Spanning Trees on Graph

Unit-4 Lecture-15

Dr. Dillip Rout, Assistant Professor, Dept. of Computer Science and Engineering

Outline

- Fundamentals of Spanning Tree
  - Breadth First Search Tree
  - Depth First Search Tree
- Spanning Tree
- Minimum Spanning Tree
- Homework
- Conclusions

Spanning Tree

- Given (connected) graph G(V,E),
- a spanning tree T(V',E'):
  - Is a subgraph of G; that is, V' ⊆ V, E' ⊆ E.
  - Spans the graph (V' = V)
  - Forms a tree (no cycle);
  - So, E' has |V| -1 edges

I. All the nodes of G are found in T
II. Some or all the edges of G are present in T such that there is no cycle
III. All the nodes of G are connected in T

Minimum Spanning Trees (MST)

Edges of G are weighted:

How much Spanning Trees T can be generated out of G? Is it just one or multiple?

Find minimum cost spanning tree

Applications

- Find cheapest way to wire your house
- Find minimum cost to send a message on the Internet

Strategy for MST

For any spanning tree T, inserting an edge e_new not in T creates a cycle

But

Removing any edge e_old from the cycle gives back a spanning tree

If e_new has a lower cost than e_old we have progressed!

Strategy for construction:

1. Add an edge of minimum cost that does not create a cycle (greedy algorithm)
2. Repeat |V| -1 times
3. Correct since if we could replace an edge with one of lower cost, the algorithm would have picked it up

Two Algorithms

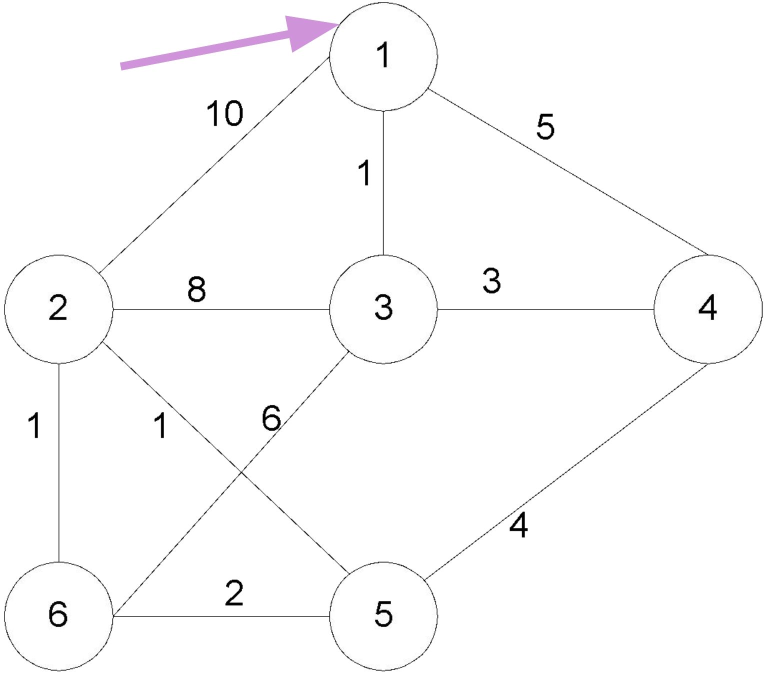
- Prim: (build tree incrementally)
    - Pick lower cost edge connected to known (incomplete) spanning tree that does not create a cycle and expand to include it in the tree
- Kruskal: (build forest that will finish as a tree)
    - Pick lowest cost edge not yet in a tree that does not create a cycle. Then expand the set of included edges to include it. (It will be somewhere in the forest.)

Prim’s Algorithm

Starting from empty T, choose a vertex at random and initialize

V = {1), E’ ={}

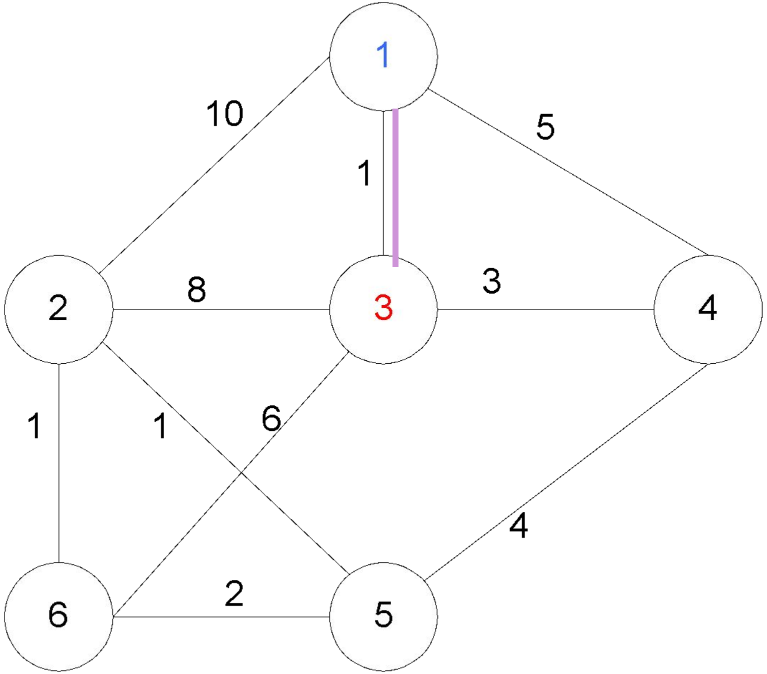
Is choosing of starting vertex affects the MST?



Prim's Algorithm

Choose the vertex u not in V such that edge weight from u to a vertex in V is minimal (greedy!)

V={1,3} E'= {(1,3) }



Prim’s Algorithm

Repeat until all vertices have been chosen

Choose the vertex u not in V such that edge weight from v to a vertex in V is minimal (greedy!)

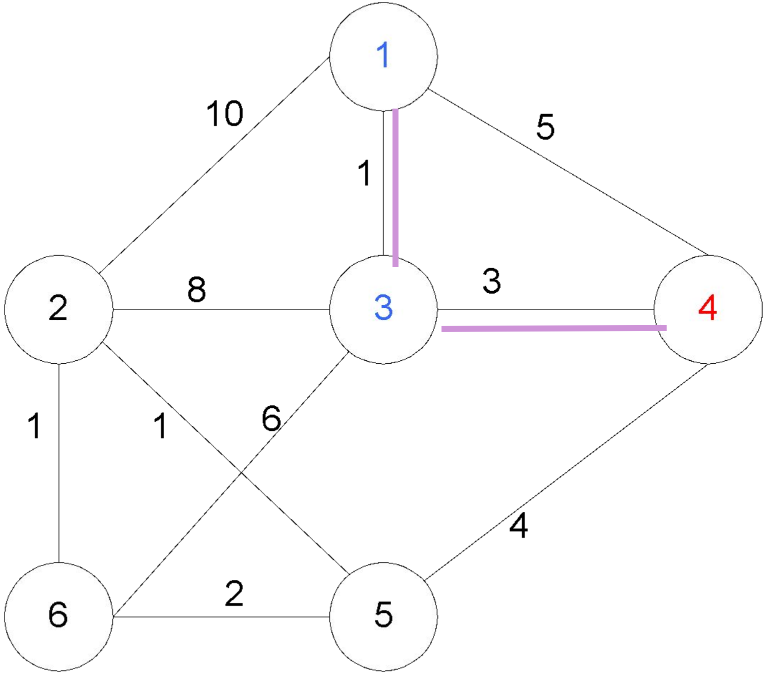
V= {1,3,4} E’= {(1,3),(3,4)}

V={1,3,4,5} E’={(1,3),(3,4),(4,5)}

….

V={1,3,4,5,2,6}

E’={(1,3),(3,4),(4,5),(5,2),(2,6)}



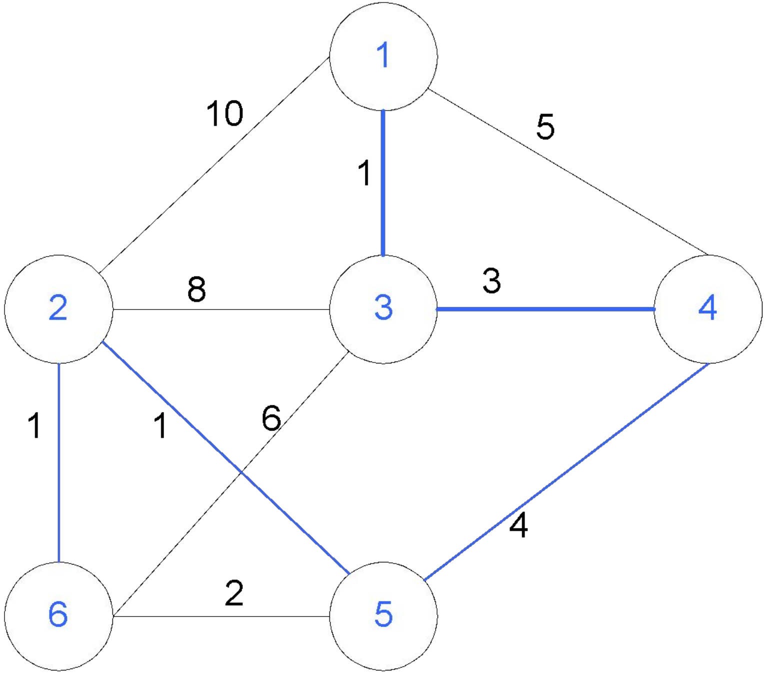
Prim's Algorithm

Repeat until all vertices have been chosen

V={1,3,4,5,2,6}

E'={(1,3),(3,4),(4,5),(5,2),(2,6)}

Final Cost: 1 + 3 + 4 + 1 + 1 = 10



Prim’s Algorithm Implementation

- Assume adjacency list representation

Initialize connection cost of each node to “inf” and “unmark” them

Choose one node, say v and set cost[v] = 0 and prev[v] =0

While they are unmarked nodes

Select the unmarked node **u** with minimum cost; mark it

For each unmarked node **w** adjacent to **u**

if cost(u,w) < cost(w) then cost(w) := cost (u,w)

prev[w] = u

Prim's algorithm Analysis

- If the "Select the unmarked node u with minimum cost" is done with binary heap then O((n+m)logn)

Kruskal’s Algorithm

- Select edges in order of increasing cost
- Accept an edge to expand tree or forest only if it does not cause a cycle
- Implementation using adjacency list, priority queues and disjoint sets

Kruskal's Algorithm

Initialize a forest of trees, each tree being a single node

Build a priority queue of edges with priority being lowest cost

Repeat until |V| -1 edges have been accepted {

    Deletemin edge from priority queue

    If it forms a cycle then discard it

    else accept the edge – It will join 2 existing trees yielding a larger tree and reducing the forest by one tree

}

The accepted edges form the minimum spanning tree

Detecting Cycles

- If the edge to be added (u,v) is such that vertices u and v belong to the same tree, then by adding (u,v) you would form a cycle
  - › Therefore to check, Find(u) and Find(v). If they are the same discard (u,v)
  - › If they are different Union(Find(u),Find(v))

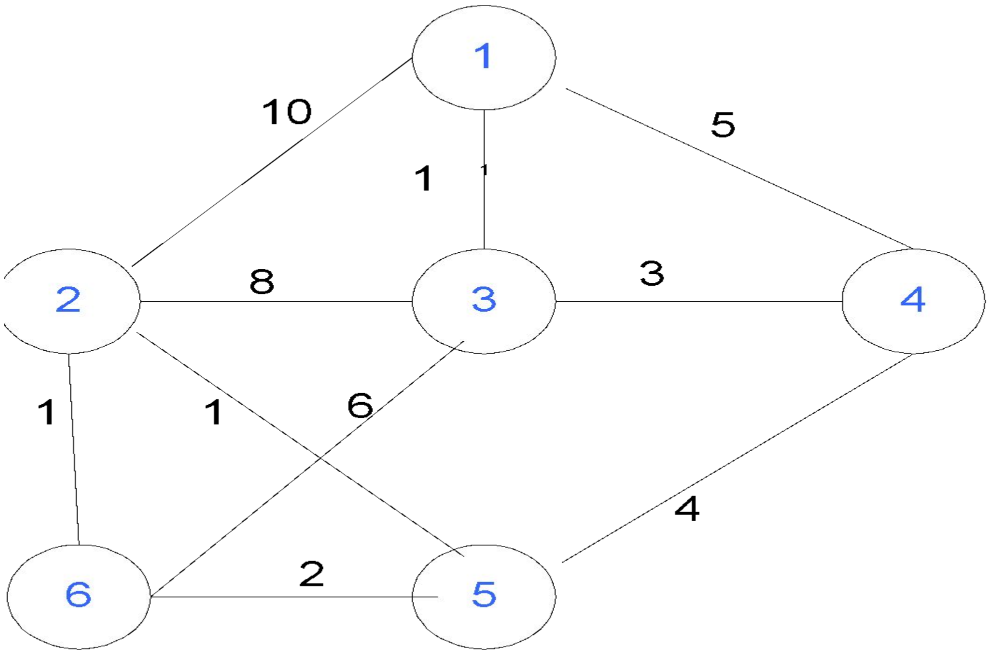
Properties of trees in K's algorithm

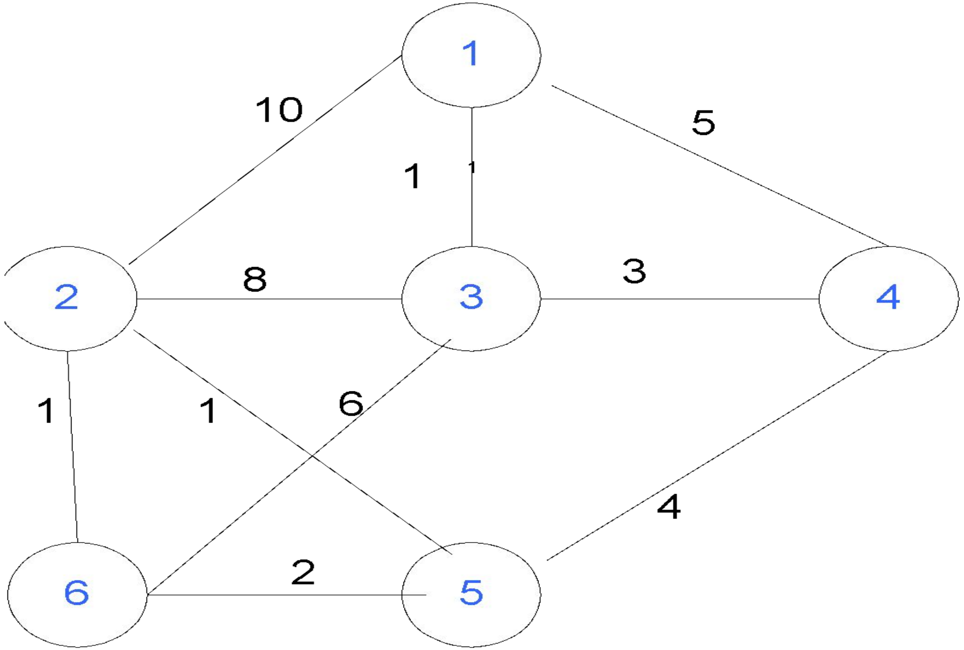
- Vertices in different trees are disjoint
  - True at initialization and Union won't modify the fact for remaining trees
- Trees form equivalent classes under the relation "is connected to"
  - u connected to u (reflexivity)
  - u connected to v implies v connected to u (symmetry)
  - u connected to v and v connected to w implies a path from u to w so u connected to w (transitivity)

K’s Algorithm Data Structures

- Adjacency list for the graph
  - To perform the initialization of the data structures below
- Disjoint Set ADT’s for the trees (recall Up tree implementation of Union-Find)
- Binary heap for edges

Example



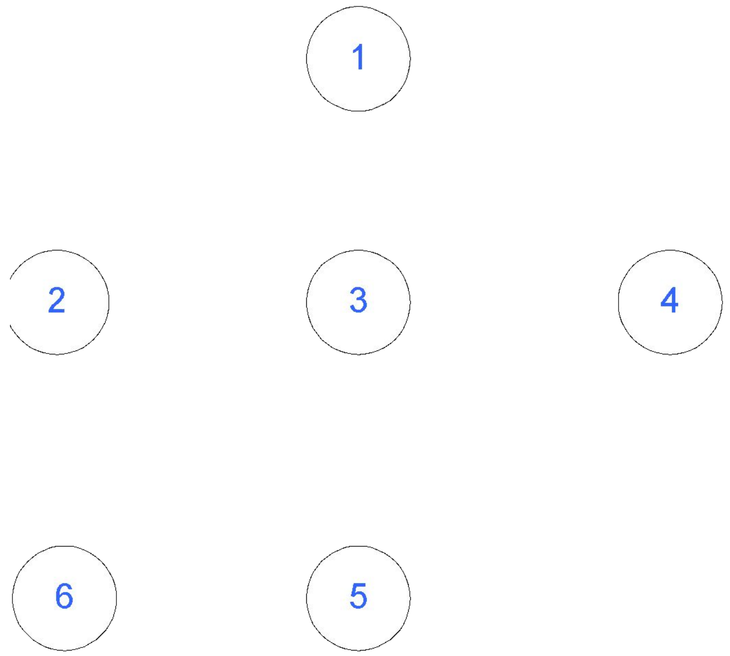

1
10
5
1
1
2
8
3
3
4
1
1
6
4
6
2
5

Initialization

Initially, Forest of 6 trees

F= {{1},{2},{3},{4},{5},{6}}

Edges in a heap (not shown)



Step 1

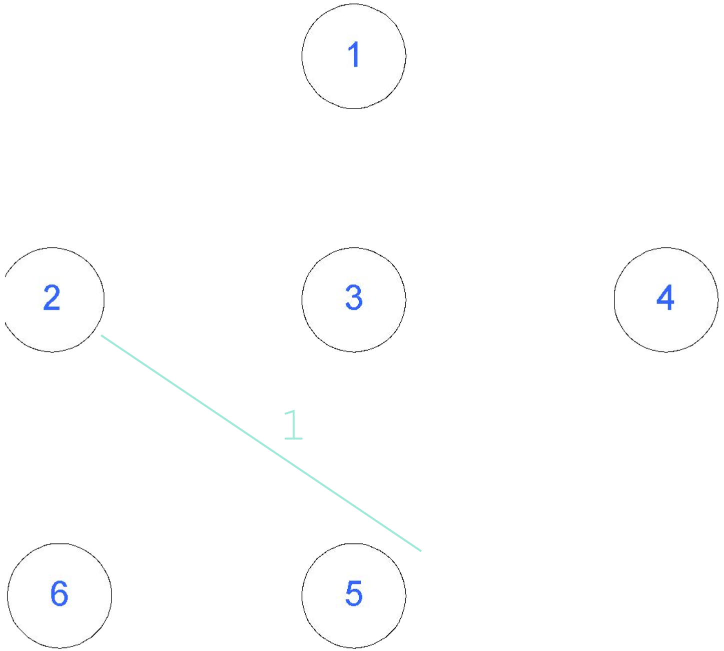
Select edge with lowest cost (2,5)

Find(2) = 2, Find (5) = 5

Union(2,5)

F= {{1},{2,5},{3},{4},{6}}

1 edge accepted



Step 2

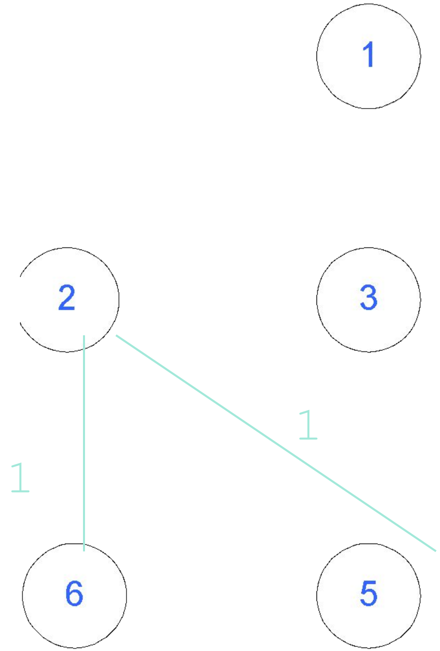
Select edge with lowest cost (2,6)

Find(2) = 2, Find (6) = 6

Union(2,6)

F= {{1},{2,5,6},{3},{4}}

2 edges accepted



Step 3

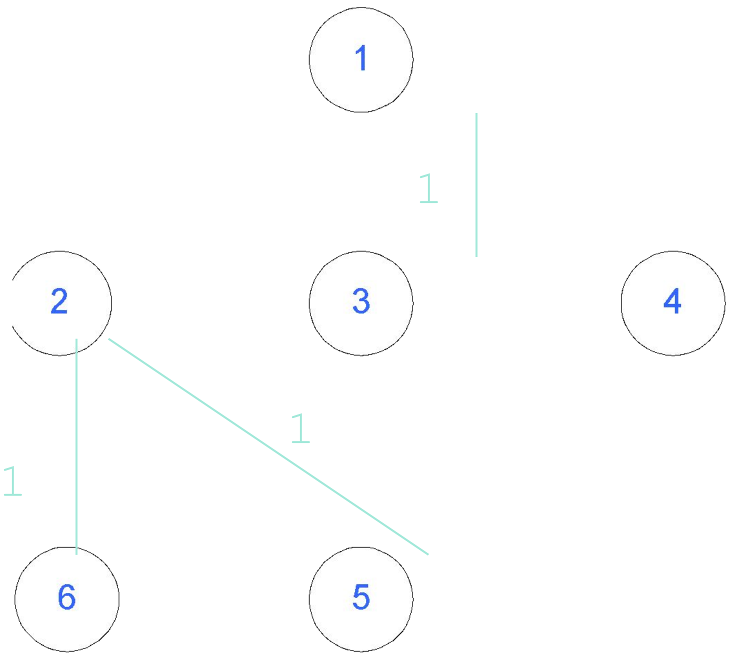
Select edge with lowest cost (1,3)

Find(1) = 1, Find (3) = 3

Union(1,3)

F= {{1,3},{2,5,6},{4}}

3 edges accepted



Step 4

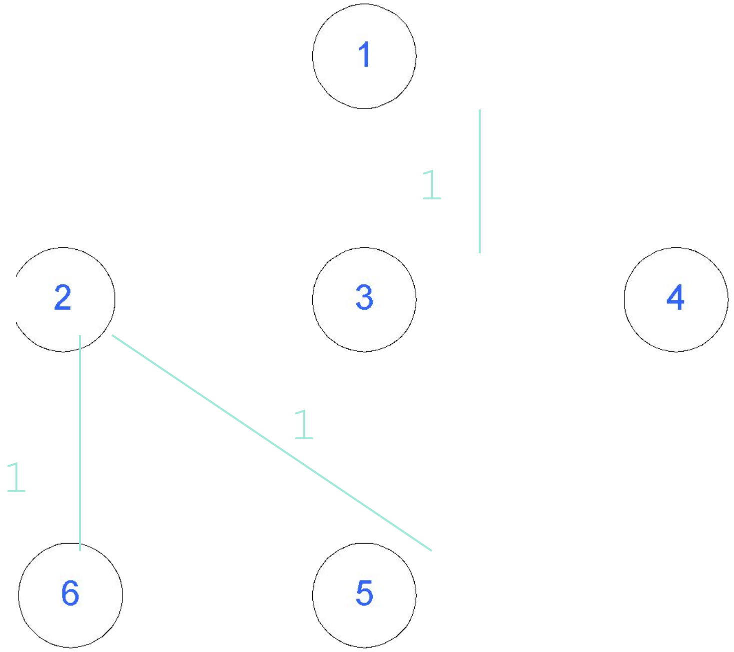
Select edge with lowest cost (5,6)

Find(5) = 2, Find (6) = 2

Do nothing

F= {{1,3},{2,5,6},{4}}

3 edges accepted



Step 5

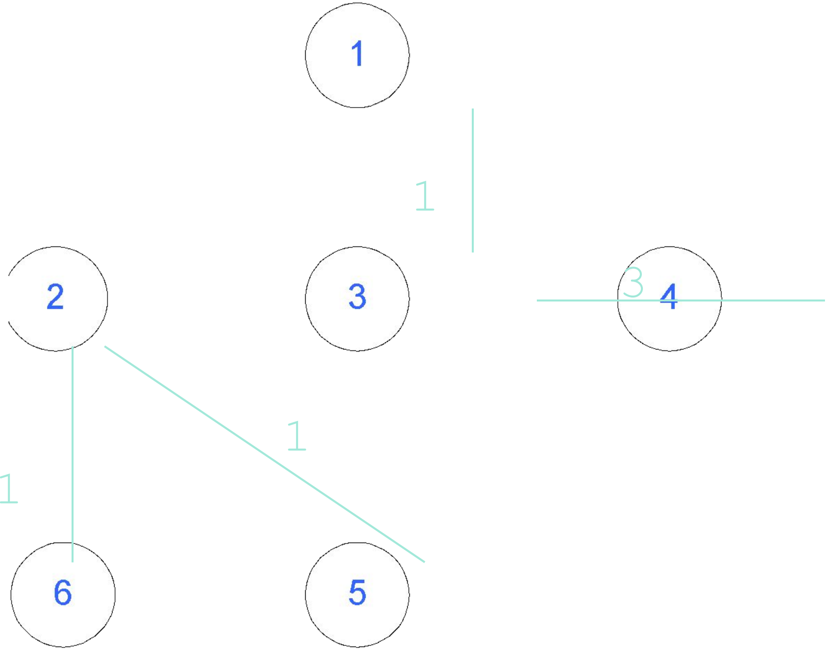
Select edge with lowest cost (3,4)

Find(3) = 1, Find (4) = 4

Union(1,4)

F= {{1,3,4},{2,5,6}}

4 edges accepted



Step 6

Select edge with lowest cost (4,5)

Find(4) = 1, Find (5) = 2

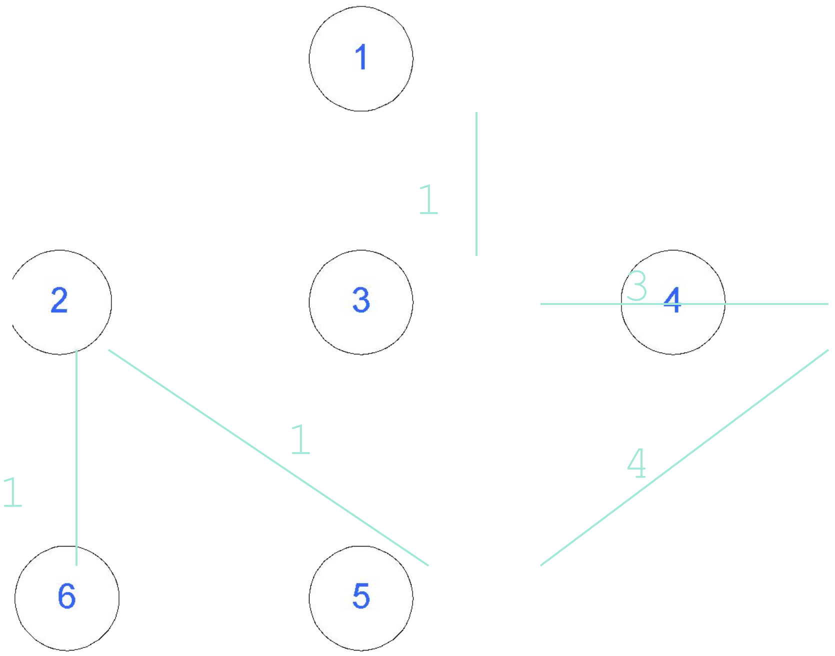
Union(1,2)

F= {{1,3,4,2,5,6}}

5 edges accepted : end

Total cost = 10

Although there is a unique spanning tree in this example, this is not



Kruskal's Algorithm Analysis

- Initialize forest $O(n)$
- Initialize heap $O(m)$, $m = |E|$
- Loop performed m times
  - In the loop one DeleteMin $O(\log m)$
  - Two Find, each $O(\log n)$
  - One Union (at most) $O(1)$
- So worst case $O(m \log m) = O(m \log n)$

Time Complexity Summary

- Recall that $m = |E| = O(V^2) = O(n^2)$
- Prim's runs in $O((n+m) \log n)$
- Kruskal runs in $O(m \log m) = O(m \log n)$
- In practice, Kruskal has a tendency to run faster since graphs might not be dense and not all edges need to be looked at in the Deletemin operations