ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 4: Sorting and Search Algorithms 2
Lecture 6: Priority queue and binary heap

Maxim Buzdalov
Saint Petersburg 2016

- Assume you are a cinema critic, and your job is to watch movies once a day
- You may process requests in the order of their appearance
- Ordinary queue (First In, First Out)

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- You have to do it before the movie premiers
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Priority queue: Interface and implementation

	Vector, Queue, Deque, ...	Same with sorting
[+]: add an element	$O(1)$	$O(n)$
[?]: query the smallest element	$O(n)$	$O(1)$
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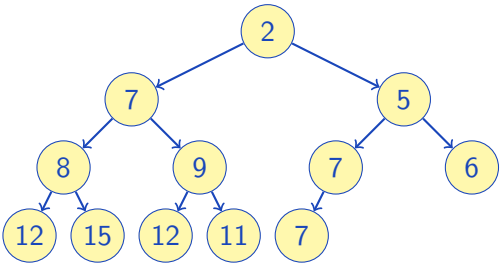
Is this the best we can do?

Binary heap: a tree-like structure



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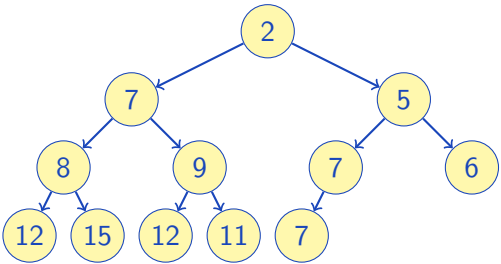
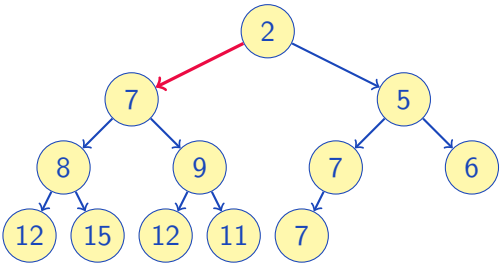


Properties of a binary heap:

- A node is not less than its parent

Binary heap: a tree-like structure



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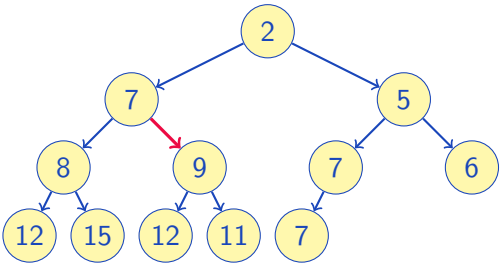

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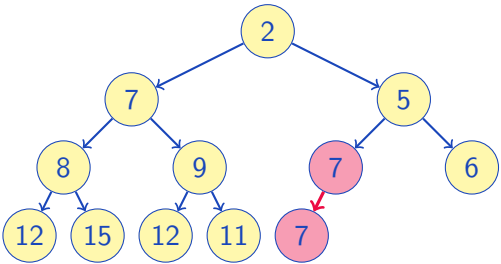

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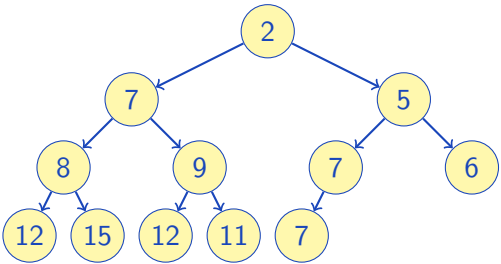
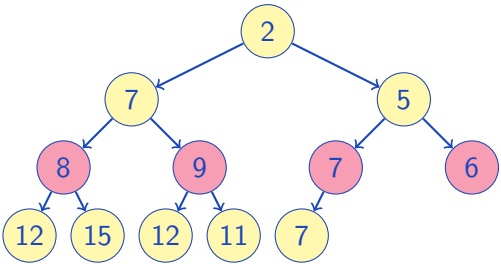


Properties of a binary heap:

- A node is not less than its parent
- Every node has at most two children

Binary heap: a tree-like structure



Properties of a binary heap:

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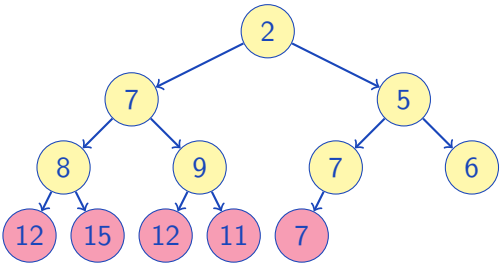
- Every node has at most two children
- All levels but the last one are complete

Binary heap: a tree-like structure



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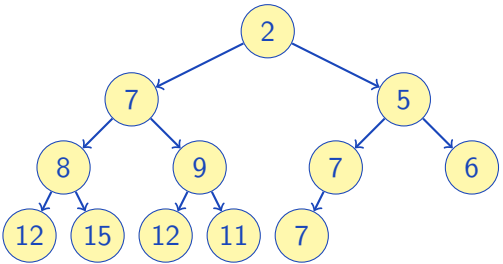
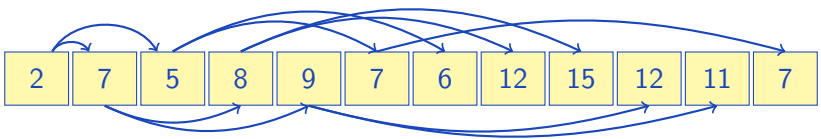
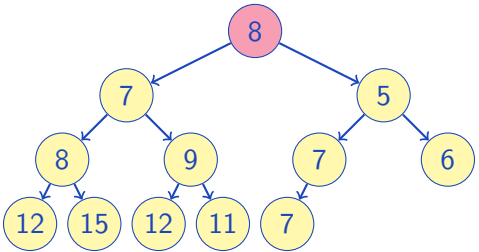


Properties of a binary heap:

- A node is not less than its parent
- Every node has at most two children
- All levels but the last one are complete
- The last level is filled from the left
- Can be stored in an array or a vector
- Node at index $i \rightarrow$ parent at index $\lfloor \frac{i}{2} \rfloor$



A node has just become greater than one of its children. How to repair the heap?



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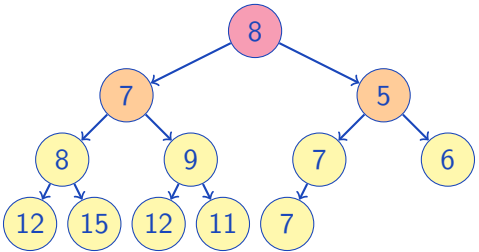

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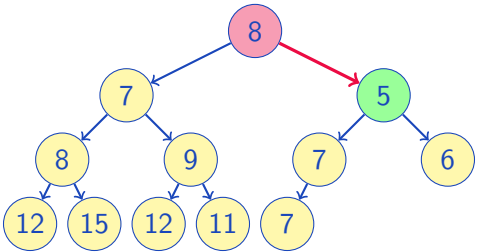

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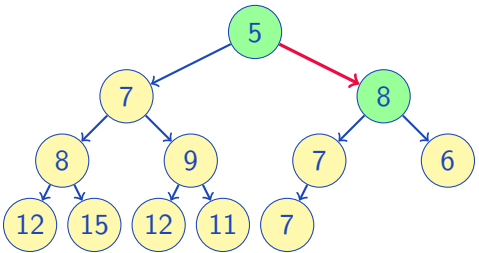

- Look at the children
- Choose the smallest
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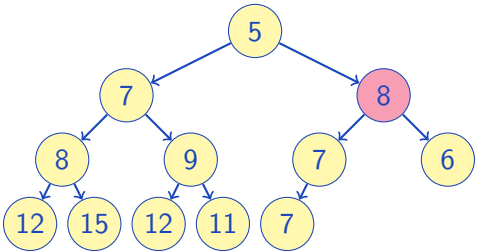

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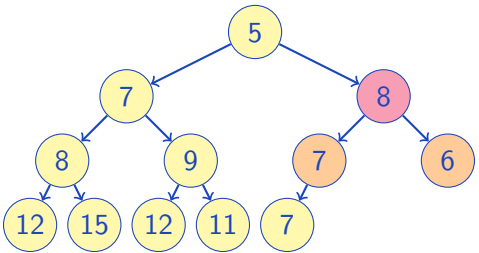

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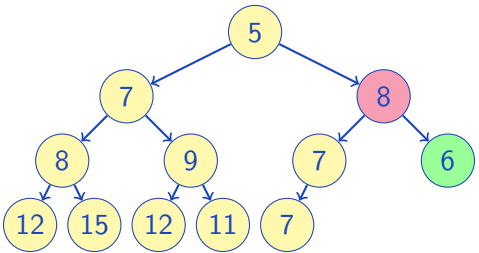

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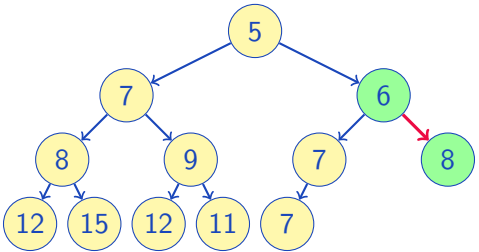

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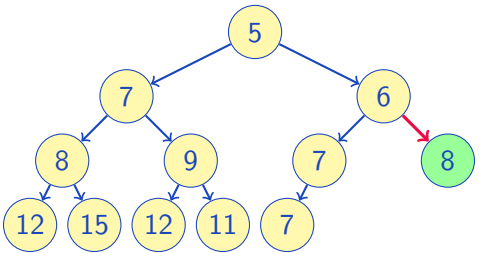

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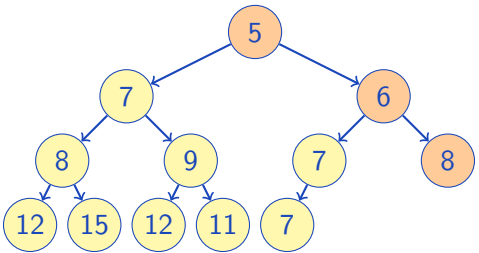

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Complexity of the operation:

$O(\text{height of a broken node}) = O(\log n)$

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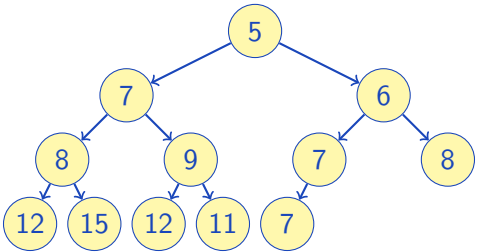

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Complexity of the operation:

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We will call this operation "Sift down"

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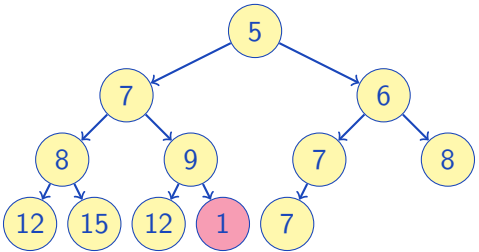

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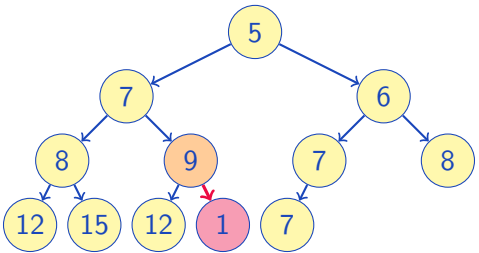

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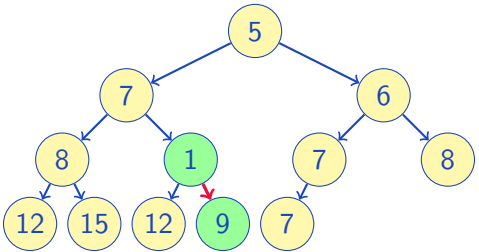

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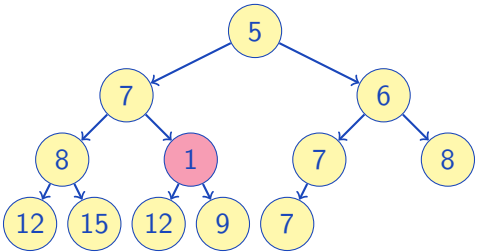

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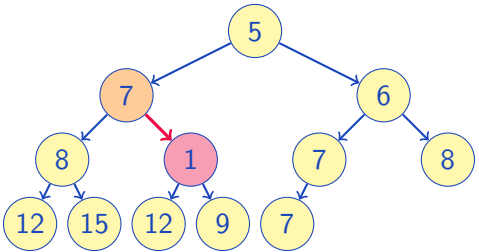

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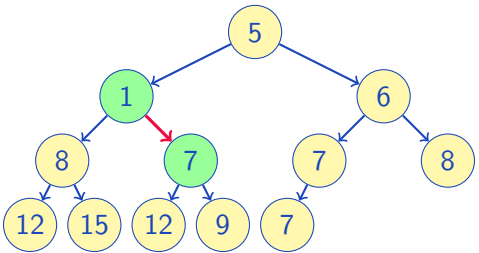

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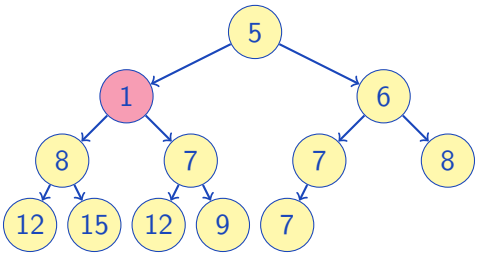

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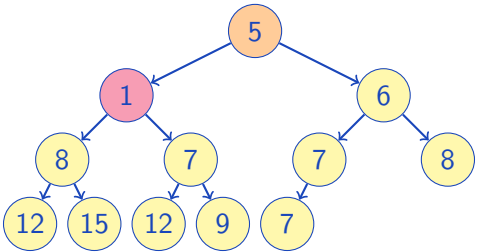

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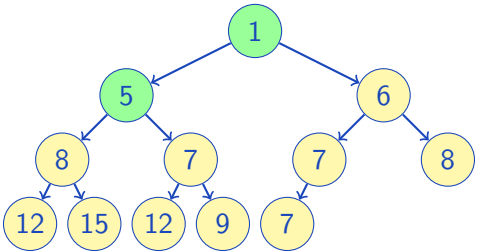

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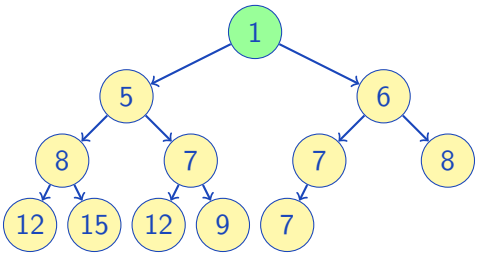

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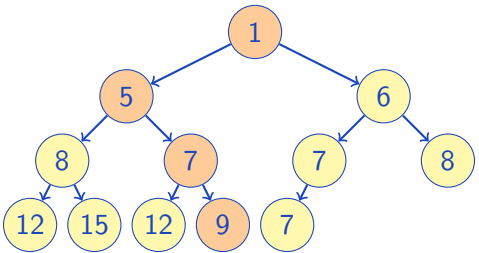

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Complexity of the operation:

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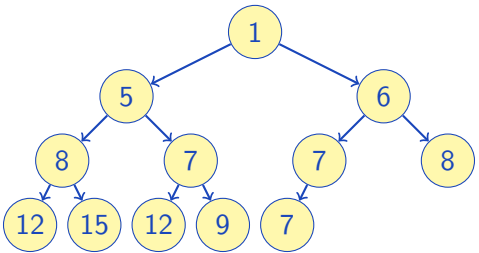

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Complexity of the operation:

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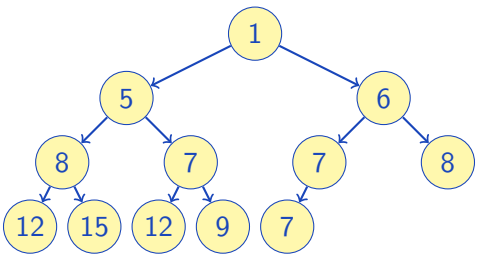
We will call this operation "Sift up"

Priority queue: Supported operations

- [+]: add an element
- [?]: query the smallest element
- [−]: remove the smallest element



Priority queue: Supported operations

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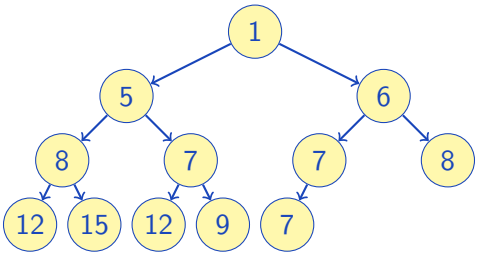

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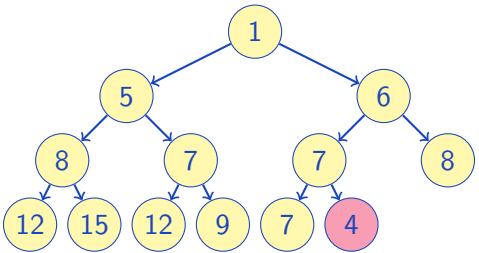
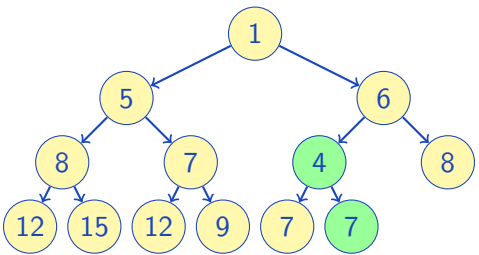


Priority queue: Supported operations

- [+]: add an element → add the new node, then sift it up
- [?]: query the smallest element
- [−]: remove the smallest element



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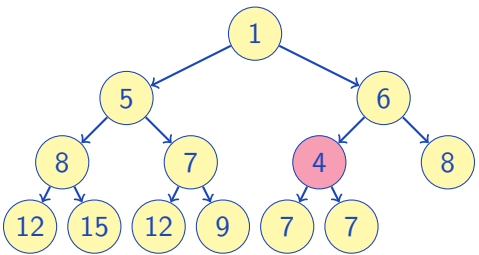

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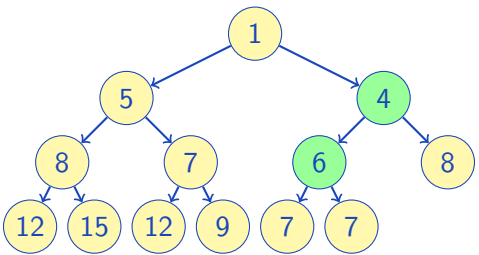

Priority queue: Supported operations

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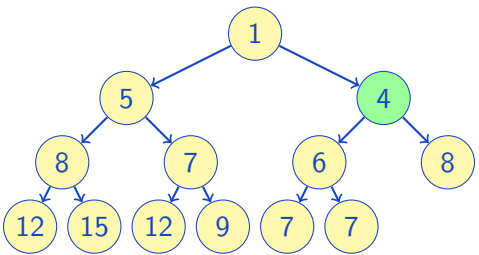
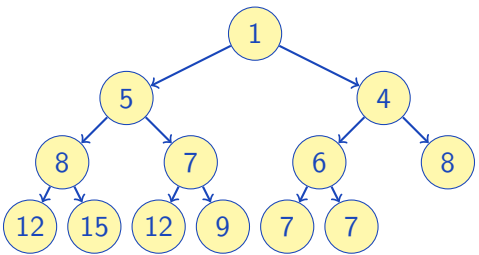
- [−]: remove the smallest element



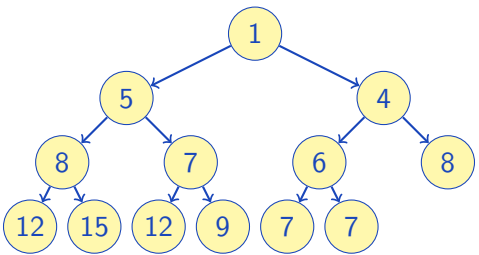
Priority queue: Supported operations

- [+]: add an element $\rightarrow$ add the new node, then sift it up $\rightarrow$ $O(\log n)$
- [?]: query the smallest element
- [−]: remove the smallest element



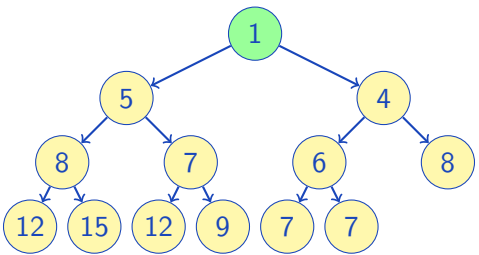
Priority queue: Supported operations

- [+]: add an element $\rightarrow$ add the new node, then sift it up $\rightarrow O(\log n)$
- [?]: query the smallest element
- [−]: remove the smallest element



Priority queue: Supported operations

- [+]: add an element → add the new node, then sift it up → $O(\log n)$
- [?]: query the smallest element → query the root
- [−]: remove the smallest element



Priority queue: Supported operations

- [+]: add an element → add the new node, then sift it up → $O(\log n)$
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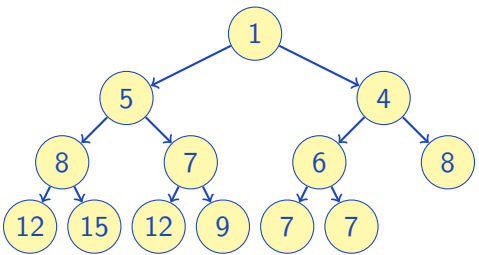
- [−]: remove the smallest element



Priority queue: Supported operations

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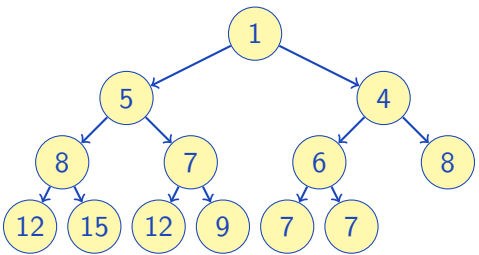
- [-]: remove the smallest element



Priority queue: Supported operations

- [+]: add an element → add the new node, then sift it up → $O(\log n)$
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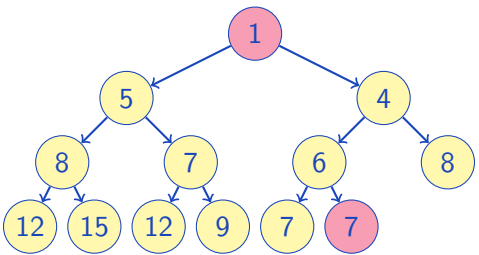
  → move the last node to the root, then sift it down



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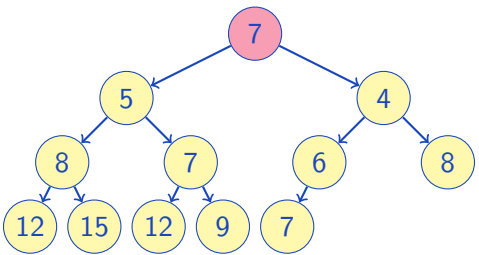

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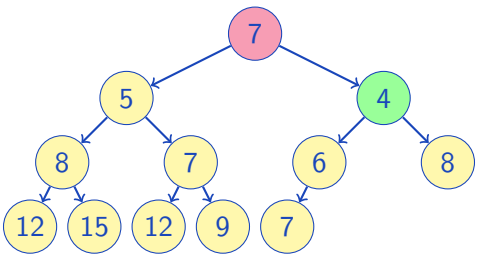

Priority queue: Supported operations

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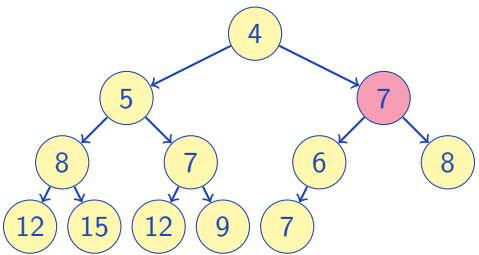

Priority queue: Supported operations

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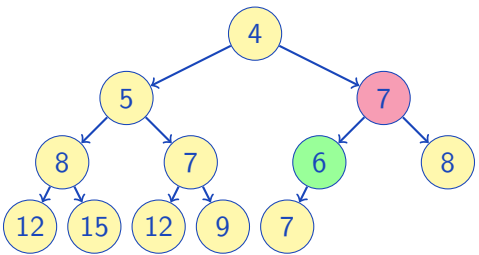

Priority queue: Supported operations

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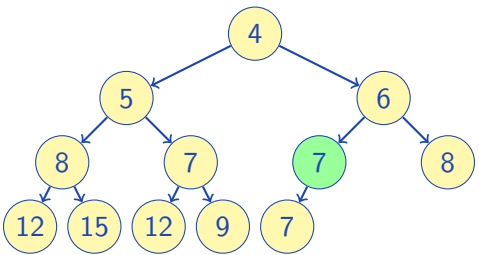

Priority queue: Supported operations

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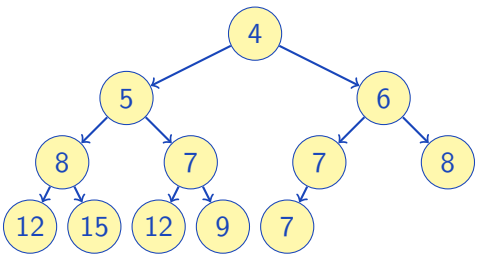

Priority queue: Implementations and their performance

	Vector, Queue, Deque, ...	Same with sorting	Binary heap
[+]	$O(1)$	$O(n)$	$O(\log n)$
[?]	$O(n)$	$O(1)$	$O(1)$
[−]	$O(n)$	$O(1)$	$O(\log n)$

- Java
  - `java.util.PriorityQueue<T>`
- C++
  - Warning: C++ assumes MAXIMUM value at root
    - everything is the same, but min $\leftrightarrow$ max, $\leq \leftrightarrow \geq$, etc.
  - High-level data structure from `<queue>`
    - `std::priority_queue<T>`
  - Low-level operations from `<algorithm>`
    - `std::make_heap`
    - `std::push_heap`
    - `std::pop_heap`
    - `std::sort_heap`
- Python
  - The `heapq` module