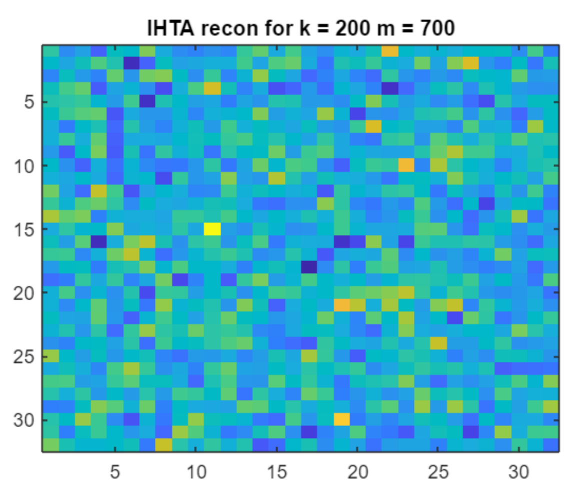

Original figure for k = 100


IHTA recon for k = 100 m = 700


Original figure for k = 150


IHTA recon for k = 150 m = 700


Original figure for k = 200


IHTA recon for k = 200 m = 700

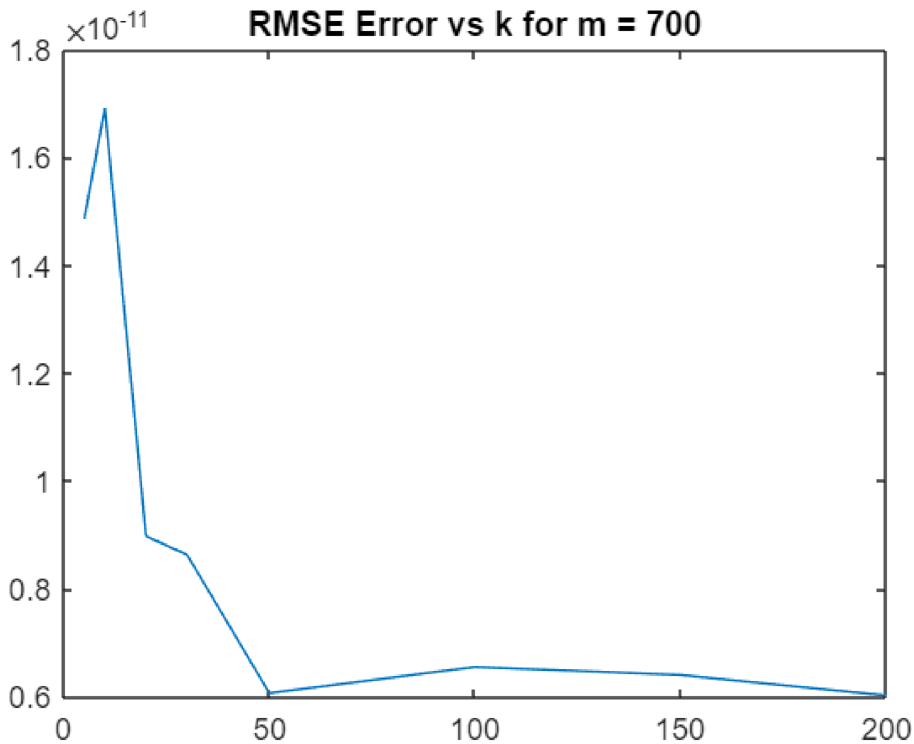


Figure 11: Caption