8

Vector Calculus

INTRODUCTION :

In previous semester we focused on differential of vector function and its algebra and important concepts like gradient, and direction derivative. In this section, we will focus on parameterization of curve, arc length, line integral, divergence and curl.

We will also discuss the relation between line integral and double integral by Green's theorem.

Universal Notation : $\vec{r} = [x, y, z] = x\hat{i} + y\hat{i} + z\hat{k}$

$\vec{a} = [a_1, a_2, a_3] = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

But (a_1, a_2, a_3) is a point not a vector.

Dot product :

If $\bar{a} = [a_1, a_2, a_3] = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = [b_1, b_2, b_3] = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then $\bar{a} \cdot \bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector product :

If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

Point Function :

A variable quantity whose value at any point in a region R of space depends upon the position of the point, is called a point function. There are two types of point functions (1) Scalar point function (2) Vector point function.

Scalar Point Function :

A function $\phi(x, y, z)$ is called scalar point function defined in the region R, if it associates a scalar quantity with every point in the region R of space.

e.g. temperature, density, potential

$f(x, y, z) = x + y + z$, $f(x, y, z) = x^2 + y^2 + z^2$

Vector Point Function :

A function $\bar{V}(x, y, z)$ is called vector point function defined in the region R, if it associates a vector quantity with every point in the region R of space.

$$\vec{V}(x, y, z) = v_1(x, y, z)\hat{i} + v_2(x, y, z)\hat{k} + v_3(x, y, z)\hat{k}$$

e.g. (i) $\bar{v}(t) = t\hat{i} + t^2\hat{j} + 3t\hat{k}$

(ii) $\bar{v}(x, y, z) = x^2\hat{i} + xy\hat{j} + (x + z)\hat{k}$

The velocity of a moving body and gravitational force are examples of vector point function.

The gradient (grad f) of a given scalar function $f(x, y, z)$ is a vector function defined by

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k},$$ here f is a differentiable function.

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$ is know as differential operator.

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \text{grad } f$$

e.g. $f(x, y, z) = x^3 + y^2 + \sqrt{z}$

$$\nabla f = \text{grad } f = \frac{\partial}{\partial x}(x^3 + y^2 + \sqrt{z})\hat{i} + \frac{\partial}{\partial y}(x^3 + y^2 + \sqrt{z})\hat{j} + \frac{\partial}{\partial z}(x^3 + y^2 + \sqrt{z})\hat{k}$$

$$= 3x^2\hat{i} + 2y\hat{j} + \frac{1}{2\sqrt{z}}\hat{k}$$

8.1 PARAMETRIZATION OF CURVES

Parametric curves in XY-Plane :

Suppose the cartesian coordinate system is given. Let $x = x(t)$, $y = y(t)$ be continuous functions of t over an interval $[a, b]$. The points $P(t) = P(x(t), y(t))$ constitutes a curve joining start point $P(a)$ and end point $P(b)$. This curve is called a parametric curve.

$x = x(t)$, $y = y(t)$ is a parametrization of the curve over $[a, b]$. These equations are called parametric equations.

(1) The point P(a) is an initial point.

(2) The parametrization of a given curve need not be unique.

Note : A curve is a set of points, where as a parametric curve is a set of points traced in a particular direction.

Parametric curve in XY plane can be written in vector form as

$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} = [x(t), y(t)],\ t \in [(a, b)]$$

Parametric curves in Space :

$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$\bar{r}(t) = [x(t), y(t), z(t)]$

$\bar{r}(t_o)$ is a position vector of point $(x(t_o), y(t_o), z(t_o))$. When t increases the locus of point $(x(t_i), y(t_i), z(t_i))$ x $t_1, t_2, \ldots$ shows positive sense on the curve. When t decreases it shows the negative sense.

$\therefore\ \bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ gives orientation of C.

Closed Curve : If end points coincide then the curve is called a closed curve.

Arc : If end points do not coincide then the curve is called an Arc.

Classification of parametric curves :

Simple curve :

If a curve does not intersect itself then it is called a simple curve.

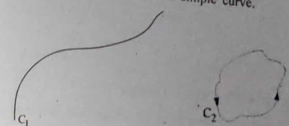


Fig. 1

There are two types of simple curves :

(1) *Simple Arc :*

An arc is said to be simple curve (arc)

if $t_1 \neq t_2 \Rightarrow P(t_1) \neq P(t_2)$, $(\bar{r}(t_1) \neq \bar{r}(t_2))$ for all $t_1, t_2 \in [a, b]$

For example, $\bar{r}(t) = \cos t\hat{i} + \sin t\hat{j},\ 0 \le t \le \pi$

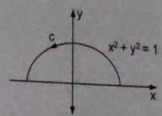


Fig. 2

(2) *Simple Closed Curve :*

A closed curve is said to be simple closed curve

if $t_1 \neq t_2 \Rightarrow P(t_1) = P\{(t_2)$, for all $t_1, t_2 \in [a, b]$

In otherwords,

Let $\bar{r}(t)$ be a parametric curve over $t \in [a,b]$.

If $\bar{r}(a) = \bar{r}(b)$ then $\bar{r}(t)$ is called closed. curve.

For example, $\bar{r}(t) = \cos t\hat{i} + \sin t\hat{j},\ 0 \le t \le 2\pi$

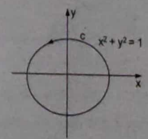


Fig. 3

Definition : Smooth curve

If a curve possesses a tangent vector which varies continuously along its length then the curve is called a smooth curve.

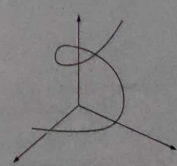


Fig. 4

Definition : Piecewise Smooth curve :

If a curve is comprised of finitely many smooth segments end to end, then it is called a piecewise smooth curve.

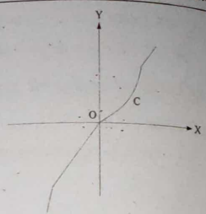


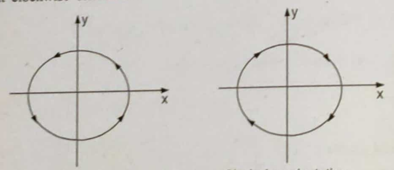
Fig. 5

Definition : Orientation of a Parametric Curve :

If specific conditions are imposed on the parametric equations then parametric curve follows specific direction as the parameter increases. This is called an orientation of a parametric curve.

For example,

The circle $x^2 + y^2 = a^2$ $(a \neq 0)$ with the parametric equations $x = a \cos t$, $y = a \sin t$; $0 \le t \le 2\pi$, which represent counter clock $\leftrightarrow$ wise orientation, whereas the parametric equations $x = a \cos t$, $y = -a \sin t$, $0 \le t \le 2\pi$ represent clockwise orientation.



Counter clockwise orientation   Clockwise orientation

Fig. 6

Parametric representation of some standard curves in the space.

Parametric equation of line

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$$

$$\bar{r}(t) = \bar{a} + \bar{b}\, t,\ t \in R$$

$$= [a_1 + b_1 t,\ a_2 + b_2 t,\ a_3 + b_3 t]$$

$$= (a_1 + b_1 t)\hat{i} + (a_2 + b_2 t)\hat{j} + (a_3 + b_3 t)\hat{k}$$

Parametric equation of circle

$x^2 + y^2 = a^2$ and $z = c$

$$\bar{r}(t) = a \cos t\ \hat{i} + a \sin t\ \hat{j} + c\hat{k}$$

For unit circle in XY-plane $x^2 + y^2 = 1$ and $z = 0$

$$\bar{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j} + 0\hat{k}$$

Parametric equation of a circular Helix

$$x^2 + y^2 = a^2 \text{ and } z = c \tan^{-1}\left(\frac{y}{x}\right)$$

$$\bar{r}(t) = a \cos t\ \hat{i} + a \sin t\ \hat{j} + ct\hat{k}$$

Note : If $c > 0$ then orientation is in upward direction of z-axis.
If $c = 0$ then it is a circle.
If $c < 0$ then orientation is in downward direction of z-axis.

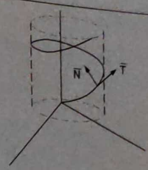


Fig. 7

EXAMPLE-1 : Write parametric representations of following curves and identify the curve.

(i) $z = x^2 + y^2$ *and* $y = 3$

$$\bar{r}(t) = t\hat{i} + 3\hat{j} + (t^2 + 9)\hat{k}$$

It is a parabola.

(ii) $z = x^2 + y^2$ *and* $z = 3$

$$\bar{r}(t) = \sqrt{3} \cos t\ \hat{i} + \sqrt{3} \sin t\ \hat{j} + 3\hat{k}$$

It is a circle.

(iii) $z^2 = x^2 + y^2$ and $z = 2$

$$\bar{r}(t) = 2 \cos t\ \hat{i} + 2 \sin t\ \hat{j} + 2\hat{k}$$

8.2 ARC LENGTH OF CURVE IN SPACE

Arc Length :

The parameterization for a curve is a set of functions depending only on a parameter t along with the bounds for the parameter. When we parameterize a curve by taking values of t from some interval $[a, b]$, the position vector $\bar{r}(t)$ of any point t on the curve can be written as

$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$\therefore \quad \frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\therefore \quad \left|\frac{d\bar{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The length of the curve is given by

$$L = \int_{t=a}^{b} \left|\frac{d\bar{r}}{dt}\right| dt = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The arc length of the curve is obtained by replacing the constant limit with a variable t.

Thus, arc length of the curve

$$S = S_{(u)} = \int_{t=t_1}^{t_2} \left|\frac{dr}{du}\right| du$$

Alternative Method :

Let $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\bar{r}'(t) = \lim_{\Delta t \to 0} \frac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t}$$ is tangent vector of curve $\bar{r}(t)$.

Tangent line at point P ($t = t_0$), on curve $\bar{r}(t)$ is $\bar{q}(t) = \bar{r}(t_0) + t\,\bar{r}'(t_0)$.

Length of curve $\bar{r}(t)$ from point $t = a$ to $t = b$

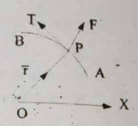


Fig. 8

$$l = \int_{t=a}^{t=b} \sqrt{\bar{r}'(t) \cdot \bar{r}'(t)}\, dt$$

$$= \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$s(t) = \int_{t=a}^{t=t} \sqrt{\bar{r}'(t) \cdot \bar{r}'(t)}\, dt$$ is called arc length function.

EXAMPLE-2 : Find the arc length of the curve

$$\bar{r}(t) = (4+3t)\hat{i} + (2-2t)\hat{j} + (5+t)\hat{k},\ 3 \le t \le 4$$

SOLUTION : Here, $\bar{r}(t) = (4+3t)\hat{i} + (2-2t)\hat{j} + (5+t)\hat{k}$

$$\therefore \frac{d\bar{r}}{dt} = 3\hat{i} - 2\hat{j} + \hat{k}$$

$$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{14}$$

The arc length, $$s = \int_{t=3}^{4} \left|\frac{d\bar{r}}{dt}\right| dt$$

$$= \int_{t=3}^{4} \sqrt{14}\, dt$$

$$= \sqrt{14}\,[t]_3^4$$

$$\therefore s = \sqrt{14}$$

EXAMPLE-3 : Find the arc length of the curve $\bar{r} = (1+3t^2)\hat{i} + (4+4t^3)\hat{j}$ between $t = 0$ to $t = 1$

SOLUTION : Here, $\bar{r} = (1+3t^2)\hat{i} + (4+4t^3)\hat{j}$

$$\frac{d\bar{r}}{dt} = 6t\hat{i} + 12t^2\hat{j}$$

$$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{(6t)^2 + (12t^2)^2}\ dt$$

$$= \sqrt{36t^2 + 144t^4}\ dt$$

The arc length $$= \int_{t=a}^{b} \left|\frac{d\bar{r}}{dt}\right| dt$$

$$= \int_{t=0}^{1} \sqrt{36t^2 + 144t^4}\ dt$$

$$= \int_{t=0}^{1} \sqrt{4t^2+1}\ (6t)\ dt$$

$$\therefore s = \frac{3}{4}\int_{t=0}^{1} \sqrt{4t^2+1}\ \frac{d}{dt}(4t^2+1)\ dt$$

$$= \frac{3}{4}\left[\frac{2}{3}(4t^2+1)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{1}{2}\left[(5)^{\frac{3}{2}} - 1\right]$$

$$\therefore s = \frac{1}{2}\left[5\sqrt{5} - 1\right]$$

EXAMPLE-4 : Find the arc length of the curve $\bar{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k},\ 0 \le t \le 2\pi$

SOLUTION : Here, $\bar{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

$$\therefore \frac{d\bar{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$$

$$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{2}$$

The arc length, $$s = \int_{t=a}^{b} \left|\frac{d\bar{r}}{dt}\right| dt$$

$$= \int_{t=0}^{2\pi} \sqrt{2}\ dt$$

$$= \sqrt{2}\,(t)_0^{2\pi}$$

$$= 2\sqrt{2}\,\pi$$

EXAMPLE-5 : Find arc length of circle $x^2 + y^2 = a^2$ and $z = 3$.

SOLUTION : We have : $x^2 + y^2 = a^2$ and $z = 3$.

Parametric equation

$$\bar{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + 3\hat{k},\ 0 \le t \le 2\pi$$

$$\therefore\ \bar{r}'(t) = -\,a\sin t\ \hat{i}\ \ a\cos t\ \hat{j}\ + 0\,\hat{k}$$

$$\bar{r}'(t)\cdot\bar{r}'(t) = a^2\sin^2 t + a^2\cos^2 t = a^2$$

$$\text{Length} = \int_{t=0}^{t=2\pi} \sqrt{\bar{r}'(t)\cdot\bar{r}'(t)}\ dt$$

$$= \int_0^{2\pi} \sqrt{a^2}\ dt$$

$$= 2\pi a$$

EXAMPLE-6 : Find length of curve of circular Helix $\bar{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + ct\,\hat{k}$ from $(a, 0, 0)$ to $(a, 0, 2\pi c)$.

SOLUTION : Here $\bar{r}(t) = [a\cos t,\ a\sin t,\ ct]$

At point $(a, 0, 0)$, $t = 0$

At point $(a, 0, 2\pi c)$, $t = 2\pi$

$$\bar{r}'(t) = [-a\sin t,\ a\cos t,\ c]$$

$$\bar{r}'(t)\cdot\bar{r}'(t) = a^2 + c^2$$

$$l = \int_{t=0}^{t=2\pi} \sqrt{\bar{r}'(t)\cdot\bar{r}'(t)}\ dt$$

$$= \int_{t=0}^{2\pi} \sqrt{a^2+c^2}\ dt$$

$$= \sqrt{a^2+c^2}\ 2\pi$$

EXAMPLE-7 : Find arc length of catenary curve $\bar{r}(t) = [t,\ \cosh t,\ 0]$ from $t = 0$ to $t = 1$.

SOLUTION : Here

$$\bar{r}(t) = [t,\ \cosh t,\ 0]$$

$$\bar{r}'(t) = [1,\ \sinh t,\ 0]$$

$$\bar{r}(t)\cdot\bar{r}'(t) = 1 + \sinh^2 t$$

$$= \cosh^2 t$$

$$\text{Arc length} = \int_{t=0}^{t=1} \sqrt{\bar{r}'(t)\cdot\bar{r}'(t)}\ dt$$

$$= \int_{t=0}^{1} \sqrt{\cosh^2(t)}\ dt$$

$$= [\sinh t]_0^1$$

$$= \sinh 1$$

$$= 1.175$$

EXAMPLE-8 : Integral $\int_C f(\bar{r})\ ds$, where $f = x^2 + y^2$,

$C \equiv y = 3x$ from $(0, 0)$ to $(2, 6)$

SOLUTION : Here s is a arc length

$$\bar{r}(t) = [t,\ 3t],\quad t = 0 \text{ to } t = 2$$

$$\text{Arc length}\ \ s = \int_0^t \sqrt{\bar{r}'(t)\cdot r'(t)}\ dt$$

$$= \int_0^t \sqrt{[1,3]\cdot[1,3]}\ dt$$

$$= \int_0^t \sqrt{10}\ dt$$

$$s = \sqrt{10}\ t$$

$$\Rightarrow ds = \sqrt{10}\ dt$$

$$\int_C f(\bar{r})\ ds = \int_{t=0}^{t=2} f(\bar{r}(t))\ \frac{ds}{dt}\ dt$$

$$= \int_0^2 (t^2 + 9t^2)\sqrt{10}\ dt$$

$$= 10\sqrt{10}\int_0^2 t^2\ dt$$

$$= 10\sqrt{10}\left[\frac{t^3}{3}\right]_0^2$$

$$= \frac{80\sqrt{10}}{3}$$

EXAMPLE-9 : Evaluate $\int_C f(\bar{r})\ ds$, where $f = x^2 + y^2 + z^2$,

$C \equiv \bar{r}(t) = [\cos t,\ \sin t,\ 2t],\ t = 0 \text{ to } \pi$

SOLUTION : Here

$$s = \int_0^t \sqrt{\bar{r}'(t)\cdot\bar{r}'(t)}\ dt$$

$$= \int_0^t \sqrt{[-\sin t, \cos t, 2]\cdot[-\sin t, \cos t, 2]}\ dt$$

$$= \int_0^1 \sqrt{\sin^2 t + \cos^2 t + 4}\ dt$$

$$= \sqrt{5}\,t$$

$$ds = \sqrt{5}\ dt$$

$$\int_C f(\bar{r})\ ds = \int_{t=0}^{\pi} (\cos^2 t + \sin^2 t + 4t^2)\cdot\sqrt{5}\ dt$$

$$= \sqrt{5}\int_0^{2\pi} (1 + 4t^2)\ dt$$

$$= \sqrt{5}\left[t + 4\frac{t^3}{3}\right]_0^{\pi}$$

$$\therefore \int_C f(\bar{r})\, ds = \sqrt{5}\left(\pi + \frac{4\pi^3}{3}\right)$$

8.3 LINE INTEGRAL

Integration of Vector Functions :

Let $\bar{F}(t)$ and $\bar{G}(t)$ be two vector functions.

If $\frac{d}{dt}\bar{F}(t) = \bar{G}(t)$, then $\bar{F}(t)$ is called an integral of $\bar{G}(t)$ w.r.to scalar variable t. It is denoted by

$$\int \bar{G}(t)\, dt = \bar{F}(t)$$

The definite integral of $\bar{G}(t)$ between $t = a$ and $t = b$ is given by

$$\int_{t=a}^{b} \bar{G}(t)\, dt = [\bar{F}(t)]_{t=a}^{t=b} = \bar{F}(b) - \bar{F}(a)$$

Line Integral :

Consider a curve C which is divided into n parts by the points $A = P_0, P_1, P_2, \ldots P_{n-1}, P_n = B$. Let $\bar{F}(\bar{r})$ be a vector point function defined at every point of a curve C. if $\bar{r}$ is the position vector of a point $P(x, y, z)$ on the curve C, then the line integral of $\bar{F}(\bar{r})$ over a curve C is defined by

$$\int_C \bar{F}(\bar{r})\cdot d\bar{r} = \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

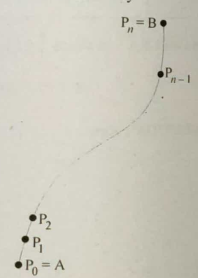


Fig. 9

If the Curve C is represented by parametric representation $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ and $t = a$ at A and $t = b$ at B, then the line integral along the curve C is

$$\int_C \bar{F}(\bar{r})\cdot d\bar{r} = \int_{t=a}^{b} \bar{F}(\bar{r})\cdot\frac{d\bar{r}}{dt}\, dt$$

$$= \int_{t=a}^{b} (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) dt$$

$$= \int_{t=a}^{b} \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right) dt$$

If the curve C (i.e. path of integration) is closed, then we use the nation $\oint_C$ instead of $\int_C$

Alternately :

Let $\bar{F}$ be a vector function and curve AB.

Line integral of a vector function $\bar{F}$ along the curve AB is defined as integral of the component of $\bar{F}$ along the tangent of the curve AB. $\frac{d\bar{r}}{ds}$ is a unit vector along tangent PT.

Component of $\bar{F}$ along a tangent PT at P = dot product of $\bar{F}$ and of unit vector along tangent PT $= \bar{F}\cdot\frac{d\bar{r}}{ds}$.

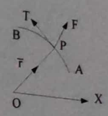


Fig. 10

Line Integral $= \oint_C \left(\bar{F}\cdot\frac{d\bar{r}}{ds}\right) ds = \oint_C \bar{F}\cdot d\bar{r}$

Note :

(1) If $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ then

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

(2) If the parametric representation of the curve C is $x = x(t)$, $y = y(t)$, $z = z(t)$ and $t = a$ at A and $t = b$ at B then

$$\int_C \bar{F}\cdot d\bar{r} = \int_{t=a}^{b} \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right) dt$$

The line integral of $\bar{F}$ between two points is equal to the potential different between these two points.

EXAMPLE-10 : If $\bar{F} = 3xy\,\hat{i} - y^2\,\hat{j}$, evaluate $\oint_C \bar{F}\cdot d\bar{r}$, where C is the arc of the parabola $y = 2x^2$ from O(0, 0) to A(1,2). **[GTU, Summer 2013]**

SOLUTION : Here, $y = 2x^2 \Rightarrow dy = 4x\, dx$

x varies from 0 to 1

Since $\bar{r} = x\hat{i} + y\hat{j}$

$$\bar{F}\cdot d\bar{r} = (3xy\,\hat{i} - y^2\,\hat{j})\cdot(dx\,\hat{i} - dy\,\hat{j})$$

$$= 3xy\, dx - y^2\, dy$$

$$= 3x(2x^2)\, dx - 4x^4\, 4x\, dx$$

$$\bar{F}\cdot d\bar{r} = (6x^3 - 16x^5)\, dx$$

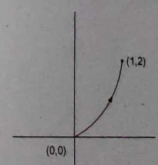


Fig. 11

Now, $$\int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} (6x^3 - 16x^5)\, dx$$

$$= \left[\frac{3x^4}{2} - \frac{8}{3}x^6\right]_0^1$$

$$= \frac{3}{2} - \frac{8}{3}$$

$$= \frac{-7}{6}$$

EXAMPLE-11 : If $\overline{F} = x^2y\hat{i} + 2yz\hat{j} + 3z^2x\hat{k}$, evaluate $\int_C \overline{F}\cdot d\overline{r}$ between A(0, 0, 0) and B(1, 2, 3) along the straight line joining A and B.

SOLUTION : The equation of line AB is $\frac{x-0}{1-0} = \frac{y-0}{2-0} = \frac{z-0}{3-0} = t$

$\therefore \quad x = t, \qquad y = 2t, \qquad z = 3t$

$\therefore \quad dx = dt, \qquad dy = 2dt, \qquad dz = 3dt$

A(x, y, z) = A(0, 0, 0) $\Rightarrow t = 0$

B(x, y, z) = B(1, 2, 3) $\Rightarrow t = 1$

$\overline{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \quad d\overline{r} = dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k}$

$$\overline{F}\cdot d\overline{r} = \left(x^2y\,\hat{i} + 2yz\,\hat{j} + 3xz^2\,\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= x^2y\ dx + 2yz\ dy + 3xz^2\ dz$$

Now, $\int_C F\cdot d\overline{r} = \int_C \left(x^2y\,dx + 2yz\,dy + 3xz^2\,dz\right)$

$$= \int_{t=0}^{1} 2t^3(dt) + 12\,t^2\,(2\ dt) + 27\,t^3\,(3\ dt)$$

$$= \int_{t=0}^{1} \left(2t^3 + 24t^2 + 81t^3\right) dt$$

$$= \int_{t=0}^{1} \left(83t^3 + 24t^2\right) dt$$

$$= \left[\frac{83t^4}{4} + \frac{24t^3}{3}\right]_0^1$$

$$= \frac{83}{4} + 8$$

$$= \frac{115}{4}$$

EXAMPLE-12 : Evaluate $\int_C \overline{F}\cdot d\overline{r}$ where $\overline{F} = (3x - 2y)\hat{i} + (y + 2z)\hat{j} - x^2\hat{k}$ and C is the parametric curve $x = t,\ y = t^2,\ z = t^3$ between A(0,0,0) and B(1,1,1).

SOLUTION : Here $x = t,\ y = t^2,\ z = t^3$

$\therefore \quad dx = dt, \qquad dy = 2t\ dt, \qquad dz = 3t^2\ dt$

A(x, y, z) = A (0, 0, 0) $\Rightarrow t = 0$

B(x, y, z) = B (1, 1, 1) $\Rightarrow t = 1$

$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \quad d\overline{r} = dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k}$

$$\overline{F}\cdot d\overline{r} = \left((3x-2y)\hat{i} + (y+2z)\hat{j} - x^2\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} - dz\,\hat{k}\right)$$

$$= (3x - 2y)\ dx + (y + 2z)\ dy - x^2\ dz$$

$$= (3t - 2t^2)\ dt + (t^2 + 2t^3)\ 2t\ dt - t^2\ 3t^2\ dt$$

$$= \left(3t - 2t^2 + 2t^3 + 4t^4 - 3t^4\right)\ dt$$

$$= \left(3t - 2t^2 + 2t^3 + t^4\right)\ dt$$

$$\int_C \overline{F}\cdot d\overline{r} = \int_{t=0}^{1} \left(3t - 2t^2 + 2t^3 + t^4\right)\ dt$$

$$= \left(\frac{3t^2}{2} - \frac{2t^3}{3} + \frac{2t^4}{4} + \frac{t^5}{5}\right)_0^1$$

$$= \frac{3}{2} - \frac{2}{3} + \frac{1}{2} + \frac{1}{5}$$

$$= \frac{23}{15}$$

EXAMPLE-13 : If $\overline{F} = x^2y\hat{i} + 2yz\hat{j} + 3z^2x\hat{k}$, evaluate $\int_C \overline{F}\cdot d\overline{r}$ between A(0, 0, 0) and B(1, 2, 3) along the straight line joining A and B.

SOLUTION : The equation of line AB is $\frac{x-0}{1-0} = \frac{y-0}{2-0} = \frac{z-0}{3-0} = t$

$\therefore \quad x = t, \qquad y = 2t, \qquad z = 3t$

$\therefore \quad dx = dt, \qquad dy = 2dt, \qquad dz = 3dt$

A(x, y, z) = A(0, 0, 0) $\Rightarrow t = 0$

B(x, y, z) = B(1, 2, 3) $\Rightarrow t = 1$

$\overline{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \quad d\overline{r} = dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k}$

$$\overline{F}\cdot d\overline{r} = \left(x^2y\,\hat{i} + 2yz\,\hat{j} + 3xz^2\,\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= x^2y\ dx + 2yz\ dy + 3xz^2\ dz$$

Now, $\int_C F\cdot d\overline{r} = \int_C \left(x^2y\,dx + 2yz\,dy + 3xz^2\,dz\right)$

$$= \int_{t=0}^{1} 2t^3(dt) + 12\,t^2\,(2\ dt) + 27\,t^3\,(3\ dt)$$

$$= \int_{t=0}^{1} \left(2t^3 + 24t^2 + 81t^3\right)\ dt$$

$$= \int_{t=0}^{1} \left(83t^3 + 24t^2\right) dt$$

$$= \left[\frac{83t^4}{4} + \frac{24t^3}{3}\right]_0^1$$

$$= \frac{83}{4} + 8$$

$$= \frac{115}{4}$$

EXAMPLE-14 : Evaluate $\int_C \bar{F} \cdot d\bar{r}$, where $\bar{F} = (3x - 2y)\hat{i} + (y + 2z)\hat{j} - x^2\hat{k}$ and C is the parametric curve $x = t,\ y = t^2,\ z = t^3$ between A(0,0,0) and B(1,1,1).

SOLUTION : Here $x = t,\ y = t^2,\ z = t^3$

$\therefore \quad dx = dt, \qquad dy = 2t\ dt, \qquad dz = 3t^2\ dt$

$A(x, y, z) = A\ (0, 0, 0) \Rightarrow t = 0$

$B(x, y, z) = B\ (1, 1, 1) \Rightarrow t = 1$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore \quad d\bar{r} = dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k}$$

$$\bar{F} \cdot d\bar{r} = \left((3x-2y)\hat{i} + (y+2z)\hat{j} - x^2\hat{k}\right) \cdot \left(dx\ \hat{i} + dy\ \hat{j} - dz\ \hat{k}\right)$$

$$= (3x - 2y)\ dx + (y + 2z)\ dy - x^2\ dz$$

$$= (3t - 2t^2)\ dt + (t^2 + 2t^3)\ 2t\ dt - t^2\ 3t^2\ dt$$

$$= \left(3t - 2t^2 + 2t^3 + 4t^4 - 3t^4\right) dt$$

$$= \left(3t - 2t^2 + 2t^3 + t^4\right) dt$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_{t=0}^{1} \left(3t - 2t^2 + 2t^3 + t^4\right) dt$$

$$= \left(\frac{3t^2}{2} - \frac{2t^3}{3} + \frac{2t^4}{4} + \frac{t^5}{5}\right)_0^1$$

$$= \frac{3}{2} - \frac{2}{3} + \frac{1}{2} + \frac{1}{5}$$

$$= \frac{23}{15}$$

EXAMPLE-15 : Find the value of line integral $\int_C \bar{F}(\bar{r}) \cdot d\bar{r}$ where $\bar{F} = -y\hat{i} - xy\hat{j} = [-y, -xy]$ C is a unit circle in 1[st] quadrant.

SOLUTION : C is a unit circle

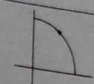


Fig. 12

$\Rightarrow \quad \bar{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j},\ t = 0 \text{ to } t = \frac{\pi}{2}$

$\Rightarrow \quad x(t) = \cos t,\ y(t) = \sin t$

$\Rightarrow \quad \bar{F}(\bar{r}(t)) = -\sin t\ \hat{i} - \cos t \sin t\ \hat{j}$

$= [-\sin t, -\cos t \sin t]$

Also, $\quad \bar{r}'(t) = [-\sin t, \cos t]$

$$\text{Now, } \int_C \bar{F}(\bar{r}) \cdot d\bar{r} = \int_{t=0}^{t=\pi/2} \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\ dt$$

$$= \int_{t=0}^{t=\pi/2} [-\sin t, -\cos t \sin t] \cdot [-\sin t, \cos t]\ dt$$

$$= \frac{1}{2}\int_{t=0}^{t=\pi/2} (1 - \cos 2t)\ dt - \int_{t=0}^{t=\pi/2} \cos^2(t) \sin t\ dt$$

$$= \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}} - \left[\frac{\cos^3 t}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} - \frac{1}{3}$$

EXAMPLE-16 : Evaluate $\int_C \bar{F}(\bar{r}) \cdot d\bar{r}$, where $\bar{F} = [z, x, y]$ $\bar{r}(t) = [\cos t, \sin t, 3t],\ 0 < t < 2\pi$

SOLUTION : Here $\quad \bar{F}(\bar{r}(t)) = [3t, \cos t, \sin t]$

$\bar{r}'(t) = [-\sin t, \cos t, 3]$

$$\int_C \bar{F}(\bar{r}) \cdot d\bar{r} = \int_{t=0}^{2\pi} \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\ dt$$

$$= \int_{t=0}^{2\pi} [3t, \cos t, \sin t] \cdot [-\sin t, \cos t, 3]\ dt$$

$$= \int_0^{2\pi} \left[-3t \sin t + \cos^2 t + 3 \sin t\right] dt$$

$$= -3\int_0^{2\pi} t \sin t\ dt + \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\ dt + 3\int_0^{2\pi} \sin t\ dt$$

$$= -3\left[t(-\cos t) - (-\sin t)\right]_0^{2\pi} + \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} + 3\left[-\cos t\right]_0^{2\pi}$$

$$= 2\pi + \pi$$

$$= 7\pi$$

EXAMPLE-17 : Find the work done by the field $\bar{F} = \left(x^2 - y^2 + x\right)^2\hat{i} - (2xy + y)\hat{j}$ in moving a particle in the XY–plane from A(0,0) to B(1,1) along the parabola $y^2 = x$ **[GTU, Summer 2014]**

SOLUTION : Here $y^2 = x$

$\Rightarrow 2y\,dy = dx$

$\Rightarrow dx = 2y\,dy$

$\bar{F} \cdot d\bar{r} = (x^2 - y^2 + x)\,dx - (2xy + y)\,dy$

y varies from 0 to 1

$$\text{Work done} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_C \left(x^2 - y^2 + x\right) dx - (2xy + y)\,dy$$

$$= \int_{y=0}^{1} \left(y^4 - y^2 - y\right) 2y\,dy - (2y^3 + y)\,dy$$

$$= \int_{y=0}^{1} \left(2y^5 - 2y^3 - y\right) dy$$

$$= \left[\frac{y^6}{3} - \frac{y^4}{2} - \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{2} - \frac{1}{2}$$

$$= \frac{-2}{3}$$

EXAMPLE-18 : Evaluate work done by force $\bar{F} = [3x^2, 2xz - y, -z]$ over the curve $\bar{r}(t) = [t, t^2, t^3], 0 \le t \le 1$.

SOLUTION : Here $\bar{r}(t) = [t, t^2, t^3]$

$\Rightarrow x(t) = t,\ y(t) = t^2,\ z(t) = t^3$

$\therefore\ \bar{F}(\bar{r}(t)) = [3t^2, 2t^4 - t^2, -t^3]$

Also, $\bar{r}'(t) = [1, 2t, 3t^2]$

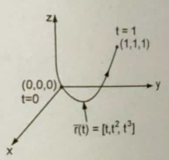


Fig. 13

$$\text{Now, } \int_C \bar{F}(\bar{r}) \cdot d\bar{r} = \int_{t=0}^{t=1} \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\,dt$$

$$= \int_0^1 \left[3t^2, 2t^4 - t^2, -t^3\right] \cdot \left[1, 2t, 3t^2\right] dt$$

$$= \int_0^1 \left[3t^2 + t^5 - 2t^3\right] dt$$

$$= \left[t^3 + \frac{t^6}{6} - \frac{2t^4}{4}\right]_0^1$$

$$= \frac{2}{3}$$

8.4 APPLICATION : WORK, CIRCULATION AND FLUX

Work done by a Force :

Let $\bar{F}$ represent the variable force acting on a particle moving along an arc AB of the curve C.

The work done by $\bar{F}$ during displacement from the point A to the point B is given by $\int_A^B \bar{F} \cdot d\bar{r}$

EXAMPLE-19 : Prove that work done equals to the gain in kinetic energy.

SOLUTION : Let $\bar{F}$ be a force

$$\text{Work done} = W = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_C \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\,dt$$

$$= \int_C m\,\bar{r}''(t) \cdot \bar{r}'(t)\,dt \quad (F = ma)$$

$$= \int_C m\bar{v}' \cdot v\,dt$$

$$= m \int_C \bar{v}' \cdot v\,dt$$

$$= m \int_C \left(\frac{\bar{v} \cdot \bar{v}}{2}\right)' dt$$

$$= \frac{m}{2} \int_C \frac{d\,|\bar{v}|^2}{dt}\,dt$$

$$= \frac{m}{2}\,|v|^2$$

= kinetic energy

EXAMPLE-20 : Find the work done in moving a particle in a force field $\bar{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = 1 + t^2$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

SOLUTION : Here, $x = 1 + t^2 \Rightarrow dx = 2t\,dt$

$y = 2t^2 \Rightarrow dy = 4t\,dt$

$z = t^3 \Rightarrow dz = 3t^2\,dt$

Now, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore\ d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\bar{F} \cdot d\bar{r} = \left(3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= 3xy\,dx - 5z\,dy + 10x\,dz$$

$$= 3\left(1 + t^2\right)\left(2t^2\right) 2t\,dt - 5t^3\,4t\,dt + 10\left(1 + t^2\right) 3t^2\,dt$$

$$= 2\left[6t^5 + 5t^4 + 6t^3 + 15t^2\right]$$

$$\therefore \text{ Work done } = \int_C \overline{F} \cdot d\bar{r}$$

$$= 2 \int_{t=1}^{2} \left[6t^5 + 5t^4 + 6t^3 + 15t^2\right] dt$$

$$= 2\left[t^6 + t^5 + \frac{3}{2}t^4 + 5t^3\right]_1^2$$

$$= 2\left[160 - \frac{17}{2}\right]$$

$$= 303$$

EXAMPLE-21 : Find the work done in moving a particle in the force field $\overline{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$, along the curves $x^2 = 4y$ and $3x^3 = 8z$ from $x = 0$ to $x = 2$.

SOLUTION : Here, $\overline{F} \cdot d\bar{r} = \left(3x^2\,\hat{i} + (2xz - y)\,\hat{j} + z\,\hat{k}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$

$$= 3x^2\,dx + (2xz - y)\,dy + z\,dz$$

$x^2 = 4y$ $\qquad$ $3x^3 = 8z$

$\therefore \; 2x\,dx = 4\,dy$ $\qquad$ $\therefore \; 9x^2\,dx = 8\,dz$

$\therefore \; dy = \frac{x}{2}\,dx$ $\qquad$ $\therefore \; dz = \frac{9}{8}x^2 dx$

x varies from $x = 0$ to $x = 2$.

$$\text{Work done } = \int_C \overline{F} \cdot d\bar{r}$$

$$= \int_C \left(3x^2\,dx + (2xz - y)dy + z\,dz\right)$$

$$= \int_{x=0}^{2} 3x^2\,dx + \left(2x\left(\frac{3x^3}{8}\right) - \frac{x^2}{4}\right)\frac{x}{2}\,dx + \frac{3}{8}\,x^3\,\frac{9}{8}\,x^2\,dx$$

$$= \int_{x=0}^{2} \left[3x^2 - \frac{x^3}{8} + \frac{51}{64}x^5\right] dx$$

$$= \left[x^3 - \frac{x^4}{32} + \frac{51}{64}\,\frac{x^6}{6}\right]_0^2$$

$$= \left[8 - \frac{1}{2} + \frac{17}{2}\right] - 0$$

$$= 16$$

EXAMPLE-22 : Find the work done by $\overline{F} = [y^2, -x^2]$ over straight line (0, 0) to (1, 4).

SOLUTION : Here $\bar{r}(t) = [t, 4t]$ $t = 0$ to $t = 1$

$\bar{r}'(t) = [1, 4]$

$\overline{F}(\bar{r}(t)) = [16t^2, -t^2]$

$$\text{Work done, W} = \int_C \overline{F} \cdot d\bar{r} = \int_0^1 \overline{F}(\bar{r}(t)) \cdot \bar{r}'(t)\,dt$$

$$= \int_0^1 [16t^2, -t^2] \cdot [1, 4]\,dt$$

$$= \int_0^1 12t^2\,dt$$

$$= 4$$

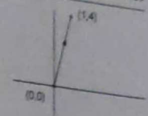


Fig. 14

EXAMPLE-23 : Find the work done by force $\overline{F} = [x-y, y-z, z-x]$ on the curve $C \equiv \bar{r}(t) = [2\cos t, t, 2\sin t]$ from (2, 0, 0) to $(2, 2\pi, 0)$

SOLUTION : Here $C \equiv \bar{r}(t) = [2\cos t, t, 2\sin t]$ from $t = 0$ to 2π

$\bar{r}'(t) = [-2\sin t, 1, 2\cos t]$

$\overline{F}(\bar{r}(t)) = [2\cos t - t, t - 2\sin t, 2\sin t - 2\cos t]$

$$\text{Work done} = \text{W} = \int_C \overline{F} \cdot d\bar{r} = \int_{t=0}^{2\pi} \overline{F}(\bar{r}(t)) \cdot \bar{r}'(t)\,dt$$

$$= \int_0^{2\pi} [2\cos t - t, t - 2\sin t, 2\sin t - 2\cos t] \cdot [-2\sin t, 1, 2\cos t]$$

$$= \int_0^{2\pi} \left(-4\cos t\,\sin t + 2t\,\sin t + t - 2\sin t + 4\sin t\,\cos t - 4\cos 2t\right) dt$$

$$= \int_0^{2\pi} \left(2t\sin t + t - 2\sin t - 4\cos 2t\right) dt$$

$$= 2\left[t\,(-\cos t) - (-\sin t)\right]_0^{2\pi} + \left[\frac{t^2}{2}\right]_0^{2\pi} + 2\left[\cos t\right]_0^{2\pi} - 4\int_0^{2\pi} \frac{1 + \cos 2t}{2}\,dt$$

$$= 2(-2\pi) + \frac{4\pi^2}{2} - 2\left[\frac{t}{2} + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -6\pi + 2\pi^2$$

EXAMPLE-24 : Find the work done by the force $\overline{F} = [e^x, e^{-y}, e^z]$ on the curve $\bar{r}(t) = [t, t^2, t]$ from (0, 0, 0) to (1, 1, 1).

SOLUTION : Here $\bar{r}(t) = [t, t^2, t]$ from t = 0 to t = 1

$\bar{r}'(t) = [1, 2t, 1]$

$\overline{F}(\bar{r}(t)) = \left[e^t, e^{-t^2}, e^t\right]$

$$\text{Workdone} = \text{W} = \int_C \overline{F} \cdot d\bar{r} = \int_{t=0}^{t=1} \overline{F}(\bar{r}(t)) \cdot \bar{r}'(t)\,dt$$

$$= \int_0^1 \left[e^t, e^{-t^2}, e^t\right] \cdot [1, 2t, 1]\,dt$$

$$= \int_0^1 \left(2e^t + 2te^{-t^2}\right) dt$$

$$= \int_0^1 2e^t - (-2t)\, e^{-t^2}\, dt$$

$$= \left[2e^t - e^{-t^2}\right]_0^1$$

$$= 2e - \frac{1}{e} - (2-1)$$

$$= 2e - \frac{1}{e} - 1$$

EXAMPLE-25 : Find $\int_C \overline{F} \cdot d\bar{r}$, where $\overline{F} = \dfrac{y\hat{i} - x\hat{j}}{x^2 + y^2}$ and C is the circle $x^2 + y^2 = 1$ transversed counter clock wise. **[G.T.U., 2009]**

SOLUTION : Here, $\overline{F} \cdot d\bar{r} = \dfrac{y\hat{i} - x\hat{j}}{x^2 + y^2} \cdot \left(dx\,\hat{i} + dy\,\hat{j}\right)$

$$= ydx - x\,dy \quad [\because x^2 + y^2 = 1]$$

The parametric equations of $x^2 + y^2 = 1$ are

$x = \cos\theta$ and $y = \sin\theta$, $\quad 0 \le \theta \le 2\pi$

$\therefore\ dx = \sin\theta\, d\theta$ and $dy = \cos\theta\, d\theta$

$\therefore\ \overline{F} \cdot d\bar{r} = \sin\theta\,(-\sin\theta\, d\theta) - \cos\theta\,(\cos\theta)\, d\theta$

$$= -\left(\sin^2\theta + \cos^2\theta\right) d\theta$$

$$= -d\theta \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

Now, $\int_C \overline{F} \cdot d\bar{r} = \int_{\theta=0}^{2\pi} -d\theta$

$$= -(\theta)_0^{2\pi}$$

$$= -2\pi$$

$\therefore$ Hence, work done $= -2\pi$

Circulation

Definition : Flow integral and Circulation

If $\bar{r}(t)$ is a smooth curve in the domain of a continuous velocity field $\overline{F}$, the flow along the curve from $t = a$ to $t = b$ is

$$\text{Flow} \int_a^b \overline{F} \cdot \overline{T}\, ds = \int_a^b \overline{F} \cdot d\bar{r}$$

If the curve is closed then flow is called the circulation.

i.e. $\oint_C \overline{F} \cdot d\bar{r}$ gives circulation.

If $\overline{V}$ represents the velocity of the fluid particle and C is a closed curve, then the line integral $\oint_C \overline{V} \cdot d\bar{r}$ is called the circulation of $\overline{V}$ around the curve C.

If the circulation $\oint_C \overline{V} \cdot d\bar{r}$ along every closed curve C in the region R is zero then $\overline{F}$ is called irrotational in R.

Thus, if $\oint_C \overline{F} \cdot d\bar{r} = 0$ then $\overline{F}$ is irrotational field.

EXAMPLE-26 : Find the circulation of the field $\overline{F} = (x - y)\hat{i} + y\hat{j} + z\hat{k}$ around the closed curve $\bar{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

SOLUTION : Here, $\bar{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

$\therefore\ \bar{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore\ x = \cos t \qquad y = \sin t \qquad z = 0$

$dx = -\sin t\, dt \qquad dy = \cos t\, dt \qquad dz = 0$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow d\bar{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\overline{F} \cdot d\bar{r} = \left((x-y)\hat{i} + y\hat{j} + z\hat{k}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$

$$= (x - y)\, dx + y\, dy + z\, dz$$

$\overline{F} \cdot d\bar{r} = \sin^2 t\, dt$

$$\text{Circulation} = \int_0^{2\pi} \overline{F} \cdot d\bar{r}$$

$$= \int_{t=0}^{2\pi} \sin^2 t\, dt$$

$$= \int_0^{2\pi} \left(\frac{1 - \cos 2t}{2}\right) dt$$

$$= \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[(2\pi - 0) - (0 - 0)\right]$$

$$= \pi$$

EXAMPLE-27 : Find the circulation of the field $\overline{F} = (y - 1)\hat{i} + x\hat{j}$ around the circle $\bar{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

SOLUTION : $\bar{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

$\bar{r}(t) = x\hat{i} + y\hat{j}$

$\therefore\ x = \cos t \qquad y = \sin t$

$dx = -\sin t\, dt \qquad dy = \cos t\, dt$

$$\bar{r}(t) = x\hat{i} + y\hat{j}$$

$$d\bar{r} = dx\ \hat{i} + dy\ \hat{j}$$

$$\bar{F} = (y-1)\hat{i} + x\hat{j}$$

$$\bar{F}\cdot d\bar{r} = \left((y-1)\hat{i} + x\hat{j}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$= (y-1)\ dx + x\ dy$$

$$= (\sin t - 1)(-\sin t\ dt) + \cos t\cos t\ dt$$

$$= (-\sin^2 t + \sin t + \cos^2 t)\ dt$$

$$= (\cos 2t + \sin t)\ dt$$

$$\text{Circulation} = \int_C \text{F}\cdot d\bar{r}$$

$$= \int_{t=0}^{2\pi} (\cos 2t + \sin t)\ dt$$

$$= \left[\frac{\sin 2t}{2} - \cos t\right]_0^{2\pi}$$

$$= (0-1) - (0-1)$$

$$= 0$$

EXAMPLE-28 : Find circulation of velocity field $\bar{F} = [x - y, x]$ along circle $x^2 + y^2 = 1$.

SOLUTION : $\bar{r}(t) = [\cos t, \sin t]$

$\bar{r}'(t) = [-\sin t, \cos t]$

$\bar{F}(\bar{r}(t)) = [\cos t - \sin t, \cos t]$

$$\text{Circulation} = \oint_C \bar{F}\cdot d\bar{r}$$

$$= \int_0^{2\pi} \bar{F}(r(t))\cdot\bar{r}'(t)\ dt$$

$$= \int_0^{2\pi} [\cos t - \sin t, \cos t]\cdot[-\sin t, \cos t]\ dt$$

$$= \int_0^{2\pi} (1 - \sin t\cos t)\ dt$$

$$= \left[t - \frac{\sin^2 t}{2}\right]_0^{2\pi}$$

$$= 2\pi$$

EXAMPLE-29 : Find the flow of $\bar{F}(t) = xy\ \hat{i} + 2x\hat{j} + \hat{k}$ along the given curve $\bar{r}(t) = t\hat{i} + t\hat{j} + \hat{k}$, $0 \le t \le 2$.

SOLUTION : Here, $\bar{r}(t) = t\hat{i} + t\hat{j} + \hat{k}$

$$\bar{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$

$\therefore\ x = t \qquad y = t \qquad z = 1$

$dx = dt \qquad dy = dt \qquad dz = 0$

$$\bar{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\bar{r} = dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k}$$

$$\bar{F}\cdot d\bar{r} = \left(xy\,\hat{i} + 2x\,\hat{j} + \hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= xy\ dx + 2x\ dy$$

$$= t^2\ dt + 2t\ dt$$

$$= (t^2 + 2t)\ dt$$

$$\text{Flow} = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_{t=0}^{2} (t^2 + 2t)\ dt$$

$$= \left(\frac{t^3}{3} + t^2\right)_0^2$$

$$= \frac{8}{3} + 4 - 0$$

$$= \frac{20}{3}$$

EXAMPLE-30 : Find the a flow of a fluid velocity field $\bar{F} = x\hat{i} + z\hat{j} + y\hat{k}$ along the helix,

$\bar{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j} + t\hat{k}\quad 0 \le t \le \pi/2$

SOLUTION : Here $\bar{r}(t) = [\cos t, \sin t, t]$

$\bar{r}'(t) = [-\sin t, \cos t, t]$

$\bar{F}(\bar{r}(t)) = [\cos t, t, \sin t]$

$$\therefore\ \text{Flow} = \int_C \bar{F}\cdot\bar{T}\ ds = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_{t=0}^{\pi/2} \bar{F}(\bar{r}(t))\cdot r'(t)\ dt$$

$$= \int_{t=0}^{\pi/2} [\cos t, t, \sin t]\cdot[-\sin t, \cos t, t]\ dt$$

$$= \int_{t=0}^{\pi/2} -\sin t\cos t + t\cos t + \sin t\ dt$$

$$= \left[\frac{\cos^2 t}{2} + t \sin t\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - \frac{1}{2}$$

Definition : Flux Across a plane curve

If C is a smooth closed curve in a domain of continuous vector field $\overline{F} = M(x, y)\hat{i} + N(x, y)\hat{j}$ in the plane and if $\vec{n}$ is the outward pointing unit normal vector on C the flux of $\overline{F}$ across C is

$$\text{Flux of } \overline{F} \text{ across C} = \oint_C \overline{F} \cdot \overline{n}\, ds$$

$$\vec{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j} \qquad \left(\because \vec{n} = \overline{T} \times \hat{k}\right)$$

$$\therefore \text{Flux of } \overline{F} = \oint_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds$$

$$= \oint_C M\,dy - N\,dx$$

EXAMPLE-31 : Find the flux of $\overline{F} = (x - y)\hat{i} + x\hat{j}$ across the circle $x^2 + y^2 = 1$ in xy plane.

SOLUTION : Here parametric equation of circle is $\overline{r}(t) = [\cos t, \sin t]$ $t \in [0, 2\pi]$

$\overline{F} = M\hat{i} + N\hat{j}$

$\Rightarrow M = x - y, \quad N = x$

$\Rightarrow M = \cos t - \sin t, \quad N = \cos t$

$x = \cos t, \quad y = \sin t$

$\Rightarrow dx = -\sin t\, dt \quad dy = \cos t\, dt$

$$\therefore \text{Flux} = \oint_C \overline{F} \cdot \overline{n}\, ds$$

$$= \oint_C (M\,dy - N\,dx)$$

$$= \int_0^{2\pi} (\cos t - \sin t) \cos t\, dt - \cos t\,(-\sin t\, dt)$$

$$= \int_0^{2\pi} (\cos^2 t)\, dt$$

$$= \int_0^{2\pi} \frac{1 + \cos 2t}{2}\, dt$$

$$= \frac{1}{2}\left[\frac{t}{2} + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \pi$$

: EXERCISE 8.1 :

(A) Find the work done by a force $\overline{F}$ along a curve C.

i.e. line integral $\int_C \overline{F} \cdot d\overline{r}$ for following.

1. $\overline{F} = [3y, 2x, 4z]$ C $\equiv \overline{r}(t) = [t, t^2, t^4]$ from (0, 0, 0) to (1, 1, 1) **Ans.** $\frac{13}{3}$
2. $\overline{F} = [\sqrt{z}, -2x, \sqrt{y}]$ C $\equiv \overline{r}(t) = [t, t, t]$ from (0, 0, 0) to (1, 1, 1) **Ans.** $\sqrt{2}$
3. $\overline{F} = [3x^2 - 3x, 3z, 1]$ $\int_C$ = (0, 0, 0) to (1, 1, 0) and (1, 1, 0) to (1,1,1) by line segment. **Ans.** $\frac{1}{2}$
4. $\overline{F} = [xy, y, -yz]$ C $= \overline{r}(t) = [t, t^2, t]$ $0 \le t \le 1$
5. $\overline{F} = [z, x, y]$ C is circular Helix. $\overline{r}(t) = [\sin t, \cos t, t]$ $0 \le t \le 2\pi$ **Ans.** $9\sqrt{14}$ **Ans.** $-\pi$
6. $\overline{F} = [4xy, -y^2]$ C $\equiv y = 2x^2$ from (0, 0) to (1, 2) **Ans.** $-\frac{2}{3}$
7. $\overline{F} = [3x^2 + 6y, -14yz, 20xz^2]$
   (i) C $\equiv \overline{r}(t) = [t, t^2, t^3]$ (0, 0, 0) to (1, 1, 1) **Ans.** (i) 5
   (ii) Straight line (0, 0, 0) to (1, 1, 1) **Ans.** (ii) $\frac{20}{3}$
8. $\overline{F} = [2x - y - z, x + y - z^2, 3x - 2y + 4z]$
   C $\equiv x^2 + y^2 = 9$ **Ans.** 18π
9. $\overline{F} = [2y + 3, xz, yz - x]$
   C $\equiv \overline{r}(t) = [2t^2, t, t^3]$ from (0, 0, 0) to (2, 1, 1) **Ans.** $\frac{288}{35}$
10. $\overline{F} = [3x^2, 2xz - y, z]$ along paths
    (i) line segment (0, 0, 0) to (2, 1, 3) **Ans.** (i) 16
    (ii) C $\equiv \overline{r}(t) = [2t^2, t, 4t^2 - t]$ $t = 0$ to $t = 1$ **Ans.** (ii) $\frac{71}{5}$
    (iii) C $\equiv \overline{r}(t) = \left[t, \frac{t^2}{4}, \frac{3t^3}{8}\right]$ $t = 0$ to $t = 2$ **Ans.** (iii) 16

(B) Find flow integral in space if $\overline{F}$ is a velocity field of a fluid find flow along curve C

i.e. $\int_C \overline{F} \cdot d\overline{r}$

1. $\overline{F} = [-4xy, 8y, 2]$ $C \equiv \overline{r}(t) = [t, t^2, 1]$ $0 \le t \le 2$ **Ans.** 48
2. $\overline{F} = [x - z, 0, x]$ C $\equiv \overline{r}(t) = [\cos t, 0, \sin t]$ $0 \le t \le \pi$ **Ans.** π

(C) Find Circulation and Flux.

1. $\overline{F} = x\hat{i} + y\hat{j}$ $C \equiv C_1 \cup C_2$ **Ans.** 0, $a^2\pi$
   $C_1 \equiv [a \cos t, a \sin t]$, $0 \le t \le \pi$
   $C_2 \equiv [t, 0]$ $-a \le t \le a$
2. $\overline{F} = -y\hat{i} + x\hat{j}$ for above curve **Ans.** $a^2\pi$, 0
3. $\overline{F} = (x + y)\hat{i} - (x^2 + y^2)\hat{j}$ $C \equiv x^2 + y^2 = 1$ upper half of the circle. **Ans.** $-\frac{\pi}{2}$

8.5 PATH INDEPENDENCE

Path Independence of Line Integral [Conservation Field and Scalar Potential Function]

For an irrotational field, the line integral of $\bar{F}$ is independent of the path of integral.

If ABCD be any closed path in an irrotational field $\bar{F}$, then

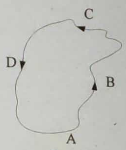


Fig. 15

$$\int_{ABCD} \bar{F}\cdot d\bar{r} = 0$$

$$\therefore \quad \int_{ABC} \bar{F}\cdot d\bar{r} + \int_{CDA} \bar{F}\cdot d\bar{r} = 0$$

$$\therefore \quad \int_{ABC} \bar{F}\cdot d\bar{r} - \int_{ADC} \bar{F}\cdot d\bar{r} = 0$$

$$\therefore \quad \int_{ABC} \bar{F}\cdot d\bar{r} = \int_{ADC} \bar{F}\cdot d\bar{r}$$

Thus, the line integral is independent of path joining the ends points.

Now, if $\bar{F}$ is covservative field, then $\bar{F} = \nabla\phi$, where ϕ is a scalar potential, then the line integral along the curve C from the points A to B is

$$\int_C \bar{F}\cdot d\bar{r} = \int_A^B \bar{F}\cdot d\bar{r} = \int_A^B \nabla\phi\cdot d\bar{r} \qquad (\because \bar{F} = \nabla\phi)$$

$$= \int_A^B \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \int_A^B d\phi$$

$$= (\phi)_A^B$$

$$= \phi(B) - \phi(A)$$

Thus, line integral depends only on the start value and end value and hence line integral is independent of the path.

EXAMPLE-32 : Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F}(\bar{r}) = [5z,\ xy,\ x^2z]$,

$C_1 \equiv$ straight line from (0, 0, 0) to (1, 1, 1)

$C_2 \equiv z = \dfrac{x^2 + y^2}{2}$ and $x = y$ from (0, 0, 0) to (1, 1, 1)

SOLUTION : **For curve C_1** parametric equation for curve is,

$\bar{r}(t) = [t,\ t,\ t]$

$r'(t) = [1,\ 1,\ 1]$

$\bar{F}(\bar{r}(t)) = [5t,\ t^2,\ t^3]$

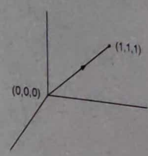


Fig. 16

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{t=0}^{t=1} \bar{F}(\bar{r}(t))\cdot\bar{r}'(t)\,dt$$

$$= \int_0^1 [5t,\ t^2,\ t^3]\cdot[1,\ 1,\ 1]\,dt$$

$$= \int_0^1 5t + t^2 + t^3\,dt$$

$$= 5\left[\frac{t^2}{2}\right]_0^1 + \left[\frac{t^3}{3}\right]_0^1 + \left[\frac{t^4}{4}\right]_0^1$$

$$= \frac{5}{2} + \frac{1}{3} + \frac{1}{4}$$

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \frac{37}{12}$$

For curve C_2 parametric equation for curve is,

$x = y = t \Rightarrow z = t^2$

$\bar{r}(t) = [t,\ t,\ t^2]$

$r'(t) = [1,\ 1,\ 2t]$

$\bar{F}(\bar{r}(t)) = [5t^2,\ t^2,\ t^4]$

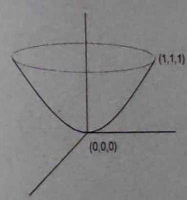


Fig. 17

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_{t=0}^{t=1} \bar{F}(\bar{r}(t))\cdot\bar{r}'(t)\,dt$$

$$= \int_0^1 [5t^2,\ t^2,\ t^4]\cdot[1,\ 1,\ 2t]\,dt$$

$$= \int_0^1 5t^2 + t^2 + 2t^5\,dt$$

$$= \frac{7}{3}$$

$$\therefore \quad \int_{C_1} \bar{F}\cdot d\bar{r} \neq \int_{C_2} \bar{F}\cdot d\bar{r}$$

Note : In general line integral between two points in space depends on the curve.

EXAMPLE-33 : Evaluate $\int_C \overline{F} \cdot d\bar{r}$ on following curves from (0, 0) to (1, 1), $\overline{F} = (x^2, -xy)$

(i) $y = x$
(ii) $y = x^2$
(iii) (0, 0) to (1, 0) and (1, 0) to (1, 1) by straight line

SOLUTION : **For curve $C_1 \equiv y = x$** parametric equation for curve is,

$\bar{r}(t) = [t, t]$ $t = 0$ to $t = 1$

$r'(t) = [1, 1]$

$F(\bar{r}(t)) = [t^2, -t^2]$

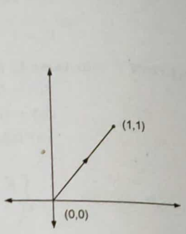


Fig. 18

$$\int_{C_1} \overline{F} \cdot d\bar{r} = \int_{t=0}^{t=1} \overline{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

$$= \int_0^1 [t^2, -t^2] \cdot [1, 1]\, dt$$

$$= \int_0^1 0\, dt$$

$$= 0$$

For curve $C_2 = y = x^2$ parametric equation for curve is,

$\bar{r}(t) = [t, t^2]$ $t = 0$ to $t = 1$

$\bar{r}'(t) = [1, 2t]$

$\overline{F}(\bar{r}(t)) = [t^2, -t^3]$

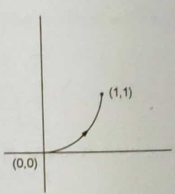


Fig. 19

$$\int_{C_2} \overline{F} \cdot d\bar{r} = \int_{t=0}^{1} \overline{F}(\bar{r}(t)) \cdot r'(t)\, dt$$

$$= \int_0^1 [t^2, -t^3] \cdot [1, 2t]\, dt$$

$$= \int_0^1 t^2 - 2t^4\, dt$$

$$= \left[\frac{t^3}{3} - \frac{2t^5}{5}\right]_0^1$$

$$= \frac{1}{3} - \frac{2}{5}$$

$$\int_{C_2} \overline{F} \cdot d\bar{r} = -\frac{1}{15}$$

For curve C_3

$$\int_{C_3} \overline{F} \cdot d\bar{r} = \int_{\overline{OA}} \overline{F} \cdot d\bar{r} + \int_{\overline{AD}} \overline{F} \cdot d\bar{r}$$

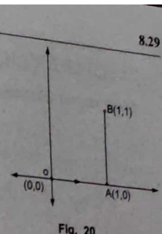


Fig. 20

Parametric equation for curve OA is,

$\overline{OA} \equiv \bar{r}(t) = [t, 0]$

$\bar{r}'(t) = [1, 0]$

$F(\bar{r}(t)) = [t^2, 0]$

$$\int_{OA} \overline{F} \cdot d\bar{r} = \int_{t=0}^{1} \overline{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

$$= \int_0^1 [t^2, 0] \cdot [1, 0]\, dt$$

$$= \int_0^1 t^2\, dt$$

$$= \frac{1}{3}$$

Parametric equation for curve AB is,

$\overline{AB} \equiv \bar{r}(t) = [1, t]$ $t = 0$ to 1

$\bar{r}'(t) = [0, 1]$

$\overline{F}(\bar{r}(t)) = [1, -t]$

$$\int_{AB} \overline{F} \cdot d\bar{r} = \int_{t=0}^{1} \overline{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

$$= \int_0^1 [1, -t] \cdot [0, 1]\, dt$$

$$= \int_0^1 -t\, dt$$

$$= -\frac{1}{2}$$

$$\therefore \int_{C_3} \overline{F} \cdot d\bar{r} = \frac{1}{3} - \frac{1}{2}$$

$$= -\frac{1}{6}$$

$$\therefore \int_{C_1} \overline{F} \cdot d\bar{r} \neq \int_{C_2} \overline{F} \cdot d\bar{r} \neq \int_{C_3} \overline{F} \cdot d\bar{r}$$

8.6 FUNDAMENTAL THEOREM OF LINE INTEGRAL

Fundamental Theorem of Line Integral

Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function ϕ such that

$$\bar{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

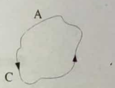
If and only if for all points A and B in D, the value of $\int_A^B \bar{F}\cdot d\bar{r}$ is independent of the path joining point A to point B in D. If the integral is independent of the path from point A to point B, its value is

$$\int_A^B \bar{F}\cdot d\bar{r} = \phi(B) - \phi(A)$$

The line integral of $\bar{F}$ between two points is equal to the potential difference between two points

* If $\bar{F}$ is conservation field and curve C is closed then

$$\oint_C \bar{F}\cdot d\bar{r} = \phi(A) - \phi(A)$$



Thus, $\oint_C \bar{F}\cdot d\bar{r} = 0$

Fig. 21

* Work done in displacing a particle from the point A to the point B under a conservative force field is work done = $\phi(B) - \phi(A)$

8.7 CONSERVATIVE FIELD

Conservative Field :

A vector field $\bar{F}$ is said to be conservative if there exists scalar potential function ϕ such that $\bar{F} = \nabla\phi$.

Component test for conservative field

If F_1, F_2, F_3 have continuous first order partial derivatives and $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a vector field. If domain of $\bar{F}$ is simply connected then vector field $\bar{F}$ is conservative if the following conditions are satisfied.

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y},\ \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

This test is called component test for conservative fields.

* If $\oint_C \bar{F}\cdot d\bar{r} = 0$, then the field is said to be conservative i.e. there is no work done in displacement from a point A to another point in the field and back to A.
* Every irrotational field is conservative.

EXAMPLE-34 : If $\bar{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ is conservative, find its scalar potential ϕ. Find also the work done in moving from a particle under this field from A(1,1,0) to B(2,0,1)

SOLUTION : Since $\bar{F}$ is a conservative field,

$$\bar{F} = \nabla\phi$$

Now, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$\therefore\quad d\phi = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= \nabla\phi\cdot d\bar{r}$$

$$\therefore\quad d\phi = \bar{F}\cdot d\bar{r} \qquad [\because \bar{F} = \nabla\phi = \text{grad}\,\phi]$$

$$= (x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)$$

$$= x^2\,dx + y^2\,dy + z^2\,dz - (yz\,dx + zx\,dy + xy\,dz)$$

$$\therefore\quad d\phi = d\left(\frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3}\right) - d\,(xyz)$$

Integrating, we get

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c, \text{ where C is the constant of integration}$$

Thus, the required scalar potential function is $\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$

Since $\bar{F}$ is conservative, the work done is independent of the path

$$\text{Work done} = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_A^B d\phi \qquad (\because d\phi = \bar{F}\cdot d\bar{r})$$

$$= (\phi)_A^B$$

$$= \left(\frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c\right)_{(1,1,0)}^{(2,0,1)} \qquad \left[\because \phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c\right]$$

$$= \left(\frac{8}{3} + 0 + \frac{1}{3} - 0 + c\right) - \left(\frac{1}{3} + \frac{1}{3} + 0 - 0 + c\right)$$

$$= 3 - \frac{2}{3}$$

$$= \frac{7}{3}$$

EXAMPLE-35 : Let $\overline{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$. Show that $\int_C \overline{F}\cdot d\bar{r}$ is independent of path of integration.

Hence find the integral when C is any path joining (1,–2,1) and (2,1,4)

OR If $\overline{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is conservative then

(i) find its scalar potential ϕ

(ii) find the work done in moving a particle under this force field from A(1, –2, 1) to B(3, 1, 4)

[GTU, Summer 2016]

SOLUTION : Here $\overline{F}$ is a conservative field

$\therefore\ \overline{F} = \nabla\phi$

Now, $d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz$

$= \nabla\phi\cdot d\bar{r}$

$= \overline{F}\cdot d\bar{r} \qquad [\because \overline{F} = \nabla\phi]$

$\therefore d\phi = \left((2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$

$= (2xy+z^3)\,dx + x^2\,dy + 3x\,z^2\,dz$

$= 2xy\,dx + z^3\,dx + x^2\,dy + 3xz^2\,dz$

$= (2xy\,dx + x^2\,dy) + z^3\,dx + 3xz^2\,dz$

$d\phi = d\,(x^2y) + d\,(xz^3)$

Integrating, we get

$\phi = x^2y + xz^3 + c$, where c is a constant of integration.

Thus, the required scalar potential function is $\phi = x^2y + xz^3 + c$,

(ii) Since $\overline{F}$ is conservative, the work done is independent of the path.

Work done $= \int_C \overline{F}\cdot d\bar{r}$

$= \int_A^B d\phi \qquad [\because d\phi = \overline{F}\cdot d\bar{r}]$

$= (\phi)_A^B$

$= \left[x^2y + xz^3 + c\right]_{(1,-2,1)}^{(3,1,4)}$

$= (9 + 192 + c) - (-2 + 1 + c)$

$= 202$

$\int_C \overline{F}\cdot d\bar{r}$ is independent of path of integration if $\overline{F} = \nabla\phi \Rightarrow \text{curl}\,\overline{F} = \bar{0}$

Here $\overline{F} = (2xy + z^3)\,\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$

Now, curl $\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$

$= \hat{i}\left[\dfrac{\partial}{\partial y}(3xz^2) - \dfrac{\partial}{\partial z}(x^2)\right] - \hat{j}\left[\dfrac{\partial}{\partial x}(3xz^2) - \dfrac{\partial}{\partial z}(2xy+z^3)\right] + \hat{k}\left[\dfrac{\partial}{\partial x}(x^2) - \dfrac{\partial}{\partial y}(2xy+z^3)\right]$

$= \hat{i}\,(0 - 0) - \hat{j}\,(3z^2 - 3z^2) + \hat{k}\,(2x - 2x)$

$\therefore$ curl $\overline{F} = \bar{0}$

$\therefore\ \overline{F}$ is an irrotational vector field.

i.e. $\overline{F}$ is a conservation field.

Hence, the integral is independent of path of integration.

EXAMPLE-36 : Show that the line integral $\int_C (2xy + 3)\,dx + (x^2 - 4z)\,dy - 4y\,dz$, where C is any path joining (0, 0, 0) to (1, –1, 3) does not depend on the path C and hence evaluate the line integral $\int_C \overline{F}\cdot d\bar{r}$

SOLUTION : Here $\overline{F} = (2xy + 3)\,\hat{i} + (x^2 - 4z)\,\hat{j} - 4y\,\hat{k}$

The line integral $\int_C \overline{F}\cdot d\bar{r}$ is independent of the path, if $\overline{F} = \nabla\phi \Rightarrow \text{curl}\,\overline{F} = \bar{0}$

Now, curl $\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + 3 & x^2 - 4z & -4y \end{vmatrix}$

$= \hat{i}\left[\dfrac{\partial}{\partial y}(-4y) - \dfrac{\partial}{\partial z}(x^2 - 4z)\right] - \hat{j}\left[\dfrac{\partial}{\partial x}(-4y) - \dfrac{\partial}{\partial z}(2xy+3)\right] + \hat{k}\left[\dfrac{\partial}{\partial x}(x^2 - 4z) - \dfrac{\partial}{\partial y}(2xy+3)\right]$

$= \hat{i}\,[-4 + 4] - \hat{j}\,[0 - 0] + \hat{k}\,[2x - 2x]$

$= \bar{0}$

Thus, curl $\overline{F} = \bar{0}$

$\therefore\ \overline{F}$ is conservative field, hence the line integral is independent of path of integration.

By total derivative,

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$= \nabla\phi \cdot d\bar{r}$$

$$= \bar{F} \cdot d\bar{r} \qquad [\because \bar{F} = \nabla\phi]$$

$$= (2xy + 3)\, dx + (x^2 - 4z)\, dy - 4y\, dz$$

$$= 2xy\, dx + 3dx + x^2\, dy - 4z\, dy - 4y\, dz$$

$$= (2xy\, dx + x^2\, dy) + 3dx - 4(z\, dy + y\, dz)$$

$$d\phi = d(x^2y) + 3dx - 4d(yz)$$

Integrating, we get

$\phi = x^2y + 3x - 4yz + c$, where c is a constant of integration

$$\int_C \bar{F} \cdot d\bar{r} = \int_A^B d\phi$$

$$= (\phi)_A^B$$

$$= \left[x^2y + 3x - 4yz + c\right]_{(0,0,0)}^{(1,-1,3)}$$

$$= (-1 + 3 + 12 + c) - (0 + 0 - 0 + c)$$

$$= 14$$

8.8 EXACT DIFFERENTIAL FORM

Exact Differential Form

A differential form $F_1(x, y, z)\, dx + F_2(x, y, z)\, dy + F_3(x, y, z)\, dz$ is an exact on a domain D in space if

$$F_1\, dx + F_2\, dy + F_3\, dz = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi$$

for some scalar function ϕ through out domain D.

Note : If $\bar{F} = \nabla\phi$

$\Rightarrow$ Curl $\bar{F} = \bar{0}$

$$\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

This is know as condition of exactness.

Theorem : Independence of Path and Exactness

If $\bar{F} = [F_1, F_2, F_3]$ with F_1, F_2, F_3 be continuous and having continuous first partial derivative in domain D in space then

(i) If $\int_C \bar{F}(\bar{r}) \cdot d\bar{r}$ independence of path in D then $F_1dx + F_2dx + F_3dy$ is exact in D and curl $\bar{F} = \bar{0}$

(ii) If curl $\bar{F} = \bar{0}$ hold in D and D is simply connected then

$\int_C \bar{F}(\bar{r}) \cdot d\bar{r}$ is independent of path.

EXAMPLE-37 : Show that $\int_C \bar{F} \cdot d\bar{r} = \int_C 2x\, dx + 2y\, dy + 4z\, dz$ is independent of path is any domain in space find value of $\int_C \bar{F} \cdot d\bar{r}$ if C is any curve with initial point A(0, 0, 0) and terminal point B(2, 2, 2).

SOLUTION : Here $\bar{F} = [2x, 2y, 4z]$

Consider $\phi(x, y, z) = x^2 + y^2 + 2z^2$

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$$

$$= [2x, 2y, 4z]$$

$$= \bar{F}$$

$\Rightarrow \int_C \bar{F} \cdot d\bar{r}$ is independence of path

$$\int_A^B \bar{F} \cdot d\bar{r} = \phi(B) - \phi(A)$$

$$= \phi(2, 2, 2) - \phi(0, 0, 0)$$

$$= \left[x^2 + y^2 + 2z^2\right]_{(2,2,2)} - \left[x^2 + y^2 + 2z^2\right]_{(0,0,0)}$$

$$= 16 - 0$$

$$= 16$$

EXAMPLE-38 : Evaluate $\displaystyle\int_{(0,1,2)}^{(1,-1,7)} (3x^2\, dx + 2yz\, dy + y^2\, dz)$

SOLUTION : Here $\bar{F} = [3x^2, 2yz, y^2]$

Consider $\phi(x, y, z) = x^3 + y^2z$

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= 3x^2\hat{i} + 2yz\hat{j} + y^2\hat{k}$$

$$= [3x^2, 2yz, y^2] = \bar{F}$$

$$\therefore \bar{F} = \text{grad } \phi$$

$$I = \int_{(0,1,2)}^{(1,-1,7)} 3x^2\, dx + 2yzdy + y^2dz = \int_{(0,1,2)}^{(1,-1,7)} \bar{F} \cdot d\bar{r}$$

is independent of path

$$I = \int_A^B \bar{F} \cdot d\bar{r} = \phi(B) - \phi(A)$$

$$= \left[x^3 + y^2z\right]_{(1,-1,7)} - \left[x^3 + y^2z\right]_{(0,1,2)}$$

$$= 8 - 2$$

$$= 6$$

EXAMPLE-39 : Evaluate $I = \int_{(0,0,1)}^{\left(0,\frac{\pi}{4},2\right)} \left[\left(2xyz^2\right) dx + \left(x^2z^2 + z\cos yz\right) dy + \left(2x^2yz + y\cos yz\right) dz\right]$

SOLUTION : Here, $\overline{F} = [2xyz^2,\ x^2z^2 + z\cos yz,\ 2x^2yz + y\cos yz]$

Consider $\phi(x, y, z) = x^2yz^2 + \sin yz$ (by observation)

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= 2xyz^2\ \hat{i} + (x^2z^2 + z\cos yz)\ \hat{j} + (2x^2yz + y\cos yz)\ \hat{k}$$

$$= \overline{F}$$

$$\therefore\ \overline{F} = \text{grad } \phi$$

$$\therefore\ I = \phi(B) - \phi(A)$$

$$= \left[x^2yz^2 + \sin yz\right]_{\left(1,\frac{\pi}{4},2\right)} - \left[x^2yz^2 + \sin yz\right]_{(0,0,1)}$$

$$= \pi + 1$$

EXAMPLE-40 : Evaluate $I = \int_{(0,-1,1)}^{(2,4,0)} (dx - dy + 2zdz)$ **[G.T.U., 2014]**

SOLUTION : $I = \int_{(0,-1,1)}^{(2,4,0)} e^{x-y+z^2}\,dx - e^{x-y+z^2}\,dy + 2z\,e^{x-y+z^2}\,dz$

Here, $\overline{F} = \left[e^{x-y+z^2},\ -e^{x-y+z^2},\ 2z\,e^{x-y+z^2}\right]$

Consider $\phi(x, y, z) = e^{x-y+z^2}$ (by observation)

$$\text{grad } \phi = e^{x-y+z^2}\ \hat{i} - e^{x-y+z^2}\ \hat{j} + 2z\,e^{x-y+z^2}\ \hat{k}$$

$$= \overline{F}$$

$$\Rightarrow\ \overline{F} = \text{grad } \phi$$

$$\therefore\ I = \phi(B) - \phi(A)$$

$$= \left(e^{x-y+z^2}\right)_{(2,4,0)} - \left(e^{x-y+z^2}\right)_{(0,-1,1)}$$

$$I = e^{-2} - e^2$$

EXAMPLE-41 : Show that $I = \int_C \overline{F} \cdot d\bar{r}$ where $\overline{F} = [2xy + z^3,\ x^2,\ 3xz^2]$

is independent of path. Hence find $\int_C \overline{F} \cdot d\bar{r}$ from A(1, –2, 1) to B(3, 1,4).

SOLUTION : Consider $\phi(x, y, z) = x^2y + xz^3$ by observation

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= 2xy\ \hat{i} + x^2\ \hat{j} + 3\ xz^2\ \hat{k}$$

$$= \overline{F}$$

$$\therefore\ \overline{F} = \nabla\phi = \text{grad } \phi$$

$$\therefore\ \int_C \overline{F} \cdot d\bar{r} = \phi(B) - \phi(A)$$

$$= \left[x^2y + xz^3\right]_{(1,-2,-1)}^{(3,1,4)}$$

$$= 202$$

EXAMPLE-42 : If $\overline{F}\ (2xy + z^3)\ \hat{i} + x^2\hat{j} + 3xz^2\hat{k}$. Show that $\int_C \overline{F} \cdot d\bar{r}$ is independent of path. Find integral when C is any path joining (1, –2, 1) and (3, 1, 4). **[G.T.U., 2016]**

SOLUTION : Consider $\phi(x, y, z) = x^2y + xz^3$

$$\Rightarrow \nabla\phi = (2xy + z^3)\ \hat{i} + x^2\hat{j} + 3xz^2\hat{k} = \overline{F}$$

$$\therefore\ \int_C \overline{F} \cdot d\bar{r} = \phi(B) - \phi(A)$$

$$= \phi\ (3, 1, 4) - \phi\ (1, -2, 1)$$

$$= 202$$

EXAMPLE-43 : Find the work done when a force $\overline{F} = (x^2 - y^2 + x)\ \hat{i} - (2xy + y)\ \hat{j}$ moves a particle in the xy-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$. Is the work done different when the path is the straight line $y = x$? **[G.T.U., 2009, 2014]**

SOLUTION : $\overline{F} = x^2 - y^2 + x\ \hat{i} - (2xy + y)\hat{j}$

Consider $\phi(x, y, z) = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2}$

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= (x^2 - y^2 + x)\hat{i} - (2xy - y)\hat{j} + 0\hat{k}$$

$$= \overline{F}$$

$$\therefore\ \overline{F} = \text{grad } \phi$$

$$\therefore\ \int_C \overline{F} \cdot d\bar{r} \text{ is independent of path}$$

$$\int_A^B \overline{F} \cdot d\bar{r} = \phi(B) - \phi(A)$$

$$= \left[\frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2}\right]_{(0,0)}^{(1,1)}$$

$$= -\frac{2}{3}$$

EXAMPLE-44 : Evaluate $I = \int_{(1,0,1)}^{(2,1,3)} (y^2 - z^2 + 3yz - 2x)\,dx + (3xz + 2xy)\,dy + (3xy - 2xz + 2z)\,dz$

SOLUTION : $\overline{F} = [y^2 - z^2 + 3yz - 2x,\ 3xz + 2xy,\ 3xy - 2xz + 2z]$

Consider $\phi(x, y, z) = xy^2 - xz^2 + 3xyz - x^2 + z^2$

$$\text{Now} \quad \text{grad}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$
$$= (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)$$
$$= \overline{F}$$
$$\therefore \overline{F} = \text{grad } \phi$$

$\therefore \int_C \overline{F} \cdot d\bar{r}$ is independent of path

$$\therefore I = \phi(B) - \phi(A)$$
$$= \left[xy^2 - xz^2 + 3xyz - x^2 + z^2\right]_{(1,0,1)}^{(2,1,3)}$$
$$= [2 - 18 + 18 - 4 + 9] - [0 - 1 + 0 - 1 + 1]$$
$$= 8$$

EXERCISE 8.2

Evaluate following integral by using

$$\int_A^B \overline{F} \cdot d\bar{r} = \phi(B) - \phi(A) \text{ if } F = \text{grad } \phi$$

1. $\displaystyle\int_{(0,0,0)}^{(a,b,c)} 2xy^2\, dx + 2x^2y\, dy + dz$ **Ans.** $a^2b^2 + c$

2. $\displaystyle\int_{(0,0,0)}^{(a,b,c)} \cos(x + yz)\, dx + z\cos(x + yz)dy + y\cos(x + yz)\, dz$ **Ans.** $\sin(a + bc)$

3. $\displaystyle\int_{(0,0,0)}^{(a,b,c)} yz\cos(xyz)\, dx + xz\cos(xyz)\, dy + xy\cos(xyz)\, dz$ **Ans.** $\sin(abc)$

4. $\displaystyle\int_{(1,1,1)}^{(1,2,3)} 3x^2\, dx + \frac{z^2}{y}\, dy + 2zlny\, dz$ **Ans.** $9\, ln(2)$

8.9 Diversion of a vector function

Definition :

Let $\vec{v}(x, y, z) = [v_1, v_2, v_3] = \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$ be a differentiable vector function, where x, y, z are cartesian coordinates and v_1, v_2, v_3 be component of $\bar{v}$ then

$$\text{div}\,\overline{V} = \nabla \cdot \bar{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of $\vec{v}$,

$$\nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial y}\hat{k}\right) \cdot \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$
$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial y}$$

e.g. $\vec{v} = 3xy^2\hat{i} + 4y^2z + x^2y^2\hat{k}$

$$\nabla \cdot \vec{v} = \frac{\partial}{\partial x}(3xy^2) + \frac{\partial}{\partial y}(4y^2z) + \frac{\partial}{\partial z}(x^2y^2)$$
$$= 3y^2 + 8yz + 0$$
$$= 3y^2 + 8yz$$

Physical Interpretation of Divergence :

Consider the flow of fluid with velocity $\overline{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ at a point $P(x, y, z)$. Let a small rectangular parallelopiped with edges $\delta x, \delta y, \delta z$ parallel to the coordinate axes in the mass of fluid with one of its corner at $P(x, y, z)$

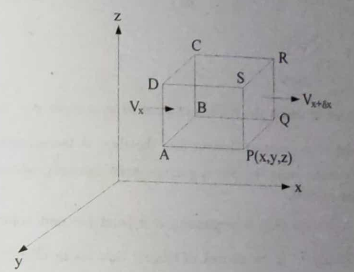


Fig. 22

The mass of the fluid flowing through the face ABCD in unit time is

$$V_x\, \delta y\, \delta z = F(x) \qquad \text{...(8.9.1)}$$

and the mass of the fluid leaving the opposite face PQRS in unit time is

$$V_{x+\delta x}\, \delta y\, \delta z = F(x + \delta x) \qquad \text{...(8.9.2)}$$

By Taylor's series, we have

$$F(x + \delta x) = F(x) + \frac{\partial F}{\partial x}\delta x$$

$$= \left(V_x + \frac{\partial V_x}{\partial x}\delta x\right)\delta y\, \delta z \qquad \text{...(8.9.3)}$$

The mass of the fluid leaving the opposite face PQRS in unit time is

$$\left(V_x + \frac{\partial V_x}{\partial x}\delta x\right)\delta y\, \delta z \qquad \text{...(8.9.4)}$$

From (8.9.4) and (8.9.2), the net decrease in the mass of the fluid in the parallelopiped corresponding to flow across the X-axis per unit time $= \dfrac{\partial V_x}{\partial x}\,\delta x\,\delta y\,\delta z.$

Similarly, net loss in the mass of fluid in the parallelopiped corresponding to flow across per unit time in the direction Y-axis $= \dfrac{\partial V_y}{\partial y}\,\delta x\,\delta y\,\delta z$ and net loss in the mass of fluid in the parallelopiped corresponding to flow across per unit time in the direction of Z-axis $= \dfrac{\partial V_z}{\partial z}\,\delta x\,\delta y\,\delta z.$

The total loss in amount of fluid per unit time $= \left(\dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}\right)\delta x\,\delta y\,\delta z$

The rate of loss of fluid per unit volume is

$$\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

$$= \nabla\cdot\overline{V}$$

$$= \operatorname{div}\overline{V}$$

Hence, $\operatorname{div}\overline{V}$ gives the rate at which the fluid is originating at a point per unit volume.

If there is no gain or loss in the volume element then the fluid is incompressible and hence $\operatorname{div}\overline{V} = 0$.

* Note that divergence of a vector point function is a scalar, point function, whereas the curl of a vector point function is a vector point function.
* $\operatorname{div}\overline{V}$ gives the rate at which the fluid is originating at a point per unit volume.

The divergence of the velocity $\overline{V}$ is the amount of electric flux which crosses a unit volume in unit time, where $\overline{V}$ represents an electric flux.

* Motion of fluid is possible if $\operatorname{div}\overline{V} = 0$
* If the fluid is incompressible, then there is no gain or loss in the volume element. In this case, $\operatorname{div}\overline{V} = 0$. This equation is called the equation of continuity for incompressible fluids.
* A vector point function $\overline{F}$ is said to be solenoidal, if $\operatorname{div}\overline{F} = 0$ everywhere in some region R.
* If $\operatorname{div}\overline{F} > 0$, then either there is a source of fluid at the point P or the fluid is expanding and its density is decreasing with the time where the fluid is entering in the field of flow.
* If $\operatorname{div}\overline{F} < 0$, then either there is a sink of fluid at the point P or the fluid is contracting and its density is increasing where fluid is leaving the field of flow.

EXAMPLE-45 : If $\overline{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$ is solenoidal, find the constant a.

SOLUTION : Since $\overline{F}$ is solenoidal, $\operatorname{div}\overline{F} = 0$

$$\therefore\quad \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$\therefore\quad \frac{\partial}{\partial x}(ax^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2x^2y^2) = 0$$

$$\therefore\quad 2axy + 0 + 2xy - 0 + 2xy - 0 = 0$$

$$\therefore\quad (2a + 4)xy = 0$$

$$\therefore\quad 2a + 4 = 0$$

$$\therefore\quad a = -2$$

EXAMPLE-46 : If $\overline{F} = (x + 3y^2)\,\hat{i} + (2y + 2z^2)\,\hat{j} + (x^2 + az)\,\hat{k}$ is solenoidal, then find the constant a.

SOLUTION : Since $\overline{F}$ is solenoidal, $\operatorname{div}\overline{F} = 0$

$$\therefore\quad \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$\therefore\quad \frac{\partial}{\partial x}(x + 3y^2) + \frac{\partial}{\partial y}(2y + 2y^2) + \frac{\partial}{\partial z}(x^2 + az) = 0$$

$$\therefore\quad 1 + 2 + a = 0$$

$$\therefore\quad a = -3$$

EXAMPLE-47 : Find divergence of following vector function $\vec{v}$

(i) $\overline{v} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\operatorname{div}\vec{v} = \nabla\cdot\vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial y}\hat{k}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial y}$$

$$= 3$$

(ii) $\vec{v} = [x^2, y^2, z^2]$

$$= x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$$

$$\nabla\cdot\vec{v} = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial y}z^2$$

$$= 2x + 2y + 2z$$

$$= 2(x + y + z)$$

(iii) $\vec{v} = xyz\,(x\hat{i} + y\hat{j} + z\hat{k})$ Here take $f = xyz$ and $\overline{u} = [x, y, z]$

$$\vec{v} = f\overline{u}$$

$$\Rightarrow \nabla\cdot\vec{v} = \nabla\cdot(f\overline{u})$$

$$= \nabla f\cdot\overline{u} + f\nabla\cdot\overline{u}$$

$$= [yz, xz, yx]\cdot[x, y, z] + xyz\,(1 + 1 + 1)$$

$$= xyz + xyz + xyz + 3xyz$$

$$= 6\,xyz$$

(v) $\vec{v} = (x^2 + y^2 + z^2)^{-3/2}\,(x\hat{i} + y\hat{j} + z\hat{k})$

$$\nabla\cdot\vec{v} = \nabla(x^2 + y^2 + z^2)^{-3/2}\cdot[x, y, z] + (x^2 + y^2 + z^2)^{-3/2}\,\nabla\cdot[x, y, z]$$

$$= \left[-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}(2x),\ -\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}(2y),\ -\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}2z\right]\cdot[x, y, z] + (x^2+y^2+z^2)^{-3/2}(1+1+1)$$

$$= -3(x^2 + y^2 + z^2)^{-5/2}\,[x, y, z]\cdot[x, y, z] + 3(x^2 + y^2 + z^2)^{-3/2}$$

$$= -3\,(x^2 + y^2 + z^2)^{-5/2}\,(x^2 + y^2 + z^2) + (3(x^2 + y^2 + z^2)^{-3/2}$$

$$= -3(x^2 + y^2 + z^2)^{-3/2} + 3(x^2 + y^2 + z^2)^{-3/2}$$

$$= 0$$

EXAMPLE-48 : Calculate Laplace $\nabla^2 f = \nabla\cdot\nabla f$

(i) $f = \dfrac{x-y}{x+y}$

$$\nabla f = \left[\frac{2y}{(x+y)^2}, \frac{-2x}{(x+y)^2}\right]$$

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{-4y}{(x+y)^3} + \frac{4x}{(x+y)^3}$$

(ii) $f = z - \sqrt{x^2+y^2}$

$$\nabla f = \left[\frac{-2x}{\sqrt{x^2+y^2}}, \frac{-2y}{\sqrt{x^2+y^2}}, 1\right]$$

$$\nabla^2 f = \nabla\cdot\nabla f$$

$$= \frac{\sqrt{x^2+y^2}\,(-2) - (-2x)\dfrac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} + \frac{\sqrt{x^2+y^2}\,(-2) - (-2y)\dfrac{2y}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^{3/2}} + \frac{y^2-x^2}{(x^2+y^2)^{3/2}}$$

$$= 0$$

EXAMPLE-49 : Incompressible flow : show that the flow with velocity vector $\vec{v} = y\hat{i}$ is incompressible show that the particles that at time $t = 0$ are in cube whose faces are portion of planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, z_1 occupy at $t = 1$ the volume 1.

SOLUTION : $\bar{v} = y\hat{i}$

$$\nabla\cdot\bar{v} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)$$

$$= 0$$

$\therefore$ $\vec{v}$ is incompressible

$$\vec{v} = \bar{r}' = \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right] = [y, 0, 0]$$

$$\Rightarrow \frac{dx}{dt} = y \qquad \frac{dy}{dt} = 0 \qquad \frac{dz}{dt} = 0$$

$$\therefore\ x = y(t) + c_1 \qquad y = c_2 z = c_3$$

$$\Rightarrow \bar{r}(t) = [c_2 t + c_1, c_2, c_3]$$

Hence as t increases from 0 to 1 the shear flow transform the cube volume 1 into a parallel pie of volume 1.

EXAMPLE-50 : If $\bar{F} = x^2z\,\hat{i} - 2y^3z^2\,\hat{j} + xy^2z\,\hat{k}$, find div $\bar{F}$ at the point $(1, -1, 1)$.

SOLUTION :

$$\text{div}\,\bar{F} = \nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2$$

$$\therefore\ (\nabla\cdot\bar{F})_{(1,-1,1)} = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2$$

$$= -3$$

EXAMPLE-51 : Show that the vector $\bar{v} = (x + 3y)\,\hat{i} + (y - 2z)\,\hat{j} + (x - 2z)\,\hat{k}$ is solenoidal.

SOLUTION :

$$\nabla\cdot\bar{v} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x - 2z)$$

$$= 1 + 1 - 2 = 0.$$

Since $\nabla\cdot\bar{v} = 0$ therefore vector $\bar{v}$ is solenoidal.

EXAMPLE-52 : Determine whether the vector field $\vec{u} = y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}$ is solenoidal at a point (1,2, 1). **[G.T.U., 2015]**

SOLUTION : Note that solenoidal vector field or divergence free vector field is a domain property not a point proerty.

Here $\vec{u} = y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}$

$$\text{div}\,\bar{u} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(y^2\hat{i} + 2xy\hat{j} - z^2\hat{k})$$

$$\text{div}\,\bar{u} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(-z^2)$$

$$\text{div}\,\bar{u} = 0 + 2x - 2z$$

$$\therefore\ (\text{div}\,\bar{u})_{(1,2,1)} = 2(1) - 2(1) = 0.$$

$\bar{u}$ is solenoidle on plane $2x - 2z = 0$, and (1, 2, 1) is on this plane.

8.10 CURL OF VECTOR FUNCTION

Definition :

Let $\vec{v}(x, y, z) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ be a differentiable vector function then the function

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$ is called curl of $\bar{v}$

i.e. $\text{curl}\,\vec{v} = \nabla\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

e.g. $\vec{v} = yz\hat{i} + zx\hat{j} + z\hat{k}$

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & z \end{vmatrix}$$

$$= \left(\left(\frac{\partial}{\partial y} z\right) - \left(\frac{\partial (zx)}{\partial x}\right)\right)\hat{i} + \left(\frac{\partial}{\partial y} yz - \frac{\partial}{\partial x} z\right)\hat{j} + \left(\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial x}(yz)\right)\hat{k}$$

$$= -x\hat{i} + y\hat{j} + (z-3)\hat{k}$$

$$= -x\hat{i} + y\hat{j} + 0\hat{k}$$

$$= [-x, y, 0]$$

Physical meaning of curl

Let $\vec{w}$ be a angular velocity of a rigid body and $\vec{v}$ be a linear velocity. We know that

$$\vec{v} = \vec{w} \times \bar{r}$$

Without loss of generality we consider angular velocity is in z-axes direction

$\vec{w} = [0, 0, w]$

$\vec{r} = [x, y, z]$, position vector of point of rigid body.

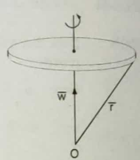


Fig. 23

$$\Rightarrow \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & w \\ x & y & z \end{vmatrix}$$

$$= -yw\hat{i} + wx\hat{j} + 0\hat{k}$$

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yw & wx & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + 2w\hat{k}$$

$$= 2\,[0, 0, w]$$

$$= 2\,\vec{w}$$

$$\Rightarrow \vec{w} = \frac{1}{2} \text{ curl } \vec{v}$$

Note : (i) The curl of the velocity field of a rotating rigid body has the direction of axis of the rotation and its magnitude equal to twice of the angular speed of the rotation.

(ii) Curl $\vec{v} = \bar{0} \Rightarrow \bar{v}$ is a irrotational fields.

(iii) Curl (grad f) = $\nabla \times \nabla = \bar{0}$

(i.e. gradient field of scalar function describe irritation at motion)

(iv) $\nabla \cdot (\nabla \times \bar{v}) = \text{div}(\text{Curl}\,\bar{v}) = 0$

(v) divergence if velocity function gives flux and curl of velocity function gives rotation.

(vi) Properties of curl

(i) Curl $(\bar{u} + \bar{v})$ = Curl $\bar{u}$ + Curl $\bar{v}$

(ii) div (curl $\bar{v}$) = 0

(iii) Curl (grad $\bar{v}$) = $\bar{0}$

(iv) Curl (div $\bar{v}$) not define

(v) Curl $(\phi\bar{v}) = \nabla \times (\phi\bar{v})$

$= \nabla\phi \times \vec{v} + \phi(\nabla \times \bar{v})$

(vi) $\nabla \cdot (\bar{u} \times \bar{v}) = \bar{v} \cdot \text{curl } \vec{u} - \bar{u} \cdot \text{curl } \bar{v}$

$= \bar{v} \cdot (\Delta \times \bar{u}) - \bar{u} \cdot (\nabla \times \bar{v})$

EXAMPLE-53 : Calculate curl $\overline{F}$ at (1, 0, 2) where $\overline{F} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$

SOLUTION : $\text{Curl } \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial z}(x^2y)\right]$$

Thus, curl $\overline{F} = \hat{i}\,[2z + 2x] - \hat{j}\,[0 - 0] + \hat{k}\,[-2z - x^2]$

At the point (1,0,2),

Curl $\overline{F} = \hat{i}(4 + 2) - \hat{j}(0) + \hat{k}(-4 - 1)$

$= 6\hat{i} - 5\hat{k}$

EXAMPLE-54 : If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that curl $\bar{r} = \bar{0}$.

SOLUTION :

$$\text{Curl}\,\bar{r} = \nabla \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \hat{i}\,(0) + \hat{j}\,(0) + \hat{k}\,(0)$$

$$= \bar{0}.$$

EXAMPLE-55 : If $\overline{F} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find curl $\overline{F}$ at the point (1, –1, 1).

SOLUTION : $\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$

$$\nabla \times \overline{F} = \hat{i}\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right] + \hat{j}\left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right] + \hat{k}\left[\frac{\partial}{\partial z}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= (2z^4 + 2x^2y)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

$\therefore$ curl $\overline{F}$ at (1, –1, 1) = $3\hat{j} + 4\hat{k}$.

EXAMPLE-56 : Find constants a, b, c so that

$\bar{v} = (x + 2y + az)\ \hat{i} + (bx - 3y - z)\ \hat{j} + (4x + cy + 2z)\ \hat{k}$ is irrotational.

[G.T.U., 2009]

SOLUTION : $\text{Curl } \bar{v} = \nabla \times \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$

$= (c + 1)\ \hat{i} + (a - 4)\ \hat{j} + (b - 2)\ \hat{k}$

This equals zero when a = 4, b = 2, c = –1 and

$\bar{v} = (x + 2y + 4z)\ \hat{i} + (2x - 3y - z)\ \hat{j} + (4x - y + 2z)\ \hat{k}$

EXAMPLE-57 : Show that $\bar{F} = (y + z)\ \hat{i} + (z + x)\ \hat{j} + (x + y)\ \hat{k}$ is irrotational and hence find its scalar potential function.

SOLUTION : Here, $\bar{F} = (y + z)\ \hat{i} + (z + x)\ \hat{j} + (x + y)\ \hat{k}$

$\therefore \text{Curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}\,(1-1) - \hat{j}\,(1-1) + \hat{k}\,(1-1)$$

$\therefore \text{Curl } \bar{F} = \bar{0}$

Thus, $\bar{F}$ is irrotational vector.

Since $\bar{F}$ is irrotational, we can find a scalar potential ϕ such that $\bar{F} = \nabla\phi$

Now, $\bar{F} = \nabla\phi$

$= \text{grad } \phi$

$$(y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

On comparison, we have

$$\frac{\partial \phi}{\partial x} = y + z, \quad \frac{\partial \phi}{\partial y} = z + x, \quad \frac{\partial \phi}{\partial z} = x + y$$

Now, $\frac{\partial \phi}{\partial x} = y + z$

Integrating, we get

$\phi = (y + z)\, x + f_1(y, z)$ ...(1)

and $\frac{\partial \phi}{\partial y} = z + x$

Integrating, we get

$\phi = (z + x)\, y + f_2\,(x, z)$ ...(2)

Finally, $\frac{\partial \phi}{\partial z} = x + y$

Integrating,

$\phi = (x + y)z + f_3\,(x, y)$ ...(3)

Equality of (1), (2) and (3) requires that $f_1\,(y, z) = yz$, $f_2\,(x, z) = xz$, $f_3\,(x, y) = xy$

Hence, $\phi = xy + yz + zx + c$, (where c is constant) is the required scalar potential function.

Alternately,

As $\bar{F} = \text{grad } \phi = \nabla\phi$

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$= \bar{F} \cdot dr \qquad (\because \bar{F} = \nabla\phi = \text{grad } \phi)$

$= (y + z)\ dx + (z + x)\ dy + (x + y)\ dz$

$= (y\ dx + z\ dx) + (z\ dy + x\ dy) + (x\ dz + y\ dz)$

$d\phi = (y\ dx + x\ dy) + (z\ dy + y\ dz) + (z\ dx + x\ dz)$

$\therefore \quad d\phi = \int \{d(xy) + d(yz) + d(xz)\}$

$\therefore \quad \phi = xy + yz + xz + c$

EXAMPLE-58 : Prove that $\bar{F} = yz\ \hat{i} + xz\ \hat{j} + xy\ \hat{k}$ is irrotational. Find a scalar function ϕ such that $\bar{F} = \text{grad } \phi$

SOLUTION : Here, $\bar{F} = yz\ \hat{i} + xz\ \hat{j} + xy\ \hat{k}$

$\text{Curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$= \hat{i}\,[x - x] - \hat{j}\,[y - y] + \hat{k}\,[z - z]$

$= \bar{0}$

Thus, curl $\bar{F} = \bar{0}$

Hence, $\bar{F}$ is an irrotational vector field

To find scalar potential function :

Here $\overline{F} = \text{grad}\ \phi$

$$\text{Now, } d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \nabla\phi \cdot d\bar{r}$$

$$= \overline{F}\ d\bar{r} \quad \left[\because \overline{F} = \text{grad}\ \phi = \nabla\phi\right]$$

$$\therefore \quad d\phi = (yz)\,dx + (xz)\,dy + (xy)\,dz$$

$$\therefore \quad \phi = \int d(xyz)$$

$\phi = xyz + c$, which is the required scalar potential function

EXAMPLE-59 : If $\overline{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$ is irrotational, then find its scalar potential function ϕ.

SOLUTION : Let $\overline{F} = \text{grad}\ \phi$

$$\overline{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$$

$$\text{Now, } d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \nabla\phi \cdot d\bar{r}$$

$$= \overline{F}\cdot d\bar{r} \quad \left[\because \overline{F} = \text{grad}\ \phi = \nabla\phi\right]$$

$$\therefore \quad d\phi = \left(y^2\cos x + z^3\right)dx + \left(2y\sin x - 4\right)dy + \left(3xz^2 + 2\right)dz$$

$$= y^2\cos x\,dx + z^3\,dx + 2y\sin x\,dy - 4\,dy + 3xz^2\,dz + 2\,dz$$

$$\therefore \quad d\phi = \left(y^2\cos x\,dx + 2y\sin x\,dy\right) + \left(z^3\,dx + 3xz^2\,dz\right) - 4dy + 2dz$$

$$\therefore \quad \phi = \int\left[d\left(y^2\sin x\right) + d\left(xz^3\right) - 4\,dy + 2\,dz\right]$$

$\therefore \quad \phi = y^2\sin x + xz^3 - 4y + 2z + c$, which is the required scalar potential function.

EXAMPLE-60 : Find curl (curl $\overline{F}$), given $\overline{F} = x^2y\hat{i} + y^2z\hat{j} + z^2y\hat{k}$

SOLUTION : Here $\overline{F} = x^2y\hat{i} + y^2z\hat{j} + z^2y\hat{k}$

$$\text{Curl}\,\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2x) - \frac{\partial}{\partial z}(y^2z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^2x) - \frac{\partial}{\partial z}(x^2y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2z) - \frac{\partial}{\partial y}(x^2y)\right]$$

$$= \hat{i}(0 - y^2) - \hat{j}(z^2 - 0) + \hat{k}(0 - x^2)$$

$$\text{Curl}\,\overline{F} = -y^2\hat{i} - z^2\hat{j} - x^2\hat{k}$$

$$\text{Curl (curl}\,\overline{F}) = \text{curl}\left(-y^2\hat{i} - z^2\hat{j} - x^2\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & -z^2 & -x^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x^2) - \frac{\partial}{\partial z}(-z^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial z}(-y^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial y}(-y^2)\right]$$

$$= \hat{i}\,[0 + 2z] - \hat{j}\,(-2x - 0) + \hat{k}\,(0 + 2y)$$

Thus, curl (curl $\overline{F}$) $= 2\left(z\hat{i} + x\hat{j} + y\hat{k}\right)$

EXAMPLE-61 : Find Curl $\vec{v}$ where $\bar{v}$ is a vector function

(i) $\vec{v} = [2y,\ 5x,\ 0]$

$$= 2y\hat{i} + 5x\hat{j} + 0\hat{k}$$

$$\text{Curl}\ \vec{v} = \nabla \times \vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 5x & 0 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(5x)\right)\hat{i} + \left(\frac{\partial}{\partial z}(2y) - \frac{\partial}{\partial x}(0)\right)\hat{j} + \left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(2y)\right)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 3\hat{k}$$

$$= [0,\ 0,\ 3]$$

(ii) $\bar{v} = (xyz)\,(x\hat{i} + y\hat{j} + z\hat{k})$

$$= x^2yz\,\hat{i} + xy^2z\,\hat{j} + xyz^2\,\hat{k}$$

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= [xz^2 - xy^2,\ x^2y - yz^2,\ y^2z - x^2z]$$

(iii) $\bar{v} = [x, y, -z]$ **[G.T.U., 2016]**

$$\text{Curl } \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

EXAMPLE-62 : Find divergence and curl of $\bar{v} = xyz\,[x, y, z]$. **[G.T.U., 2013]**

SOLUTION : $\bar{v} = x^2yz\ \hat{i} + xy^2z\ \hat{j} + xyz^2\ \hat{k}$

$$\text{Div } \bar{v} = \nabla\cdot\bar{v} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial x}(xyz^2)$$

$$= 2xyz + 2xyz + 2xyz = 6xyz$$

$$\text{Curl } \bar{v} = \nabla\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \hat{i}\,[xz^2 - xy^2] + \hat{j}\,[x^2y - yz^2] + \hat{k}\,[y^2z - x^2z]$$

$$= x\,(z^2 - y^2)\,\hat{i} + y\,(x^2 - z^2)\,\hat{j} + z\,(y^2 - x^2)\,\hat{k}$$

EXAMPLE-63 : Obtain *curl* $\bar{F}$ at the point (2, 0, 3), where $\bar{F} = z\,e^{2xy}\,\hat{i} + 2xy\cos y\ \hat{j} + (x + 2y)\,\hat{k}$ **[G.T.U., 2011, 2015]**

SOLUTION :

$$\text{curl } \bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\,e^{2xy} & 2xy\cos y & (x+2y) \end{vmatrix}$$

$$= \hat{i}\,[2 - 0] + \hat{j}\,[e^{2xy} - 1] + \hat{k}\,[2y\cos y - z\,e^{2xy}(2x)]$$

$$(\nabla\times\bar{F})_{\text{at }(2,\,0,\,3)} = 2\,\hat{i} + \hat{j}\,(0) + \hat{k}\,[0 - 3(4)] = 2\hat{i} - 12\hat{k}.$$

EXAMPLE-64 : A vector field is given by $\bar{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. Show that $\bar{F}$ is irrotational and find its scalar potential. **[G.T.U., 2015]**

SOLUTION :

$$\text{Curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0}$$

$\therefore$ Vector field $\bar{F}$ is irrotational.

$$\text{Consider } \phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + \frac{y^2}{3} + c$$

$$\nabla\phi = \frac{\partial}{\partial x}\phi\,\hat{i} + \frac{\partial}{\partial y}\phi\,\hat{j} + \frac{\partial}{\partial z}\phi\,\hat{k}$$

$$= (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}.$$

$\therefore$ ϕ is require scalar potential.

EXAMPLE-65 : If a vector $\bar{F}$ is irrotational, then show that $\bar{F} = \nabla\phi$, = grad ϕ, where ϕ is a scalar point function of $\bar{F}$ or scalar potential of $\bar{F}$.

SOLUTION : Let curl $\bar{F} = \bar{0}$ where $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \bar{0}$$

$$\text{i.e. } \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \bar{0}$$

$$\therefore \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0,\quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = 0,\quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \bar{0}$$

$$\text{i.e. } \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

These three conditions are satisfied when

$$F_1 = \frac{\partial\phi}{\partial x},\ F_2 = \frac{\partial\phi}{\partial y},\ F_3 = \frac{\partial\phi}{\partial z},\ \text{where } \phi \text{ is a function of x, y, z}$$

$$\therefore \bar{F} = \hat{i}\,\frac{\partial\phi}{\partial x} + \hat{j}\,\frac{\partial\phi}{\partial y} + \hat{k}\,\frac{\partial\phi}{\partial z} = \nabla\phi$$

ϕ is known as the scalar point function of $\bar{F}$.

A vector field $\bar{F}$ which can be derived from a scalar field ϕ so that $\bar{F} = \nabla\phi$ is called a **conservative vector field** and ϕ is called the **scalar potential**. Note that conversely if $\bar{F} = \nabla\phi$, then $\nabla\times\bar{F} = \bar{0}$.

EXAMPLE-66 : Prove that $\bar{F} = (y^2\cos x + z^3)\,\hat{i} + (2y\sin x - 4)\,\hat{j} + 3xz^2\,\hat{k}$ is irrotational. **[G.T.U., 2016]**

SOLUTION :

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix} = \bar{0} \text{ on simplification}$$

$$\text{Hence } \bar{F} = \nabla\phi = \hat{i}\,\frac{\partial\phi}{\partial x} + \hat{j}\,\frac{\partial\phi}{\partial y} + \hat{k}\,\frac{\partial\phi}{\partial z}$$

$$\therefore \frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \quad \text{...(i)}$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \quad \text{...(ii)}$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \quad \text{...(iii)}$$

$\therefore$ from (i), $\phi = y^2\sin x + xz^3 + f_1(y, z)$.

from (ii), $\phi = y^2\sin x - 4y + f_2(z, x)$.

from (iii), $\phi = xz^3 + f_3(x, y)$.

$\therefore \phi = y^2\sin x + xz^3 - 4y + c$, where c is an arbitrary constant.

EXAMPLE-67 : A fluid motion is given by $\bar{v} = (y \sin z - \sin x)\,\hat{i} + (x \sin z + 2yz)\,\hat{j} + (xy \cos z + y^2)\,\hat{k}$. Is motion irrotational? If so, find the velocity potential. **[G.T.U., 2010]**

OR

Show that the vector field $\bar{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$ is conservative and find the corresponding scalar potential. **[G.T.U., 2014]**

SOLUTION :

$$\text{Curl } \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= \hat{i}\,[x\cos z + 2y - x\cos z - 2y] + \hat{j}\,[y\cos z - y\cos z] + \hat{k}\,[\sin z - \sin z]$$

$$= \hat{i}\,(0) + \hat{j}\,(0) + \hat{k}\,(0) = \bar{0}$$

Since curl $\bar{v} = \bar{0}$, the motional is irrotational.

Hence $\quad \bar{v} = \nabla\phi = \hat{i}\,\dfrac{\partial\phi}{\partial x} + \hat{j}\,\dfrac{\partial\phi}{\partial y} + \hat{k}\,\dfrac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = y\sin z - \sin x \qquad \text{(i)}$$

$$\frac{\partial\phi}{\partial y} = y\sin z + 2yz \qquad \text{(ii)}$$

$$\frac{\partial\phi}{\partial z} = xy\cos z + y^2 \qquad \text{(iii)}$$

From (i), $\quad \phi = xy\sin z + \cos x + f_1(y, z)$

From (ii), $\quad \phi = xy\sin z + y^2 z + f_2(z, x)$

From (iii), $\quad \phi = xy\sin z + zy^2 + f_3(x, y)$

$\therefore\ \phi = xy\sin z + y^2 z + \cos x + c$, where c is an arbitrary constant is required velocity potential.

EXAMPLE-68 : Determine f(r) so that the vector $f(r)\,\bar{r}$ is both solenoidal and irrotational.
where $r = \sqrt{x^2 + y^2 + z^2}$, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ **[G.T.U., 2011]**

SOLUTION :

$$f(r)\,\bar{r} = f(r)\,(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= xf(r)\,\hat{i} + yf(r)\,\hat{j} + zf(r)\,\hat{k}$$

$$\text{div }\{f(r)\bar{r}\} = \frac{\partial}{\partial x}\{x\,f(r)\} + \frac{\partial}{\partial y}\{y\,f(r)\} + \frac{\partial}{\partial z}\{z\,f(r)\}$$

$$= f(r) + x\,f'(r)\cdot\frac{\partial r}{\partial x} + f(r) + y\,f'(r)\,\frac{\partial r}{\partial y} + f(r) + zf'(r)\,\frac{\partial r}{\partial z}.$$

$$\text{div }\{f(r)\bar{r}\} = 3\,f(r) + f'(r)\left\{x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right\}$$

$$= 3\,f(r) + f'(r)\left(\frac{x^2 + y^2 + z^2}{r}\right) \qquad (\because r^2 = x^2 + y^2 + z^2)$$

$\Rightarrow$ div $\{f(r)\bar{r}\} = 3\,f(r) + rf'(r)$

since the vector is solenoidal

div $\{f(r)\bar{r}\} = 0$

$$\Rightarrow\ 3f(r) + f'(r)r = 0 \text{ i.e. } \frac{f'(r)}{f(r)} = -\frac{3}{r}$$

Integrating both sides w.r.t. r, we get

$$\log f(r) = -3\log r + \log c = \log r^{-3} c$$

$$\Rightarrow f(r) = \frac{c}{r^3}, \text{ where c is a constant.}$$

Now

$$\text{curl }\{f(r)\bar{r}\} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\,f(r) & y\,f(r) & z\,f(r) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\{z\,f(r)\} - \frac{\partial}{\partial z}\{y\,f(r)\}\right] + \ldots$$

$$= \hat{i}\left\{z\,f'(r)\frac{\partial r}{\partial y} - y\,f'(r)\frac{\partial r}{\partial z}\right\} + \ldots$$

$$= \hat{i}\ f'(r)\left\{z\frac{\partial r}{\partial y} - y\frac{\partial r}{\partial z}\right\} + \ldots$$

$$= \hat{i}\ f'(r)\left\{z\cdot\frac{y}{r} - y\cdot\frac{z}{r}\right\} + \ldots$$

$$= \bar{0} \text{ since } r^2 = x^2 + y^2 + z^2$$

Hence curl $\{f(r)\bar{r}\}$ is irrotational vector for all f(r).

EXAMPLE-69 : Find the scalar potential function f for $\bar{A} = y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}$. **[G.T.U., 2008]**

SOLUTION :

$$\text{Curl } \bar{A} = \nabla \times \bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & 2xy & -z^2 \end{vmatrix}$$

$$= \hat{i}\,(0) + \hat{j}\,(0) + \hat{k}\,(2y - 2y) = \bar{0}$$

Since curl $\bar{A} = \bar{0}$, $\bar{A} = \nabla\phi$, where ϕ is a scalar potential function of $\bar{A}$

$$\bar{A} = \nabla\phi \Rightarrow (y^2\hat{i} + 2xy\hat{j} - z^2\hat{j}) = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

Compare components,

$$\Rightarrow \quad \frac{\partial\phi}{\partial x} = y^2 \ \ldots\text{(i)} \quad \frac{\partial\phi}{\partial y} = 2xy \ \ldots\text{(ii)}, \quad \frac{\partial\phi}{\partial z} = -z^2 \ \ldots\text{(iii)}$$

from (i) $\qquad \phi = xy^2 + f_1(y, z)$

from (ii) $\qquad \phi = \dfrac{2xy^2}{2} + f_2(z, x) = xy^2 + f_2(y, z)$

from (iii) $\qquad \phi = -\dfrac{z^3}{3} + f_3(x, y)$

$$\therefore \quad \phi = xy^2 - \frac{z^3}{3} + C, \text{ where C is an arbitrary constant.}$$

EXAMPLE-70 : Prove that $r^n \bar{r}$ is irrotational, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = \sqrt{x^2+y^2+z^2}$ **[G.T.U., 2009]**

SOLUTION : In order to prove $r^n \bar{r}$ irrotational, find $\nabla \times (r^n \bar{r})$ i.e. curl $(r^n \bar{r})$.

$$\text{Curl}\,(r^n \bar{r}) = \sum i \times \frac{\partial}{\partial x}(r^n r)$$
$$= \sum i \times \left\{ n\,r^{n-1}\,r\,\frac{x}{r} + r^n\,i \right\}$$
$$= \sum \left\{ n\,r^{n-2} x\,(i \times r) + r^n (i \times i) \right\}$$
$$= \sum n\,r^{n-2}\,x\,(y\hat{k} - z\hat{j}) + 0$$
$$= n\,r^{n-2}\,\{x\,(y\hat{k} - z\hat{j}) + y\,(z\hat{i} - x\hat{k}) + z\,(x\hat{j} - y\hat{i})\}$$
$$= \bar{0}$$

$\therefore$ $r^n \bar{r}$ is irrotational.

EXAMPLE-71 : Show that $\bar{F} = 2xy\,z\hat{i} + (x^2z + 2y)\,\hat{j} + x^2y\,\hat{k}$ is irrotational and find a scalar function ϕ such that $\bar{F} = \text{grad}\,\phi$. **[G.T.U., 2009]**

SOLUTION :

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z + 2y & x^2y \end{vmatrix}$$
$$= \hat{i}\,(x^2 - x^2) + \hat{j}\,(2xy - 2xy) + \hat{k}\,(2xz - 2xz) = \bar{0}$$

$\Rightarrow$ $\bar{F}$ is irrotational

Now, $\bar{F} = \text{grad}\,\phi = \dfrac{\partial \phi}{\partial x}\hat{i} + \dfrac{\partial \phi}{\partial y}\hat{j} + \dfrac{\partial \phi}{\partial z}\hat{k}$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = 2xyz \qquad \text{...(i)}$$
$$\frac{\partial \phi}{\partial y} = x^2z + 2y \qquad \text{...(ii)}$$
$$\frac{\partial \phi}{\partial z} = x^2y \qquad \text{...(iii)}$$

Integrating (i), (ii) and (iii) w.r.t. x, y and z respectively, we get

$$\phi = x^2yz + f_1(y, z) \qquad \text{(iv)}$$
$$\phi = x^2yz + y^2 + f(z, x) \qquad \text{(v)}$$
$$\phi = x^2yz + y^2 + f_3\,(x, y) \qquad \text{(vi)}$$
$$\therefore \phi = x^2yz + y^2 + C$$

EXAMPLE-72 : For a solenoidal vector field $\bar{H}$ show that curl curl curl curl $\bar{H} = \nabla^4 \bar{H}$.

SOLUTION : $\bar{H}$ being solenoidal $\nabla \cdot \bar{H} = 0$

$$\text{curl curl } \bar{H} = \nabla \times (\nabla \times \bar{H})$$
$$= \nabla\,(\nabla \cdot \bar{H}) - (\nabla \cdot \nabla)\,\bar{H}$$
$$= -\nabla^2 \bar{H} \qquad (\because \nabla \cdot \bar{H} = 0)$$

Let $\bar{F} = -\nabla^2 \bar{H}$

$$\therefore \text{curl curl curl curl } \bar{H} = \text{curl curl } \bar{F}$$
$$= \nabla \times (\nabla \times \bar{F})$$
$$= \nabla\,(\nabla \cdot \bar{F}) - (\nabla \cdot \nabla)\,\bar{F}$$
$$= \nabla\,[\nabla \cdot (-\nabla^2 \bar{H})] - \nabla^2 \bar{F}$$
$$= \nabla\,[-\nabla^2\,(\nabla \cdot \bar{H})] - \nabla^2\,(-\nabla^2 \bar{H})$$
$$= 0 + \nabla^4 \bar{H}$$
$$\therefore \text{curl curl curl curl } \bar{H} = \nabla^4 \bar{H}$$

EXAMPLE-73 : Find curl $\bar{F}$, where $\bar{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$

SOLUTION : Let $x^3 + y^3 + z^3 - 3xyz = f$

$\therefore$ $\bar{F} = \text{grad}\,f = \nabla f$

Since curl $\bar{F} = \nabla \times \bar{F} = \nabla \times (\nabla f) = \bar{0}$ $\qquad (\because$ curl (grad f) = 0)

EXAMPLE-74 : Prove that $\nabla^2\,f(r) = f''(r) + \dfrac{2}{r}\,f'(r)$ where $\bar{r} = \sqrt{x^2+y^2+z^2}$ **[G.T.U., 2012]**

SOLUTION :
$$\nabla^2 f(r) = \nabla \cdot \{\nabla\,f(r)\} = \text{div}\,\{\text{grad}\,f(r)\}$$
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r)$$

Now $\dfrac{\partial}{\partial x}\,f(r) = \dfrac{\partial}{\partial x}\,f(r) \cdot \dfrac{\partial r}{\partial x} = f'(r) \cdot \dfrac{x}{r}$ since $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{\partial^2}{\partial x^2}\,f(r) = \frac{\partial}{\partial x}\,[f'(r) \cdot x \cdot r^{-1}]$$
$$= f'(r)\,r^{-1} \cdot 1 + f''(r) \cdot \frac{x}{r} \cdot x\,r^{-1} - f'(r)\,x^2\,r^{-3}$$
$$= \frac{f'(r)}{r} + \frac{x^2 f''(r)}{r^2} - \frac{x^2 f'(r)}{r^3}$$

Similarly, $\dfrac{\partial^2}{\partial y^2}\,f(r) = \dfrac{f'(r)}{r} + \dfrac{y^2 f''(r)}{r^2} - \dfrac{y^2 f'(r)}{r^3}$

and $\dfrac{\partial^2}{\partial z^2}\,f(r) = \dfrac{f'(r)}{r} + \dfrac{z^2 f''(r)}{r^2} - \dfrac{z^2 f'(r)}{r^3}$

Adding, we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r) = \frac{3\,f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$
$$= \frac{2\,f'(r)}{r} + f''(r)$$

$$\therefore \nabla^2\,f(r) = \text{div grad } f(r) = \frac{2\,f'(r)}{r} + f''(r).$$

EXAMPLE-75 : Maxwell's equations of electromagnetic theory are

$$\nabla \cdot \bar{E} = 0,\ \nabla \cdot \bar{H} = 0,\ \nabla \times \mathbf{E} = -\frac{\partial \bar{H}}{\partial t},\quad \nabla \times \bar{H} = \frac{\partial \bar{E}}{\partial t}$$

Show that $\bar{E}$ and $\bar{H}$ satisfy wave equations

(i) $\nabla^2 \bar{E} = \dfrac{\partial^2 \bar{E}}{\partial t^2}$ and (ii) $\nabla^2 \bar{H} = \dfrac{\partial^2 \bar{H}}{\partial t^2}$

SOLUTION : $\nabla \times (\nabla \times \bar{E}) = \nabla \times \left(-\dfrac{\partial \bar{H}}{\partial t}\right) = -\dfrac{\partial}{\partial t}(\nabla \times \bar{H})$

$$= -\frac{\partial}{\partial t}\left(\frac{\partial \bar{E}}{\partial t}\right) = -\frac{\partial^2 \bar{E}}{\partial t^2} \qquad \text{...(i)}$$

But $\nabla \times (\nabla \times \bar{E}) = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -\nabla^2 \bar{E}$ ...(ii)

Then from (i) are (ii), we have $\nabla^2 \bar{E} = \dfrac{\partial^2 \bar{E}}{\partial t^2}$

Now $\nabla \times (\nabla \times \bar{H}) = \nabla \times \left(\dfrac{\partial \bar{E}}{\partial t}\right) = \dfrac{\partial}{\partial t}(\nabla \times \bar{E})$

$$= \frac{\partial}{\partial t}\left(-\frac{\partial \bar{H}}{\partial t}\right) = -\frac{\partial^2 \bar{E}}{\partial t^2} \qquad \text{...(iii)}$$

but $\nabla \times (\nabla \times \bar{H}) = \nabla(\nabla \cdot \bar{H}) - \nabla^2 \bar{H} = -\nabla^2 \bar{H}$ ...(iv)

from (iii) and (iv), we have $\nabla^2 \bar{H} = \dfrac{\partial^2 \bar{H}}{\partial t^2}$

The equation $\nabla^2 u = \dfrac{\partial^2 u}{\partial t^2}$

i.e. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{\partial^2 u}{\partial t^2}$ is called the wave equation.

EXAMPLE-76 : Show that if the necessary partial derivatives of the components of the field $\bar{G} = M\hat{i} + N\hat{j} + P\hat{k}$ are continuous, then $\nabla \cdot (\nabla \times \bar{G}) = 0$. **[G.T.U., 2010]**

SOLUTION : $\bar{G} = M\hat{i} + N\hat{j} + P\hat{k}$

$$\nabla \times \bar{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \hat{j}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \bar{G}) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 N}{\partial z \partial y}$$

$= 0.$

EXERCISE 8.3

1. Find divergence and curl
   (i) $\bar{v} = [x^2 - y^2, 2xy, y^2 - xy]$ **Ans.** $4x$, $[2y - x, y, 4y]$
   (ii) $\bar{v} = [y + z, z + x, x + y]$ **Ans.** 0, $\bar{0}$
   (iii) $\bar{v} = [x^2, 2yz, 1 + 2z]$ **Ans.** $2x + 2y + 2$, $\bar{0}$
   (iv) $\bar{v} = [y^2 - z^2 + 3yz - 2x, 3xz + 2xy, 3xy - 2xz + 2z]$ **Ans.** 0, $\bar{0}$.
   (v) $\bar{v} = [z^2 + 2x + 3y, 3x + 2y + z, y + 2xz]$ **Ans.** $4 + 2x$, $\bar{0}$
2. Show that $\vec{v} = [3x^2y, x^3 - 2yz^2, 3z^2 - 2y^2z]$ is irrotational but not solenoidal find f for which $\vec{v} = \nabla f$
   Ans. $f = x^3y - y^2z^2 + z^3 + C$
3. Find value of a for div $\bar{v} = 0$
   (i) $\bar{v} = [z + 3y, x - 2z, x + az]$ **Ans.** $a = 0$
   (ii) $\bar{v} = [3x, x + y, -az]$ **Ans.** $a = 4$
4. Find value of a, b, c for curl $\bar{v} = 0$
   $\bar{v} = [x + y + az, bx + 2y - z, -x + cy + 2z]$ **Ans.** $a = -1$, $b = 1$, $c = -1$
5. Show that following function $\bar{F}$ are irrotational find ϕ for which $\bar{F} = \nabla\phi$
   (i) $\bar{F} = [6xy + z^3, 3x^2 - z, 3xz^2 - y]$ **Ans.** $3x^2y + z^3x - yz + C$
   (ii) $\bar{F} = [2xyz, x^2z + 2y, x^2y]$ **Ans.** $\phi = x^2yz + y^2 + C$
   (iii) $\bar{F} = [y^2z, 2xyz, xy^2 + 2z]$ **Ans.** $\phi = xy^2z + z^2 + C$
   (iv) $\bar{F} = [y \sin z - \sin x, x \sin z + 2yz, xy \cos z + y^2)$ **Ans.** $\phi = xy \sin z + \cos x + y^2z$

8.11 GREEN's THEOREM FOR PLANE

Relation between line integral of close plane curve with double integration on domain bounded by curve.

Theorem-3.1. If M (x, y), N (x, y), $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ be continuous everywhere in a region R of xy–plane bounded by a closed curve C, then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

The integration being taken along the entire boundary C of domain R such that R is on the left as we orient in the direction of integration (See Fig.)

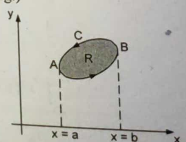


Fig. 24

Green's Thoerem in Vector Form

Let $\bar{F} = M\hat{i} + N\hat{j}$ and $\bar{r} = x\hat{i} + y\hat{j}$, then $\bar{F} \cdot d\bar{r} = Mdx + Ndy$.

$$\text{Also curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & O \end{vmatrix} = \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

The component of curl $\bar{F}$ which is normal to a region R in xy-plane is

$$(\nabla \times \bar{F}) \cdot \hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Hence, Green's Theorem in the plane can be written in the vector form as

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_R (\text{curl } \bar{F}) \cdot \hat{k}\ dx\ dy = \iint_R (\nabla \times \bar{F}) \cdot d\mathbf{A}$$

where $d\mathbf{A} = dA\ \hat{k} = \hat{k}\ dx\ dy$ is the vector normal to the domain R in xy–plane and is of magnitude $|d\mathbf{A}| = dx\ dy$. We shall later, extend this result to more general curves and surfaces in the form of a theorem known as **Stokes's Theorem**.

EXAMPLE-77 : Using Green's theorem, evaluate $\oint_C \left[(x^2 + xy)\,dx + (x^2 + y^2)\,dy\right]$, where C is the square formed by the four lines $x = \pm 1,\ y = \pm 1$

SOLUTION : Here $\oint_C (M\ dx + N\ dy) = \oint_C \left[(x^2 + xy)\,dx + (x^2 + y^2)\,dy\right]$

$\therefore\quad M = x^2 + xy, \qquad N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\therefore\quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x$$

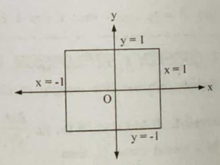


Fig. 25

The square is formed by $x = \pm 1,\ y = \pm 1$

$$\text{By Green's theorem, } \oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} x\ dy\ dx$$

$$= \int_{x=-1}^{1} x\,(y)_{-1}^{1}\ dx$$

$$= 2\int_{x=-1}^{1} x\ dx$$

$$= (x^2)_{-1}^{1}$$

$$= 1 - 1$$

$$= 0$$

$$\text{Thus, } \oint_C \left[(x^2 + xy)\,dx + (x^2 + y^2)\,dy\right] = 0$$

EXAMPLE-78 : Use Green's theorem to evaluate

$\oint_C (x^2 y\ dx + y^3\ dy)$, where C is bounded by $y = x$ and $y = x^3$ from (0, 0) to (1, 1)

[GTU, Summer 2015]

SOLUTION : Here, $\oint_C (Mdx + Ndy) = \oint_C (x^2 y\ dx + y^3\ dy)$

$\therefore\quad M = x^2 y \qquad N = y^3$

$$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = 0$$

$$\therefore\quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 - x^2 = -x^2$$

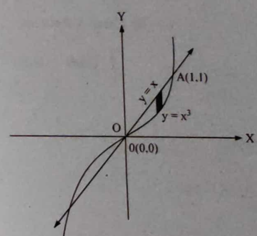


Fig. 26

By Green's theorem,

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy$$

$$= \int_{x=0}^{1} \int_{y=x^3}^{x} (-x^2)\ dy\ dx$$

$$= -\int_{x=0}^{1} x^2 (y)_{x^3}^{x}\ dx$$

$$= -\int_{x=0}^{1} x^2 (x - x^3)\ dx$$

$$= \left(\frac{x^6}{6} - \frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{6} - \frac{1}{4}$$

$$= \frac{-1}{12}$$

$$\text{Hence, } \oint_C (x^2 y\ dx + y^3\ dy) = \frac{-1}{12}$$

EXAMPLE-79 : Using Green's theorem, evaluate $\oint_C (x - y)\,dx + (x + y)\,dy$, where C is region enclosed by $y = x^2$ and $y^2 = x$.

SOLUTION : Here $\oint_C (\mathrm{M}dx + \mathrm{N}dy) = \oint_C (x-y)\,dx + (x+y)\,dy$

$\mathrm{M} = x - y \qquad \mathrm{N} = x + y$

$\therefore \quad \dfrac{\partial \mathrm{M}}{\partial y} = -1 \qquad \dfrac{\partial \mathrm{N}}{\partial x} = 1$

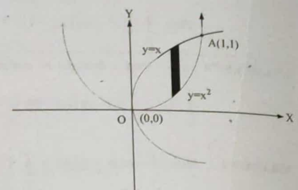


Fig. 27

$\therefore \quad \dfrac{\partial \mathrm{N}}{\partial x} - \dfrac{\partial \mathrm{M}}{\partial y} = 1 - (-1) = 2$

By Green's theorem,

$$\oint_C (\mathrm{M}dx + \mathrm{N}dy) = \iint_R \left(\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 2\,dx\,dy$$

$$= 2 \int_{x=0}^{1} (y)_{x^2}^{\sqrt{x}}\,dx$$

$$= 2 \int_{x=0}^{1} (\sqrt{x} - x^2)\,dx$$

$$= 2\left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3}[2 - 1]$$

$$= \frac{2}{3}$$

Hence, $\oint_C ((x-y)\,dx + (x+y)\,dy) = \dfrac{2}{3}$

EXAMPLE-80 : Use Green's theorem to evaluate $\oint_C \left((2x^2 - y^2)\,dx + (x^2 + y^2)\,dy\right)$, where C is the boundary of the area enclosed by the X-axis and the upper half of the circle $x^2 + y^2 = a^2$

SOLUTION : Here $\oint_C (\mathrm{M}dx + \mathrm{N}dy) = \oint_C \left((2x^2 - y^2)\,dx + (x^2 + y^2)\,dy\right)$

$\therefore \quad \mathrm{M} = 2x^2 - y^2 \qquad \mathrm{N} = x^2 + y^2$

$\therefore \quad \dfrac{\partial \mathrm{M}}{\partial y} = -2y \qquad \dfrac{\partial \mathrm{N}}{\partial x} = 2x$

$\therefore \quad \dfrac{\partial \mathrm{N}}{\partial x} - \dfrac{\partial \mathrm{M}}{\partial y} = 2x + 2y$

By Green's theorem

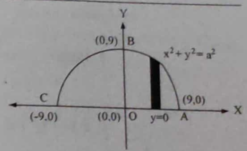


Fig. 28

$$\oint_C (\mathrm{M}dx + \mathrm{N}dy) = \iint_R \left(\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y}\right) dx\,dy$$

$$= \int_{x=-a}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (2x + 2y)\,dy\,dx$$

$$= 2\int_{x=-a}^{a} \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{x=-a}^{a} \left[x\sqrt{a^2-x^2} + \frac{1}{2}(a^2 - x^2) - 0\right] dx$$

$$= 2\int_{x=0}^{a} (a^2 - x^2)\,dx \qquad \left[\because f(x) = x\sqrt{a^2-x^2} \text{ is an odd function}\right]$$

$$= 2\left(a^2x - \frac{x^3}{3}\right)_0^a$$

$$= \frac{2}{3}[3a^3 - a^3]$$

$\therefore \quad \oint_C \left((2x^2 - y^2)\,dx + (x^2 + y^2)\,dy\right) = \dfrac{4a^3}{3}$

EXAMPLE-81 : Use Green's theorem to evaluate $\oint_C \left((3x - 8y^2)\,dx + (4y - 6xy)\,dy\right)$, where C is the boundary of the triangle with vertices (0,0), (1,0) and (0,1) **[GTU, Summer 2014]**

SOLUTION : Here $\oint_C (\mathrm{M}dx + \mathrm{N}dy) = \oint_C \left((3x - 8y^2)\,dx + (4y - 6xy)\,dy\right)$

$\therefore \quad \mathrm{M} = 3x - 8y^2 \qquad \mathrm{N} = 4y - 6xy$

$\therefore \quad \dfrac{\partial \mathrm{M}}{\partial y} = -16y \qquad \dfrac{\partial \mathrm{N}}{\partial x} = -6y$

$\therefore \quad \dfrac{\partial \mathrm{N}}{\partial x} - \dfrac{\partial \mathrm{M}}{\partial y} = -6y + 16y = 10y$

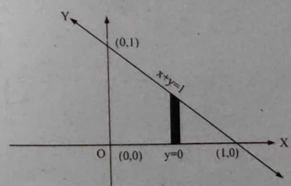


Fig. 29

By Green's theorem

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$\therefore \oint_C \left((3x - 8y^2)\, dx + (4y - 6xy)\, dy\right) = \int_{x=0}^{1} \int_{y=0}^{1-x} 10y\; dy\, dx$$

$$= 10 \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= 5 \int_{x=0}^{1} (1-x)^2\; dx$$

$$= 5 \left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -\frac{5}{3}\; [0 - 1]$$

$$= \frac{5}{3}$$

Hence, $\oint_C \left((3x - 8y^2)\, dx + (4y - 6xy)\, dy\right) = \frac{5}{3}$

EXAMPLE-82 : Using Green's Theorem, evaluate $\oint_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ where C is the boundary of the region bounded by $y^2 = x$ and $y = x^2$.

SOLUTION : $y^2 = x$ and $y = x^2$ are two parabolas intersecting at (0, 0) and (1, 1)

Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

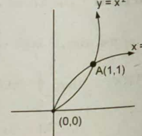


Fig. 30

$$\frac{\partial M}{\partial y} = -16y, \qquad \frac{\partial N}{\partial x} = -6y$$

$$\therefore \oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\; dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\; dx\; dy$$

$$= \int_0^1 5\left(y^2\right)_{x^2}^{\sqrt{x}}\; dx$$

$$= 5 \int_0^1 (x - x^4)\; dx$$

$$= 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$$

EXAMPLE-83 : Applying Green's theorem, evaluate $\oint_C [(y - \sin x)\, dx + \cos x\; dy]$, where C is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2}{\pi}x$.

SOLUTION : Here $M = y - \sin x$,

$N = \cos x$

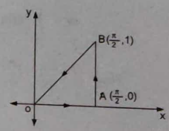


Fig. 31

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x \text{ and}$$

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\; dy$$

$$= \iint_R (-\sin x - 1)\; dy\; dx$$

$$= \int_{x=0}^{\frac{\pi}{2}} \left[\int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)\; dy\right] dx = \int_{x=0}^{\frac{\pi}{2}} [-y \sin x - y]_0^{\frac{2x}{\pi}}\; dx$$

$$= \int_{x=0}^{\frac{\pi}{2}} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right) dx$$

$$= -\frac{2}{\pi}[-x\cos x + \sin x]_0^{\frac{\pi}{2}} - \left[\frac{x^2}{\pi}\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi} - \frac{\pi}{4}$$

EXAMPLE-84 : Verify Green's theorem for the function $\overline{F} = (x + y)\,\hat{i} + 2xy\,\hat{j}$ and C is the rectangle in the xy-plane bounded by $x = 0$, $y = 0$, $x = a$, $y = b$. **[G.T.U., 2016]**

SOLUTION : $\oint_C Mdx + Ndy = \oint_C \overline{F} \cdot d\overline{r}$

$$= \oint_C (x + y)\; dx + 2xy\; dy$$

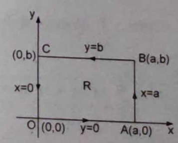


Fig. 32

Along line segment OA, $y = 0 \Rightarrow dy = 0$

$$\therefore \int_{OA} Mdx + Ndy = \int_0^a x dx = \frac{a^2}{2} \qquad \text{...(i)}$$

Along line segment AB, $x = a \Rightarrow dx = 0$

$$\therefore \int_{AB} Mdx + Ndy = \int_0^b 2ay\; dy = \frac{2ab^2}{2} = ab^2 \qquad \text{...(ii)}$$

Along line segment BC, $y = b \Rightarrow dy = 0$

$$\therefore \int_{BC} Mdx + Ndy = \int_a^0 (x + b)\; dx = -\frac{a^2}{2} - ab \qquad \text{...(iii)}$$

Along line segment CO, $x = 0 \Rightarrow dx = 0$

$$\therefore \quad \int_{CO} Mdx + Ndy = \int_b^0 0 \; dy = 0 \qquad ...(iv)$$

Adding (i), (ii), (iii) and (iv)

$$\Rightarrow \quad \oint_C Mdx + Ndy = ab^2 - ab \qquad ...(A)$$

$$\text{Now } \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^a \int_0^b (2y - 1)\, dx\, dy$$

$$= \int_0^a \left(\frac{2y^2}{2} - y\right]_0^b dx$$

$$= \int_0^a (b^2 - b)\, dx = (b^2 - b) \int_0^a dx$$

$$= (b^2 - b)\,(x)_0^a = (b^2 - b)\, a = ab^2 - ab \qquad ...(B)$$

From (A) and (B) we see that

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's Theorem is verified.

EXAMPLE-85 : Verify Green's theorem for $\overline{F} = x^2\hat{i} + xy\hat{j}$, under the square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$ [G.T.U., 2015]

SOLUTION : $\overline{F} = x^2\hat{i} + xy\hat{j}$

$\therefore M = x^2$ and $N = xy$

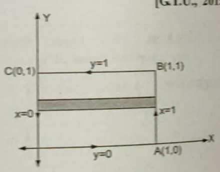


Fig. 33

$$\text{Claim : } \oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dxdy$$

$$\text{Now } \oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AB} M\,dx + N\,dy + \int_{BC} M\,dx + N\,dy + \int_{CD} M\,dx + N\,dy$$

For OA $y = 0 \Rightarrow dy = 0$

For AB $x = 1 \Rightarrow dx = 0$

For BC $y = 1 \Rightarrow dy = 0$

For CD $x = 0 \Rightarrow dx = 0$

$$\therefore \oint M\,dx + N\,dy = \int_0^1 x^2 dx + \int_0^1 y\,dy + \int_1^0 x^2\,dx + \int_1^0 0\,dy$$

$$= \frac{1}{3} + \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{2}$$

$$\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dxdy = \int_0^1 \int_0^1 (y - 0)\, dxdy$$

$$= \int_0^1 y\,[x]_0^1\, dy$$

$$= \int_0^1 y\, dy = \frac{1}{2}$$

$$\therefore \oint M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dxdy$$

EXAMPLE-86 : Evaluate $\oint_C \left[-\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy\right]$, where $C = C_1 \cup C_2$ with $C_1 : x^2 + y^2 = 1$ and $C_2 : x = \pm 2$, $y = \pm 2$ [G.T.U., 2008, 2011]

SOLUTION : Here $M = -\frac{y}{x^2+y^2}$, $N = \frac{x}{x^2+y^2}$

$$\therefore \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}, \quad \frac{\partial N}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

which are continuous on the region R bounded by C.

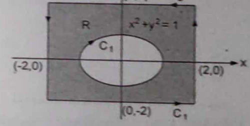


Fig. 34

$$\oint_C \left[-\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy\right] = \iint_R \left[\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right] dx\,dy$$

$$= 0$$

EXAMPLE-87 : Use Green's theorem to evaluate $\oint x^2ydx + y^3\,dy$, where C is the closed path formed by $y = x$ and $y = x^3$ from (0, 0) to (1, 1). [G.T.U., 2015]

SOLUTION : $M = x^2y \qquad N = y^3$

$$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = 0$$

$$\oint x^2y\,dy + y^3\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dydx$$

$$= \int_0^1 \int_{y=x^3}^{y=x} -x^2\, dydx$$

$$= -\int_0^1 x^2\,[x - x^3]\, dx$$

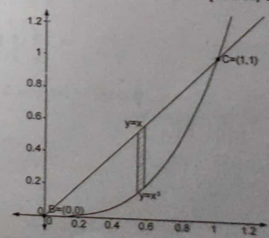


Fig. 35

$$= -\int_0^1 x^3 - x^5\, dx$$

$$= -\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$$

$$= -\left(\frac{1}{4} - \frac{1}{6}\right)$$

$$= \frac{-1}{12}$$

EXAMPLE-88 : State Green's theorem and also evaluate the integral $\oint_C (6y + x)\, dx + (y + 2x)\, dy$ where C : The circle $(x - 2)^2 + (y - 3)^2 = 4$ **[G.T.U., 2010]**

SOLUTION : Here $M = 6y + x$, $N = y + 2x$

$$\therefore \frac{\partial M}{\partial y} = 6, \quad \frac{\partial N}{\partial x} = 2$$

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R [2 - 6]\, dxdy$$

$$= -4 \iint_R dxdy$$

$$= -4\,(4\pi) \qquad [\because \text{Area of the circle} = 2\pi r = 2\pi(2) = 4\pi$$

$$= -16\,\pi$$

EXAMPLE-89 : Use Green's theorem to evaluate the integral $\oint_C (y^2dx + x^2dy)$, where C : The triangle bounded by $x = 0$, $x + y = 1$, $y = 0$. **[G.T.U., 2010]**

SOLUTION : Here $M = y^2$ and $N = x^2$

$$\therefore \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2x$$

From Green's Theorem

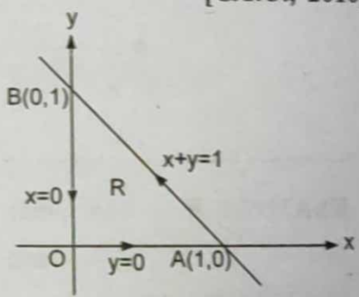


Fig. 35

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= 2\int_0^1 \int_0^{1-y} (x - y)\, dxdy$$

$$\therefore \oint_C (Mdx + Ndy) = 2\int_0^1 \left[\frac{x^2}{2} - xy\right]_0^{1-y} dy$$

$$= 2\int_0^1 \left[\frac{(1-y)^2}{2} - y(1-y)\right] dy$$

$$= 2\int_0^1 \left[\frac{1}{2} - 2y + \frac{3}{2}y^2\right] dy$$

EXAMPLE-90 : Obtain Green's theorem Area formula and use it to obtain the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

[G.T.U., 2010]

SOLUTION : From Green's Theorem, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (Mdx + Ndy) \quad ...(1)$$

Let M = 0 and N = x, then from equation (1) we get

$$A = \iint_R dx\, dy = \oint_C x\, dy \quad ...(2)$$

The integral on the left is the area A of the region R. Next let M = –y, N = 0, then, from the relation (1), we get

$$A = \iint_R dx\, dy = -\oint_C y\, dx \quad ...(3)$$

Adding (2) and (3), we get

$$A = \frac{1}{2}\oint_C (x\, dy - y\, dx) \quad ...(4)$$

For an **ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or x = a cos t, y = b sin t, we get from (4) the familar result

$$A = \frac{1}{2}\int_C [(a \cos t)(b \cos t)\, dt - (b \sin t)(-a \sin t)\, dt]$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2 t + \sin^2 t)\, dt$$

$$= \frac{1}{2} ab\, [t]_0^{2\pi}$$

$$= \pi ab.$$

Area of a plane region in polar coordinates

Let r and θ be polar coordinates defined by

$x = r\cos\theta,\ y = r\sin\theta$. Then

$dx = \cos\theta\, dr - r\sin\theta\, d\theta,\ dy = \sin\theta\, dr + r\cos\theta\, d\theta.$

The relation (4) reduces to

$$A = \frac{1}{2}\oint_C [r\cos\theta(\sin\theta\, dr + r\cos\theta\, d\theta) - r\sin\theta(\cos\theta\, dr - r\sin d\theta)]$$

or $$A = \frac{1}{2}\oint_C r^2\, d\theta \quad ...(5)$$

EXAMPLE-91 : Find area bounded y r = a(1 – cosθ)

SOLUTION :

$$A = \frac{a^2}{2}\int_0^{2\pi} (1 - \cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{a^2}{2} \int_0^{2\pi} \left[1 - 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= \frac{3\pi}{2} a^2$$

8.12 Second Order Differential Operator

→ Div (grad ϕ) = $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$. The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

The equation $\nabla^2\phi = 0$ is called Laplace's equation.

(1) curl (grad ϕ) = $\bar{0}$, i.e. $\nabla \times (\nabla\phi) = \bar{0}$

(2) div (curl $\bar{F}$) = 0, i.e $\nabla \cdot (\nabla \times \bar{F}) = 0$

(3) curl (curl $\bar{F}$) = grad (div $\bar{F}$) $- \nabla^2\bar{F} = \nabla(\nabla \cdot \bar{F}) - \nabla^2\bar{F}$

EXAMPLE-92 : Calculate $\nabla^2 f$ at the point (1,2,0), where $f = 2x^2z - y^2z^3 + 4x^3y + 2x - 3y - 4$

SOLUTION : As we know, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Here $f = 2x^2z - y^2z^3 + 4x^3y + 2x - 3y - 4$.

$$\therefore \quad \frac{\partial f}{\partial x} = 4xz - 0 + 12x^2y + 2$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = 4z + 24xy \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial y} = 0 - 2yz^3 + 4x^3 - 3$$

$$\therefore \quad \frac{\partial^2 f}{\partial y^2} = -2z^3 + 0 - 0 \qquad \text{... (2)}$$

$$\frac{\partial f}{\partial z} = 2x^2 - 3y^2z^2$$

$$\frac{\partial^2 f}{\partial z^2} = 0 - 6y^2z \qquad \text{... (3)}$$

From (1), (2) and (3),

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 4z + 24xy - 2z^3 - 6y^2z$$

At the point (1,2,0), $\nabla^2 f = 0 + 48 - 0 - 0 = 48$

EXERCISE 8.4

(A). Verify Green's Theorem.

1. $\oint_C [(xy + y^2)\, dx + x^2\, dy]$, where C is bounded by $y = x$ and $y = x^2$.

   Ans. $-\dfrac{1}{20}$

2. $\oint_C [(3x - 8y^2)\, dx + (4y - 6xy)\, dy]$ where C is the boundary of the region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

3. $\oint_C (2x - y^2)\, dx - xy\, dy$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

   Ans. 60π

(B) Use Green's theorem to evaluate line integral.

1. $\oint_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$ where C is the boundary of the area enclosed by the x–axis and the upper–half of the circle $x^2 + y^2 = a^2$.

   Ans. $\dfrac{4a^3}{3}$

2. $\oint_C [(2x^2 + y^2)\, dx + (x^2 + y^2)\, dy]$, where C is the boundary of the surface in xy–plane enclosed by the x–axis and the semi–circle $y = \sqrt{1 - x^2}$.

   Ans. $\dfrac{4}{3}$

3. $\oint_C [(x^2 + xy)\, dx + (x^2 + y^2)\, dy]$, where C is the boundary of the square $y = \pm 1$, $x = \pm 1$.

   Ans. 0

4. $\oint_C [(x^2 - 2xy)\, dx + (x^2 y + 3)\, dy]$ around the boundary C of the region $y^2 = 8x$, $x = 2$.

   Ans. $\dfrac{125}{5}$

5. $\oint_C e^{-x} (\sin y\, dx + \cos y\, dy)$, C being the

   rectangle with vertices $(0, 0)$, $(\pi, 0)$, $\left(\pi, \dfrac{\pi}{2}\right)$, $\left(0, \dfrac{\pi}{2}\right)$.

   Ans. $2\left(e^{-\pi} - 1\right)$

6. $\oint_C [\cos y\hat{i} + x(1 - \sin y)\hat{j}] \cdot d\bar{r}$ for a closed curve which is given by $x^2 + y^2 = 1,\ z = 0.$

Ans. π

7. A vector field $\bar{F}$ is given by $\bar{F} = (\sin y)\ \hat{i} + x(1 + \cos y)\ \hat{j}$, Evaluate $\oint_C \bar{F} \cdot d\bar{r}$, where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0.$

Ans. πab

8. $\oint_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = y^3\hat{i} - x^3\hat{j}$ and C : $x^2 + y^2 = a^2,\ z = 0.$

Ans. $-\frac{3a^4\pi}{2}$

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