A REVIEW OF HINDU MATHEMATICS UP TO THE 12th CENTURY

Amongst the mathematical works of the Hindus from which we reconstruct the history of Hindu Mathematics are :

(1) The *Śulba-sûtras.* These works from a part of the six vedangas (« members of the Veda »). They are contained in the *Śrauta sûtras.* The earliest of the *Śulba-sûtras* were composed by BAUDHAYANA about 800 B. C. The others that are available to us now are those of ÂPASTAMBA, KÂTYÂYANA, MÂNAVA, MAITRÂYAṆA, VÂRÂHA and VÂDHULA.

(2) The *Bakhshâlî Manuscript.* It is a copy of a work composed about the third century A. D. A lithograph of the work has been published by G. R. KAYE whose estimate of the date of the manuscript is incorrect, likewise his notes and comments are misleading.

(3) The *Âryabhaṭîya.* It is an astronomical treatise composed by ÂRYABHAṬA I (499). It contains a chapter on mathematics.

(4) The *Brâhma-sphuṭa-siddhânta* by BRAHMAGUPTA (628). It is an astronomical work with chapters on arithmetic mensuration and algebra.

(5) The works of BHÂSKARA I (522). These are (i) Commentary on the *Âryabhaṭîya,* (ii) *Mahâ-Bhâskarîya* and (iii) *Laghu-Bhâskarîya.* These works are unpublished.

(6) The *Triśatikâ* by ŚRÎDHARA (750). It is a short treatise on arithmetic and mensuration. It is said to be an abridgement of a bigger work by the same author.

(7) The *Gaṇita-sâra-saṁgraha* by MAHÂVÎRA (850). It is a treatise on arithmetic and mensuration ; it contains also a section dealing with indeterminate equations.

(8) The commentary of PṚTHUDAKASVÂMÎ (860) on the *Brâhma-sphuṭa-siddhânta.* Only a portion of this commentary is available.

(9) *Laghumânasa* by MAÑJULA (932). It is an astronomical treatise.

(10) The *Maha-siddhânta* by ÂRYABHAṬA II (950). It is an astronomical treatise containing a chapter on mathematics.

(11) The works of ŚRÎPATI (1039). These are (i) *Siddhânta Śekhara* and (ii) *Gaṇita-tilaka*.

(12) The works of BHÂSKARA II (1150). These are (i) *Lîlâvatî*, (ii) *Bîjagaṇita*, (iii) *Siddhânta Śiromaṇi*, and (iv) *Karaṇa-kutûhala*.

Besides the above works there are a number of commentaries on them as well as a number of purely astronomical works which help us in forming a true estimate of the progress made by the ancient Hindus in the science of mathematics.

I shall now briefly enumerate some of the important discoveries made by the Hindu mathematicians — discoveries which were communicated to Europe through the Arabs.

Arithmetic

Notation. The place-value notation which has been adopted by the whole of the civilised world was perfected in India about the beginning of the Christian era. The symbol *śunya* (« zero ») occurs in the Piṇgala *Chandaḥ Sûtra*, a work belonging to the third century B. C. The *Bakhshâlî Manuscript* (c. 200), the *Sûrya-siddhânta*, the *Âryabhaṭîya* (499), the *Pañca-siddhantikâ* (505), and all later mathematical works use the place-value numerals. In fact no mathematical work exist which does not use these numerals The earliest epigraphic use of the numerals is found in an inscription of 595 A. D. We find that by the middle of the seventh century these numerals had reached as far east as Cambodia where we find inscriptions dated 605, 606, 608 of the Śaka era corresponding to 683, 684 and 686 A. D. In the west the fame of these numerals appears to have reached Syria, for they are mentioned by the Syrian scholar SEVERUS SEBOKHT (c. 650) [1].

Multiplication. The Hindus had several methods of multiplication. Of these I shall mention two : (i) *kapâṭa-sandhi* method (ii) *sthânakhaṇḍa* method. In the *kapâṭa-sandhi* (« door-

[1] Cf. DATTA and SINGH : *History of Hindu Mathematics* I, Lahore (1935) Chap. i.

junction ») method the multiplicand and multiplier are arranged as in the junction of a door and the method proceeds as below ;

Ex. To multiply 135 by 12.

The numbers are placed as

$$\begin{array}{r} 12 \\ 135 \end{array}$$

Multiplying 12 by 5 and setting down the result below, we have

$$\begin{array}{r} 12 \\ 1360 \end{array}$$

The multiplier is moved one place to the left, thus

$$\begin{array}{r} 12 \\ 1360 \end{array}$$

Multiplying by 3, adding the result and then moving the multiplier, we have

$$\begin{array}{l} 12 \\ 1420 \end{array}$$

Then, after multiplication and addition as before, we have

$$1620$$

as the result.

This method was performed on a *pâṭi* (« wooden board ») by means of a piece of chalk, and figures were rubbed out and new ones substituted in their places as the work proceeded.

There were several variations of this method. In Arabia the method occurs in the works of AL-ḪWÂRIZMÎ (825), AL-NASAVÎ (1025), AL-HAṢṢÂR (1175) and several others. It was called by AL-NASAVÎ *tarik al-hindi* (« the method of the Hindus »). In Europe it occurs in the works of MAXIMOS PLANUDES.

(ii) The *sthâna-khaṇḍa* method is the same as our present method. BHÂSKARA II (1150) describes it thus :

« Multiply separately by the places of figures and add together ».

The method occurs in all the Hindu mathematical works from BRAMHAGUBTA (628) onwards.

Division. The modern method of long division is of Hindu origin. It occurs in all the treatises of the Hindus. Thus ŚRÎDHARA (750) says:

« Having removed the common factor, if any, from the divisor and dividend, divide by the divisor the digits of the dividend one after another in the inverse order »[2].

Extraction of roots. Methods of extracting square and cube roots are found in the *Âryabhaṭîya* (499) and all later works. Our modern methods are contractions of ÂRYABHAṬA'S methods[2].

The Hindu methods travelled to Europe through Arabia and occur precisely in the same form in the works of PEURBACH (1423-1461), CHUQUET (1448), LA ROCHE (1520), CATANEO (1546) and others.

The Rule of Three. The Rule of Three owes its origin to the Hindus. In India it was called *trairâśika*, which translated literally means « three terms ». Thus the name of the method in the European languages is a translation of the Sanskrit name. The rule was stated by ÂRYABHATA I (499) as follows:

« In the Rule of Three the *phala* (« fruit ») being multiplied by the *icchâ* (« requisition ») is divided by the *pramâṇa* (« argument »). Almost all other Hindu writers use the same terminology and state the rule in similar words. ÂRYABHAṬA II (950), uses a different terminology. He says:

« The first term is called *mâna*, the middle term *vinimaya* and the last one *icchâ*. The first and the last are of the same denomination. The last multiplied by the middle and divided by the first gives the result »[3].

With the above may be compared the following statement of the method given by DIGGES (1572), an English writer:

« Worke by the rule ensueing . . . . . Multiplie the last number by the second and divide the product by the first number » . . . . . « In the placing of these numbers this must be observed that the first and third be of one denomination ».

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[2] See DATTA and SINGH, l. c pp. 143 f.

[3] *Ibid*, p. 205.

Problems. [4] Many of the problems of our present arithmetic are of Hindu origin. I shall quote two typical problems whose solutions were given by BRAHMAGUPTA (628):

(1) In what time will a given sum s, the interest on which for t months is r, become k times itself?

(2) In what time will four fountains, being let loose together, fill a cister which they would severally fill in a day, in half a day, in a quarter and in a fifth part of a day?

Algebra

It is generally admitted that the science of algebra owes its origin to the Hindus. In this science the Hindus made remarkable progress at a very early date. I shall enumerate some of the important results obtained by them.

Name. BRAHMAGUPTA (628) called algebra *kuṭṭaka gaṇita* or simply *kuṭṭaka*. The term *kuṭṭaka* meaning « pulveriser » refers to a branch of algebra dealing particularly with the subject of indeterminate equations of the first degree. It is interesting to find that this subject was considered so important by the ancient Hindus that the whole science was named after it. PRTHUDAKASVÂMÎ (860) uses the term *bîjagaṇita* for algebra. *Bîja* means « element » and also « analysis » and *gaṇita* means « the science of calculation ». Hence *bîjagaṇita* literally means « the science of calculation with elements » as opposed to numerical calculation, or « the science of analytical calculation ». All later writers use the term *bîjagaṇita*. Sometimes algebra was called by the name *avyakta-gaṇita* i. e., « the science of calculation by the unknown ».

Determinate Equations. BRAHMAGUPTA used the terms *samakaraṇa, samî-karaṇa* (« making equal ») or more simply *sama* (« equation ») for equations. It thus appears that our present term equation is a translation of the Hindu term.

The geometrical solution of a linear equation in one unknown is found in the Śulba sûtras of BAUDHÂYANA (c. 800 B. C.). A number of linear equations are found in the *Bakhshâlî Manus-*

[4] For references see DATTA and SINGH, l. c., p. 222 and p. 234.

cript. They are solved by the method of « false position ». In the works of later Hindu mathematicians beginning from that of ÂRYABHAṬA I (499), the solution is algebraic. It may be mentioned here that the rule of « false position » does not occur in Hindu algebra, as the Hindus possessed a well developed algebraic symbolism at a very early date.

The *Bakhshâlî manuscript* as well as later treatises contain linear equations of the form

$$x_1 + x_2 = a_1 \; ; \; x_2 + x_3 = a_2 \; ; \; \ldots\ldots \; x_n + x_1 = a_n \, ,$$

which seems to have been very popular.

Quadratic equations. In the Śulba sûtras we come across the quadratic equation

$$7\,x^2 + \frac{1}{2}x = 7\frac{1}{2} + m$$

with its approximate solution given by KÂTYÂYANA [5] as

$$x^2 = 1 + \frac{m}{7} \, .$$

The general solution of the simple quadratic equation

$$4\,h^2 - 4\,d\,h = -\,c^2$$

is found in the early Jaina canonical works (500-300 B. C.), and also in the *Tatvarthâdhigama sûtra* [6] (c. 150 B. C.), as

$$h = \frac{1}{2}\left(d - \sqrt{d^2 - c^2}\right) .$$

A formula for determining the number of terms of an arithmetic series whose first term (a), common difference (b) and sum (s) are known, is found in all the mathematical treatises. The *Bakhshâlî manuscript* [7] gives the solution as

$$n = \frac{\sqrt{8\,b\,s + (2\,a - b)^2} - (2\,a - b)}{2\,b}$$

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[5] DATTA, *The Science of the Sulba*, Calcutta, 1932, p. 166.

[6] DATTA, *Quellen und Studien zur Ges. d. Math.* (1931), pp. 245-54.

[7] Folio 65 verso, continued on folios 64 recto and 56 recto and verso.

ÂRYABHAṬA I (499) gives the result in the form

$$n = \frac{\sqrt{8\,b\,s + (2\,a - b)^2} - 2\,a + 1}{b}$$

BRAHMAGUPTA (628) states the solution of the equation

$$a\,x^2 + b\,x = c$$

as follows :

« The quadratic : the absolute quantity multiplied by four times the coefficient of the square of the unknown is increased by the square of the coefficient of the middle (*i. e.*, x) ; the square-root of the result being diminished by the coefficient of the middle and divided by twice the coefficient of the square of the unknown, is (the value of) the middle (*i. e.*, x) [8].

The method by which the above result is obtained was explained by ŚRÎDHARA (750) thus :

« Multiply both the sides (of the equation) by a known quantity equal to four times the coefficient of the square of the unknown ; add to both sides a known quantity equal to the square of the coefficient of the unknown ; then (extract) the square-root ».

Writing the equation as

$$a\,x^2 + b\,x = c\,,$$

we have

$$4\,a^2\,x^2 + 4\,a\,b\,x = 4\,a\,c$$

or $$(2\,a\,x + b)^2 = 4\,a\,c + b^2$$

Therefore $$2\,a\,x + b = \pm\sqrt{4\,a\,c + b^2}$$

giving $$x = \pm\frac{\sqrt{4\,a\,c + b^2} - b}{2\,a}$$

The original treatise of ŚRÎDHARA in wich the above rule occurs is lost. The rule is, however, preserved in quotations by BHÂSKARA II by JÑANARÂJA and by SURYADÂSA. It is difficult to say whether ŚRÎDHARA used both the signs of the radical or not. There is, however, definite evidence of the use of both the signs

[8] *Brâhma-sphuṭa-siddhânta* of BRAHMAGUPTA ed. by S. DVIVEDI, Benares, 1902, xii, 15.

of the radical by MAHÂVÎRA[9] (850), who gives the solution of the equation

$$\frac{a}{b} x^2 - x + c = 0$$

as

$$x = \frac{\frac{b}{a} \pm \sqrt{\left(\frac{b}{a} - 4c\right)\frac{b}{a}}}{2}$$

Various problems involving simultaneous quadratic equations of the following forms are found in the works of Hindu mathematicians :

$$\left.\begin{matrix} x - y = d \\ x\,y = b \end{matrix}\right\} \text{(i)} \quad \left.\begin{matrix} x + y = a \\ x\,y = b \end{matrix}\right\} \text{(ii)} \quad \left.\begin{matrix} x^2 + y^2 = c \\ x\,y = b \end{matrix}\right\} \text{(iii)} \quad \left.\begin{matrix} x^2 + y^2 = c \\ x + y = a \end{matrix}\right\} \text{(iv)}$$

ÂRYABHAṬA I (499) gives (i) and (iv), BRAHMAGUPTA (628) gives (i), (iii) and (iv), and MAHÂVÎRA (850) gives all of them. These equations are also found in the works of later writers.

The equations

$$\left.\begin{matrix} x^2 - y^2 = m \\ x - y = n \end{matrix}\right\} \qquad \left.\begin{matrix} x^2 - y^2 = m \\ x + y = p \end{matrix}\right\}$$

were given great prominence by all Hindu mathematicians. BRAHMAGUPTA gives to the process the name *viṣama karana*.

Indeterminate equations of first degree. The earliest Hindu algebraist to treat of indeterminate equations of the first degree, as far as we know, was ÂRYABHAṬA I (499). He gave a method for finding the general solution in positive integers of the simple indeterminate equation

$$b\,y - a\,x = c$$

for integral values of a, b, c and further indicated how to extend his method to get positive integral solutions of simultaneous indeterminate equations of the first degree. BHÂSKARA I (522) showed that the same method might be applied to solve $b\,y - a\,x = -c$ and further that the solution of this equation would

[9] *Gaṇita-sâra-samgraha* of MAHÂVÎRA, iv. 59. Also iv. 57 ; iv. 61-4.

follow from that of $b\,y - a\,x = -1$. BRAHMAGUPTA and later writers simply adopted the methods of ÂRYABHAṬA I and BHÂSKARA I. About the middle of the tenth century ÂRYABHAṬA II[10] improved them by pointing out how the operations can in certain cases be abridged considerably. He also noticed the cases of failure with respect to the equation of the form $b\,x - a\,y = +c$. These results reappear in the works of posterior writers.

Problems for whose solution the ancient Hindus investigated indeterminate equations of the first degree may be classified into three varieties. The problem of one varity is : to find a number (N) which when divided by two given numbers a, b will leave as remainders two other given numbers R_1, R_2. Thus we have

$$N = a\,x + R_1 = b\,y + R_2$$

Hence $$a\,x - b\,y = (R_2 - R_1)$$

Or $$a\,x - b\,y = \pm\, c\,,$$

where $c = R_1 - R_2$. In problems of the second kind we are required to find a number x such that its product with a given number α being increased or decreased by another number γ and then divided by a third number β will leave no remainder. In other words we shall have to solve

$$\frac{\alpha\,x \pm \gamma}{\beta} = y$$

in positive integers. The third variety of problems lead to equations of the form

$$a\,x + b\,y = \pm\, c.$$

The particular case $a\,x - b\,y = \pm 1$ was given prominence and was considered separately by BHÂSKARA I, BRAHMAGUPTA and BHÂSKARA II, who showed how the solution in positive integers of the general equation $a\,x - b\,y = \pm\, c$ could be derived from the particular case.

Rules for, the solution of simultaneous indeterminate equations of the first degree are found in the works of BRAHMAGUPTA[11]

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[10] For details see DATTA and SINGH, l.c., Pt. II (to appear soon).

[11] *Brâhma-sphuṭa-siddhânta*, xviii. 51.

Srîpati[12] (1039) and Bhâskara II[13] (1150). Such equations are furnished by problems of the following type:

« (Four merchants) who have horses 5, 3, 6 and 8 respectively, camels 2, 7, 4 and 1; mules 8, 2, 1 and 3, and oxen 7, 1, 2 and 1 in number, are all owners of equal wealth. Tell me instantly the prices of the horses etc ».

Find a number N which when severally divided by α_1, α_2, ... α_n leaves remainders β_1, β_2, ... β_n respectively.

Indeterminate equations of second degree. The equations

$$N x^2 + c = y^2$$

and

$$N x^2 + 1 = y^2$$

were considered by Brahmagupta (628) who established the following lemmas to obtain their general solutions in integers:

(i) If $x = \alpha$, $y = \beta$ be a solution of the equation

$$N x^2 + k = y^2$$

and $x = \alpha'$, $y = \beta'$ be a solution of

$$N x^2 + k' = y^2$$

then $x = \alpha \beta' \pm \alpha' \beta$, $y = \beta \beta' \pm N \alpha \alpha'$ is a solution of the equation

$$N x^2 + k k' = y^2$$

In other words, if

$$N \alpha^2 + k = \beta^2$$

and

$$N \alpha'^2 + k' = \beta'^2$$

then

$$N (\alpha \beta' \pm \alpha' \beta)^2 + k k' = (\beta \beta' \pm N \alpha \alpha')$$

(ii) If $x = \alpha$, $y = \beta$ be a solution of

$$N x^2 + k = y^2,$$

then $x = 2 \alpha \beta$, $y = \beta^2 + N \alpha^2$ is a solution of

$$N x^2 + k^2 = y^2 .$$

(iii) If $x = \alpha$, $y = \beta$ be a solution of

$$N x^2 + k^2 = y^2$$

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12 *Siddhânta Śekhara*, xiv. 15-16.

13 *Bîjagaṇita* of Bhâskara II ed. by S. Dvivedi and revised by Murlidhara Jha, Benares, 1927, p. 76.

then $x = \alpha/k$, $y = \beta/k$ is a solution of

$$N x^2 + 1 = y^2 .$$

The above lemmas were rediscovered in Europe by EULER in 1764 and by LAGRANGE in 1768.

BRAHMAGUPTA's method for solving $N x^2 + 1 = y^2$ consists in obtaining impirically α, k and β, such that

$$N \alpha^2 \pm k = \beta^2 ,$$

and then using his lemmas to get the general solution of $N x^2 + 1 = y^2$.

ŚRÎPATI [14] (1039) seems to have been the first to give the solutions

$$x = \frac{2\, m}{m^2 - N} , \quad y = \frac{m^2 + N}{m^2 - N} ,$$

where m is any rational number. This appears in the works of later Hindu mathematicians. This solutions was rediscovered in Europe by BROUNCKER (1657).

To obtain solutions in positive integers BRAHMAGUPTA uses the auxiliary equation $Nx^2 + 4 = y^2$, and obtains

$$x = \frac{1}{2}\, \alpha\, \beta\, (\beta^2 + 3)\, (\beta^2 + 1)$$

$$y = (\beta^2 + 2)\, \frac{1}{2}\, (\beta^2 + 3)\, (\beta^2 + 1) - 1$$

Putting $p = \alpha\, \beta$, and $q = \beta^2 + 2$, we can write the above as

$$x = \frac{1}{2}\, p\, (q^2 - 1)$$

$$y = \frac{1}{2}\, q\, (q^2 - 3).$$

This solution was rediscovered by EULER.

ŚRÎPATI [15] expressly observes that if $k = \pm 1, \pm 2,$ or ± 4, the roots obtained by BRAHMAGUPTA's method are integral, but no method seems to have been known to him for finding a root of

$$N \alpha^2 \pm k = \beta^2$$

k having one of the above values. BHÂSKARA II (1150) however,

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14 *Siddhânta Śekhara*, XIV. 33.

15 *Ibid.*, XIV. 32.

succeeded in evolving a simple method of getting two integral solutions of the above. This method is called by him *cakravâla* (« the cyclic method »). Thus BHASKARA II succeeded in solving

$$N x^2 \pm c = y^2$$

completely.

BHÂSKARA II also succeeded in obtaining the general solutions of the following equations :

(i) $a x^2 + b x + c = y^2$

(ii) $a x^2 + b x + c = \alpha y^2 + \beta y + \gamma$

(iii) $a x^2 + b y^2 \pm c = z^2$

(iv) $a x^2 + b x y + c y^2 = z^2$

There are many other types of equations that occur in the works of BHÂSKARA II. These can not be mentioned here. But before I conclude this topic I wish to point out that BHÂSKARA II obtained the solution of the double equation

$$a x^2 + b y^2 + c = u^2$$
$$\alpha x^2 + \beta y^2 = \gamma + v^2$$

He takes the example

$$x^2 + y^2 - 1 = u^2$$
$$x^2 - y^2 - 1 = v^2$$

and gives its solution as

$$x = \frac{(4 m^4 + n^4) + r^2}{(4 m^4 + n^4) - r^2}, \qquad u = \frac{2 r (2 m^2 + n^2)}{(4 m^4 + n^4) - r^2},$$

$$y = \frac{4 m n r}{(4 m^4 + n^4) - r^2}, \qquad v = \frac{2 r (2 m^2 - n^2)}{(4 m^4 + n^4) - r 2^2}$$

where m. n and r are any rational numbers.

A particular case of the above solution, for $r = s/t$, was obtained by GENOCCHI [16] (1851). Another particular case was solved by E. CLERE [17] (1850). A third easily deducible solution was given by DRUMMOND [18] in 1902.

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16 *Nouv. Ann. Math.* X (1851), pp. 80-85.
17 *Nouv. Ann. Math.* IX (1850), pp. 116-118.
18 *American Math. Monthly*, IX (1902), p. 232.

Geometry

It is said that geometry originated in Egypt, near about the second millinium before Christ, in connection with land survey. The Greeks who learnt the science from the Egyptians, carried it to a very high degree of perfection. Hindu geometry commenced at a very early period, certainly not later than that in Egypt, in connection with the construction of altars for Vedic sacrifices. In course of time it outgrew its original purpose and was cultivated for its own sake also. Indeed there is no doubt that the study of geometry as a science began very early in India. This early Hindu geometry was in advance of contemporary Egyptian or Chinese geometry. Though the Hindu achievements in geometry were completely overshadowed by the success of the Greeks at a later stage, « it should never be forgotten that the world owes its first lessons in geometry not to Greece, but to India »[4]. In fact the early Hindus obtained many interesting and ingenious results, results which are generally attributed to the Greeks but which should be credited to the Hindus. In their treatises, the ancient writers have not left any proofs or demonstrations of the propositions discovered by them. Only the bare results are left recorded.

The pythagorean theorem. The *Śulba sûtras* are the earliest geometrical works of the Hindus. The following results are either expressly stated or implied in the various constructions found in the *Śulba sûtras* :

(1) The diagonals of a rectangle bisect each other. They divide the rectangle into four parts two and two (vertically opposite) which are equal in all respects.

(2) The diagonals of a rhombus bisect each other at right angles.

(3) An isosceles triangle is divided into two equal halves by the line joining the vertex to the middle point of the base.

(4) The area of a square formed by joining the middle points of the sides of another square is half that of the original one.

(5) A quadrilateral formed by the lines joining the middle points of the sides of a rectangle is a rhombus whose area is half that of the rectangle.

(6) A parallelogram and a rectangle on the same base and within the same parallels have the same area.

(7) The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.

(8) If the sum of the squares on two sides of a triangle be equal to the square on the third side, then the triangle is right angled.

Theorem (7) above has been stated by BAUDHÂYANA (*c.* 800 B. C.) in the following words:

« The diagonal of a rectangle produces both areas which its length and breadth produce separately »[19].

ÂPASTAMBA[20] and KÂTYÂYANA[21] give the above theorem in almost identical terms. The theorem is now universally attributed to the Greek PYTHAGORAS (*c.* 540 B. C.) though « no really trustworthy proof exists that it was actually discovered by him »[22]. The Chinese knew the numerical relation for the particular case $3^2 + 4^2 = 5^2$, probably in the time of CHOU-KONG (died 1105 B. C.). The Kahun Papyrus (*c.* 2000 B. C.) contains four similar numerical relations, all of which can be derived from the above one. As for the Hindus, one instance of that kind $39^2 = 36^2 + 25^2$, occurs in the *Taittirîya Saṁhitâ*[23] (before 2000 B. C.). It should be noted that this instance is different from that known to other early nations.

Although particular instances of the theorem are found amongst several ancient nations, the first enunciation of the theorem in its general form occurs first in India. It is not improbable that BAUDHÂYANA possessed a proof of the theorem. But what this proof was will never be known with certainty. BÜRK, HANKEL, THIBAUT and DATTA are of opinion that BAUDHÂYANA knew a proof of the theorem[24].

Values of π. The Hindus studied geometry mainly with a view to its applications. Thus they devoted attention to finding formulae for areas etc. For finding the area of the circle the Hin-

19 *Baudhâyana Śulba*, i. 48.
20 *Apastamba Śulba*, i. 4.
21 *Kâtyâyana Śulba*, ii. 11.
22 HEATH, *History of Greek Mathematics*, Vol. I, p. 144 f.
23 vi. 2.4.6.; it occurs also in the *Śatapatha Brâhmaṇa*, x. 2.3.4.
24 See DATTA, *The Science of the Śulba*, Calcutta, 1932, Ch. ix.

dus gave several values of π. In the *Śulba sûtras* we find the values

$$\pi = 4 \Big/ \left\{ 1 + \frac{1}{3} (\sqrt{2} - 1) \right\}^2 = 3.0883 \ldots\ldots$$

$$\pi = 4 \left\{ 1 - \frac{1}{8} - \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8} \right\} = 3.0885 \ldots\ldots$$

$$\pi = 4 \left(\frac{13}{15}\right)^2 = 3.004 \ldots\ldots$$

In 499 A. D., ÂRYABHAṬA I determined an extremely accurate value of π which was better than all the Greek values:

$$\pi = \frac{62,832}{20,000} = 3\ 1416$$

The value $\pi = \sqrt{10}$ is due to the Hindus and is first recorded in the *Sûryaprajñapti* (sûtra 20). It also occurs in the *Tattvârthâdhigama sûtra*[25] (*c.* 150 A. D.). ÂRYABHAṬA'S value of π was improved upon in the fifteenth and sixteenth centuries.

Application of algebra. BRAHMAGUPTA made some very remarkable contributions to the properties of an inscribed convex quadrilateral. If Σ denote the area and m, n the diagonals of an inscribed quadrilateral whose sides are a, b, c, d, then his results are

(1)[3] $$\Sigma = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $2s = a + b + c + d$.

(2)[4] $$m = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}},$$

$$n = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$$

The Hindus made use of their knowledge of indeterminate equations to propose and solve a variety of problems concerning the sides and areas of plane figures. And thus they appl ed their algebra to geometry.

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25 iii. 2.

The earliest attempt to obtain right angled triangles having a given side is found in the *Śulba*. In particular, we find two such triangles having the sides (a, $3\,a/4$, $5\,a/4$) and (a, $5\,a/12$, $15\,a/12$). BRAHMAGUPTA (628) proposed to find all right angled triangles having the side a and the others sides *rational*. His solution is

$$a\,,\ \frac{1}{2}\left(\frac{a^2}{n} - n\right),\ \frac{1}{2}\left(\frac{a^2}{n} + n\right)$$

MAHÂVÎRA [26] (850) also gives the above solution. BHÂSKARA II has found another solution [27]

$$a\,,\ \frac{2\,n\,a}{n^2 - 1}\,,\qquad n\left(\frac{2\,n\,a}{n^2 - 1}\right) - a\,.$$

Similar reults occur for finding all right angled triangles having a given hypotenuse.

MAHÂVÎRA proposes and solves the following problems :

(1) « In a rectangle the area is numerically equal to the perimeter ; in another the area is numerically equal to the diagonal. What are the sides (in each of these cases) ? » [28].

(2) Find a rectangle of which twice the diagonal, thrice the base, four times the upright and twice the perimeter together equal the area (numerically) » [29].

(3) The perimeter of a rectangle is unity. Tell me quickly, after calculating, what are its base and upright » [30].

(4) Find a rectangle in which twice the diagonal, thrice the base, four times the upright and the perimeter together equal unity » [31].

(5) Find all isosceles triangles with rational integral sides and areas [32].

(6) Find two isosceles triangles whose perimeters, as also their areas, are equal or related in a given proportion [33].

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26 *Gaṇita-sâra-saṁgraha*, vii. 97 f.

27

28 Ibid. vii. 115.

29 Ibid. vii. 117.

30 Ibid. vii. 118.

31 Ibid. vii. 119.

32 Ibid. vii. 108. Given also by BRAHMAGUPTA, l. c., xii. 33.

33 Ibid. vii. 137.

(7) Find all rational scalene triangles [34].

(8) Find all triangles having a given area [35].

BRAHMAGUPTA has shown how to find an isosceles trapezium whose sides, diagonals, altitude, segments and areas can all be expressed in rational numbers [36]. He further formulates the following remarkable proposition :

Find all quadrilaterals which will be inscribable within circles, whose sides, diagonals, perpendiculars, segments, areas and also the diameter of the circumscribed circles will be expressible in rational *integers* [37].

Solutions of the above have also been given by MAHÂVÎRA, ŚRÎPATI, BHÂSKARA II and others. Finally MAHÂVÎRA has given the solution of the following remarkable problem :

Find all rational triangles and quadrilaterals inscribable in a circle of given diameter.

Trigonometry

The science of trigonometry is said to have originated with the Greek HIPPARCHOS (*c.* 140 B. C.). But the Greeks used the chord of the angle instead of the sine or cosine functions. The chord of a given arc, which corresponds to the side of a certain polygon inscribed in a certain circle naturally attracted more the attention of a race of geometers. The function *sine* is in reality an invention of the Hindus. The change from the chord of an arc to the semi chord may appear to be simple, but the first conception of this change did not come easily as will be realised by the fact that it did not occur to the Greek mind. The modern term *sine* is itself derived from Hindu sources. The Sanskrit term for the function is *jyâ* or *jîvâ*. The early Arab writers adopted this term and called it *jîba*. This latter term was subsequently corrupted in their tongue into *jaib*. The early latin translators of the Arabic works, such as GHERARDO OF CREMONA (*c.* 1150) rendered it as *sinus* which means « bosom » or « bay ». The Hindu term

34 Ibid. vii. 110. Given by BRAHMAGUPTA, l. *c.*, xii. 34.

35 Ibid. vii. 160-62.

36 BRAHMAGUPTA, l. *c.*, xii. 36.

37 Ibid. xii. 38.

for the cosine was *koṭi-jyâ*, often abbreviated into *ko-jyâ*. When jyâ became *sinus*, ko-jyâ was naturally rendered as *co-sinus*.

The Hindus used three functions of an angle, namely, *jyâ*, *koti-jyâ* and *utkrama-jyâ*, which correspond respectively to our modern sine, co-sine and versed sine functions.

The study of functions of complements or supplements of arcs, is a contribution of the Hindus to trigonometry. MAÑJULA (932) states :

« The *jyâ* is positive or negative in the quadrants above or below (the prime line) ; and the *koṭi* is positive, negative, negative, and positive successively ».

The Hindus knew trigonometrical formulae corresponding to the following :

(1) $$\sin^2\theta + \cos^2\theta = 1$$

(2) $$\sin(\theta/2) = \sqrt{\frac{1 - \cos\theta}{2}}$$

(3) $$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\,.$$

(4) $$\sin^2 2\theta + \operatorname{versin}^2 2\theta = \sin^2\theta\,,$$

(5) $$\sin(45 \pm \theta) = \frac{1 \pm \sin 2\theta}{2}\,,$$

(6) $$\frac{1}{2}\sin(\alpha - \beta) = \frac{1}{2}(\sin\alpha - \sin\beta)^2 + (\cos\alpha - \cos\beta)^2\,.$$

Of the above the first three were also known to the Greeks ; the fourth was stated by VARÂHAMIHIRA (505) ; the remaining two are due to BHÂSKARA II (1150).

The Hindu astronomers were also acquainted with the following formulae of spherical trigonometry :

$$\cos c = \cos a\cos b + \sin a\sin b\cos C,$$

$$\cos A\sin c = \cos a\sin b - \sin a\cos b\cos C,$$

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}\,.$$

Every Indian treatise on Astronomy contains a table of s nes and versed sines as also of the'r d fferences, calculated for the angle $3\frac{3}{4}^{\circ}$ and its multiples. In this connection we have the following formula given in the *Sûrya-siddhânta* (*c.* 400) :

$$\sin (n+1)\,\theta - \sin n\,\theta = \sin n\,\theta - \sin (n-1)\,\theta - \frac{\sin n\,\theta}{225}.$$

The above formula is used for calculating the table of sines. It depends on the calculation of their second differences. DELAMBRE thinks it to be curious and remarks : « This differential process has not up to now been employed except by BRIGGS who himself did not know that the constant factor was the square of the chord or of the interval, and who could not obtain it except by the fact by comparing the second differences obtained in a different manner . . . » Here then is a method which the Hindus possessed but which is found neither amongst the Greeks nor amongst the Arabs » [38].

Calculus

MAÑJULA (932) has given the differential formula

$$\delta u = \delta v \pm e \cos \theta\, \delta\theta$$

corresponding to the quation

$$u = v \pm e \sin \theta\,.$$

This formula he obtained in connection with the determination of the true motion of a planet. He says :

« True motion in *minutes* is equal to the cosine (of the mean anomaly) multiplied by the difference (of the mean anomalies) and divided by the *cheda*, added or subtracted contrarily (to the mean motion) » [39].

The differential of $\sin\theta$ is termed by BHÂSKARA II as the *tâtkâlika bhogya-khaṇḍa*, and the differential formula

$$\delta(\sin\theta) = \cos\theta\,\delta\theta$$

has been proved by him. It has been used by him for calculating the *ayana-valana* (« angle of position »). He has further made use of the following two theorems :

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[38] DELAMBRE, *Histoire de l'Astronomie Ancienne*, I, Paris, 1817, pp. 457-60.

[39] *Laghumânansa*, ii. 7. Here *cheda* (« divisor ») $= \frac{1}{e}$.

(1) When a variable attains its maximum value, its differential vanishes.

(2) When a planet is either in apogee or in perigree the equation of the centre vanishes and, therefore, for some intermediate position the increment of the equation of centre also vanishes [40].

Another very remarkable formula for the differential of a function involving the inverse sine function as well as the quotient of two functions one of which is under the radical sign, is the following.

$$\delta \left\{ \sin' \left(\frac{a \sin \theta}{\sqrt{b^2 + 2\,a\,b \cos \theta + a^2}} \right) \right\}$$

$$= \left(\sqrt{b^2 + 2\,a\,b \cos \theta + a^2} - \frac{b\,(b + a \cos \theta)}{\sqrt{b^2 + 2\,a\,b \cos \theta + a^2}} \right) \frac{\delta\,\theta}{\sqrt{b^2 + 2\,a\,b \cos \theta + a^2}}.$$

This result occurs in the works of ÂRYABHAṬA [41] II (950) and BHÂSKARA II (1150) [42].

Lucknow, University AVADHESH NARAYAN SINGH

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40 These results occur in the Golâdhyâya Spastâdhikâra vâsanâ of the *Siddhânta Śiromaṇi*, and were first noted by SUDHAKARA DVIVEDI.

41 *Mahâ-siddhânta*, iii. 27.

42 *Siddhânta Śiromaṇi*, Grahagaṇita, Spaṣṭâdhikâra, 39.

REVISTA DE MATHEMATICA HINDU ANTE SECULO XII.

A. memora operes fundamentale de mathematica hindu ante seculo XII. Postea illo refer principale resultatus acquisito. *Arithmetica* : nos debe ad mathematicos hindu moderno notatione de numeros cum symbolo zero ; methodos de multiplicatione, divisione et extractione de radices quadrato et cubico, etc. *Algebra* : Hindus cognosce et solve aequationes lineare et quadratico, aequationes indeterminato de primo et de secundo gradu. *Geometria* : plure theorema, quem A. refer, es noto ad Hindus et ultra illos cognosce aliquo valore de π et fac applicatione de algebra ad geometria. *Trigonometria* : A. refer plure formula noto ad Hindus. *Calculo differentiale* : mathematicos hindu cognosce aliquo aequatione differentiale et aliquo theorema de calculo.