Shortest Paths on Graph

Unit-4 Lecture-16

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Outline

- Fundamentals of Shortest Path
- Dijkstra Algorithm
- Bellman Ford Algorithm
- Real-life Problem Solving using Graphs
  - Delivery Agent
- Homework
- Conclusions

Fundamentals of Shortest Path

Shortest Path - Problem Statement

Given a weighted directed graph (G=(V,E)), one common problem is finding the shortest path between two given vertices.

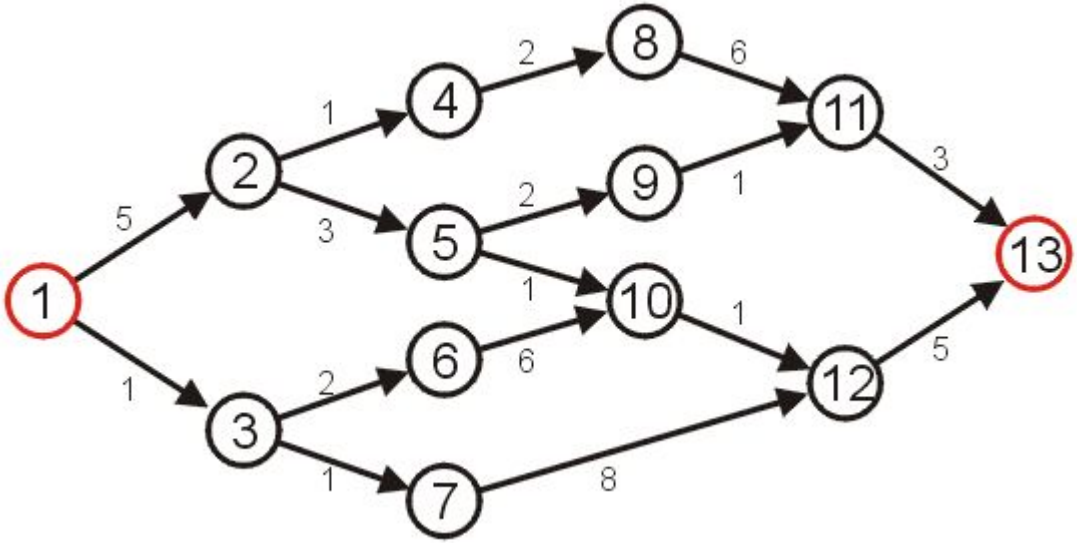
Recall that in a weighted graph, the length of a path is the sum of the weights of each of the edges in that path.

Types:

- Single Pair Shortest Path
- **Single Source Shortest Path**
- Single Destination Shortest Path
- All Pair Shortest Path

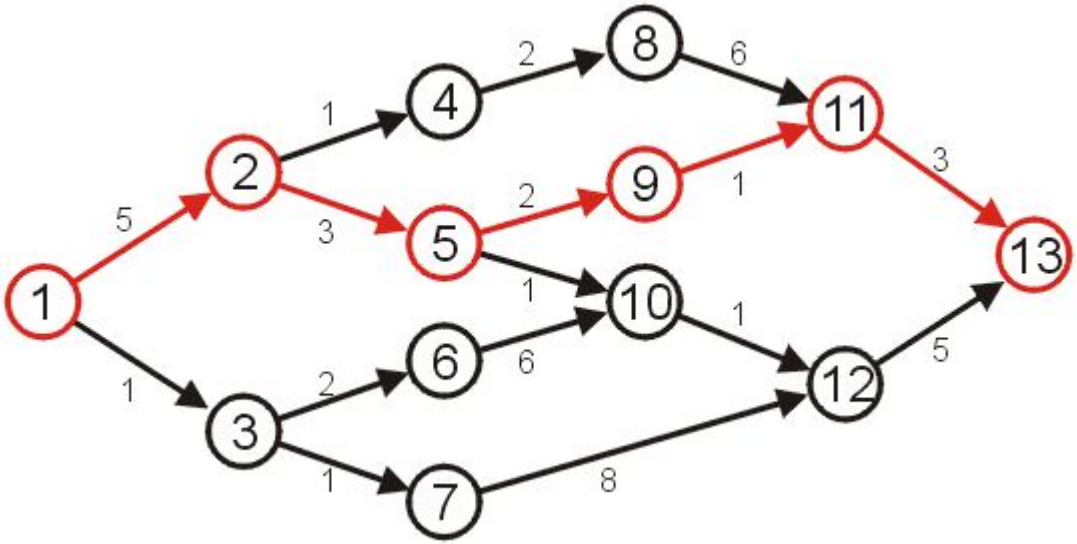
Shortest Path - Example (Single Pair)

Given the graph below, suppose we wish to find the shortest path from vertex 1 to vertex 13



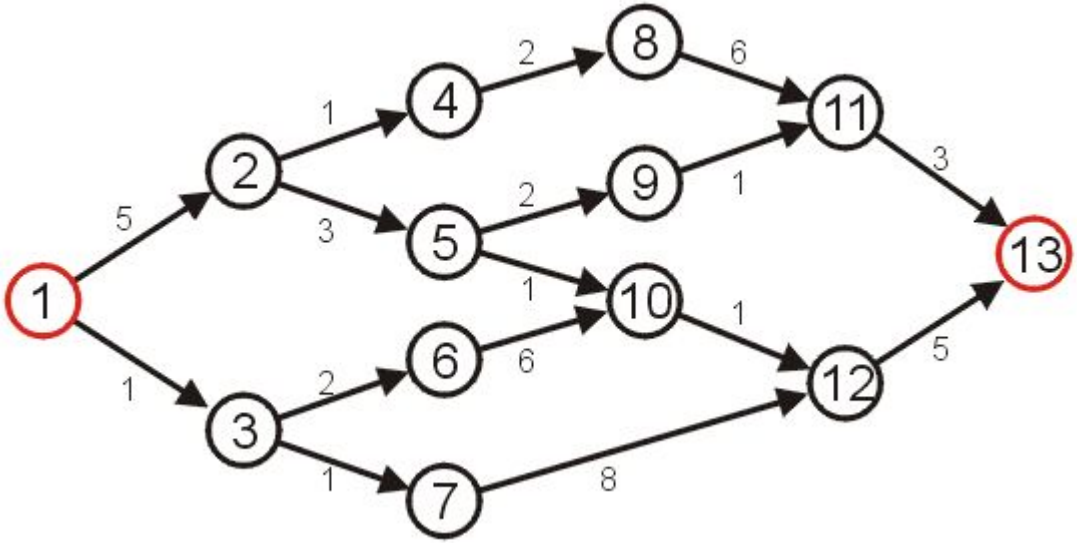
Shortest Path - Example (Single Pair) - Solution

- After some consideration, we may determine that the shortest path is as follows, with length 14
- Other paths exists, but they are longer



Shortest Path - Example (Single Source)

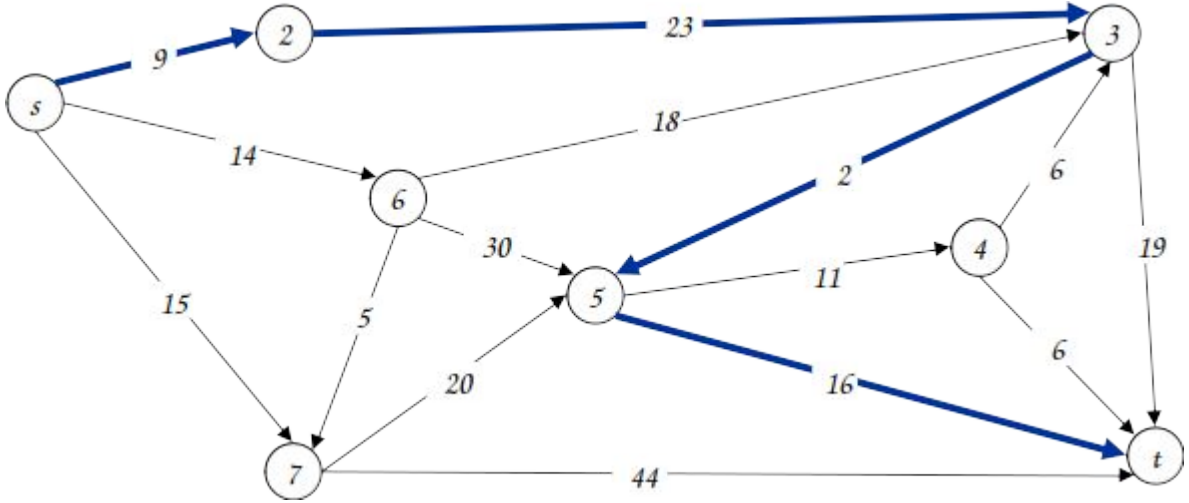
Given the graph below, suppose we wish to find the shortest paths from vertex 1 to every other vertices



Shortest Path - More Example

Given: Weighted Directed graph G = (V, E). Source s, destination t.

Find shortest directed path from s to t.



Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 48.

Barrier to Shortest Path - Negative Weight Cycle

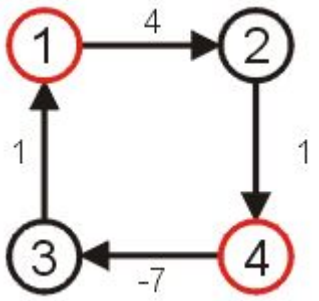
Clearly, if we have negative vertices, it may be possible to end up in a cycle whereby each pass through the cycle decreases the total length

Thus, a shortest length would be undefined for such a graph

Consider the shortest path from vertex 1 to 4…

If you go around the loop then the cost will reduce.

We will only consider non-negative weights.



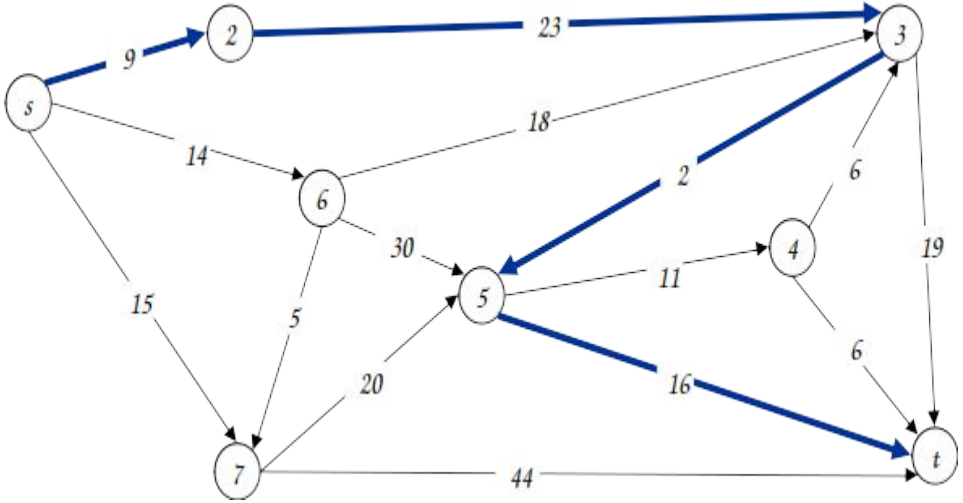
Discussion Points

How many possible paths are there from s to t?

Can we safely ignore cycles? If so, how?

Any suggestions on how to reduce the set of possibilities?

Can we determine a **lower bound** on the complexity like we did for comparison sorting?

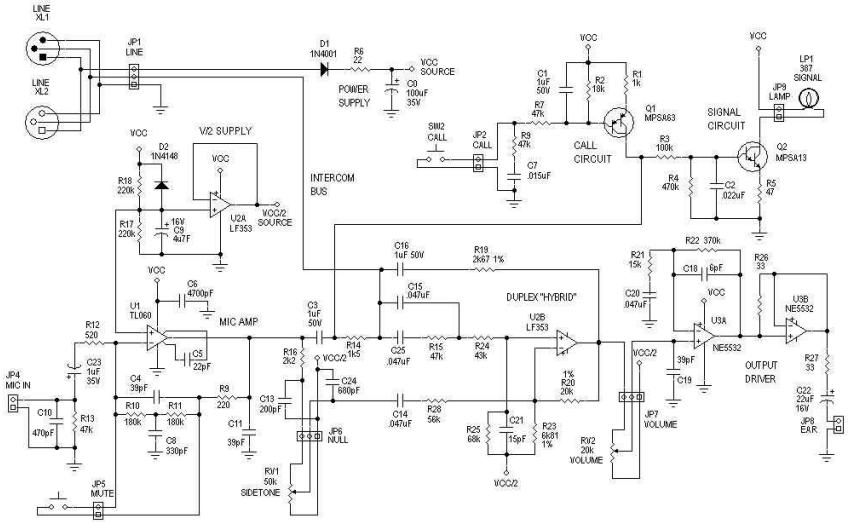


Key Observations

- A key observation is that if the shortest path contains the node v, then:
  - It will only contain v once, as any cycles will only add to the length.
  - The path from s to v must be the shortest path to v from s.
  - The path from v to t must be the shortest path to t from v.
- Thus, if we can determine the shortest path to all other vertices that are incident to the target vertex we can easily compute the shortest path.
  - Implies a set of subproblems on the graph with the target vertex removed.

Applications of Shortest Paths

One application is circuit design: the time it takes for a change in input to affect an output depends on the shortest path



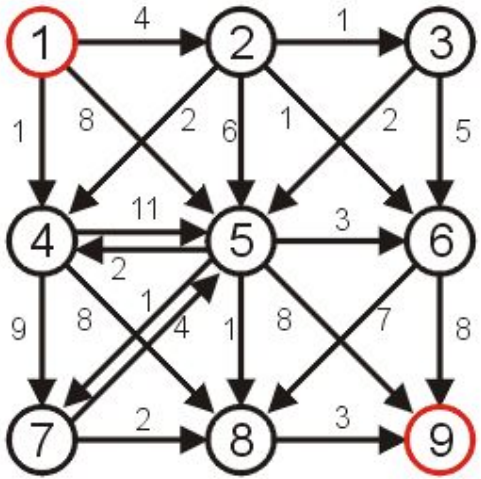
Dijkstra Algorithm

Dijkstra's Algorithm

- Works when all of the weights are positive.
- Provides the shortest paths from a source to **all** other vertices in the graph.
  - Can be terminated early once the shortest path to t is found if desired.

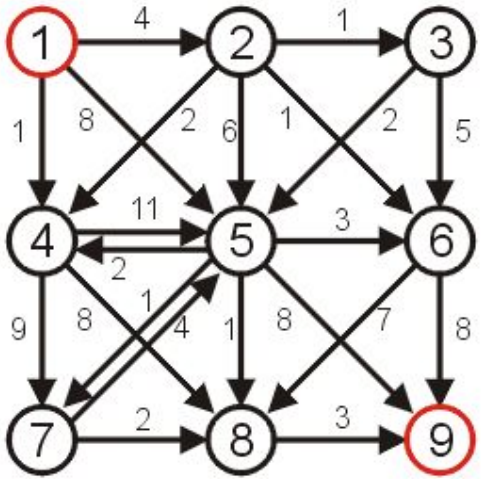
Shortest Path

- Consider the following graph with positive weights and cycles.



Dijkstra's Algorithm

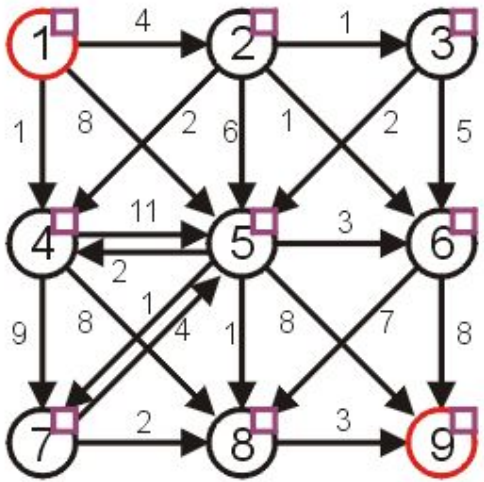
- A first attempt at solving this problem might require an array of Boolean values, all initially false, that indicate whether we have found a[illegible]rce.



1	F
2	F
3	F
4	F
5	F
6	F
7	F
8	F
9	F

Dijkstra's Algorithm

- Graphically, we will denote this with check boxes next to each of the vertices (initially unchecked)

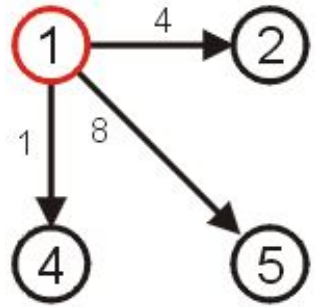


Dijkstra's Algorithm

- We will work bottom up.
  - Note that if the starting vertex has any adjacent edges, then there will be one vertex that is the shortest distance from the starting vertex. This is the shortest reachable vertex of the graph.
- We will then try to extend any ***existing*** paths to new vertices.
- Initially, we will start with the path of length 0
  - this is the trivial path from vertex 1 to itself

Dijkstra's Algorithm

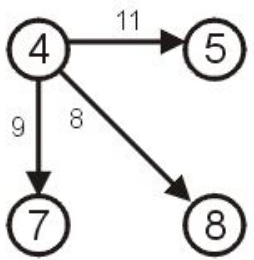
- If we now extend this path, we should consider the paths
  - (1, 2) length 4
  - (1, 4) length 1
  - (1, 5) length 8



The *shortest* path so far is (1, 4) which is of length 1.

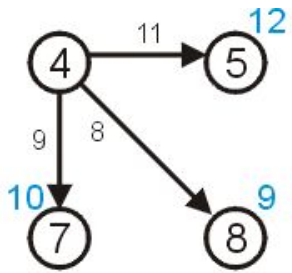
Dijkstra's Algorithm

- Thus, if we now examine vertex 4, we may deduce that there exist the following paths:
  - (1, 4, 5) length 12
  - (1, 4, 7) length 10
  - (1, 4, 8) length 9



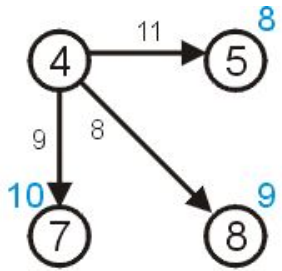
Dijkstra's Algorithm

- We need to remember that the length of that path from node 1 to node 4 is 1
- Thus, we need to store the length of a path that goes through node 4:
  - 5 of length 12
  - 7 of length 10
  - 8 of length 9



Dijkstra's Algorithm

- We have already discovered that there is a path of length 8 to vertex 5 with the path (1, 5).
- Thus, we can safely ignore this longer path.



Dijkstra's Algorithm

- We now know that:
  - There exist paths from vertex 1 to vertices {2,4,5,7,8}.
  - We know that the shortest path from vertex 1 to vertex 4 is of length 1.
  - We know that the shortest path to the other vertices {2,5,7,8} is at most the length listed in the table to the right.

Vertex	Length
1	0
2	4
4	1
5	8
7	10
8	9

Dijkstra's Algorithm

- There cannot exist a shorter path to either of the vertices 1 or 4, since the distances can only increase at each iteration.
- We consider these vertices to be ***visited***

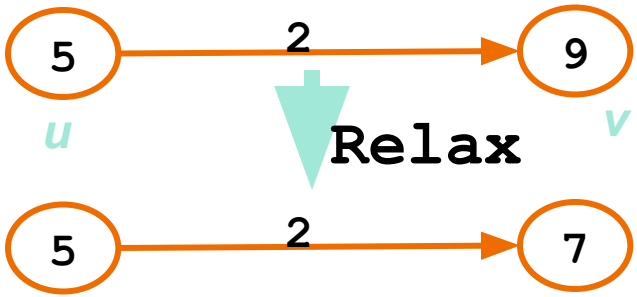
If you only knew this information and nothing else about the graph, what is the possible lengths from vertex 1 to vertex 2? What about to vertex 7?

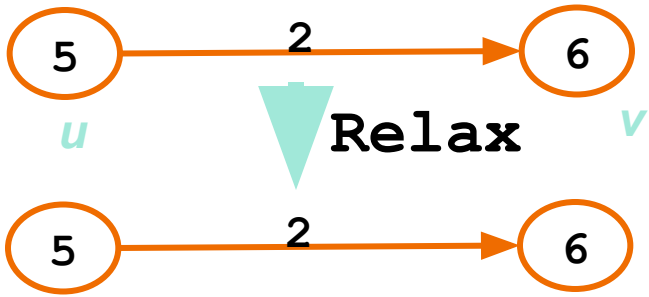
Vertex	Length
1	0
2	4
4	1
5	8
7	10
8	9

Relaxation

- Maintaining this shortest discovered distance d[*v*] is called **relaxation**:

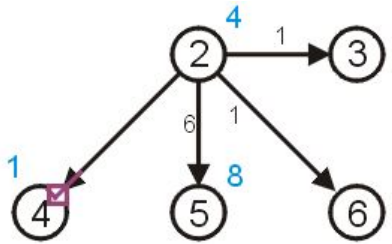
```
Relax(u,v,w) {
    if (d[v] > d[u]+w) then
        d[v]=d[u]+w;
}
```





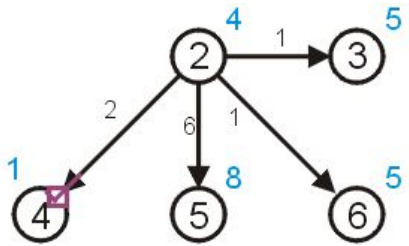
Dijkstra's Algorithm

- In Dijkstra's algorithm, we always take the next unvisited vertex which has the current shortest path from the starting vertex in the table.
- This is vertex 2



Dijkstra's Algorithm

- We can try to update the shortest paths to vertices 3 and 6 (both of length 5) however:
  - there already exists a path of length 8 < 10 to vertex 5 (10 = 4 + 6)
  - we already know the shortest path to 4 is 1



Dijkstra's Algorithm

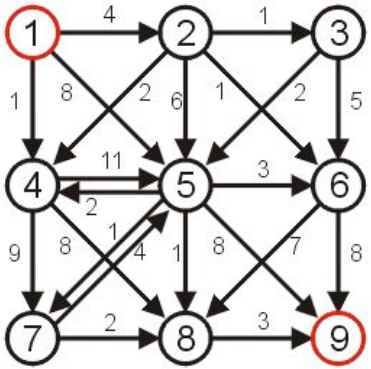
- To keep track of those vertices to which no path has reached, we can assign those vertices an initial distance of either
  - infinity (∞),
  - a number larger than any possible path, or
  - a negative number
- For demonstration purposes, we will use ∞

Dijkstra's Algorithm

- As well as finding the length of the shortest path, we'd like to find the corresponding shortest path
- Each time we update the shortest distance to a particular vertex, we will keep track of the predecessor used to reach this vertex on the shortest path.

Dijkstra's Algorithm

- We will store a table of pointers, each initially 0
- This table will be updated each time a distance is updated



1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0

Dijkstra's Algorithm

- Graphically, we will display the reference to the preceding vertex by a red arrow
  - if the distance to a vertex is ∞, there will be no preceding vertex
  - otherwise, there will be exactly one preceding vertex

Dijkstra's Algorithm

- Thus, for our initialization:
  - we set the current distance to the initial vertex as 0
  - for all other vertices, we set the current distance to ∞
  - all vertices are initially marked as unvisited
  - set the previous pointer for all vertices to null

Dijkstra's Algorithm

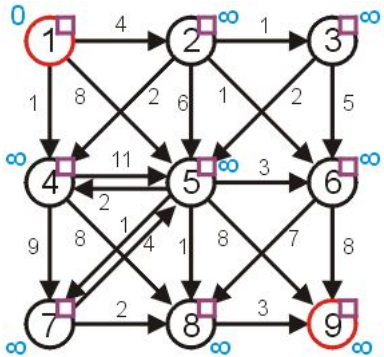
- Thus, we iterate:
  - find an unvisited vertex which has the shortest distance to it
  - mark it as visited
  - for each unvisited vertex which is adjacent to the current vertex:
    - add the distance to the current vertex to the weight of the connecting edge
    - if this is less than the current distance to that vertex, update the distance and set the parent vertex of the adjacent vertex to be the current vertex

Dijkstra's Algorithm

- Halting condition:
  - we successfully halt when the vertex we are visiting is the target vertex
  - if at some point, all remaining unvisited vertices have distance ∞, then no path from the starting vertex to the end vertex exits
- Note: We do not halt just because we have updated the distance to the end vertex, we have to **visit** the target vertex.

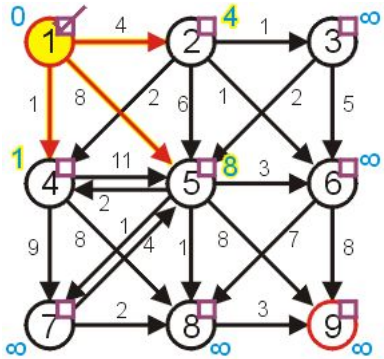
Example

- Consider the graph:
  - the distances are appropriately initialized
  - all vertices are marked as being unvisited



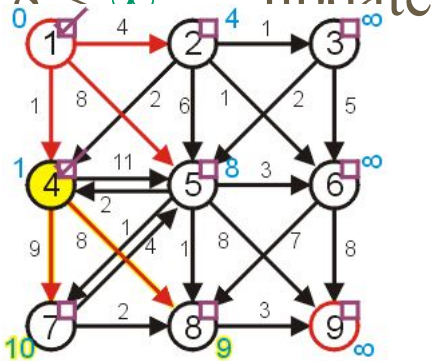
Example

- Visit vertex 1 and update its neighbours, marking it as visited
  - the shortest paths to 2, 4, and 5 are updated



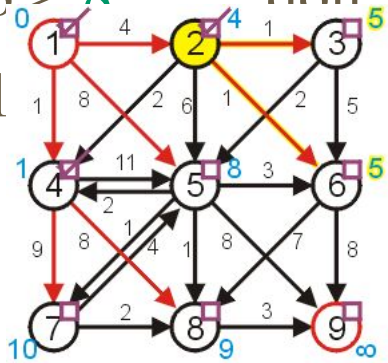
Example

- The next vertex we visit is vertex 4
  - vertex 5 $1 + 11 \geq 8$ don't update
  - vertex 7 $1 + 9 < \infty$ update
  - vertex 8 $1 + 8 < \infty$ update



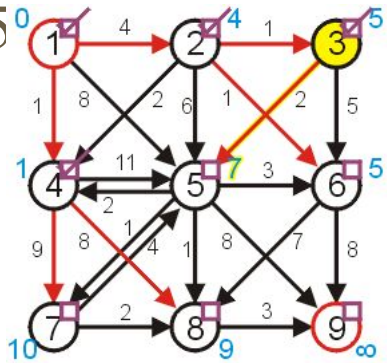
Example

- Next, visit vertex 2
  - vertex 3 $4 + 1 < \infty$ update
  - vertex 4 already visited
  - vertex 5 $4 + 6 > 8$ don't update
  - vertex 6 $4 + 1$ te



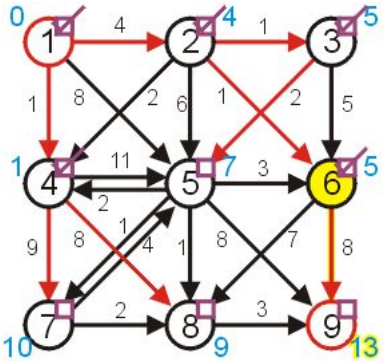
Example

- Next, we have a choice of either 3 or 6
- We will choose to visit 3
  - vertex 5 $5 + 2 < 8$ update
  - vertex 6 $5 + 5$ t update



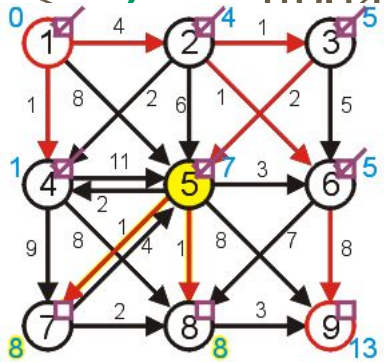
Example

- We then visit 6
  - vertex 8 $5 + 7 \geq 9$ don't update
  - vertex 9 $5 + 8 < \infty$ update



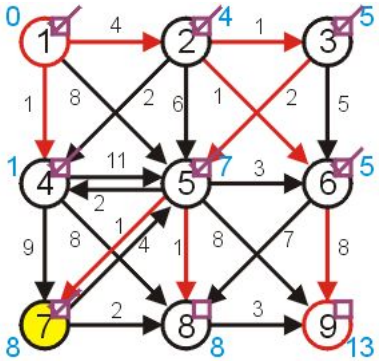
Example

- Next, we finally visit vertex 5:
  - vertices 4 and 6 have already been visited
  - vertex 7 $7 + 1 < 10$ update
  - vertex 8 $7 + 1 < 9$ update
  - vertex 9 $7 + 8$ t update



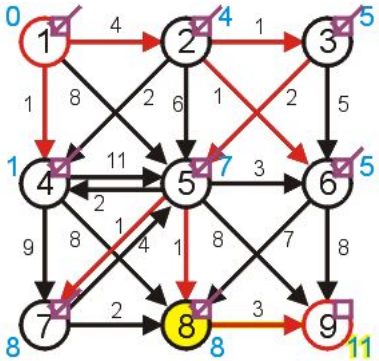
Example

- Given a choice between vertices 7 and 8, we choose vertex 7
  - vertices 5 has already been visited
  - vertex 8 $8 + 2 \geq 8$ don't update



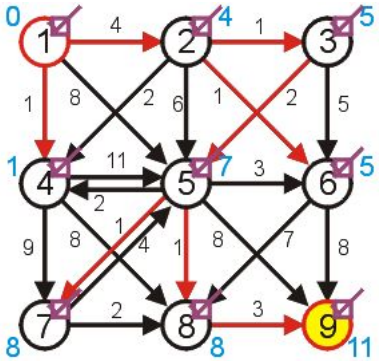
Example

- Next, we visit vertex 8:
  - vertex 9 8 + 3 < 13 update



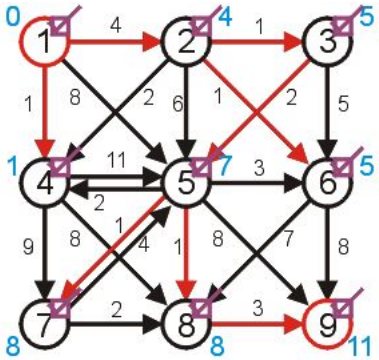
Example

- Finally, we visit the end vertex
- Therefore, the shortest path from 1 to 9 has length 11



Example

- We can find the shortest path by working back from the final vertex:
  - 9, 8, 5, 3, 2, 1
- Thus, the shortest path is (1, 2, 3, 5, 8, 9)

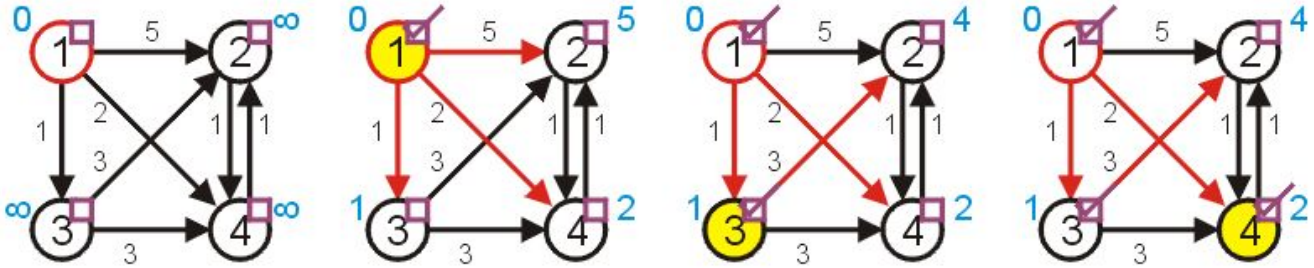


Example

- In the example, we visited all vertices in the graph before we finished
- This is not always the case, consider the next example

Example

- Find the shortest path from 1 to 4:
  - the shortest path is found after only three vertices are visited
  - we terminated the algorithm as soon as we reached vertex 4
  - we only have useful information about 1, 3, 4



Dijkstra's algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$
$S \leftarrow \varnothing$
$Q \leftarrow V$
while $Q \neq \varnothing$ Q is a priority queue maintaining $V - S$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$
          $p[v] \leftarrow u$

Dijkstra's algorithm

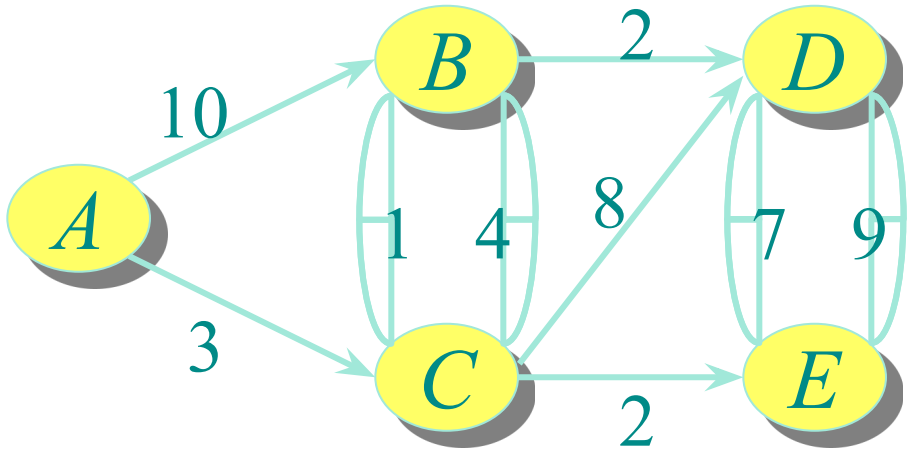
$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$
$S \leftarrow \varnothing$
$Q \leftarrow V$
while $Q \neq \varnothing$ Q is a priority queue maintaining $V - S$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$
          $p[v] \leftarrow u$

relaxation step

Implicit DECREASE-KEY

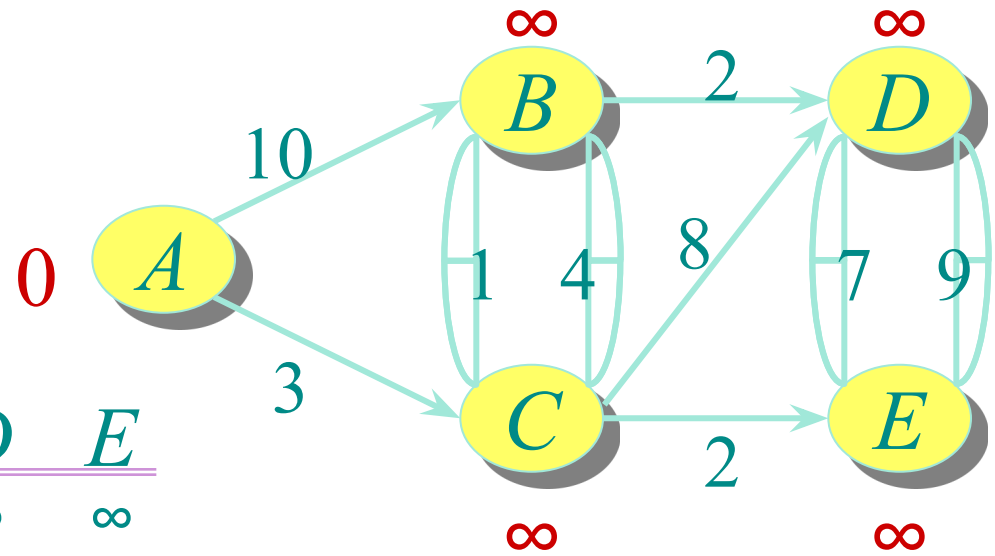
Example of Dijkstra's algorithm

Graph with nonnegative edge weights:



Example of Dijkstra's algorithm

Initialize:

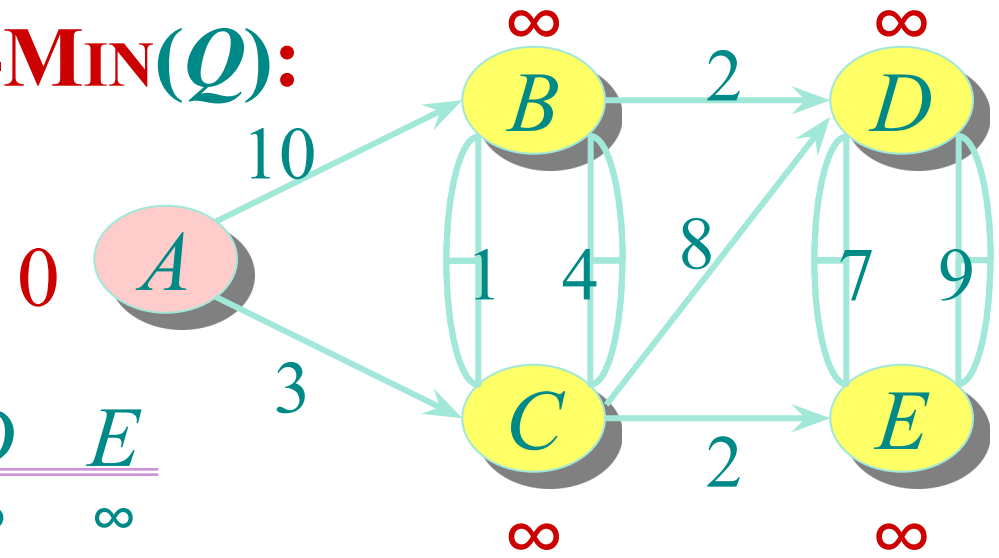


Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: {}

Example of Dijkstra's algorithm

"A" ← EXTRACT-MIN(Q):



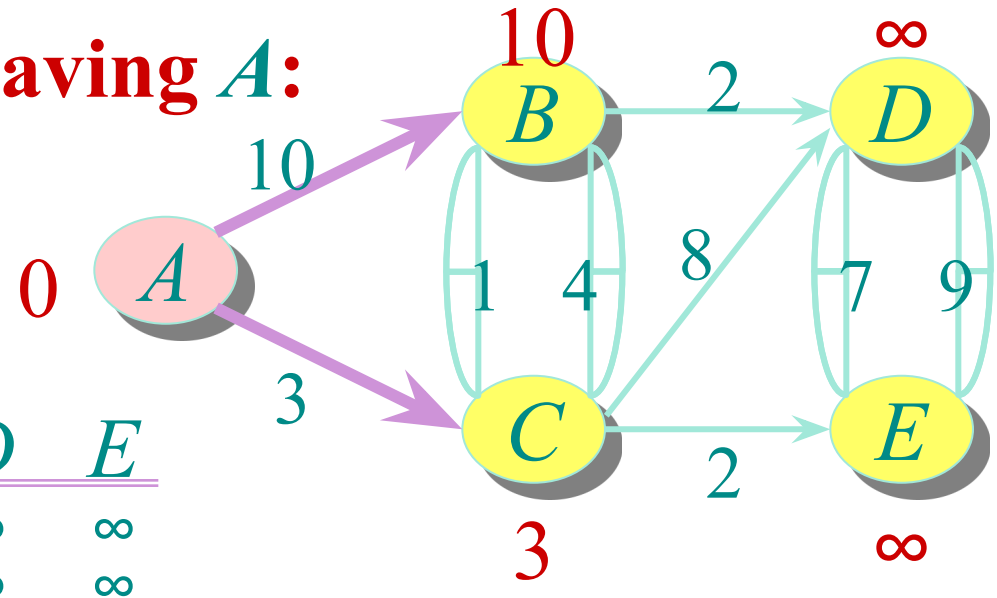
Q:

A	B	C	D	E
0	∞	∞	∞	∞

S: { A }

Example of Dijkstra's algorithm

Relax all edges leaving A:

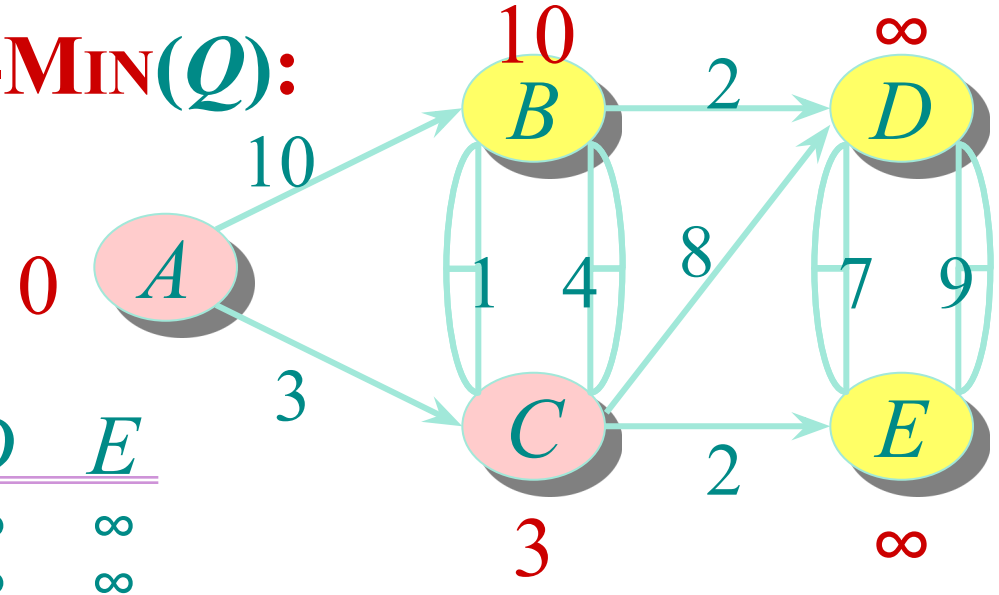


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A }

Example of Dijkstra's algorithm

"C" ← EXTRACT-MIN(Q):



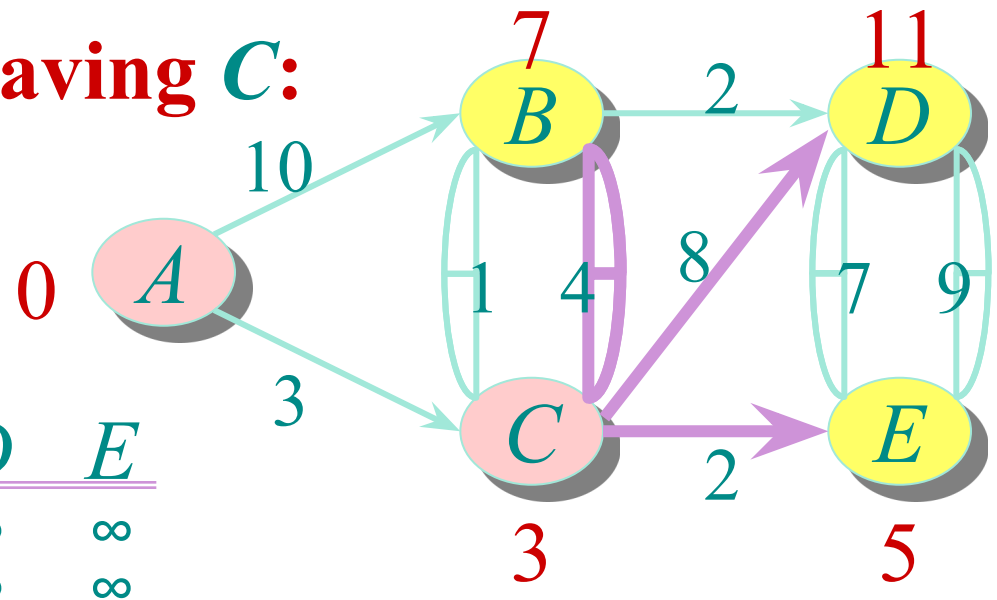
Q:

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞

S: { A, C }

Example of Dijkstra's algorithm

Relax all edges leaving *C*:

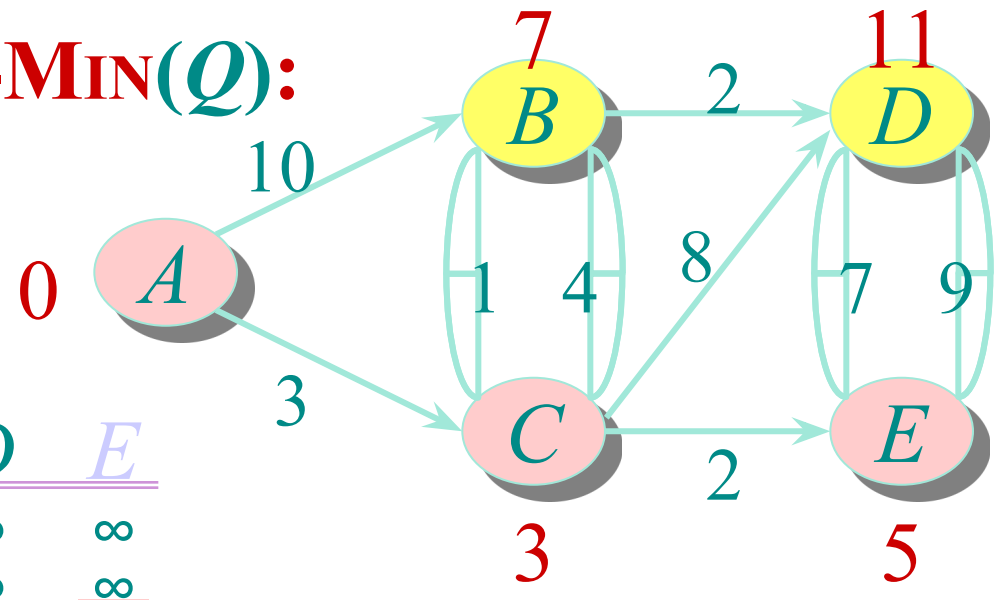


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { *A*, *C* }

Example of Dijkstra's algorithm

"E" ← EXTRACT-MIN(Q):



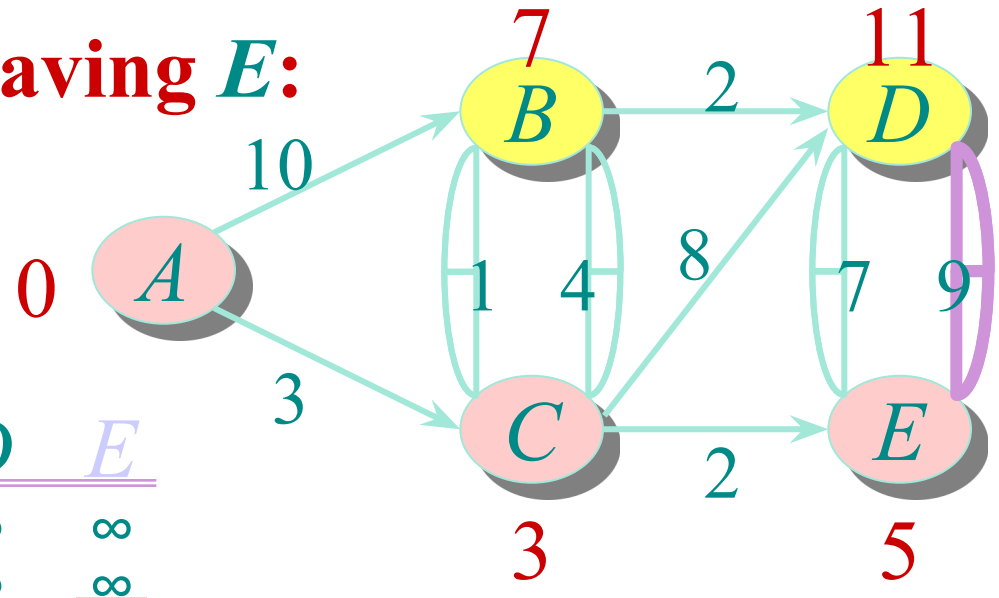
Q:

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5

S: { A, C, E }

Example of Dijkstra's algorithm

Relax all edges leaving *E*:

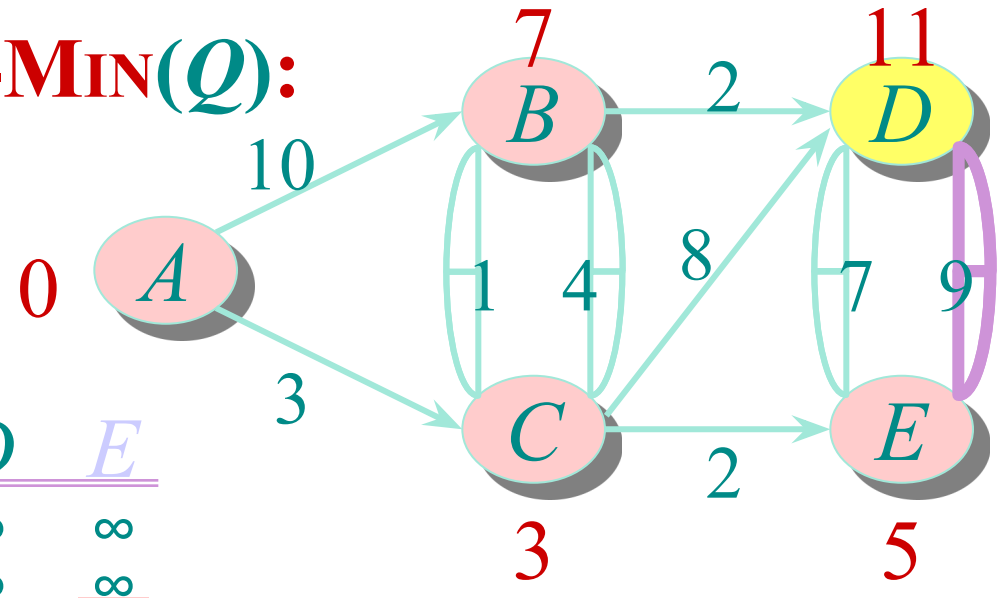


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E* }

Example of Dijkstra's algorithm

"*B*" ← EXTRACT-MIN(*Q*):



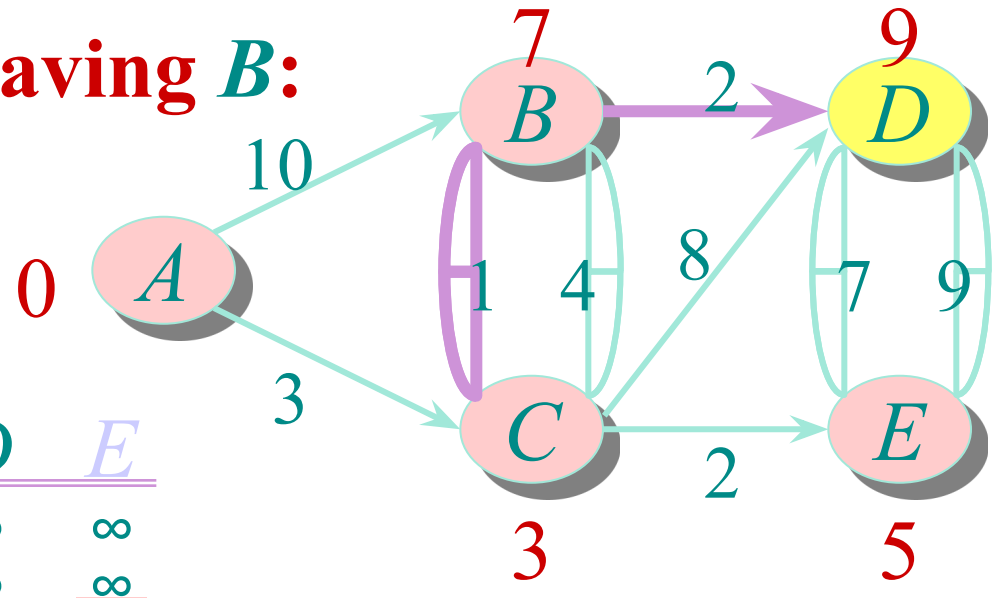
Q:

A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	

S: { *A*, *C*, *E*, *B* }

Example of Dijkstra's algorithm

Relax all edges leaving *B*:

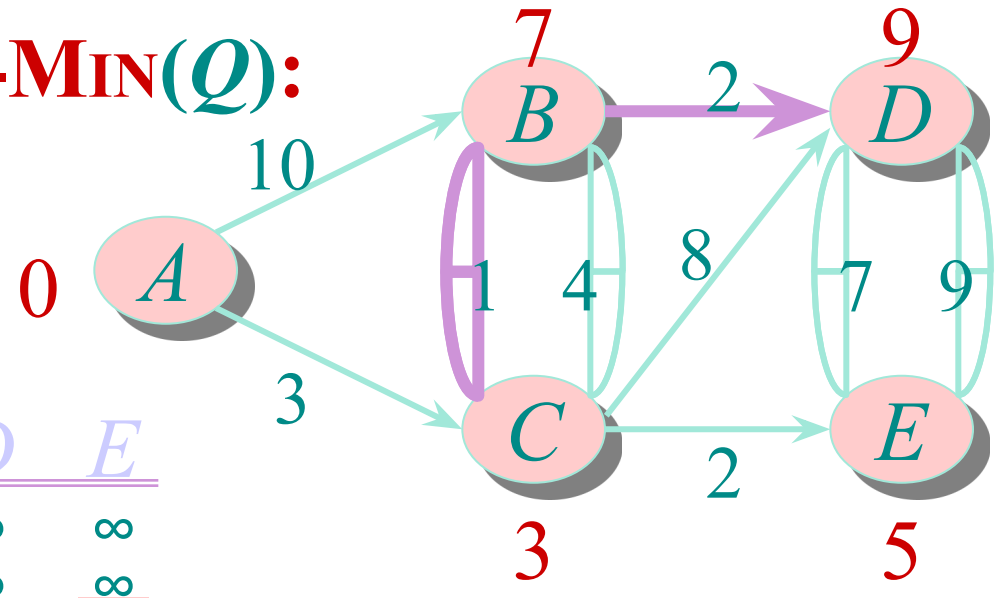


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { *A*, *C*, *E*, *B* }

Example of Dijkstra's algorithm

"*D*" ← EXTRACT-MIN(*Q*):



Q:

A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	

S: { *A*, *C*, *E*, *B*, *D* }

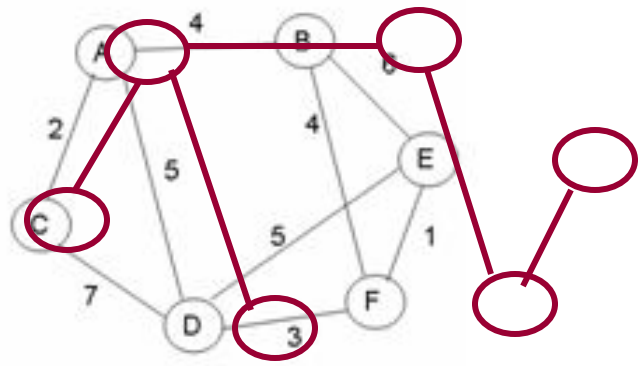
Summary

- Given a weighted directed graph, we can find the shortest distance between two vertices by:
  - starting with a trivial path containing the initial vertex
  - growing this path by always going to the next vertex which has the shortest current path

Practice

Node	Included	Distance	Path
A	t	-	-
B	f t	4	A
C	f t	2	A
D	f t	5	A
E	f t	~~∞~~ ~~10~~ 9	~~-~~ ~~B~~ F
F	f t	~~∞~~ 8	~~-~~ B

Give the shortest path tree for node A for this graph using Dijkstra's shortest path algorithm. Show your work with the 3 arrays given and draw the resultant shortest path tree with edge weights included.



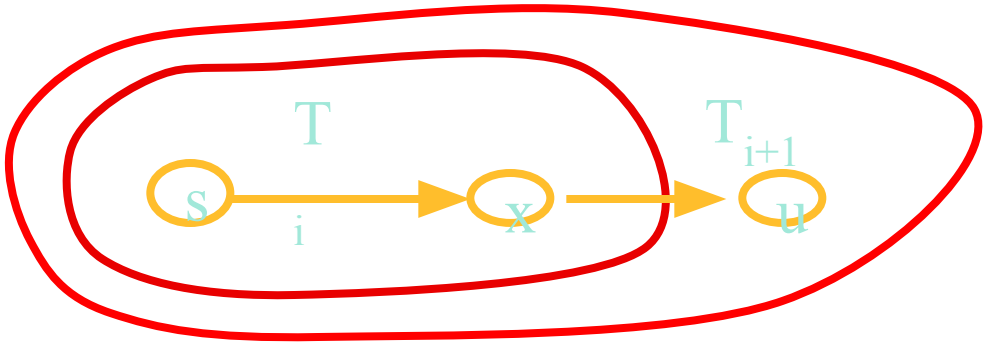
Correctness of Dijkstra's Alg.

- Let T_i be the tree constructed after i-th iteration of the while loop:
  - The nodes in T_i are not in Q
  - The edges in T_i are indicated by parent variables
- Show by induction on i that the path in T_i from s to u is a shortest path and has distance d[u], for all u in T_i.
- **Basis:** $i = 1$.

  s is the only node in T_1 and $d[s] = 0$.

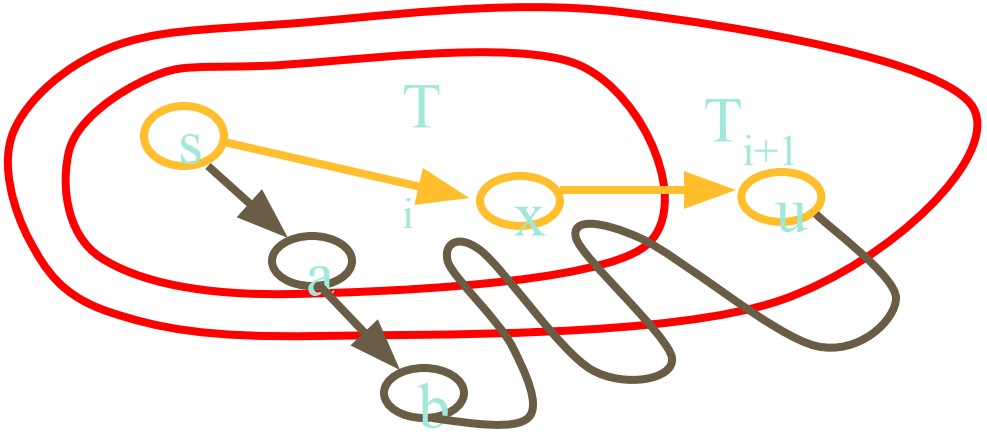
Correctness of Dijkstra's Alg.

- **Induction:** Assume Ti is a correct shortest path tree and show for Ti.
- Let u be the node added in iteration i.
- Let x = parent(u).



Need to show path in T_{i+1} from s to u is a shortest path, and has distance d[u]

Correctness of Dijkstra's Alg

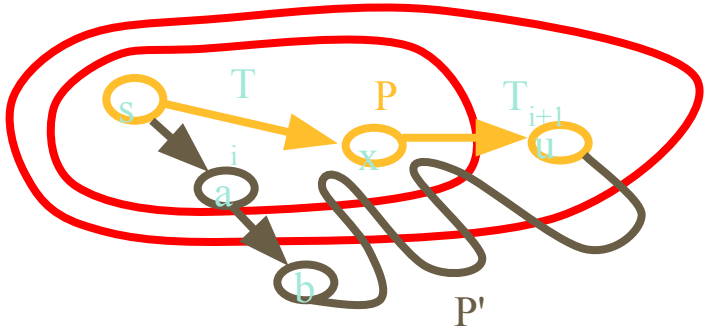


P, path in T_{i+1} from s to u

P', another path from s to u

(a,b) is first edge in P' that leaves T_i

Correctness of Dijkstra's Alg



Let P1 be part of P' before (a,b.)

Let P2 be part of P' after (a,b).

w(P') = w(P1) + w(a,b) + w(P2)

≥ w(P1) + w(a,b) (nonneg wts)

≥ wt of path in T_i from s to a + w(a,b) (inductive hypothesis)

≥ w(s->x path in T_i) + w(x,u) (alg chose u in iteration i and d-values are accurate, by inductive hypothesis

= w(P).

So P is a shortest path, and d[u] is accurate after iteration i+1

Running Time of Dijstra's Alg.

- initialization: insert each node once
  - $O(V\ T_{ins})$
- $O(V)$ iterations of while loop
  - one extract-min per iteration => $O(V\ T_{ex})$
  - for loop inside while loop has variable number of iterations…
- For loop has $O(E)$ iterations total
  - one decrease-key per iteration => $O(E\ T_{dec})$
- Total is $O(V\ (T_{ins} + T_{ex}) + E\ T_{dec})$

Running Time using Binary Heaps and Fibonacci Heaps

- $O(V(T_{ins} + T_{ex}) + E \cdot T_{dec})$
- If priority queue is implemented with a binary heap, then
  - $T_{ins} = T_{ex} = T_{dec} = O(\log V)$
  - total time is $O(E \log V)$
- There are fancier implementations of the priority queue, such as Fibonacci heap:
  - $T_{ins} = O(1)$, $T_{ex} = O(\log V)$, $T_{dec} = O(1)$ (amortized)
  - total time is $O(V \log V + E)$

Using Simpler Heap Implementations

- $O(V(T_{ins} + T_{ex}) + E \cdot T_{dec})$
- If graph is dense, so that $|E| = \Theta(V^2)$, then it doesn't help to make T_{ins} and T_{ex} to be at most $O(V)$.
- Instead, focus on making T_{dec} be small, say constant.
- Implement priority queue with an unsorted array:
  - $T_{ins} = O(1)$, $T_{ex} = O(V)$, $T_{dec} = O(1)$
  - total is $O(V^2)$

Bellman-Ford Algorithm

Basic Idea

- Consider each edge (u,v) and see if u offers v a cheaper path from s
  - compare d[v] to d[u] + w(u,v)
- Repeat this process |V| - 1 times to ensure that accurate information propagates from s, no matter what order the edges are considered in
- It is a dynamic programming approach.
  - It builds from the bottom, and goes up.
  - It first checks shortest paths with only 1 edge, then shortest paths with 2 edges, next shortest paths with 3 edges, and so on. In between it relaxes the edges and path whenever a better (shorter distance) path is found.
  - It is a label correcting algorithm. So, the distances to any of the nodes are finalized only after completing the entire loop, not in the middle. [Note- Dijkstra is a label setting algorithm which may fix the distance of a node in the middle]

Bellman-Ford SSSP Algorithm

- input: directed or undirected graph G = (V,E,w)

//initialization

- initialize d[v] to infinity and parent[v] to nil for all v in V other than the source
- initialize d[s] to 0 and parent[s] to s

// main body

- for i := 1 to |V| - 1 do
  - for each (u,v) in E do // consider in arbitrary order
    - Relax(u,v, w(u,v))

// cycle detection / correctness check

- for each edge (u,v) ∈ E
  - if (d[v] > d[u] + w(u,v))
    - return "no solution";

Bellman-Ford Algorithm

```
BellmanFord(G, s)
for each v ∈ V
   d[v] = ∞;
   Parent[v] = NA;
d[s] = 0;
for each (u,v) in E do// lexicography order
   if d[u] + w(u,v) < d[v] then // Relaxation
       d[v] := d[u] + w(u,v)
       parent[v] := u
for each edge (u,v) ∈ E
   if (d[v] > d[u] + w(u,v))
      return "no solution";
```

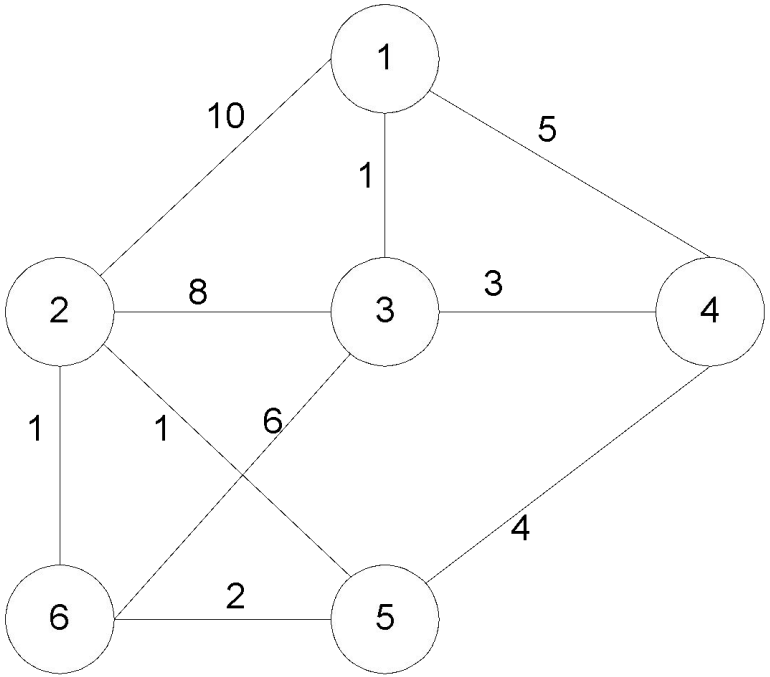
Initialize d[] which will converge to shortest-path value

Relaxation: Make |V|-1 passes, relaxing each edge

Test for solution: have we converged yet? Ie, ∃ negative cycle?

Example

- Represent it in adjacency cost matrix
- Then, run Bellman-Ford algorithm



Detecting Negative Weight Cycles

Just run the loop one more time, i.e., instead of V-1, run it for V times.

- for each (u,v) in E do
  - if d[u] + w(u,v) < d[v] then
    - output "negative weight cycle exists"

If you observe that any of the nodes is still relaxed then the input graph contains at least one negative weight cycle. Then, discard the results of the algorithm.

Running Time of Bellman-Ford

- $O(V)$ iterations of outer for loop
- $O(E)$ iterations of inner for loop
- $O(VE)$ time total

Space complexity $O(1) + O(V) = O(V)$

Correctness of Bellman-Ford

- Suppose there is a negative weight cycle.
- Then the distance will decrease even after iteration |V| - 1
  - shortest path distance is negative infinity
- This is what the last part of the code checks for.

Correctness of Bellman-Ford

Assume no negative-weight cycles.

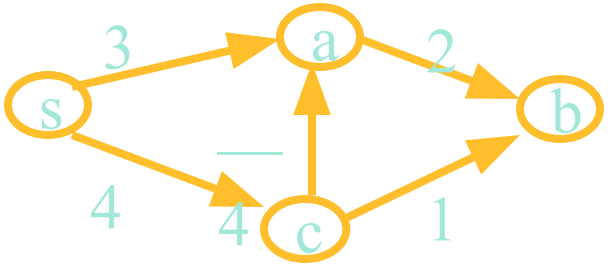
Lemma: d[v] is never an underestimate of the actual shortest path distance from s to v.

Lemma: If there is a shortest s-to-v path containing at most i edges, then after iteration i of the outer for loop, d[v] is at most the actual shortest path distance from s to v.

Theorem: Bellman-Ford is correct.

This follows from the two lemmas and the fact that every shortest path has at most |V| - 1 edges.

Bellman-Ford Example



process edges in order
(c,b)
(a,b)
(c,a)
(s,a)
(s,c)

<board work>

The Boost Graph Library

The BGL contains generic implementations of all the graph algorithms that we have discussed:

- Breadth-First-Search
- Depth-First-Search
- Kruskal's MST algorithm
- Prim's MST algorithm
- Strongly Connected Components
- Dijkstra's SSSP algorithm
- Bellman-Ford SSSP algorithm

I recommend that you gain experience with this useful library.
Recommended reading: The Boost Graph Library by J.G. Siek, L.-Q. Lee, and A. Lumsdaine, Addison-Wesley, 2002.

Shortest Path in Directed Acyclic Graph (DAG)

DAG Shortest Paths

- Bellman-Ford takes O(VE) time.
- For finding shortest paths in a DAG, we can do much better by using a topological sort.
- If we process vertices in topological order, we are guaranteed to never relax a vertex unless the adjacent edge is already finalized. Thus: just one pass. O(V+E)

DAG-Shortest-Paths(G, w, s)

1. ***topologically sort the vertices of G***
2. **INITIALIZE-SINGLE-SOURCE(G, s)**
3. **for *each vertex u, taken in topologically sorted order***

DAG Shortest Paths

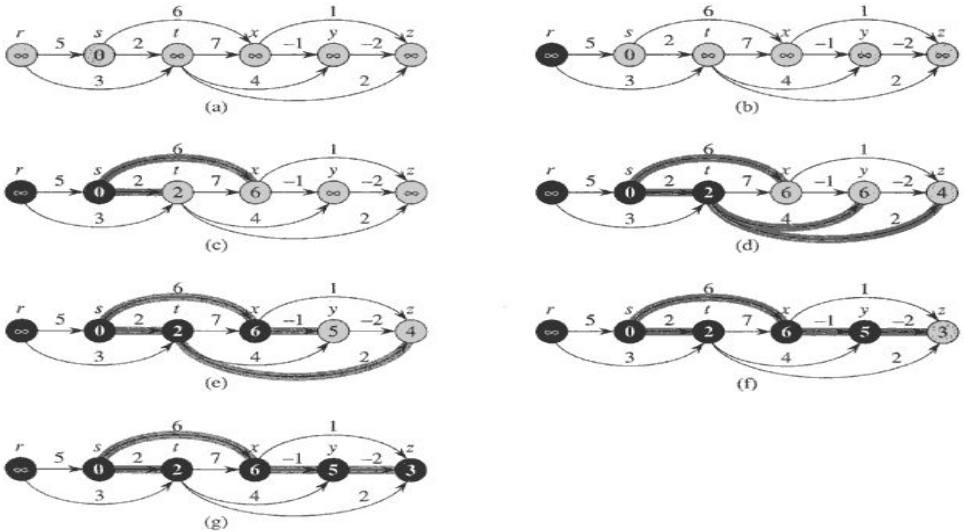


Figure 24.5 The execution of the algorithm for shortest paths in a directed acyclic graph. The vertices are topologically sorted from left to right. The source vertex is s. The d values are shown within the vertices, and shaded edges indicate the π values. **(a)** The situation before the first iteration of the **for** loop of lines 3–5. **(b)–(g)** The situation after each iteration of the **for** loop of lines 3–5. The newly blackened vertex in each iteration was used as u in that iteration. The values shown in part (g) are the final values.

```
private IEnumerable<int> TraverseComponent( int startingNode )
    {
        activeList.Put(startingNode);
        while (activeList.Count > 0)
        {
            int currentNode = activeList.GetNext();
            if (!visited[currentNode])
            {
                visited[currentNode] = true;
                if (this.TraversalOrder == Graph.TraversalOrder.PreOrder)
                    yield return currentNode;
                foreach (int node in indexedGraph.Neighbors(currentNode))
                {
                    if (!visited[node])
                    {
                        activeList.Put(node);
                    }
                }
                if (this.TraversalOrder == Graph.TraversalOrder.PostOrder)
```