HISTORIA MATHEMATICA 19 (1992), 371–384

Mathematical Foundations of a Cultural Project or Ramchandra's Treatise "Through the Unsentimentalised Light of Mathematics"

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The nineteenth century witnessed a number of projects of cultural rapprochement between the knowledge traditions of the East and West. This paper discusses the attempt to render elementary calculus amenable to an Indian audience in the indigenous mathematical idiom, undertaken by an Indian polymath, Ramchandra. The exercise is specifically located in his book *A Treatise on the Problems of Maxima and Minima*. The paper goes on to discuss the "vocation of failure" of the book within the context of encounter and the pedagogy of mathematics. © 1992 Academic Press, Inc.

Das 19. Jahrhundert erlebte eine Anzahl von Versuchen der kulturellen Annäherung zwischen den Wissenstraditionen des Osten und des Westens. Der Aufsatz erörtert den Versuch, den Calculus für eine indische Zuhörerschaft im einheimischen mathematischen Idiom elementar darzustellen, ein Versuch, der von Ramchandra unternommen wurde. Dies trifft vor allem auf sein Buch *"A treatise on the problems of Maxima and Minima"* zu. Der Aufsatz behandelt den Fehlschlag des Buches im Rahmen des kulturellen Kontextes und der Mathematikausbildung. © 1992 Academic Press, Inc.

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کتاب A Treatise on the Problems of Maxima and Minima سے وابستہ مندرجہ بالا سوالوں
کی تلاش سے منسلک ہے۔ اس عمل کو جاری رکھتے ہوئے یہ مقالہ حلقۂ علم الحساب میں علمی ہدائتوں
کے روبرو ٹکراؤ اور درسی فوائد کی روشنی میں اس کتاب کی وجوہات مذمت ناکامی کو بھی تلاش
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AMS 1991 subject classifications: 01A32, 01A55
KEY WORDS: maxima, algebra, calculus, *Bija-Ganita*, pedagogy, Ramchandra

The introduction of modern science into India toward the end of the 19th century triggered a number of cognitive encounters, defined as conflicts between different/distinct knowledge systems, such as the traditional and the modern, as well as Eastern and Western. One of the forms of this encounter resulted in an attempt to

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revitalize traditional knowledge systems, the science of the ancient and medieval periods included [Kopf 1970, Habib & Raina 1989a]. A significant section of the emerging Indian intelligentsia had taken upon itself the task of bridging this gap. In this paper we examine the mathematical basis of one such cognitive encounter in the efforts of a nineteenth century Indian polymath, Y. Ramchandra (1821–1880) of Delhi. Scientific and literary writings (Habib & Raina 1989b) and the cultural roots of his book on mathematics *A Treatise on the Problems of Maxima and Minima* have been discussed elsewhere [Raina & Habib 1990].

The emphasis of this paper will be more on the mathematical distinctiveness and limitations of *A Treatise on the Problems of Maxima and Minima* (hereafter referred to as *The Treatise*) first published in Delhi in 1850 [Ramchandra 1850] and then in London in 1859 [Ramchandra 1859]. We also peripherally touch upon the concerns of his second book on mathematics, *A Specimen of a New Method of the Differential Calculus called the Method of Constant Ratios* (hereafter referred to as *The Method*), published in Calcutta in 1863 [Ramchandra 1863]. But before we proceed with an investigation of *The Treatise*, a word about the history of calculus in general, and the development of elementary calculus in particular, is in order.

THE ALGEBRAIC PROGRAM FOR CALCULUS

A major mathematical program was undertaken in Europe, during the last few decades of the 18th and early decades of the 19th century: to install both differential and integral calculus as autonomous enterprises [Fraser 1989]. Boyer [1969] has periodized the history of calculus in the following way:

—the geometric stage, where geometric problems and concepts predominate.

—the algebraic stage, commencing about 1740 with Euler, and closing around 1800 with Lagrange.

—the stage of classical analysis, beginning in the early 19th century, and inaugurated by the textbooks of Cauchy [Boyer 1969, 317].

By the end of the 18th century, the algebraic stage in the development of calculus was drawing to a close. In this stage the underlying program was rooted in a commitment to a formalism for calculus that did not take any recourse to geometrical notions. There was thus a shift in the emphasis of the program of calculus, for now the major problem was not to define a curve mathematically, but to reduce calculus to a study of formulas [Scott 1958, 140] [1]. Cauchy, on the other hand, installed calculus in its modern form as analysis, thus affecting a schism between mathematics and theoretical physics, while he went on to return the curve to calculus via an arithmetical theory of the numerical continuum [Fraser 1989, 332].

Though Ramchandra's *The Treatise* was published in India in 1850, the commitment of the work, it will be shown, is philosophically and mathematically grounded in the algebraic stage of the history of calculus. Though the project was undertaken in India, 50 years after its closure in Europe, its origins were quite different. Two external factors provided the major impetus for *The Treatise*. To begin with the two factors serving as a conceptual rationale for the work were (1) the alleged predisposition of the Indian people to algebraic methods, and (2) the desire to resuscitate this native mathematical ability [Raina & Habib 1990, 455–472]. Ramchandra felt that this orientation could be internalized, as what Glas would call "methodological insight" [Glas 1989, 115–131], and thus in turn could influence the internal development of algebra, even toward calculus. This distinction between the algebraic program of calculus in Europe and Ramchandra's program in India is particularly important, for the European program was motivated by the desire to develop formal methods in analysis (Fraser 1989) and to reduce analysis to manipulations.

A BACKGROUND TO *THE TREATISE*

Ramchandra was born in 1821 in the small town of Panipat near Delhi. He was schooled in the Indian tradition, while he received his higher education in a "modern school." In 1843 he was appointed Science Teacher at Delhi College, whence commenced his career as an Urdu journalist. This was also the time when under the aegis of the Vernacular Translation Society, he was involved in translating books on science from English to Urdu. In 1845, he commenced work on *The Treatise* [Jacob 1902]. The purport of *The Treatise* was to advance the standard of education in India, and the larger cause of science among the Indian populace. Ramchandra dedicates *The Treatise* to reviving the spirit of algebra, "so as to resuscitate 'the native disposition of these people' that had been eroded over the centuries" [Raina & Habib 1990, 459]. Ramchandra's contribution to the history of the pedagogy of mathematics lay in the application of the theory of equations as developed in the Indian tradition to the solution of elementary problems of maxima and minima, in particular for obtaining the maxima and minima of a function. Even though Ramchandra commences from a twelfth century Indian text, Bhaskaracharya's *Bija-Ganita* (hereafter BG), the attempt to bridge the gap between the two mathematical traditions was not revivalist, for he was looking for a familiar tradition from which to develop his pedagogy for mathematics teaching. Bhaskaracharya's *Bija-Ganita* and *Lilavati,* were very important textbooks in mathematics in the traditional Indian curriculum. It was therefore imperative that Ramchandra locate the developments in mathematics with respect to them, and then outline a heuristic to proceed with advances from the point where these texts left off. In 1845, Ramchandra wrote a mathematical primer in Urdu called the *Sari-ul-Fahm:* he was then a teacher of English at the Sircari Madrasa. The book consisted of eight chapters, including one on mathematical riddles. Though the *Sari-ul-Fahm* professed to encompass the contents of the *Lilavati* and the *Khulasat-ul-Hisab* and to extend far beyond, the text retained a stylistic allegiance to the lyrical *Lilavati* (Ramchandra 1849).

The method developed by Ramchandra could be used for functions involving quadratics or higher order expressions, as well as for expressions involving two or more variables. The method is based on a theorem (or that is what De Morgan refers to it as, in the introduction to Ramchandra's *Treatise*) [Ramchandra 1859] from which he derives methods for obtaining the maxima and minima of a function without introducing the concept of differentiation. The first method does not allow for imaginary roots, and some of the other related methods discussed by Ramchandra had also been discussed in some 19th-century elementary mathematical textbooks such as Wood's *Algebra* and the *Encyclopedia Metropolitana* [Jacob 1902]. Ramchandra also discusses another method that also does not involve imaginary roots since imaginary quantities might "appear somewhat mysterious and unintelligible" for beginners in mathematics [Jacob 1902, 17]. In the Preface to both the Indian and the British edition of the *The Treatise,* Ramchandra writes, "This latter method I may venture to call a new method, because in all the mathematical works I have had access to, I have never seen a single problem of maxima and minima solved by it, though it is used to reduce an adfected quadratic to a pure one in a great many works on algebra." In the 1859 edition of *the Treatise,* De Morgan wrote a long introduction to the book, where he further clarified "*Ramchandra's problem*—I think it ought to go by that name, for I cannot find that it was ever current as an exercise of ingenuity in Europe—is to find the maximum and minimum" [Ramchandra 1859, p. xiii]. In a letter dated 1901, Mary Everest Boole, widow of the British algebraist George Boole, wrote to Dr. Bose of India that English youths were being taught to solve problems in maxima and minima by "other simple devices similar in essence to Ramchandra's and probably superior in efficiency" [Boole 1911, 11]. It is likely that these "other simple devices" were developed after the publication of *The Treatise,* for neither De Morgan nor Ramchandra was familiar with any such device in the 1850s.

When the book first appeared in India in 1850, it was subjected to rough treatment at the hands of the critics. However, even the most severe of them finally did acknowledge that "the author that gave birth to this mathematical idea is capable of producing something much better" [Anonymous 1850]. Rather than discussing the merit of the work, the critics were more agitated about Ramchandra's temerity in publishing a book in English [Jacob 1902]. Six years later, when Ramchandra came to be acquainted with De Morgan, he wrote to him saying, "When I composed my work on the Problem of Maxima and Minima I built my castles in the air, but *Calcutta Review* &c. destroyed the empty phantasms of my brain. . . ." Jacob, son-in-law of Ramchandra, goes on to point out that Ramchandra was "subjected to kind rebukes from some of the best friends of native education in the North-West Provinces, for his ambition in publishing works in English" [Jacob 1902, 18]. In 1856, E. Drinkwater Bethune, member of the Supreme Council and Chairman of the Education Commission, Calcutta, thought differently, and forwarded the book to De Morgan for comment [Home Public Letters from Court, 1856]. De Morgan thus received a copy of the Indian edition of *The Treatise* sometime in 1850. De Morgan proceeded to canvass for its publication in England and its subsequent distribution in England and Europe [De Morgan 1858].

De Morgan was quick to realize the relevance of the work to the pedagogy of mathematics, for he writes of the book, ". . . a short paper with a few examples, would have sufficed to have put the whole matter before a scientific society. *But it was Ramchandra's object to found an elementary work upon his theorem, for the use of beginners with a large store of examples*" [Ramchandra 1859, xiv]. De Morgan also shared the 19th century view of historians of mathematics that the Indian tradition of mathematics was an algebraic one [2]. On account of his interest in indeterminate equations, De Morgan was familiar with the *Bija-Ganita* through

Colebrooke's translation [Colebrooke 1817]. *The Treatise,* as pointed out earlier, was published at De Morgan's instance by Unwin Publishers, London, in 1859. The cover of the book carried the following notification: "Reprinted by the order of the Honourable Court of Directors of the East India Company for circulation in Europe and in India. Under the Superintendence of Augustus De Morgan, FRAS, FCPS of Trinity College, London." De Morgan was also associated with the development of mathematical syllabi for British schools and colleges [Pycior 1983, 211–226], and was sensitive to the intrinsic pedagogical worthiness of the book for an Indian audience, as well as a segment of the book that was seen to be relevant to the British educational program. In particular he recommends the chapter on quadratic equations (Chapter 2 of *The Treatise*): "which . . . could advantageously supersede some of the conundrums which are manufactured under the name of problems producing equations" [Ramchandra 1859, xv].

A little has to be said about De Morgan's interest in the work. It must be recognized that De Morgan was a mathematician who had a particularly important role to play in the revival of mathematics in Britain, and as much with the courses on mathematics in British schools and colleges. In fact, one of his tasks was in securing the place of algebra in a "liberal education" [Pycior 1983, 211–226]. As part of his efforts on clarifying pedagogical approaches to mathematics teaching, he authored two books on elementary calculus. In 1835, he published his *The Elements of Algebra Preliminary to the Differential Calculus and Fit for the Higher Classes of Schools* [Dubbey 1980, 36], and then in 1842 a second book on *The Differential Calculus* where he clarifies the concept of limit. The concern with the pedagogy of mathematics was also transmitted by De Morgan to his students, and two of Issac Todhunter's books were translated into Urdu by Ramchandra [Habib & Raina 1989b].

THE TREATISE BY ITSELF

To begin with, the problems discussed in *The Treatise* fall in the domain of elementary calculus. *The Treatise* does not concern itself with analysis, wherein the numerical continuum becomes particularly important. Second, the problems normally encountered in elementary calculus do not extend to the domain of complex variables. Consequently, for one nurtured in the tradition of the theory of equations, the BG offered itself as a convenient text [Datta & Singh 1962] whence to commence bridging the gap between traditional mathematics and the new mathematics of change, the mathematics of a dynamic society, calculus [Struik 1989, 3–4]. Ramchandra was convinced that Indians familiar with the solution of quadratics as set forth in the BG would encounter no difficulty in solving problems of maxima and minima without taking recourse to the technique of differentiation. Bhaskara in the BG specifies that "the square root of a positive number is positive as well as negative. *There is no square root of a negative number because it is non-square*" [Datta & Singh 1962 II, 24]. Thus the equation $ax^2 + bx + c = 0$ will have real roots provided $b^2 - 4ac \geq 0$. Ramchandra uses this fact to determine maxima and minima of expressions of the form $ax^2 + bx$.

Thus in a problem from Chapter 1 of *The Treatise* (see below) it is necessary to maximize $f(x) = ax - x^2$ $(a > 0)$. Letting $r = ax - x^2$, and solving for x in terms of r, we obtain

$$x = a/2 \pm \sqrt{a^2/4 - r}.$$

The maximum value of r that yields a positive value of x is evidently $r = a^2/4$, corresponding to the value $x = a/2$.

Ramchandra's method is also applicable to cases where $f(x)$ is a higher order polynomial. Here he lets $r = f(x)$ and factors the equation $r - f(x) = 0$ in a way that allows him to determine the maximum or minimum value of r that is consistent with the requirement that x be real. The resulting procedure yields a value of x which makes the expression

$$f(x) = -(x^3 + Ax^2 + Bx) \tag{1}$$

a maximum. Let $r = -(x^3 + Ax^2 + Bx)$, so that

$$x^3 + Ax^2 + Bx + r = 0. \tag{2}$$

Dividing (2) by $x + a$ we obtain

$$(x^3 + Ax^2 + Bx + r) = (x^2 + (A - a)x + a^2 + B - aA)(x + a) + (r - a(a^2 + B - aA). \tag{3}$$

(In his commentary De Morgan would refer to the possibility of factoring polynomials in this way as "Ramchandra's theorem.") Suppose now that $-a$ $(a > 0)$ is a root of (2). It follows that the remainder in (3) is zero and so

$$r - a(a^2 + B - aA) = 0 \tag{4}$$

$$\Rightarrow x^3 + Ax^2 + Bx + r = \{x^2 + (A - a)x + a^2 + B - aA\}(x + a). \tag{5}$$

Now let $x = a^*$ be a second root of (2) distinct from a. It follows from (5) that a^* is a root of

$$x^2 + (A - a)x + (a^2 + B - aA) = 0,$$

which using (4) may be written

$$x^2 + (A - a)x + r/a = 0. \tag{6}$$

Solving (6) we obtain

$$a^* = -(A - a)/2 \pm [(A - a)^2/4 - r/a]^{1/2}. \tag{7}$$

Because $a > 0$ it follows that the largest value of r consistent with a real solution of (7) is given by

$$(A - a)^2/4 = r/a. \tag{8}$$

Using (4) and (8) we have

$$(A - a)^2/4 = a^2 + B - aA. \tag{9}$$

Solving (9) for a gives

$$a = \{A + (4A^2 - 12B)^{1/2}\}/3. \tag{10}$$

Substituting into equation (7) gives the final solution

$$x_{\max} = a^* = -(2A - \sqrt{4A^2 - 12B}/6. \tag{11}$$

As far as the form of presentation is concerned, *The Treatise* shares certain basic features with mathematical textbooks published in Europe toward the end of the 18th and the early decades of the 19th century. The first feature was that any new method or technique proposed was considered to be readily adaptable to any set of problems. The failure of a theorem at isolated values, or the unsuitability of the method in a particular domain was not construed as an invalidation of the method. The practice of establishing a priori the existence of solutions to a general class of equations had not yet become the norm (Glas 1989). However, it was absolutely necessary to demonstrate the applicability of the method to a broadly defined class of problems, and this is what Ramchandra embarks upon to establish his method.

The introductory chapter of *The Treatise* discusses the theorem and outlines the method. In the second chapter we have a large number of examples illustrating the method for obtaining the maxima and minima of quadratic functions. The third chapter discusses the same for cubic equations. The fourth chapter contains solved examples for fourth, fifth, sixth, and seventh degree equations. Problems of maxima and minima involving equations with two or more variable quantities are discussed in the fifth chapter. In the supplementary section of the book Ramchandra solves "a few new problems" by this method; e.g., that of determining the area enclosed between four given straight lines. We now discuss some examples from Ramchandra's *The Treatise*.

Problem (Chapter 1). Divide a given number into two parts, so that the product is a maximum.

Let the number be a, one part be x, and the maximum product be r:

$$\Rightarrow r = x(a - x)$$

$$\Rightarrow x = a/2 + \sqrt{a^2/4 - r}.$$

From the method of impossible roots r cannot be greater than $a^2/4$, or else the value of the discriminant would be prohibited

$$\Rightarrow x(a - x) \text{ is a maximum when } a^2/4 = r$$

$$\Rightarrow x_{\max} = a/2.$$

Ramchandra solves the same problem by another method without having recourse again to the condition of impossibility of imaginary roots.

Let the maximum product be $ax - x^2$, and let $x = y + a/2$

$$\Rightarrow ax - x^2 = a^2/4 - y^2.$$

Now $a^2/4 - y^2$ is a maximum when $y = 0$; hence the maximum value of x is $a/2$.

At this point it would be worthwhile to briefly discuss Fermat's geometrical solution to the same problem. In 1629, Fermat solved a problem generally having two solutions, but a single one for a maximum or minimum. Given a line of length a, divided by a point P into lengths x and $a - x$, there exists only one point such that the area of the rectangle A formed by x and $a - x$ is a maximum. The area on the segments x and $a - x$ is given by $A = x(a - x)$. Instead of distance x mark off $x + E$, hence $A = (x + E)(a - x - E)$. For a maximum x and $x + E$ must coincide. Thus the two values are equal: $x = a/2$. In modern differential calculus E is replaced by h or Δx [Fermat 1629, 63].

Problem (Chapter 1). Divide a number a into two fractions such that the sum of their squares is a minimum.

Let one of the fractions be x and the other a/x, and the minimum of their squares r such that

$$r = x^2 + (a/x)^2 \Rightarrow x^4 - rx^2 = -a^2$$

$$x^2 = r/2 + \sqrt{\{r^2/4 - a^2\}}$$

For r to be a minimum $r^2/4 = a^2$, or $a = r/2$

$$\Rightarrow x_{\min} = \sqrt{a}.$$

Ramchandra also solves the problem without using imaginary roots. As in the previous example, he puts $x^2 = y^2 + r/2$; substituting for x we get $y^2 + a^2 = r^2/4$.

For r to be a minimum $y = 0$, $\Rightarrow x_{\max} = \sqrt{a}$.

Problem (Chapter 1). The method is also demonstrated for other quadratics: Determine the value of x for which the expression $a^4 + b^3x - c^2x^2$ is the maximum.

The above expression can be written as

$$c^2(a^4/c^2 + b^3x/c^2 - x^2).$$

Since a^4/c^2 is a constant, the expression $b^3x/c^2 - x^2 = r$, is to be maximized.

$$x = b^3/(2c^2) + \sqrt{(b^6/4c^4 - r)}. \tag{12}$$

r cannot be greater than $b^6/4c^4$, $\Rightarrow x_{\max} = b^3/2c^2$.

Solving by the other method, rewrite Eq. (12) as $x^2 - xb^3/c^2 = r$, and let $x = y + b^3/2c^2$

$$\Rightarrow r = b^6/4c^4 - y^2, \qquad \text{for } r \text{ to be a maximum } y = 0$$

$$\Rightarrow x_{\max} = b^3/2c^2.$$

Problem (Chapter 2). Obtain the fraction, the cube of which being subtracted from it, the remainder is the greatest possible.

Let the fraction be x, and the greatest remainder r, where $r = x - x^3 \Rightarrow x^3 - x + r = 0$. Assume $-a$ to be a negative root

$$\Rightarrow (x^3 - x + r) = (x^2 - ax + a^2 + 1)(x + a) + \text{rem.}$$

Following the procedure as outlined above, we obtain the value of

$$r = a^3 - a \Rightarrow a^2 - a = r/a$$

$$\Rightarrow x^2 - ax + r/a = 0 \Rightarrow x = a/2 + \sqrt{(a^3 - 4r)/4a}.$$

r will be a maximum when $a^3 = 4r \Rightarrow r = a^3/4$.

Hence $x_{max} = a/2 = 1/\sqrt{3}$. This is a special case of a problem discussed earlier where $A = 0$ and $B = -1$.

Problem (Chapter 3). Find the value of angle x when $m \sin(x - a)\cos x$ is a maximum.

Since m is a constant, the task is that of finding the maximum of $\sin(x - a)$ $\cos x$.

Let $y = \cos x$, $b = \cos a$, $c = \sin a = \sqrt{(1 - b^2)}$, and let the maximum value be r.

$r = by\sqrt{(1 - y^2)} - cy^2$; now square both sides and we have

$$(c^2 + b^2)y^4 + (2cr - b^2)y^2 + r^2 = 0$$

$$y^2 = (b^2 - 2rc^2)/2 + \sqrt{b^4 - 4b^2c^2r - 4r^2 2c^2(1 - c^2)}/2.$$

For r to be a maximum $1 - c^2 > 0$ and $c < 1$

$$\Rightarrow 4r^2c^2(1 - c^2) + 4b^2c^2r = b^4$$

$$\Rightarrow r = b^2(1 - c)/(2c(1 + c))$$

$$\Rightarrow y^2 = (b^2 - 2rc^2 2)/c^2 = (\cos(\alpha/2) - \sin(\alpha/2)^2/2$$

$$\Rightarrow y = \cos(45 + \alpha/2) = \cos x \Rightarrow x_{max} = \alpha/2 + 45.$$

The method outlined above is then extended by Ramchandra to obtain the general form for solving fourth degree equations. For example consider a polynomial of the form $x^4 + Ax^3 + Bx^2 + Cx + r$. The polynomial can be expressed as the product of two quadratics; let one of them be of the form

$$ax^2 + bx + c.$$

Thus $(x^4 + Bx^3 + Cx + r) = Q(x)(ax^2 + bx + c) + R$, where $Q(x) = x^2 + (A - a)x + B + a^2 - Aa - b = 0$ and $R = r - c(B + a^2 - Aa - b) = 0$

$$\Rightarrow r/c = B + a^2 - Aa - b. \tag{13}$$

In (13) once the value of r is obtained, it is then possible to solve for the value of x.

THE TREATISE'S VOCATION OF FAILURE

The examples discussed above are all taken from *The Treatise*, and illustrate how elementary problems of maxima and minima could be solved without utilizing the techniques of differentiation, a knowledge of which in turn required a minimum grounding in geometry. In an earlier paper we discussed the external factors that came in the way of the acceptance of *The Treatise* as a textbook for mathematics teaching in India [Raina & Habib 1990]. In what follows, we attempt to elucidate the factors relevant to the practice of this method as mathematics, and that could have possibly come in the way of the realization of *The Treatise's* pedagogic objectives. As pointed out earlier Ramchandra's method is predicated on the solution of equations having only real roots, a class of equations that does not entirely encompass the study of modern analysis. Even though the book conforms to the spirit of calculus as algebraic analysis, the method from the point of view of the general methods being introduced into calculus in the mid-19th century was beset with limitations. *The Treatise* itself did not aspire to be a book on elementary calculus, but offered a method for solving elementary problems of maxima and minima without using methods of the calculus.

Further, for higher order polynomial expressions, the method required that they be reduced to the quadratic form. The maxima or minima would then correspond to the real and equal roots of this reduced quadratic. The method becomes problematic with higher order expressions which are not factorizable into products of quadratic expressions, or products of quadratic expressions and some other lower order polynomial expression. Finally, the method, unlike the calculus, would generate only the global maxima or minima, but would not find other maxima or minima of the expression.

Arguing from within the framework of presentation of Ramchandra's theorem, i.e., the statement of the theorem followed by validation through a large number of examples, in this case taken from Simpson's book on fluxions, the textbooks on differential calculus written by Hall, Young, and Ritchie, and Hirsch's book on geometry and mixed mathematics [Ramchandra 1859], it is evident that most of the examples chosen were numerical, while a few taken over from coordinate geometry. Further, there is no reference in *The Treatise* to the *tangent problem* or the *quadrature problem*. In fact, the relation between a function and its representation as a curve is not discussed. Consequently, the concept of a maximum and minimum is a purely arithmetical one, and does not offer any insight into the manner in which a function behaves, or the evolution of the gradient or slope of a curve as it approaches either a maximum or a minimum value. Hence, the method was ideally suited to examples where one was merely interested in obtaining the maximum or minimum and not in the slope.

In the formalism of differential calculus, the equation deciding the maximum or minimum of a function is that the gradient given by the first differential be zero. The sign of the second differential decides whether the corresponding point is a maximum or a minimum. In Ramchandra's method the critical or deciding condition is that it provides the maximum or minimum value of the expression for real values of x, and for equal roots of the reduced quadratic form. In addition, despite its limitation as an arithmetical device, it provides the global maxima for a polynomial expression.

But then *The Treatise* was the first step in the algebraic program that Ramchandra intended to undertake for calculus. Though this cannot be inferred from *The Treatise* itself, there is a continuity in the problematic addressed in his second book, *The Method*. This is also evident, from the cautious closing lines of *The Treatise*, "*I had to say something more regarding the Algebraical theory of*

Maxima and Minima, but being afraid of enlarging too much, I conclude these sheets'' [Ramchandra 1859, 185] (emphasis added), but surfaces clearly in *The Method.* This limitation of the work is also recognized by De Morgan, for he writes that Ramchandra has ''a much stronger leaning towards geometry than could have been expected by a person acquainted with the *Bija-Ganita,* but he has not the power in geometry which he has in algebra'' [Ramchandra 1859, xi]. But the intent of *The Treatise* was pedagogical, and Ramchandra responded to criticism that declared its worthiness as an original criticism with ''it is manifest from my preface and introduction that the object of my Treatise is not to make additions to what is already known in mathematics, but to lay before the public a new method of Differential Calculus for establishing the new conclusions'' [Jacob 1902, 33–34]. This point is also repeatedly emphasized by De Morgan: ''Europe must remember that his [Ramchandra's] purpose is to teach Hindoos, and that probably he knows better how to do this than they could tell him'' [De Morgan 1858]. Ramchandra primarily viewed his work as that of a pedagogue seeking a device that would bring out the best in his students. Nevertheless, in addition to De Morgan's interest in the pedagogy of mathematics teaching, he was also receptive to other consequences of Ramchandra's *Treatise.* For as Mary Boole writes, De Morgan ''caused a Treatise'' by Ramchandra to be published in England, ''in order to prove to the English men of science that the Hindu mind masters, without the aid of differential calculus, problems that had hitherto been solved only by calculus'' [Boole 1911, 7]. This is itself the fruit of a realization that while mathematics is done one way, ''it could as well be done another'' way [Hodgkin 1986, 174], a realization that did find concrete expression in *The Treatise,* as well as in De Morgan's appreciation of it.

THE TREATISE: THE TEXT AND CONTEXT

While not subscribing to a strong delineation between the internal and external history of science, this paper has largely worked within the paradigm of the mathematical content of the program, while at each stage emphasizing the possible source of such a ''conceptual variation'' [Glas 1989]. But it must also be clarified that the algebraic program of *The Treatise,* while having its roots in an alternate cultural and pedagogical program, was certaintly influenced by the spirit of the Euler–Lagrange program. This influence can be located both textually and contextually. It would not be out of place here to remark that the French Encyclopedists had a significant influence on Ramchandra, and that he was familiar with the work of Lagrange, as well as other mathematicians of the time, such as D'Alembert and Laplace. This is evident in the newspaper articles written by Ramchandra on these mathematicians, in Urdu, some of which were also published in a short biography of renowned scientists [3]. Further, the objective Ramchandra set himself in his second book, *The Method,* takes cognizance of the Lagrangian calculus developed as an algebra of finite quantities, the derivatives being expressed as coefficients of the Taylor expansion. In *The Method* Ramchandra seeks to go beyond the limitations of Lagrange's method. The methods that were then employed in the differential calculus, such as the fluxional method, the infinitesimal method, the calculus of functions, and the method of limits, were beset with difficulties since they all required a knowledge of the concept of limits and infinitesimals [Ramchandra 1863]. But then that has to do with the hard core problems of calculus, and is beyond the scope of the present essay concerned as it is with the pedagogy of school mathematics.

In order to underline the continuity of the problematic that Ramchandra was concerned with it may be pointed out that *The Method* was the more serious articulation of a modest idea that took seed in *The Treatise,* that of grounding calculus differently. But the methods developed in both books were seen as pedagogic variations. In a letter dated 1st December, 1861, De Morgan called Ramchandra's method of constant ratios ingenious, though he also voiced his reservations concerning them [Jacob 1902, 33–34]. The book was also reviewed by Phillip Killand and Rev. Skinner of Edinburgh University, Professor Fischer of St. Andrews, and Mr. Reynolds, a high wrangler of Cambridge. A letter from Killand and Skinner to Dr. B. B. Smith, Civil Surgeon, Delhi, reads, "His solutions are ingenious and, to some extent, original, but *they cannot be called additions to our knowledge. They are rather variations on it* [Jacob 1902, 33] (emphasis added). Which in a way was also Ramchandra's evaluation of his work, his hope being that his labors "might be found of some real benefit in the department of mathematical education" [Ramchandra 1863, preface]. The hope remained unfulfilled since both books were ignored. *The Method* is still to be studied in detail. This paper has merely attempted to identify the mathematical limitations of Ramchandra's theorem illustrated in *The Treatise. The Treatise* could certainly have served as an elementary text for school curricula in Indian schools. We have elsewhere tried to identify external factors that proved to be unfavorable for the overall reception of *The Treatise*. But apart from the fact that the method could not be extended to the entire range of problems that calculus dealt with, and the bind imposed by growing imperial design, that accorded no place to theories of knowledge that allowed for the transplantation of modern science on a Sanskritic base, it is also necessary to examine the reorganization of mathematical syllabi for schools and colleges, to understand why *The Treatise,* despite the support of mathematical heavyweights like De Morgan and Boole, was aborted as a minor episode in the history of mathematics teaching in India.

ACKNOWLEDGMENTS

The author is grateful to the referees of earlier drafts of this paper submitted to *Historia Mathematica*. The errors of interpretation continue to be those of the author.

NOTES

1. Further Fraser [Fraser 1989] points out that while both Euler and Lagrange had emphasised the algebraic character of differential and integral calculus, they pursued the separation of calculus from geometry, while simultaneously cultivating geometric and mechanical applications.

2. A representative set of works expressing the view is [Bell 1940] and [Cajori 1991]. Cajori writes, [1991, 94f] "Incomparably greater progress than in the solution of determinate equations was made by the Hindus in the treatment of indeterminate equations. Indeterminate analysis was a subject to which the Hindu mind showed a happy adaptation." Burton writes, "In the period AD 400 to 1200 the Indians developed a system of mathematics superior in everything except geometry to the Greeks' [Burton

1985, 245]. De Morgan in the Introduction referred to above writes, "The (Hindu) sought refuge from arithmetic in algebra, the Greek sought refuge from arithmetic in geometry. The greatness of Hindu invention is algebra . . ." [Ramchandra 1859].

3. Some of the articles by Ramchandra on these mathematicians are as follows: In the *Tazkirat-ul-Kamileen,* Delhi, 1849, 163, *Zikr Lagrange Ka* ("On Lagrange"), where Ramchandra discusses Lagrange's contribution to mathematics (ilm-i-riyazi) and fluid dynamics and mechanics (harkat-aur-tamuj adsame sayal); p. 163, *Hyat dan laplace,* ("The Astronomer Laplace") discusses Laplace's contribution to celestial mechanics (ilm-i-hiyat riyazi); p. 160, *Zikr Muhandis Euler Ka* ("On the Contributions of the 'Philosopher-Mathematician' Euler"). Also see two articles published in the Urdu paper edited by Ramchandra, *Fawaid-ul-Nazarin,* 29th Dec. 1845, *Usul ilm-i-hisab juziyat-o-kuliyat* ("On the Principle of Maxima and Minima") and *Jabr-o-Muqabla* ("On Algebra"). Interestingly, in the 29th December 1845 article, Ramchandra considered maxima and minima a discipline in itself, concerned with the determination of unknown quantities (thereby subsuming it as part of algebra), and not a part of differential calculus. *Asool-i-jabr-o-Muqabla,* another mathematical primer in Urdu, was published by Ramchandra in 1845.

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