Zero Sum games and LP Duality
A CS 602 presentation
Derivation of Strong LP Duality from Minimax Theorem of Zero Sum Games

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Notations used

- m and n are positive integers, $[n] = \{1,...n\}$.
- All vectors are column vectors unless specified otherwise. The jth component of a vector x is written as x_j.
- All matrices have real entries.
- The transpose of a matrix A is written A^T.
- The all-zero and the all-one vector are written as $\mathbf{0} = (0, \ldots, 0)^T$ and $\mathbf{1} = (1, \ldots, 1)^T$, their dimension depending on the context, and the all-zero matrix as just 0.
- Inequalities between vectors or matrices such as $x \geq \mathbf{0}$ represent inequality between all respective components.

LP and Dual LP

- A linear program (LP) in *inequality* form, given by an $m \times n$ matrix A and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ for a vector $x \in \mathbb{R}^n$ is:

$$\max_x c^T x, \text{ subject to } Ax \leq b, x \geq \mathbf{0} \tag{1}$$

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- The *dual* of the above *primal* LP for a vector $y \in \mathbb{R}^m$ is given by:

$$\min_y b^T y, \text{ subject to } A^T y \geq c, y \geq \mathbf{0} \tag{2}$$

LP duality

- **Weak LP duality**
  It states that if both *primal LP* (1) and the *dual LP* (2) have feasible solutions x and y, respectively, then their objective function values are mutual bounds, that is,

$$c^T x \leq b^T y \tag{3}$$

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- **Strong LP duality**
  If the *primal LP* (1) and the *dual LP* (2) are feasible, then there exist feasible x and y with $c^T x = b^T y$, which are therefore optimal solutions.

Definition - Zero Sum Game

- A **zero sum game** given by an $m \times n$ matrix A, is played between a row player who secretly chooses a row i of A and a column player who secretly chooses a column j.
- Then both players reveal their choices, after which the row player receives the *payoff* a_{ij} from the column player (and since it is a zero-sum game is like a *cost* to the column player, or *payoff* for column player is $-a_{ij}$).
- Such games are called *zero-sum games* since whatever one player gains is what the other player loses.
- A common example is *Rock-Paper-Scissors*.

Definition - Strategy

- A *strategy* refers to a player's plan specifying which choices it will make in every possible situation, leading to an eventual outcome.
- The rows and columns are called the players' **pure strategies**.
- The players can *randomize* their strategies by choosing actions according to a probability distribution called a *mixed strategy*. The row player is then assumed to maximize their *expected payoff* and the column player to minimize their *expected cost*.

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- We denote the set of mixed strategies of the row player by:

$$Y = \{y \in \mathbb{R}^m | y \geq \mathbf{0}, y^T\mathbf{1} = 1\}$$

and the column player by:

$$X = \{x \in \mathbb{R}^n | x \geq \mathbf{0}, x^T\mathbf{1} = 1\}$$

- With the mixed strategies of the row and column player, the expected payoff of the row player and the expected cost to the column player is $y^T Ax$.

Von Neumann's Minimax theorem

- A is the matrix associated with some *zero-sum game*.
- Suppose column player plays first with mixed strategy x and then row player plays with mixed strategy y, then expected payoff for row player is given by:

$$\min_{x} \max_{y} y^T Ax \tag{4}$$

  In other words choose $x \in X$ such that the maximum over $y \in Y$ of $y^T Ax$ is minimized.

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- The **minimax** theorem due Von Neumann states that optimum value of (4) and (5) are equal to some unique value ν called the **value** of the game.

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Reconstructing mini-max and maxi-min as LPs

We rewrite the optimization problem $\max_y y^T Ax$, where $y \in Y$ for a given x as

$$\min_v v, \;\; Ax \leq \mathbf{1}v$$

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Then, (4) corresponds to minimizing this over $x \in X$

$$\min_{x,v} v \text{ subject to } Ax \leq \mathbf{1}v, \; x \in X \tag{6}$$

and (5) corresponds, similarly to

$$\max_{y,u} u \text{ subject to } A^T y \geq \mathbf{1}u, \; y \in Y \tag{7}$$

Strong LP Duality proves Minimax Theorem for Zero Sum Games

- We modify LP(6) to:

$$\max_{x,v} -v \text{ subject to } Ax - \mathbf{1}v \leq \mathbf{0},\ -\mathbf{1}^T x = -1 \tag{8}$$

and LP(7) to:

$$\min_{y,u} -u \text{ subject to } A^T y - \mathbf{1}u \geq \mathbf{0},\ -\mathbf{1}^T y = -1 \tag{9}$$

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Now, (8) and (9) form a primal-dual pair in general LP form. Since both LPs are feasible, the strong LP duality theorem (which also holds for LPs in general) implies that their optimal values are equal ($-v = -u$), which proves **Minimax theorem for zero sum games**.

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- Our aim now is to prove the converse. (Note that the min-max and max-min values in (6) and (7) exist without having to assume LP-duality)

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Dantzig's game (a fake proof!)

This portion assumes minimax theorem.

Dantzig's Theorem

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. Consider the zero sum game with the payoff matrix $B \in \mathbb{R}^{(m+n+1)\times(m+n+1)}$, defined as:

$$B = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix} \tag{10}$$

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Then B has game value 0, with a min-max strategy $z = (y, x, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ which is also a max-min strategy, with $Bz \leq \mathbf{0}$. If $t > 0$ then $\frac{1}{t}x$ is a *optimal* solution to the primal LP (1) and $\frac{1}{t}y$ is a *optimal* solution to the dual LP (2). If $(Bz)_{(m+n+1)} < 0$, then $t = 0$ and both the primal and dual are *infeasible*.

Proof by Dantzig

- $B = -B^T$
  $\therefore$ The value of the game is 0 (refer Appendix B).
- $Bz \leq \mathbf{0}$ (using the definition of min-max).
  In other words $Ax - bt \leq \mathbf{0}$, $-A^T y + ct \leq \mathbf{0}$ and $b^T y - c^T x \leq 0$.

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- If $t > 0$, then $\frac{1}{t}x$ and $\frac{1}{t}y$ are primal and dual feasible solutions respectively with $c^T x \geq b^T y$, but by weak duality $c^T x \leq b^T y$, therefore $c^T x = b^T y$ and we have $\frac{1}{t}x$ and $\frac{1}{t}y$ as respective optimal solutions.

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- If $(Bz)_{(m+n+1)} < 0$, i.e., $c^T x < b^T x$, weak duality is violated if $t > 0$, so $t = 0$. Moreover, $Ax \leq \mathbf{0}$ and $A^T y \geq \mathbf{0}$ and, $b^T y \leq \mathbf{0}$ or $c^T x \geq \mathbf{0}$. This leads to infeasiblity of atleast one of the LPs (1) or (2).

Loophole

- This seems to prove strong LP duality from minimax theorem by the construction of a game that gives us either the optimal solutions or proves infeasibility of atleast one of the LPs.

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- This seems to prove strong LP duality from minimax theorem by the construction of a game that gives us either the optimal solutions or proves infeasibility of atleast one of the LPs.
- However, this does not cover the case when $(Bz)_{(m+n+1)} = 0$ and $t = 0$.
- If we assume $t - (Bz)_{(m+n+1)} > 0$, then the above proof by Dantzig works!!!.

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- However, this does not cover the case when $(Bz)_{(m+n+1)} = 0$ and $t = 0$.
- If we assume $t - (Bz)_{(m+n+1)} > 0$, then the above proof by Dantzig works!!!.
- However, this assumption turned out to be equivalent[1] to assuming *Farkas' lemma*! This defeats the point of proving LP Duality as we already know the proof of strong LP duality from Farkas' Lemma.

[1] Proof is omitted

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4. Tucker's Theorem **proves** Farkas' Lemma

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3. Gordan's Theorem **proves** Tucker's Theorem
4. Tucker's Theorem **proves** Farkas' Lemma
5. Farkas' Lemma **proves** strong LP duality

Therefore, we have **minimax theorem of zero sum games** proving **strong LP duality**.

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The Theorems of Gordan and Ville

Let $A \in \mathbb{R}^{m \times n}$. The following Theorem of Gordan (11) proves the Theorem of Ville (12) and vice versa and (12) proves the minimax theorem and vice versa.

$$\nexists x \in \mathbb{R}^n : Ax = \mathbf{0},\ x \geq \mathbf{0},\ x \neq \mathbf{0} \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A > \mathbf{0}^T \tag{11}$$

$$\nexists x \in \mathbb{R}^n : Ax \leq \mathbf{0},\ x \geq \mathbf{0},\ x \neq \mathbf{0} \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A > \mathbf{0}^T,\ y \geq \mathbf{0} \tag{12}$$

Proof of Ville's theorem from Gordan's Theorem

- Assume (11). We prove (12).
- (12)'s ($\Leftarrow$) direction is trivial (multiply y^T to both sides of the inequality $Ax \leq \mathbf{0}$ to get $y^T Ax > 0$ which contradicts $y^T Ax \leq y^T \mathbf{0} = 0$).

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- We now prove the other direction.
- Assume $\nexists x \geq \mathbf{0}, x \neq \mathbf{0}, Ax \leq \mathbf{0}$, then $\nexists x, s \geq \mathbf{0}, (x, s) \neq (\mathbf{0}, \mathbf{0}), Ax + s = \mathbf{0}$. Now consider the matrix $B = (A\ I)$ where I is the $m \times m$ identity and the vector $z = (x, s)^T$.
- Equivalently, $\nexists z \geq \mathbf{0}, Bz = 0, z \neq \mathbf{0}$, then by (11), $\exists y', y'^T B > \mathbf{0}^T$ and thus $y = y'$, gives us the y in (12) (note $y' > \mathbf{0}$ as $y'^T I > \mathbf{0}^T$).
- Hence proved $\square$

Proof of Gordan's theorem from Ville's Theorem

- Assume (12). We prove (11).
- Similar to before, the ($\Leftarrow$) is trivial, so we prove the ($\Rightarrow$) direction.
- Assume $\nexists x, Ax = \mathbf{0}, x \geq \mathbf{0}, x \neq \mathbf{0}$. Then $\nexists x, Ax \leq \mathbf{0}, -Ax \leq \mathbf{0}, x \geq \mathbf{0}, x \neq \mathbf{0}$.
- Then by (12), $\exists y^+, y^- \geq \mathbf{0}$ such that ${y^+}^T A > \mathbf{0}^T, -{y^-}^T A > \mathbf{0}^T$. Therefore, ${y^+}^T A - {y^-}^T A > \mathbf{0}^T$ and $y^+ - y^-$ gives us our y.
- Hence proved $\square$

Proof of Ville's theorem from Minimax Theorem

1. Now we prove the theorem of Ville from minimax theorem.
2. Assume minimax theorem holds on $A \in \mathbb{R}^{m \times n}$. We once again prove the $(\Rightarrow)$ side since the other side $(\Leftarrow)$ is trivial.
3. Assume $\nexists x \in \mathbb{R}^n : Ax \leq \mathbf{0},\ x \geq \mathbf{0}, x \neq \mathbf{0}$, then by the formulation of the min-max strategy in (6) we see that the value of the game, which is $\min_{x,v} v, Ax \leq \mathbf{1}v$, must be positive. Otherwise, $\exists x \in X$ due to minimax theorem such that $Ax \leq \mathbf{0}$ which contradicts the assumption.

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4. Then, by minimax theorem there is an optimal $y \in Y$ and $u = v$ such that $y^T A \geq \mathbf{1}^T u > \mathbf{0}^T$.
   $\therefore y^T A > \mathbf{0}^T$.
5. Hence proved $\square$

Proof of Minimax theorem form Ville's Theorem

1. To prove minimax theorem from the theorem of Ville.
2. Assume (12).
3. Consider the game on A. It has a max-min payoff u and according to (7) a max-min strategy $y \in Y$. Let $A' = A - \mathbf{1}u\mathbf{1}^T$.
4. $y^T A' = y^T A - u\mathbf{1}^T \geq \mathbf{0}^T$ (by (7)).
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5. $\exists x \geq \mathbf{0}, x \neq \mathbf{0}$ such that $A'x \leq \mathbf{0}$.
6. If not, by Ville's theorem, $\exists z, z^T A' > \mathbf{0}^T, z \geq \mathbf{0}$ and we can (because $z \neq 0$) scale z so that it belongs to Y. $z^T A' > \mathbf{0}^T \Rightarrow z^T A' \geq \varepsilon \mathbf{1}^T$ for some ε or $z^T A \geq (u + \varepsilon)\mathbf{1}^T$ which means u is not optimal $\#$.

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7. Therefore $A'x \leq \mathbf{0}$ for some $x \in X$ (by scaling). So $Ax \leq \mathbf{1}u$, or the min-max value is atmost u. But min-max is trivially greater than or equal to max-min. Thus, the min-max value is u, proving minimax theorem.
8. Hence proved $\square$

The Lemma of Tucker

The following is the lemma of Tucker for $A \in \mathbb{R}^{m \times n}$:

$$\exists y \in \mathbb{R}^m, x \in \mathbb{R}^n : y^T A \geq \mathbf{0}^T,\ x \geq \mathbf{0},\ Ax = \mathbf{0}, x_n + (A^T y)_n > 0 \quad (13)$$

and it has an equivalent inequality form

$$\exists y \in \mathbb{R}^m, x \in \mathbb{R}^n : y^T A \geq \mathbf{0}^T,\ y \geq \mathbf{0},\ x \geq \mathbf{0},\ Ax \leq \mathbf{0}, x_n + (A^T y)_n > 0 \quad (14)$$

It has a form for skew symmetric matrices $B \in \mathbb{R}^{k \times k}$ (provable from the lemma itself),

$$\exists z \in \mathbb{R}^k : z \geq \mathbf{0},\ Bz \leq \mathbf{0},\ z_k - (Bz)_k > 0 \quad (15)$$

This can be proved[2] using induction on n, the number of columns of A.

[2]We omit the proof in this presentation

The Theorem of Tucker

1. The last column of A in Tucker's lemma plays a special role which can be taken by any other column. We have the stronger version, the theorem of Tucker, provable from his lemma:

$$\exists y \in \mathbb{R}^m, x \in \mathbb{R}^n : y^T A \geq \mathbf{0}^T,\ x \geq \mathbf{0},\ Ax = \mathbf{0}, x + A^T y > 0 \quad (16)$$

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2. *Proof,*
Choose $x^{(i)}$, $y^{(i)}$ in (13) such that $x_i^{(i)} + (A^T y^{(i)})_i > 0$ (these come from Tucker's Lemma).
Then $x = \sum_{i=1}^n x^{(i)}$ and $y = \sum_{i=1}^n y^{(i)}$ satisfy (16).
3. We now move to the climax, an unexpected proof ... (*or was it?*)

Proof of Tucker's theorem from Gordan's theorem

Observation

If $Ax = \mathbf{0}$ and $x \geq \mathbf{0}$, then $\forall y$ such that $y^T A \geq \mathbf{0}^T$: if $x_j > 0$ then $(y^T A)_j = 0$ because otherwise, $y^T Ax = 0 = \sum_{j \in [n]} (y^T A)_j x_j > 0$

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1. Hence, (for a given A) for any x satisfying the conditions of (16), the set:

$$S = \mathrm{supp}(x) = \{\, j \in [n] \,\big|\, x_j > 0\}$$

   is unique.

2. The main idea is that the nonnegativity constraints for the variables x_j, $j \in S$ can be dropped and these variables therefore be eliminated, which allows applying Gordan's Theorem to the remaining variables.

Proof of Tucker's theorem from Gordan's theorem

1. Let $A = [A_1 \dots A_n]$. For any $S \subseteq [n]$ and $J = [n] - S$, we write $A = [A_J \, A_S]$ and $x = [x_J \, x_S]$ for $x \geq \mathbf{0}$.

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2. $Ax = 0, x \geq \mathbf{0}$ and $Ax' = 0, x' \geq \mathbf{0} \Rightarrow A(x + x') = 0$ and $x + x' \geq \mathbf{0}$.
3. $\text{supp}(x + x') = \text{supp}(x) \cup \text{supp}(x')$.

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3. $\mathrm{supp}(x + x') = \mathrm{supp}(x) \cup \mathrm{supp}(x')$.
4. Choose S as a maximal support under inclusion. i.e., $\forall x \geq \mathbf{0}\ (Ax = 0 \implies S \not\subset \mathrm{supp}(x))$. This exists since $\mathrm{supp}(x)$ is finite.

Proof of Tucker's theorem from Gordan's theorem

1. Let $A = [A_1 \dots A_n]$. For any $S \subseteq [n]$ and $J = [n] - S$, we write $A = [A_J \, A_S]$ and $x = [x_J \, x_S]$ for $x \geq \mathbf{0}$.
2. $Ax = 0, x \geq \mathbf{0}$ and $Ax' = 0, x' \geq \mathbf{0} \Rightarrow A(x + x') = 0$ and $x + x' \geq \mathbf{0}$.
3. $\text{supp}(x + x') = \text{supp}(x) \cup \text{supp}(x')$.
4. Choose S as a maximal support under inclusion. i.e., $\forall x \geq \mathbf{0} \; (Ax = 0 \implies S \not\subset \text{supp}(x))$. This exists since $\text{supp}(x)$ is finite.
5. $\forall y$ such that $A^T y \geq \mathbf{0}$: $y^T A_S = \mathbf{0}^T$ (by observation)

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5. $\forall y$ such that $A^T y \geq \mathbf{0}$: $y^T A_S = \mathbf{0}^T$ (by observation)
6. We now show that

$$\exists y \in \mathbb{R}^m, x = (\mathbf{0} \; x_S) : y^T A_J > \mathbf{0}^T, y^T A_S = \mathbf{0}^T, Ax = 0, x_s > \mathbf{0} \tag{17}$$

which implies Tucker's theorem (we will use Gordan's theorem).

Proof ctd[1]

1. Consider some $\tilde{x} \geq \mathbf{0}, A\tilde{x} = \mathbf{0}$ with maximal support (S), i.e., $x_S > \mathbf{0}$. If $S = [n]$ we are done, as then $\tilde{x} > 0$, giving (16).
2. Let k be the rank of A_S. If $k = m$, we claim that $S = [n]$.
3. If $k = m$, then A_S is full rank, hence for any $j \in J = [n] - S$, $A_j = A_S \hat{x_S}$, for some $\hat{x_S}$.
4. Consider x' which is 1 at the jth position and $x'_S = \alpha(\tilde{x_S}) - \hat{x_S}$ for a sufficiently large α so that $x'_S > \mathbf{0}$. Then $Ax' = 0, x' \geq \mathbf{0}$ and $\text{supp}(x') = S \cup j$ contradicting the maximality of S if $S \neq [n]$.

Proof ctd^2

1. Hence, let $k < m$. In order to apply Gordan's theorem, eliminate x_S from $Ax = A_J x_J + A_S x_S = \mathbf{0}$ by replacing it with an equivalent system $CAx = \mathbf{0}$ for a suitable invertible matrix $C \in \mathbb{R}^{m \times m}$.
2. Note that any solution of $A_J x_J + A_S x_S = \mathbf{0}$ is a solution of $CA_J x_J + CA_S x_S = \mathbf{0}$ and since C is invertible, vice versa.

Proof ctd[3]

1. WLOG let the last k rows of A_S be linearly independent and form the matrix F. Let the ith row of A_S be a_{iS}. Then $\forall i \in \{1, ...m-k\}$, $a_{iS} = z^{(i)}F$ for some $z^{(i)} \in \mathbb{R}^{(1 \times k)}$.
2. Then we construct C as:

$$C = \begin{bmatrix} 1 & \dots & 0 & & -z^1 & \\ & \ddots & & & \vdots & \\ 0 & \dots & 1 & & -z^{(m-k)} & \\ 0 & \dots & 0 & 1 & \dots & 0 \\ & \vdots & & & \ddots & \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix} \tag{18}$$

C is upper triangular and hence invertible ($\det(C) = 1$)

Proof ctd[4]

We have:

$$CA_J = \begin{bmatrix} D \\ E \end{bmatrix}, CA_S = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

Where $D \in \mathbb{R}^{(m-k)\times|J|}$, $E \in \mathbb{R}^{k\times|J|}$ and $F \in \mathbb{R}^{k\times|S|}$ is as defined earlier.

Proof ctd[5]

1. Suppose $\exists x_J \in \mathbb{R}^{|J|}$ such that

$$Dx_J = \mathbf{0}, x_J \geq \mathbf{0}, x_J \neq \mathbf{0} \tag{19}$$

2. Because F has rank k, $\exists x_S$ such that $Ex_J = -Fx_S$ or $Ex_J + Fx_S = \mathbf{0}$ hence $CA_Jx_J + CA_Sx_S = \mathbf{0}$ and hence $A_Jx_J + A_Sx_S = \mathbf{0}$.
3. Let $x(\alpha) = (x_J, x_S + \alpha\tilde{x_S})$, we have $Ax(\alpha) = \mathbf{0}$ since $A\tilde{x_S} = \mathbf{0}$ and $x(\alpha) \geq \mathbf{0}$ for arbitrarily large α.
4. But $x(\alpha)$ has a larger support than $\tilde{x_S}$ because $x_J \neq \mathbf{0}$. $\#$.
5. Hence, no such x_J as in (19) exists and for the finishing blow, by **Gordan's theorem** $\exists w \in \mathbb{R}^{m-k}, w^T D > \mathbf{0}^T$ i.e.,

$$(w^T, \mathbf{0}^T) \begin{bmatrix} D \\ E \end{bmatrix} > \mathbf{0}^T, (w^T, \mathbf{0}^T) \begin{bmatrix} 0 \\ F \end{bmatrix} = \mathbf{0}^T$$

   With $y = (w^T, \mathbf{0}^T)C$ and $x = \tilde{x}$, this implies (17) and we're done
6. Hence proved $\square$

Farkas' Lemma

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$: The following three are versions of Farkas' lemma (and are provably equivalent)[3]

$$\nexists x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0} \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T,\ y^T b < 0 \tag{20}$$

$$\nexists x \in \mathbb{R}^n : Ax \leq b,\ x \geq \mathbf{0} \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T,\ y \geq \mathbf{0},\ y^T b < 0 \tag{21}$$

$$\nexists x \in \mathbb{R}^n : Ax \leq b,\ \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A = \mathbf{0}^T,\ y \geq \mathbf{0},\ y^T b < 0 \tag{22}$$

[3] We've shown in class that (21) proves strong LP duality and thus we omit the proof.

Proof of Farkas' Lemma from Tucker's Lemma

To prove Farkas' lemma (20) from Tucker's lemma [2] (we once again prove only the ($\Rightarrow$) side as the other side ($\Leftarrow$) is trivial),

$$(\nexists x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0}\ ) \Rightarrow (\exists y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T,\ y^T b < 0)$$

Rewriting in contrapositive form,

$$(\forall y \in \mathbb{R}^m : \neg(y^T A \geq \mathbf{0}^T) \vee y^T b \geq 0) \Rightarrow (\exists x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0})$$

OR

$$(\forall y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T \Rightarrow y^T b \geq 0) \Rightarrow (\exists x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0}) \quad (23)$$

Proof of Farkas' Lemma from Tucker's Lemma

Proof,

1. Consider the matrix $A' = [A \;\; -b]$ such that it is concatenation of A and $-b$.

Proof of Farkas' Lemma from Tucker's Lemma

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1. Consider the matrix $A' = [A \ \ -b]$ such that it is concatenation of A and $-b$.
2. By Tucker's lemma we are guaranteed that $\exists x_0, y_0$ with
   $x_0 = ({x'}^T, (x_0)_{n+1})^T$,(where $x' \in \mathbb{R}^n$) such that
   $y_0^T A' \geq \mathbf{0}^T$ (which is equivalent to saying $y_0^T A \geq \mathbf{0}^T \wedge y_0^T b \leq 0$,)
   $x_0 \geq \mathbf{0}$,
   $A' x_0 = \mathbf{0}$ (i.e., $Ax' - b(x_0)_{n+1} = \mathbf{0}$) and
   $(x_0)_{n+1} - y_0^T b > 0$

Proof of Farkas' Lemma from Tucker's Lemma

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   $A' x_0 = \mathbf{0}$ (i.e., $Ax' - b(x_0)_{n+1} = \mathbf{0}$) and
   $(x_0)_{n+1} - y_0^T b > 0$
3. Now by the hypothesis of Farkas' lemma (23), i.e.
   $\forall y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T \Rightarrow y^T b \geq 0$
   and since $y_0 \in \mathbb{R}^m$, and $y_0^T A \geq \mathbf{0}^T$, we have $y_0^T b \geq 0$,
   Also, we have $y_0^T b \leq 0$ and thus $y_0^T b = 0 \Rightarrow (x_0)_{n+1} > 0$
4. This gives $Ax = b$ for $x = x'/(x_0)_{n+1} \geq \mathbf{0}$

Hence proved $\square$

Conclusion

- This brings us to the end of the presentation.

[4] $\xrightarrow{\star}$ is obvious and we have not shown proof for $\xrightarrow{\star\star}$

Conclusion

- This brings us to the end of the presentation.
- We have shown the following derivations:
  Minimax Theorem $\rightarrow$ Theorem of Ville $\rightarrow$ Gordan's theorem $\rightarrow$ Theorem of Tucker $\stackrel{\star}{\rightarrow}$ Tucker's Lemma $\rightarrow$ Farkas' lemma $\stackrel{\star\star}{\rightarrow}$ Strong LP Duality [4]
- Thus, we have proved that **Strong LP Duality can be derived from Minimax Theorem of Zero Sum Games.** $\square$

[4] $\stackrel{\star}{\rightarrow}$ is obvious and we have not shown proof for $\stackrel{\star\star}{\rightarrow}$

Thank You

Thank you for attending the presentation!!!

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Appendix A

- Since Tucker's lemma (or theorem) proves Farkas' lemma, it isn't surprising that it proves *strict complementary slackness*.
- Recall that for the LP and its dual, the feasible pair x and y respectively is optimal iff $c^T x = y^T Ax = y^T b$ which means $y^T(b - Ax) = (c^T - y^T A)x = 0$.

Appendix A

- Since Tucker's lemma (or theorem) proves Farkas' lemma, it isn't surprising that it proves *strict complementary slackness*.
- Recall that for the LP and its dual, the feasible pair x and y respectively is optimal iff $c^T x = y^T Ax = y^T b$ which means $y^T(b - Ax) = (c^T - y^T A)x = 0$.
- This means, in each component of say $y^T(b - Ax)$ atleast one of y_j^T and $(b - Ax)_j$ ($j \in [n]$) is 0 and similarly for $(c^T - y^T A)x$. This is **complementary slackness**.

Strict Complementary Slackness

For the LP and dual LP defined earlier, $\exists x, y$, both optimal, that is, they satisfy complementary slackness conditions and

$$y + (b - Ax) > \mathbf{0}, \quad x + (y^T A - c^T) > \mathbf{0} \tag{24}$$

Appendix A

Proof:

Tucker's lemma for skew symmetric matrices (15) shows that for a skew symmetric matrix B, there is some z such that

$$z \geq \mathbf{0}, \ \ Bz \leq \mathbf{0}, \ \ z_k - (Bz)_k > 0$$

This applied to Dantzig's game gives a $z = (x', y', t')$, with $t' > 0$.
$x = \frac{1}{t}x'$ and $y = \frac{1}{t}y'$ satisfy (24).

Appendix A

- The proof of strict complementary slackness demonstrates a very good use of Dantzig's game B.
- Geometrically, the LP solutions x and y are then in the relative interior of the set of optimal solutions.
- Unless this set is a singleton, x and y are not unique, but their supports $\text{supp}(x)$ and $\text{supp}(y)$ are unique, shown similarly to the initial argument in the proof of Tucker's theorem from Gordan's theorem.

Appendix B

A zero sum game formed by a skew-symmetric matrix B has value 0 and the min-max strategy is the same as the max-min strategy. This comes from the fact that:

$$\begin{aligned}
&\min_v v, Bx \leq \mathbf{1}v, x \in X \\
=&\min_v v, -Bx \geq -\mathbf{1}v \\
=&\min_v v, B^T x \geq -\mathbf{1}v \\
=&\max_v -v, B^T x \geq \mathbf{1}(-v) \\
=&\max_u u, B^T y \geq \mathbf{1}u, y \in Y
\end{aligned}$$

Thus, we obtain $x = y$.
Then, the value of the game $u = x^T Bx = -x^T B^T x = -(x^T Bx)^T = -u$.
$\therefore\ u = 0$

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References

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[2] Tucker, A. W. (1956). Dual systems of homogeneous linear relations. In: H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, volume 38 of *Annals of Mathematics Studies*, 1–18. Princeton University Press, Princeton, NJ.