New Indian Values of π from the Mānava Śulba Sūtra

by

R. C. GUPTA*

1. Introduction

India's oldest written works are the *Vedas*. To assist their proper study, there are six ancillary texts, called *Vedāṅgas* ("limbs of the veda"), namely *Śikṣā* (Phonetics), *Kalpa* (Ritualistics), *Vyākaraṇa* (Grammar), *Nirukta* (Etymology), *Chandas* (Prosody), and *Jyotiṣa* (Astronomy and Astrology). The *Kalpa* deals with rules and methods for performing vedic rituals, sacrifices and ceremonies, and is divided into three categories, namely *Śrauta, Grahya,* and *Dharma.* The *Śrauta Sūtras,* especially those which belong to the various *Saṁhitās* of the *Yajur Veda,* often include treatises that give rules concerning the mensuration and construction of fire-places and altars, and also deal with allied ecclesiastical matters.

These treatises are often found as separate works and are called *Śulba Sūtras*. They represent, in the coded form, the much older and traditional Indian mathematics developed for construction and transformation of vedic altars of various forms. The *Śulba Sūtras* are thus the oldest geometrical treatises which are also simply called *Śulbas*. The word *śulba* literally means a cord, rope or string and is derived from the basic root *śulb* (or *śulv*), "to mete out" or "to measure".

The names of the 10 *Śulba sūtras* are known, namely *Baudhāyana, Āpastamba, Satyāṣāḍa* (whose text is said to be identical with that of *Āpastamba*), *Kātyāyana, Mānava, Maitrāyaṇī* (which is said to be another recension of the *Mānava*), *Vārāha, Vādhūla, Maśaka,* and *Hiraṇyakeśī.* They are variously dated and their exact times of composition or compilation are controversial. The *Baudhāyana Śulba Sūtra* (= *BSS*) is the oldest of them and is generally placed between 800

*Birla Institute of Technology, P.O. Mesra, Ranchi, 835215 India.

B.C. and 500 B.C. The Āpastamba Śulba Sūtra (= *ASS*), *Kātyāyana Śulba Sūtra* (= *KSS*), and the *Mānava Śulba Sūtra* (= *MSS*) are the other important old works.[1]

Following A. J. E. M. Smeur[2] I shall distinguish between π and π', defined respectively by

$$\frac{\text{area of circle}}{\text{square on the radius}} = \pi$$

and

$$\frac{\text{circumference of a circle}}{\text{diameter}} = \pi'.$$

2. *Some Rules Related to Circle and Square*

For transforming a square into a circle of equal area (approximately), the *BSS* 2.9 (p. 19) prescribes the following rule:

> If it is desired to turn a square into a circle, half the diagonal-cord is stretched from the centre towards the east (which is along the right bisector of a side). By one-third of that which lies beyond (the side) combined with the remainder the desired circle is drawn.

That is, if

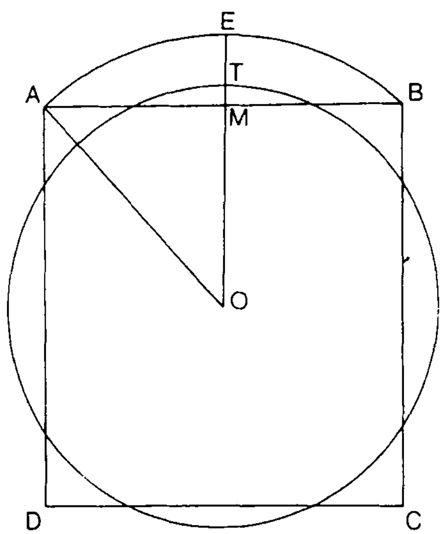

Figure 1.

$$MT = ME/3, \tag{1}$$

where ME is that part of OE (= OA, the half diagonal) which lies beyond the side AB (= a) then OT (= r) is the radius of the required circle. It can be easily seen that

$$r = a(2+\sqrt{2})/6. \tag{2}$$

The same rule (but in different wordings) is found in the *ASS* 3.2 (p. 40), *KSS* 3.11 (p. 56), and *MSS* 1.8 (p. 58). Since the true area of the obtained-circle will be πr^2, the above rule implies the following approximation of π

$$\pi = a^2/r^2 = 18\ (3-2\sqrt{2}). \tag{3}$$

It may be pointed out that the same rule is repeated in *MSS* 11.10 (p. 66) although the language used here is not so clear as in *MSS* 1.8.

The *MSS* 11.13 (p. 66) gives a rough rule for getting the circumference of a circle quickly. It says

> ***viṣkambha pañcabhāgaśca viṣkambhastriguṇaśca yaḥ / sa maṇḍalaparikṣepo na vālamatiricyate //*** (The fifth part of the diameter and three times the diameter is the perimeter of the circle. Not even a hair's length is extra.)

That is, the perimeter p is related to the diameter d by

$$p = (d/5) + 3d = (16/5)d \tag{4}$$

which is an upper bound of the circle's circumference. The relation (4) implies the approximation

$$\pi' = p/d = 16/5. \tag{5}$$

Although simple, the above rule is quite differently (and wrongly) interpreted by van Gelder[3] and Kulkarni.[4] According to these scholars the rule states that the side length of a square of an area equal to that of a given circle is 13 parts out of the 15 parts into which the diameter of the circle is divided. That is,

$$(13d/15)^2 = \pi d^2/4 \tag{6}$$

which implies

$$\pi = 676/225. \tag{7}$$

The *BSS* 2.11 (p. 19), *ASS* 3.3 (p. 41), and *KSS* 3.12 (p. 56) all contain rules which will give (6), and the two above scholars perhaps thought (incorrectly) that the same should be the case with *MSS* 11.13 which otherwise gives (4), and not (6).

About the *Gārhapatya* fire-altar, the *MSS* 13.6 (p. 68) says

> A square or a circle is the twofold form of the *Gārhapatya*. Construct the square of (side) one *vyāyāma*, and the circle of (radius) half *puruṣa*.

Now according to *MSS* 4.4 and 4.5 (p. 60) itself, a *vyāyāma* is equal to 96 *aṅgulas* (finger-breadths) and a *puruṣa* is equal to 120 *aṅgulas*. So that the areas of the two forms of the *Gārhapatya* fire-altar will be as follows.

$$\text{Square form} = 96^2 = 9216 \text{ sq. } \textit{ang}.$$
$$\text{Circle form} = \pi \cdot 60^2 = 3600\,\pi \text{ sq. } \textit{aṅg}.$$

Even with the simple and low value $\pi = 3$, the above two areas are not equal (even approximately).[5] It is also possible to interpret the second part of the above rule (*MSS* 13.6) as follows:

> Construct the square of measure one (square) *vyāyāma*, and the circle of (area) half (square) *puruṣa*.

This will yield the areas:

$$\text{Square form} = 96 \times 96 = 9216 \text{ sq. } \textit{aṅg}.$$
$$\text{Circle form} = (120 \times 120)/2 = 7200 \text{ sq. } \textit{aṅg}.$$

These cannot either be regarded approximately equal.[6] The *BSS* 7.4, and 7.5 (p. 25), and *ASS* 7.3 (p. 44) state that the *Gārhapatya* fire-altar is known to be of one *vyāyāma* measure, and that it is a square according to some and a circle according to others. This ancient Indian tradition of a twofold form is also very well expressed in the *Āpastamba Śrauta Sūtra*, XVI, 14.1, as[7]

> *Gārhapatyaciterāyatanaṁ vyāyāmamātraṁ caturasraṁ parimaṇḍalaṁvā.* (The extent of the *Gārhapatya* altar is a square or circle and measures one square *vyāyāma*.)

These statements from the two ancient works show that the areas of the two forms of the *Gārhapatya* should be equal. Modern scholars such as Thibaut,[8] Datta,[9] and others, also believe in the equality of the areas. Inspite of all this, the *MSS* 13.6 rule does not, as shown above, lead to equal areas. In fact the equalization of the area will mean

$$3600\,\pi = 9212$$

which will imply the very unlikely value

$$\pi = 64/25 = 2.56. \tag{8}$$

However, if we equate the perimeters (instead of areas) of the two forms, namely a square of side one *vyāyāma* and a circle of radius half *puruṣa*, we get

$$4 \times 96 = 2\pi' \cdot 60$$

giving exactly the same value of π' as in (5). Thus we find that the rules in *MSS* 11.13 and *MSS* 13.6 use the same approximation ($\pi' = 3.2$). It may also be pointed out that besides the equality of areas and perimeters, there seems to be a third type, namely equality of breadths. For example, the ancient scholiast Dvārkānātha Yajva, while commenting on *BSS* 7.5, says[10]

> *Parimaṇḍalapakṣe vyāyāmārdhena parimaṇḍalakaraṇam. Tatra caturaratnyāyāmaḥ.* (In the case of circular form [of the *Gārhapatya*], the circle should be drawn with half *vyāyāma* (as radius). There the breadth is four *aratnis* [$= 4 \times 24$ *aṅgulas*].)

Thus the diameter of the circle will be equal to the side of the square but neither their areas nor perimeters will be equal.

Without quoting the Sanskrit text, Datta writes[11] that the "*MSS*

(I.27) states that a square of two by two cubits is equivalent to a circle of one cubit and three *aṅgulas*". Since a cubit (*aratni*) is taken to have 24 *aṅgulas*, the rule gives

$$4 = \pi(1 + 3/24)^2 \tag{9}$$

which yields the approximation

$$\pi = 256/81. \tag{10}$$

This value is also implied in some old Indian rules which are quoted by Khadilkar[12] and which are equivalent to

$$a = d - (2d/18) = d - d/9 \tag{11}$$

where a is the side of the square equal in area to a circle of diameter d. Khadilkar adds that the same value ($\pi = 3.1605$ nearly) is also given by the *MSS* but does not quote any reference. It may be pointed out that the Egyptian *Rhind Mathematical Papyrus* (circa 1650 B.C.), Problem 50, uses the second form of the rule (11) for finding the area of a round field of diameter 9 *khet.*[13]

According to Mazumdar's account of *MSS*[14]

(i) *Gārhapatya* altar is a circle of radius 14 *aṅgulas* less 1 *yava.*
(ii) Āhvanīya altar is a square of side 24 *aṅgulas.*
(iii) *Dakṣinī* altar is a semi-circle of radius $19^1/_2$ *aṅgulas.*

Apparently these statements are from the commentary of Śivadāsa (see below) in an illustration to *MSS* 1.8 and not from the text or *MSS*. The areas of the above three altars are to be the same. But Mazumdar made a mistake in stating[15] that an *aṅgula* has 8 *yavas*. This mistake led Kulkarni to take the radius of a circle as 13 plus 7/8 *aṅgulas* and get, by[16]

$$\pi(111/8)^2 = 24^2 \tag{12}$$

the wrong value

$$\pi = 36864/12321 = 2.99 \text{ nearly.} \tag{13}$$

The correct relation is, according to *MSS* 4.4 (p. 60), that one *aṅgula* equals 6 *yavas*. So that the radius of the circle will be 13 plus 5/6 *aṅgulas* as given by Śivadāsa.[17] Hence the correct equation will be

$$\pi(83/6)^2 = 24^2 \tag{14}$$

which gives

$$\pi = 20736/6889 = 3.01 \text{ nearly.} \tag{15}$$

Consideration of the semicircular altar will give

$$(\pi/2) \cdot (39/2)^2 = 24^2 \tag{16}$$

which yields

$$\pi = 512/169 = 3.03 \text{ nearly.} \tag{17}$$

3. *New Values of* π

Sen and Bag's remark[18] that in *MSS* 11.14 and 11.15 "ordinary squares are drawn without any mathematical significance" shows that they have not understood the rules fully. We shall present a simple and meaningful interpretation of these two verses. *MSS* 11.14 (p. 66) states:

> *Daśadhā chidya viṣkambhaṁ tribhāgānuddharettataḥ / Tena yaccaturasraṁ syānmaṇḍale tadapaprathiḥ //* (After dividing the diameter (of a circle) into ten (equal) parts, leave out three parts therefrom. The square which is drawn with the remainder (as a side) has its extension upto the circle.)

That is, if d is the diameter of a circle, then $7(d/10)$ will be (approximately) the side of the inscribed square or

$$AB = 7d/10. \tag{18}$$

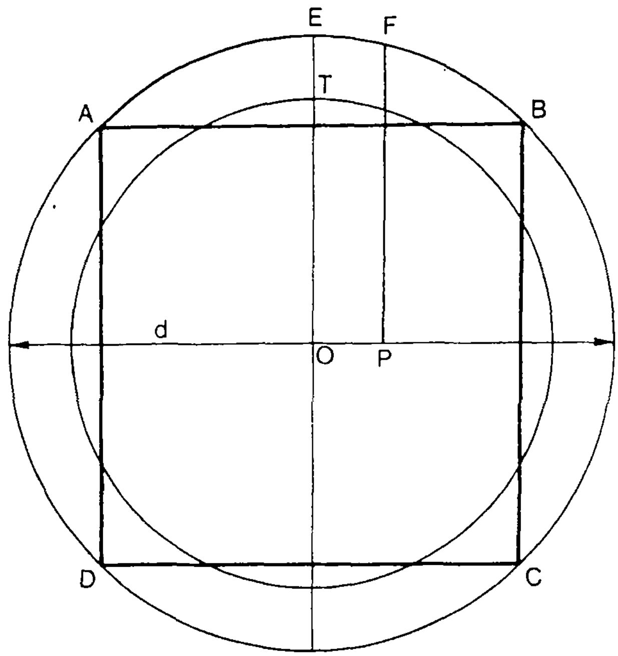

Figure 2.

Since the actual value of AB is $d/\sqrt{2}$, the above implies the simple approximation

$$\sqrt{2} = 10/7. \tag{19}$$

It may be remarked that, since $1/\sqrt{2}$ is the same as $\sqrt{2}/2$, the above approximation may be considered equivalent with $\sqrt{2} = 7/5$ also.

The next verse *MSS* 11.15 (p. 66) states:

> *Caturasraṁ navadhā kuryād-dhanuḥkoṭyastridhātridhā / Utsedhātpañcamaṁ lumpetpuriṣeṇeha tāvatsamam //* (Make a ninefold division of the square and [the elongated trisecting lines will] divide the arcual segments [of the circumscribing circle] into three parts each. Leave out the fifth part from the altitude [OE]. The circular disk formed by the remainder [OT] as radius is equal to that [square].)

That is, if

$$r = OT = OE - (OE/5) = (4/5)\,OE \tag{20}$$

then (approximately)

$$\pi r^2 = \text{area of the square} = a^2 \tag{21}$$

where a is the actual side of the square. Now the actual value of OE is $a/\sqrt{2}$, so that the above rule (20) gives

$$r = 4a/5\sqrt{2}. \tag{22}$$

Hence, by (21) we get the approximation

$$\pi = a^2/r^2 = (5\sqrt{2}/4)^2 = 25/8. \tag{23}$$

However, if we use the approximation (18) of the previous verse with $OE = d/2$, then from (20) $r = (4/5) \cdot (d/2)$, and from (18) $a = 7d/10$. Hence

$$\pi = a^2/r^2 = 49/16. \tag{24}$$

Another value is obtained if we use the approximation (19) in (22). Thus (22) will be

$$r = 14a/25 \tag{25}$$

yielding

$$\pi = a^2/r^2 = 625/196 \tag{26}$$

which gives a value in excess, but better than (5).

Now we shall give a possible and simple derivation of the basic relation (20) for the suggested method of circling the square. The traditional older method, found in all the four important *Śulba Sūtras* (*Section* 2) is represented by the relation (2)

$$r = a(2+\sqrt{2})/6.$$

Using the approximation (19) for $\sqrt{2}$, we get from (2)

$$r = 4a/7 = OT. \tag{27}$$

But (figure 1) $OE = OA = AC/2 = (\sqrt{2}a)/2$ which, by using the same approximation (19), becomes

$$OE = 5a/7. \tag{28}$$

Hence by (27) and (28)

$$OT = (4/5)\ OE = OE - (OE/5)$$

which is height diminished by its own fifth part, and is the required relation (20). Moreover, the new rule is far better than the older traditional rule because the approximation (23) is more accurate than (3).

The translation of the *MSS* 11.15 as given by Sen and Bag[19] is incomplete and does not serve any purpose. Kulkarni has quoted the better translation of Van Gelder but he takes *PF* (instead of *OE*) for the height (*utsedha*).[20] After making some tedious calculations (and an assumption for $\sqrt{17}$) he gets the approximation[21]

$$r = 11a/20 \tag{29}$$

which yields

$$\pi = 400/121 = 3.3 \text{ nearly} \tag{30}$$

instead of our (23). Since this value is quite high, he thinks that the original reading in the *MSS* text may be equivalent to

$$r = (utsedha) - (utsedha)/6$$

by which he finds r to be $55a/96$, and π as nearly 3.047.[22]

4. *Table of values of π*

Serial Number	*Value of π*	*Reference/Remark*
1.	$64/25 = 2.56$	*MSS* 13.6 (cf. Serial No. 13)
2.	$(192/111)^2 = 2.99$ nearly	Wrong calculation by Mazumdar and Kulkarni (see Serial No. 4)
3.	$676/225 = 3.004$ nearly	*MSS* 11.13 misinterpreted by van Gelder (see Serial No. 12)
4.	$(144/83)^2 = 3.01$ nearly	Śivadāsa's example (for Serial No. 2)
5.	$512/169 = 3.03$ nearly	Rule quoted by Mazumdar
6.	$(96/55)^2 = 3.047$ nearly	Kulkarni's guess for *MSS* 11.15
7.	$1296/425 = 3.049$ nearly	Exact calculation for Kulkarni's guess (Serial No. 6)
8.	$49/16 = 3.063$ nearly	*MSS* = 11.15 with *MSS* 11.14
9.	$18(3-2\sqrt{2}) = 3.094$ nearly	*MSS* 1.8 and *MSS* 11.10
10.	$25/8 = 3.125$	Our interpretation of *MSS* 11.15
11.	$256/81 = 3.1605$ nearly	Rule quoted by Datta
12.	$625/196 = 3.18$ nearly	*MSS* 11.15 with *MSS* 11.14
13.	$16/5 = 3.2 = (\pi')$	*MSS* 11.13; also our new interpretation of *MSS* 13.6
14.	$400/121 = 3.3$ nearly	Kulkarni's interpretation and calculation for *MSS* 11.15
15.	$225/68 = 3.31$ nearly	Accurate calculation for Kulkarni's interpretation of *MSS 11.15* (Serial No. 14)

Details and exact location of references are given in the main body of the present paper. It will be seen from the Table that the best value of π therein is 25/8 (Serial No. 10). In France this value was given by La Comme in 1836, and in England by James Smith in 1860 (see *Mathematics Magazine*, Vol. 23, pp. 226–227). It is an old Babylonian value.

ABBREVIATIONS

ASS: *Āpastamba Śulba Sūtra*
BSS: *Baudhāyana Śulba Sūtra*
KSS: *Kātyāyana Śulba Sūtra*
MSS: *Mānava Śulba Sūtra*

REFERENCES AND NOTES

1. There exist various editions and translations of these four Śulba Sūtras. Unless otherwise mentioned, text and page references in the present paper are according to S. N. Sen and A. K. Bag, *The Śulbasūtras,* INSA, New Delhi, 1983.
2. A. J. E. M. Smeur, "On the Value equivalent to π in Ancient Mathematical Texts. A New Interpretation", *Archive for History of Exact Sciences,* Vol. 6 (1970), 249–270, p. 252.
3. J. M. Van Gelder (translator), *The Mānava Śrautasūtra,* New Delhi, 1960; p. 300.
4. R. P. Kulkarni, "The Value of π known to Śulbasūtrakāras", *Indian Jour. Hist. Science,* Vol. 13 (1978), 32–41; p. 34.
5. Van Gelder (ref. 3), p. 303, has given the two areas as 9216 and 11307 *aṅg.*[2] respectively.
6. Of course the mean of 3600 π and 7200 will give a value nearly equal to 9216.
7. Quoted in S. D. Khadilkar (editor), *Kātyāyana Śulba Śūtra,* Poona, 1974, p. 60.
8. G. Thibaut, *Mathematics in the Making in Ancient India,* edited by D. Chattopadhyaya, K. P. Bagchi & Co., Calcutta and New Delhi, 1984, p. 93.
9. B. Datta, *The Science of the Sulba,* Calcutta, 1932, pp. 21 and 150.
10. V. Bhattacarya (editor), *Baudhāyanaśulbasūtram* (with two commentaries), Varanasi, 1979, p. 70. Also see Datta (ref. 9) pp. 180–181.
11. Datta (ref. 9), p. 149. The same rule is quoted by Kulkarni (ref. 4), p. 37, and by K. S. Shukla in the *Indian Jour. Hist. Science,* Vol. 15 (1980), p. 151. It may be that the rule belonged to the commentary part of the manuscript (C.U. 23 = No. I.E. 17 of Asiatic Society of Bengal) consulted by Datta (ref. 9, p. 230).
12. Khadilkar (ref. 7), p. 66. The rules are quoted from the commentary of Mahīdhara on the *KSS.*
13. A. B. Chace (translator), *The Rhind Mathematical Papyrus,* Reprinted by the N.C.T.M., Reston, Virginia 1979, p. 49 or column 92.
14. N. K. Mazumdar, "Mānava Śulba Sūtram", *Jour. Dept. of Letters* (Calcutta University), Vol. 8 (1922), 327–342, p. 335.
15. *Ibid.,* p. 333.
16. Kulkarni (ref. 4), p. 37. There is a printing error also (R.H.S. should be 576).
17. See Van Gelder (ref. 5), p. 287.
18. Sen and Bag (ref. 1), p. 278.
19. *Ibid.,* p. 136.
20. Kulkarni (ref. 4), pp. 38–39. *FP* is parallel to *EO* and trisects *AB*.
21. *Ibid.* p. 39. The actual or exact value (i.e. without assuming any approximation) of r will be $(2\sqrt{17}\,a)/15$ and that of π will be 225/68 (=3.31 nearly).
22. Ibid. p. 40. He has given r in the unsimplified form as 165 a/288. The exact values of r and π will be $(5\sqrt{17}\,a)/36$ and1296/425 (=3.049 nearly).