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Man Mohan
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Dedicated to the memory of
U. N. SINGH
The Founder of the Indian Society for History of Mathematics

U. N. Singh (November 19, 1920 – April 9, 1989)

Professor B. S. Yadav
(31 July 1931 – 24 February 2010)

B. S. Yadav was born in Mathura, India. He dedicated his whole life to the cause of mathematics and the history of mathematical sciences. After obtaining his B Sc (1953) and M Sc (1957) degrees from Agra University and Aligarh Muslim University, respectively, he pursued research in mathematics at M. S. University, Baroda, for which he was awarded the Ph.D. in 1965. Through his extended research career he made significant contributions to functional analysis, operator theory, Fourier analysis, and studies in the history of mathematics. His teaching career started at M. S. University, Baroda (Lecturer, 1956–1964). He then moved to Sardar Patel University, Vallabh Vidya Nagar, Gujarat (Reader, 1964–1970), and later to Delhi University (Reader, 1970–1976 & Professor, 1976–1996). He played an influential role at various universities and Indian boards of studies. Besides serving on numerous important committees such as the Academic Council and Research Degree Committee, he also worked as Head of the Department of Mathematics and as Dean of the Faculty of Mathematical Sciences in Delhi University for several years. He was widely traveled, with over 25 visits to various universities abroad, and was a Visiting Professor of Mathematics at Cleveland State University, Cleveland, Ohio, USA during 1987–1988. He was associated with several academic bodies and research journals. He was a founding member of the Indian Society for History of Mathematics, founded by his guide and mentor, the late Professor Udita Narayana Singh in 1978. He was deeply and emotionally involved with this society and was the main driving force behind its activities for over two decades. He served the society as its Administrative Secretary over a long period and was the Editor of the society's bulletin *Ganita Bhāratī* in recent years. His tireless efforts contributed a good deal in creating a new awareness in the study and research of the history of mathematics in the Indian context. He inspired many, young and old, to enter into new ventures in the study of ancient, medieval, and modern history of mathematics. He continued to be engaged with mathematics to his last breath.

B. S. Yadav was an organizer par excellence. He organized many national and international conferences, seminars, and workshops. He always encouraged the publication of their proceedings and had been anxiously awaiting an early release of this book before his untimely death, but the cruel hands of destiny did not allow the fulfillment of his wish.

B. S. Yadav (31 July 1931 – 24 February 2010)

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Foreword

In recent years, the study of the history of Indian mathematics has accelerated dramatically. This is illustrated not only by the holding of international meetings, including the 2001 meeting initiated by the Indian Society for History of Mathematics and the 2003 joint meeting of the Indian Mathematical Society and the American Mathematical Society, one of whose themes was the history of Indian mathematics, but also by the increasing number of research articles on the subject being published in international journals. However, it is still true that the history of mathematics in India has not yet achieved the recognition that it deserves.

The knowledge of ancient and medieval Indian mathematics was always kept alive in India, but once the British ruled the subcontinent, the education system they introduced did not value native Indian contributions to mathematics. Thus the knowledge of these ideas tended to be buried in the rush to learn modern European mathematics. Nevertheless, there were British scholars, and later scholars from elsewhere in Europe, who attempted to bring some knowledge of this mathematics back to Europe.

Probably the earliest substantive knowledge in Europe of the history of Indian mathematics — in the early nineteenth century — was due to Henry Thomas Colebrooke, who, after collecting various Sanskrit mathematical and astronomical texts, published in 1817 his *Algebra with Arithmetic and Mensuration* from the Sanskrit of Brahmagupta and Bhāskara. Thus, parts of the major texts of two of the most important medieval Indian mathematicians became available in English, along with excerpts from Sanskrit commentaries on these works. Then in 1835, Charles Whish published a paper dealing with the fifteenth-century work in Kerala on infinite series, and in 1860 Ebenezer Burgess published an English translation of the *Sūrya-Siddhānta*, a major early Indian work on mathematical astronomy. In 1874 Hendrik Kern produced an edition of the *Āryabhaṭīya* by Āryabhaṭa, while George Thibaut wrote a detailed essay on the *Śulbasūtras*, which was published, along with his translation of the *Baudhāyana-Śulbasūtra*, in the late 1870s. Indian researchers around the same time, including Bapu Deva Sastri, Sudhakara Dvivedi, and

S. B. Dikshit, began looking again into their mathematical heritage. Although their own work was originally published in Sanskrit or Hindi, their research paved the way for additional translations into English.

Despite the availability of some Sanskrit mathematical texts in English, it still took many years before Indian contributions to the world of mathematics were recognized in major European historical works. Of course, European scholars knew about the Indian origins of the decimal place-value system. But in part because of a tendency in Europe to attribute Indian mathematical ideas to the Greeks and also because of the sometimes-exaggerated claims by Indian historians about Indian accomplishments, a balanced treatment of the history of mathematics in India was difficult to achieve. Perhaps the best of such works in the first half of the twentieth century was the *History of Indian Mathematics: A Source Book*, published in two volumes by the Indian mathematicians Bibhutibhusan Datta and Avadhesh Narayan Singh in 1935 and 1938. More recent book-length surveys of Indian mathematics include *The History of Ancient Indian Mathematics*, by C. N. Srinivasiengar (1967) and *Mathematics in Ancient and Medieval India*, by A. K. Bag (1979). Briefer surveys as parts of longer works include the article by S. N. Sen on "Mathematics," which appears in *A Concise History of Science in India*, edited by D. M. Bose, S. N. Sen, and B. V. Subbarayappa (1971) and several chapters in *The Crest of the Peacock: Non-European Roots of Mathematics*, by George Gheverghese Joseph (1991, 2000). But given that a large number of ancient and medieval Indian manuscripts have not yet been translated from the original languages, there is still much to learn before a comprehensive story of Indian mathematics can be told. Therefore, in recent years, several Indian scholars have produced new Sanskrit editions of ancient texts, some of which have never before been published. And new translations, generally into English, are also being produced regularly, both in India and elsewhere.

The availability of new sources has led to an increasing number of research articles dealing with Indian mathematics, both in Indian journals, such as *Gaṇitā Bhārti* (the Bulletin of the Indian Society for History of Mathematics) and the *Indian Journal of the History of Science*, and in the standard international journals on the history of mathematics. One published collection of research and survey articles is the 2005 volume *Contributions to the History of Indian Mathematics*, published by the Hindustan Book Agency.

Ancient Indian Leaps into Mathematics aims to continue this process of making the history of Indian mathematics known to the global mathematical community. It contains a wide-ranging collection of articles, exemplifying the nature of recent work in Indian mathematics. Among the articles are ones dealing with the mathematical work of two reputed Indian mathematicians, Brahmagupta and Bhāskarācārya. There are also articles about the relationship of Indian mathematics to the mathematics of both China and Greece. There is a study of Keralese mathematics and a study of the technique of application of areas as seen in the *Śulbasūtras*. And there is a detailed article on Indian calendrical calculations, complete with computer programs enabling readers to determine Indian dates on their own.

The net result of books such as this one is that the history of Indian mathematics is now becoming a part of the world history of mathematics. It is possible to see common threads in Indian mathematics and mathematics in other civilizations, and therefore scholars are examining the transmission of mathematical ideas among India, China, the Islamic world, and Western Europe. It is certainly too early to come to definite conclusions about whether the algorithms for root extraction were transmitted among India, China, and the Islamic world, for example, or whether the Jesuits transmitted Keralese ideas on power series to Europe. But with continued research in the history of Indian mathematics, we will get closer to answers to such questions. We will also be better able to understand how the particular nature of a civilization affects the development of its mathematics.

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Preface

If you talk of the history of mathematics to any Indian mathematician, you are most likely to have two types of reactions. Firstly, his mind would immediately flash back to the great works of ancient Indian mathematicians, including even the mathematical contents of the Vedas which he might hardly have had a chance to look at. This is but natural. Apart from Indians, the whole world feels rightly proud of the importance of the great mathematical achievements in ancient India. Secondly, he remonstrates (and this may happen even in the case of a respectable Indian mathematician) that he does not know the history of mathematics, nor is he interested in knowing it; in fact, he has no time to learn it. However, the fact remains that as soon as a new mathematical work originates, its history begins along with it and keeps taking shape as the subject develops. Thus the history of mathematics cannot be separated from mathematics. As André Weil put it, the history of mathematics is in itself mathematics and no one should venture to enter the field unless he knows enough of mathematics.

Placed between these two extraordinary adventitious and redundant situations, the studies of the history of mathematics in India have suffered hopelessly. In fact, they have yet to be initiated in the right perspective. While a good university in every part of the world has a department/institute of study in the history and philosophy of mathematics, it is disappointing that there is hardly a university in India that provides facilities for such studies.

This has led to a deadening disposition: History of mathematics in India has come to mean the 'history of ancient Indian mathematics'. While there is a vast ocean of world history of mathematics spread before us, Indians cannot afford to row in the backwaters of this vast ocean, however rich these may be. Moreover, how long could we continue doing so? After all, studies in the history of mathematics in India cannot be cramped to, say, up to the twelfth century C.E.

Regarding the objectives of this volume, the proposal was to publish a collection of contributed articles in the form of a book entitled *Ancient Indian Leaps into Mathematics*. The aim was to highlight significant, positive, and concrete contributions made by ancient Indian mathematicians in the initial advancement of mathematics and possibly relate them to the developments elsewhere in the world in those days, particularly to those in Greece, the Middle East, China, and Japan. The author of an article to be included in the book was expected to take care of the following:

The article should not contain any vacuous, pompous or pretentious statements, and at no stage be verbose. Statements like the 'Vedas contain all ancient knowledge' and 'there exists no knowledge outside the Vedas' should be considered out of place. The article should not directly or indirectly be based on the contents or spirit of the book *Vedic Mathematics* by Jagadguru Swami Sri Bharati Krisna Tirathji Maharaj, for the simple reason that the contents of that book are neither Vedic nor ancient Indian.[1,2]

Unnecessary praise for ancient Indian mathematics or any part thereof has to be avoided. The article need not be a research article on ancient Indian mathematics (mathematicians) and could very well be a 'revisitation' of the subject, but the exposition has to be very well-defined and concrete. A literary style of the exposition is welcome.

Articles describing concrete illustrations of the influence or connections of ancient Indian mathematics on Greek and Middle Eastern countries will be preferred. They should appear as one unit of the world history of mathematics, rather than belonging to one sector or civilization having nothing to do with the rest of the world.

Finally, and most importantly:

Unfortunately it is customary among most Indian scholars to exaggerate the achievements of a particular Indian mathematician to claim that he was better than the greatest of his time, without bothering in the least about the existence of others during that period. As many such claims are not actually true and it is almost impossible to prove them, their efforts in aggrandizing his real achievements ultimately result in belittling his work in the overall context of the world history of mathematics. Again, even more unfortunately, there has been a consistent tendency on the part of Western historians of mathematics and the Euro-centric scholars of history of mathematics, to ignore, let alone undermine, the mathematical achievements of ancient India. Their belief that except for the discovery of the concept of zero and the decimal representation of numbers, which, of course, is now universally accepted, everything else great in mathematics was done outside India is really untenable. The reason for such thinking is, in fact, somewhat easy to understand: The whole of Europe

[1] The interested reader is recommended to refer to: Dani, S. G.: Myth and Reality: on 'Vedic Mathematics.' An updated version of a two-part article in Frontline (Vol. 10, No. 21, October 22, 1993, pp. 90–92, and Vol. 10, No. 22, November 5, 1993, pp. 91–93).

[2] Neither Vedic Nor Mathematics: A statement signed by S. G. Dani, and et al. http://www.sacw.net/DC/CommunalismCollection/ArticlesArchive/NoVedic.html.

learned mathematics only through the Greeks. The present book aims to dispel this notion and place in a proper perspective the significant achievements of ancient India in the world history of mathematics.

The royalties from the publication of the work will go to the Indian Society for History of Mathematics to help overcome its financial infirmities.

What is it that we have done? Almost nothing! In the civil administration of every city or town, there are offices of 'Assessors and Collectors'. Their job is to assess the properties of the residents in the area of their jurisdiction, evaluate taxes on these properties according to the rules, and collect the taxes for the government. This is precisely what we have done. Once the objectives were decided, we contacted a number of reputed scholars of the history of ancient Indian mathematics for their contributions. Not all came forward. However, those who did constitute a good (a well-defined) subset of the undefined collection. It is only for the reader to judge how far we have been successful in our humble mission.

As it is a unique endeavor, we had to tap many sources for assistance. Our first thanks go to the authors who were so enthusiastic to contribute their articles to the volume. We appreciate their patience to bear with us for almost 2 years to wait to see their work in print. Next, we are grateful to Professor Victor Katz who took pleasure, and pains too, to go through the whole manuscript, make significant suggestions to improve the text, including even rejecting an article on pertinent grounds, before writing the Foreword. Again, our thanks are also due to Satish Verma, SGTB Khalsa College, Delhi University, endowed with unusually skilled expertise, who helped us in preparing camera-ready copy of the manuscript. His knack of converting Sanskrit ślokās into their Roman rendering, and vice versa, was remarkable.

Finally we thank Birkhäuser Boston, and in particular, Ann Kostant and Avanti Paranjpye, for their help in preparing the manuscript in its final form.

New Delhi, India
August 15, 2008

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Indian Calendrical Calculations

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> The months of the Hindus are lunar, their years are solar;
> therefore their new year's day must in each solar year
> fall by so much earlier as the lunar year is shorter than the solar....
> If this precession makes up one complete month,
> they act in the same way as the Jews,
> who make the year a leap year of 13 months...,
> and in a similar way to the heathen Arabs.
>
> — Alberuni's *India*

1 Introduction

The world's many calendars are of three primary types: diurnal, solar, and lunar; see the third edition of our *Calendrical Calculations* [3] (henceforth *CC*). All three are represented among the many calendars of the Indian subcontinent.

- A *diurnal* calendar is a day count, either a simple count, like the Julian day number used by astronomers, or a complex, mixed-radix count, like the Mayan long count (see Sect. 10.1 of *CC*). The classical Indian day count (*ahargaṇa*) is used for calendrical purposes.
- *Solar* calendars have a year length that corresponds to the solar year. All modern solar calendars add leap days at regular intervals to adjust the

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mean length of the calendar year to better approximate the true solar year. The solar (*saura*) calendar is more popular in northern India; a similar one is in use in Nepal.[3]

- A *lunar* (*cāndra*) calendar has as its primary component a mensural unit that corresponds to the lunar synodic month. It can be purely lunar, as in the 12-month Islamic calendar year (see Chap. 6 of *CC*), or it can incorporate occasional leap months, as in the Hebrew lunisolar calendar (see Chap. 7 of *CC*). Several forms of the lunisolar calendar are in use in India today; the Tibetan *Phugpa* or *Phug-lugs* calendar is somewhat similar (see Chap. 19 of *CC*).

In general, a date is determined by the occurrence of a cyclical event (equinox, lunar conjunction, and so on) before some "critical" time of day, not necessarily during the same day. For a "mean" (*madhyama*) calendar, the event occurs at fixed intervals; for a "true" (*spaṣṭa*) calendar, the (approximate or precise) time of each actual occurrence of the event must be calculated. Various astronomical values were used by the Indian astronomers Āryabhaṭa (circa 300 C.E.), Brahmagupta (circa 630 C.E.), the author of *Sūrya-Siddhānta* (circa 1000 C.E.), and others.

We systematically apply the formulæ for cyclic events in Chap. 1 of *CC* to derive formulæ for generic mean single-cycle and dual-cycle calendars; see Chap. 12 of *CC* for more details. Solar calendars are based on the motion of the sun alone, so they fit a single cycle pattern; lunisolar calendars, on the other hand, take both the solar and lunar cycles into account, so they require double-cycle formulæ. We apply these generic algorithms to the old Indian solar and lunisolar calendars, which are based on mean values (see Chap. 9 of *CC*). We also use the code in *CC* to compare the values obtained by the much more complicated true Indian calendars (Chap. 18 of *CC*) with their modern astronomical counterparts. Unless noted otherwise, we centre our astronomical calculations on the year 1000 C.E.

We ignore many trivial differences between alternative calendars, such as eras (year count). Some Indian calendars count "elapsed" years, beginning with year 0; others use "current," 1-based years. The offsets of some common eras from the Gregorian year are summarized in Table 1. Indian month names are given in Table 2. Tamil names are different. There are also regional differences as to which is the first month of the year. Finally, calendars are local, in the sense that they depend on local times of sunrise.

The next brief section describes the Indian day count. Section 3 presents a generic solar calendar and shows how the mean Indian solar calendar fits the pattern. It is followed by a section that compares the later, true calendar with modern astronomical calculations. Similarly, Sect. 5 presents a generic

[3] We have been unable to ascertain the precise rules of the Nepalese solar calendar.

Table 1. Some eras, given as the offset from the Gregorian year

Era	Current year	Elapsed year
Kali Yuga	+3102	+3101
Nepalese		+877
Kollam	+823	
Vikrama	+58	+57
Śaka	−77	−78
Bengal		−593
Caitanya		−1486

Table 2. Indian month names

	Vedic	Zodiacal sign			Month	
1	Madhu	Meṣa	(Aries)	मेष	Vaiśākha	वैशाख
2	Mādhava	Vṛṣabha	(Taurus)	वृषभ	Jyeṣṭha	ज्येष्ठ
3	Śukra	Mithuna	(Gemini)	मिथुन	Āṣāḍha	आषाढ
4	Śuchi	Karka	(Cancer)	कर्क	Śrāvaṇa	श्रावण
5	Nabhas	Siṃha	(Leo)	सिंह	Bhādrapada	भाद्रपद
6	Nabhasya	Kanyā	(Virgo)	कन्या	Āśvina	आश्विन
7	Issa	Tulā	(Libra)	तुला	Kārtika	कार्तिक
8	Ūrja	Vṛścika	(Scorpio)	वृश्चिक	Mārgaśīrṣa	मार्गशीर्ष
9	Sahas	Dhanus	(Sagittarius)	धनुस्	Pauṣa	पौष
10	Sahasya	Makara	(Capricorn)	मकर	Māgha	माघ
11	Tapas	Kumbha	(Aquarius)	कुम्भ	Phālguna	फाल्गुन
12	Tapasya	Mīna	(Pisces)	मीन	Caitra	चैत्र

lunisolar calendar and its application to the Indian version, and is followed by a section on the true and astronomical versions. Section 7 discusses aspects of the traditional calculation of the time of sunrise. Finally, Sect. 8 outlines the difficulty of computing the day of observance of holidays based on the lunisolar calendar.

Following the style of *CC*, the algorithms in this paper are presented as mathematical function definitions in standard mathematical format. All calendar functions were automatically typeset directly from the Common Lisp functions listed in the appendix.

2 Diurnal Calendars

In most cases, a calendar date is a triple $\langle y, m, d\rangle$, where year y can be any positive or negative integer, and month m and day d are positive integers, possibly designated "leap." A day count is convenient as an intermediate

Table 3. Day count correlations

Date (Julian)		J.A.D.	Ahargaṇa	R.D.
2 January 4713 B.C.E.	(noon)	1	−588,464.5	−1,721,423.5
18 February 3102 B.C.E.	(midnight)	588,465.5	0	−1,132,959
3 January 1 C.E.	(midnight)	1,721,425.5	1,132,960	1

device for conversions between calendars. One can count days from the first day of a calendar, normally $\langle 1, 1, 1\rangle$, called the *epoch*. So $\langle 1, 1, 1\rangle$, $\langle 1, 1, 2\rangle$, and so on correspond to (elapsed) day count 0, 1, and so on. Days prior to the epoch have negative day counts and nonpositive year numbers. Day counts can be 0-based or 1-based; that is, the epoch may be assigned 0 or 1. In *CC*, we use the *Rata Die* (R.D.) count, day 1 of which is 1 January 1 (Gregorian).

The *ahargaṇa* ("heap of days") is a 0-based day count of the Kali Yuga (K.Y.) era, used in Indian calendrical calculations. The K.Y. epoch, day 0 of the *ahargaṇa* count, is Friday, 18 February 3102 B.C.E. (Julian). Its correlations with R.D. and with the midday-to-midday Julian day number (J.A.D., popular among astronomers) are given in Table 3. An earlier count, with much larger numbers, was used by Āryabhaṭa. We use the onset of the Kali Yuga, R.D. −1,143,959, for our **hindu-epoch**.

3 Mean Solar Calendars

The modern Indian solar calendar is based on a close approximation to the *true* times of the sun's entrance into the signs of the sidereal zodiac. The Hindu names of the zodiac are given in Table 2. Traditional calendarists employ medieval epicyclic theory (see [11] and Sect. 18.1 of *CC*); others use a modern ephemeris. Before about 1100 C.E., however, Hindu calendars were usually based on *average* times. The basic structure of the calendar is similar for both the mean (*madhyama*) and true (*spaṣṭa*) calendars. The Gregorian, Julian, and Coptic calendars are other examples of mean solar calendars; the Persian and French Revolutionary are two examples of true solar calendars. In this and the following section, we examine these two solar calendar schemes.

The Indian mean solar calendar, though only of historical interest, has a uniform and mathematically pleasing structure. (Connections between leap-year structures and other mathematical tasks are explored in [4].) Using the astronomical constants of the *Ārya-Siddhānta* yields 149 leap years of 366 days, which are distributed evenly in a cycle of 576 years. Similarly, 30-day and 31-day months alternate in a perfectly evenhanded manner.

3.1 Single-Cycle Calendars

The mean solar calendar is an instance of a general single-cycle calendar scheme. Consider a calendar with mean year length of Y days, where Y is a positive real number. If Y is not a whole number, then there will be common years of length $\lfloor Y \rfloor$ and leap years of length $\lceil Y \rceil$, with a leap-year frequency of $Y \bmod 1$.

To convert between R.D. dates and single-cycle dates, we apply formulæ (1.73) and (1.77) from Sect. 1.12 of *CC*. Suppose that a year is divided into months of length as close to equal as possible. For a standard 12-month year, the average length of a month would be $M = Y/12$. Some months, then, should be $\lfloor M \rfloor$ days long, and the rest $\lceil M \rceil$ days. Alternatively, a year may include a 13th short month, in which case M is the mean length of the first 12 months. (In any case, we may assume that $Y \geq M \geq 1$.)

A day is declared "New Year" if the solar event occurs *before* some critical moment. In other words, if t_n is the critical moment of day n, then n is New Year if and only if the event occurs during the interval $[t_n - 1, t_n)$. The beginnings of new months may be handled similarly or may be determined by simpler schemes, depending on the calendar; we discuss this below.

Suppose that the sun was at the critical longitude at the critical time t_{-1} of day -1, the day before the epoch, so that day -1 just missed being New Year. Finally, assume that a leap day, when there is one, is added at year's end. The number n of elapsed days from the calendar's epoch $\langle 0, 0, 0 \rangle$ until a date $\langle y, m, d \rangle$ (all three components are for now 0-based) is simply

$$\lfloor yY \rfloor + \lfloor mM \rfloor + d, \tag{1}$$

with inverse

$$\begin{aligned} y &= \lceil (n+1)/Y \rceil - 1, \\ n' &= n - \lfloor yY \rfloor, \\ m &= \lceil (n'+1)/M \rceil - 1, \\ d &= n' - \lfloor mM \rfloor. \end{aligned} \tag{2}$$

If the rule is that the event may occur up to *and including* the critical moment, then n is New Year if and only if the event occurs during the interval $(t_n - 1, t_n]$. Accordingly, we need to change some ceilings and floors in the above formulæ. Supposing that the event transpired exactly at that critical moment t_0 of the epoch, the elapsed-day calculation becomes:

$$\lceil yY \rceil + \lceil mM \rceil + d. \tag{3}$$

The inverse function, assuming $Y \geq M \geq 1$, converting a day count n into a 0-based date, $\langle y, m, d \rangle$, is

$$\begin{aligned} y &= \lfloor n/Y \rfloor, \\ n' &= \lfloor n \bmod Y \rfloor, \\ m &= \lfloor n'/M \rfloor, \\ d &= \lfloor n' \bmod M \rfloor. \end{aligned} \tag{4}$$

The above four formulæ (1–4) assume that months are determined in the same way as are years, from a specified average value M, and, therefore, follow the same pattern every year (except for the leap day in the final month of leap years).[4] There is an alternative version of mean solar calendars in which month lengths can vary by 1 day, and are determined by the mean position of the sun each month. In this case, we combine the calculation of the number of days in the elapsed years and those of the elapsed months. Assuming a 12-month year with $M = Y/12$, we have

$$\lfloor yY + mM \rfloor + d \tag{5}$$

or

$$\lceil yY + mM \rceil + d, \tag{6}$$

depending on whether the "before" or "not after" version is required. The inverses for these two variable-month versions are

$$\begin{aligned} y &= \lceil (n+1)/Y \rceil - 1, \\ m' &= \lceil (n+1)/M \rceil - 1, \\ m &= m' \bmod 12, \\ d &= n - \lfloor m'M \rfloor, \end{aligned} \tag{7}$$

and

$$\begin{aligned} y &= \lfloor n/Y \rfloor, \\ m &= \lfloor n/M \rfloor \bmod 12, \\ d &= \lfloor n \bmod M \rfloor, \end{aligned} \tag{8}$$

respectively.

3.2 Generic Single-Cycle Calendars

The critical event for a calendar sometimes occurs exactly at the calendar's epoch (K.Y., in the Indian case). However, often an additional complication is introduced, wherein the relevant critical event occurred some fraction of a day before the critical time for the epoch. Furthermore, the cyclical month pattern may have its own starting point. Accordingly, for the fixed-month calendar, we are given the following constants:

1. The calendar epoch, **single-cycle-epoch**, an integer.
2. The offset of the first critical event, **delta-year**, a number in the range $[0, 1)$.
3. The average year length, **average-year-length**, of at least 1 day.

[4] In *CC*, formulæ are given for the hybrid case where years are determined by the "not after" convention, but months by a "before" rule. There are also various cosmetic differences between the formulæ given here and in *CC*.

4. The average month length, **average-month-length**, at least 1 day long, but no longer than an average year.
5. The offset for the first month, **delta-month**, also in the range $[0, 1)$.

In the "before" version of the rules, the critical yearly event for the epochal year occurred **delta-year** days after the earliest possible moment, which is $1 -$ **delta-year** days before the critical time. Similarly, the critical monthly event for the first month of the calendar occurred **delta-month** days after its earliest possible time.

To convert a single-cycle 1-based date to an R.D. date, we add to the epoch the days before the *year*, the days before *month* in *year*, and the days before *day* in *month*, taking the initial offsets into account:

$$\textbf{fixed-from-single-cycle}\left(\boxed{\mathit{year}\,|\,\mathit{month}\,|\,\mathit{day}}\right) \stackrel{\text{def}}{=} \tag{9}$$

$$\begin{aligned}&\textbf{single-cycle-epoch}\\&+\lfloor(\mathit{year}-1)\times\textbf{average-year-length}+\textbf{delta-year}\rfloor\\&+\lfloor(\mathit{month}-1)\times\textbf{average-month-length}+\textbf{delta-month}\rfloor+\mathit{day}-1\end{aligned}$$

In the other direction, we compute the single-cycle date from an R.D. date by determining the year from the start of the mean year using (1.68) from *CC*, the month from (1.68) applied to the month parameters, and the day by calculating the remainder:

$$\textbf{single-cycle-from-fixed}\,(\mathit{date}) \stackrel{\text{def}}{=} \tag{10}$$

$$\boxed{\mathit{year}\,|\,\mathit{month}\,|\,\mathit{day}}$$

where

$$\begin{aligned}
\mathit{days} &= \mathit{date} - \textbf{single-cycle-epoch}\\
\mathit{year} &= \left\lceil \frac{\mathit{days}+1-\textbf{delta-year}}{\textbf{average-year-length}} \right\rceil\\
n &= \mathit{days}\\
&\quad - \left\lfloor \textbf{delta-year} + (\mathit{year}-1)\times\textbf{average-year-length} \right\rfloor\\
\mathit{month} &= \left\lceil \frac{n+1-\textbf{delta-month}}{\textbf{average-month-length}} \right\rceil\\
\mathit{day} &= n+1\\
&\quad -\lfloor \textbf{delta-month}\\
&\qquad + (\mathit{month}-1)\times\textbf{average-month-length} \rfloor,
\end{aligned}$$

The Coptic calendar, with **average-year-length** of $365\frac{1}{4}$ days, is also a single-cycle calendar, but we need to use an artificial **average-month-length** of 30 to accommodate its twelve 30-day months, which are followed by an extra "month" of epagomenē lasting 5–6 days. Also, **single-cycle-epoch** = R.D. 103,605, **delta-year** = 1/4, and **delta-month** = 0. Compare Table 1.4 and Chap. 4 of *CC*.

The Julian (old style) calendar, on the other hand, even though it has the same year length as the Coptic, does not fit our scheme, because of its irregular month lengths; see Chap. 3 of *CC*.

It should be stressed that for these functions to operate correctly for rational parameters, precise arithmetic is incumbent. Otherwise, 4 years, say, of average length $365\frac{1}{4}$ might not add up to an integral number of days, wreaking havoc on functions using floors, ceilings, and modular arithmetic.

For the alternate version, where the critical event may occur at the critical time, **delta-year** is the fraction of the day *before* the critical moment of the epoch at which the event occurred. The same is true for **delta-month**. So, we have, instead,

$$\textbf{alt-fixed-from-single-cycle}\left(\begin{array}{|c|c|c|}\hline \mathit{year} & \mathit{month} & \mathit{day} \\ \hline\end{array}\right) \stackrel{\text{def}}{=} \tag{11}$$

$$\begin{aligned}&\textbf{single-cycle-epoch}\\ &+ \lceil(\mathit{year}-1)\times\textbf{average-year-length}-\textbf{delta-year}\rceil\\ &+ \lceil(\mathit{month}-1)\times\textbf{average-month-length}-\textbf{delta-month}\rceil + \mathit{day}-1\end{aligned}$$

$$\textbf{alt-single-cycle-from-fixed}\,(\mathit{date}) \stackrel{\text{def}}{=} \tag{12}$$

$$\begin{array}{|c|c|c|}\hline \mathit{year} & \mathit{month} & \mathit{day} \\ \hline\end{array}$$

where

$$\begin{aligned}\mathit{days} &= \mathit{date}-\textbf{single-cycle-epoch}+\textbf{delta-year}\\ \mathit{year} &= \left\lfloor\frac{\mathit{days}}{\textbf{average-year-length}}\right\rfloor+1,\\ n &= \lfloor \mathit{days} \bmod \textbf{average-year-length}\rfloor+\textbf{delta-month}\\ \mathit{month} &= \left\lfloor\frac{n}{\textbf{average-month-length}}\right\rfloor+1,\\ \mathit{day} &= \lfloor n \bmod \textbf{average-month-length}\rfloor+1.\end{aligned}$$

This version, too, works for the Coptic calendar, but with **delta-year** = $\frac{1}{2}$.

Now for the variable-month version. As before, we have the epoch of the calendar **single-cycle-epoch**, the average year length **average-year-length**, and the initial offset **delta-year**. However, instead of the fixed-month structure given by **average-month-length** and **delta-month**, we simply specify the (integral) number of months in the year, **months-per-year**. To convert between R.D. dates and dates on this single-cycle mean calendar, we again apply formulæ (1.65) and (1.68) from Sect. 1.12 of *CC*, but with minor variations.

In this case, to convert a single-cycle date to an R.D. date, we add the days before the mean *month* in *year*, and the days before *day* in *month*:

$$\textbf{var-fixed-from-single-cycle}\left(\boxed{\textit{year}\,|\,\textit{month}\,|\,\textit{day}}\right) \stackrel{\text{def}}{=} \tag{13}$$

$$\begin{aligned}&\textbf{single-cycle-epoch}\\ &+\lfloor (\textit{year}-1)\times\textbf{average-year-length}+\textbf{delta-year}\\ &\qquad + (\textit{month}-1)\times\textit{mean-month-length}\rfloor\\ &+\textit{day}-1\end{aligned}$$

where

$$\textit{mean-month-length} = \frac{\textbf{average-year-length}}{\textbf{months-per-year}}$$

In the other direction, we compute the single-cycle date from an R.D. date by determining the year from the start of the mean year using (1.68), the month from (1.68) applied to the month parameters, and the day by subtraction:

$$\textbf{var-single-cycle-from-fixed}\,(\textit{date}) \stackrel{\text{def}}{=} \tag{14}$$

$$\boxed{\textit{year}\,|\,\textit{month}\,|\,\textit{day}}$$

where

$$\begin{aligned}\textit{days} &= \textit{date}-\textbf{single-cycle-epoch}\\ \textit{offset} &= \textit{days}+1-\textbf{delta-year}\\ \textit{year} &= \left\lceil\frac{\textit{offset}}{\textbf{average-year-length}}\right\rceil,\\ \textit{mean-month-length} &= \frac{\textbf{average-year-length}}{\textbf{months-per-year}},\\ m' &= \left\lceil\frac{\textit{offset}}{\textit{mean-month-length}}\right\rceil-1,\\ \textit{month} &= 1+(m' \bmod \textbf{months-per-year})\\ \textit{day} &= \textit{days}+1\\ &\quad-\lfloor\textbf{delta-year}+m'\times\textit{mean-month-length}\rfloor\end{aligned}$$

3.3 Indian Mean Solar Calendar

Unlike other solar calendars, especially the universally used Gregorian, the Indian calendars are based on the sidereal (*nakshatra*) year. The old Hindu mean (*madhyama*) solar calendar is an example of the second version of our generic solar calendar, using an estimate of the length of the sidereal year and mean sunrise as the critical time.

However, we need the fourth version of the formulæ, with the determining event occurring before or at the critical time:

$$\begin{aligned}&\textbf{alt-var-fixed-from-single-cycle}\left(\boxed{\mathit{year}\,|\,\mathit{month}\,|\,\mathit{day}}\right) \stackrel{\text{def}}{=}\\ &\quad\textbf{single-cycle-epoch}\\ &\quad+\Big\lceil (\mathit{year}-1)\times\textbf{average-year-length}-\textbf{delta-year}\\ &\qquad+(\mathit{month}-1)\times \mathit{mean\text{-}month\text{-}length}\Big\rceil\\ &\quad+\mathit{day}-1\end{aligned} \tag{15}$$

where

$$\mathit{mean\text{-}month\text{-}length} = \frac{\textbf{average-year-length}}{\textbf{months-per-year}}$$

and

$$\textbf{alt-var-single-cycle-from-fixed}\,(\mathit{date}) \stackrel{\text{def}}{=} \boxed{\mathit{year}\,|\,\mathit{month}\,|\,\mathit{day}} \tag{16}$$

where

$$\begin{aligned}\mathit{days} &= \mathit{date}-\textbf{single-cycle-epoch}+\textbf{delta-year}\\ \mathit{mean\text{-}month\text{-}length} &= \frac{\textbf{average-year-length}}{\textbf{months-per-year}}\\ \mathit{year} &= \left\lfloor\frac{\mathit{days}}{\textbf{average-year-length}}\right\rfloor+1\\ \mathit{month} &= 1+\left(\left\lfloor\frac{\mathit{days}}{\mathit{mean\text{-}month\text{-}length}}\right\rfloor \bmod \textbf{months-per-year}\right)\\ \mathit{day} &= \lfloor \mathit{days} \bmod \mathit{mean\text{-}month\text{-}length}\rfloor+1\end{aligned}$$

Following the *First Ārya Siddhānta* regarding year length, the constants we would need are:

$$\textbf{single-cycle-epoch} \stackrel{\text{def}}{=} \textbf{hindu-epoch} \tag{17}$$

$$\textbf{average-year-length} \stackrel{\text{def}}{=} 365\frac{149}{576} \tag{18}$$

$$\textbf{delta-year} \stackrel{\text{def}}{=} \frac{1}{4} \tag{19}$$

$$\textbf{months-per-year} \stackrel{\text{def}}{=} 12 \tag{20}$$

The above algorithms give a 1-based year number. The necessary changes for versions of the Indian calendar that use elapsed years (including those in *CC*) are trivial.

4 True Solar Calendars

One may say that a solar calendar is astronomical if the start of its years is determined by the actual time of a solar event. The event is usually an equinox or solstice, so we presume that it is the moment at which the true solar longitude attains some given value, named **critical-longitude** below, and can assume that the true and mean times of the event differ by at most 5 days.

The astronomical Persian calendar (Chap. 14 of *CC*) uses a critical solar longitude of 0° for the vernal equinox and apparent noon in Tehran as its critical moment. The defunct French Revolutionary calendar (Chap. 16 of *CC*) used a critical solar longitude of 180° for the autumnal equinox and apparent midnight in Paris as its critical moment.[5]

4.1 Generic Solar Calendars

Fixed-month versions of the true calendar usually have idiosyncratic month lengths. This is true of both the Persian and Bahá'í calendars; see Chaps. 14 and 15 of *CC*. So we restrict ourselves to the determination of New Year. First, we define a function to determine the true longitude at the critical time of any given day, where the critical time is determined by some function **critical-time**:

$$\textbf{true-longitude}\,(\mathit{date}) \stackrel{\text{def}}{=} \tag{21}$$
$$\textbf{solar-longitude}\,(\textbf{critical-time}\,(\mathit{date}))$$

[5] A generic version of such calendars was mentioned in our paper [2] as Lisp macros.

Since solar longitude increases at different paces during different seasons, we search for the first day the sun attains the **critical-longitude**, beginning 5 days prior to the mean time:

$$\textbf{solar-new-year-on-or-after}\,(date) \stackrel{\text{def}}{=} \tag{22}$$

$$\underset{d \geq start}{\text{MIN}} \left\{ \begin{array}{l} \textbf{critical-longitude} \leq \textbf{true-longitude}\,(d) \\ \leq \textbf{critical-longitude} + 2 \end{array} \right\}$$

where

$$\lambda = \textbf{true-longitude}\,(date)$$

$$\begin{aligned} start = {} & date - 5 \\ & + \Big\lfloor \textbf{average-year-length} \times \tfrac{1}{360} \\ & \quad \times ((\textbf{critical-longitude} - \lambda) \mod 360) \Big\rfloor \end{aligned}$$

The initial estimate is based on the current solar longitude λ, with an average daily increase of $360°/Y$.

Solar New Year (*Sowramana Ugadi*) in a given Gregorian year is then computed as follows:

$$\textbf{hindu-solar-new-year}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{23}$$

$$\textbf{solar-new-year-on-or-after}\left(\textbf{fixed-from-gregorian}\left(\boxed{g\text{-}year \,|\, \textbf{january} \,|\, 1} \right) \right)$$

which uses the R.D. from Gregorian conversion function **fixed-from-gregorian** (2.17) of *CC*.

For a variable-month version of the true calendar, such as the Indian solar calendar and its relatives, the start of *each* month is also determined by the true solar longitude:

$$\textbf{solar-from-fixed}\,(date) \stackrel{\text{def}}{=} \boxed{year \,|\, m+1 \,|\, date - begin + 1} \tag{24}$$

where

$$\lambda = \textbf{true-longitude}\,(date)$$

$$m = \left\lfloor \frac{\lambda}{30°} \right\rfloor$$

$$year = \text{round}\left(\frac{\textbf{critical-time}\,(date) - \textbf{solar-epoch}}{\textbf{average-year-length}} - \frac{\lambda}{360°} \right)$$

Table 4. Sidereal year values

Source	Length
First Ārya-Siddhānta	365.258681 $365^{\mathrm{d}}6^{\mathrm{h}}12^{\mathrm{m}}30^{\mathrm{s}}$
Brahma-Siddhānta	365.258438 $365^{\mathrm{d}}6^{\mathrm{h}}12^{\mathrm{m}}\ 9^{\mathrm{s}}$
Original Sūrya-Siddhānta	365.258750 $365^{\mathrm{d}}6^{\mathrm{h}}12^{\mathrm{m}}36^{\mathrm{s}}$
Present Sūrya-Siddhānta	365.258756 $365^{\mathrm{d}}6^{\mathrm{h}}12^{\mathrm{m}}36.56^{\mathrm{s}}$
Modern Value (for 1000 C.E.)	365.256362 $365^{\mathrm{d}}6^{\mathrm{h}}\ 9^{\mathrm{m}}\ 8.44^{\mathrm{s}}$

$$\begin{aligned} approx &= date - 3 - (\lfloor \lambda \rfloor \mod 30°) \\ begin &= \underset{i \geq approx}{\text{MIN}} \left\{ m = \left\lfloor \frac{\textbf{true-longitude}\,(i)}{30°} \right\rfloor \right\} \end{aligned}$$

This function can be inverted using the methods of Sect. 18.5 of *CC*.

4.2 True Indian Solar Calendar

For the Indian solar calendar, we need to use the Indian sidereal longitude function (**hindu-solar-longitude** in *CC*) in place of **solar-longitude** (in the **true-longitude** function). The length of the sidereal year according to the *Sūrya-Siddhānta* is

$$\textbf{average-year-length} = 365\frac{279457}{1080000}\,.$$

(See Table 4.) The year begins when the sun returns to sidereal longitude 0°. There are various critical times of day that are used to determine exactly which day is New Year.

- According to the Orissa rule (followed also in Punjab and Haryana), sunrise is used. In other words, the solar month is determined by the stellar position of the sun the following morning:

$$\textbf{orissa-critical}\,(date) \stackrel{\text{def}}{=} \textbf{hindu-sunrise}\,(date + 1) \tag{25}$$

  where **hindu-sunrise** is sunrise according to the Indian rule or practice [(18.33) in *CC*]. See Sect. 7 for details.
- According to the Tamil rule, sunset of the current day is used:

$$\textbf{tamil-critical}\,(date) \stackrel{\text{def}}{=} \textbf{hindu-sunset}\,(date)\,, \tag{26}$$

  where **hindu-sunset** is sunset according to the Indian rule or practice.
- According to the Malayali (Kerala) rule, 1:12 P.M. (seasonal time) on the current day is used:

$$\textbf{malayali-critical}\,(\mathit{date}) \stackrel{\text{def}}{=} \tag{27}$$

$$\textbf{hindu-sunrise}\,(\mathit{date}) + \tfrac{3}{5} \times (\textbf{hindu-sunset}\,(\mathit{date}) - \textbf{hindu-sunrise}\,(\mathit{date}))\,,$$

Kerala also uses a different critical longitude.

- According to some calendars from Madras, apparent midnight at the end of the day is used:

$$\textbf{madras-critical}\,(\mathit{date}) \stackrel{\text{def}}{=} \tag{28}$$

$$\textbf{hindu-sunset}\,(\mathit{date}) + \tfrac{1}{2} \times (\textbf{hindu-sunrise}\,(\mathit{date}+1) - \textbf{hindu-sunset}\,(\mathit{date}))\,.$$

- According to the Bengal rule (also in Assam and Tripura), midnight at the start of the civil day is usually used, unless the zodiac sign changes between 11:36 P.M. and 12:24 A.M. (temporal time), in which case various special rules apply, depending on the month and on the day of the week.

See [8, p. 12] and [1, p. 282]. The function **critical-time** should be set to one of these.

4.3 Indian Astronomical Solar Calendar

For an astronomical Indian solar calendar, we need to substitute an astronomical calculation of *sidereal* longitude for **solar-longitude** in **true-longitude**. We should also use astronomical geometric sunrise (and/or sunset) for **hindu-sunrise** (and **hindu-sunset**) in **critical-time**; see Sect. 7.

The difference between the equinoctal and sidereal longitude (the *ayanāmsha*) changes with time, as a direct consequence of precession of the equinoxes. It is uncertain what the zero point of Indian sidereal longitude is, but it is customary to say that the two measurements coincided circa 285 C.E., the so-called "Lahiri *ayanāmsha*." Others (for example [10, Sect. 18]) suggest that the two measurements coincided around 560 C.E. Either way, the overestimate of the length of the mean sidereal year used by the siddhantas leads to a growing discrepancy in the calculation of solar longitude; see Table 4. (The length of the sidereal year is increasing by about 10^{-4} s/year.)

The Indian vernal equinox, when the sun returns to the sidereal longitude 0°, is called Mesha saṃkrānti. Solar New Year, the day of Mesha saṃkrānti, as computed by **hindu-solar-new-year** with traditional year lengths, is nowadays about 4 days later than that which would be obtained by astronomical calculation (assuming the Lahiri value).

To calculate the sidereal longitude, we use the algorithm for precession in [7, pp. 136–137], as coded in (13.39) of *CC*, **precession**. The values given by this function need to be compared with its value when the *ayanāmsha* was 0, given, according to some authorities, by the following:

$$\textbf{sidereal-start} \stackrel{\text{def}}{=} \tag{29}$$

$$\textbf{precession}\left(\textbf{universal-from-local}\left(\textbf{mesha-samkranti}\,(285\ \text{C.E.})\,,\textbf{hindu-locale}\right)\right),$$

where **mesha-samkranti** (18.51) of *CC* gives the local time of the (sidereal) equinox, Ujjain is our **hindu-locale**, and **universal-from-local** is one of the time conversion functions ((13.8) of *CC*). Then:

$$\textbf{sidereal-solar-longitude}\,(t) \stackrel{\text{def}}{=} \tag{30}$$

$$\left(\textbf{solar-longitude}\,(t) - \textbf{precession}\,(t) + \textbf{sidereal-start}\right) \bmod 360$$

as in (13.40) of *CC*. That done, we can compare the astronomical calendar with the approximations used in the true Indian calendar.

The cumulative effect over the centuries of the difference in length of the sidereal year on the time of Mesha saṃkrānti, and on the sidereal longitude at that time, is shown in Fig. 1. In 1000 C.E., it stood at about $1°37'$.

The average difference between the calculated sidereal longitude and the astronomical values was $2°3'$ during 1000–1002 C.E. In addition, Fig. 2 shows a periodic discrepancy of up to $\pm 12'$ between the siddhāntic estimate of solar longitude and the true values. The figure also suggests that neither the

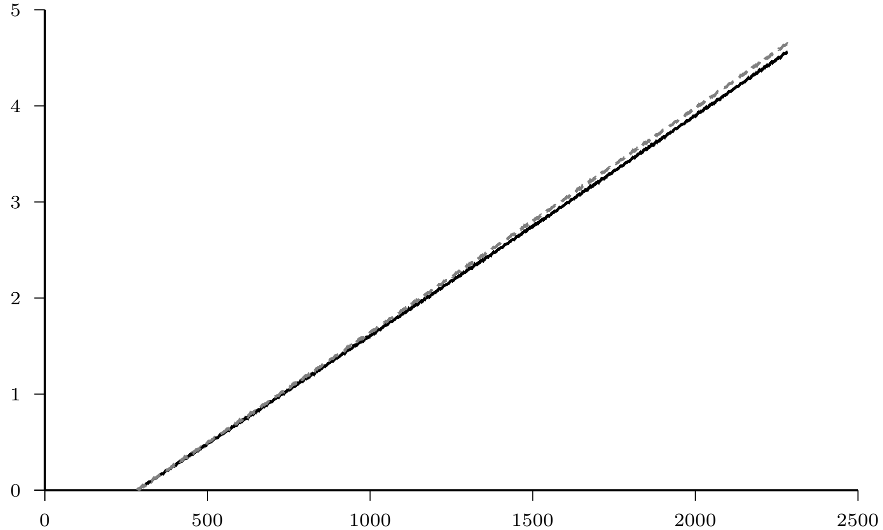

Fig. 1. Increasing difference, for 285–2285 C.E., between siddhāntic and astronomical sidereal longitudes in degrees (*solid line*) and days (*dashed line*), assuming coincidence in 285

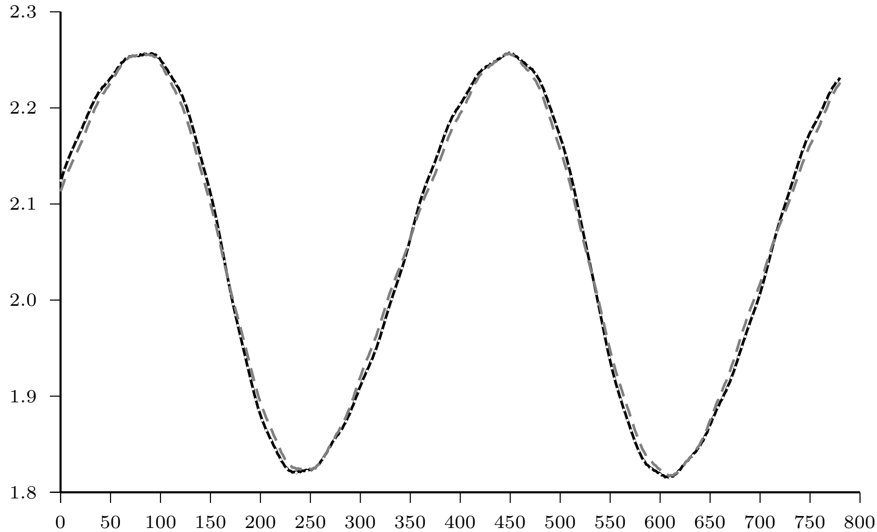

Fig. 2. Difference (1000–1001 C.E.) between siddhāntic and astronomical sidereal solar longitude (*solid black line*) in hours. *Dotted white line* (atop the *black line*) uses the mathematical sine function, rather than an interpolated tabular sine; *dashed line* uses a fixed epicycle; they are virtually indistinguishable

interpolated stepped sine function used in traditional astronomy nor the fluctuating epicycle of Indian theory (using the smallest [best] size, instead) make a noticeable difference for the sun. In other words, the tabular sine and arcsine functions (see Table 18.2 in *CC*) are precise enough for the purpose, while the theory of changing epicycle (see Fig. 18.2 in *CC*) is unnecessary for the sun.

The difference in longitude for a given moment t, Ujjain local time, is calculated as:

$$\begin{aligned}&\textbf{hindu-solar-longitude}(t)\\&-\textbf{sidereal-solar-longitude}(\textbf{universal-from-local}(t, \textbf{hindu-locale}))\end{aligned}$$

5 Lunisolar Calendars

The lunisolar calendar type is represented by the Chinese (see Chap. 17 of *CC*), Hebrew (see Chap. 7 of *CC*), and Easter (see Chap. 8 of *CC*) calendars today, as well as those of some of the Indian and other Asian cultures (for example, the Tibetan *Phugpa* calendar; see Chap. 19 of *CC*), and was historically very popular. The basic idea is that months follow the lunar cycle, with leap months added every 2–3 years, so that the average year length matches the sun's apparent celestial revolution. Indian lunisolar calendars can be further subdivided into those whose months begin with each new moon (the *amānta* scheme) and those that go from full moon to full moon (*pūrṇimānta*).

The Hebrew and Easter calendars follow a fixed leap-year cycle, as did the old Hindu mean lunisolar calendar; the Chinese and modern Hindu calendars determine each month and year according to the true positions of the sun and moon. Unlike the Hebrew lunisolar calendar, with its 19-year cycle of 7 leap years, Indian intercalated months, in the mean scheme, do not follow a short cyclical pattern. Rather, in the *Ārya-Siddhānta* version, there are 66,389 leap years in every 180,000-year cycle. The Hebrew, Easter, and mean Indian leap-year rules all distribute leap years evenly over the length of the cycle.

Lunisolar calendars can also come in the same two flavours, fixed- and variable-month patterns. The Indian mean lunisolar calendar has variable months, like its solar sister; the Hebrew calendar has a more-or-less fixed scheme (see Chap. 7 of *CC* for details).

In the fixed-month scheme, one fixed month (usually the last) of the 13 months of a leap year is considered the leap month, so one can just number them consecutively. This is not true of the Indian calendar, in which any month can be leap, if it starts and ends within the same (sidereal) zodiacal sign.

Unlike other calendars, a day on the mean Indian calendar can be *omitted* any time in a lunar month, since the day number is determined by the phase of the mean moon. Here we concentrate on the leap year structure; see *CC* for other details.

5.1 A Generic Dual-Cycle Calendar

Let Y and M be the lengths of the mean solar year and lunar month in days, respectively, where $Y \geq M \geq 1$ are positive real numbers. If Y is not a multiple of M, then there will be common years of $\lfloor Y/M \rfloor$ months and leap years of length $\lceil Y/M \rceil$ months. Then a year has Y/M months on average, with a leap-year frequency of $(Y \bmod M)/M$.

The basic idea of the dual-cycle calendar is to first aggregate days into months and then months into years. Elapsed months are counted in the same way as years are on the single-cycle calendar, using an average length of M instead of Y. Then, years are built from multiple units of months, rather than days, again in a similar fashion to a single-cycle calendar.

For the Indian mean lunar calendar, according to the *Ārya Siddhānta*, we would use the values

$$M = 29\tfrac{2362563}{4452778},$$
$$Y = 365\tfrac{149}{576},$$

and sunrise as the critical time of day. The Hebrew calendar also follows a dual-cycle pattern, with

$$M = 29\tfrac{13753}{25920},$$
$$Y = \tfrac{285}{19}M,$$

and noon as critical moment, but exceptions can lead to a difference of up to 3 days.

To convert from a 0-based lunisolar date $\langle y, m, d \rangle$ to a day count, use

$$\lceil (\lfloor yY/M \rfloor + m)M \rceil + d. \tag{31}$$

In the other direction, we have:

$$\begin{aligned} m' &= \lfloor n/M \rfloor, \\ y &= \lceil (m'+1)M/Y \rceil - 1, \\ m &= m' - \lfloor yY/M \rfloor, \\ d &= n - \lceil m'M \rceil. \end{aligned} \tag{32}$$

When the leap month is not fixed and any month can be leap as in the Indian calendar, we would use an extra Boolean component for dates $\langle y, m, \ell, d \rangle$, and would need to determine which month is leap. On the Chinese calendar, the first lunar month in a leap year during which the sun does not change its zodiacal sign (counting from month 11 to month 11) is deemed leap. In the Indian scheme, the rule is similar: any month in which the sidereal sign does not change is leap.

As was the case for the solar calendars, there are variants corresponding to whether the critical events may also occur at the critical moments. See Sect. 12.2 of *CC*.

6 True Lunisolar Calendar

The general form of the determination of New Year on a lunisolar calendar is as follows:

1. Find the moment s when the sun reaches its critical longitude.
2. Find the moment p when the moon attains its critical phase before (or after) s.
3. Choose the day d before (or after) p satisfying additional criteria.

Some examples include:

- The Nicæan rule for Easter is the first Sunday after the first full moon on or after the day of the vernal equinox; see Chap. 8 of *CC*.
- The classical rule for the first month (*Nisan*) of the Hebrew year was that it starts on the eve of the first observable crescent moon no more than a fortnight before the vernal equinox; see Sect. 20.4 of *CC*.
- The 11th month of the Chinese calendar almost always begins with the new moon on or before the day of the winter solstice (270°). The Chinese New Year is almost always the new moon on or after the day the sun reaches 300°; see Chap. 17 of *CC*.

- The Indian Lunar New Year is the (sunrise-to-sunrise) day of the last new moon before the sun reaches the edge of the constellation Aries ($0°$ sidereal); see Chap. 18 of *CC*.

Using the functions provided in *CC*:

1. The moment s can be found with **solar-longitude-after** (13.33).
2. Finding the moment p can be accomplished with **lunar-phase-at-or-before** (13.54) or **lunar-phase-at-or-after** (13.55).
3. Choosing the day is facilitated by **kday-on-or-after** and its siblings (Sect. 1.10).

For example, the winter-solstice-to-winter-solstice period is called a *suì* on the Chinese calendar. Hence, the start of the Chinese month in which a *suì* begins, that is, the month containing the winter solstice (almost always the 11th month, but on occasion a leap 11th month) is determined by:

$$\textbf{sui-month-start-on-or-after}\,(date) \stackrel{\text{def}}{=} \tag{33}$$

$$\left\lfloor \textbf{standard-from-universal} \left(moon, \textbf{chinese-location}\,(date)\right) \right\rfloor$$

where

$$\begin{aligned} sun &= \textbf{universal-from-standard} \\ &\quad \Big(\Big\lfloor \textbf{standard-from-universal} \\ &\qquad \Big(\textbf{solar-longitude-after}\,(270°, date), \\ &\qquad\ \ \textbf{chinese-location}\,(date) \Big) \Big\rfloor, \\ &\quad\ \ \textbf{chinese-location}\,(date) \Big) \\ moon &= \textbf{lunar-phase-at-or-before}\,(0°, sun + 1) \end{aligned}$$

For the Indian calendars, the functions should use sidereal longitudes and can be traditional or astronomical, as desired. Using the astronomical code of *CC*, we can define:

$$\textbf{sidereal-solar-longitude-after}\,(\phi, t) \stackrel{\text{def}}{=} \tag{34}$$

$$\underset{x \in [a,b]}{\overset{u-l<\varepsilon}{\text{MIN}}} \left\{ ((\textbf{sidereal-solar-longitude}\,(x) - \phi) \bmod 360) < 180° \right\}$$

where

$$\begin{aligned} \varepsilon &= 10^{-5}, \\ rate &= \frac{\textbf{average-year-length}}{360°}, \end{aligned}$$

$$\begin{aligned}
\tau &= t + \mathit{rate} \times \big(\,\big(\,\phi - \textbf{sidereal-solar-longitude}\,(t)\,\big) \mod 360\,\big)\,,\\
a &= \max\{t, \tau - 5\}\,,\\
b &= \tau + 5.
\end{aligned}$$

The function MIN performs a bisection search in $[a, b]$ with accuracy ε.

For the traditional Hindu calendar, we would use (18.50) in *CC*, **hindu-solar-longitude-at-or-after** instead of **sidereal-solar-longitude-after**, and use the following in place of **lunar-phase-at-or-before**:

$$\textbf{hindu-lunar-phase-at-or-before}\,(\phi, t) \stackrel{\text{def}}{=} \tag{35}$$

$$\underset{x\in[a,b]}{\overset{u-l<\varepsilon}{\mathrm{MIN}}} \left\{ ((\textbf{hindu-lunar-phase}\,(x) - \phi) \mod 360) < 180^\circ \right\},$$

where

$$\begin{aligned}
\varepsilon &= 2^{-17},\\
\tau &= t - \textbf{hindu-synodic-month} \times \tfrac{1}{360}\\
&\qquad \times ((\textbf{hindu-lunar-phase}\,(t) - \phi) \mod 360)\,,\\
a &= \tau - 2,\\
b &= \min\{t, \tau + 2\}\,.
\end{aligned}$$

Then we can use the following to compute the start of Indian lunisolar month m:

$$\textbf{hindu-lunar-month-on-or-after}\,(m, \mathit{date}) \stackrel{\text{def}}{=} \tag{36}$$

$$\begin{cases} \mathit{date} & \textbf{if } \mathit{moon} \le \textbf{hindu-sunrise}\,(\mathit{date}),\\ \mathit{date} + 1 & \textbf{otherwise}, \end{cases}$$

where

$$\begin{aligned}
\lambda &= (m - 1) \times 30^\circ,\\
\mathit{sun} &= \textbf{hindu-solar-longitude-after}\,(\lambda, \mathit{date})\,,\\
\mathit{moon} &= \textbf{hindu-lunar-phase-at-or-before}\,(0^\circ, \mathit{sun})\,,\\
\mathit{date} &= \lfloor \mathit{moon} \rfloor\,,
\end{aligned}$$

The time of the *tithis* (lunar "days," corresponding to 30ths of the lunar phase cycle) differs an average of less than 13 min between the traditional and astronomical calculations (again in 1000 C.E.). See Fig. 3.

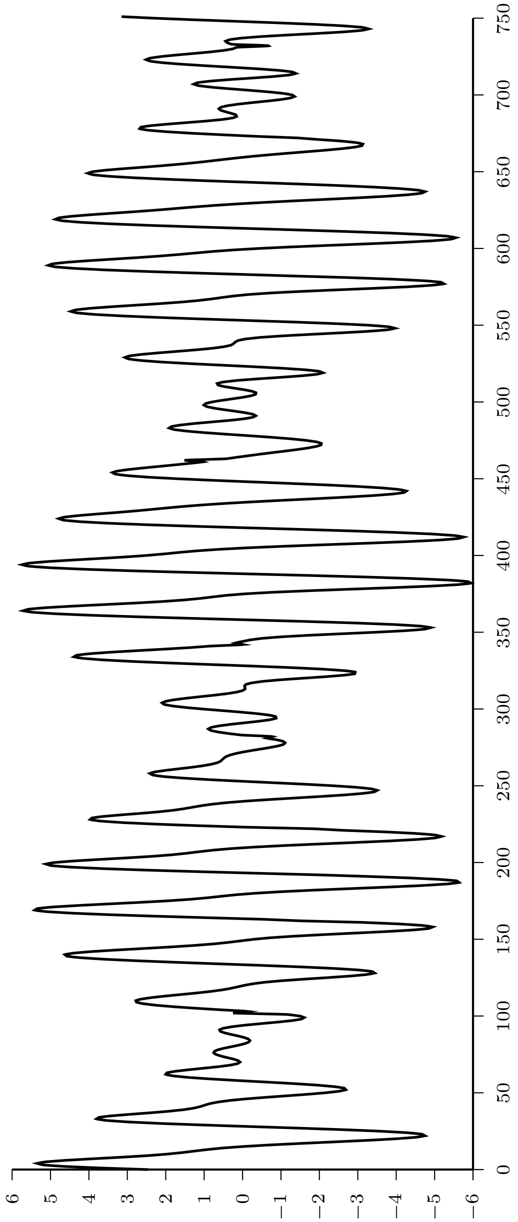

Fig. 3. Tithi time differences, 1000–1001 C.E. in hours

Figure 4 shows the same (nil) impact of sine and epicycle (using the biggest epicycle) on the calculation of lunar sidereal longitude as we found in the solar case. Moreover, the sixteenth century correction (*bīja*) of Gannṇesa Daivajna for the length of the anomalistic months (from 488,203 revolutions of the apogee in a *yuga* to 488,199) also is of no consequence (in the sixteenth century as well as in the 11th). The difference between the calculated longitude and astronomical values was $1°56' \pm 3°25'$.

In the true version of the Indian lunisolar calendar, months, called *kshaya*, may also be expunged when two zodiacal sign transitions occur in one lunar month. Thus, even a 12-month year can have a leap month (as was the case in 1963–1964), and a leap year can even have two (as in 1982–1983). See our *Calendrical Tabulations* [6]. The code above does not check whether month m is expunged.

There are several competing conventions as to the placement and naming of leap months and excision of suppressed months; see [8, p. 26].

7 Sunrise

Generally, Indian calendarists advocate the use of geometric sunrise for calendrical determinations:[6]

$$\textbf{hindu-sunrise}\,(\mathit{date}) \stackrel{\text{def}}{=} \textbf{dawn}\,(\mathit{date}, \textbf{hindu-locale}, 0°)\,. \tag{37}$$

Lahiri, however, suggests a depression angle of $47'$ (including $31'$ for refraction); astronomers typically use $50'$.

As is well known, the original siddhāntic calculation for sunrise uses a simple approximation for the equation of time. Figure 5 compares the two versions. Using an accurate equation of time, but otherwise following the siddhāntic method for sunrise, gives close agreement with geometric sunrise. See Fig. 6.

8 Holidays

Many of the holidays in India depend on the local lunisolar calendar. Table 5 lists some of the more popular holidays. (For a comprehensive list in English, see [9].) There is a very wide regional variance in timing and duration of holidays.

In general, holidays do not occur in leap months or on leap days. If a month is skipped, as happens intermittently (with gaps of 19–141 years between occurrences), then the "lost" holidays are moved to the next month,

[6] Pal Singh Purewal [personal communication, April 29, 2002]: "Most Indian almanac editors give and advocate the use of the centre of the solar disk for sunrise without refraction."

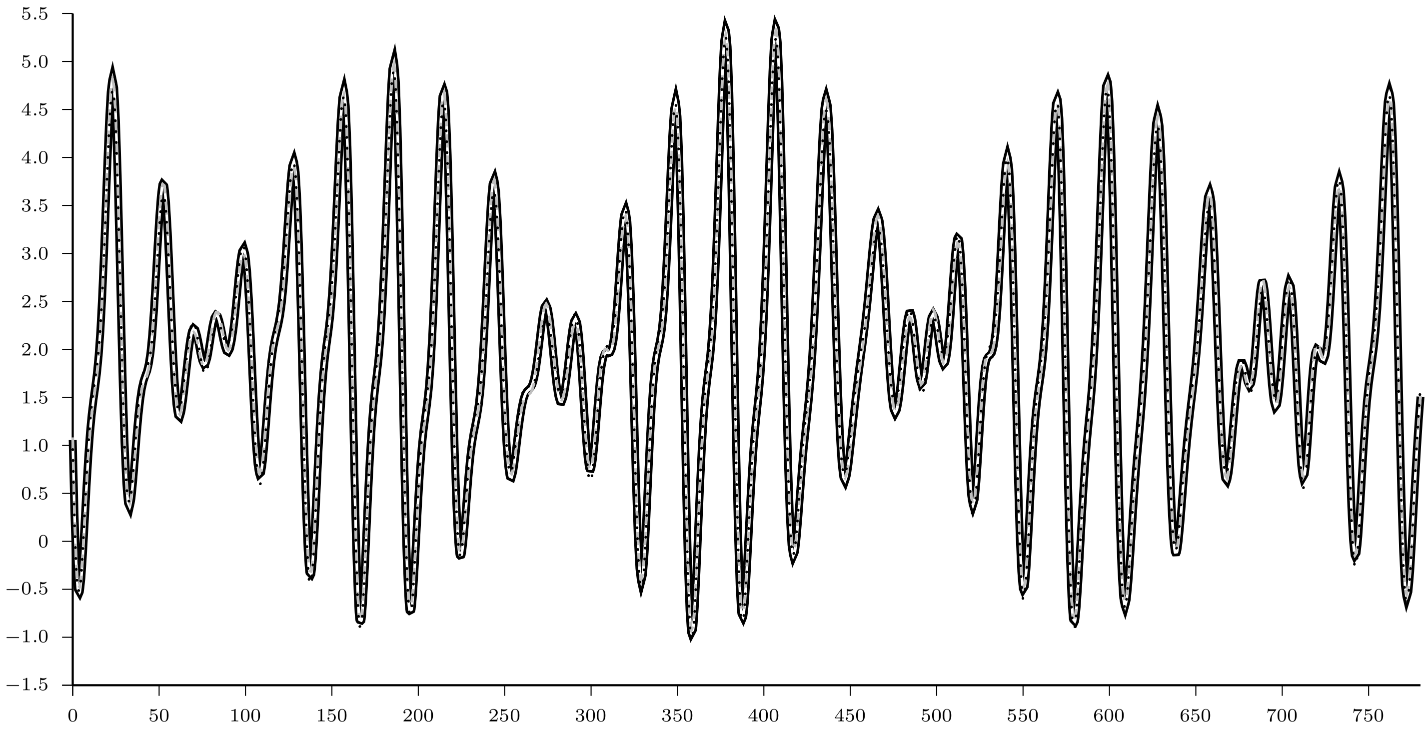

Fig. 4. Difference (1000–1001 C.E.) between siddhāntic and astronomical sidereal lunar longitude (*thick solid black line*). *Thinner white line* (largely atop the *black line*) uses the mathematical sine function, rather than an interpolated tabular sine; *dashed line* (largely overlying the *white line*) uses a fixed epicycle; the *black dotted line* (largely overlying the *white line*) is sans *bīja* correction; they are virtually indistinguishable

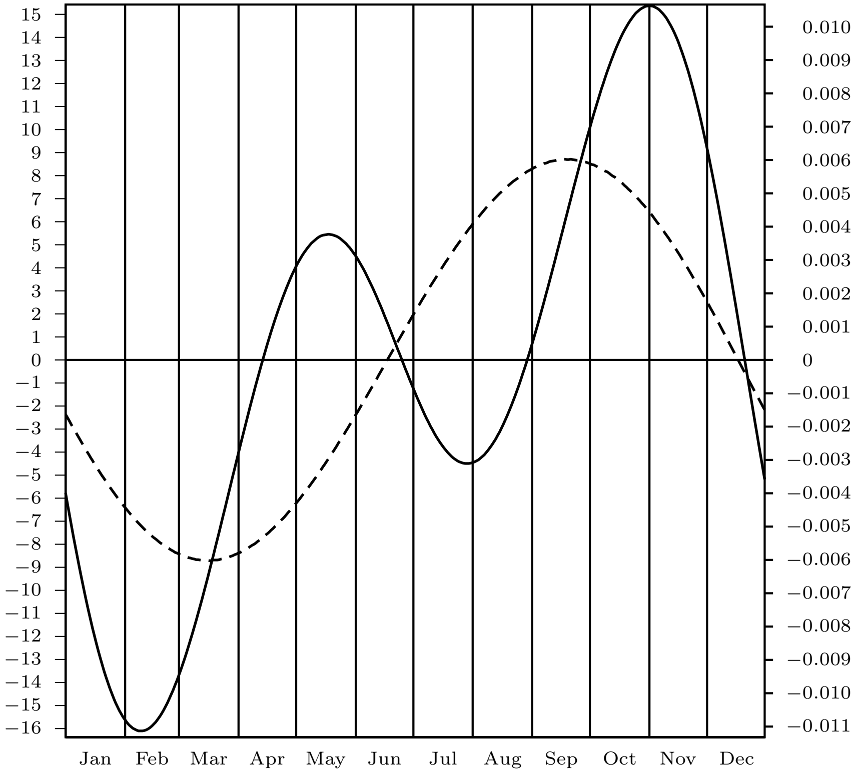

Fig. 5. The equation of time in 1000 C.E. The astronomical version is shown as a *solid line*; the Hindu version is shown as a *dashed line*. The left vertical axis is marked in minutes and the right vertical axis is marked in fractions of a day

depending again on regional conventions. In many places, rather than skip a whole month, two half months are skipped and their holidays are moved backward or forward, depending on which lost half-month they are meant to occur in.

The precise day of observance of a lunisolar event usually depends on the time of day (sunrise, noon, midnight, etc.) at which the moon reaches a critical phase (*tithi*). According to [5], for example, Ganēśa Chaturthī is celebrated on the day in which *tithi* (lunar day) 4 is current in whole or in part during the midday period from 10:48 A.M. to 1:12 P.M. (temporal time). If that lunar day is current during that time frame on two consecutive days, or if it misses that time frame on both days, then it is celebrated on the former day.[7]

Some functions for holiday calculations are provided in Sect. 19.6 of *CC*.

[7] Precise details for the individual holidays are difficult to come by.

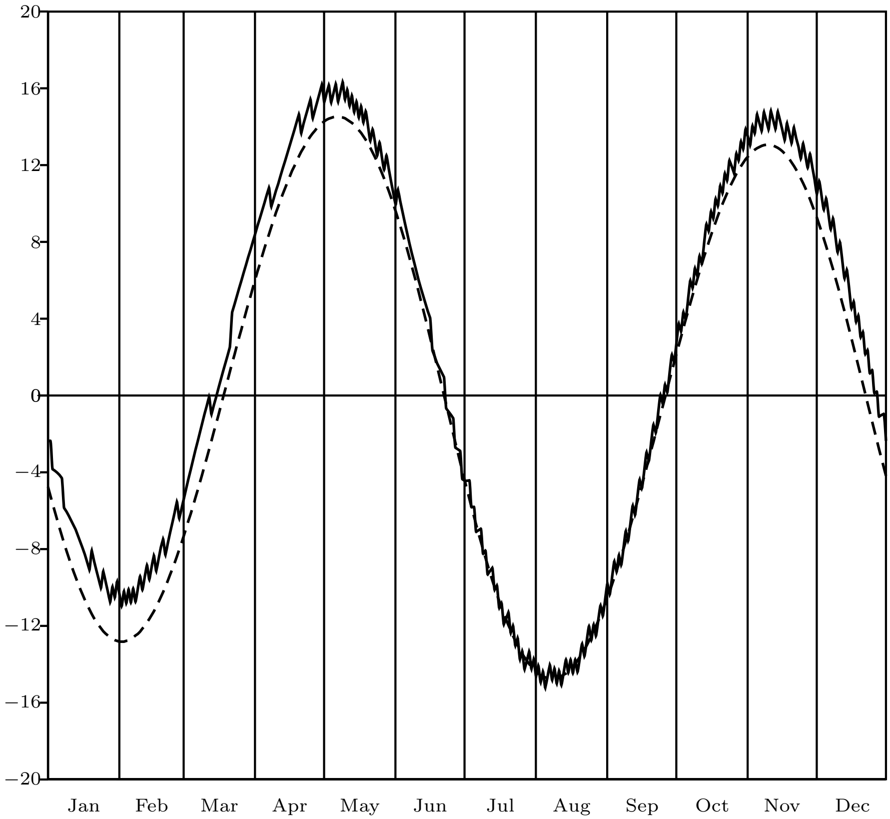

Fig. 6. Sunrise, Hindu and astronomical, 1000 C.E. difference in sunrise times is shown as a *solid line*; the difference of just the equation of time calculation is shown as a *dashed line*. The vertical scale is in minutes

Table 5. Some Hindu lunisolar holidays

Holiday	Lunar date(s)
New Year (Chandramana Ugadi)	Caitra 1
Birthday of Rāma	Caitra 9
Birthday of Krishna (Janmāshṭamī)	Śrāvaṇa 23
Ganēśa Chaturthī	Bhādrapada 3 or 4
Dashara (Nava Rathri), last 3 days	Āśvina 8–10
Diwali, last day	Kārtika 1
Birthday of Vishnu (Ekadashi)	Mārgaśīrṣa 11
Night of Śiva	Māgha 28 or 29
Holi	Phālguna 15

Appendix: The Lisp Code

This appendix contains the Common Lisp source code for the algorithms presented in the preceding sections. *CC* should be consulted for undefined functions and macros.

```
(defun fixed-from-single-cycle (s-date)
  ;; TYPE single-cycle-date -> fixed-date
  (let* ((year (standard-year s-date))
         (month (standard-month s-date))
         (day (standard-day s-date)))
    (+ single-cycle-epoch
       (floor (+ (* (1- year) average-year-length)
                 delta-year))
       (floor (+ (* (1- month) average-month-length)
                 delta-month))
       day -1)))
```

```
(defun single-cycle-from-fixed (date)
  ;; TYPE fixed-date -> single-cycle-date
  (let* ((days (- date single-cycle-epoch))
         (year (ceiling (- days -1 delta-year)
                        average-year-length))
         (n (- days (floor (+ delta-year
                              (* (1- year) average-year-length)))))
         (month (ceiling (- n -1 delta-month) average-month-length))
         (day (- n -1 (floor (+ delta-month
                                (* (1- month)
                                   average-month-length))))))
    (hindu-solar-date year month day)))
```

```
(defun alt-fixed-from-single-cycle (s-date)
  ;; TYPE single-cycle-date -> fixed-date
  (let* ((year (standard-year s-date))
         (month (standard-month s-date))
         (day (standard-day s-date)))
    (+ single-cycle-epoch
       (ceiling (- (* (1- year) average-year-length)
                   delta-year))
       (ceiling (- (* (1- month) average-month-length)
                   delta-month))
       day -1)))
```

```
(defun alt-single-cycle-from-fixed (date)
  ;; TYPE fixed-date -> single-cycle-date
  (let* ((days (+ (- date single-cycle-epoch) delta-year))
         (year (1+ (quotient days average-year-length)))
         (n (+ (floor (mod days average-year-length)) delta-month))
         (month (1+ (quotient n average-month-length)))
```

```
        (day (1+ (floor (mod n average-month-length)))))
    (hindu-solar-date year month day)))
```

```
(defun var-fixed-from-single-cycle (s-date)
  ;; TYPE single-cycle-date -> fixed-date
  (let* ((year (standard-year s-date))
         (month (standard-month s-date))
         (day (standard-day s-date))
         (mean-month-length (/ average-year-length
                               months-per-year)))
    (+ single-cycle-epoch
       (floor (+ (* (1- year) average-year-length)
                 delta-year
                 (* (1- month) mean-month-length)))
       day -1)))
```

```
(defun var-single-cycle-from-fixed (date)
  ;; TYPE fixed-date -> single-cycle-date
  (let* ((days (- date single-cycle-epoch))
         (offset (- days -1 delta-year))
         (year (ceiling offset average-year-length))
         (mean-month-length (/ average-year-length
                               months-per-year))
         (m-prime (1- (ceiling offset mean-month-length)))
         (month (+ 1 (mod m-prime months-per-year)))
         (day (- days -1
                 (floor
                  (+ delta-year
                     (* m-prime mean-month-length))))))
    (hindu-solar-date year month day)))
```

```
(defun alt-var-fixed-from-single-cycle (s-date)
  ;; TYPE single-cycle-date -> fixed-date
  (let* ((year (standard-year s-date))
         (month (standard-month s-date))
         (day (standard-day s-date))
         (mean-month-length (/ average-year-length
                                  months-per-year)))
    (+ single-cycle-epoch
       (ceiling (+ (* (1- year) average-year-length)
                   (- delta-year)
                   (* (1- month) mean-month-length)))
       day -1)))
```

```
(defun alt-var-single-cycle-from-fixed (date)
  ;; TYPE fixed-date -> single-cycle-date
  (let* ((days (+ (- date single-cycle-epoch) delta-year))
         (mean-month-length (/ average-year-length
                               months-per-year))
         (year (1+ (quotient days average-year-length)))
```

```
          (month (+ 1 (mod (quotient days mean-month-length)
                           months-per-year)))
          (day (1+ (floor (mod days mean-month-length)))))
     (hindu-solar-date year month day)))
```

```
(defun true-longitude (date)
  ;; TYPE moment -> longitude
  (solar-longitude (critical-time date)))
```

```
(defun solar-new-year-on-or-after (date)
  ;; TYPE fixed-date -> fixed-date
  ;; Fixed date of solar new year on or after fixed date.
  (let* ((lambda (true-longitude date))
         (start
          (+ date -5
             (floor (* average-year-length 1/360
                       (mod (- critical-longitude lambda) 360))))))
    (next d start
              (<= critical-longitude
                  (true-longitude d)
                  (+ critical-longitude 2)))))
```

```
(defun hindu-solar-new-year (g-year)
  ;; TYPE gregorian-year -> fixed-date
  ;; Fixed date of Hindu solar New Year in Gregorian year.
   (solar-new-year-on-or-after
     (fixed-from-gregorian
       (gregorian-date g-year january 1))))
```

```
(defun solar-from-fixed (date)
  ;; TYPE fixed-date -> solar-date
  ;; Solar date equivalent to fixed date.
  (let* ((lambda (true-longitude date))
         (m (quotient lambda (deg 30)))
         (year (round (- (/ (- (critical-time date) solar-epoch)
                            average-year-length)
                         (/ lambda (deg 360)))))
         (approx ; 3 days before start of mean month.
          (- date 3
             (mod (floor lambda) (deg 30))))
         (begin ; Search forward for beginning...
          (next i approx ; ... of month.
                (= m (quotient (true-longitude i)
                               (deg 30))))))
     (hindu-solar-date year (1+ m)  (- date begin -1))))
```

```
(defun orissa-critical (date)
  ;; TYPE fixed-date -> moment
  ;; Universal time of critical moment on or after date
  ;; according to the Orissa rule
  (hindu-sunrise (1+ date)))
```

```
(defun tamil-critical (date)
  ;; TYPE fixed-date -> moment
  ;; Universal time of critical moment on or after date
  ;; according to the Tamil rule
  (hindu-sunset date))
```

```
(defun malayali-critical (date)
  ;; TYPE fixed-date -> moment
  ;; Universal time of critical moment on or after date
  ;; according to the Malayali rule
  (+ (hindu-sunrise date)
     (* 3/5 (- (hindu-sunset date) (hindu-sunrise date)))))
```

```
(defun madras-critical (date)
  ;; TYPE fixed-date -> moment
  ;; Universal time of critical moment on or after date
  ;; according to the Madras rule
  (+ (hindu-sunset date)
     (* 1/2 (- (hindu-sunrise (1+ date)) (hindu-sunset date)))))
```

```
(defconstant sidereal-start
  (precession (universal-from-local
               (mesha-samkranti (ce 285))
               hindu-locale)))
```

```
(defun sidereal-solar-longitude (tee)
  ;; TYPE moment -> angle
  ;; Sidereal solar longitude at moment tee
  (mod (+ (solar-longitude tee)
          (- (precession tee))
          sidereal-start)
       360))
```

```
(defun sui-month-start-on-or-after (date)
  ;; TYPE fixed-date -> fixed-date
  ;; Fixed date of start of Chinese month containing solstice
  ;; occurring on or after date.
  (let* ((sun (universal-from-standard
               (floor (standard-from-universal
                       (solar-longitude-after (deg 270) date)
                       (chinese-location date)))
               (chinese-location date)))
         (moon (lunar-phase-at-or-before (deg 0) (1+ sun))))
    (floor (standard-from-universal moon (chinese-location date)))))
```

```
(defun sidereal-solar-longitude-after (phi tee)
  ;; TYPE (season moment) -> moment
  ;; Moment UT of the first time at or after tee
  ;; when the sidereal solar longitude will be phi degrees.
  (let* ((varepsilon 1d-5) ; Accuracy of solar-longitude.
         (rate ; Mean days for 1 degree change.
```

```
          (/ average-year-length (deg 360)))
         (tau ; Estimate (within 5 days).
          (+ tee
             (* rate
                (mod (- phi (sidereal-solar-longitude tee)) 360))))
         (a (max tee (- tau 5))) ; At or after tee.
         (b (+ tau 5)))
    (binary-search ; Bisection search.
     l a
     u b
     x (< (mod (- (sidereal-solar-longitude x) phi) 360)
          (deg 180l0))
     (< (- u l) varepsilon))))
```

```
(defun hindu-lunar-phase-at-or-before (phi tee)
  ;; TYPE (phase moment) -> moment
  ;; Moment UT of the last time at or before tee
  ;; when the Hindu lunar-phase was phi degrees.
  (let* ((varepsilon (expt 2 -17)) ; Accuracy.
         (tau ; Estimate.
          (- tee
             (* hindu-synodic-month 1/360
                (mod (- (hindu-lunar-phase tee) phi) 360))))
         (a (- tau 2))
         (b (min tee (+ tau 2)))) ; At or before tee.
    (binary-search ; Bisection search.
     l a
     u b
     x (< (mod (- (hindu-lunar-phase x) phi) 360)
          (deg 180l0))
     (< (- u l) varepsilon))))
```

```
(defun hindu-lunar-month-on-or-after (m date)
  ;; TYPE (hindu-lunar-month fixed-date) -> fixed-date
  ;; Fixed date of first lunar moon on or after fixed date.
  (let* ((lambda (* (1- m) (deg 30)))
         (sun (hindu-solar-longitude-after lambda date))
         (moon (hindu-lunar-phase-at-or-before (deg 0) sun))
         (date (floor moon)))
    (if (<= moon (hindu-sunrise date))
        date (1+ date))))
```

```
(defun hindu-sunrise (date)
  ;; TYPE fixed-date -> moment
  ;; Geometrical sunrise at Hindu locale on date.
  (dawn date hindu-locale (deg 0)))
```

Acknowledgments

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India's Contributions to Chinese Mathematics Through the Eighth Century C.E.*

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1 Buddhism: The Medium of Interaction

The rock edicts of King Aśoka (third century B.C.E.) show that he had already paved the way for the expansion of Buddhism outside India.[2] Subsequently, Buddhist missionaries took Buddhism to Central Asia, China, Korea, Japan, and Tibet in the north, and to Burma, Ceylon, Thailand, Cambodia, and other countries of the south. This helped in spreading Indian culture to these countries. It is aptly observed that "Buddhism was, in fact, a spring wind blowing from one end of the garden of Asia to the other end causing to bloom not only the lotus of India, but the rose of Persia, the temple flower of Ceylon, the zebina of Tibet, the chrysanthemum of China and the cherry of Japan. It is also said that Asian culture is, as a whole, Buddhist culture."[3] Moreover, some of these countries received with Buddhism not only their religion but practically the whole of their civilization and culture.

The generally accepted view is that China received Buddhism from the nomadic tribes of Eastern Turkestan toward the end of the first century B.C.E., although there is evidence to show that Indians had gone there earlier to propagate the faith.[4] The Chinese tradition narrates that the Han emperor, Ming-Ti (first century C.E.), had sent an embassy to India to

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[2] Bapat, P. V. (general editor): *2500 Years of Buddhism*. Publications Division, Delhi, p. 53 (1964).

[3] Ibid., p. 397.

[4] Ibid., pp. 59 and 110.

B.S. Yadav and M. Mohan (eds.), *Ancient Indian Leaps into Mathematics*,
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bring back Buddhist priests and scriptures.[5] Consequently, two Indian monks, Kia-yeh Mo-than (Kāśyapa Mātaṅga) and Chu-fa-lan (probably Dharmaratna or Gobharana), reached the Han capital, Loyang. They learned Chinese and translated Buddhist books, the first of which was *Foshuo-ssu-shih-erh-cheng-ching* (the Sūtra of 42 Sections Spoken by Buddha).[6] With the arrival of more monks, both from India and Central Asia, the Loyang monastery became a centre of Indian culture. A large number of Indian books were translated, and people began to adopt Buddhist monastic rituals. Buddhism prevailed so extensively that by the sixth century, the number of monasteries had rise to about 30,000, and the number of monks and nuns to two million.[7]

The tradition of the Buddhist educational system gave birth to large-scale monastic universities. Some of these famous universities were Nālandā, Valabhī, Vikramśilā, Jagaddala, and Odantapurī. They attracted students and scholars from all parts of Asia. Of these, the Nālandā university was most famous, with about ten thousand students and fifteen hundred teachers. The range of studies covered both sacred and secular subjects of Buddhist as well as Brahminical learning. The monks eagerly studied, besides Buddhist works (including *Abhidharma-kośa*), the Vedas, medicine, arithmetic, occult sciences, and other popular subjects.[8] There was special provision for the study of astronomy, and it is said that the university included an astronomical observatory.[9]

According to the findings of a modern Chinese historian (Liang Chi-Chao), more than 160 Chinese pilgrims and scholars came to India between the fifth and eighth centuries.[10] Of these, Fa-Hien (fifth century), Yuan Chwang (seventh century), and I-tsing (eighth century) are the most famous. Some of them stayed and studied in India for several years. They returned to their homeland with many Pali and Sanskrit works, hundreds of which were translated into Chinese.

2 Indian Astronomy and Mathematics in Ancient China

We have seen that Buddhism was the medium for cultural exchange between India and China, providing opportunities for the exchange of ideas. Buddhism exerted great influence in various fields in China and was the main vehicle for transmission of Indian scientific ideas to that land. The influence was so great

[5] Mukherjee, P. K.: *Indian Literature Abroad* (*China*). Calcutta Oriental Press, Calcutta, p. 1 (1928).

[6] Ibid., pp. 2–3.

[7] Chou Hsiang-Kuang: *The History of Chinese Culture*. Central Book Depot, Allahabad, p. 106 (1958).

[8] Bapat (ref. 2), p. 239, and Mukherjee (ref. 5), pp. 78–79.

[9] See K. S. Shukla, *Āryabhaṭa* (booklet), New Delhi, p. 5 (1976).

[10] Bapat (ref. 2), pp. 163–164.

that even scientists embraced the new faith. For instance, the astronomer Han Chai and the mathematician Wang Fan (about 200 C.E.) both became Buddhists (Mikami, p. 57). A great deal of Indian astronomy and mathematics became known in China through the translation of Indian works, and through the visits of Indian scholars. We shall briefly outline the broad facts in this section.

The *Mātaṅga-avadāna* was translated (or retranslated) into Chinese in about the third century C.E., although the original is believed to date earlier.[11] It gives the lengths of monthly shadows of a 12-inch gnomon, which is the standard parameter of Indian astronomy. The work also mentions the 28 Indian *nakṣatraṣ*.

Śārdūlakarṇāvadāna was translated into Chinese several times, beginning in the second century. This work contains the usual Sanskrit names of the 28 *nakṣatraṣ* starting with *kṛttikā*, but the number of *grahas* mentioned is only seven, excluding thereby Rāhu and Ketu, which were often added in the manuscripts and translations.[12] The measures of shadows for various parts of the day mentioned in the work (pp. 54–55) are the same as in the *Atharva Vedāṅga Jyotiṣa*, verses 6 to 11.

Lalitavistara is another work that was translated into Chinese several times from the first century onward. It is in this work that the famous Buddhist centesimal-scale counting occurs during the dialogue between Prince Gautamaand the mathematician Arjuna. The first series of counts ends with *tallakṣaṇa* ($= 10^{53}$), beyond which eight more *gaṇanā* series are mentioned.[13] Atomic-scale counting is also mentioned (there being 7^{10} *paramaṇus* in one aṅgulaparva) (p. 104).

Vasubandhu (fourth century) was so honoured for his work that he was known as the Second Buddha. His *Abhidharma-kośa*, in which he wrote his own commentary, is an encyclopedic work that played an important role in propagating Buddhist philosophy and thought in Asia. It was translated into Chinese and Tibetan. It contains early Buddhist ideas in cosmography (Jambūdvīpa being given the form of a *śakaṭa*) and astronomy (sun and moon revolving around the Meru).[14] It is through this work that we know that the Buddhist school used 60 decuple terms in decimal counting.[15]

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[11] Yabuuti, K.: *Indian and Arabian Astronomy in China*. In: The Silver Jubilee volume of the Zinbun-Kagaku-Kenkyusyo, Kyoto, pp. 585–603 (1954).

[12] Mukhopadhyay, S. K. (ed.): The *Śārdūlakarāvadāna*. Visvabharati, Santiniketan, pp. 46–53 and p. 104 (1954).

[13] Vaidya, P. L. (ed.): *Lalitavistara*. Darbhanga, p. 103 (1958). The last number in the final count will be equal to $10^{7+9\times 46} = 10^{421}$.

[14] *Abhidharmakośa* edited by Dvārikadas Sastri, 2 Volumes, Varanasi, III, 45–60 (1981) (Vol. I, pp. 506–518).

[15] Ibid., p. 544.

The *Mahāprajñā-pāramitā Śāstra* (of Nāgarjuna, second century) was translated into Chinese by Kumārajīva in the early fifth century.[16] The astronomical parameters mentioned in this translation are comparable to those given in the *Vedāṅga Jyotiṣa*.[17]

Bodhiruci I arrived in China (from central India) in 508 C.E., and is said to have translated several Indian astronomical books into Chinese.[18]

An Indian system of numeration appeared in the Chinese work *Ta Pao Chi Ching* (*Mahāratnakūta Sūtra*), translated by *Upaśūnya* (in 541 C.E.).[19] Paramārtha (Po-lo-mo-tho), a native of Ujjain, arrived in China in 548 C.E. and translated about 70 works including the *Abhidharmakośa* (*vyākhyā*)-*śāstra* and the *Lokasthiti-abhidharmaśāstra* (which has astronomical content).[20]

There was a brief setback to Indian activities in China when Wu-Ti came to power in 557 C.E., but they were resumed during the Sui Dynasty (581–618 C.E.). The Indian paṇḍita, Narendrayaśas, was recalled from exile in 582 C.E. Among the works he translated was the *Mahāvaipulya Mahāsannipāta Sūtra*, from Sanskrit. It contains *nakṣatras*, the zodiacal cycle, calendrical material, and other Indian astronomical theories.[21]

The Chinese translations of the following works are mentioned in the *Sui Shu*, or *Official History of the Sui Dynasty* (seventh century):[22]

1. *Po-lo-mên Thien Wên Ching* (Brahminical Astronomical Classic) in 21 books.
2. *Po-lo-mên Chieh-Chhieh Hsien-jen Thien Wên Shuo* (Astronomical Theories of Brāhmaṇa Chieh-Chhieh Hsienjen) in 30 books.
3. *Po-lo-mên Thien Ching* (Brahminical Heavenly Theory) in one book.
4. *Mo-têng-Chia Ching Huang-thu* (Map of Heaven in the Mātaṅgī Sūtra) in one book.
5. *Po-lo-mên Suan Ching* (Brahminical Arithmetical classic) in three books.
6. *Po-lo-mên Suan Fa* (Brahminical Arithmetical Rules) in one book.
7. *Po-lo-mên Ying Yang Suan Ching* (Brahminical Method of Calculating Time) in one book.

16 Bapat (ref. 2), p. 115.

17 Chin Keh-mu, "India and China: Scientific Exchange" in D. Chattopadhyaya (ed.): *Studies in History of Science in India*. Vol. II, pp. 776–790, (1982) (Solar month $30\frac{1}{2}$ days (year = 366 d.), $P = 27\frac{21}{60}$ (cf. $27\frac{21}{67}$), and $S = 29\frac{30}{62}$ (cf. $29\frac{32}{62}$).

18 Mukherjee (ref. 6), p. 38.

19 Needham, J.: *Science and Civilization on China*. Vol. III, Cambridge, UK, p. 88 (1959).

20 See Bapat (ref. 2), p. 214; Mukherjee (ref. 5), p. 34; and Needham (ref. 19), p. 707, where the Chinese title of the second work appears as *Li Shih A-Pi-Than Lun* (Philosophical Treatise on the Preservation of the World).

21 Needham, J. (ref. 19), p. 716, and *Chin Keh-mu* (ref. 17), p. 784.

22 Gupta, R. C.: *Indian Astronomy in China During Ancient Times*. Vishveshvaranand Indological Journal, XIX, 266–276, p. 270 (1981).

Although these translations are lost, they were also mentioned in other sources.

More vigorous contacts and activities took place during the glorious period of the Tang Dynasty (618–907 C.E.). In response to an envoy sent by the Indian king Harṣavardhana in 641 C.E. to China, two missions came from there to India. Hiuen Tsang (or Yuan Chwang) needed 22 horses to carry the works that he took from India to China in 645. He translated 75 of these, including *Abhidharmakośa*.

The great influence of Indian astronomy at that time can be seen by the presence of a number of Indian astronomers in the Chinese capital Chang-Nan, where there was a school in which Indian *sidhāntas* were taught.[23] In fact, there were three clans of Indian astronomers, namely Kāśyapa, Gotama, and Kumāra. These Indians were employed in the Chinese National Astronomical Bureau and helped in improving the local calendar.

The greatest of these was Gotama Siddha (or Gautama Siddhārtha). He became the president of the Chinese Astronomical Board and director of the royal observatory. Under imperial order (from Hsuan-tsung) he translated the famous *Chiu Chih Li* ("Navagraha Karaṇa") from Indian astronomical material in 718 C.E. A few years later, he compiled the *Khai-Yuan Chan Ching* (the Khai Yuan Treatise on Astronomy and Astrology) in 120 volumes, of which the 104th is the *Chiu Chih Li*. It includes the Indian sine table ($R = 3, 438$, $h = 225\,\text{min}$) and Indian methods of calculation with nine numerals and zero (denoted by a thick dot $\bullet$). The astronomy was based on nine planets, including *Lo-hou* and *Chi-tu* (which are Chinese forms of the Sanskrit names Rāhu and Ketu).[24]

3 Earlier Chinese Parallels of Indian Mathematical Pieces

Before addressing the question of mutual transmissions further, we shall first mention the close resemblances that exist between some mathematical problems, rules, and formulas as found in China and India.

(I) The Broken Bamboo Problem (वंशभङ्गोद्देशकः)

In China this is found in the famous *Chiu Chang Suan Shu* (Nine Chapters on the Mathematical Art), whose present text is placed in the first century C.E. Its ninth chapter, entitled "kou ku" (Right Triangles), contains the following problem:[25]

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[23] Ibid., pp. 271–273.

[24] The work has been fully translated with notes by Kiyori Yabuuti in his paper "Researches on the *Chiu-Chih Li* Indian Astronomy under the Thang Dynasty" Acta Asiatica, Vol. 36, pp. 7–48 (1979).

[25] Waerden, B. L. van der: *Geometry and Algebra in Ancient Civilization*. Springer–Verlag, Berlin, p. 53 (1983).

Problem 13: A bamboo is 1 chang (= 10 *Chhih*) tall. It is broken, and the top touches the ground 3 *chhih* from the root. What is the height of the break?

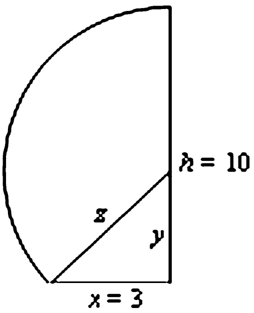

Fig. 1. The bamboo problem

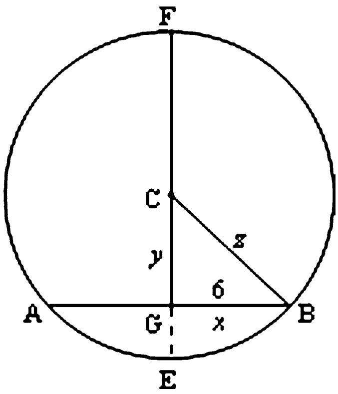

Fig. 2. Solution to the bamboo problem

The solution to the problem is (see Fig. 1)

$$y = (h - x^2/h)/2 = 4\frac{11}{20} \text{ } chhih.$$

It is understood that the solution is based on the Pythagorean property, so that

$$y + z = h \text{ and } z^2 - y^2 = x^2.$$

One of the two similar examples given by Bhāskara I (629 C.E.) reads[26]

अष्टादशकोच्छायो वंशो वातेन पातितो मूलात् ।
पड़्गत्वाऽसौ पतितस्त्रिभुजं कृत्वाक्व भग्नःस्यात् ॥

aṣṭādaśakocchāyo vaṁśo vātena pātito mūlāt
ṣaḍgatvāạsau patitastribhujaṁ kṛtvākv bhagnaḥsyāt

A bamboo of height 18 is felled by the wind. It falls at (a distance of) 6 from the root (thus) forming a triangle. Where is the break?

[26] Shukla, K. S. (ed.); *Āryabhaṭīya* with the commentary of Bhāskara I and *Someśvara*, INSA, New Delhi, India, pp. 99–100 (1976).

Bhāskara's solution is based on applying the relation (see Fig. 2)

$$GF \cdot GE \;=\; GB^2,$$

which is given in *Āryabhaṭīya* II, 17 (second half), on which he is commenting. He gets

$$GE \;=\; x^2/h \;=\; 2 \;=\; z - y.$$

Then doing *saṁkramaṇa* with $z + y \;=\; 18$, he found z and y to be 10 and 8.

(II) Problem of a Reed in a Pond (कमलोद्देशकः)

This is problem no. 6 in the ninth chapter of the *Chiu Chang Suan Shu*:[27]

> There is a pond whose section is a square of side 1 *chang* (= 10 *chhih*). A reed grows at its centre and extends 1 *chhih* above the water. If the reed is pulled to the side (of the pond), it reaches the back precisely. What are the depth of the water and the length of the reed?

The solution given[28] is $x = (z^2 - e^2)/2e$, where z is half the side of the pond, and $y = x + e$ (see Fig. 3).

Bhāskara I's first similar example (out of two) reads[29]

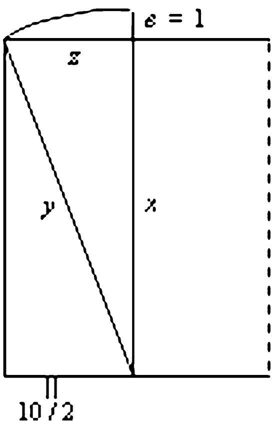

Fig. 3. The reed problem

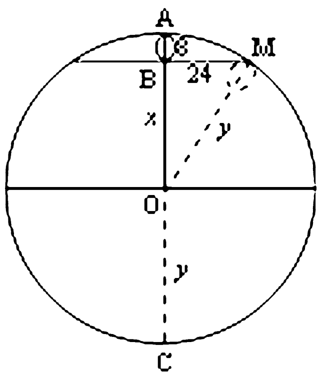

Fig. 4. Solution to the reed problem

[27] Waerden, B. L. van der (ref. 25), pp. 50–51.

[28] Swetz, Frank: The Amazing *Chiu Chang Suan Shu*. Math. Teacher, 65, 423–430, p. 429. Translation kindly supplied by D. B. Wagner.

[29] Shukla (ed.), op. cit. (ref. 26), pp. 100–102. Shukla's remark (p. 299) that the Chinese and Hindu solutions are "quite different" is not justified, since both are ultimately based on the Pythagorean property. The relation $BC = y + x = z^2/e$

कमलं जलात्प्रदृश्यं विकसितमष्टाङ्ग्गुलं निवातेन।
नीतं मज्जति हस्ते, शीघ्रं कमलाम्भसीवाच्ये॥

kamalṁ jalātpradṛśyaṁ vikasitamaṣṭāṅggulaṁ nivāten
nītaṁ majjati haste, śīghraṁ kamalāmbhasī vācye

A lotus in full bloom of 8 *aṇgulas* is visible (just) above the water. When carried away by the wind, it submerges just at the distance of 1 *hasta* (= 24 aṇgulas). Tell quickly (the height of) the lotus plant and (the depth of) the water.

His solution is again based on the same property of chords, namely (Fig. 4)

$$BC \;=\; BM^2/AB \;=\; z^2/e.$$

And then applying *saṁkramaṇa* to $y+x \;=\; z^2/e$ and $y-x \;=\; e$, he gets the height of the lotus y and the depth of the water x as 40 and 32 (aṇgulas). On simplification, *Bhāskara's* solution

$$x \;=\; \frac{1}{2}\left(\frac{z^2}{e} - e\right)$$

becomes the same as the Chinese solution

(III) Approximate Volumes of a Sphere

The *Chiu Chang Suan Shu* (first century C.E.) used the approximate rule:

$$V \;=\; \frac{9}{2}r^3 \tag{11}$$

for calculating the diameter of a sphere when its volume V is known.[30] In India, Bhāskara I quotes a rule that gives (11) directly:[31]

व्यासार्धधनं भित्वा नवगुणितमयो गुडस्य घनगणितम्।

vyāsārdhadhanaṁ bhitvā navaguṇitamayo guḍasya ghanagaṇitam

The product of 9 and half the cube of the radius is the ball's volume.

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follows from the property of chords (which itself is based on the Pythagorean property) or from $y^2 - x^2 = z^2$ and $y - x = e$. The slight difference in methods is not significant.

[30] Mikami, Y.: *The Development of Mathematics in China and Japan*, reprinted by Chelsea, New York, p. 14 (1961).

[31] Shukla (ref. 26), p. 61.

Two centuries later, *Mahāvīra* (about 850 C.E.) gave the same rule and regarded it, like Bhāskara, as only a *vyāvahārika*, or practical (not exact), rule.[32] The same is also found in other Jaina works such as *Tiloyasāra* (*gāthā* 19) of Nemicandra (about 975 C.E.) and the *Gaṇitasāra* (V. 25) of Ṭhakkura Pheru (about 1300 C.E.). This shows a Jaina tradition for (11).

Another item of interest is that in China, *Liu Hui* (third century) interpreted (11) wrongly as equivalent to[33]

$$V = \frac{\pi^2}{2} r^3. \tag{12}$$

In India also, Mahāvīra seems to have thought that (11) was based on (12) with the practical value $\pi = 3$. He further derived a better formula by taking $\pi = \sqrt{10}$, which he considered to be sūkṣma.[34]

(IV) The Problem of 100 Chickens

In China, the earliest statement of the problem of a hundred chickens is found in the *Chang Chhiu-Chien Suan Ching* (Arithmetical Classic of Chang Chhiu-Chien), which is generally placed in the second half of the fifth century. It runs as follows:[35]

> A cock costs 5 pieces (*wên*) of money, a hen 3 pieces, and 3 chickens 1 piece. If we buy, with 100 pieces, 100 birds, what will be their respective numbers?
> (Answers: 4 + 18 + 78; 8 + 11 + 81; 12 + 4 + 84.)

A century later, Chen Luan gave two similar problems with cost 5, 4, 1/4, (Answer: 15 + 1 + 84), and 4, 3, 1/3 (Answer: 8 + 14 + 78)[36].

In India such problems appear in the *Bakshāli Manuscript* (whose exact date is uncertain or controversial). One problem relates to buying a total of 20 animals (monkeys, horses, and deer) for a total of 20 *paṇas* at costs 1/4 (say), 4 and 1/2. (Answer: 2 + 5 + 15.)[37]

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32 Jain, L. C. (ed.): *Gaṇitasārasaṅgraha* (with Hindi translation), Sholapur, III, 28, p. 259 (1963).

33 Wagner, D. B.: "*Liu Hui and Tsu Keng-chih* on the Volume of a Sphere," Chinese Science, No. 3, 59–79, p. 60 (1978).

34 Gupta, R. C.: "Volume of a Sphere in Ancient India," paper presented at the Seminar on Astronomy and Mathematics in Ancient India, Calcutta, May 19–21, 1987, has details.

35 Mikami (ref. 30), p. 43. On p. 39 he says that the work "probably belongs to latter half of the sixth century."

36 Ibid., p. 44.

37 Hayashi, Takao: The Bakshali Manuscript, Ph.D. thesis, Brown University, p. 649 (1985). He places the work in the seventh century, which is somewhere in the middle of the early (fourth century) and late (tenth century) dates assigned to it.

Another similar example relates to prices or earnings of men, women, and *śūdras* or children at rates 3, 3/2, and 1/2. (Answer: 2 + 5 + 13.)[38]

An example of buying 100 birds (pigeons, cranes, swans, and peacocks) with 100 *rūpas* (or *paṇas*) with rates 3/5, 5/7, 7/9, 9/3 occurs in *Śrīdhara's Pāṭīgaṇita* (Ex. 78–79) (eighth century) as well as in *Mahāvīra's Gaṇitasārasaṅgraha* (VI, 152–153) (ninth century).[39] This problem was quite popular in India, and one of the many solutions is 15 pigeons, 28 cranes, 45 swans, 12 peacocks.[40] Similar problems were also popular in other parts of the world, as shown by works, of various authors starting with *Alcuin* (ninth century).[41]

In simple matters, like the use of $\pi = 3$, we may accept independent discoveries or inventions by different cultural groups. But when specific characteristic rules and problems, such as (I)–(IV) considered above, are found to occur in different cultural areas, we have to favor a theory of diffusion. Of course, there may have been an older common source from which material was possibly transmitted to the various cultural areas. B.L. van der Waerden (p. 66) considers a pre-Babylonian common source for Chinese and Babylonian algebra. In fact, he has formulated the thesis of a common Indo-European origin of mathematics that flowed to China, India, Babylonia, Greece, and Egypt (pp. 67–69). We have evidence that some peculiar rules such as the "surveyor's rule" for the area of a quadrilateral[42] and the use of $h(c + h)/2$ (or its other derived forms) for the area of a segment of a circle were widely diffused.

Regarding pieces of (I)–(IV) discussed above, we have not come across specific earlier instances in which these are found as such. It is therefore to be presumed that there was some interaction that ultimately led to transmission between China and India. We have already noted above that even Chinese mathematicians, such as Wag Fan (about 200 C.E.), became Buddhists (Mikami, ref. 28, 57). Needham[43] mentions the monk Than Ying (about 440 C.E.), who could have been a teacher of *Chiu Chang Suan Shu* and commentary by *Liu Hui*.

References to Buddhism and Buddhist works are found even in the mathematical treatises of China such as the *Sun Tzu Suan Ching* or Arithmetical Manual of Master Sun, which is placed[44] between 280 and 473 C.E. Master

[38] Ibid., p. 650; and David Singmaster, *Sources in Recreational Mathematics*, 3rd Preliminary Edition, p. 139, June 1988.

[39] Shukla, K. S. (ed.): The Patiganita of Sridharacarya, Lucknow, pp. 80–83 (1959)(text) and 50–51 (transl.), Jain (ref. 30), p. 131.

[40] Shukla (ref. 39) has given all the 16 solutions. Also see Hayashi (ref. 37), p. 650, for more references.

[41] Singmaster, op. cit. (ref. 38), pp. 139–144.

[42] Gupta, R. C.: The Process of Averaging in Ancient and Medieval Mathematics. *Gaṇita Bhāratī*, III, 32–42 (1981).

[43] Needham (ref. 19), p. 149.

[44] Mikami (ref. 30), p. 26.

Sun's work is important for early indeterminate analysis in China. *Chen Luan* (sixth century), who was also interested in indeterminate analysis, was an ardent believer of Buddhism. He read Buddhist works profoundly and mentioned them in his writings.

At least some of the Indian scholars who visited China must have become familiar to some extent with the local mathematical traditions, especially the more popular common and recreational types of problems. Some of these Indians frequently returned to India (if only temporarily). In addition, Chinese pilgrims, scholars, and envoys (including diplomats) who visited India may have taken some Chinese mathematical classics, such as the famous *Chiu Chang Suan Shu*, with them. Books may have been part of gifts that may have been presented to the kings or universities. All such things indicate a strong possibility of mathematical interaction between China and India. But while these were documented in Chinese sources, there is no similar positive literary or other documentary evidence known from Indian sources that specifies clearly the arrival of any Chinese mathematical material in India.[45]

4 I-Hsing (683–727 C.E.): The Great Chinese Astronomer–Mathematician

By the end of the seventh century C.E., much Indian mathematics and mathematical astronomy was known in China. The compilation of *Chiu Chih Li* in Chinese by Gautama Siddha from Sanskrit sources represents the culmination of such transmissions in 718 C.E. Through this work, Indian methods of computation based on the decimal place-value system (with a zero symbol) and Indian trigonometry (based on sines) were formally introduced in China. The analysis of the contents of *Chiu Chih Li* by Yabuuti (ref. 22 at the end) shows that mathematical astronomy as found in *Sūryasidhānta* and in the works of *Varaḥmihira* (sixth century C.E.) and *Brahmagupta* (seventh century) was known in China at the beginning of the eighth century.

At this time I-Hsing appeared on the Chinese scene. He was an able mathematician, deeply learned in astronomy, and was well-versed in Sanskrit (Mikami, ref. 28, p. 60). He combined in himself the traditions of Chinese and Indian mathematical sciences. He became a Buddhist monk, attended convocations of monks and *śramaṇas*, and traveled widely to acquire knowledge (Needham, ref. 17, p. 38).

[45] There are similarities in many other mathematical works that we have not discussed here. Some of these are treated by B. Datta in his paper "On the Supposed Indebtedness of Brahmagupta to *Chiu Chang Suan Shu*," *Bulletin of the Calcutta Math. Soc.*, Vol. XXII, pp. 39–51 (1930). Datta does not mention Bhāskara I. Also see van der Waerden (ref. 23), pp. 196–208, for $\pi = 3.1416$, and L.C. Jain, "Jaina School of Mathematics (A Study in Chinese Influences and Transmissions)," *in Contribution of Jainism to Indian Culture* (ed. by R.C. Dwivedi), Delhi, India, 206–220 (1975).

I-Hsing won a great reputation for his combinatorial calculations. Due to his Buddhist training, he could easily handle large numbers such as 3^{361} or 10^{172}. His methods were capable of enumerating all possible changes and transformations occurring on a go or chess board (Needham, ibid., p. 139). He could also handle indeterminate problems involving large numbers (ibid. pp. 119–120). In India, similar problems had already been solved by Bhāskara I (early seventh century). Some scholars have confused him with I-Hsing, the pilgrim.[46]

Between 721 and 727 C.E., I-Hsing prepared, by imperial order, a calendar known as *Ta Yen Li* (Needham, ref. 17, p. 37), in which he applied higher mathematics. Out of the 23 different systems of calendars known by that time, I-Hsing's was found to be accurate and has stood the test of time (Mikami, ref. 28, p. 60).

Gautama Chuan (of the Kumāra clan) probably knew that one of his Indian collegues had taught I-Hsing the method (say as given in the *Sūrya-Siddhānta*) for relating gnomon shadows and solar zenith distance (or altitude) by means of *Chiu Chih Li's* sine table.[47] I-Hsing fully used this knowledge.

Greatly influenced by Indian astronomy, I-Hsing made measurements in ecliptic coordinates, which had previously played a minor role (Needham, ref. 17, p. 202). He was associated in training officials and observers for the great meridian survey of 724 C.E.[48] The observed data were also analyzed by him. He developed a tangent table that is the earliest of its kind in the world. This development was based on Indian information about the use and values of sines, from which his tangent table was derived.[49] He used methods of finite differences, fitting of polynomials, and interpolation.[50]

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[46] Shukla (ref. 26), p. 311.

[47] Cullen, C.: "An Eighth Century Chinese Table of Tangents," *Chinese Science*, No. 5, 1–33, p. 32 (1982).

[48] Beer, A., et al.: An Eighth Century Meridian Line: I-Hsing's Chain of Gnomons. Vistas in Astronomy, Vol. 4, 3–28, p. 14 (1961).

[49] Cullen (ref. 47), p. 32.

[50] See Cullen's paper (ref. 47) and *Historia Mathematica*, Vol. 11, pp. 45–46 (1984), where it is stated that *Liu Ch′uo* (about 600 C.E.) knew the formula for interpolation for equal intervals and *Li Ch′un-feng* (665 C.E.) had studied finite differences up to the second order, and interpolation for equal as well as for unequal intervals. See R. C. Gupta.: Second Order Interpolation in Indian Mathematics, etc. *Indian J. Hist. Sci.,* IV, 86–98 (1969).

The Influence of Indian Trigonometry on Chinese Calendar-Calculations in the Tang Dynasty

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1 The Impact of Indian Trigonometry on Mathematics in Ancient China

It is an important event in the history of Chinese mathematics that Indian trigonometry was introduced into China together with the *Chiuchi* calendar translated by Gautama Siddha in 718 C.E.[3] A question arises, then, as to how Chinese mathematics was influenced by Indian trigonometry. Although the *Chiuchi* calendar introduced Indian trigonometry and Western mathematical astronomy to traditional Chinese mathematics, it cannot be treated as part of ancient Chinese mathematics, since neither the Chinese mathematicians at that time nor their successors adopted the algorithms contained in their work. It is to be mentioned here that this paper discusses only the impact of Indian trigonometry, focusing on the impact of the related calculation of $R \sin \theta$ based on the sexagesimal measure of an angle, though it is known that Chinese astronomy was influenced by Indian astronomy.[4]

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[3] Yang-Xiu, Ou and Qi, Song: *Xin Tangshu*. Zhong Hua Shu Ju, Vols. 28–36, pp. 637–662.

[4] Yao-Yong, Duan: The Discussion that Indian Trigonometry affected Chinese Calendar-Calculation in the Tang Dynasty. Ganita-Bharati (*Bull. Indian Soc. Hist. Math.*) Vol. 20(4), pp. 110–112 (1998).

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1.1 The Impact of the Basic Concept and $R \sin\theta$

1.1.1 The Basic Concept

There are few signs of influence on Chinese mathematical astronomy by Indian concepts such as angle based on sexagesimal measure. In Chinese calendars the circumference was divided into $365\frac{1}{4}^\circ$, and the sun traveled 1° per day. This method of division adopted by Chinese calendar-makers was never changed due to its effectiveness. Moreover, the calendars depended on the relationship between angle and degree, for example, $1^\circ = 100'$ in *Dayanli*, $1^\circ = 17'$ in *Qianxiangli*, and $1^\circ = 10'$ in *Lindeli*, etc.[5] The calculation of eclipses also did not use the concept of angle, which led to the determination of an eclipse according to the phase angles of the planets. There did appear traces of angle–degree in the *Dayanli* and *Futianli*, but that was only for the convenience of the difference calculation. As for the table of the course of the sun ($0-182^\circ$), it indeed belongs to the Chinese tradition of mathematics, in which one quadrant amounts to 91° and $1^\circ = 100'$. It shows to a certain extent that Chinese scholars had never adopted the concepts of degree, minute, phase, etc. and the relations of their conversion in Indian trigonometry.

1.1.2 The Table of $R \sin\theta$

The table of $R \sin\theta$ in the *Chiuchi* [1] calendar was applied to determine interval quantities.[6] Since $3438 \cdot \sin(\lambda - \Omega)$, and the interval quantities of the moon can then be determined, which amounts to the "departure from the ecliptic of the moon" in Chinese mathematical astronomy:

$$3438 \sin\beta = \frac{4 \cdot 3438 \sin(\lambda - \Omega)}{(40341/\Delta\lambda)}.$$

In which we mark $\beta, \Omega,$ and λ as celestial latitude of the moon, longitude of ascending node of the moon and celestial longitude of the moon, respectively (for example, the actual movement of the moon per day). In the *Dayanli* and *Lindeli*, the solar course and the lunar departure were calculated first by consulting the tables of solar course and lunar departure, and then by means of secondary interpolation.[7] Moreover, one can see $1^\circ = 10'$ in *Lindeli* and $1^\circ = 120'$ in *Dayanli*. All those are apparently different from the *Chiuchi* calendar.

[5] Zhi-Gang, Ji: *The Creative Change in the Calendar of the Sui and Tang Dynasties*. Studies of Mathematical Astronomy in Ancient China, Xian: Northwestern University Press, p. 19 (1994).

[6] Gupta, R. C.: *Early Indians on Second Order Sine Differences*. Indian Journal of History of Science. Vol. 7, No. 2, pp. 81–86 (1972).

[7] Rong-Bin, Wang: *The Principle of Designing for the Method of Interpolation in the Chinese Ancient Calendar*. Studies of Mathematical Astronomy in Ancient China, Northwestern University Press, Xian, p. 198 (1994).

1.2 Yi Xing and the Table of Tangents in *Dayanli*

1.2.1 A Short Biography of Yi Xing

Yi Xing, originally named Zhang Sui (683–727), was born in Changle, Henan province. He was a child prodigy and proved highly intelligent while growing up. To avoid Empress Wu Zetian's nephew, Wu Sansi, Yi Xing became a monk and lived in seclusion on Son Mountain. He acknowledged the Zenmaster Pu Ji as his teacher of Buddhism. He later moved to Tiantai, Zhejiang province, and rejected several appointments from the royal court. In 717, Yi Xing was eventually forced into service at the court by order of Emperor Tang Xuanzong, and he was given responsibility for calendar reform in 721. The new calendar was completed in 727, the year of Yi Xing's death. It was said that Yi Xing studied Vedic literature while living as a hermit in Dangyang, Hubei province.[8]

Yi Xing's *Dayanli* was a great contribution to traditional Chinese science, and his work can be considered a milestone in astronomy of medieval China. Having made extensive observations and difficult measurements for several years, Yi Xing obtained the necessary data for the new calendar, at the same time he carefully investigating earlier calendars. He paid great attention to the *Huangjili*, which was most probably the chief source of the *Dayanli*.

1.2.2 The Case of Dayan Plagiarizing Chiuchi

Early in the Tang Dynasty, there were debates about whether *Dayanli* [2] had plagiarized *Chiuchi*.[9] According to the *New History of Tang, Annals of Calendar III*, "Mathematician Qutan Zhuan was unhappy about not being able to participate in calendar making, and therefore presented together with Chen Xuanjing a memorial to the Emperor in 733 criticizing *Dayanli*, saying that it had plagiarized the *Chiuchi* calendar, but the methods were not complete." *Dayanli* was also reproached by the master of Prince Nangong Yue. The Emperor therefore asked the shiyushi (supervisor royal) Li Lin and the taishiling (astronomer royal) Heng to make a comparison by using the records of eclipses at the observatory. The result was that the degree of accuracy in predicting eclipses was 70–80% for the *Dayanli*, 20–30% for *Lindeli*, and 10–20% only for the *Chiuchi* calendar. Therefore the objections to *Dayanli* were rejected by the emperor, and Nangong Yue and the others were convicted.

In fact, one could hardly find a paragraph in *Dayanli* that was directly taken from the *Chiuchi* calendar. At that time, the accuracy of a calendar was

[8] Se, Ang Tain: *A Biography of Yi Xing* (683–727 C.E.). Kertas-Kertas Penggajian Jionghua Papers on Chinese Studies, Vol. III, Jabatan Penggajian Jionghua Unversity, Malaya, pp. 31–58 (1989).

[9] Jiu-Jin, Chen: *The Collection of Chen Jiu-Jin*. Heilongjang Education Press, Harbin, pp. 371–372 (1993).

mainly checked from records of eclipses. *Dayan* was a new calendar, while the data in *Chiuchi* were at least partially out of date. It is therefore not surprising that *Chiuchi* would lose in the controversy over accuracy. However, it was not totally baseless for Chen and Nangong to support Zhuan in accusing Yi Xing of "incompleteness" in using *Chiuchi*'s methods in his *Dayanli*.

(a) The method of Ri Yue Shi Jin (apparent radius of the Sun and Moon) and the Di Ying Ban Jing (radius of the Earth's shadow) for determining eclipses had its origin in *Chiuchi*. *Dayanli* mentioned it in remarks, but did not use it in the calendar itself. This is "incomplete."

(b) Having adopted the concept of Jiu Fu Shi Cha (eclipse differences of nine places) in *Chiuchi*, *Dayanli* used less-convenient interpolation method instead of the trigonometric approach. This is "incomplete."

(c) To correct the unreasonable jump data of the excess and deficiency of the moon's daily path in *Huangjili*, *Dayanli* reduced the minute value of excess and deficiency $2.77°$–$2.42°$, which was still less accurate than the value 2 degrees 14 min in *Chiuchi*.

(d) The nine-road method of daily path to determine the moving period of the intersection point of the ecliptic and the Moon's path in *Dayanli* had a source in *Chiuchi*, the Axiu method, but it is not as accurate as the latter.

It is significant that Yi Xing adopted some of the strong points in *Chiuchi*, but unfortunately he was critical of *Chiuchi* on the whole. That caused the controversy that led Chen, Nangong, and Zhuan to be convicted. Zhuan was reappointed at the royal observatory only 25 years later. Eventually *Chiuchi* was forgotten. As a result of the case of the so-called plagiarizing of *Chiuchi*, it was often concluded that Chinese mathematics was influenced by Indian trigonometry. However, such was not the case: *Dayanli* had not used Indian trigonometry.

1.2.3 Yi Xing's Table of Tangents

Indian trigonometry was transmitted to China in the Tang Dynasty, and Yi Xing knew the *Chiuchi*. Moreover, in *Dayanli* there appeared a table of tangents. One may therefore draw the conclusion that the table of tangents was influenced by Indian trigonometry. However, that was actually not the case. Let us have a look at how Yi Xing calculated his table of tangents. Yi Xing's method was recorded in the *New History of the Tang Dynasty*:

> There would be no gnomon (shadow) right under the sun (*Dairi*). To one degree north, the length of the gnomon shadow is 1379. Take this as the initial difference, which will increase by 1 for every degree up to $25°$, the last increment is 26; it will increase by 2 for every degree up to $40°$, the last increment is 56; it will increase by 3 for every degree up to $44°$, the last increment is 68; it will increase by 5 for every degree up to $50°$, ... it will increase by 19 for every degree up to $60°$, the last increment is 160; ... it will increase 39 for every degree up

to 72°, the last increment is 260;... Take the Ducha (i.e., difference for every degree), sum up all the differences, and add the sum to the initial difference, you will get the Guicha (i.e., difference of the length of the gnomon shadow) at the given degree; sum all the Guicha up, you will get the Guishu (i.e., the length of the gnomon shadow) at the given degree.

According to the above paragraph, quoted from the *New History of the Tang Dynasty*, Liu Jinyi and C. Cullen gave tables, among which Cullen's table seems not to agree with the original source in its treatment of degree increments.[10] The following is a table based on Liu's table by checking against the source from the *The Ancient History of the Tang Dynasty*:[11] Liu's table, though it has been collated, is not yet completely consistent with the original text (Table 1).

Table 1. The part of Richan (the course of the sun) table

North to Dairi ($X°$)	Guishu (L)	Guicha	Diff. of degree	Adding rate	North to Dairi ($X°$)	Guishu (L)	Guicha	Diff. of degree	Adding rate
0	0	1379	1	1	21	3.0499	1610	22	1
1	0.1379	1380	2	1	22	3.2109	1632	23	1
2	0.2759	1782	3	1	23	3.3741	1655	24	1
3	0.4141	1385	4	1	24	3.5396	1679	25	1
4	0.5526	1389	5	1	25	3.7075	1704	26	2
5	0.6915	1394	6	1	26	3.8771	1730	28	2
6	0.8309	1400	7	1	27	4.0509	1758	30	2
7	0.9709	1407	8	1	28	4.2267	1788	32	2
8	1.1116	1415	9	1	29	4.4055	1820	34	2
9	1.2531	1424	10	1	30	4.5875	1854	36	2
10	1.3955	1434	11	1	31	4.7729	1890	38	2
11	1.5389	1445	12	1	32	4.9691	1928	40	2
12	1.6834	1457	13	1	33	5.1547	1968	42	2
13	1.8291	1470	14	1	34	5.3515	2010	44	2
14	1.9761	1484	15	1	35	5.5525	2054	46	2
15	2.1245	1499	16	1	36	5.7591	2100	48	2
16	2.2744	1515	17	1	37	5.9679	2148	50	2
17	2.4259	1532	18	1	38	6.1827	2198	52	2
18	2.5791	1550	19	1	39	6.4025	2250	54	2
19	2.7341	1569	20	1	40	6.6275	2304	56	2
20	2.8910	1589	21	1	41	6.8579	2360	59	3

[10] Cullen, Christopher: An Eighth Century Chinese Table of Tangets. *Chinese Science*, 192, Vol. 5, p. 1.

[11] Xu, Liu: *The Ancient History of the Tang Dynasty*. VI (Vol. 34) Zhonghua Shuju, Beijing, p. 1254.

There are two differences between Cullen's and Liu's tables. First, Liu's table, having been collated, is able to explain the whole of Yi Xing's method in the *New History of the Tang Dynasty* except the phrase "up to 72°, the last increment is 260," while Cullen did not make the necessary collation, but gave the values of the increments at the interval points. Secondly, they differ from each other in the treatment of the increments from one interval to another. A problem is how to understand "increase 1 for every degree up till 25°," i.e., what is the increment from 25 to 26°? Taking 1 as the increment at 25°, "the last increment is 26" will be explainable. It seems to the author that Cullen's explanation is unreasonable at this point. Moreover, it is known from the *The Ancient History of the Tang Dynasty* that for the interval $1°-45°$ the last increment is $68'$. There followed three intervals after that: $45°-61°-160°$. *The Ancient History of the Tang Dynasty* added the phrase "up to 25°, the last increment is 26," implying that one should take the increment at 25° as previously. Thus the whole method will be fully understandable only if we revise the sentence "up to 72°, the last increment is $260''$ by the substitution of "423" for the figure "260."

The author supports Liu's approach for making the table. According to Cullen, Yi Xing's table was affected by the Indian table of $R\sin\theta$ and was obtained by the following calculating steps:

(a) Finding the values of sine for each degree by the table of $R\sin\theta$ (using the first interpolation)
(b) Finding the values of $3484 \cdot R\sin(90° - x)$ for each degree by a similar method
(c) Producing the table of tangents by use of the formula

$$8\frac{3484\sin x}{3484\ R\sin(90° - x)}.$$

Unfortunately, there is no evidence to show that Yi Xing made the transformation of the angle systems between 360° and $365\frac{1}{4}°$. It is also unbelievable that a Chinese scholar in the eighth century would know the relation

$$\tan a = \frac{\sin a}{\cos a}.$$

Moreover, by Cullen's calculating process, the result for the initial difference would be 1378, not 1379. All these things show that it seems impossible for Yi Xing to have calculated his table by means of Cullen's process, and the inference made by Cullen is groundless. As a matter of fact, it is more likely that Yi Xing calculated his table of tangents by means of the table of differences for the purpose of working out the length of the gnomon shadow for the given places. In practical measurement, the angle of the sun at the northernmost city in ancient China, Tiele, to the pole at the winter solstice is 76°, therefore

Yi Xing reasonably stopped the calculation of his table at 80°. It looks at first as if the figures for the last column appear quite irregularly, but one would find a very regular table of differences if one carried the calculation up to the second and third differences.

Table Extract

Adding rate	1st-Difference	Higher differences			
440					
	620				
1060		180			
	800		0		
1860		180		0	
	980		0		0
2840		180		0	
	1160		0		
4000		180			
	1240				
5340					

It is no coincidence, but in fact, there is strong evidence to show that Yi Xing constructed his table of tangents by means of higher differences.

1.3 The Influence of *Futianli*

The author of *Futianli* was Cao Shikui, who lived in the middle Tang. Cao Shikui's method for determining the actual distance in degrees traveled by the sun was recorded in a book titled *Futianjing Richanbiao Licheng* which was not known until the 1960s.[12,13]

In modern symbols, denoting the actual longitude of the sun by λ, the perigee angle by θ, the average perigee angle by l, the length of a tropic year by T, and eccentricity by e, Cao Shikui's method described in the *Licheng* amounts to:

$$\begin{aligned} \lambda - l &= 2e\sin\theta, \\ f(l) &= \frac{1}{3300}(182 - l) \quad (0 \le l \le 91), \end{aligned}$$

and $f(l)$ satisfies the following relations:

(a) $\quad f(T+l) = f(l)$

[12] Yu-xing, Tao: *Futianli* (the study of), *Study of History of Science*, Yanbo Press, Japan, No. 71, pp. 118–120 (1964).

[13] Mao, Zhongshan: *The Position of Futianli in the History of Astronomy. Study of History of Science,* Yanbo Press, Japan, No. 71, pp. 120–122 (1964).

(b) $f(\frac{T}{2}+l) \ = \ -f(l)$

(c) $f(\frac{T}{4}+l) \ = \ f(\frac{T}{2}-l)$.

By the definition of $f(l)$ it follows that $f(l) \ \Leftrightarrow \ A\sin l$.

We do not know even now how Cao Shikui's method was constructed. At any rate, the formula for the central differences created by *Futianli* was further developed by the astronomer Bian Gang at the end of the Tang. The formula was of the form $f(x)=ax(b-x)$, and was therefore called the "method of subtraction and multiplication." Cao Shikui used a single interpolation polynomial instead of the traditional calendar tables and the corresponding interpolations for different intervals. Cao was a pioneer in China who opened the way to formulizing calendar tables and their algorithms. *Futianli*'s other contribution was that for the first time it gave up the long and complicated tradition of calculating the Shang Yuan Ji Nian (grand beginning) for calendar-making in China, and took the Yushui (rain water, the first of the 24 solar terms) as Li Yuan (beginning of the calendar), another important attempt to reform calendar-making after the *Yuanjiali* (443).

A significant fact regarding *Futianli* is that it differed from all the previous Chinese calendars in relation to the Indian calendars. In fact, the *Futianli* bore some analogy with Indian calendars. For instance, it used Luohou and Jidu to calculate eclipses, which were apparently influenced by Indian astronomy. In addition, the *Licheng*'s table listed the differences of the course of the sun in solar terms by degree, which also was similar to Indian calendars. Therefore Wang Yinglin, of the Song Dynasty, said in his *Kunxue Jiwen* (Vol. 9) that the *Futianli* had its origin in the Indian calendar.

In short, there is considerable content in the *Futianli* that showed the influence of Indian calendars. Nevertheless, the *Futianli* did not adopt the Indian carry system: $1° = 60^{'}$ and 1 Zhou (cycle) = 360°.

2 Conclusions and Some Remarks

2.1 A Comparison Between Calendar Systems

By comparing ancient calendar systems of China and the West, one may see that from the *Huangjili* onwards that the Chinese had noticed the variability of the velocity of the visual movement of the sun and had worked out tables of the course of the sun. Chinese scholars formed a calendar system that was based on observational data and used interpolation as the main approach to calculation. In the time of Yi Xing, the interpolation approach became more sophisticated, and Yi Xing and others built up more systematic algorithms that they applied in constructing calendars. In the middle Tang, Cao Shikui made further progress to formulize calendar tables and their algorithms, which had great impact on calendar-making thereafter. It would have been difficult for Chinese scholars to give up such a long tradition of a sophisticated and effective algorithm system. The different angle systems represent an example of such a difficulty. Cao Shikui adopted considerable Indian calendrical knowledge in his *Futianli*, but, still, he transformed it into a form adapted to the Chinese angle system (1 Zhou $= 365\frac{1}{4}^{\circ}$).

2.2 Equivalence of the Chinese *Gou–Gu Method* and Indian Trigonometry

Indian trigonometry is revealed in Indian astronomy and calendars, in particular as an application to the astronomical and calendrical calculations of elements of trigonometry with the correct concept of the angle and a rational carry system of all Indian trigonometric knowledge, only the table of $R\sin\theta$ was transmitted to China. Since in practical calculations, $R\sin\theta$ represents one side of a right triangle, it can therefore be seen as equivalent to the so-called *Gou–Gu method*: a calculational approach using the Pythagorian theorem, which Chinese mathematicians and astronomers had mastered. However, Chinese scholars continued to use the *Gou–Gu method* in making calendars and astronomical calculations, but they did not link the side values of a right triangle with its angles, which prevented them from accepting true trigonometry.

2.3 Conclusion for Exchanges

Both Chinese mathematics and Indian mathematics belong to the oriental tradition, which has a common inclination to pay major attention to calculations. On the other hand, of course, each of them kept its own characteristics. Indian trigonometry was a product of the improvement and re-creation of Greek astronomy. This led India to be a country where trigonometry developed, and, in particular, the origin and application of the table of $R\sin\theta$, which symbolized to a certain extent the appearance of trigonometry. In contrast, Chinese mathematicians and astronomers did not give up their own system, and did not take advantage of trigonometry in their exchanges with their Indian colleagues. Yi Xing used interpolation but not trigonometry in his calendrical calculations. Even Cao Shikui, who went further in studying Indian trigonometry, still transformed the $360°$ system to the $365\frac{1}{4}^{\circ}$ system.

For a long time, Chinese scholars accepted foreign culture only under the paradigm of Jiu Zhang (*Nine Chapters* on the Mathematical Art), and were often reluctant to accept new ideas of mathematics from other countries and areas. That is, of course, inevitable to some extent. If there had been no complete calendar system in China at that time, and if it had been the *Chiuchi*, not the *Dayanli* in of which the accuracy for predicting eclipses was 70–80%, then trigonometry would have appeared and developed in China.

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André Weil: His Book on Number Theory and Indian References

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1 André Weil

After the death of André Weil, one of the leading mathematicians of the twentieth century, in 1998, six articles appeared on his life and work in the *Notices of the American Mathematical Society* in 1999. These were written by some of his close associates, vividly picturing his charismatic personality and the extraordinary breadth and exceptional depth in the mathematics that he created.

> H. Cartan, one of his great friends and collaborators of Bourbaki wrote, "His contributions to enriching the heritage of mankind are enormous."

However, one of the most interesting cornerstones of Weil's multifaceted life and splendid deeds is scarcely mentioned. This was that he embarked on his professional career at Aligarh Muslim University, India, when he was just a young man of 23, almost immediately after finishing his D.Sc. from the University of Paris. How it all happened makes an incredibly fascinating story. In fact, Weil was associated from his school days with Sylvain Levy, the f amous French linguist occupying the "Indian Chair" at the Collège de France. Levy aroused in him an immense interest in Indian culture and civilization by suggesting a supplementary study of Sanskrit and great works like the *Bhagavadgītā*, *Mahābhārata*, and *Meghadūta*. Noting his unabated interest in going to India, Levy was ultimately instrumental in Weil's appointment as professor and head of the Department of Mathematics of Aligarh Muslim University in 1930.

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The period of Weil's stay in India, a little more than two years, left such an indelible mark on his intellectual upbringing that the whole philosophy of his life and mathematical research could be viewed as a manifestation of the ramifications of this period; see Yadav [10]. He himself says that he had a "second birth" (dvija) at the end of this period. There is nothing surprising that out of the six chapters of his autobiography, *Apprenticeship of a Mathematician* [8], he devotes one full Chapter entitled "India," to this period. Weil's autobiography, provides several glimpses into different phases of his life in which his Indian background (of just two years) was decisive in crucial decisions and sustained him in trying situations, troubles, and tribulations. For example, when the Second World War broke out, he was supposed to join the army as a reserve officer, but he decided to refuse service and fled to Finland. He says that he was justified in his decision because his paramount duty at that time was to continue serving mathematics rather than the army. He felt had been strongly that the Second World War had been forced on France and therefore it was not his country's war. Moreover, mathematics had already suffered in his country because of the First World War, and not many mathematicians were left. Thus he felt morally determined to perform his duty to continue serving mathematics as best he could.

In fact, he came to this crucial decision because of his firm belief in Indian mythology and thought. He was inspired by the world-famous gospel of the *Bhagavadgītā* (Chap. 2, ślokā 47):

karmaṇyevādhikāraste mā phaleśu kadācana

(To perform your duty only is your right but never on its fruit.)

Another justification that Weil gives for his decision is based on Gandhi's concept of the "Satyāgraha" (civil disobedience), which he successfully used in the freedom fight against British rule in India. In fact, Gandhi felt that it is one's duty to disobey laws whenever one is convinced that they are fundamentally unjust, regardless of the consequences. Weil felt that it was his duty to devote himself to mathematics and it would have been a sin to let himself be diverted from it.

One of the best parts of Weil's autobiography relates to his poetic description of his courage against the sufferings of the solitude of his prison cell at Rouen, where he was imprisoned after his return from Finland. His contact with Indian thought again helped to sustain him, and he appears as the metamorphosed "Yakṣa" of the Rouen prison. Yakṣa is the main character in *Meghadūta*, one of the finest poems in Sanskrit by the Indian poet Kalidāsa, who is universally recognized as one of the greatest poets the world has ever produced. Weil had studied *Meghadūta* so deeply that he could recite some of its verses from memory. He wrote heartbreaking letters to his wife, Eveline, describing his life and mental state in the solitude of the prison cell, so touching, so deep in introspection, and so emotionally charged.

Whenever he was not doing mathematics in the prison cell, he read the *Chandogya Upanishada* and *Bhagavadgītā*, which he always kept by his side. André Weil's fundamental work penetrates into at least nineteen areas of pure and applied mathematics recognized by the International Union of Mathematics, including algebraic geometry, algebraic topology, analysis, Lie groups and Lie algebras, differential equations, number theory, and history of mathe matics.

Some of the landmarks that he established in his creative research are:

- The proof of Riemann hypothesis for smooth projective curves over a finite field.
- The construction of the compactification in the theory of almost periodic functions.
- The development of harmonic analysis on locally compact groups.
- A Cauchy integral formula in several complex variables that anticipates the Silov boundary (this was established when he was at Aligarh).
- Weil's conjectures for the number of points on a nonsingular projective variety.
- The fundamental Mordell–Weil theorem for elliptic curves.
- The introduction of the Weil group in class field theory and that of fiber bundles in algebraic geometry.
- The use of holomorphic fiber bundles in several complex variables.

See his collected papers [9].

To quote Jean-Pierre Serre from his address to the Academie des Sciences de Paris in March 1999, dedicated to the memory of André Weil,

> What makes his work unique in the twentieth century is the prophetic aspect (Weil sees into the future) combined with the most classic precision. Reading and studying his work and discussing it with him have been my greatest joy as a mathematician.

2 His Book *Number Theory*

André Weil wrote more than twenty books, ranging from textbooks to research monographs and sets of unpublished lecture notes. However, we are concerned here with his book *Number Theory: An Approach Through History* [7].

This is an unusual book by an unusual author, in the sense that it does not simply deal with certain topics in number theory or works of some authors, giving the proper context and references and possibly highlighting their importance and standing in the overall area of investigation. It essentially studies four masters who are known as the founders of the modern number theory: Fermat, Euler, Lagrange, and Legendre, but while doing so it actually describes in detail the salient features of the subject, covering 36 centuries starting from 1900 B.C.E. [5]. The book serves as a sound testament to Weil's abiding interest in the history of mathematics. It shows that Weil was endowed with the knack of efficiently combining a variety of ideas in different directions. He presents the subject in such a fashion that it appears as if he himself were studying the subject along with the masters. In his own words:

> Our main task will be to take the reader, so far as practicable, into the workshop of authors, watch them at work, share their successes and perceive their failures.

We take an example from his book in which he highlights a result of Lagrange and invokes the work done by ancient Indian mathematicians in its historical context.

3 The Square-Nature (Varga-Prakrti)

We consider the Diophantine equation

$$Ny^2 + 1 \ = \ x^2, \tag{1}$$

where $N > 1$, and seek its integer solutions (x, y). We assume without any loss of generality that N is free from square factors. This is known in ancient Indian mathematics as 'Square-Nature' (Varga-Prakrti), see [4], and is well known as "Pell's equation" in general mathematical literature after the British mathematician, John Pell. There is nothing on record that he worked on the equation, but he did have correspondence with Euler (1707–1783) on the subject. No one knows the exact origin of the equation, but it is known to be associated with the algebraic formulation of a problem posed by Archimedes (287–212 B.C.E.) to the Alexandrians [2]. André Weil devotes Chap. IX of his book to a discussion of Pell's equation (1), giving a brief history of the problem and refering to the works of Colebrooke [3] and Datta and Singh [4]. True to his spirit of "watch them at work," he explains Brahmagupta's method of solving the equation, showing that if the equation (1) has a nontrivial solution (other than the trivial one $(1, 0)$), then it has infinitely many solutions. Datta and Singh devote 41 pages of their book to Square-Nature. However, over and above what is given in Colebrooke [3], Datta and Singh [4], and Weil [7], we shall show how Brahmagupta's result can be proved by an elegant use of elementary group theory; see Chahal [2].

We observe first that if (x, y) is a solution of (1), then so also are $(\pm x, \pm y)$. Thus, in what follows, we shall assume that $x, y \geq 0$.

Brahmagupta's Theorem. *If* (1) *has a nontrivial solution, then it has infinitely many solutions.*

Proof. We shall follow Chahal [2]. Let us consider the set G of all solutions (x, y) of (1). Then G is nonempty; Since $(1, 0) \in G$. It was Brahmagupta who defined for the first time a binary composition $*$ on G by

$$(x, y) * (x', y') \ = \ (xx' + Nyy', xy' + x'y). \tag{2}$$

See Weil [7, p. 21]. We see that $*$ is associative and has $(1,0)$ as an identity. Moreover, for each $(1,0) \in G$, $(x,-y)$ serves as its inverse $(x,y)^{-1}$. Thus G is a group. Now suppose that $(x_1,y_1) \in G$, where $x_1, y_1 \geq 1$. For each positive integer n, we put $(x_n,y_n) = (x_1,y_1)^n$, then $(x_n,y_n) \in G$. Since $N, x_1, y_1 \geq 1$, we see from the binary composition (2) that

$$y_1 < y_2 < y_3 < \cdots ,$$

and hence G is infinite. Thus (1) has infinitely many solutions.

Though Brahmagupta's definition of binary composition was of genuine ingenuity, his result did not produce any nontrivial solution of (1). It is interesting to observe that such situations have arisen elsewhere in the history of mathematics. For example, his argument can be fairly compared with that of G. Cantor: After showing that the set of all real numbers is uncountable and that the set of all algebraic numbers is countable, Cantor concluded that the set of all real transcendental numbers is infinite (in fact, uncountable). Mathematicians initially rejected Cantor's argument, since it did not show the existence of even a single transcendental number. Brahmagupta's work was rediscovered by Euler in 1764 and recognized as important by Lagrange in 1768.

It was left to the genius of *Bhāskarācarya* II (1114–1185) and his less-well-known contemporary Jayadeva [6] to solve the equation completely for some particular values of $N = 61, 67, 83, 92$, etc. The task was, in fact, very difficult for such numerical solutions and, as such, was no mean achievement.

It was Lagrange (1736–1813), however, who first showed the existence of a nontrivial solution of the Square-Nature. We say that a (general) group G is cyclic if there exists an element $g \in G$ such that

$$G \;=\; \{g^n : n \in \mathbf{Z}\},$$

and g, in that case, is called a generator of G. Finally, it was Dirichlet who proved that the set G of all solutions of the Square-Nature forms an infinite cyclic group. Studies concerning the Square-Nature in its own right and with applications in various mathematical sciences cover a wide spectrum and have acquired great significance in mathematics. See Chahal [2].

Before we close, I would like to highlight yet another important aspect of Brahmagupta's attempt which has hardly been emphasized earlier. A little digression will perhaps provide a better context.

If we trace back the history of structuralism in the Western world, it originated in literature when the Russian linguist Roman Jacobson started using the term 'structure' for the first time in his works in 1929. However, the credit for its mathematical incarnation goes to Bourbaki, who aimed at building the whole edifice of mathematics to base on the theory of sets. Three binding principles were the axiomatic method, the study of structures, and the unity of mathematics. Bourbaki's structuralism movement was not confined to mathematics alone, but started having ramifications in diverse disciplines like anthropology, philosophy, psychology, economics, and even linguistics. A society called 'Oulipo' was even founded in order to deconstruct and rebuild literature using Bourbaki's mathematical structuralism. The structuralism reigned all through the middle of the twentieth century C.E.

All that we have seen above was initiated by Brahmagupta in the seventh century C.E. (see Chap. 12 for his life and work) in order to solve the Square-Nature. A particular case of the problem was raised by no less a sovereign mind than Archimedes' in the third century C.E. The most important step in the whole development was his definition of the binary composition on the set of all solutions of the Square-Nature. The concept of 'binary composition' is all prevailing and omnipresent in mathematical structuralism and its applications, as it serves as a thread intertwining any kind of mathematical structure. Brahmagupta combined two solutions of the Square-Nature to give another, that is, defined a binary composition on the set of solutions of the Square-Nature, an achievement of genuine ingenuity with far-reaching consequences.

Regarding André Weil, the unquestioned leader of Bourbaki's mathematical structuralism, his researches led to the formulation of sweeping generalizations about zeta functions of general algebraic varieties over finite fields, interpreting the Riemann hypothesis in the new context. He even proved the Riemann hypothesis for smooth projective curves over a finite field, a work which he himself would say it had engaged him most all through his research activities. P. Deligne obtained the proof of the corresponding Riemann hypothesis for zeta functions of arbitrary varieties over finite fields.

If a history of twentieth-century mathematics is written, the work of Weil and Deligne on the Riemann hypothesis must easily be reckoned as one of the top-ranking achievements in mathematical research. Its numerous applications to the solution of long-standing problems in number theory, algebraic geometry, and discrete mathematics are witness to the significance of these general Riemann hypotheses; see Bombieri [1].

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On the Application of Areas in the *Śulbasūtras*

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1 The *Śulbasūtras*

The earliest known texts from ancient India that deal directly with mathematics are the *Śulbasūtras.* The Sanskrit word *śulba* (sometimes written *śulva*) means a rope or a cord, and these texts thus contain the rules of the cord, the knowledge necessary for measuring the arenas and altars used in ancient Indian rituals. Generally dated to the period between 800 and 200 B.C.E., the *Śulbasūtras* are important documents giving information about mathematical knowledge at an early stage in Indian history.

There are a number of *Śulbasūtra* texts, the main ones being the *Baudhāyana-śulbasūtra*, the *Āpastamba-śulbasūtra*, the *Kātyāyana-śulbasūtra*, and the *Mānava-śulbasūtra*, all of which have been published more than once. In the following, all references to these texts refer to the edition of Sen and Bag cited in the bibliography.

2 Mathematics in the *Śulbasūtras*

The mathematical propositions and methods delineated in the *Śulbasūtras* are aimed at practical applications. Although passages such as *Āpastamba-śulbasūtra* 5.7 show that reasoning comparable to what one might call a proof is employed, proofs, in the Euclidean sense of the term, are not found in them. The mathematical content of the *Śulbasūtras* is rich, though, and a brief overview will be given here.

The Pythagorean theorem is given in a geometrical formulation in both the *Baudhāyana-śulbasūtra* (1.9–1.12) and the *Āpastamba-śulbasūtra* (1.4–1.5). After the statement of the theorem, the *Baudhāyana-śulbasūtra* (1.13) gives

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a list of sides of rectangles, all corresponding to Pythagorean triples, to illustrate the theorem; most of these Pythagorean triples are used in the *Āpastamba-śulbasūtra* (5.2–5.6) for measuring out the *mahāvedi*, a sacrificial arena, using cords and stakes. Methods for construction of various geometrical figures, most importantly squares and rectangles, but also isosceles triangles and rhombi, using cords and stakes are given. It is explained how to construct a square equal in area to the sum of the areas of two given squares, and, similarly, how to construct a square with an area equal to the difference of the areas of two given squares. The construction of a figure equal in area to a second figure and similar to a third is presented for a number of figures: transforming a square into a rectangle, a rectangle into a square, and a square into an isosceles trapezoid, as well as attempts at transforming a square into a circle (circling the square) and a circle into a square (squaring the circle). A value of $\sqrt{2}$, namely

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 34},$$

correct to five decimal places, is given. A passage in the *Baudhāyana-śulba-sūtra* (4.15) states that a pit of diameter 1 *pada* (a linear measure) has a circumference of 3 *pada*, which implies $\pi = 3$, while the methods for circling the square and squaring the circle imply a value of π of roughly 3.088. The *Mānava-śulbasūtra* (11.13) further gives a formula for computing the circumference, c, of a circle with diameter d, namely $c = \frac{16}{5} \cdot d$, which implies $\pi = 3.2$. The *Mānava-śulbasūtra* (10.9) gives a formula for computing a volume by multiplying length, breadth and height.

3 The *Agnicayana*

In addition to the above, the texts also provide directions for covering various altars with bricks according to certain rules laid down by the ritual texts. One of the rituals for which the *Śulbasūtras* provide such directions is the *agnicayana* (literally, "the piling up of the fire-altar"). In the *agnicayana*, a fire-altar is erected according to specific rules. It is constructed in five layers with 200 bricks of equal height in each; the edges of the bricks in one layer are furthermore not permitted to coincide with the edges of the bricks in the adjacent layer or layers, except along the border of the fire-altar, where it cannot be avoided. The total area of the altar has to be $7\frac{1}{2}$ square *puruṣa* (a linear measure corresponding to the height of a man with his arms stretched upward), but its shape can be varied according to the desired outcome of the ritual. An older ritual text, the *Taittirīya-saṃhitā* (5.4.11; see [3, Vol. 2, p. 438–439]), gives a list of shapes and their corresponding outcomes, including the shape of a falcon for flight to heaven, and an isosceles triangle or a chariot wheel for the destruction of enemies. The *Baudhāyana-śulbasūtra* (20–21) contains one shape that is not mentioned in the *Taittirīya-saṃhitā*, namely that of a

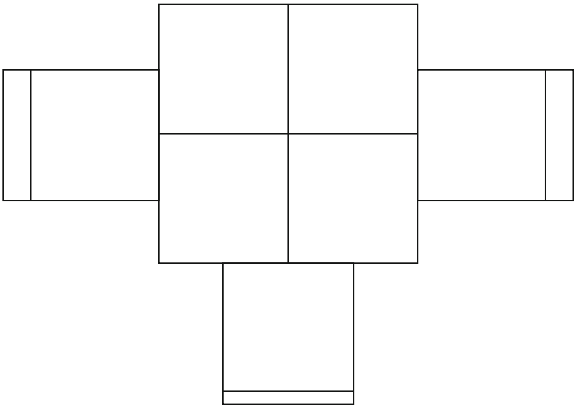

Fig. 1. Rectangular falcon

tortoise for winning *brahmaloka*, the world of the spirit. The practical matters concerning measuring out these altars and erecting them with bricks is taken up in the *Śulbasūtras.*

If a patron desired to perform the *agnicayana* more than once, the area of the altar was to be increased by 1 square *puruṣa* at each successive performance. The geometrical problem involved here is to increase the area of a given figure while preserving its shape. In the *Śulbasūtras* (*Baudhāyana-śulbasūtra* 5.6 and *Āpastamba-śulbasūtra* 8.6) this is accomplished in essentially the following way. The area to be added to the original $7\frac{1}{2}$ square *puruṣa*, say n square *puruṣa*, is divided into 15 parts. Two such parts are added to a square *puruṣa* and the side of the resulting square gives a new unit length that is then used instead of the *puruṣa* to construct the altar according to the rules given previously.

Of particular interest here is the *agnicayana* altar shaped like a falcon. In its simplest form, the falcon-shaped altar is composed of squares and rectangles: four squares with side 1 *puruṣa* form the body, two such squares form the two wings, and a seventh square forms the tail. In addition, the two wings are extended by $\frac{1}{5}$ *puruṣa*, and the tail by $\frac{1}{10}$ *puruṣa*. The altar is shown in Fig. 1. The altar together with instructions for erecting it with bricks in different ways is given in *Baudhāyana-śulbasūtra* 8–9 and *Āpastamba-śulbasūtra* 9–11. In the following this particular falcon-shaped altar will be referred to as the rectangular falcon.

In addition to this form, two falcon-shaped altars that better resemble actual birds of prey are described in both the *Baudhāyana-śulbasūtra* (10 and 11) and the *Āpastamba-śulbasūtra* (15–17 and 18–20). These falcon-shaped altars, which will be referred to as realistic falcons in the following, have heads, curved wings and tail, and, with the exception of the first realistic falcon in the *Āpastamba-śulbasūtra*, plumage. As an example of one of these, Fig. 2 shows the second realistic falcon in the *Baudhāyana-śulbasūtra*.

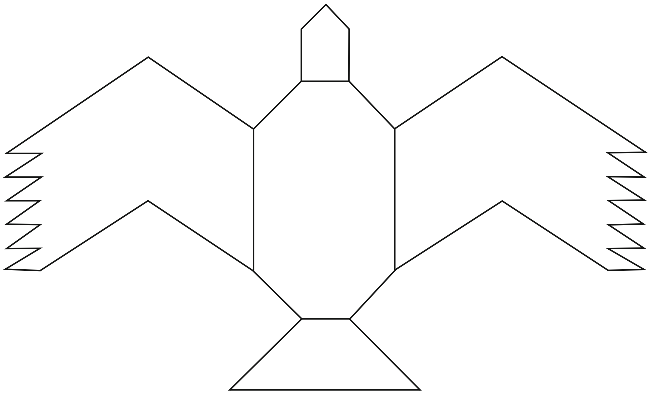

Fig. 2. Realistic falcon

4 Relationship Between the *Śulbasūtras* and Older Literature

Seidenberg noted that the *Śulbasūtras* never claim originality, but merely assert that they follow earlier teachings [9, p. 105]. However, as has been pointed out by Chattopadhyaya, such assertions always occur in connection with theological or ritual considerations and not in connection with the practical aspects of altar construction [11, p. vi–vii]. In other words, when it comes to the question of what is to be done, the *Śulbasūtras* will cite an authoritative ritual text, but when it comes to how it is to be done, no authorities are cited. In this connection, Pingree's strong suspicion that "the geometry was not invented to provide the priests with a technical means of meeting their rather arbitrary rules for the construction of altars, but rather that the rules were devised to utilize an existing constructive geometry" [7, p. 184] is important. This raises the important question whether the practical knowledge of the *Śulbasūtras* was the property of a class of craftsmen (Chattopadhyaya argues that the tradition could have ties with the Harappan civilization [11, p. xviii]), rather than of the priestly class. As has been pointed out by David Pingree, the origin of the practice of applying geometry to constructing altars as well as the development of this science within the Indian tradition are topics that need to be addressed [7, p. 184], but these important issues aside, it can be shown that much of the mathematics in the *Śulbasūtras* predates the texts themselves.

Seidenberg has argued convincingly that some of the mathematical results and procedures found in the *Śulbasūtras*, such as the Pythagorean theorem and the increase of the area of the *agnicayana* altar while preserving its shape, must have been known in the period of the *Śatapatha-brāhmaṇa* and, in the case of the Pythagorean theorem, perhaps even in the period of the *Taittirīya-saṃhitā* [9, pp. 105–108]. Kulkarni has similarly explored mathematics as known before the composition of the *Śulbasūtras* [5, pp. 9–29].

Considering all this, there can be little doubt that much of the mathematical knowledge of the *Śulbasūtras* is not original to them and was known in previous periods. The *Śulbasūtras* are, however, the first known texts to codify this knowledge. Mathematical results and methods, such as the Pythagorean theorem, and the conversion of a given rectangle into a square of the same area, are given a general formulation in them, while other results are given in connection with specific constructions. It is possible that certain procedures, such as the squaring of the circle or the circling of the square, are original. In other words, the *Śulbasūtras* mark the point in Indian history where mathematical knowledge is for the first time presented directly in sections of the sacred texts. It is in this sense that the *Śulbasūtras* represent a leap in the history of ancient Indian mathematics.

5 Application of Areas

Van der Waerden believed that three mathematical propositions, namely the Pythagorean theorem, the application of areas (i.e., the problems in Euclid's *Elements* VI. 28–29, which are generalizations of II. 5–6 and enable one to solve quadratic equations geometrically), and the problem of constructing a figure equal in area to a second figure and similar to a third, were not thrown together arbitrarily, but were part of a Pythagorean mathematical textbook, "The Tradition of Pythagoras" [12, pp. 117–118]. Investigating the relationship between Greek and Indian mathematics, Seidenberg pointed out that while the third problem is the central problem of the *Śulbasūtras* and the Pythagorean theorem is necessary for its solution, "there is no clear evidence in the Śulvasūtras on the application of areas, but it has been suggested that the Vedic priests could solve quadratic equations, and there are some grounds, not very solid to be sure, for this opinion" [9, p. 100]. What Seidenberg is referring to here is Datta's investigation of the algebraic significance of the *Śulbasūtra* method for enlarging an *agnicayana* altar, connecting it with quadratic equations [1, pp. 165–177]. As we saw above, the *Śulbasūtra* method for enlarging an altar is indeed ingenious, but as a geometrical procedure it does not go beyond the rules and methods given in the *Śulbasūtras*, and there is no evidence, nor does it seem likely, that the *Śulbasūtra* geometers were aware of such algebraic interpretations of the procedure. Datta often gave an algebraic interpretation of *Śulbasūtra* material that goes beyond what is reasonable to assume on the part of the ancient *Śulbasūtra* geometers; see [4] for a critique of this in connection with the problem of covering altars with square bricks.

Nevertheless, the question of the application of areas in the *Śulbasūtras* is an intriguing one, and Seidenberg's statement that "it is conceivable that contemplation of the bird-altar generated questions and answers that did not make their way into the Śulvasūtras" [8, p. 337] has merit. In this spirit the question will be explored further.

6 Transition from Rectangular Falcon to Realistic Falcon

Of the falcon-shaped altars found in the *Śulbasūtras*, the rectangular one is the oldest. Its shape is a gross approximation to the shape of an actual bird of prey, a defect that is rectified by the realistic falcons. Exactly how the given shapes of the realistic falcons were found by the *Śulbasūtra* geometers is not stated in the texts. Perhaps a general outline of the new falcon shape was conceived and it was then sought to accomplish it by modifying the rectangular falcon. The first realistic falcon in the *Āpastamba-śulbasūtra*, where explicit instructions on how to redistribute the area of the rectangular falcon in order to obtain the shape of the altar is given (see *Āpastamba-śulbasūtra* 15.3), could indicate this. In any case, finding the given shapes must have involved shifting areas around between the various parts of the falcon.

7 The Tail of the Falcon

For the two realistic falcons in the *Āpastamba-śulbasūtra* and the first realistic falcon in the *Baudhāyana-śulbasūtra*, clear instructions are given on how to measure the various parts of the altar. In the case of the second realistic falcon in the *Baudhāyana-śulbasūtra*, however, the information that we are given in the text is much more scarce. We are given the area of each of its parts expressed as a number of a certain type of brick and a few more instructions regarding the body and the wings.

From this information one can with relative ease find the shapes of the falcon's body, head, and wings using methods found in the *Śulbasūtras*. For the tail, however, we have only the area and also, since the shape of the body can be deduced, the length of the side where it touches the body. However, finding the shape of the tail from only these two values is not straightforward.

The two realistic falcons in the *Baudhāyana-śulbasūtra* have the same tails, which again are the same as the tail of the second realistic falcon in the *Āpastamba-śulbasūtra* as well as the tails of the *agnicayana* altars in the shapes of a heron and an *alaja* bird given in the *Baudhāyana-śulbasūtra* (12 and 13, respectively), a fact that may explain the scarcity of information.

The tail is an isosceles trapezoid as shown in Fig. 3 (and also Fig. 2). When partitioned into two triangles and a rectangle, the two triangles are right-angled and isosceles.

The shape of the tail can further be found from information given in the text, namely that its area equals that of 15 of a certain type of brick: two of

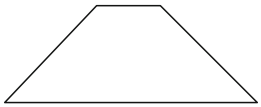

Fig. 3. Tail of realistic falcon

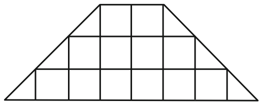

Fig. 4. Tail of realistic falcon covered with bricks

these brick fit along the side of the body where the tail is to be attached, and building from there, cutting three bricks diagonally to form the slope, the tail emerges (see Fig. 4).

However, let us for a moment consider the problem from the point of view of knowing only the area, A, of the tail as well as the length, a, of the side where it is to be attached to the body. It was suggested above that the *Śulbasūtra* geometers would probably have had to experiment with the distribution of the total area between the parts of the falcon before finding a suitable form for the realistic falcon, and in that process one can imagine them ending up with a remaining area to be used for the tail as well as an idea about its shape. This would have led to the problem that we are now considering. How is the shape of the tail determined from this information?

We can start by partitioning the tail into two right isosceles triangles and a rectangle. The left isosceles triangle is then moved to the right side and combined with the other isosceles triangle to form a square. This square together with the rectangle form a larger rectangle whose area is equal to that of the tail. This is illustrated in Fig. 5.

That the *Śulbasūtra* geometers were familiar with such procedures is clear from the method for converting a rectangle into an isosceles trapezoid found in the *Baudhāyana-śulbasūtra* (2.6), a method that, as pointed out by Sen and Bag, was known at the time of the *Śatapatha-brāhmaṇa* [10, p. 160], as well as a passage in the *Āpastamba-śulbasūtra* (5.7) where the area of the *mahāvedi*, which, like the tail of the realistic falcon, is an isosceles trapezoid, is derived by cutting off a right triangle from one side and attaching it on the other, thus transforming the *mahāvedi* into a rectangle of the same area.

Let us consider the problem algebraically. If the longer side of the rectangle is y and the shorter x, then, since $x \cdot y = A$ and $y - x = a$, $x^2 + a \cdot x = A$, which shows that the problem is essentially that of solving a quadratic equation.

If we instead consider the problem geometrically, the transformation outlined in Fig. 5 brings to mind Proposition II.6 of Euclid's *Elements*, which can be used to solve it.

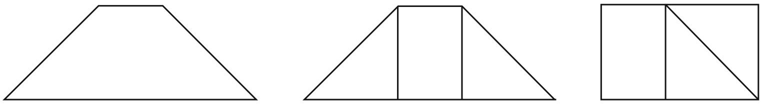

Fig. 5. Tail transformed into a rectangle

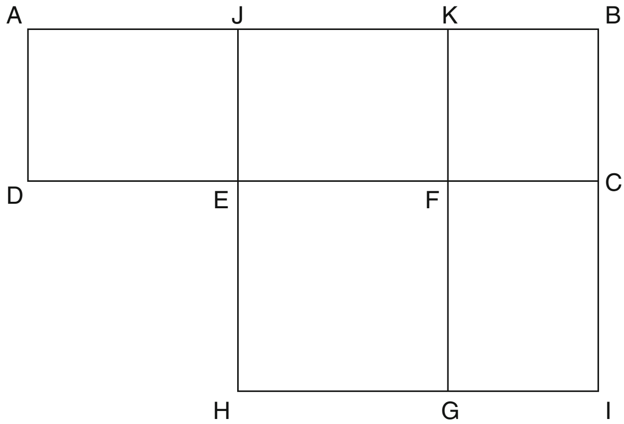

Fig. 6. Euclid's Proposition II.6

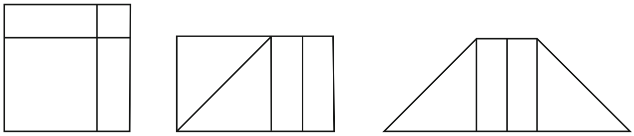

Fig. 7. Construction of the tail of a realistic falcon

In Fig. 6, ABCD indicates an original rectangle that has been partitioned into a square (KBCF) and two equal rectangles (AJED and JKFE). The rectangle FCIG is equal in shape and area to these two rectangles, but is situated on the adjacent side of the square KBCF. According to the proposition, the combined area of the rectangle ABCD and the square EFGH equals the area of the square JBIH.

With this in mind, the problem can now be solved as follows. First a square of area $A + (\frac{a}{2})^2$ is constructed. Two lines parallel to two adjacent sides of the square are drawn at a distance of $\frac{a}{2}$ from the sides. This partitions the original square into two squares and two rectangles. The square of side $\frac{a}{2}$ is removed and the two rectangles are placed on the same side of the square. Then all that remains to be done is to bisect the square diagonally and relocate one half on the other side. The whole procedure is shown in Fig. 7.

There is nothing in the procedure that goes beyond what is found directly in the *Śulbasūtras*, and had a *Śulbasūtra* geometer approached the problem in this way, it is likely that he would have been able to solve the problem as outlined.

This does not, of course, demonstrate that this was how the dimensions of the tail were determined in ancient times. In fact, it is more likely that it is not the case. That all the realistic falcons, except one, as well as the altars shaped like a heron and an *alaja* bird in the *Baudhāyana-śulbasūtra* have the same tail speaks against the idea. It is more likely that a trial-and-error process was carried out, or that a number of desired bricks were simply arranged to form the tail and the rest of the falcon was suited to accommodate it. Once a suitable tail had been arrived at, it was used again in other realistic falcons.

Still, the above considerations are worth pointing out because they provide an example of how the solution of a quadratic equation via the application of areas could have come about practically, i.e., not merely as a mathematical exercise, but in response to an actual problem. Throughout the history of what we now call quadratic equations, from their origins in Mesopotamia some 4,000 years ago to the Renaissance, they have never been accompanied by convincing problems related to practical matters. The problems are given more as an exercise in abstract problem-solving than as a tool for handling practical matters. (I am grateful to Professor Victor Katz for these observations.) The problem of the tail of the falcon would be a convincing practical backdrop from which a quadratic equation could have arisen, if indeed this is how the *Śulbasūtra* geometers solved it. An interesting question is whether later Indian geometers arrived at this procedure from an investigation of *Śulbasūtra* material.

8 Quadratic Equations in Ancient Mesopotamia

The first interpreters of Old Babylonian mathematics tried to understand it in terms of modern mathematics. This led to an interpretation that was decidedly algebraic. However, a more recent study of the material by Høyrup provides us with an interpretation that is geometric. That this geometric interpretation has some similarity with *Śulbasūtra* methods was noted by Høyrup, who also, and rightly so, warned against drawing premature conclusions based on it:

> The general design of some of their (the *Śulbasūtras*') constructions is quite similar to the naive cut-and-paste procedures of the surveyors' tradition; but arguments solely from the "general design" of a mathematical procedure are dangerous unless this design is very complex or presupposes a fallacy that is not likely to occur often-both the number system and plane geometry have their inner constraints that may easily give rise to parallel developments. It is not to be excluded that closer analysis of the *Śulbasūtras* will substantiate links to the

> Near Eastern surveyors; but for the moment this hypothesis is not substantiated . . . [2, pp. 408–409]

Keeping this in mind, it is nevertheless of interest to make a comparison between the Old Babylonian methods for solving quadratic equations and the one outlined above in connection with the tail of the falcon. As will be seen, there is a clear similarity. Here we will focus on one example, a problem from tablet YBC 6967, which is published in [6, p. 129]. According to the interpretation of Neugebauer and Sachs, the problem is algebraic, and amounts to solving the simultaneous equations: $x \cdot y = 60$ and $y - x = 7$, which yields

$$x = \sqrt{\left(\frac{7}{2}\right)^2 + 60} - \frac{7}{2} = 5 \text{ and } y = \sqrt{\left(\frac{7}{2}\right)^2 + 60} + \frac{7}{2} = 12.$$

Høyrup [2, pp. 55–58], however, convincingly interprets the problem geometrically, according to what he calls cut-and-paste geometry. The approach is as follows:

In Fig. 8 we start with an original rectangle about which it is known that its area is 60 and that the difference of its sides is 7. The rectangle is partitioned into a square and two equal rectangles of width $\frac{7}{2} = 3\frac{1}{2}$. One of these rectangles is moved on to the adjacent side of the square, and a square of side $\frac{7}{2}$ is added in the corner to form a larger square. This larger square has area $60+(\frac{7}{2})^2 = 72\frac{1}{4}$, and thus its side is $\sqrt{72\frac{1}{4}} = 8\frac{1}{2}$. The longer and the shorter sides of the original rectangle can now easily be found as $8\frac{1}{2} + 3\frac{1}{2} = 12$ and $8\frac{1}{2} - 3\frac{1}{2} = 5$.

As noted above, the similarity between this approach and the one outlined in connection with the tail of the falcon is clear. In both cases, the difference between the two sides of a rectangle as well as its area are known, and by moving and adding areas, a square, from which the sides of the rectangle can be deduced, is created. In fact, the Old Babylonian method, as reconstructed by Høyrup, is exactly what is required to solve the problem of determining the tail of the falcon. The approach is intuitive, simple, and in harmony with the methods we find in the *Śulbasūtras.*

In 1988, Pingree remarked that "the question of the origin of this (*Śulbasūtra*) geometry . . . and its relations both to Mesopotamian mathematics and to a Vedic craft tradition seems . . . to be yet unanswered" [7, p. 184]. This is an important question, and further detailed analyses need to be carried out before it can begin to be answered.

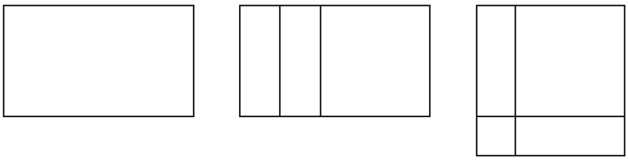

Fig. 8. Mesopotamian cut-and-paste geometry

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Divisions of Time and Measuring Instruments of Varaḥmihira

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1 Introduction

The Vedic scholars were keen observers of the sky and they associated Rigvedic the concept of time with the motion of the sun. During the early Rigvedic age it had been recognized that one year was composed of 360 days and 360 nights as in the following hymn [24, I, 164, 11]:

द्वादशारं नहिताज्जराय
वर्वति चक्रं परिद्यामृतस्य ।
आ पुत्रा अग्ने मिथुना सो
अत्र सप्त शतानि विंशतिश्च तस्युः ॥

dvādaśāraṁ nahitājjarāya
varvarti cakraṁ paridyāmṛtasya |
ā putrā agne mithunāso
atra sapta śatāni viṁśatiśca tasyuḥ ||

> That is, the sun's wheel consisting of twelve spokes (months) revolves in the sky and is never destroyed. Oh! Fire, on this wheel are mounted seven hundred and twenty people (360 days and 360 nights).

The above hymn implies that for ordinary purposes the year consisted of 12 months and 360 days, i.e., each month was composed of 30 days.

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The *Atharvaveda*, on the other hand, categorically states that the earth revolves around the sun and the latter divides day and night into thirty parts according to the following hymns [4, VI, 30, 1 and 3]:

आयं गौः पृश्निर क्रमीदस दन्मातरं पुरः।
पितरं च प्रयन्त्स्वङ ॥१॥
त्रिशद् धामा वि राजति वाक् पतंगो अशिश्रियत्।
प्रति वस्तोरहर्द्युभिः ॥३॥

āyaṁ gauḥ pṛśnir kramīdasa danmātaraṁ puraḥ |
pitaraṁ ca prayantsvń || 1 ||
triśad dhāmā vi rājati vāk pataṁgo aśiśriyat |
prati vastorahardyubhiḥ || 3 ||

> That is, this earth revolves in space, it revolves with its mother, water, in its orbit. It moves round its father, the sun. Thirty divisions of the day (and night) are illuminated by the rays of the sun. The sun alone is the shelter and support of our speech.

The above hymns affirm that the sun divides a day or a night into thirty parts. This implies that one such part is equal to 24 min, which was later called a "naḍī." Since *Atharvaveda* was composed around the year 3000 B.C.E. (cf. [21, Sect. 2]), it seems that "naḍī" is the oldest known unit of time on this planet.

It may be mentioned here that in a more recent work, Hunger and Pingree have pointed out that this unit of time was "probably" borrowed from Babylonian sources. They, in fact, write [11, p. 46]:

> This (Babylonian) measure of time occurs again in the zigpustar texts, and is the equivalent and probably the source of the Indian division of the nychthemeron into 30 *muhūrtas*, and then into 60 *ghatikās* or *nādikās*, a unit of time measurement that first appears, with many other Mesopotamian features, in the *Jyotiṣavedāṅga* composed by Lagadha in ca.–400 (400 B.C.E.).

Also, they write (loc. cit.):

> Babylonian omens as well as astronomical knowledge was transmitted to India, beginning probably around the middle of the first millennium B.C.E.

It is necessary to emphasize here that Lagadha composed "*Vedāṅga Jyotiṣa*" around the year 1500 B.C.E. (for details see [9, p. 124] and [2]) and the term "*ghatikā*" does not appear anywhere in this text. *Ghatikā*, of course, is a colloquial word, which came into use in India after the composition of *Pancasiddhāntikā* by Varaḥmihira in the year 505 C.E. We shall discuss the measurement of *nāḍī* in subsequent sections.

Since the *Atharvaveda* was composed much earlier than the appearance of the Babylonian civilization in Mesopotamia, the statement that the Indians borrowed the unit of time in the form of *nāḍī* is untenable. Moreover, Hunger and Pingree have mentioned that the astronomical material presented by them belongs to a period later than the first half of the second millennium B.C.E.

They write [11, p. 1]:

> What this volume does attempt to cover is the astronomical material found in the tablets of both omen texts and purely astronomical texts from the earliest times – the Old Babylonian period of, probably, the first half of the second millennium B.C.E. – down to the latest – the period of the Parthian control of Mesopotamia in the late first century C.E.

The above statements clearly ensure that the unit of time *nāḍī*, which has been described in detail by *Lagadha* in *Vedāṅga Jyotiṣa*, is an indigenous Indian product and not borrowed from Babylonian sources.

This paper is divided into nine sections. Section 2 deals with the divisions of time before Varaḥmihira, and in Sect. 3 we present the partitions of time in *Bṛahatsamhita* composed by Varaḥmihira. Section 4 is devoted to the study of divisions of time in *Brahmasphuṭa-Siddhānta* composed by Brahmagupta in 628 C.E., while Sect. 5 deals with the partitions of time in the modern *Sūrya-Siddhānta*.

Section 6 deals with the measurement of time prior to Varaḥmihira, while Sect. 7 is devoted to the study of *Ambu-Yantras* (water-clocks) designed by him. In Sect. 8, we present *Ambu-Yantras* described by various astronomers on the lines of Varaḥmihira.

The last section of this paper deals with the *Śanku-Yantras* before Varaḥmihira and also designed by him, which were in common use for many centuries.

2 Divisions of Time Prior to Varaḥmihira

2.1 Measures of Time in *Vedāṅga Jyotiṣa*

Lagadha was a renowned astronomer of ancient India, who composed an astronomical text entitled *Vedāṅga Jyotiṣa* in a systematic and coherent way around the year 1500 B.C.E. (for details see [9], p. 124, [16], pp. 13–15, and [2]). He was, in fact, the first person to have developed new techniques for calculating unknown quantities from known quantities and used them in some astronomical calculations. *Vedāṅga Jyotiṣa* is extant in two recensions, namely the *Ṛgveda–Jyotiṣa* (or *Ṛgjyotiṣa)* and *Yajurveda–Jyotiṣa* (or *Yajuṣjyotiṣa*). Of course, a number of verses in these scriptures are similar. In *Ṛgjyotiṣa*, which is extant, there are 36 verses, while *Yajuṣjyotiṣa* consists of 44 verses.

Also, there is an *Atharvaveda-Jyotiṣa* consisting of 162 verses divided into 14 chapters, but this is not considered as a part of *Vedāṅga Jyotiṣa* (for details see [26, p. 468]).

Svamī Satya Prakash Sarasvatī, using the following couplet of the *ṚgJyotiṣa* (cf. [15]; 7 and [16], 8), has pointed out that Lagadha was a Kashmiri Brāhmaṇa:

धर्मवृधिरपां प्रस्थः
क्षपा ह्रास उदग्गतौ ।
दक्षिणेतौ विपर्यासः
षण्मुहूर्त्ययनेन तु ॥

dharmavṛdhirapāaṁ prasthaḥ
kṣapā hrāsa udaggatau |
dakṣiṇetau viparyāsaḥ
ṣaṇmuhūrtyayanena tu ||

> That is, during the northward course of the sun the increase of the day and the decrease of the night is one *prastha* of water, while the reverse is the case during the southward course. During the whole course the difference between the day and night is of six *muhūrtas*.

Svamī categorically writes [26, p. 471]:

> We have nowhere in the plains of India an increase of 6 *muhūrtas* in the days or in the nights. Such an observed increase is seen only in the north-western corner of this country. This very much means that Lagadha belonged to somewhere near Kashmir where he observed such a difference between the lengths of days and nights. It corresponds to the 34° 46′ or 34° 55′. These latitudes correspond to somewhere not far from Srinagar, Kashmir.

Vedāṅga Jyotiṣa, motivated from the description in *Atharvaveda* of the division of a day into thirty parts, has prescribed a practical unit of time and named it *nāḍī* as in the following verse (cf. [15, 23] and [16], p. 29):

पलानि पंचशदपां धृतानि
तदाढकं द्रोणमतः प्रमेयम् ।
त्रिभिर्विहीनं कुडवैस्तु कार्य
तन्नाडिकायास्तु भवेत प्रमाणम्॥ २४ ॥

palāni paṁcaśadapāṁ dhṛtāni
tadāḍhakaṁ droṇamataḥ prameyam |
tribhirvihīnaṁ kuḍavaistu kārya
tannāḍikāyāstu bhaveta pramāṇam ||24||

> That is, a vessel with the capacity of 50 *palas* of water is the measure of one *ādhaka* and the measure of a *droṇa* is obtained from it (which is equal to 4 *ādhakas*). Three *kudavas* deducted from a *droṇa* provides the measure of a *nāḍīkā* (time).

In other words, this verse implies that

$$\begin{aligned} 50\ \textit{palas} &= 1\ \textit{ādhaka}, \\ 4\ \textit{ādhaka} &= 1\ \textit{droṇa} = 200\ \textit{palas}, \\ 1\ \textit{nāḍīkā} &= 1\ \textit{droṇa} - 3\ \textit{kudavas}. \end{aligned}$$

Svamī Satya Prakash Sarasvatī, in order to define a *nāḍīkā*, writes [26, p. 501]:

> The ancient practice in this country was to have a *ghatikā* vessel of the capacity of $6\frac{1}{4}$ *Prastha* of water with a hole at the bottom. When floated upon water, it sank with a sound (नाद = *nāda*) as soon as it was full of water which entered the vessel. This indicated the lapse of one *nāḍīkā* or one *ghatikā* time.

In the above statement "the ancient practice" reflects that the *ghatikā* vessel was being used even before the composition of the *Vedāṅga Jyotiṣa*. Since Svamī has not given any proof for the validity of his statement, it is still an open problem. Much later, however, the *Sūrya-Siddhānta* provides a description of the *ghatikā* vessel, which will be discussed in a subsequent section.

As explained by Svamī (loc. cit.), the unit of time *nāḍīkā* is related to *droṇa* in the following way

$$\begin{aligned} 50\ \textit{palas} &= 1\ \textit{ādhaka}, \\ 4\ \textit{ādhakas} &= 1\ \textit{droṇa} = 200\ \textit{palas}, \\ 4\ \textit{prasthas} &= 1\ \textit{ādhaka} = 50\ \textit{palas}, \\ 4\ \textit{kudavas} &= 1\ \textit{prastha} = 12\frac{1}{2}\ \textit{palas}, \\ \longrightarrow \quad 1\ \textit{kudava} &= 3\frac{1}{8}\ \textit{palas} \ \text{ and } \ 3\ \textit{kudavas} = 9\frac{3}{8}\ \textit{palas}. \end{aligned} \tag{2.1}$$

Hence, according to the above verse, we see that

$$\begin{aligned} 1\ \textit{nāḍīkā} &= 1\ \textit{droṇa} - 3\ \textit{kudavas}, \\ &= 200\ \textit{palas} - 9\frac{3}{8}\ \textit{palas}, \\ &= 190\frac{5}{8}\ \textit{palas} = 190\frac{5}{8} \div 12\frac{1}{2}\ \textit{prasthas}, \\ &= 15\frac{1}{4}\ \textit{prasthas}, \\ \longrightarrow \quad 12\ \textit{nāḍīkās}, &= 183\ \textit{prasthas}. \end{aligned} \tag{2.2}$$

Since during the northward course of the sun the increase of the day is one *prastha* of water, one *ayana* is composed of 183 days.

$$\longrightarrow \quad 1 \text{ year} \quad = \quad 366 \text{ days.}$$

An analogous verse in the *Ṛgjyotiṣa* also partially affirms the above measurements [15, 16]:

नाडिके द्वे मुहूर्तस्तु
    पंचाशत्पलमाढकम् ।
आढकात्कुम्भिका द्रोणः
    कुडवैर्वर्धते त्रिभिः ॥ १७ ॥

nāḍike dve muhūrtastu
    paṁcāśatpalamāḍhakam |
āḍhakātkumbhikā droṇaḥ
    kuḍavairvardhate tribhiḥ ||17||

> That is, two *nāḍīkās* form one *muhūrta*, while one *ādhaka* consists of fifty *palas*. One *nāḍī* increased by three *kudavas* is a *kumbhaka* or *droṇa*.

The *Vedāṅga Jyotiṣa* also divides a civil (*savan*) day into *muhūrtas*, *nāḍīkās*, and *kalās* as in the following verse (cf. [15], and [16, p. 24]):

कला दश सविशां स्यात्
    द्वे मुहूर्तस्य नाडिके ।
द्युत्रिंशत तत्कालानां
    तु षट छतीत्र्यधिका भवेत् ॥ १६ ॥

kalā daśa saviśāṁ syāt
    dve muhūrtasya nāḍike |
dyutriṁśata tatkālānāṁ
    tu ṣaṭa chatītryadhikā bhavet ||16||

> That is, ten plus a twentieth *kalās* of time makes one *nāḍī*, and one *muhūrta* is equal to two *nāḍīs*. Thirty *muhūrtas* form a day (= *ahorātra*, i.e., day and night) and there are 603 *kalās* in a day.

Thus, in other words, we have

$$\begin{aligned}
1 \textit{ nāḍīkā} &= 10\frac{1}{20} \textit{ kalās}, \\
2 \textit{ nāḍīkās} &= 1 \textit{ muhūrta}, \text{ and} \\
30 \textit{ muhūrtas} &= 60 \times \frac{201}{20} \textit{ kalās} = 603 \textit{ kalās} = 1 \text{ day } (\textit{ahorātra}). \qquad (2.3)
\end{aligned}$$

The unit *kāstha* is the 124th part of a day as in the following couplet [15, *Yajurjyotiṣa* 12]:

द्युहेयं पर्व चेत्पादे
पादस्त्रिंशत्तु सैकिका।
भागात्मनाऽपवृज्यांशान्
निर्दिशेदधिको यदि॥

dyuheyaṁ parva cetpāde
pādastriṁśattu saikikā |
bhāgātmanāapavṛjyāṁśān
nirdiśedadhiko yadi ||

The first part of the above couplet implies that the first *pāda* or a quarter of a *kalā* is equal to 31 *kāsthās*, which ensures that

$$1 \textit{ kalā} = 124 \textit{ kāsthās}.$$

The relationship between *kāsthās* and *kalā* is also provided by the following verse [15, *Yajuṛjyotiṣa* 30]:

पंचत्रिंशं शतं
पौष्णम एकोनम् अयनान्यृषेः।
पर्वणां स्याच्चतुष्पादो
काष्ठानां चैव ताः कलाः॥

paṁcatriṁśaṁ śataṁ
pauṣṇam ekonam ayanānyṛṣeḥ |
parvaṇāṁ syāčatuṣpādo
kāṣṭhānāṃ caiva tāḥ kalāḥ ||

> That is, the total number of the sidereal revolutions of the sun (in a *yuga*) is 135, the *ayanas* of the moon are one less (i.e., 134). One-fourth of the number of (lunar) *parvas* (in a *yuga*) is called a *pāda* and a similar number of *kāsthās* (i.e., 124) is a *kalā*.

The above couplet, therefore, prescribes that

$$\text{Number of lunar } \textit{parvas} \text{ in a } \textit{yuga} = 124, \text{ and}$$
$$1 \textit{ kalā} = 124 \textit{ kāsthās}. \tag{2.4}$$

The relationship between a *kāsthā* and *guruvakṣaras* (letters of double *mātrās* (long syllables)) is prescribed by the following couplet (cf. [15], *Ṛgjyotiṣa* 18 and [16, p. 30]):

ससप्तकं भयुक् सोमः
सूर्यो द्यूनि त्रयोदश।
नवमानि च पन्चाह्नः
काष्ठा पन्चाक्षरा भवेत्॥

sasaptakaṁ bhayuk somaḥ
sūryo dyūni trayodaśa |
navamāni ca pañcāhnaḥ
kāṣṭhā pancākṣarā bhavet ||

That is, the moon is in possession of each *naksatra* (asterism) sixty-seven times in a *yuga*, while the sun remains in each *naksatra* for 13 days and 5/9 part of a day. A *kāsthā* is the time taken to pronounce five *akṣaras.*

The last line of the above verse ensures that the time taken to pronounce 5 *akṣaras* or 10 *mātrās* is equal to 1 *kāsthā*. (2.5)

Thus, finally, combining (2.3), (2.4), and (2.5), these ancient units of time may be arranged in the following order:

$$\begin{array}{rcl} 124\ \textit{kāsthās} & = & 1\ \textit{kalā}, \\ 10\dfrac{1}{20}\ \textit{kalās} & = & 1\ \textit{nāḍīkā}, \\ 2\ \textit{nāḍīkās} & = & 1\ \textit{muhūrta}, \\ 30\ \textit{muhūrtas} & = & 60\ \textit{nāḍīkās} \ = \ 603\ \textit{kalās} \ = \ 1\ \text{(civil) day.} \end{array}$$

The *Yajuṣjyotiṣa* prescribes that one solar year is composed of three hundred and sixty-six days as in the following verse [15, 27]:

त्रिशत्यह्नां सषटषष्टिरब्दः
षट चर्तवोऽयने ।
मासा द्वादश सौरास्त्युः
एतत्पन्च गुणं युगम् ॥ २८ ॥

triśatyahanaaṁ saṣaṭaṣaṣṭirabdaḥ
ṣaṭa cartavoạyane |
māsā dvādaśa saurāstyuḥ
etatpanca guṇaṁ yugam ||28||

That is, three hundred and sixty-six days form one year (solar), six seasons (*ṛtus*) and two *ayanas* (the northern and the southern progress of the sun). Twelve solar months form a year and five years make a *yuga.*

Hence it is clear that:

$$\begin{array}{rrcl} & 1\ \text{solar year} & = & 366\ \text{days}, \\ & & = & 12\ \text{months (solar)}, \\ & & = & 6\ \text{seasons } (\textit{rtus}), \\ & & = & 2\ \textit{ayanas}\ (\textit{Uttarayana}, \text{ and } \textit{Daksināyana}), \\ \longrightarrow & 1\ \text{solar month} & = & 30\dfrac{1}{2}\ \text{days.} \end{array}$$

It is interesting to mention here that the year consisting of 360 days and 360 nights was fairly known during the *Ṛgvedic* age (see [21], Sect. 1). Also, the concept of a *yuga* was known to early Vedic scholars. In the following hymn of the *Yajurveda*, for instance, the names of all the five years of a *yuga* are clearly given [30, Chap. XXVII, p. 45]:

संवत्सरो ऽसि परिवत्सरो ऽसि
इदावत्सरोऽसि इदवत्सरोऽसि वत्सरोऽसि ।

saṁvatsaro ạsi parivatsaro ạsi
idāvatsaroạsi idavatsaroạsi vatsaroạsi |

Griffith writes (loc. cit., p. 388):

> *Saṁvatsara* and the rest (i.e., *Parivatsura*, *Idāvatsara*, *Idvatsara* and *Vatsara*) are the names given to the years of the five-year cycle intended, with the aid of an intercalary month, to adjust the difference between the lunar and the solar year.

During the time of the composition of the *Vedāṅga Jyotiṣa* a *yuga* consisting of five years was fully recognized and the names of these constituent years were the same as described in the *Yajurveda*.

2.2 The Concept of Moment (*Kṣaṇa*)

The concept of the minimal unit of time appears in the *Yoga-Sutras* of *Patañjali* in connection with the attainment of knowledge by "Saṁyama." *Patañjali* ordains in aphoristic form (cf. [23], Book III, Hymn 52, p. 335):

क्षण तत्क्रमयोः संयमाद्विवेकजं ज्ञानम् ।

kṣaṇa tatkramayoḥ saṁyamādvivekajaṁ jñānam |

Thus, according to *Patañjali*, knowledge is acquired from *samyama* on a moment and its sequence.

Vyāsa, writing a commentary on the above maxim, provides two definitions of a kṣaṇa (moment). He writes [loc. cit., p. 335]:

यथापकर्षपर्यनां द्रव्यं परमणुरेवं
परमापकर्ष पर्यनाः कालः क्षणः ।
यावता वा समयेन चलितः परमाणुः
पूर्व देशं जह्यादुत्तर देशमुपसम्पद्येत ;
सकालः क्षणः ॥

yathāpakarṣaparyantaṁ dravyaṁ paramaṇurevaṁ
paramāpakarṣa paryantaḥ kālaḥ kṣaṇaḥ |
yāvatā vā samayena calitaḥ paramāṇuḥ
pūrva deśaṁ jahyāduttara deśamupasampadyet;
sakālaḥ kṣaṇaḥ ||

As mentioned above, Vyāsa has given two definitions of a *kṣaṇa* (moment). According to the second definition, moment is the time taken by an atom in motion on leaving one point in space and reaching the adjacent point. The time interval of this displacement is known as a moment and the continuous flow of moments is its sequence.

In the first definition, Vyāsa asserts that since the minimal object (or smallest particle of cognition) is an atom, as such a moment is the minimal unit of time. This implies that a moment is the smallest part of time in which the minutest mutation is recognized by a *yogī*.

2.3 Reckoning of Time in the *Arthaśāstra*

It is said that Kautilya (or Cāṇakya), the renowned author of the *Arthaśāstra*, was born at a village, Cāṇaka (at present known as Canakā) situated between Pātaliputra (Patna) and Gayā. He was the prime minister and mentor of Emperor Candragupta Maurya. He composed the *Arthaśāstra* around the year 317 B.C.E. In Book II, Chap. 20, he has described various divisions of time in the form of thirty-nine Sutras (from 28 to 66) (cf. [12, pp. 423–429] and [13, pp. 139–141]). In Sutras 28 and 29 he writes:

कालमानमत ऊर्ध्वम् । २८ ।
तुटो लवो निमिषः काष्ठा कला नालिका
मुहूर्त्तः पूर्वापर भागौ दिवसो रात्रिः
पक्षो मास ऋतुरययं संवत्सरो
युगमिति कालाः । २९ ।

kālamānamat-ūrdhvam |28|
tuṭo lavo nimiṣaḥ kāṣṭhā kalā nālikā
muhūrttaḥ pūrvāpar bhāgau divaso rātriḥ
pakṣo maas ṛturayayaṁ saṁvatsaro
yugamiti kālāḥ |29|

That is, hereafter measurement of time is explained. These are *tuta*, *lava*, *nimesa*, *kāsthā*, *kalā*, *nalika*, *muhurta*, forenoon, afternoon, day, night, fortnight, month, season, *ayana*, year, and *yuga* (cycle of years).

Kautilya defines these divisions in the following form (loc. cit., pp. 424–425):

द्वौ तुटौ लवः ३०

dvau tuṭau lavaḥ

i.e., 2 *tutas* = 1 *lava.*

द्वौ लवौ निमेषः ३१

dvau lavau nimeṣaḥ

i.e., 2 *lavas* = 1 *nimesa.*

पंच निमेषाः काष्ठा ३२

paṁca nimeṣāḥ kāṣṭhā

i.e., 5 *nimesās* = 1 *kāṣṭhā.*

त्रिंशत्काष्ठाः कला ३३

triṁśatkāṣṭhāḥ kalā

i.e., 30 *kāṣṭhā* = 1 *kalā.*

चत्वारिंशत्कलाः नालिका ३४

catvāriṁśatkalāḥ nālikā

i.e., 40 *kalās* = 1 *nālikā.*

द्विनालिको मुहूर्त्तः ३६

dvināliko muhūrttaḥ

i.e., 2 *nālikās* = 1 *muhūrttaḥ.*

पंचदश मुहूर्तो दिवसो रात्रिश्च
चैत्रे चाश्वयुजे च मासि भवतः। ३७

paṁcadaśa muhūrto divaso rātriśca
caitre cāśvayuje ca māsi bhavataḥ |

That is, a day and night of fifteen *muhūrtas* occur in the months of Caitra and Aśvayuja (Aśvin).

ततः परं त्रिभिमुहूर्तैरन्यतरः
षण्मासं वर्धते ह्रसते चेति। ३८

tataḥ paraṁ tribhimuhūrtairanyataraḥ
ṣaṇmāsaṁ vardhate hṛsate ceti |

That is, after that, one of them (the day) increases by three *muhūrtas* for six months and then decreases in the same way, and vice versa the other (night).

पंचदशाहोरात्राः पक्षः ४३

paṁcadaśāhorātraaḥ pakṣaḥ

That is, fifteen days and nights make a fortnight.

द्विपक्षो मासः ४६

dvipakṣo māsaḥ

That is, two fortnights make a month.

Although Kautilya describes seven types of months, only the following three are associated with the calendar system:

त्रिंशदहोरात्रः कर्म मासैः ४७

triṁśadahorātraḥ karma māsaiḥ

That is, thirty days and nights make a work (*savan*) month.

सार्धः सौरः ४८

sārdhaḥ sauraḥ

That is, a half day more (than a work month) makes a solar month.

अर्धन्यूनश्चान्द्र मासः ४९

ardhanyuunaścāndra māsaḥ

That is, a half day less (than a work month) makes a lunar month.

As in the *Vedāṅga Jyotiṣa*, during the time of Kautilya too, a *yuga* was composed of five years. He writes (loc. cit., 63 and 64):

द्वअयनः संवत्सरः ।
पंचसंवतरो युगम्॥

dvayanaḥ saṁvatsaraḥ |
paṁcasaṁvataro yugam ||

That is, two *ayans* form a year and five years make a cycle (*yuga*).

Kautilya, discarding the *yuga* system of the *Manusmṛti* and the *Puraṇas*, has followed the pattern of *Vedāṅga Jyotiṣa*. The modern *Sūrya-Siddhānta*, on the other hand, has adopted the Paurānic *yuga* system, which will be discussed in a subsequent section.

2.4 Divisions of Time in *Āryabhaṭīya*

It is beyond discussion that Āryabhaṭa was one of the greatest mathematicians and astronomers of ancient India. According to the first stanza of his famous work *Gaṇitapādaḥ*, he received knowledge of mathematics and astronomy at Kusumapura. He writes [3, p. 45]:

ब्रह्म - कु - शशि - बुध - भृगु - रवि -
कुज - गुरु - कोण - भगणान् नमस्कृत्य ।
आर्यभटस्तिवह निगदित
कुसुम पुरे ऽभ्यर्चितं ज्ञानम्॥ १ ॥

brahama-ku-śaśi-budha-bhṛgu-ravi-
kuja-guru-koṇa-bhagaṇān namaskṛtya |
āryabhaṭastivah nigadita
kusum pure ạbhyarcitaṁ jñānam ||1||

> That is, after doing obeisance to Brahma, the Earth, the Moon, Mercury, Venus, the Sun, Mars, Jupiter, Saturn and to the constellations, Āryabhaṭa sets forth the venerable knowledge (of astronomy) at Kusumapura.

The above verse clearly affirms that Āryabhaṭa composed *Āryabhaṭīya* at Kusumapura. Almost all commentators of *Āryabhaṭīya* have accepted Kusumapura as the workplace of Āryabhaṭīya. Smith [25, p. 156] has mentioned that Kusumapura cannot be identified with Patna (Pataliputra), but it is a place not far from modern Patna. Of course, a number of mathematicians of Bihar firmly assert that the present-day Phulwari (Garden of Flowers) Sharīf is the place "Kusumapura" described in *Āryabhaṭīya*, which is about four kilometres from Patna. Since the meaning of Kusumapura is synonymous with Phulwari, the above assertion seems to be appropriate. For other details see [14, pp. 14–17].

There is no dispute about the year of his birth. He writes in "Kālakriyāpadaḥ" of *Āryabhaṭīya* (cf. [3], pp. 201–202):

षष्ट्यब्दानां षष्टिर्यदा
व्यतीतास्त्रयश्च युगपादाः ।
त्र्यधिका विंशतिरब्दास -
तदेह मम जन्मनो ऽ तीताः॥ १० ॥

ṣaṣṭyabdānāṁ ṣaṣṭiryadā
vyatītāstrayaśca yugapādāḥ |
tryadhikā viṁśatirabdāsa -
tadeh mam janmano ạ tītāḥ ||10||

The above verse states that at the time of the composition of *Āryabhaṭīya*, sixty *yugas* of sixty years and three-quarter yugas had elapsed and at that time Āryabhaṭa was 23 years of age. This implies that at that time Kṛtayuga, Tretā, and Dvāpara had elapsed and 3600 years of Kaliyuga also had passed, and Āryabhaṭa was 23 years old.

Since, according to the Hindu calendar system, Kaliyuga commenced from February 18, 3102 B.C.E., Āryabhaṭa composed *Āryabhaṭīya* in the year (3600–3101) = 499 C.E. and he was born in 476 C.E.

Āryabhaṭa rejected the highly artificial scheme of time-division prevailing at that time and replaced it with the following:

$$\begin{aligned} \text{1 day of Brahma or Kalpa} &= 14\ \textit{manus}, \\ 1\ \textit{manu} &= 72\ \textit{yugas}, \\ 1\ \textit{yuga} &= 43,20,000\ \text{years}. \end{aligned}$$

In one section of *Āryabhaṭīya* entitled:

कालक्रिया पादः

kālakriyā pādaḥ

That is, In "Reckoning of Time," he provides the divisions of time in detail in almost the same way as in earlier works. In the first couplet of this section, he writes [3. p. 172]:

वर्ष द्वादश मासास्-
त्रिंशद दिवसो भवेत् स मासस्तु ।
षष्टिर्नाड्यो दिवसः
पष्टिश्च विनाडिका नाडी॥

varṣa dvādaśa māsās -
trimśad divaso bhavet sa māsastu |
ṣaṣṭirnāḍyo divasaḥ
ṣaṣṭiśca vināḍikā nāḍī ||

That is, one year consists of twelve months, and thirty days form a month. A day (*ahorātra*) is composed of sixty *nāḍīs*, and sixty *vināḍikās* make up a *nāḍī*.

Aryabhata further writes (loc. cit., p. 173):

गुर्वक्षराणि षष्टिर्
विनाडिकार्क्षी षडेव वा प्राणाः ।

gurvakṣarāṇi ṣaṣṭir
vināḍikārkṣii ṣaḍeva vā prāṇāḥ |

That is, sixty *guruvaksaras* form a *vinādikā*, or the time taken for six respirations (*prāna*) make a *vinādi* (or *vinādikā*).

Following the traditions of the *Manusmṛti* and the Purānas, Āryabhaṭa has given the length of time equal to one *yuga* of Brahmā. In the following verse he writes [3, p. 196]:

रविवर्षं मानुष्यं तदपि
त्रिंशद्गुणं भवति पित्र्यम्।
पित्र्यं द्वादशं गुणितं
दिव्यं वर्षं विनिर्दिष्टम्॥ ७॥

ravivarṣaṁ mānuṣyaṁ tadapi
triṁśadguṇaṁ bhavati pitryam |
pitryaṁ dvādaśaṃ guṇitaṁ
divyaṁ varṣa vinirdiṣṭam ||7||

That is, the solar year is a human year, and thirty human years form one *pitṛ* year. Twelve *pitṛ* years make up one *divya* or divine year.

He further writes (loc. cit., p. 197):

दिव्यं वर्षं सहस्त्रं ग्रह सामान्यं
युगं द्विषट्क गुणम्।
अष्टोत्तरं सहस्त्रं ब्राह्मो
दिवसो ग्रहयुगानाम्॥ ८॥

divyaṁ varṣa sahastraṁ graha sāmānyaṁ
yugaṁ dviṣaṭka guṇam |
aṣtottaraṁ sahastraṁ brāhmo
divaso grahayugānām ||8||

That is, 12 × 1000 *divya varas* form a *yuga*, and 1008 *yugas* constitute a day of Brahmā.

In other words, we have the following divisions of a day of Brahmā:

$$\begin{aligned} 30 \text{ (solar) years} &= 1\ \textit{pitṛ} \text{ year},\\ 12\ \textit{pitṛ} \text{ years} &= 1\ \textit{divya} \text{ year},\\ 12,000\ \textit{divya} \text{ years} &= 1\ \textit{yuga},\\ \longrightarrow \quad 1\ \textit{yuga} &= 4,320,000 \text{ years}. \end{aligned}$$

As in the Purāṇas, these 4,320,000 years of a *yuga* are shared by Kṛta, Tretā, Dvāpara and Kāli yugas in the ratios 4:3:2:1, respectively. It seems that for practical purposes Āryabhaṭa considers a *yuga* of 60 years.

About the beginning of a day, Āryabhaṭa has propounded two postulates, which have been criticized by Varaḥmihira in the following verse of the *Pancasiddhāntikā* (cf. [28, XV, 20] and [29, pp. 420–421]):

लङ्कार्ध रात्र समये
दिन प्रवृत्तिं जगाद चार्यभटः ।
भूयः स एव सूर्यो
दयात्प्रभृत्याः लङ्कायाम् ॥ २० ॥

laṅkārdha rātra samaye
dina pravṛtiṁ jagāda cāryabhaṭaḥ |
bhūyaḥ sa eva sūryo
dayātprabhṛtyāḥ laṅkāyām ||20||

> That is, Āryabhaṭa has stated that the day begins from midnight at Lankā, and again he says that the day begins from sunrise at Lankā.

He further writes (loc. cit., XV, 25):

लङ्कार्धरात्र समया दन्यत् सूर्योदयाच्चैव ।

laṅkārdharātra samayā danyat sūryodayāčaiva |

> That is, counting time from midnight at Lankā is different from the reckoning from sunrise.

Āryabhaṭa has given the divisions of time on the pattern of the Purānic system. Although he has not described in the *Āryabhaṭīya* the instruments to measure them, Ôhashi has mentioned that in an other work of his entitled *Āryabhaṭa Siddhānta*, which is no longer extant, he has given the descriptions of a number of astronomical instruments, including Śanku-yantra and a water instrument to measure time. For details see [18].

3 Divisions of Time in the *Bṛhatsaṁhitā*

Varaḥmihira composed the *Bṛhatsaṁhitā* around the year 550 C.E. It was his last work, covering almost all aspects of human life in beautiful poetic forms. Commenting on his poetic art in the *Bṛhatsaṁhitā*, Bhat writes [30, p. XV]:

> When we go through this work, we are reminded of Homeric similes and the linguistic elegance and charm of Vālmīki, Vyāsa, Bhāsa and Kālidāsa.

Chapter II of *Bṛhatsaṁhitā*, entitled "सांवत्सरसूत्राध्यायः" (*sāṁvatsaras-ūtrādhyāyaḥ*), presents a brief description of the qualifications of an astrologer. In the following paragraph it describes the various divisions of time that must be fully known to an astrologer [30. p. 8]:

तत्र ग्रह गणिते पौलिश रोमक वासिष्ठ
सौर पैतामहेषु पंचस्वेतेषु सिद्धान्तेषु
युग वर्षायनर्तुमास पक्षाहोरात्रयाम
मुर्हुत नाडी प्राण त्रुटि त्रुट्याद्यवयवादिकस्य
कालस्य क्षेत्रस्य च वेत्ता॥ ४ ॥

tatra graha gaṇite pauliśa romaka vāsiṣṭha
saur paitāmaheṣu paṁcasveteṣu siddhānteṣu
yuga varṣāyanartumāsa pakṣāhorātrayāma
murhuta nāḍī prāṇa truṭi truṭyā dyavayavādikasya
kālasya kṣetrasya ca vettā ||4||

That is, an astronomer must have studied the works of Pauliśa, Romaka, Vāsiṣtha, Saur, and Paitāmaha. He must be well-versed in the various subdivisions of time such as *yuga* (=43,20,000 solar years), year, solstice, *rtu* (consisting of two solar months), month, fortnight, ahorātra (a solar day), *yāma* (one-eighth of a solar day = 3 hours), *muhūrta*, *nāḍī*, *prāṇa*, *truṭi*, and other divisions of time.

Bhat (loc. cit.) has given the following table for the relationships between various measures of time:

Time taken to pronounce one syllable is a Nimeśa,

$$\begin{aligned}
2 \textit{ Nimeśas} &= 1 \textit{ Truṭi},\\
2 \textit{ Truṭis} &= 2 \textit{ Lava},\\
2 \textit{ Lavas} &= 1 \textit{ Kṣāna},\\
10 \textit{ Kṣānas} &= 1 \textit{ Kāsthā},\\
10 \textit{ Kāsthās} &= 1 \textit{ Kalā},\\
10 \textit{ Kalās} &= 1 \textit{ Nādikā},\\
60 \text{ Long syllables} &= 1 \textit{ Vinādi},
\end{aligned} \tag{3.1}$$

or

6 *prānas* (Breaths = one inspiration and one expiration) = 1 *vinādi.*

The above statements of Varaḥmihira imply that in those days, knowledge of various divisions of time was compulsory for all astronomers and astrologers.

4 Partitions of Time in the *Brahmasphuṭa Siddhānta*

Brahmagupta was one of the prominent mathematicians of "Varaḥmihira Gurukula." He composed the Brahmasphuṭa Siddhānta at the age of thirty. He writes (cf. [6], XXIV, pp. 7–8):

श्री चापवंशतिलके श्रीव्याघ्र मुखे नृपे शकनृपाणाम्।
पंचाशत् संयुक्तैर्वर्षशतैः पंचभिरतीतैः॥ ७॥
ब्रहम् स्फुट सिद्धान्तः सज्जन् गणितज्ञ गोल वित् प्रीत्यै।
त्रिंशद् वर्षेण कृतो जिष्णुसुत ब्रह्म गुप्ते न॥ ८॥

śrī cāpavaṁśatilake śrīvyā-ghra mukhe nṛpe śakanṛpāṇām |
paṁcāśat saṁyuktairvarṣaśataiḥ paṁcabhiratītaiḥ ||7||
braham sphuṭa siddhaanth sajjan gaṇitajña gol vit prītyai |
triṁśad varṣeṇa kṛto jiṣṇusuta brahm gupte n ||8||

> That is, during the rule of Vyāghramukha, a great king of the Cāpa clan, when 550 years of the Śaka era had passed (i.e., 628 C.E.), Brahmagupta, son of Jisnu, at the age of thirty, composed *Brahmasphuṭa Siddhānta* for the benefit of good mathematicians and astronomers.

For the life history of Brahmagupta see [26], Chap. III and [7], Chap. I. About the partitions of time, Brahmagupta writes [6; I, 5]:

प्राणैर्विनाडिकार्क्षी षड्भिर्घटिका षष्ट्या।
घटिका षष्ट्या दिवसो दिवसानां त्रिंशता मासाः॥

prāṇairvināḍikārkṣī ṣaḍbhirghaṭikā ṣaṣṭyā |
ghaṭikā ṣaṣṭyā divaso divasānāṁ triṁśatā māsāḥ ||

The above verse provides the following units of time:

$$\begin{aligned} 6\ \textit{Prānas} &= 1\ \textit{Ṛksa-vināḍīkā}\ (\text{or } 1\ \textit{pala})\ (= 24 \text{ seconds}),\\ 60\ \textit{Vināḍīkās} &= 1\ \textit{ghatikā}\ (\text{or } \textit{Nāḍī}), \text{and}\\ 60\ \textit{Ghatikās} &= 1\ \text{Day}\ (\textit{ahorātra}). \end{aligned}$$

The above divisions of time are similar to those pointed out by Varaḥmihira in the *Bṛhatsamhita*.

5 Reckoning of Time in the Modern *Sūrya-Siddhānta*

It is well-known that the extant *Sūrya-Siddhānta* at present is somewhat different from that presented by Varaḥmihira in his *Pañcasīddhāntikā*.

Although the *Sūrya-Siddhānta* known to Varaḥmihira resembles its present form in fundamental features, the two differ significantly at a number of points (for details see [28], pp. xii–xvi). It is also speculated that Varaḥmihira too introduced some changes in the old *Sūrya-Siddhānta*, but as pointed out by Thibaut and Dvivedi, these changes were made for the sake of convenience in calculations. Thibaut and Dvivedi write [28, p. xv]:

> The investigation of special cases thus certainly favours the conclusion that the changes which the old *Sūrya-Siddhānta* has undergone in Varaḥamihir's representation are purely formal, and that convenience of calculation is held by him to be a consideration of altogether secondary importance.

Shukla [27, pp. 15–29] has discussed in detail a number of differences between the results obtained by Varaḥamihir's *Sūrya-Siddhānta* and its present form.

The various units of time are described in the first chapter of the modern *Sūrya-Siddhānta*. The following verse provides the basic concept of time (cf. [27], Chap. I, 10; p. 2 and [8], p. 5):

भूतानामन्तकृत्कालः
        कालो ऽन्यः कलनात्मकः ।
स द्विधा स्थूल सूक्ष्मत्वान्
        मूर्तश्चामूर्त एव च ॥

bhūtānāmantakṛtkālaḥ
        kālo ạnyaḥ kalanātmakaḥ |
sa dvidhā sthūla sūkṣmatvān
        mūrtaścāmūrta eva ca ||

> That is, time is the destroyer of the world, and another time makes it move. This latter (time), depending on whether it is gross or minute, is known as real (*mūrta*) or unreal respectively.

In the next stanza we find the difference between real and unreal time and its basic division into units [27, Chaps. I, II]:

प्राणदिः कथितो मूर्तः
        त्र्युट्याद्यो ऽमूर्त संज्ञकः ।
पडभिः प्राणैः विनाडी
        स्यात्तत्षष्ट्या नाडिका स्मृता ॥

prāṇadiḥ kathito mūrtaḥ
        tryuṭyādyo ạmūrta saṁjñakaḥ |
ṣaḍabhiḥ prāṇaiḥ vināḍii
        syāttatṣaṣṭyā nāḍikā smṛtā ||

> That is, (the time) that begins with respirations (*prāna*) is real and that which begins with atoms is called unreal or unembodied. Six respirations (*prāna*) make a *vināḍī* and sixty of these a *nāḍī*.

Thus, according to modern the *Sūrya-Siddhānta*, we have the following table of the divisions of sidereal time:

$$\begin{aligned} 6 \text{ respirations} &= 1\ \textit{vināḍī}, \\ 60\ \textit{vināḍīs} &= 1 \text{ nadī}. \end{aligned}$$

Referring to the Purānic divisions of a day, Burgess [8, p. 6], in his commentary on the *Sūrya-Siddhānta*, has given the following divisions of a day (*ahorātra*):

$$\begin{aligned} 15 \text{ twinklings } (\textit{nimesha}) &= 1 \text{ bit } (\textit{kāshthā}), \\ 30 \text{ bits} &= 1 \text{ minute } (\textit{kalā}), \\ 30 \text{ minutes} &= 1 \text{ hour } (\textit{muhūrta}), \\ 30 \text{ hour} &= 1 \text{ day}. \end{aligned}$$

Here, of course, 1 hour = 1 *muhūrta* = 48 minutes of modern time, which entails that 1 day = 1,440 minutes.

Shukla [27, p. 2] has pointed out that the following verse on divisions of time is given in the form of a commentary on the *Sūrya-Siddhānta* edited by Rāmakṛṣṇa Ārādhya (1472 C.E.):

तीक्ष्ण सूच्याब्ज पुष्पस्य
दल बेधस्त्रुटिर्भवते।
तच्छतं लव इत्युक्तं
तत्त्रिंशस्तु निमेषकः॥
निमेषैसप्त विंशत्या
कालो गुर्वक्षरस्तु सः।
दश गुर्वक्षरोच्चार कालः
प्राणो ऽभिधीयते॥

tiikṣṇa sūcyābja puṣpasya
dala bedhastruṭirbhavate |
tacchataṁ lava ityuktaṁ
tattriṁśastu nimeṣakaḥ ||
nimeṣaisapta viṁśatyā
kālo gurvakṣarastu saḥ |
daśa gurvakṣaroccāra kālaḥ
prāṇo ạbhidhīyate ||

That is, the time taken by a sharp needle in piercing a lotus petal is called a "*truti*," hundred *truties* form a "*lava*," and thirty *lavas* are equal to a "*nimeṣa*." Twenty-seven *nimeṣas* are equal to the time taken in pronouncing a "*gurvakṣara*" (long syllable), and then *guruvakṣaras* form the time equivalent to one "*prāṇa*."

In other words, the table of the above divisions may be expressed in the following form:

1 *truti*	=	Time taken by a sharp needle to pierce a lotus petal,
100 *truties*	=	1 *lava*, 30 *lavas* = 1 *nimeṣa* (twinkling),
27 *nimeṣas*	=	1 *guruvakṣara*, and
10 *guruvakṣaras*	=	1 *prāṇa* (respiration).

The modern *Sūrya-Siddhānta* provides the division of a month in the following couplet [27, I, 12]:

नाडी षष्ट्या तु नाक्षत्रम्
अहोरात्रं प्रकीर्तितम् ।
तत्त्रिंशता भवेन्मासः
सावनो ऽ र्कोदयैः स्मृतः ॥

nāḍī ṣaṣṭyā tu nākṣatram
ahorātraṁ prakīrtitam |
tatriṁśatā bhavenmāsaḥ
sāvano ạ rkodayaiḥ smṛtaḥ ||

That is, sixty *nāḍīs* make a (sidereal) *ahorātra* (day + night). A month is composed of thirty such sidereal days and has as many (thirty) sunrises.

Thus, we have

60 *nāḍīs*	=	1 day (civil),
30 days	=	1 month,

and the sun rises thirty times in a month (civil).

In the stanza given below, the *Sūrya-Siddhānta* makes a distinction between a lunar month and a solar month [27, I, 13]:

ऐन्दवस्तिथिभिः तद्वत्संक्रान्त्या सौर उच्यते ।
मासैर्द्वादशभिर्वर्षं दिव्यं तदह उच्यते ॥

aindavastithibhiḥ tadvatsaṁkrāntyaa saura ucyate |
māsairdvādaśabhirvaṣaṁ divyaṁ tadaha ucyate ||

That is, a lunar month is composed of as many (thirty) lunar days (*tithis*), and a solar month is ascertained by the entrance of the sun into a sign of the zodiac, while a year is made of twelve months.

The above divisions of a year were in common use during the time of the *Vedāṅga Jyotiṣa*, which describes them in much detail, as given in Sect. 2. This clearly demonstrates that the *Vedāṅga Jyotiṣa* provided a foundation for the complex structure of the *Sūrya-Siddhānta* (ancient and modern both). In the *Sūrya-Siddhānta* too, as in the *Vedāṅga Jyoyiṣa*, there are the following three types of days:

1. Sāvan or civil day, from sunrise to the next sunrise as described in the verse [27, I, 36]

   उदयादुदयं भानोः भूमि सावन वासराः ।

   udayādudayaṁ bhānoḥ bhuumi sāvana vāsarāḥ |

   That is, the terrestrial civil days are counted from one sunrise to the next sunrise.
2. Sidereal day = 60 *nāḍīs* = the time for one rotation of the earth on its axis.

   The *Sūrya-Siddhānta* provides a method to calculate the length of a sidereal day (cf. [8, pp. 28–29]).
3. Lunar day = time between one new moon and the next divided by thirty as mentioned above.

It may be recalled here again that during the age of the *Vedas* and the *Vedāṅga Jyotiṣa* a *yuga* was composed of five *samvatsaras* (years), but, following the Purānic traditions, the *Sūrya-Siddhānta* has developed the concept of Caturyuga (Quadruple Age) consisting of 4,320,000 solar years. This enormous length of time is divided into four yugas, namely Kṛta yuga (Golden Age), Tretā yuga (Silver Age), Dvāpara yuga (Bronze Age), and Kali yuga (Iron Age) in the ratio of 4:3:2:1 respectively. The present age of Caturyuga is Kali yuga with its total length of 4,320,00 solar years, which, according to Hindu reckoning, began in the year 3102 B.C.E.

According to *Sūrya-Siddhānta*, 71 Caturyuga plus at its end a Sandhi Kāla (twilight time) is, equivalent to a Kṛta yuga "*manvantara*," after which there is a deluge (great flood) as described in the following stanza [27, I, 18]:

युगानां सप्ततिस्सैका
मन्वन्तरमिहोच्यते ।
कृताब्द संख्या तस्यान्ते
सन्धिः प्रोक्तो जलप्लवः ॥

yugānaaṁ saptatissaikā
manvantaramihocyate |
kṛtābda saṁkhyā tasyānte
sandhiḥ prokto jalaplavaḥ ||

> That is, seventy-one *yugas* form one *manvantara* (patriarchate), and at its end is a *sandhikāla* (twilight), equivalent to a Kṛtayuga, consisting of a deluge (great flood).

The next verse provides the length of time to form a *kalpa* [27, I, 19]:

ससन्धयस्ते मनवः
कल्पे ज्ञेयाः चतुर्दश।
कृत प्रमाणः कल्पादौ
सन्धिः पन्चदश स्मृताः॥

sasandhayaste manavaḥ
kalpe jñeyāḥ caturdaśa|
kṛta pramāṇaḥ kalpādau
sandhiḥ pancadaśa smṛtāḥ ||

> That is, a *kalpa* consists of fourteen *manavantaras* with their respective twilights and at the beginning of the *kalpa* there is a fifteenth dawn equal to the length of a *Krtayuga* (Golden Age).

It is interesting to mention here that Burgess [8, p. 11] considers a *kalpa* equal to an eon in English, which is not appropriate in this situation, because *kalpa* is properly measured in solar years, while an eon is a long period of time that cannot be measured.

The *Sūrya-Siddhānta* further describes the length of time that constitutes a day of Brahma, the creator, as in the following couplet [27, I, 20]:

इत्यं युग सहस्रेण
भूत संहार कारकः।
कल्पो ब्राह्ममहः प्रोक्तं
शर्वरी तस्य तावती॥

ityaṁ yuga sahasreṇa
bhūta saṁhāra kārakaḥ |
kalpo brāhmamahaḥ proktaṁ
śarvarī tasya tāvatī ||

> That is: one thousand *kalpas* form a day of Brahmā, after which all of creation is destroyed. A night of Brahmā is also of the same length of time.

About the longevity of Brahmā, the *Sūrya-Siddhānta* provides the following details (cf. [27, I, 21] and [8, p. 12]):

परमायुश्शतं तस्य
तया ऽहोरात्र संख्यया।

आयुषोऽर्धं मितं तस्य
शेषात्कल्पो ऽयमा दिमः॥

paramāyuśśataṁ tasya
tayā ạhorātra saṁkhyayā |
āyuṣoạrdha mitaṁ tasya
śeṣātkalpo ạyamā dimaḥ ||

> That is, his (Brahmā's) extreme age is a hundred years (i.e. 360 days and 360 nights of Brahmā). One half of his life has elapsed and of the remainder this is the first *kalpa.*

Burgess (loc. cit.) has calculated the length of Brahmā's life as 311,040,000,000,000 solar years. Bewildered by the reckoning of these long periods of time, Burgess observes [8, p. 11]:

> Vast as this period is, however, it is far from satisfying the Hindu craving after infinity.

For other details about the reckoning of time in the modern *Sūrya-Siddhānta* see [8], Chap. I.

Although the *Vedāṅga Jyotiṣa*, *Arthaśāstra*, *Āryabhaṭīya*, *Bṛhatsamhita*, and the modern *Sūrya-Siddhānta* provide very minute divisions and subdivisions of time, they do not provide precise methods for their measurement. In this respect, Burgess has observed [8, pp. 6–7]:

> These minute subdivisions are... curiously illustrative of a fundamental trait of Hindu character: a fantastic imaginativeness, which delights itself with arbitrary theorizings, and is unrestrained by and careless of, actual realities. Thus, having no instruments by which they could measure even seconds with any tolerable precision, they vied with one another in dividing the second down to the farthest conceivable limit of minuteness.

Burgess, of course, is only partially correct, because even in those early days Hindus had devised a number of useful instruments to measure the time for practical purposes, which will be described in subsequent sections.

6 Measurement of Time Prior to Varaḥmihira

The earliest use of water in measuring time is described in the *Vedāṅga Jyotiṣa*, composed by Lagadha around the year 1500 B.C.E. Although in implicit form, the Vedāṅga Jyotiṣa associates the daily increase of daytime when the sun moves from the winter to the summer solstice with (the flow of) one *prastha* of water (from a jar), as in the following verse (cf. [7, 15], and [10], pp. 216–217 for details):

घर्मवृद्धिरपां प्रस्थः
  क्षपाह्रास उदग्गतौ ।
दक्षिणे तौ विपर्यासः
  षण्मुहूर्त्ययनेनतु ॥

gharmavṛddhirapāṃ prasthaḥ
  kṣapāhrāsa udaggatau |
dakṣiṇe tau viparyāsaḥ
  ṣaṇmuhūrtyayanenatu ||

> That is, during the northward course of the sun the daily increase in the daytime or decrease in the nighttime is equal to a *prastha* of water, while during the southward course the opposite is the case. The total increase or decrease during each such course is equal to six *muhūrtas.*

Since each solstice period is composed of 183 days, it implies that

$$6 \textit{ muhūrtas } = 183 \textit{ prastha} \text{ of water } = 12 \text{ nāḍīkās.}$$

Of course, the *Vedāṅga Jyotiṣa* does not prescribe any mechanism to measure one *nāḍīkā* of time with the flow of $15\frac{1}{4}$ *prastha* of water from a jar. It appears, as pointed out by Fleet (loc. cit., p. 217), that the process was "too familiar to be mentioned."

As described in Sect. 2(c), Kautilya has given minute divisions and subdivisions of time. But for practical purposes he describes a process for the measurement of a *nāḍīkā* (cf. [12, p. 424] and [13, p. 139]):

चत्वा रिंशत्कला कलाः नालिका
  सुवर्णमाषकाश्चत्वारश्चतुरङ्गुलायामाः ।
कुम्भच्छिद्रमाढकम्-ऽ म्भसो वा नालिका ॥

catvā riṁśatkalāa kalāaḥ nālikā
  suvarṇamāṣakāścatvāraścaturaḍgul āyāmāḥ |
kumbhacchidramāḍhakam-ạ mbhaso vā naalikaa ||

> That is, forty *kalās* form a *nāḍīkā*, which is determined by the flow of one *ādhak* of water from a jar (*kumbha*) through an aperture (at the bottom) made by a wire of four *māśas* of gold and four *angulas* (3 inches) in length.

The above measurement of time implies that

$$\begin{aligned} 1 \textit{ nāḍīkā} &= 1 \textit{ ādhaka} \\ &= 40 \textit{ kalās} \\ &= 1/4 \textit{ droṇa of water.} \end{aligned}$$

Fleet, referring to a Buddhist work of the first century C.E. entitled *Divyāvadān,* has inferred that one *nāḍīkā* of time was considered equal to the flow of one *droṇa* of water from an aperture pierced by a wire of 4 *angulas* length and made of one *suvarna* of gold. As pointed out by Fleet, this difference between the measurement of a *nāḍīkā* in the *Arthaśāstra* and the Buddhist text is due to the size of the apertures for the flow of water. Although the lengths of the wires in both cases are the same, i.e., 4 *angulas*, the weight of gold prescribed in the *Arthaśāstra* is one-fourth that of the *Divyāvadān*, because (cf. [10, p. 222]):

1 *suvarṇa* = 16 *māsakas* (in weight).

Fleet, referring to the *Vāyu Purāna*, has observed that according to the contemporary Māgadha measure, 1 *nāḍī* is equal to the time taken for the flow of one prastha of water through an aperture of the same size as prescribed in the *Arthaśāstra* (cf. [10, p. 221]). He adds:

> In any case, since 4 *prastha* = 1 *ādhaka*, this description gives a water-clock of the same kind and size with that of Kautilīya. Thus, from the Purānas also, we have
> 1 *nāḍīkā* = 1 *ādhaka* = 1/4 *droṇa* of water.

But, of course, the divisions of a *nāḍī* into *kalās* are quite different in the *Arthaśāstra* and the *Divyāvadān.* According to the *Arthaśāstra*:

चत्वारिंशत् कलाः नाडीका।

catvaariṁśat kalāaḥ naaḍīkā |

That is, 1 *nāḍīkā* = 40 *kalās.*

The *Divyāvadān*, on the other hand, prescribes [10, p. 218]

कलानाम एकत्रिंशद् एका नालिका।

kalāanām ekatriṁśad ekā nālikā|

That is, 1 *nālikā* (or *nāḍīkā*) = 31 *kalās.*

Referring to the *Vāyu Purāna*, Fleet writes that according to “the Māgadh measure” [10, p. 221]:

नाडिका तुं प्रमाणेन कला दश च पंच च।

nāḍikā tuṁ pramāṇena kalāa daśa ca paṁca ca |

That is, 1 *nāḍīkā* = 15 *kalās.*

These differences in the divisions of a *nāḍī* may be accounted due to regional variations, while *nāḍī* remained a standard measure of time in ancient India.

Fleet (loc. cit., p. 228) has compared the units for weighting gold in ancient and modern times and found that 1 *suvarṇamāśaka* = 5 *ratis* = 9 grain, which implies that;

1 *suvarṇa* = 16 *suvarṇa māśakas* = 144 grains.

Finally, he notes:

> It hardly seems practicable to determine by calculation the respective sizes of the holes which would be made by the two piercing-tools of these weights and sizes. But the holes were evidently very small ones.

7 The *Ambu-Yantra* of Varaḥmihira

In order to measure time, Varaḥmihira usually recommends two instruments, namely a gnomon (or shadow) and a *yantra* (water-appliance). He writes [30, p. 12]:

तन्त्रे सुपरिज्ञाते लग्ने
छायाम्बुयन्त्रसंविदिते ।
होरार्थे च सुरूढे
नादेष्टुर्भारती वन्ध्या॥

tantre suparijñaate lagne
chāyāmbuyantrasaṁvidite |
horārthe ca surūḍhe
nādeṣṭurbhāratī vandhyā ||

> That is, the predictions of one who knows astronomy well, can calculate the exact *lagna* using the gnomon instrument and the *ambu-yantra* (water-appliance) and is well versed in horoscopy, will never be fruitless.

In this section we study in some detail the *ambu-yantra* mentioned above. Varaḥmihira in Chap. XIV of the *Panca-siddhantika*, entitled "छेद्यक-यंत्राणि" (chedyaka-yaṁtraaṇi), has described in detail a number of instruments to measure the various astronomical quantities. This chapter contains instruments designed or improved by Varaḥmihira himself for the development of astronomy and the day-to-day work of the people. In the following stanza, entitled "कालमान-यंत्राणि" (kaalamaana-yaṁtraaṇi), he describes the basic requirements for the construction of astronomical instruments (cf. [28], XIV, [26], and [29, p. 274]):

गुणसलिल्पांशुभिर्यो-
जितानि बीजानि सर्वयंत्राणाम् ।
तैः फलके कूर्म मानव-
यथेष्ट रूपाणि कार्याणि॥

guṇasalilpāṁśubhiryo-
jitāni bījāni sarvayaṁtrāṇām |
taiḥ phalake kūrma mānav-
yatheṣṭa rūpāṇi kāryāṇi ||

That is, the seeds of all instruments (for the measurement of time) are furnished by string, water, and sand. Using them, one may construct instruments of any shape, such as a tortoise or a man, and mount them on a wooden board.

Varaḥmihira forbids a teacher of astronomy to disclose the secrets of these instruments except to a devoted pupil. He writes (loc. cit.):

गुरुरचपलाय दद्या-
च्छिष्यायैतान्य वाप्य शिष्यो ऽपि।
पुत्रेणा ऽप्यज्ञातं
बीजं संयोजयेद् यंत्रे॥

gururacapalāya dadyā-
cchiṣyāyaitānya vāpya śiṣyo ạpi |
putreṇā ạpyajñātaṁ
bījaṁ saṁyojayed yaṁtre ||

That is, a teacher should impart this knowledge only to a devoted pupil, and the latter should use it properly, keeping the secrets (of constructing these instruments) unknown even to his son.

Varaḥmihira has described two types of *ambu-yantras* for the measurement of time. About the first instrument, he writes (cf. [28], XIV, [30], and [29, p. 275]):

द्युनिशि विनिःसृत तोया
दिष्टच्छिद्रेण षष्टि भागो यः।
सा नाडी स्वमतो वा
श्वासाशीतिः शतं पुंसः॥ ३१॥

dyuniśi viniḥsṛta toyā
diṣṭacchidreṇa ṣaṣṭi bhāgo yaḥ |
sā nāḍī svamato vā
śvāsāśītiḥ śataṁ puṁsaḥ ||31||

That is, the sixtieth part of water contained in a *nychthemeron* (clepsydra), which escapes from an aperture, defines the duration of one *nāḍī*, which is the same as the time taken by a man to make one hundred eighty respirations.

Since the duration of a *nāḍī* has been measured by means of the flow of water from a *kumbha* or *ghata* (clepsydra), this unit of time, in colloquium, was also known as "*ghaṭī*." This *ambu-yantra* of Varaḥmihira, being convenient for the measurement of time, became immensely popular in India, so that it was installed in almost every village inside the Śiva temples, and water used to trinkle down over the idol. Although, nowadays, it has become a part of the rituals, it is the *ambu-yantra* (or ghati-yantra) of Varaḥmihira for the measurement of time.

Varaḥmihira designed another instrument, commonly known as *kapālaka-yantra*, for the reckoning of time. He writes in the *Pañcasiddhāntika* [28, XIV, 32]:

कुम्भार्धाकारं ताम्रं पात्रं कार्यं मूले छिद्रं
स्वच्छे तोये कुण्डे न्यस्तं तस्मिन् पूर्णे नाडी स्यात्।
मूलाल्पत्वाद्वेधो वा षष्टिर्योज्या चह्ना रात्र्या
वर्णाः षष्टिर्वक्राः श्लोको यत्तत् षष्टया वा सा स्यात्॥ ३२॥

kumbhārdhākāraṁ tāmraṁ pātraṁ kāryaṁ mūle chidraṁ
svacche toye kuṇḍe nyastaṁ tasmin pūrṇe nāḍī syāt |
mūlālpatvādvedho vā ṣaṣṭiryojyā cahnā rātryā
varṇāaḥ ṣaṣṭirvakrāḥ śloko yattat ṣaṣṭyā vā sā syāt ||32||

> That is, construct a copper vessel shaped like a hemispherical jar and pierce a hole at its bottom. Place it in a basin filled with pure water. When it is filled with water, a *nāḍī* has elapsed. The hole at the bottom has to be made in such a way that the vessel may have sixty immersions in one *nychthemeron*. Or, it is the time in which sixty ślokas (verses), each composed of sixty long syllables, can be recited.

As pointed out by Thibaut and Dvivedi (loc. cit., p. 82):

> Stanza 32 consists of 60 long syllables, thus constitutes a śloka such as – according to Varaḥmihira – may be recited in the sixtieth part of a *nāḍīaka*.

Varaḥmihira has named the above hemispherical bowl made of copper as "अर्ध कपालं" (*ardha-kapālaṁ*) (cf. [28, XIV, 19]), which in the modern *Sūrya-Siddhānta* was called "*kapālaka-yantra*."

8 The *Ambu-Yantra* After Varaḥmihira

Bṛahmagupta made some modifications with the *ambu-yantra* of Varaḥmihira. In place of a *kumbha*, he prefers a graduated cylindrical vessel. He writes [6, XXII, 46]:

नलको मूले विद्धस्तत् स्रुति घटिकोद्धृतः समुच्छ्रायः।
लब्धाङ्गुलैस्तु तैर्नाडिका क्रियायन्त्र सिद्धिरतः॥

nalako mūle viddhastat sruti ghaṭikoddhtaḥ samucchrāyaḥ |
labdhāṅgulaistu tairnāḍikā kriyāyantra siddhirataḥ ||

> That is, a cylinder with a hole at its bottom is taken. Its height is divided into *ghatikās* during which water flows out. A *nāḍīkā* (i.e. *ghatikā*) is graduated into *angulas*, so that the instrument is properly set up.

Bṛahmagupta has closely followed Varaḥmihira in describing a "*kapālaka-yantra*" (hemispherical water-clock). He writes [6, XXII, 41]:

घटिका कलशार्धाकृति ताम्रंपात्रं तले ऽ पृथुच्छिद्रम् ।
मध्ये तज्जलमज्जन षष्ट्या द्युनिशं यथा भवति॥

ghaṭikā kalāśārdhākṛti tāmraṁpātraṁ tale a pṛthucchidram |
madhye tajjalamajjana ṣaṣṭya dyuniśaṁ yathā bhavati ||

> That is: a *ghaṭikā* (*yantra*) is a hemispherical vessel made of copper with a small aperture at the bottom so that it sinks into the water sixty times in one day and night.

In the modern *Sūrya-Siddhānta* too we find a description of a *ghaṭī-yantra* of the same type as designed by Varaḥmihira (cf. [27], 23). Bhāskarācarya II, who belonged to "Varaḥmihira Gurukula," has described around the year 1150 C.E., a "*ghaṭi-yantra*" in exactly the same way as designed by Varaḥmihira in the *Pancasiddhāntikā* (XIV, [31]). He writes [5, Golādhyāya, XI, 8]:

घटदल रूपा घटिका ताम्रि तले पृथुच्छिद्रा ।
द्युनिशि निमज्जनमित्या भक्तं द्युनिशिं घटी मानम्॥

ghaṭadala rūpā ghaṭikā tāmri tale pṛthucchidrā |
dyuniśi nimajjanamityā bhaktaṁ dyuniśiṁ ghaṭī mānam ||

> That is, a *ghatikā* is a hemispherical vessel made of copper with a hole at its bottom, so that the duration of a day and night is divided by the number of times it sinks into water, which is the measure of a *ghaṭī* (*nāḍī* = 24 minutes).

The *ghati-yantras*, as designed by Varaḥamihir, were in common use for more than 600 years, and a number of renowned astronomers, including Lalla and Śrīpati have described them in detail.

As noted by Ôhashi [18, p. 277]:

> The clepsydra (*ambu-yantra*) was probably the most popular astronomical instrument in India until recently, and there are several historical records of this instrument.

For details of these historical records, see Fleet [10], and [18, pp. 276–279].

9 Measurement of Time by *Śanku-Yantra*

As described in Sect. 1, the earliest mention of a unit of time is found in the *Atharvaveda*, which asserts that the sun divides a day into thirty parts. It seems that it was the initial attempt to ascertain time by observing the position of the sun in the sky that led to the use of the shadow of the sun for more accurate measurements. The thirtieth part of a day, as mentioned above, was called a "*nāḍī*" or "*nāḍīkā*," which has been described in detail in the *Ṛgveda* and the *Yajurveda* recensions of the *Vedāṅga Jyotiṣa*.

The earliest use of the shadow of a *gnomon* at any time of the day to calculate the *lagna* (i.e., ecliptic point, which is on the eastern horizon at any given time) and vice versa is found in the *Vasistha Siddhānta* as described by Varaḥmihira in the *Pañcasiddhāntikā*, Chap. II, 11–13. *Vāsiṣṭha-Siddhānta* also provides rules for the calculation for midday shadows from the longitude of the sun. For details of these calculations see [22]. This ensures that the construction of *śanku-yantras* and their use for the measurement of time, longitude of the sun, and determination of the *lagna* at any time from a gnomon's shadow was fairly well known during the time of the composition of *Vāsiṣṭha-Siddhānta* (cf. [9, p. 233] and [19, 21] for details about the time of *Vāsiṣṭha-Siddhānta*).

The use of the shadow of a gnomon for the measurement of time is found in the *Arthaśāstra*, composed by Kautilyā around the year 317 B.C.E. He writes (cf. [12, p. 425] and [13, p. 140]):

छायायामष्ट पौरुष्यमश्टादश भाग-छेदः,
    षट पौरुष्यां चतुर्दश भागः,
त्रिपौरुष्यामष्ट भागः,
    द्विपौरुष्यां षड्भागः,
पौरुष्यां चतुर्भागः,
    अष्टाङ्गुलायां त्र्योदश भागाः,
चतुरङ्गुलायां त्रयोऽष्टभागाः,
    अच्छायो मध्यान्ह इति।

chāyāyāmaṣṭa pauruṣyamaś.ṭādaśa bhāga-chedaḥ,
    ṣaṭ pauruṣyāṁ caturdaśa bhāgaḥ,
tripauruṣyāmaṣṭa bhāgaḥ,
    dvipauruṣyāṁ ṣaḍbhāgaḥ,
pauruṣyāṁ caturbhāgaḥ,
    aṣṭāṅgulāyāṁ tryodaśa bhāgāḥ,
caturaṅgulāyāṁ trayoạṣṭ ạbhāgāḥ,
    acchāyo madhyānha iti |

That is, when the shadow (of the gnomon) is eight *paurusas*, one-eighteenth part of the day is passed; when six *paurusas*, one-fourteenth part (is past); when three *paurusas*, one-eighth part; when two *paurusas*, one-sixth part;

> when one *paurusas*, one-fourth part; when eight *angulas*, three-tenths part (is past); when four *angulas*, three-eighths part; and when there is no shadow, it is midday. (Translation by Kangle [13, p. 140]).

Kautilya adds [12, p. 425]:

परावृत्ते दिवसे शेषमेवं विधात् ।

parāavṛtte divase śeṣamevaṁ vidhāt |

> That is, in the afternoon the above-mentioned rules work in the same way.

Varaḥmihira, in *Pañcasiddhāntikā*, has given an elegant method for the calculation of time from the gnomon's shadow. He writes (cf. [28], IV, 48; [29], p. 121):

षड्घ्ने स्वद्युमिते
छिन्ने सद्वादशैर्विमाध्यान्हैः ।
छायाङ्गलैर्गतास्ता
नाड्यः प्राक् पृश्ठतः शेषाः ॥

ṣaḍghne svadyumite
chinne sadvādaśairvimādhyānhaiḥ |
chāyāṅgalairgatāstā
nāḍyaḥ prāk pṛśṭhataḥ śeṣāḥ ||

> That is, multiply the measure of daytime in *nāḍīs* by 6, and divide by the *angulis* (length) of the shadow, after having added 12 and subtracted the length of the midday shadow of the date. The results are the *nāḍikās* from sunrise in the forenoon and the remaining time to sunset, in the afternoon.

The above rule can be expressed in the following mathematical form:

$$n \;=\; \frac{6 \times d}{12 + s + s_0},$$

where n = *nāḍīs*, d = duration of the day, 12 = length of the gnomon in *angulas*, s = length of the shadow (in *angulas*), and s_0 = length of the midday shadow.

In the following couplet, Varaḥmihira provides a rule to calculate the length of the shadow, when the time is known. He writes (cf. [28], IV, 49, and [29], p. 122):

छाया ऽऽर्की नाडीभि
र्दिनमानं षडघ्नमुद्धरेत्तत्र ।
लब्धं द्वादश हीनं
मध्यान्हच्छायया सहितम् ॥

chāyā ạạrkī nāḍībhi
rdinamānaṁ ṣaḍaghnamuddharettatra |
labdhaṁ dvādaśa hīnaṁ
madhyānhacchāyayā sahitam ||

> That is, multiply the duration of the day by six and divide it by the given time in *nāḍīkās*; subtract 12 from the quotient and add the midday shadow. The result is the gnomon's shadow due to the sun at the given time.

Abraham [1, p. 215] has expressed the above rules in the following form:

$$\frac{d}{2t} = \frac{s - s_0}{g} + 1,$$

where t/d is the fraction of daytime, s is the shadow of the gnomon of length g, and s_0 is the length of the noon shadow.

About the increase and decrease of a day when the sun moves toward the summer and winter solstices respectively, Kautilya writes (cf. [12, p. 425] and [13, p. 140]):

पंचदशमुहूर्तो दिवसो रात्रिश्च
चैत्रे चाश्वयुजे च मासि भवतः॥ ३७॥
ततः परं त्रिभिर्मुहूर्त्तैरन्यतरः
षण्मासं वर्धते ह्रसते चेति॥ ३८॥

paṁcadaśamuhūrto divaso rātriśca
caitre cāśvayuje ca māsi bhavataḥ ||
tataḥ paraṁ tribhirmuhūrttairanyataraḥ
ṣaṇmāsaṁ vardhate hrasate ceti ||

> That is: the day and night of fifteen *muhūrtas* occur in the months of Caitra and Āśvayuja (Āsvin). After this the former increases and (then) decreases by three *muhūrtas* during a period of six months, and in the same way the latter too.

Abraham, reading the above lines, writes [1, p. 216]:

> The rule for the uniform variation of the length of daylight 12–18 *muhūrtas*, implies a latitude of about 35°, and so seems to have been uncritically borrowed from Babylonia.

The claim of Abraham that the method of the construction of the *śanku-yantra* was borrowed from "Babylonia" seems to be totally untenable. Lagadh, in fact, in *Vedāṅga Jyotiṣa*, has written about this variation of six *muhūrtas*. But he was a Kashmiri Brāhmaṇa. For details see Sect. 2.

Bṛahmgupta, one of the top-ranking mathematicians of "Varaḥmihira Gurukula," has given the following rule for the calculation of time by *śanku-yantra* (cf. [6], XII, 52):

छायानर सैकहृतं द्युदलं
प्राग् परयोर्द्युगत शेषम्।
दिनगत शेषांशहृतं द्युदलं
छाया नर व्येकम्॥

chāyānar saikahṛtaṁ dyudalaṁ
prāg parayordyugata śeṣam |
dinagata śeṣāṁśahṛtaṁ dyudalaṁ
chāyā nara vyekam ||

> That is, the half-day divided by the ratio of the shadow with the (length of the) gnomon and one added to it provides the elapsed or remaining part of the day in the forenoon or afternoon.

Conversely, the length of the shadow is obtained by dividing the half-day by the time (given), subtracting 1, and multiplying the result by the length of the gnomon. In concrete mathematical terms, the above rules can be expressed in the forms

$$t = \frac{d/2}{(s/g) + 1} \quad \text{and} \quad s = \left(\frac{d/2}{t} - 1\right) \times g.$$

Of course, these formulas can be easily derived from the corresponding results of Varaḥmihira. For details see [6, pp. 997–998].

Mahāvīrācārya also, in his famous work *Ganitasara-Samgraha*, has used *śanku-yantra* for the measurement of time (cf. [17]; IX, 15–16, and [18, p. 190] for details). It may be mentioned here that Varaḥmihira and Bṛahmagupta had developed some more sophisticated methods to ascertain exact time. For details see [18, pp. 190–194].

During the first quarter of the eighteenth century, Savai Rājā Jai Singh established five observatories for research work in astronomy. *Śanku-yantras* were installed in these observatories for the measurement of time. One such *śanku-yantra* established by Jaisingh at the observatory in Ujjain is still in good condition and visitors can record time from it.

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The Golden Mean and the Physics of Aesthetics

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1 Introduction

In 2002, the celebrated Philippine composer José Maceda [11] organized a symposium on "A Search in Asia for a New Theory of Music" in Manila. His objective was to explore Asian mathematical sources that may be helpful in providing a direction to world music everywhere. The papers at the symposium have since appeared as a book [1]. At this meeting, Maceda had many conversations with me regarding what he called the "aesthetics of Asian court musics which seek permanence with little change." I told Maceda that I considered the *dhvani* approach to aesthetics particularly insightful, since it made the audience central to the process of illuminating the essential sentiment behind the artistic creation (e.g., [5]). According to this view the permanence being sought is the "universal" that can only be approached and never quite reached by the diverse paths that represent different cultural experiences.

Maceda appreciated the role that physiological geometry plays in perception and aesthetics, but he was emphatic that there was a cultural component to it and that there could be no unique canon of beauty in art. I take this thought as the starting point of my discussion of the golden proportion ($\Phi = 1.618\ldots$), defined by

$$\Phi \;=\; \frac{1}{\Phi - 1},$$

which some theorists have claimed is fundamental to aesthetics. The conception of it as a pleasing proportion arose in the late 1800s, and there are no written texts that support such usage in the ancient world. Indeed, many non-Western cultures either do not speak of a unique ideal or consider other ratios as ideal, such as $\sqrt{2}$ of Islamic architecture.

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Zeising [19] appears to have been the first to propose the romantic idea that in the golden ratio "is contained the fundamental principle of all formation striving to beauty and totality in the realm of nature and in the field of the pictorial arts." Two prominent twentieth century architects, Ernst Neufert and Le Corbusier, used this ratio deliberately in their designs. More recently, many industrial designs (such as the dimensions of the ubiquitous credit card) have been based on it. *Avant-garde* musicians such as Debussy, Bartok, and Xenakis have composed melodic lines with intervals chosen according to this ratio. Salvador Dali used this proportion in some canvases. Cartwright et al. [3] argue its importance in music.

But such usage does not make it an aesthetic law, and Livio [9] debunks this pretension effectively, showing that the research on which this claim has been made is ambiguous at best. I agree with Livio's warning that:

> Literature is bursting with false claims and misconceptions about the appearance of the Golden Ratio in the arts (e.g., in the works of Giotto, Seurat, Mondrian). The history of art has nevertheless shown that artists who have produced works of truly lasting value are precisely those who have departed from any formal canon for aesthetics. In spite of the Golden Ratio's truly amazing mathematical properties, and its propensity to pop up where least expected in natural phenomena, I believe that we should abandon its application as some sort of universal standard for "beauty," either in the human face or in the arts [10].

In this note I first present the background to the golden mean and its relationship to Mount Meru (*Meru-Prastāra*) of Piṅgala (c. 200 B.C.E. but perhaps 450 B.C.E. if the many textual notices about him being the younger brother of the great grammarian Pāṇini are right) and its subsequent exposition by other mathematicians. The term "golden section" (*goldene schnitt*) seems to have been used first by Martin Ohm in the 1835 edition of the textbook *Die Reine Elementar-Mathematik*. The first known use of this term in English is in James Sulley's 1875 article on aesthetics in the ninth edition of the *Encyclopaedia Britannica*.

Next, I present multiplicative variants of Mount Meru and the Fibonacci sequence that may explain why the octave of Indian music has 22 micronotes (*śruti*), a question that interested José Maceda. I also discuss other sequences related to Mount Meru, including those that were described by the musicologist and musical-scale theorist, Ervin Wilson. I conclude with a discussion of the question of aesthetic universals that was raised by José Maceda.

2 Historical Background

As historical background, one must go to Piṅgala's *Chandaḥśāstra* 8.32–8.33 (see, e.g., [16],[14],[15],[13]), in which, while classifying poetic meters of long and short syllables, Piṅgala presents Mount Meru (*Meru-Prastāra*, "Steps of Mount Meru," and *Meru-Khaṇḍa*, "Portion of Mount Meru" by Bhāskara II in his *Līlāvatī* written in 1150), which is also known as *Pascal's triangle* (Fig. 1). The shallow diagonals of Mount Meru sum to the Fibonacci series, whose limiting ratio is the golden mean (Fig. 2). Piṅgala's cryptic rules were explained by later commentators such as Kedāra (seventh century) and Halāyudha (tenth century).

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Fig. 1. Mount Meru

The Fibonacci numbers are described by several Indian mathematicians in the centuries following Piṅgala as being produced by the rule

$$F(n+1) = F(n) + F(n-1).$$

Virahānka (seventh century) explicitly gives the sequence $3, 5, 8, 13, 21$. Further explanations regarding the numbers were presented by Gopāla (c. 1135 C.E.) and the polymath Hemacandra Sūrī (1089–1172). If we assume that long syllables take twice as long to say as short ones, Hemachandra (to use the more common spelling of his name) was dealing with the following problem: "If each line takes the same amount of time, what combination of short

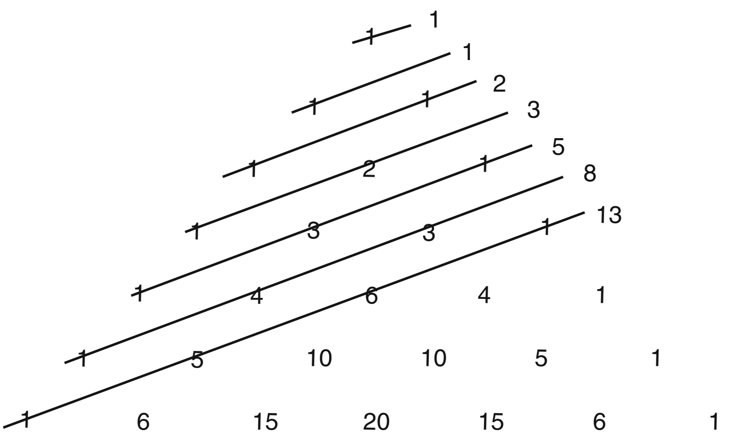

Fig. 2. Fibonacci series from Mount Meru

(S) and long (L) syllables can it have?" The general answer is that a line that takes n time units can be formed in $F(n)$ ways. This was explicitly stated by Hemachandra in about 1150 C.E.

In Europe, Fibonacci's *Liber Abaci* in 1202 describes these numbers; the book was meant to introduce the Indian number system and its mathematics, which he had learned in North Africa from Arab teachers while a young man growing up there. Fibonacci speaks of his education in North Africa thus:

> My father, who had been appointed by his country as public notary in the customs at Bugia acting for the Pisan merchants going there, summoned me to him while I was still a child, and having an eye to usefulness and future convenience, desired me to stay there and receive instruction in the school of accounting. There, when I had been introduced to the art of the Indians' nine symbols through remarkable teaching, knowledge of the art very soon pleased me above all else and I came to understand it.

It has been suggested that the name "Gopāla-Hemachandra numbers" be used for the general sequence

$$a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ldots$$

for any pair $a,\ b$, which for the case $a=1,\ b=1$ represents the Fibonacci numbers. This would then include the Lucas series, for which $a=2$ and $b=1$. It would also include other series such as the one for which $a=1$ and $b=21$, which generates the numbers

$$1,\ 21,\ 22,\ 43,\ 65,\ 108,\ldots.$$

Nārāyaṇa Paṇḍita's *Gaṇita Kaumudi* (1356) studies additive sequences in which each term is the sum of the previous q terms. He poses the problem thus:

> A cow gives birth to a calf every year. The calves become young and they begin giving birth to calves when they are three years old. Tell me, O learned man, the number of progeny produced during twenty years by one cow. (See [14],[15] for details).

Mount Meru was also known in other countries. The Chinese call it "Yang Hui's triangle" after Yang Hui (c. 1238–1298), and the Italians know it as "Tartaglia's triangle," after the Italian algebraist Tartaglia, born Niccolo Fontana (1499–1557), who lived a century before Pascal. In western European mathematical literature, it is called Pascal's triangle after the *Traité du triangle arithmétique* (1655), by Blaise Pascal, in which several results then known about the triangle were used to solve problems in probability theory. Interesting historical anecdotes regarding the combinatoric aspects of these numbers are given in Sect. 7.2.1.7 of the classic *The Art of Computer Programming* by Knuth [8].

Here we use the name Mount Meru, since it has also had fairly wide currency in English-language literature (see, e.g., [12]), and because it was used by Ervin Wilson, whose work will be discussed here. Wilson describes recurrence sequences that he calls Meru 1 through 9.

3 A Multiplicative Mount Meru and a Multiplicative Sequence of Notes

The problem of why Indian musicologists speak of a sequence of 3, 7, and 22 notes has been of long-standing interest (see, e.g., [4], [6]). Although Clough et al. [4] present it as the problem of the passage from a chromatic universe of 22 divisions to a "diatonic" set of seven degrees, there is no textual evidence in support of a prior conception of 22 micronotes. For example, Bharata Muni's *Nāṭya Śāstra* speaks both of the 7 and the 22 divisions.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 10 & & 4 & & 1 & & \\
 & 1 & & 5 & & 41 & & 41 & & 5 & & 1 & \\
1 & & 6 & & 206 & & 1682 & & 206 & & 6 & & 1
\end{array}
$$

Fig. 3. A multiplicative Mount Meru

We propose that a combinatoric consideration may have been behind the choice. Given that the study of poetic meters was an important part of education at that time, it is likely that variants of Mount Meru were examined. An interesting variant on the standard Mount Meru is a multiplicative Meru in which each descendent number is a product of the numbers above it plus one, with the further condition that when the multiplicands are both 0, the product remains 0. One can see in Fig. 3 that all other numbers are 0's, and therefore the product at the edges is $0 \times 1 + 1 = 1$.

An interesting sequence emerging from a similar multiplicative logic is

$$M(n+1) = M(n) \times \; M(n-1) + 1,$$

which gives us

$$0,\ 1,\ 1,\ 2,\ 3,\ 7,\ 22,\ 155,\ 3411,\ 528706, \ldots.$$

We propose that the occurrence of the numbers 3, 7, and 22 in this series is not a coincidence and the above construction was behind the identification of the 22 micronotes in the Indian theory of music. The original number is taken to be three basic notes that become the standard seven notes and the 22 micronotes (*śruti*). Although there is no way to confirm that this was indeed the case, it appears plausible.

4 General Recurrence Sequences

In general, we can write the sequence using the linear recurrence relation

$$F(n) \ = \ \sum_{i=1}^{K} a(i)F(n-i),$$

where $a(i)$ values represent suitable constants. The limiting ratio, in this case, will be a solution to the algebraic equation

$$x^K - a(1)x^{K-1} - a(2)x^{K-2} - \cdots - a(K) \ = \ 0.$$

It is obvious that any number of ratios can be obtained by a proper choice of K and $a(i)$'s. For the case that the $a(i)$'s are each 1, the characteristic equation for the ratio may be simplified to

$$x^{K+1} - 2x^K + 1 \ = \ 0.$$

Less radical variants of Mount Meru and of Fibonacci sequences may also be conceived. For example, Wilson's variations of Fibonacci sequences [17], [18] are obtained using different slopes of Mount Meru by combining two earlier terms in the sequence. They were named by him Meru 1 through Meru 9.

5 Wilson's Meru 1 Through Meru 9

1. Meru 1: $A_n = A_{n-1} + A_{n-2}$ with limiting ratio 1.618033... (golden mean)
2. Meru 2: $B_n = B_{n-1} + B_{n-3}$ with limiting ratio 1.465571...
3. Meru 3: $C_n = C_{n-2} + C_{n-3}$ with limiting ratio 1.324717...
4. Meru 4: $D_n = D_{n-1} + D_{n-4}$ with limiting ratio 1.380277...
5. Meru 5: $E_n = E_{n-3} + E_{n-4}$ with limiting ratio 1.220744...
6. Meru 6: $F_n = F_{n-1} + F_{n-5}$ with limiting ratio the same as Meru 3
7. Meru 7: $G_n = G_{n-2} + G_{n-5}$ with limiting ratio 1.236505...
8. Meru 8: $H_n = H_{n-3} + H_{n-5}$ with limiting ratio 1.193859...
9. Meru 9: $I_n = I_{n-4} + I_{n-5}$ with limiting ratio 1.167303...

The explicit use of these exotic sequences to create musical scales has been presented by the composer Burt [2]. His *Wilson Installations*, a set of three live solo electronic music performances, dealt "with the notions of extended tuning systems, the relation of tuning to timbre and spatiality of sound, and the concentration of attention for extended time periods."

One may also generalize the Mount Meru structure. In Fig. 4 we show a triplicate Mount Meru, for which one of the characteristic sequences would be

					1					
				1	1	1				
			1	2	3	2	1			
		1	3	6	7	6	3	1		
	1	4	10	16	19	16	10	4	1	
1	5	15	30	45	51	45	30	15	5	1

Fig. 4. A triplicate Mount Meru

$F(n+1) = F(n) + F(n-1) + F(n-2)$, that is, $1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots$. Another example is the sequence $2, 2, 3, 7, 12, 22, 41, 75, 138, \ldots$, which generates not only 3, 7, and 22, but also 12, another characteristic number of Indian music.

6 Structural Considerations

If a golden rectangle is drawn and a square is removed, the remaining rectangle is also a golden rectangle (Fig. 5a). Continuing this process and drawing circular arcs yields a curve that approximates the logarithmic spiral, which is a form found in nature (Fig. 5b). These numbers are encountered in many plants in the arrangement of leaves around the stem, pine cones, seed head packing, and flower petals owing to optimality conditions.

Since physical reasons underlie this occurrence of the golden mean in nature, it is tempting to look at the structural basis of why it is a pleasing proportion cognitively, although we must remember that it is not the only one.

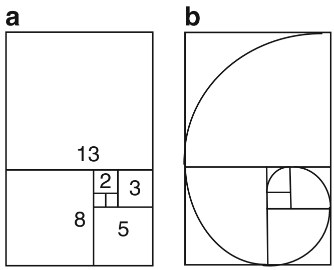

Fig. 5. The Fibonacci sequence in (**a**) two dimensions and (**b**) as a spiral

If the stimulation inside the brain corresponding to an input spreads through a spiral function, then the pleasantness of the golden mean would be clear. On the other hand, neurophysiological structures are not quite two-dimensional, and therefore, there would be a component of the spiral that would go into the third dimension, providing a departure from the "normative" golden mean to many other similar numbers, such as Wilson's Meru 2 through 9, or other more general numbers. Since the details of the structures are unique to the individual, there is further variability regarding what would be optimal. Nevertheless, inanimate, living, and cognitive systems show similar behavior, which is a consequence of pervasive recursionism [7].

7 Concluding Remarks

This note has reviewed some variants of Fibonacci sequences that are of interest to musicologists. The Fibonacci sequence corresponds to a spreading function that is two-dimensional,

$$x^2 = x + 1,$$

that is, with the next step as the square of the previous one (Fig. 5). By analogy, in a purely three-dimensional spreading function, the operative formula should be

$$x^3 = x + 1,$$

which corresponds to the recurrence relation of Meru 3. This may be written, alternatively, as the solution of the balanced-looking equation

$$\Psi = \frac{1}{(\Psi - 1)(\Psi + 1)}.$$

Since the neurophysiological cognitive structures do not have a symmetry across all three dimensions (there is also variation across individuals), proportions intermediate to the golden mean and Meru 3 (1.324717...) would also be aesthetically pleasing, especially in traditions that are contemplative, as in Asia.

Returning to the question of aesthetic universals that Maceda posed, I agree with him that it is most likely that they do not exist. It is cultural authority and tradition that creates them, although they may be shaped by "universals" associated with our cognitive systems.

Acknowledgement

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Piṅgala Binary Numbers

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1 Introduction

According to Sanskrit scholars, Piṅgala Nāga is supposed to be the younger brother of the eminent grammarian Pāṇini (Upadhyaya [7]). According to the Mīmānsaka [4], they lived in Shalatur village of Peshawar around 2850 B.C.E. Piṅgala's *Chandas Śāstram* is the oldest available authoritative work on prosody. He has referred to his predecessors in his work frequently, but no earlier treatises dealing exclusively with Vedic or classical Sanskrit meters have survived. His work deals with various aspects of prosody. Poetry is considered the crowning glory of Sanskrit literature mainly because of its perfect poetical structure due to the mathematics of metrics. Therefore, the construction of a poetical verse needs an excellent knowledge of prosody. The basic rules of Piṅgala's *Chandas Śāstram* are presented only in *sūtras* (formulas). *Sūtra* is a Sanskrit word and it literally means (coded or terse) formula. So, it is understood only through commentaries. There are a couple of commentaries available on *Chandas Śāstram*. In all that follows, we adhere to the most authoritative commentary by the tenth century Sanskrit scholar and mathematician, Halāyudha Bhaṭṭa (see [3]). Piṅgala used binary codes to ascertain the position of an even verse with a certain syllabic arrangement in the list of all possible verses with the same number of syllables. We limit ourselves to a brief discussion on the methods of writing Piṅgala binary numbers and mapping rules between binary and decimal numbers.

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2 Fundamentals

In Sanskrit, there is no poetry without *chandas* or meter. To understand and appreciate any poetical composition, a good knowledge and a feel of the meter is essential. "The lore of the meter is the boat for those who desire to cross the deep ocean of poetry," says the great poet *Daṇḍī* (eighth century C.E.), in his *Kāvyādarśa*:

सा विद्या नौस्तितीर्षूणां गम्भीरं काव्यसागरम्।

Sā vidyā naustitīrṣūṇāṁ gambhīraṁ kāvyasāgaram.

To express or describe any experience, feeling, emotion, or action, the choice of the appropriate meter is of vital importance, because each meter has its own mood, rhythm and movement.

While using a particular meter, says Kshemendra (*circa* 1025–1075 C.E.), in his *Suvṛttatilaka*, "One has to see the *rasa*, the mood, the nature of the description and context":

काव्ये रसानुसारेण वर्णनानुगुणेन च।
कुर्वीत सर्ववृत्तानां विनियोगं विभागवित्॥

Kāvye rasānusāreṇa varṇanānuguṇena ca,
Kurvīta sarvavṛttānāṁ viniyogaṁ vibhāgavit.

Therefore one should have knowledge of meters. Describing the importance of meters, *Bhaṭṭa* (see [3]) writes:

वेदानां प्रथमांगस्य कवीनां नयनस्य च।
पिंगलाचार्य सूत्रस्य मयावृत्तिर्विधास्यते॥

Vedānāṁ prathamāṅgasya kavīnāṁ nayanasya ca,
Piṅgalācārya sūtrasya mayāvṛttirvidhāsyate.

This merely means: I present a commentary on the metrics, the prime limb of the *Vedas* and the eyes of poets.

In Sanskrit and in Indian languages derived from Sanskrit, the meter is determined by the arrangement of short and long syllables. The large number of possible permutations and combinations has given rise to a large variety of meters.

Almost all the *padyakāvyas* or poetical compositions in Sanskrit follow a metrical structure. Therefore, to understand and appreciate them, a knowledge of metrics or *chandas* is essential.

The vast body of Sanskrit literature that we possess is the legacy of the *riṣis* (great scholars) both ancient and modern. It is an inexhaustible wealth of inspiration for a student of poetic *chandas*.

The study of metrics has a long tradition as an important branch of Vedic learning. The text that deals with the rules of metrics is called *Chandas Śāstram* and is one of the six *Vedāṅgas*, or limbs, of the *Vedas*. Indeed, the *chandas* itself is considered to be the two legs of the *Vedas* by the eminent grammarian, Pāṇini, the great mathematician and astronomer, Bhāskarācārya (b. 1114 C.E.), and other top-ranking scholars.

2.1 *Chandas* or Meter

Literary compositions (*kāvyas*) in Sanskrit may be in the form of prose (*gadya*) or in the form of verse (*padya*). A poetical stanza or verse in Sanskrit is called a *padya*. Generally a *padya* or a verse contains four *pādas*, or quarters or metrical lines.

Sanskrit verses are classified into groups and subgroups according to

(a) The number of syllables or syllabic instants that they contain in each quarter, and
(b) The position or placement of short and long syllables within the verse.

These groups and subgroups are called meters.

2.2 *Pāda* or Quarter

All verses in Sanskrit generally contain four lines. Each line is called a *pāda* (also called quarter if the verse has four metrical lines). The first two *pādas* form the first half of a verse and the remaining two *pādas* the latter half of the verse.

A *pāda* or quarter is regulated either by the number of syllables (*akṣara*) or by the number of syllabic instants (*mātrās*) or moras.

2.3 *Akṣara* or Syllable

An *akṣara* is as much of a word as can be pronounced distinctly at once or by one effort of the voice. So a vowel with or without one or more consonant is considered as one syllable. A syllable can be short (*laghu*) or long (*guru*), depending on whether its vowel is short or long.

2.4 *Laghu* or Short Syllables

The vowels अ (*a*), इ (*i*), उ (*u*), ऋ (*ṛ*), ऌ (*ḷ*) are short. Whenever any of these are used in a verse separately or with one or more consonants, it will be considered as a short syllable. For example, क (*ka*), कि (*ki*), etc. are short syllables.

The vertical bar (।) is used to represent a short syllable in scansion and metrical analysis.

2.5 *Guru* or Long Syllables

The vowels आ ($\bar{a}$), ई ($\bar{i}$), ऊ ($\bar{u}$), ॠ ($\bar{\d{r}}$), ए (e), ऐ (ai), ओ (o), औ (au) are long. Whenever any of these is used in a verse separately or with one or more consonants, it will be considered as long. For example, का ($k\bar{a}$), की ($k\bar{i}$), etc. are long syllables.

The symbol (ऽ) is used to represent a long syllable in scansion and metrical analysis. A short vowel gets the practical status of long under the following three conditions:

(a) If a vowel is followed by an *anusvāra*, for example, तं (*taṁ*), गं (*gaṁ*), etc.

(b) If a vowel is followed by a *visarga*, for example, तः (*taḥ*), गः (*gaḥ*), etc.

(c) If a vowel is followed by a conjunct consonant, for example, बन्ध (*bandha*).

Notice that the short syllable *ba* has to be counted long, since it is followed by न्ध (*ndha*).

As a final remark about the counting of syllables in a meter, a short syllable at the end of a quarter of a meter may be considered a long syllable and vice versa according to the chanting or singing requirements of the meter.

As an illustration, we present a verse in *anuṣṭup* meter along with its scansion into short (।) and long (ऽ) syllables.

Example 2.1:

। ऽ । ऽ । ऽ । ऽ
पुनातु भक्तिरच्युता
। ऽ । ऽ । ऽ । ऽ
सदाऽच्युताङ्घ्रिपद्मयोः ।
। ऽ । ऽ । ऽ । ऽ
श्रुतिस्मृतिप्रमाणिका
। ऽ । ऽ । ऽ । ऽ
भवाम्बुराशितारिका ॥

Notice that nothing is lost mathematically if instead of reading the actual verse and counting syllables, one looks at only the scansion, showing the arrangement involving short (।) and long (ऽ) ones. So, from a mathematical analysis point of view, the above scansion may be presented by the following matrix, wherein we have not followed the modern taste of using brackets or a similar notation.

। ऽ । ऽ । ऽ । ऽ
। ऽ । ऽ । ऽ । ऽ
। ऽ । ऽ । ऽ । ऽ
। ऽ । ऽ । ऽ । ऽ

Indeed, an even meter may be defined as $4 \times n$ matrix that is, a matrix consisting of four rows and n columns, wherein each element of a row is either । or ऽ and all four rows have the same syllabic arrangement. (The modern symbols for । and ऽ are 1 and 0 respectively.) Notice that in the above illustration, the number of elements in each row is 8, and all four rows are identical. This motivates us to designate an even meter by an even matrix of order $4 \times n$. In many cases of even meters, we may consider only one row, since all four rows of an even meter are identical.

2.6 *Mātrā* or Metrical Unit

A unit of metrical quantity is called a *mātrā* or mora. A mora denotes the time required to utter a short vowel. All short vowels are regarded as consisting of one mora. All long vowels and diphthongs are regarded as consisting of two moras.

2.7 *Verse* Classification

All *padyas* or verses in Sanskrit may be classified as either *vṛtta* or *jāti*. In all that follows, we adhere to Piṅgala's discussion on *vṛttas* or verses with four quarters.

A *vṛtta chanda* is one that is regulated by the number and positions of syllables in each *pāda* or quarter. *Vṛtta* meters are further divided into three categories, such as:

(a) *Samavṛtta* or even meters.

(b) *Ardhasamavṛtta* or half-even meters.

(c) *Viṣamavṛtta* or uneven meters.

Verses in which all (four) quarters contain an equal number of syllables and the same syllabic arrangement is called *samavṛtta* or even meters. Thus if the metrical scansion of a meter has all four rows identical then the meter is called even. If alternate rows of the scansion are identical, then the meter is categorized as half-even. An uneven meter is neither even nor half-even. It is important to note that the class of even meters is contained in the class of half-even meters (Table 1).

Note that English poetry is regulated by accent, whereas Sanskrit (and other languages derived from Sanskrit) poetry is regulated by quantity (*m̄atra*).

Apart from these eight *Varṇic Gaṇas*, Piṅgala uses two other *gaṅas*, viz. *La* (ल) for short (*laghu*) and *Ga* (ग) for long (*guru*). The system consisting of eight *gaṅas*, that is, *Ma*, *Ya*, *Ra*, *Sa*, *Ta*, *Ja*, *Bha*, *Na*, and the other two, *La*, *Ga* are called the ten syllables of Piṅgala (TSP). They pervade the whole creation of meters. The best way to remember them (and their elements) is to read them in the following musical order:

Table 1. The eight trisyllabic feet or *gaṇas*

Serial Number	1	2	3	4	5	6	7	8
Name of gaṇas in Roman script	*Magaṇa*	*Yagaṇa*	*Ragaṇa*	*Sagaṇa*	*Tagaṇa*	*Jagaṇa*	*Bhagaṇa*	*Nagaṇa*
Name of gaṇas in Hindi script	मगण	यगण	रगण	सगण	तगण	जगण	भगण	नगण
Symbols denoting gaṇas	ऽ ऽ ऽ	। ऽ ऽ	ऽ । ऽ	। । ऽ	ऽ ऽ ।	। ऽ ।	ऽ । ।	। । ।
Symbolic initial letters	*Ma* म	*Ya* य	*Ra* र	*Sa* स	*Ta* त	*Ja* ज	*Bha* भ	*Na* न

(TSP) *Ya - Mā - Tā - Rā - Ja - Bhā - Na - Sa - La - Gaṃ*

This needs a brief explanation. Suppose we wish to know *Yagaṇa*. We start with *Y* and notice that *Ya* is a short (।) syllable, *Mā* (the next entry in the (TSP)) is long (ऽ) and *Tā* (the third entry in the system (TSP)) is also long (ऽ). Thus *Yagaṇa* stands for ।ऽऽ. Similarly, to know *Ragaṇa* we start with *R* and notice that *Rā* is long (ऽ) and the next two entries in the (TSP), viz. *Ja* and *Bhā*, are short (।) and long (ऽ) respectively. So *Ragaṇa* stands for ऽ । ऽ. In order to know *Sagaṇa* quickly, we see that *Sa-La-Gaṃ* gives । । ऽ. Recall that *La* and *Ga* (which is written as *Gaṃ* in (TSP)) stand for short (।) and long (ऽ) respectively. Thus one can find any of the *gaṇas*. For an excellent discussion on the combinatoric nature of (TSP), one may refer to Kak [2].

One of the most surprising aspects of the masterly composition of various examples and definitions of different meters is that in many cases of Sanskrit meters, the definition of each meter is itself composed in that particular example of the meter (see [3], for instance, *Sūtras* 4.40 and 8.2 and its examples or any other *sūtra* and its example).

In the next example, we consider an even meter or even matrix of order $4 \times n$ with $n = 12$.

Example 2.2: Consider the following **even meter** or even matrix of order 4×12 along with its scansion:

Varṇic Vṛitta (वर्णिक वृत्त)

। । ऽ। । ऽ । । ऽ। । ऽ
जय राम सदा सुख धाम हरे 12 *varṇas* (12 वर्ण)
रघुनायक सायक चाप धरे 12 *varṇas* (12 वर्ण)
भव बारन दारन सिंह प्रभो 12 *varṇas* (12 वर्ण)
गुन सागर नागर नाथ बिभो 12 *varṇas* (12 वर्ण)

Śrī Rāmacaritamānasa–Laṅkākāṇḍa (श्री रामचरितमानस-लङ्काकांड) – 110.1. Notice that the order and number of syllables are the same in all four quarters.

This verse may be represented by the following even matrix of order 4×12 or simply by (। । ऽ । । ऽ । । ऽ । । ऽ):

। । ऽ । । ऽ । । ऽ । । ऽ
। । ऽ । । ऽ । । ऽ । । ऽ
। । ऽ । । ऽ । । ऽ । । ऽ
। । ऽ । । ऽ । । ऽ । । ऽ

Note 1: Number and arrangement of syllables in this example are the same in all four quarters of this stanza.

Note 2: According to binary numbers, its position is 1756th in the Piṅgala list of 4096 ($= 2^{12}$) **meters, each consisting of 12 syllables**. We follow Piṅgala to explain it below (see Example N1).

3 *Pratyayas:* Methods of Cognitions

The eighth chapter of the Piṅgala *Chandas Śāstram* (cf. [3]) discusses six types of *pratyayas* (cognitions).

प्रस्तारो नष्ट उद्दिष्ट एकद्व्यादिलगक्रिया।
संख्यानमध्वयोगश्च षडेते प्रत्ययाः मताः॥

Prastāro naṣṭa uddiṣṭa ekadvyādilagakriyā,
saṅkhyānamadhvayogaśca ṣaḍete pratyayāḥ matāḥ.

Here we discuss only the following:

3.1 *Varṇic prastāra* (expansion)

3.2 *Naṣṭa*

3.3 *Uddiṣṭa*

3.1 *Varṇic* Expansion

This prescribes two methods to write Piṅgala binary numbers. The first method is the usual way of writing binary numbers, but from left to right. The second method says:

(a) Write long (ऽ) and short (।) alternately in the first column.

(b) Write two longs (ऽऽ) and two shorts (। ।) alternately in the second column.

Table 2. *Varṇic* expansion (1 *varṇa*)/Piṅgala binary numbers

1	ऽ	0
2	।	1

Table 3. *Varṇic* expansion (2 *varṇas*)/Piṅgala binary numbers

1	ऽ ऽ	00
2	। ऽ	10
3	ऽ ।	01
4	। ।	11

Table 4. *Varṇic* expansion (3 *varṇas*)/Piṅgala binary numbers

1	*Ma*	ऽ ऽ ऽ	000
2	*Ya*	। ऽ ऽ	100
3	*Ra*	ऽ । ऽ	010
4	*Sa*	। । ऽ	110
5	*Ta*	ऽ ऽ ।	001
6	*Ja*	। ऽ ।	101
7	*Bha*	ऽ । ।	011
8	*Na*	। । ।	111

(c) Write four longs (ऽऽऽऽ) and four shorts (। । । ।) alternately in the third column; and so on.

In conformity with modern taste, one may write 0 for ऽ (long) and 1 for । (short). This is best illustrated through Tables 2–5 given below for 1, 2, 3, and 4 syllables.

We note that in metrical analysis, *mātrās* are counted from left to right. This is the main reason that Piṅgala prescribes the generation of binary numbers from left to right in various metrical scansions.

3.2 *Naṣṭa*

This gives a method to convert a decimal number into the equivalent Piṅgala binary number and is best described by the following example.

Example N1: Find the syllabic arrangement of the meter at the 14th place in the list of Piṅgala expansion of even meters having five *varṇas*. That is, convert 14 into a Piṅgala binary number consisting of five digits. (Mathematically, we are converting 13 into a 5-digit Piṅgala binary number.)

Table 5. *Varṇic* expansion (4 *varṇas*)/Piṅgala binary numbers

1	ऽ ऽ ऽ ऽ	0000
2	। ऽ ऽ ऽ	1000
3	ऽ । ऽ ऽ	0100
4	। । ऽ ऽ	1100
5	ऽऽ । ऽ	0010
6	। ऽ । ऽ	1010
7	ऽ । । ऽ	0110
8	। । । ऽ	1110
9	ऽ ऽ ऽ ।	0001
10	। ऽ ऽ ।	1001
11	ऽ । ऽ ।	0101
12	। । ऽ ।	1101
13	ऽ ऽ । ।	0011
14	। ऽ । ।	1011
15	ऽ । । ।	0111
16	। । । ।	1111

Solution: By the formula *Lardhe* (लर्धे) (8/24, i.e., formula 24 of Chap. 8 in [3]), since 14 is even, write a short (।). Again, since $14 \div 2 = 7$ (odd), write a long (ऽ) next to the previous entry. Again $\frac{7+1}{2} = 4$ (even), write a short (।) as the next entry. (Indeed, whenever we get an odd number in this process, add 1 to make it divisible by 2.). Again $4 \div 2 = 2$ (even), write a short (।) as the next entry.

Finally, we obtain $2 \div 2 = 1$ (odd), place a long (ऽ) as the next entry. This way the final outcome is । ऽ । । ऽ.

Example N2: Convert 14 into Piṅgala binary numbers consisting of 5, 6, 7, and 8 bits. In view of the previous example, we have

$(14)_{Pb(5)}$ = । ऽ । । ऽ = 10110 (i)
$(14)_{Pb(6)}$ = । ऽ । । ऽ ऽ = 101100 (ii)
$(14)_{Pb(7)}$ = । ऽ । । ऽ ऽ ऽ = 1011000 (iii)
$(14)_{Pb(8)}$ = । ऽ । । ऽ ऽ ऽ ऽ ऽ = 10110000 (iv)

[Here $(14)_{Pb(n)}$ stands for the Piṅgala binary number involving n bits corresponding to the number 14.]

The above method is equivalent to the **modern system** of conversion (given below). Further, reversing the sequence of bits in (i)–(iv), one can easily see that the modern binary representations of the natural number 13 in 5, 6, 7, and 8 bits are 01101, 001101, 0001101, 00001101 respectively.

Subtract 1 from the decimal value and then divide the result by 2, and the remainder is written in the ones place (in the sense of Piṅgala). The result is again divided by 2, its remainder written in the next place, i.e., to the right of the previous entry. This process is repeated until the number becomes zero. Reading the sequence of remainders from the top to the bottom gives the corresponding Piṅgala binary number. If the count of remainders obtained this way is less in the number of *verṇas*, then that many zeros should be postfixed while reading from top to bottom. For example, let's convert 1756 to a Piṅgala binary number. Subtracting one gives $1756 - 1 = 1755$.

Operation	**Remainder**
$1755 \div 2 = 877$	1
$877 \div 2 = 438$	1
$438 \div 2 = 219$	0
$219 \div 2 = 109$	1
$109 \div 2 = 54$	1
$54 \div 2 = 27$	0
$27 \div 2 = 13$	1
$13 \div 2 = 6$	1
$6 \div 2 = 3$	0
$3 \div 2 = 1$	1
$1 \div 2 = 0$	1

Reading the sequence of remainders from top to bottom gives the Piṅgala binary numeral 11011011011. Since the count obtained for the sequence is 11, to represent in the 12 *verṇas*, place a zero (0) at the end, giving the final result as 110110110110. So the final outcome is | | ऽ | | ऽ | | ऽ | | ऽ.

Example N3: Consider another example to convert 12 to a Piṅgala binary number of 4 *verṇas*, i.e., of 4 bits. Subtracting 1 gives $12 - 1 = 11$. Reading

Operation	**Remainder**
$11 \div 2 = 5$	1
$5 \div 2 = 2$	1
$2 \div 2 = 1$	0
$1 \div 2 = 0$	1

the sequence of remainders from top to bottom gives the binary numeral 1101. So the final outcome is | | ऽ |.

3.3 *Uddiṣṭa*: Conversion from a Piṅgala Binary Number to Decimals

Method I

In the context of metrical analysis, *uddiṣṭa* is the decimal equivalent of a (Piṅgala) binary number. This is discussed by Piṅgala (cf. formulas 8.26 and

8.27 in [3]). We give the literal meaning of these formulas for the sake of original taste. In order to know the *uddiṣṭa*, i.e., the desired number (corresponding to a particular meter having a certain syllabic arrangement of bits in the listing of meters), first write the (Piṅgala) binary number, and then proceed from right to left. Write 2 beneath the first syllable if the same is short (।), or 1 (i.e., $2 - 1$) if the same is long (ऽ). Next, multiply the first (numerical) entry (viz. 2 or 1 as the case may be) by 2, and write the product beneath the next syllable if the same is short, otherwise, subtract 1 from the product and write the outcome beneath the second syllable as the second entry. Continue this process until the last syllable. The final outcome is the *uddiṣṭa*, that is, the desired number.

Evidently, this formula gives **an ingenious quick method to convert a** Piṅgala **binary number into its decimal equivalent**. The following examples illustrate the method intended by the formulas.

(a)	0	0	1	1				
	ऽ	ऽ	।	।				Its place is 13th in the list of
	13	7	4	2				meters with four *varṇas* or bits.
(b)	1	1	0	1	0			
	।	।	ऽ	।	ऽ			Its place is 12th in the list of
	12	6	3	2	1			meters with five *varṇas* or bits.
(c)	0	0	0	0	0	0	1	
	ऽ	ऽ	ऽ	ऽ	ऽ	ऽ	।	Its place is 65th in the
	65	33	17	9	5	3	2	list of meters with seven bits.

Method II

In order to convert a Piṅgala binary number into its decimal equivalent, we write 1, 2, 4, 8, 16, 32, etc. beneath the Piṅgala binary codes, and discard the numbers that are beneath the long ones, and add the undiscarded numbers. The outcome plus one gives the decimal equivalent.

For example, consider a stanza (cf. Example 2.2) having 12 *varṇas*, the following scheme explains the method:

1	1	0	1	1	0	1	1	0	1	1	0
।	।	ऽ	।	।	ऽ	।	।	ऽ	।	।	ऽ
1	2	4	8	16	32	64	128	256	512	1024	2048
		×			×			×			×

Sum of the undiscarded numbers

$$= 1 + 2 + 8 + 16 + 64 + 128 + 512 + 1024 \ = \ 1755.$$

Hence the decimal equivalent is $1755 + 1 \ = \ 1756$.

Further, Piṅgala (see [3]) says that 2048 multiplied by 2, that is 2^{12}, will be the number of all possible even matrices of order 4×12. The conversion rules discussed above may easily be verified for even matrices of order $4 \times n$, $n = 1, 2, 3, 4$, from Tables 2–5 given above. As regards the *uddiṣṭa*, we give a **comparative chart** of calculation by both methods. For the sake of simplicity, consider examples (a), (b), and (c) above:

(a)	0	0	1	1				
	ऽ	ऽ	।	।				
	13	7	4	2				⇐ Superfast **method I**
	1	2	4	8				⇒ **Method II**
	×	×						The final answer is $(4+8)+1 = 13$
(b)	1	1	0	1	0			
	।	।	ऽ	।	ऽ			
	12	6	3	2	1			⇐ Superfast **method I**
	1	2	4	8	16			⇒ **Method II**
			×		×			The final answer is $(1+2+8)+1 = 12$
(c)	0	0	0	0	0	0	1	
	ऽ	ऽ	ऽ	ऽ	ऽ	ऽ	।	
	65	33	17	9	5	3	2	⇐ Superfast **method I**
	1	2	4	8	16	32	64	⇒ **Method II**
	×	×	×	×	×	×		The final answer is $64+1 = 65$

We remark that Method II is available in current textbooks of computer science, while the superfast method to find the *uddiṣṭa* is not found in modern mathematics and computer science. Evidently, the advantage that it has over Method II for persons using slates or dust boards, as in classical times, for computation, no numbers need to be stored for a final outcome.

4 Concluding Remarks

Gottfried Leibniz (1646–1716) documented fully the modern binary number system in the eighteenth century. Leibniz's system used 0 and 1, like the modern binary numeral system. However, in 1854, the British mathematician George Boole published a landmark work detailing a system of logic, now popularly called Boolean algebra. His logical system was responsible for the development of the binary system, particularly in its implementation in electronic theory (see, for instance, Eves [1]). Post-Piṅgala works on *Chandaśāstra* usually discuss the applications of Piṅgala binary numbers in metrical analysis. Variants of *Varṇic Meru* (popularly called Pascal's triangle) are widely discussed in relation to various aspects of *vṛtta* and *jāti* meters. It seems that *Meru* and some of its variants were very well known among the ancient prosodists, and the binary system was not very popular among students. This assertion is based on the fact that almost every commentary on *Piṅgala's Chandaśāstram* gives fundamental details of the binary system, while not many details of *Meru* are given, presumably because the relationship between binomial expansions and *Varṇic Meru* was taught to them at school. This view is also supported by the fact that textbooks dealing with school mathematics used to incorporate a discussion on *Meru*. For example, one may refer to the ninth-century Jain mathematician *Mahāvīrācārya's* magnum opus *Gaṇita Sāra Saṅgrha* (Compilation of the Essence of Mathematics) and *Bhāskarācārya's Līlāvatī* (composed in 1150 C.E.) (see [5]). As regards the invention of binary numbers, nothing definitive can be said. We can say only that knowledge of metrics was considered essential for the study of the Vedas. According to Subhas Kak, extant Vedas date back (at least) to 8000 B.C.E. (see Pearce [8]). Compositions on prosody before Piṅgala are not available. However, it seems that Vedic and ancient prosodists had considerable knowledge about computational mathematics. For a discussion on the concept of zero in prosody, one may refer to Sarma [6]. There remains much work on the mathematical aspect of prosody to be done.

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The Reception of Ancient Indian Mathematics by Western Historians

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1 The Context of Renaissance Humanism

Western reception of ancient Indian mathematics during the nineteenth century is very biased by the humanist tradition. Reflections and statements of Western historians on Indian mathematics can be fully understood only if this context is known and acknowledged.

During the Middle Ages, mathematics was hardly practised or appreciated by the intellectual elite. The Middle Ages knew two traditions of mathematical practice. On the one hand, there was the scholarly tradition of arithmetic theory, taught at universities as part of the *quadrivium*. The basic text on arithmetic, presented as one of the seven liberal arts, was Boethius's *De Institutione Arithmetica* [12]. The Boethian arithmetic strongly relies on Nichomachus of Gerasa's *Arithmetica* from the second century [29] This basically qualitative arithmetic deals with properties of numbers and ratios. All ratios have a name, and operations or propositions on ratios are expressed in a purely rhetorical form. The qualitative aspect is well illustrated by the following proposition from Jordanus de Nemore's *De Elementis Arithmeticae Artis* (c. 1250, Book IX, Proposition LXXI; [3], p. 199):

> Datis superparticularibus vel multiplicibus superparticularibus multiplices superparticulares et superpartientes et datis superpartientibus aut multiplicibus superpartientibus superpartientes et multiplices superpartientes procreare.

A superparticular has the form $\frac{n+1}{n}$ and thus covers proportions such as the common sesquialter (3/2) and sesquiter (4/3) proportions; a superpartient proportion has the form $\frac{m+n}{n}$ with $m > 1$ and includes proportions such as 8/3. The proposition describes how to create multiple superparticular proportions from a given one. As may be clear from this example, treated in the

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most extensive treatise of the period, arithmetic served little practical purpose and was not applied outside monasteries and universities. It was intended mainly for aesthetic and intellectual pursuit. During the eleventh century, a board game named *Rhythmomachia* was designed to meet these aesthetic aspirations. Originated as the subject of a competition on the knowledge of Boethian arithmetic among cathedral schools in Germany [1], the game was played until the sixteenth century, when the arithmetic tradition passed into oblivion. Despite its limited applicability, Boethian arithmetic evolved into a specific kind of mathematics, typical for the European Middle Ages, and left its mark on early natural philosophy. Carl Boyer's book on the history of calculus demonstrates how fourteenth-century thinkers such as Bradwardine and Richard Suiseth developed ideas on continuity and acceleration within this framework that influenced the later development of mathematics and natural philosophy [2, Chap. 3].

A second tradition concerned arithmetical problem solving, of which Alcuin's *Propositiones ad Acuendos Juvenes* (Propositions for Sharpening Youths) from the ninth century provides us with an extant witness. This collection contains 53 problems, many of which are repeated over and over in medieval and Renaissance works. Translations are quite recent. Folkerts [11] translated Alcuin into German. Hadley provided an English translation, annotated by Singmaster [34]. As the title suggests, the problems were to be used for educational purposes and to be read aloud, copied, and solved by students. Arithmetical problem-solving became much more advanced with the introduction of Arabic algebra through the Latin translations of al-Khwārizmī's *Algebra* by Robert of Chester (c. 1145), Gerard of Cremona (c. 1150), Guglielmo de Lunis (c. 1215). With the possible exception of Jean de Murs's *Quadripartitum numerorum* at the Sorbonne (1343, [20]), algebra was not practiced or even spoken about at universities for the next three centuries. However, algebra flourished and continuously developed within the vernacular tradition of abacus schools in fourteenth and fifteenth century Italy. Algebra was not only a foreign invention by its Arabic origin, it was also completely foreign to the scholarly tradition.

During the fifteenth century, Italian humanists eagerly started collecting editions of Greek mathematics. One of the most industrious was Cardinal Bessarion, who lived in Venice. By the time of his death in 1472, he had accumulated over 500 Greek manuscripts [32, pp. 44–46 and 90–109]. Regiomontanus, who had befriended Bessarion, began to study these Greek texts around 1463, including Diophantus's *Arithmetica*. He reported his find of the six books of the *Arithmetica* in a letter to Giovanni Bianchini [9, pp. 256–257]. By then he was well acquainted with Arabic algebra. He owned a copy of the manuscript on algebra by al-Khwārizmī, possibly from his own pen (MS. Plimpton, 188). Highly receptive to influences between traditions, he immediately conjectured a relation. In his *Oratio*, a series of lectures at the University of Padua in 1464, he introduced the idea that Arabic algebra descended from Diophantus's *Arithmetica* [28]. This heralded the initiation of a myth cultivated by humanists for centuries. Diophantus, first considered to be the source of inspiration for Arabic algebra, became the alleged origin of European algebra. Several humanist writers, such as Ramus, chose to neglect or reject the Arabic roots of Renaissance algebra altogether [18]. As a matter of fact, Diophantus had almost no impact on European mathematical practice before the late sixteenth century. Diophantus inspired authors on algebra such as Stevin, Bombelli, and Viète, because by then symbolic algebra was well established. By overrating the importance of Diophantus and downgrading the achievements of Arabic algebra, humanist writers created a new mythical identity of European mathematics. Suddenly Greek mathematics became European mathematics. However, most Greek sources were unavailable before the sixteenth century. In fact, Greek mathematics was more foreign to the European mathematical practice than Arabic mathematics was; the latter was slowly but surely appropriated with the abacus tradition. Ironically, the medieval qualitative arithmetic, which was a genuine European tradition, became completely forgotten.

Only later did European historians learn about ancient Indian mathematics, and what they learned was strongly influenced by the humanist mathematical tradition. We will now give a brief overview of the first assessments of Indian algebra in the West.

2 The First Descriptions of Indian Algebra

In some sense, Wallis's *Treatise on Algebra* (1685) can be considered the first serious historical investigation of the history of algebra. John Wallis was well informed about Arabic writings through Vossius and was one of the first to attribute, correctly, the name algebra to *al-jābr* in *Kitāb fī al-jābr wa'l-muqābala.* He also pointed out the mistaken origin of algebra as Geber's name, which was a common misconception before the seventeenth century [44, p. 5]. Unprecedented, Wallis casts doubts on Diophantus's contribution to modern algebra. He even launched the idea that Arabic algebra may have originated from India [44, p. 4]:

> However, it is not unlikely that the Arabs, who received from the Indians the numeral figures (which the Greeks knew not), did from them also receive the use of them, and many profound speculations concerning them, which neither Latins nor Greeks know, till that now of late we have learned them from thence. From the Indians also they might learn their algebra, rather than from Diophantus.

So, while in the seventeenth century no Sanskrit mathematics had yet been introduced into Europe, scholars by then were aware of the existence of Indian algebra. Wallis's view persisted in eighteenth-century historical studies, which reiterated the influence of Indian mathematics. Pietro Cossali, who wrote an extensive monograph on the history of algebra, concluded his discussion on al-Khwārizmī's *Algebra* with al-Khwārizmī "not having taken algebra from the

Greeks,... must have either invented it himself, or taken it from the Indians. Of the two, the second appears to me the most probable" [8, I, pp. 216–219] Hutton, who included a long entry on algebra in his *Mathematical and Philosophical Dictionary*, wrote [19, I, p. 66]:

> But although Diophantus was the first author on algebra that we know of, it was not from him, but from the Moors or Arabians that we received the knowledge of algebra in Europe, as well as that of most other sciences. And it is matter of dispute who were the first inventors of it; some ascribing the invention to the Greeks, while others say that the Arabians had it from the Persians, and these from the Indians.

In the early nineteenth century, the English orientalist, Henry Thomas Colebrooke, who previously published his *Sanskrit Grammar* (1805), undertook the task of translating three classics of Indian mathematics, the *Brahmasphuta Siddhānta* of Brahmagupta (628) and the *Līlāvatī* and the *Bījagaṇita* of Bhāskara II (1150) [7]. At once European historians had something to reflect upon. In a period when mathematics was hardly practised in Europe and in the Islamic regions, there appeared to have existed an Indian tradition in which algebraic problems were solved with multiple unknowns, in which zero and negative quantities were accepted, and in which sophisticated methods were used to solve indeterminate problems. In general, nineteenth-century historians showed an admiration for the Hindu tradition. However, whenever explanations were required, scholars were divided into two opposing camps, which we could call the believers and the nonbelievers. Nonbelievers did not grant Indian mathematicians the status of original thought. Indian knowledge must have stemmed from the Greeks, the cradle of Western mathematics, or even mathematics as such. The major nonbeliever was Moritz Cantor, who published an influential four-volume work on the history of mathematics (1880–1908). Cantor [5, II] takes every opportunity to point out the Greek influences on Hindu algebra. Some examples: the Indians learned algebra through traces of algebra within Greek geometry ("Spuren griechischer Algebra müssen mit griechischer Geometrie nach Indien gedrungen sein und werden sich dort nachweisen lassen," [5, II, p. 562]); Brahmagupta's solution to quadratic equations has Greek origins ("So glauben wir auch deutlich die griechische Auflösung der quadratischen Gleichung, wie Heron, wie Diophant sie übte, in der mit ihr nicht bloss zufällig übereinstimmenden Regel des Brahmagupta zu erkennen," [5, II, p. 584]); or the *Epanthema* as discussed below.

The believers were not convinced by accidental resemblances between Greek and Hindu solution methods and did not see why Indian mathematics could not have been an independent development. In particular, Hankel [13, p. 204] touches the sore spot when he writes:

> That humanist education's deeply inculcated prejudice that all higher intellectual culture in the Orient, in particular all science, is risen from Greek soil and that the only mentally, truly productive people have been the Greeks, makes it difficult for us to turn around the direction of influence for

> one instant. (Das uns durch die humanistische Erziehung tief eingeprägte Vorurtheil, dass alle höhere geistige Cultur im Orient, insbesondere alle Wissenschaft aus griechischem Boden entsprungen und das einzige geistig wahrhaft productive Volk das griechische gewesen sei, kann uns zwar einen Augenblick geneigt machen, das Verhältniss umzukehren [my translation]).

Soon after Kern [23] published the Sanskrit edition of the *Āryabhaṭīya* (*AB*), the French orientalist, Léon Rodet was the first to provide a translation in a Western language (1877, published in Rodet [30]). Rodet wrote several articles and monographs on Indian mathematics and its relation with earlier and later developments in the Arab and Western world, published in the French *Journal Asiatiques*. He is the scholar who displays the most balanced and subtle views on the relations between traditions. In particular, his appraisal of Hindu and Arabic algebra as two independent traditions is still of value today (see [16] for an assessment). He certainly was a believer. Concerning Āryabhaṭa's inadequate approximation of the volume of a sphere (Prop. 7), he writes somewhat cynically that if Āryabhaṭa got his knowledge from the Greeks, then apparently he chose to ignore Archimedes ("Mais elle a, pour l'histoire des mathématiques, d'autant plus de valeur, parce qu'elle nous démonstre que si Āryabhaṭa avait reçu quelque enseignement des Grecs, il ignorait au moins les travaux d'Archimède," [31, p. 409]).

George Thibaut, who translated several Sanskrit works on astronomy, such as Varāhamihira's *Pañcasiddhāntikā* (1889), also wrote an article on Indian mathematics and astronomy in the *Encylopedia of Indo-Aryan Research*. Concerning influences from Greek mathematics, he takes a middle position. In discussing Hindu algebra he writes that "in all these correspondences does Indian algebra surpass Diophantus?" ("In allen diesen Beziehungen erhebt sich die indische algebra erheblich über das von Diophant Geleistete" [38, p. 73]). As on the origins of Indian mathematics, he points out that Indian algebra, especially indeterminate analysis, is closely intertwined with its astronomy. As he argued on the Greek roots of Indian "scientific" astronomy, his evaluation is that Indian mathematics is influenced by the Greeks through astronomy. However, he adds that several arithmetical and algebraic methods are truly Indian [38, pp. 76–78].

Despite the existence of several studies and opinions that should provide sufficient counterbalance for Cantor's position as a nonbeliever, his views remained influential well into the twentieth century. We may say that the "humanist prejudice" is still alive today. The myth that Greek mathematics is our (Western) mathematics has become intertwined with our cultural identity so strongly, that it becomes difficult to understand intellectual achievements within mathematics foreign to the Greek tradition.

We will now look in detail at an example that has been one of the main arguments for the advocates of Greek influence. The example clearly shows how historical investigation can be misled through prejudice.

3 A Case Study: The Bloom of Thymaridas

We have demonstrated elsewhere that if there is an influence between Indian algebra and European arithmetic, it should be situated on the level of protoalgebraic solution recipes, orally disseminated through riddles and recreational problems [15]. One interesting example in this respect is a class of determinate linear problems in which the partial sums are given and the individual quantities are unknown. We found strong similarities in the rules for solving this type of problem both in Hindu algebra and in Renaissance arithmetic. These rules have a special interest for our discussion, since we have both a Greek and a Hindu tradition of their use. There has been a controversy about the possible influence of Greek mathematics on Indian algebra, as defended by Cantor and Kaye and disputed by Rodet. We will here shed more light on the controversy and explain the dispute as a misunderstanding of the rule. We will demonstrate in detail that the Greek and Indian versions are in fact two different rules and that the alleged influence from Greece to India is therefore highly disputable.

3.1 The Original Formulation in Hindu Sources

The first Indian source for a formulation of this rule is from Āryabhaṭa I, 499, (*AB*, ii, 29; [6, p. 40]) as follows:

> If you know the results obtained by subtracting successively from a sum of quantities, each one of these quantities set these results down separately. Add them all together and divide by the number of terms less one. The result will be the sum of all the quantities.

The rule is somewhat obscure and difficult to understand without examples, but some observations can be drawn from the formulation that are central to our further discussion. Firstly, the rule is valid for any number of quantities. It is not limited to two or three quantities. Secondly, the sum of all the quantities is unknown and is provided by the rule. Furthermore, and not evident from the rule as cited above, is that the partial sums relate to the total of all the quantities, except one. In modern symbolism the general structure of the problem thus is as follows:

Suppose n amounts $(a_1, a_2, \ldots, a_n)$ with unknown sum S and with the partial sums $(s_1, s_2, \ldots, s_n)$ are given, where $s_i = S - a_i$. Then

$$S = \frac{1}{n-1} \sum_{i=1}^{n} s_i.$$

The rule and the problems it applies to should not be confused with a similar problem in which the partial sums of two consecutive quantities are given. For three numbers, the problems are evidently the same, but they diverge for more than three quantities. For example, for five quantities the corresponding equations are

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= s_1, \\ a_1 + a_3 + a_4 + a_5 &= s_2, \\ a_1 + a_2 + a_4 + a_5 &= s_3, \\ a_1 + a_2 + a_3 + a_5 &= s_4, \\ a_2 + a_3 + a_4 + a_5 &= s_5, \end{aligned} \quad \text{and} \quad \begin{aligned} a_1 + a_2 &= s_1, \\ a_2 + a_3 &= s_2, \\ a_3 + a_4 &= s_3, \\ a_4 + a_5 &= s_4, \\ a_5 + a_1 &= s_5. \end{aligned}$$

Let us apply the rule to a simple problem (not discussed by Āryabhaṭa) that can be formulated symbolically as

$$\begin{aligned} x_1 + x_2 &= 13, \\ x_2 + x_3 &= 14, \\ x_1 + x_3 &= 15. \end{aligned}$$

Applying Āryabhaṭa's rule, the solution would be based on the rule for deriving the sum of all three unknown quantities as follows:

$$x_1 + x_2 + x_3 = \frac{13 + 14 + 15}{3 - 1} = 21.$$

This allows us to determine the value of the quantities by subtracting the partial sums from the total with the solution (7, 6, 8). A commentator of the *Āryabhaṭiya*, called Bhāskara I (written 629, not to be confused with Bhāskara II), gives two examples of problems that can be solved with Āryabhaṭa's rule with the partial sums (30, 36, 49, 50) and (28, 27, 26, 25, 24, 23, 21) [33, pp. 307–308].

3.2 The Derived Problem in Hindu Sources

From the ninth century we find a derived version of the previous problem in Hindu sources. Mahāvīra gives an elaborate description of the rule in the *Gaṇitasārasaṃgraha* (*GSS*, stanzas 233–235, [26], 357–359) which we here reproduce:

> The rule for arriving at [the value of the money contents of] a purse which [when added to what is on hand with each of certain persons] becomes a specified multiple [of the sum of what is on hand with the others]:
> The quantities obtained by adding one to [each of the specified] multiple numbers [in the problem and then] multiplying these sums with each other, giving up in each case the sum relating to the particular specified multiple, are to be reduced to their lowest terms by the removal of common factors. [These reduced quantities are then] to be added. [Thereafter] the square root [of this resulting sum] is to be obtained, from which one is [to be subsequently] subtracted. Then the reduced quantities referred to above are to

> be multiplied by [this] square root as diminished by one. After this, these are to be separately subtracted from the sum of those same reduced quantities. Thus the moneys on hand with each [of the several persons] are arrived at. These [quantities measuring the moneys on hand] have to be added to one another, excluding from the addition in each case the value of the money on the hand of one of the persons and the several sums so obtained are to be written down separately. These are [to be respectively] multiplied by [the specified] multiple quantities [mentioned above]; from the several products so obtained the [already found out] values of the moneys on hand are [to be separately subtracted]. Then the [same] value of the money in the purse is obtained [separately in relation to each of the several moneys on hand].

The introductory sentence states that the rule is to be used for determining the value of a purse. The rule is followed by a number of problems that begin as "Four men saw on their way a purse containing money" (ibid. stanzas $245\frac{1}{2}$, 367). This is the earliest instance, in our investigation of the sources, in which the popular problem of men finding a purse is discussed. While problems with the same structure and numerical values have been formulated before, the context of men finding a purse seems to have originated in India before 850 C.E. Formulations with the purse turn up in Arabic algebra with al-Karkhī's *Fakhrī* (c. 1050) and in the *Miftāh al-mu`āmalāt* of al-Tabari (c. 1075). Fibonacci has many variations of it in the *Liber Abaci* (1202) and after that it becomes the most common problem in Western arithmetic until the later sixteenth century. For an understanding of the rule, let us look at its application to a given problem (*GSS*, [26], p. 360):

> Three merchants saw [dropped] on the way a purse [containing money]. One [of them] said [to the others], "If I secure this purse, I shall become twice as rich as both of you with your moneys on hand." Then the second [of them] said, "I shall become three times as rich." Then the other, [the third], said, "I shall become five times as rich." What is the value of the money in the purse, as also the money on hand [with each of the three merchants]?

We can represent the problem in symbolic equations as follows:

$$\begin{aligned} x + p &= 2(y + z), \\ y + p &= 3(x + z), \\ z + p &= 5(x + y). \end{aligned}$$

Let us apply the recipe of Mahāvira to this problem, step by step. By "adding one to [each of the specified] multiple numbers", we have 3, 4, and 6. "Multiplying these sums with each other" we get 72. This has to be "reduced to their lowest terms by the removal of common factors." This least common multiple is 12. The reduced quantities are then 4, 3, and 2 respectively. Adding these together gives 9. From this the square root is 3. Then the reduced quantities "are to be multiplied by the square root as diminished by one," which is 2.

This leads to 8, 6, and 4. The money on hand for each of the merchants now is the difference of these values with the sum of the reduced quantities, being 9. The solution thus is 1, 3, and 5. The rest of the rule is an elaborate way to derive the value of the purse. Using the values in any one of the equations immediately leads to 15 for the value of the purse. Mahāvīra provides no explanation or derivation of the rule. For a mathematical argument for the validity of the rules see Heeffer [15].

3.3 The Problem in Greek Sources

3.3.1 The Bloom of Thymaridas

We know almost nothing about Thymaridas of Paros, but he is supposed to have lived between 400 and 350 B.C.E. [36, pp. 385–386]). The only extant witness is Iamblichus, in his comments on the *Introduction to Arithmetic* by Nichomachus of Gerasa. The best known source for *The Bloom of Thymaridas* is Heath's classic on Greek mathematics. Heath [14, p. 94] does not formulate the rule, he only observes that "the rule is very obscurely worded" and writes out the equations. The text from Iamblichus was first published in Holland with a Latin translation by Tennulius [37] from the Paris manuscript BNF Gr. 2093. A critical edition, based on multiple manuscripts, was published by Pistelli [27]. Nesselmann [25, p. 233] quotes the Greek text and the Latin translation from Tennulius, who translated the method as *florida sententia*. We give here our own literal translation from Pistelli [27, p. 62]:

> From this we are also acquainted with the method of the Epanthema, passed down to us by Thymaridas. Indeed, when a given quantity divides into determined and unknown parts, and the unknown quantity is paired with each of the others, so will the sum of these pairs, diminished by the sum [of all the quantities] be equal to the unknown quantity in case of three quantities. With four quantities it will be half of it, with five it will be a third, with six, a fourth, and so on.

The rule is not as obscure as considered by Heath. Let us extract the basic elements of the rule, and compare these with the version of Āryabhaṭa:

- The rule applies to any number of quantities, as does Āryabhaṭa's.
- The sum is given in the problem. The rule is described as the division of a known quantity in determined and undetermined parts. In Aryabhaṭa's rule the sum is what is looked for.
- The partial sums are the sums of the pairs of the unknown part with each of the known quantities. In Āryabhaṭa's rule the partial sums include all the numbers except one.

In short, this rule is different from Āryabhaṭa's in two important aspects. Its intention is to find one unknown part of a determined quantity. Āryabhaṭa's rule is meant for finding the sum of numbers of which the partial sums of all minus one is given. Even in the case of three numbers, when the partial sums are the same, the rules have different applications. To make it clear to the modern eye, here is a symbolic version in the general case:

$$\left.\begin{cases} x + a_1 + a_2 + \ldots + a_{n-1} = s \\ x + a_1 = s_1 \\ x + a_2 = s_2 \\ \vdots \\ x + a_{n-1} = s_{n-1} \end{cases}\right\} \quad x = \frac{1}{n-2}\sum_{i=1}^{n-1} s_i - s$$

3.3.2 Diophantus

In the first book of the *Arithmetica* of Diophantus, we find four instances of the problem type. Problems 16 and 17 are of the original type as covered by Āryabhaṭa's rule. Let us first look at problem 17 with four unknown quantities. We use Ver Eecke [43, p. 22] as the best translation of the *Arithmetica*:

> Trouver quatre nombres qui, additionnés trois à trois, forment des nombres proposés. Il faut toutefois que le tiers de la somme des quatre nombres soit plus grand que chacun d'eux. Proposons donc que les trois nombres, additionnés à la suite à partir du premier, forment 20 unités; que les trois à partir du second forment 22 unités, que les trois à partir du troisième forment 24 unités, et que les trois à partir du quatrième forment 27 unités.

In modern symbolism, the problem reads as follows:

$$\begin{aligned} a + b + c &= 20, \\ b + c + d &= 22, \\ a + c + d &= 24, \\ a + b + d &= 27. \end{aligned}$$

Diophantus's solution is not based on a protoalgebraic rule but has all the characteristics of algebra. He uses the *arithmos* as an abstract quantity for the unknown, to represent the sum of the four quantities [43, p. 22]:

> Posons que la somme des quatre nombres est 1 arithme. Dès lors, si nous retranchons les trois premiers nombres, c'est-à-dire 20 unités, de 1 arithme, il nous restera, comme quatrième nombre, 1 arithme moins 20 unités. Pour les mêmes raisons, le premier nombre sera 1 arithme moins 22 unités; lé second sera 1 arithme moins 24 unités, et le troisième 1 arithme moins 27 unités. Il faut enfin que les quatre nombres additionnés deviennent égaux à 1 arithme. Mais, les quatre nombres additionnés forment 4 arithmes moins 93 unités; ce que nous égalons à 1 arithme, et l'arithme devient 31 unités.

If $a + b + c + d = x$, then the four numbers not included in the partial sums are $x - 20$, $x - 22$, $x - 24$, and $x - 27$ respectively. Adding these four together is equal to their sum or x; thus $4x - 93 = x$ and $x = 31$. This problem in the *Arithmetica* is followed by problems 18 and 19, of a related type, but not the one covered by Mahāvīra's formulation. We show here only the symbolic translation of problem 19:

$$\begin{aligned} a+b+c &= d+20, \\ b+c+d &= a+30, \\ a+c+d &= b+40, \\ a+b+d &= c+50. \end{aligned}$$

The solution is similar to the previous problem but depends on the choice of $2x$ for the sum of the four numbers.

3.3.3 The Extended Rule from Iamblichus

Iamblichus extends the rule of Thymaridas to another problem type that was to become very popular during the following centuries. In modern symbolism this amounts to the set of equations

$$x+p = a(y+z), \tag{1}$$
$$y+p = b(x+z), \tag{2}$$
$$z+p = c(x+y). \tag{3}$$

Iamblichus gives two examples of the problem. The first example can be formulated symbolically as follows. Nesselmann [25, pp. 234–235] gives the literal German translation from the Greek. We will follow Nesselmann's rather than Heath's reconstruction:

$$\begin{aligned} a+b &= 2(c+d), \\ a+c &= 3(b+d), \\ a+d &= 4(b+c), \\ a+b+c+d &= 5(b+c). \end{aligned}$$

The problem is formulated in a way that reminds us of Diophantus: "Find four numbers such that..." Although Diophantus's *Arithmetica* has no problems like this, problems 18–20 of the first book are variations on the original *Epanthema* problem. Iamblichus's own variation is in some way analogous to the versions of the *Arithmetica* and might be influenced by it. However, while Diophantus's solution is algebraic, this one depends on a protoalgebraic rule. The fourth expression in the problem formulation is superfluous and is recognized as such by Iamblichus, where he adds, "this follows directly from the previous statements." It is added to facilitate the application of the rule. The procedure is explained by Iamblichus in three steps:

(1) Set the sum of the four numbers equal to the number found by multiplying the four factors together. Thus $2\;.\;3\;.\;4\;.\;5 = 120$.
Iamblichus does not explain why this is necessary, but it can be demonstrated in the following way: Completing the left side of the equations (1)–(3) to the sum of the four numbers, we arrive at

$$\begin{aligned} x+y+z+p &= (a+1)(y+z), \\ x+y+z+p &= (b+1)(x+z), \\ x+y+z+p &= (c+1)(x+y). \end{aligned}$$

Therefore, the sum of the four integers must be divisible by $(a+1)$, $(b+1)$, and $(c+1)$. This can be represented by means of the least common multiple s. Now, Iamblichus does not use s but $2s$ for a reason that will become apparent later. In the example, the least common multiple is 60, therefore $2s$ is 120. So, let us suppose that $x+y+z+p = 2s$.

(2) The sum of each pair can be found by taking $\frac{a}{a+1}$, $\frac{b}{b+1}$, and $\frac{c}{c+1}$ from the sum $2s$ respectively. This becomes apparent from:

$$\begin{aligned} x+p &= a(y+z), \\ (a+1)(x+p) &= a(x+y+z+p). \end{aligned}$$

The three sums $(x+p)$, $(y+p)$, and $(z+p)$ in the example become $80, 90$, and 96.

(3) Only now does Iamblichus refer to the use of the *Epanthema* rule. Indeed, we have the partial sums $(x+p), (y+p), (z+p)$ and we have the total sum $2s$. The *Epanthema* therefore determines the common part p as follows:

$$p = \frac{(x+p)+(y+p)+(z+p)-2s}{2},$$

or

$$p = \frac{80+90+96-120}{2} = 73,$$

which leads to the other values as 7, 17, and 23. The reason why Iamblichus used $2s$ instead of the least common multiple s is that s would lead to the nonintegral solution

$$p = \frac{40+45+48-60}{2} = 36\frac{1}{2}.$$

In summary, we discern two important factors that are relevant for the understanding of the controversy that follows.

(1) Our only source for the *Epanthema* is Iamblichus. There are at least six centuries between Thymaridas and the extant witness. In the absence of any written source, we should consider Iamblichus's discussion of the method as a late interpretation of the Pythagorean number theory.

The formulation of the rule with determined and unknown quantities suits the context of third-century Greek analysis better than it would fit in Pythagorean number mysticism.

(2) The extended problem, which has become known as the problem of men finding a purse, is in itself quite different from the original problem to which the *Epanthema* rule applies. The problem, devised by Iamblichus, could be considered a variation like several others in the *Arithmetica* of Diophantus. Iamblichus gives the rules to reduce the problem to a form in which the *Epanthema* can be used. This distinction is important because many have wrongly identified the men-find-a-purse problem with the *Bloom of Thymaridas*.

3.3.4 The Controversy

We now come to the discussion on the relevance of the *Epanthema* method and the controversy about the influences on and from Indian mathematics. Since there are two aspects of the discussion, we will deal with the issues separately. Firstly, we address the historical question of the main source of the men-find-a-purse problem. Secondly, we discuss the more philosophical question of the relevance of the *Bloom* on the conceptual development of algebra.

3.3.4.1 The Origin of Linear Problems of Men Finding a Purse

Nesselmann [25] refrains from comments on the *Bloom of Thymaridas* in his *Algebra of the Greeks*. He treats the method with full respect for the extant Greek text by Iamblichus. After Nesselmann, the problem was discussed, by several scholars, in relation to Hindu algebra. Rodet [31], in his French translation of Āryabhaṭa's treatise, does not mention the *Epanthema*. Rodet was no believer in the influence of Greek mathematics in Asia. We can assume that he did not discuss the *Epanthema* because, in his point of view, there simply is no relation to Āryabhaṭa's rule.

On the other hand, Cantor [5, II, p.584], after discussing Āryabhaṭa's stanza 29, observes, "We do not fear any disagreement, if in this problem and in the *Epanthema* of the Thymaridas, we recognize a relation which is so close that a coincidence is not imaginable" ("Wir fürchten keinen Widerspruch, wenn wir in dieser Aufgabe und in dem Epantheme des Thymaridas so nahe Verwandte erkenne, dass an einen Zufall nicht zu denken ist"). Citing Cantor and Heath, Kaye [22, p. 40, note 2] writes "The examples in the text are undoubtedly akin to the '*Epanthema.*' " Tropfke [41, p. 399] words it more sharply and considers the formulation of Āryabhata's stanza 29 "equivalent with the *Epanthema* of Thymaridas" and states that the *BM* "contains problems of the same sort." However, in the original edition, Tropfke [41, III, p. 42] is more prudent: "Āryabhaṭa bietet einige solcher Wortgleichungen, unter denen uns eine wegen ihren Ähnlichkeit mit dem Epanthem des Thymaridas ausfällt." Apparently it is Kurt Vogel, who edited the 1980 edition, who believes in a strong connection.

All the suppositions of the Greek influence are based solely on the alleged resemblance of the problems. As shown above, Āryabhaṭa's rule is very different from the *Epanthema*. The argument that both are equivalent is plainly false. The suggestion that the *Epanthema* provides evidence of an influence of Greek mathematics on Hindu algebra has very little substance. Instead, it seems that the argument is biased by normative beliefs about the superiority of Greek culture. Let us now proceed to the second question on conceptual influences.

3.3.4.2 A Case of Pythagorean Algebra?

This single problem, which became known to us through Iamblichus, six centuries after Thymaridas, has convinced many that Greek algebra originated with the Pythagoreans. After writing out the equations, Cantor [5, I, p. 148] concludes:

> This is, as one can see, all rhetorical algebra, in which only the symbols are missing in order to agree completely with the modern way of solving equations, and specifically the expressions of the given and unknown quantities was rightly emphasized. (Das ist, wie man sieht, vollständig gesprochene Algebra, welcher nur Symbole fehlen, um mit einer modernen Gleich ungsauflösung durchaus übereinzustimmen, und insbesondere ist mit Recht auf die beiden Kunstausdrücke der *gegebene* und *unbekannten* Grösse aufmerksam gemacht worden.)

Heath's interpretation is copied in many other works including Smith [35, p. 91], Cajori [4, p. 59], van der Waerden [42, p. 116], Flegg [10, p. 205], and Kaplan [21, p. 62]. Cajori finds in the Thymaridas "investigations of subjects which are really algebraic in their nature." Van der Waerden goes as far as to claim that "we see from this that the Pythagoreans, like the Babylonians, occupied themselves with the solution of systems of equations with more than one unknown." Instead, Klein [24, p. 36] sees in the problem an intent to "determine special relations between numbers" and places it as "the counterparts in the realm of 'pure' units of the computational problems proper to practical logic." We agree with Klein's interpretation. Even if Iamblichus's depiction of the problem from Thymaridas is faithful, the six centuries separating these two mathematicians require an interpretation that accounts for two different contexts. Pythagoreans were concerned with the properties of numbers and with the relations between numbers. Lacking any further evidence, we cannot attribute an algebraic interpretation to the Pythagorean number theory. On the other hand, in the context of the late Greek period of Diophantus and Iamblichus, an algebraic reading is warranted. Thus, the *Bloom* is an old number-theoretic problem, revived and extended in an algebraic context.

4 Conclusion: The Ground Was Wet Everywhere

The humanist project of reviving ancient Greek science and mathematics played a crucial role in the creation of an identity for the European intellectual tradition. While Greek mathematics was hardly known or practised before the fifteenth century, humanist mathematicians identified themselves with this tradition. When Regiomontanus declared that algebra was invented by Diophantus, humanist writers rejected the Arabic roots of algebra, though it was practised and turned into an independent tradition for two centuries in Italian cities such as Florence and Siena. The newly-created identity of mathematics descending from ancient Greek thinkers blurred historical perception. When Indian algebra and arithmetic were introduced into Europe, the leading historians of the nineteenth century could only see its alleged relation with Greek mathematics. The *Bloom of Thymaridas* is an excellent illustration of distorted historical investigation. Not only was it wrongly inferred that the Indian method for solving determined linear problems depended on Iamblichus, historians forced a connection between third century Greek analysis and Pythagorean number theory. The origin of the algebra of Diophantus still needs an explanation, but it is very doubtful that it is to be found in Pythagoras.

Apparently, nineteenth century historians found it difficult to accept that mathematics is a human intellectual activity practiced across cultures within societies that needed and supported the achievements of mathematical practice. A true history of mathematics should take into account contributions of all origins. Jens Høyrup, who studied the evolution and transmission of mathematics between cultures, formulates it as follows [17, p. 98]:

> Diophantos would use the rhetorical algebra, the Chinese *Nine Chapters on Arithmetic* would manipulate matrices, and the *Liber abbaci* would find the answer by means of proportions. We should hence not ask, as commonly done, whether Diophantos (or the Greek arithmetical environment) was the source of the Chinese or vice versa. There was *no specific source: The ground was wet everywhere.*

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The Indian Mathematical Tradition with Special Reference to Kerala: Methodology and Motivation

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1 Introduction

Indian mathematical tradition is found to be as old as Indian civilization, and some of the important mathematical concepts contributed by Indians are found to have roots running deep into the earliest strata of Indian scientific thought. Like every other science in India, mathematics too, is seen to have grown in close connection with religion, and the pace of progress in the discipline was in accordance with the rate at which the user of the discoveries was refining himself. Consequently, classical literature containing well-developed systems of metaphysical, social, and religious philosophies and arts of the ancient Indians also contains the seeds of various mathematical concepts developed by them to meet their religious and other needs. Through various casual references, some of the important mathematical concepts developed and acquired through the ages are brought to us through classical literature, and in this sense classical literature may be said to have rendered untold service to the progress of mathematical sciences, especially to the science of numerals. However, it was Āryabhaṭa I (born 476 C.E.) who systematized and synthesized the astronomical and mathematical knowledge acquired by him through oral tradition. His two masterly treatises, viz. the *Āryabhaṭasiddhānta* (known only through citations in later works) and the *Āryabhaṭīya* may be said to have opened the doors to a scientific approach to astronomy and mathematics. Several scholars of this period of systematization (fifth to twelfth century C.E) who contributed their invaluable share to the development of the twin sciences include Varāhamihira, Bhāskara I, Brahmagupta, Haridatta, Lalla, Skandasena, Śrīdhara, Mahāvīra, Govindasvāmin, Śaṅkaranārāyaṇa, Vaṭeśvara, Pṛthūḍakasvāmin, Āryabhaṭa II, Bhaṭṭotpala, Muñjāla, Śrīpati, Udayadivākara, Sūryadeva and Bhāskara II.

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After Bhāskara II, some influential scholars in various parts of the country continued their astronomical and mathematical activities and produced some important works. They include Thakkura Pheru (1265–1330 C.E.), of the court of the Delhi Sultanate, who composed the *Gaṇitasāra*; Nārāyaṇa Paṇḍita, who authored the *Gaṇitakaumudī* in 1356 C.E.; Mahendra Suri, who wrote the *Yantrarāja* in 1370 C.E.; Jñānarāja, who wrote the *Siddhāntasundara* in 1503 C.E.; Nityānanda, who authored the *Siddhāntarāja* in 1639 C.E.; Munīśvara, who authored the *Siddhānta-sarva-bhauma* in 1646 C.E.; Kamalākara, who composed the *Siddhānta Tattva Viveka* in 1658 C.E.; and Jagannātha Sāmrāt, who prepared a Sanskrit translation of Arabic version of Euclid's *Elements* entitled the *Rekhāgaṇita* in 1718 C.E. and another work in 1732 C.E. by the name *Siddhānta Sāmrāṭ*, which is a translation of Ptolemy's *Almagest*. During this post-Bhāskara II period there was a great spurt in astronomical and mathematical activity on the narrow strip of land called Kerala along the southwest coast of India. With a silent takeoff the twin sciences attained new heights through the hands of erudite scholars such as Saṅgamagrāma Mādhava (1340–1425), Vaṭaśreṇi Parameśvara (1360–1455), Gārgya Kerala Nīlakaṇṭha Somayāji (1444–1545), Jyeṣṭhadeva and Tṛkkuṭṭaveli Śaṅkara Vāriyar, both of sixteenth century, Acyuta Piṣāroṭi (1550–1621), Putumana Somayāji (1660–1740) and Śaṅkara Varman (1774–1839). Their contributions have a quite different flavor.

2 Some Significant Developments and Their Motivations

One branch that attained great heights through the hands of Kerala mathematicians is trigonometry, or "*jyotpatti*" (*jyā*+*utpatti* = source of *R*sines). Indian mathematicians made notable contributions in this field in general. The concept of *jyā* (*R*sines) is seen to have evolved from their astronomical needs such as those for computation of latitudes, position and movements of planets in their respective orbits on the stellar sphere, and so on. Referring to such astronomical needs, Nīlakaṇṭha Somayāji, in his *Golasāra*, briefly states why the concept of *jyā* should be known.[1] Owing its vast utility in astronomical computations, the concept of *jyā* continued to attract the attention of Indian astronomers and mathematicians. Introduction of the terminology *jyā* by Āryabhaṭa I in his *Āryabhaṭīya* along with his methods for computation marked the beginning of the scientific treatment of the concept. He set the tradition of using the radius value 3,438′ for computing tabular *R*sines, and this value was obtained by dividing the circumference of a circle into 21,600 parts. This practice was followed by most of his successors. However, the

[1] Sastry, Parameśvara: Transcript copy C.1024.E of the manuscript *Golasāra Siddhāntadarpaṇamca* of *Gārgya Kerala Nīlakaṇṭha* Mss No. T 846. B, (KUORI and Mss Library, Thiruvananthapuram), iii vs.2. Also ref. K. V. Sarma (ed.): *Golasāra*, Cr. Ed. with English Translation, VVRI, Hoshiarpur, (p. 14) 1970.

value of R was improved later by Vaṭeśvara to 3,437′44″ and further refined to 3,437′44″48‴ by Mādhava, which is equivalent to 1 radian in modern terms. Computation of sine tables and interpolation of sine values needed for astronomical calculations is found to be a general feature in Indian astronomy and mathematics. As such, several methods for construction of tabular sines, including the most innovative idea of using power series, along with techniques for interpolation of desired values using tabular values and differences can be had from various astronomical and mathematical works. In this context mention may be made of:

1. Govindasvāmin's second order difference interpolation formula in the form
$$f(x+nh) \;=\; f(x)+n\Delta f(x)+\frac{n(n-1)}{2}\{\Delta f(x)-\Delta f(x-h)\},$$
which occurs in his *Mahābhāskarīya bhāṣya*.[2]

2. Parameśvara's third-order interpolation formula stated in the *Siddhāntadīpika*, which is equivalent to Gregory's expansion of the third-order Taylor Series approximation for sine.[3]

3. Parameśvara's formula for inverse interpolation of sine stated in the *Siddhāntadīpika*, which is a forerunner of the mean value theorem of differential calculus.[4]

4. Mādhava's Taylor series for sine and cosine in the form:
$$f(x+nh) \;=\; f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots,$$
found in the *Āryabhaṭīya Bhāṣya*[5] of Nīlakaṇṭha Somayāji, and

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[2] Sastri, T. S. Kuppanna (ed.): *Mahābhāskarīya* of *Bhāskarācārya* with the *Bhāṣya* of *Govindasvamin* and the super commentary *Siddhāntadīpika* of *Parameśvara*, Madras, 1957 (iv. 22, pp. 201–202); For details ref. R. C. Gupta: Second Order Interpolation in Indian Mathematics up to the Fifteenth Century, *IJHS*, Vol. 4, (pp. 86–98) 1969.

[3] Sastri, T. S. Kuppanna (ed.): *Mahābhāskarīya* of *Bhāskarācārya* with the *Bhāṣya* of *Govindasvāmin* and the super commentary *Siddhāntadīpika* of *Parameśvara*, Madras, 1957 (iv. 22, p. 205, vs. 14–16); For details see R. C. Gupta: An Indian Form of Third Order Taylor Series Approximation for Sine, *Historia Mathematica*, Vol. 1, (pp. 287–289) 1974.

[4] Sastri, T. S. Kuppanna (ed.): *Mahābhāskarīya* of *Bhāskarācārya* with the *Bhāṣya* of *Govindasvāmin* and the super commentary *Siddhāntadīpika* of *Parameśvara*, Madras, 1957 (iv. 22, p. 205, vs. 18–19). For details see R. C. Gupta: A Mean Value Type Formula for Inverse Interpolation of Sine, *Mathematics Education*, Vol. 10, No. 1 (pp. 17–20) 1976.

[5] Sastry, K. Sambasiva: *Āryabhaṭīyam* with *Bhāṣya* of *Nīlakaṇṭha*, Part I, TSS No. 101, p. 55, Thiruvananthapuram (1930).

5. Power series for sine, cosine, and square of sine, viz:

$$
\begin{aligned}
R\sin\theta &= R\theta + \frac{(R\theta)^3}{3!R^2} + \frac{(R\theta)^5}{5!R^4} + \cdots \\
R\cos\theta &= R - \frac{(R\theta)^2}{2!R} + \frac{(R\theta)^4}{4!R^3} + \cdots,
\end{aligned}
$$

and

$$
(R\sin\theta)^2 = (R\theta)^2 - \frac{(R\theta)^4}{(2^2-\frac{2}{2})R^2} + \frac{(R\theta)^6}{(2^2-\frac{2}{2})(3^2-\frac{3}{2})R^4} + \cdots,
$$

discussed in the *Yuktibhāṣā*[6] of Jyeṣṭhadeva.

The invention of such trigonometric series attributed to Mādhava is an important milestone in the development of the discipline. Desire for more and more accuracy in astronomical computations gradually paved the way for their entry into the field of such infinite series expansions for trigonometric functions. They developed such infinite series expressions hoping to attain the desired degree of accuracy by taking as many terms as needed from the infinite number of terms. Such power series expansions also have great significance from a historical point of view because of their introduction for the first time in the world of mathematics.

Among various methods for computation of sine tables, a brief mention may be made of Nīlakaṇṭha's procedure given in the *Goḷasāra*[7] that is capable of giving highly accurate values.[8] Using this method we can construct accurate sine and cosine tables consisting of $l = 3 \times 2^m$ values at the arc interval of angular measure $h = \theta/2^m$ minutes for $m = 0, 1, 2, 3, \ldots$, initiating the procedure from $\theta = 30° = 1{,}800'$. Denoting:

$$
R\sin\left(\frac{\theta}{2^i}\right),\ R\,\text{versin}\left(\frac{\theta}{2^i}\right),\ \text{and}\ R\cos\left(\frac{\theta}{2^i}\right)
$$

respectively by S_i, V_i, and C_i for $i = 0, 1, 2, 3, \ldots, m$ and the i^{th} tabular Rsine and Rcosine respectively by $J_i = R\sin(ih)$ and $K_i = R\cos(ih)$ for $i = 1, 2, 3, \ldots, l$, then the following algorithm is embedded in the enunciation of Nīlakaṇṭha:

[6] Thampuran, Rama Varma Maru and Akileswara Aiyar (ed): *Yuktībhāṣā*, Part I, Mangalodayam, Trichur, 1947, (Ch. vii). Also see C. T. Rajagopal and A. Venkataraman: The Sine And Cosine Power Series in Hindu Mathematics, *Journal of Royal Asiatic Society of Bengal* – Kolkata, Vol. XV. (pp. 1–13) 1949.

[7] *Golasāra* (iii. vs. 6–14).

[8] Mallayya, V. Madhukar: An Interesting Algorithm for Computation of Sine Tables From the Golasāra of Nīlakaṇṭha, *Gaṇitā Bhārati*, Vol. 26, Nos. 1–4 (pp. 40–55) 2004. For details, ref. Mallayya, V. Madhukar: *The Golasāra Concept of Jyā: A Study in Modern Perspective*, SSVV, Thiruvananthapuram (Chaps. 3 and 4), 2004.

Step I: Starting from $S_0 = R/2$ derived in a geometrical background for $\theta = 30^\circ = 1{,}800'$ obtain the value of $S_i = R\sin(\theta/2^i)$ for $i = 1, 2, 3$ etc. in succession using the geometrically described formula

$$S_i = \frac{1}{2}\sqrt{S_{i-1}^2 + V_{i-1}^2},$$

where

$$V_{i-1} = R - C_{i-1} \text{ and } C_{i-1} = \sqrt{R^2 - S_{i-1}^2}.$$

The value of S_{i-1} so obtained after (say) m repetitions (viz; S_m) is to be taken as the first tabular $R\sin$ value. Then the table under construction will be at arc interval of $h = \theta/2^m$ minutes, where $\theta = 1{,}800'$, and will contain l values, where

$$l = \frac{90 \times 60}{h} = \frac{90 \times 60}{\theta/2^m} = 3 \times 2^m \text{ for } m = 0, 1, 2, 3, \ldots.$$

Step II: Compute $R = \frac{21{,}600 \times 113}{2 \times 355}$ using the ratio $\pi = \frac{355}{113}$ obtained from the circumference to diameter values stated in the rule.

Step III: With $J_1 = R\,\mathrm{sinh} = S_m$ as the first tabular $R\sin$ and $J_l = R\sin(lh) = R$ as the last tabular $R\sin$ compute

$$J_{l-1} = \sqrt{R^2 - J_1^2}.$$

Step IV: Using J_l and J_{l-1} compute

$$\Delta J_{l-1} = J_l - J_{l-1},$$
$$\text{and } \lambda = 2\left(\frac{\Delta J_{l-1}}{R}\right).$$

Step V: Now compute

$$\Delta J_{l-i} = \lambda \times J_{l-(i-1)} + \Delta J_{l-(i-1)},$$
$$J_{l-i} = J_{l-(i-1)} - \Delta J_{l-i},$$
$$\text{and } K_{l-i} = \sqrt{R^2 - J_{l-i}^2}$$

for $i = 2, 3, 4, \ldots, l-2$.

Nilakaṇṭha's geometrically described method to find the first tabular Rsine value dates back to the period of Āryabhaṭa I.

Another significant development that took place in the history of Kerala mathematics is in the field of the infinite series. Kerala mathematicians soared into the so-far unexplored field of infinite series and made several inroads in this highly fertile area while analysing various infinite series discovered by them for the evaluation of the circumference of a circle from its diameter. Let us have a look at what motivated Kerala mathematicians to break the barriers of the finite. The peculiar relation between the circumference and diameter had continued to attract the attention of Indian mathematicians from ancient times. Their awareness of the incommensurability of the circumference-diameter ratio and their eagerness to find the value of the circumference from the diameter with as much accuracy as possible prompted them to formulate various methods for achieving their goal. Their continued efforts to attain the desired degree of accuracy gradually steered them into the interesting field of the infinite series.

Āryabhaṭa I's usage of the term "*āsannaḥ*", in the sense "*approaching*" or "*close to*" or "*very near to*" or "*almost equal to*", while giving the value of the circumference of a circle with a given diameter, attracted the attention of later mathematicians. Āryabhaṭa I, in the *Gaṇitapāda* of his astronomical treatise *Āryabhaṭīya*, states:[9]

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।
अयुतद्वयविष्कम्भस्यासन्नो वृत्तपरिणाहः॥

caturadhikam śatamaṣṭaguṇam dvāṣaṣṭistathā sahasrāṇām |
ayutadvayaviṣkambhasyāsanno vṛttapariṇāhaḥ ||

> Four added to one hundred, multiplied by eight and then added to sixty-two thousand is nearly (or very close to) the value of the circumference of a circle of diameter twenty thousand.

According to this, the circumference is $(4 + 100) \times 8 = 62{,}000$ for a circle whose diameter is 20,000, which gives the value 3.1416 for π (symbolic notation $\approx$ is used here for *āsannaḥ* in modern terms). Commenting on this technical term, Nīlakaṇṭha Somayāji in his *Āryabhaṭīya Bhāṣya* says:[10]

कुतः पुनर्वास्तवीं संख्यामुत्सृज्यासन्नैवेहोक्ता। उच्यते। तस्या वक्तुमशक्यत्वात्।कुतः।येन मानेन मीयमानो व्यासो निरवयवः स्यात्, तेनैव मीयमानः परिधिः पुनः सावयव एव स्यात्। येन च मीयमानः परिधिर्निरवयवस्तेनैव मीयमानो व्यासोऽपि सावयव एव, इत्येकेनैव मानेन मीयमानयोरुभयोः क्वापि न निरवयवत्वं स्यात्। महान्तमध्वानं गत्वाप्यल्पावयवत्वमेव लभ्यम्। निरवयवत्वं तु क्वापि न लभ्यमिति भावः॥

[9] Sastry, K. Sambasiva: *Āryabhaṭīyam* with *Bhāṣya* of *Nīlakaṇṭha*, Part I, Thiruvananthapuram Sanskrit series No. 101, (p. 41) Thiruvananthapuram (1930).

[10] ibid(pp. 41–42).

kutaḥ punarvāstavīm saṁkhyāmutsṛjyāsannaivehoktā | ucyate | tasyā vaktumaśakyatvāt | kutaḥ | yena mānena mīyamāno vyāso niravayavaḥ syāt, tenaiva mīyamānaḥ paridhiḥ punaḥ sāvayava eva syāt | yena ca mīyamānaḥ paridhirniravayavastenaiva mīyamāno vyāso'pisāvayava eva, ityekenaiva mānena mīyamānayorubhayoḥ kvāpi na niravayavatvam syāt | mahāntamadhvānam gatvāpyalpāvayavatvameva labhyam | niravayavatvam tu kvāpi na labhyamiti bhāvaḥ ||

Why is it that the actual value is left out and this *very near* value stated? Let me say. It is impossible to state (the actual value). Why? That unit which leaves no remainder when the diameter is measured will leave a remainder if used again for measuring circumference. Likewise, the unit which leaves no remainder in the measure of the circumference will leave a remainder in the diameter if measured by the same unit. Hence if both (the diameter and circumference) are measured by the same unit, a remainderless state (*niravayavatvam*) is never attained. Even if this is carried out farther to a great extent only diminution of the remainder (*alpāvayavatvam*) can be obtained but absence of remainder can never be obtained—this is the meaning.

Noticing the impossibility of attaining the remainderless state in the evaluation of both the circumference and diameter using the same unit from each other, Kerala mathematicians went on with their efforts to find better and better approximations by reducing the remainder. Saṅgamagrāma Mādhava is found to have succeeded in arriving at a good approximation for the circumference given the diameter that corresponds to the ratio, correct up to ten decimal places. Nīlakaṇṭha refers to Mādhava's enunciation regarding this in his commentary on the *Āryabhaṭīya*:[11]

विबुध नेत्र गजाहिहुताशन त्रिगुणवेद भवारण बाहवः
नवनिखर्व मिते वृत्तिविस्तरे परिधिमानमिदं जगदुर्बुधाः ।

Vibudha netra gajāhihutāśana triguṇaveda bhavāraṇa bāhavaḥ
navanikharva mite vṛttivistare paridhimānamidam jagadurbudhāḥ |

It has been stated by the learned that the circumference of a circle with diameter 900000000000 is 2827433388233.

This gives $\pi = 3.14159265359$, which is correct to 10 decimal places. Later, in the *Līlāvatī* commentary, *Kriyākramakarī*, the commentator Śaṅkara Vāriyar has given different approximations citing various authorities. According to one statement the circumference is 355 for a diameter measure of 113, and according to the another, the value is 1,04,348 for 33,215. This second estimate gives the ratio 3.1415926539211. Like Nīlakaṇṭha, Śaṅkara also refers to the more accurate value attributed to Mādhava.[12] A much better value appears later, in the *Sadratnamālā* of Śaṅkara Varman, according to which the

[11] ibid(p. 42).

[12] Sarma, K. V.: *Līlāvatī* of *Bhāskarācārya* with *Kriyākramakarī* of *Śaṅkara* and *Nārāyaṇa*, V.V.R.I. Hoshiarpur (p. 377), 1975.

circumference corresponding to the diameter measure 1 *parārdha* (i.e., 10^{17}) is 314159265358979324, which gives the ratio correct to 17 decimal places.[13]

Kerala mathematicians not only gave such good estimates for the circumference given the diameter but also derived some exact expressions for the circumference, which allowed them to compute the value with as much accuracy as desired. For this they introduced some ingenious geometrico-analytic techniques based on the abstract concepts of continuous and repeated summations after the subdivision of tangents and arcs into a large number of infinitesimally small segments.[14] Their ingenious efforts in this direction, by sowing seeds of several advanced ideas of integral calculus, limits, and infinitesimals finally resulted in breaking the barriers of the finite, paving the way for their entry into the field of infinite series. The discovery of several infinite series expansions for the circumference in terms of diameter is attributed to Mādhava by later authorities on the subject, including his disciples. Incidentally, while computing the circumference using the Mādhava series $C = 4d - \frac{4d}{3} + \frac{4d}{5} - \cdots$, they realized the nature of the slow convergence of the series, since it demands a large number of terms to give a good approximation. This prompted Mādhava and his successors to make a further study and analysis of series, which resulted in their discovery of several rapidly convergent series and opened up new vistas in this unexplored field.[15] Some of the infinite series thus discovered by Kerala mathematicians and gathered from their works such as the *Tantrasaṅgraha* of Nīlakaṇṭha, and the commentary *Yuktidīpika* by Śaṅkara Vāriyar, the *Yuktibhāṣā* of Jyeṣṭhadeva, the *Kriyākramakarī* of Śaṅkara Vāriyar, the *Karaṇapaddhati* of Putumana Somayāji, and the *Sadratnamālā* of Śaṅkara Varman are given below in modern notation:

$$
\begin{aligned}
C &= 4d - \frac{4d}{3} + \frac{4d}{5} - \ldots, \\
C &= 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \ldots + (-1)^{n-1}\frac{4d}{2n-1} + (-1)^n \frac{4dn}{(2n)^2+1}, \\
C &= 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \ldots + (-1)^{n-1}\frac{4d}{2n-1} + (-1)^n \frac{4d(n^2+1)}{[(n^2+1)4+1]n}, \\
C &= \frac{16d}{1^5+4\times 1} - \frac{16d}{3^5+4\times 3} + \frac{16d}{5^5+4\times 5} - \ldots, \\
x &= \left(\frac{\sin x}{\cos x}\right) - \frac{1}{3}\left(\frac{\sin^3 x}{\cos^3 x}\right) + \frac{1}{5}\left(\frac{\sin^5 x}{\cos^5 x}\right) - \ldots,
\end{aligned}
$$

or in other words,

$$\tan^{-1} s = s - \frac{s^3}{3} + \frac{s^5}{5} - \ldots,$$

[13] Sarma, K. V.: *Sadratnamala* of *Śaṅkara Varman*, INSA, Delhi, iv. 2 (p. 26), 2001.

[14] Thampuran, Maru: *Yūktibhāṣā*, Part I, Ch. vi, (pp. 84–116).

[15] ibid(pp. 116–142).

$$
\begin{aligned}
C &= d\sqrt{12}\left\{1 - \frac{1}{3 \times 3} + \frac{1}{3^2 \times 5} - \frac{1}{3^3 \times 7} + \dots\right\}, \\
C &= \frac{8d}{2^2 - 1} + \frac{8d}{6^2 - 1} + \frac{8d}{10^2 - 1} + \dots, \\
C &= 3d + \frac{4d}{3^3 - 3} - \frac{4d}{5^3 - 5} + \frac{4d}{7^3 - 7} - \dots, \\
C - 3d &= \frac{6d}{(2 \times 2^2 - 1)^2 - 2^2} + \frac{6d}{(2 \times 4^2 - 1)^2 - 4^2} \\
&\quad + \frac{6d}{(2 \times 6^2 - 1)^2 - 6^2} + \dots, \\
C &= 2d + \frac{4d}{2^2 - 1} - \frac{4d}{4^2 - 1} + \frac{4d}{6^2 - 1} - \dots + (-1)^{n-1}\frac{4d}{(2n)^2 - 1} \\
&\quad + (-1)^n \frac{4d}{2[(2n+1)^2 + 2]}, \\
C &= \frac{4d}{4^2 - 1} + \frac{4d}{8^2 - 1} + \frac{4d}{12^2 - 1} + \dots .
\end{aligned}
$$

Commenting on the computational aspects, Śaṅkara Vāriyar observes that it is impossible to get the exact value, and thus possible to just find an approximation by terminating the process at the stage that gives the desired degree of accuracy. Śaṅkara says:[16]

> एवं मुहुः फलानयने कृतेऽपि युक्तितः क्वापि न समाप्तिः।
> तथापि यावदपेक्षं सूक्ष्मतामापाद्य पाश्चात्यान्युपेक्ष्य फलानयनं समापनीयम्॥
>
> *evam muhuḥ phalānayane kṛte'pi yuktitaḥ kvāpi na samāptiḥ |*
> *tathāpi yāvadapekṣam sūkṣmatāmāpādya pāśćātyānyupekṣya phalānayanam samāpanīyam ||*
>
> Thus by computing the results successively, it is theoretically impossible to come to an end. Hence the computation has to be terminated when the desired accuracy is attained, and the final value taken, ignoring the remainder.

To stop the computation at the desired stage, the infinite series has to be truncated, and such truncation causes some error in the resulting value. In order to compensate for the loss of terms due to truncation and to minimize this truncation error, Kerala mathematicians introduced the concept of some sort of transformation of the partial sums of the slowly convergent series by appending some remainder term, or rather, correction term to it, depending on the number of terms in the partial sum. The introduction of a correction term is an effective tool for getting a better approximation. The value computed

[16] Sarma, K. V.: *Kriyākramakarī* (p. 385).

after the introduction of a correction term is found to be better than that computed without it, and the number of terms needed to get a desired degree of accuracy is comparatively less when the correction term is introduced. The *Yuktibhāṣā* gives a demonstration of the improvement caused in the calculated value of the circumference by comparing the values obtained before and after applying a correction term to partial sums of the series $C = 4d - \frac{4d}{3} + \frac{4d}{5} - \ldots$. It is found that the use of correction terms, not only improves the value to a considerable extent but also expedites the convergence of the series. Realizing this practical utility of correction terms, Kerala mathematicians introduced the concept, throwing open the doors for further studies in this interesting field of series approximation. In order to check the degree of accuracy attained, they evaluated the corresponding error. In order to reduce the error and increase the degree of accuracy they went on modifying the correction term. Their search for better and more efficient correction terms naturally resulted in an interesting analysis of correction terms. Detailed analytic derivations of different correction terms and the corresponding error analysis can be had from the *Kriyākramakarī* of Śaṅkara Vāriyar[17] and the *Yuktibhāṣā* of Jyeṣṭhadeva.[18] Incidentally, several correction terms arrived at are found to be successive convergents of certain special continued fractions.[19] Another important development in this direction that is to be taken note of is that from the estimated error expressions obtained by them during their analysis of correction terms, they succeeded in deriving some other useful infinite series for computation of the circumference given the diameter.[20]

3 Notion of Proof: Forms, Nature, Style, and Purpose

Now, regarding the notion of proof and reasoning, it may be noted that Kerala mathematicians placed considerable emphasis on providing an elaborate exposition of various results, by discussing their reasoning, supported by several numerical illustrations and various kinds of proofs in algebraic and geometrical backgrounds that were necessary for the benefit of all kinds of students. Their exposition often started from the elementary level and was presented in an instructive form that could be easily followed and understood by all. In certain commentaries and other expository works some of the important useful discoveries and demonstrations from ancient works like the *Śulbasūtra*s (of about 800 B.C.E) are also included and presented in a systematic manner.

[17] ibid(pp. 387–391).

[18] Thampuran, Maru: *Yuktībhāṣā*, Part I Trichur (Ch. vi, pp. 120–142), 1947.

[19] Hayashi, T., Kusuba, T., and Yano, M.: The Correction of Madhava Series for the Circumference of a Circle, *Centaurus*, Vol. 33, pp. 149–174 (1990).

[20] Mallayya, V. Madhukar: Śaṅkara's Correction Functions for Series Approximations, *Recent Trends in Mathematical Analysis*, Allied Publishers, Delhi (pp. 176–189) 2003.

Some examples are: the detailed exposition of Baudhāyana's demonstrations of the method for evaluation of the square root of 2 in a geometrical context, given by Nilakaṇṭha in his commentary on the *Āryabhaṭiya*; the demonstrations of Baudhāyana's theorem on the square of diagonals (popularly known as the Pythagorean theorem) explained in the *Yuktibhāṣā*; and some of the commentaries on the *Līlāvatī* and *Āryabhaṭīya*. Detailed demonstrations of such important mathematical truths formulated periodically are brought to us through various commentaries and other expository works. The derivations pertaining to the power series expansions for trigonometric functions, various infinite series for the circumference in terms of the diameter, correction terms for approximating series, error estimation and so on found in the *Yuktibhāṣā*, *Kriyākramakarī*, etc., bear much historical significance in the sense that one can hear heavy knockings on the doors of calculus and infinitesimal analysis several centuries before their use or discovery elsewhere. In the course of their elaborate expositions they used various mathematical tools such as the relation between arcs and the corresponding chords, similarity of triangles, which is a geometrical manifestation of the rule of three, the base–altitude–hypotenuse theorem or Baudhāyana's theorem, the concept of summation of finite and infinite geometric series, the concept of *saṅkalita*, *saṅkalita_saṅkalita*, or *vārasaṅkalita*, and so on. The elaborate demonstrations given by them reveal their awareness of various advanced mathematical concepts such as certain integrals, differentials, infinitesimals, and operations with small or vanishing quantities (*śūnya prāyamāya saṅkhya*, which means a number that has become almost zero). They devised and used certain special mathematical techniques that are equivalent to the ε-δ techniques of modern analysis, the Cauchy–Stolz theorem on limits, integration as a limiting sum, etc. Such abstract ideas are found to have been described in verbal form in the *Yuktibhāṣā*, *Kriyākramakarī*, etc. Moreover, their discussions on cyclic quadrilaterals give us some useful and important results such as Paramesvara's formula for the circumradius of a cyclic quadrilateral popularly known by the name L'huiller's formula,[21] and the interesting concept of a third diagonal for a cyclic quadrilateral,[22] along with the formula for finding its area.

Indian mathematicians in general, and Kerala mathematicians in particular, were very fond of providing geometrical proofs for various arithmetic and algebraic truths. Even progressive series were treated in geometrical contexts. Such demonstrations are capable of providing a clear picture of the abstract ideas under discussion. Taking into consideration the type of results and the nature as well as the background of the learners, the learned ācāryas

[21] Sarma, N. Anantakrishna: Transcript copy of the Ms. *Parameśvarakṛta Līlāvatīvyākhyā*, p. 89. Also see T. A. Sarasvati Amma: *Geometry in Ancient and Medieval India*, Delhi (pp. 108–109) 1991.

[22] Dvivedi, Padmakara (ed): The *Gaṇita Kaumudi* of *Nārāyaṇa Paṇḍita*, Part II, Benaras, 1942 (p. 58, vs. 48). For more details see K. V. Sarma: *Kriyākramakarī*, pp. 348–362 and Maru Thampuran: *Yuktībhāṣā*, Part I, Ch. vii (pp. 228–237).

adopted different kinds of proofs for establishing the validity of the results. They often employed algebraic or analytic proofs, numerical illustrations, geometrical demonstrations, indirect methods, and even the method of direct observation, experimentation, and contemplation. An algebraic or analytic argument helps one to understand and analyze the underlying abstract mathematical concepts. A numerical illustration serves to verify or exemplify the result. However, through a geometrical demonstration one can visualize the underlying concepts and thus be convinced at once. For the benefit of those who do not comprehend an algebraic argument or an analytic proof, a geometric demonstration or a numerical illustration will serve the purpose, and vice versa.

Apart from these types of demonstrations, it is seen that some sort of indirect proofs resembling the *reductio ad absurdum* style were also employed for proving certain statements regarding the nonexistence of some mathematical entities (such as the square root of a negative number) by assuming the corresponding alternative hypothesis and negating it after a logically-based argument using already known and established facts. Kṛṣṇa Daivajña, in his commentary, *Bījāṅkura*, on the *Bījagaṇita*, of Bhāskarācārya,[23] establishes that a negative number cannot have a square root in the following manner, which in fact has a flavor of the *reductio ad absurdum*:

न मूलं क्षयस्यास्तीति। तत्र हेतुमाह। तस्याकृतित्वादिति। वर्गस्य हि मूलं लभ्यते। ऋणाङ्कस्तु न वर्गः कथमतस्तस्य मूलं लभ्यते। ननु, ऋणाङ्कः कुतो वर्गो न भवति। न हि राजनिदेशः। किंच यदि न वर्गस्तर्हि वर्गत्वं निषेद्धुमप्वनुचितमप्रसक्तेः। सत्यम्। ऋणाङ्कं वर्गं वदता भवता कस्य स वर्गं इति वक्तव्यम्। न तावद्धनाङ्कस्य समद्विघातो हि वर्गः। तत्र धनाङ्केन धनाङ्के गुणिते यो वर्गो भवेत् स धनमेव। स्वयोर्वधः स्वमित्युक्तत्वात्। नाप्यृणाङ्कस्य। तत्रापि समद्विघातार्थमृणोङ्केनर्णाङ्के गुणिते धनमेव वर्गो भवेत् अस्वयोर्वधः स्वमित्युक्तत्वात्। एवं सति कमपि तमङ्कं न पश्यामो यस्य वर्गः क्षयो भवेत्॥

na mūlam kṣayasyāstīti| tatra hetumāha| tasyākṛtitvāditi| vargasya hi mūlam labhyate| ṛṇāṅkastu na vargaḥ kathamatastasya mūlam labhyate| nanu, ṛṇāṅkaḥ kuto vargo na bhavati| na hi rājanideśaḥ| kimca yadi na vargastarhi vargatvam niṣeddhumapvanucitamaprasakteḥ | satyam| ṛṇāṅkam vargam vadatā bhavatā kasya sa vargam iti vaktavyam | na tāvaddhanāṅkasya samadvighāto hi vargaḥ| tatra dhanāṅkena dhanāṅke guṇite yo vargo bhavet sa dhanameva|

[23] Vasista, Viharilal: *Bījagaṇīta*, with commentary, *Bījāṅkura* of *Kṛṣṇa Daivajña*, Jammu (p. 16, vs. 13), 1977.

> *svayorvadhaḥ svamityuktatvāt| nāpyṛṇāṅkasya| tatrāpi samadvighātārthamṛṇoṅkenarṇāṅke guṇite dhanameva vargo bhavet asvayor vadhaḥ svamityuktatvāt| evam sati kamapi tamaṅkam na paśyāmo yasya vargaḥ kṣayo bhavet|*
>
> There is no square root for a negative number. The reason is stated here. It is because of its non squareness. The square root is obtained from the square. A negative number cannot be a square. Then how can its square root be obtained? All right, one may say this, but WHY is it that a negative number is not a square. There is no royal injunction here (that rules out the possibility of a negative number being a square). Moreover, if it is not (said to be) a square, then it is also not said that squareness (*vargatvam*) is prohibited–by reason of inapplicability (of any rule that would tell us what to think in advance about negative numbers being squares). Very well, it must be stated by the gentleman (*bhavata*) here who is saying that a negative number is a square: of WHAT number is it a square? Definitely, not of a positive number because its square, being the product of two equals, is the result of multiplication of the positive number with the positive number, which will be only a positive number. Also, not of a negative number because here also the square, being the product of two equals, is the result of multiplication of the negative number with the negative number, which will be only a positive number. As such there is no way seen whatsoever by which we can find a number whose square is negative.

In this manner it is established that a negative number cannot be a square of any number and by the method of inversion (*vyastavidhi*) no number can become a square root of a negative number.

Another accepted form for checking the validity of certain enunciations and making corrections if necessary was direct practical observation and experimentation (*nirīkṣaṇa parīkṣaṇam*). The results and theories that sprouted from such scientific observations were then treated as established truths. The famous *Dṛggaṇita* system of astronomical computation promulgated by Vaṭaśreṇi Parameśvara is stated to have been developed by him on the basis of his observational and contemplative studies for a very long period. Commenting on Verse 48 of the *Golapāda* of the *Āryabhaṭīya*, *Nīlakaṇṭha* observes[24]: *cāvadhārya śāstrāṇyapi bahūnyālocya pañca pañcaśadvarṣakālam nirīkṣya grahaṇagrahayogādiṣu parīkṣya samadṛggaṇitam karaṇam cakāra.* Parameśvara's famous *Dṛggaṇita* system is thus the result of his fifty-five years of rigorous observations, experimentation, and contemplation. This shows that the results and theories derived from such practical observations were accepted by all, and such results form the *pramāṇa*s, or established truths, for future use.

Various types of methods of proof were thus used, depending on the nature of the result and taking into consideration the nature of the learner. As such,

[24] Pillai, Suranad Kunjan: *Āryabhaṭīya* with *Bhāṣya* of *Nīlakaṇṭha*, Part III, *Golapādā*, Trivandrum Sanskrit series No. 185, Thiruvananthapuram (vs. 48), 1957.

they are found to be quite informal in style and flexible in nature. They are not based on any rigid formal deductive system starting from a set of self-evident axioms. However, there was no compromise with rigor, and this is evident from the proofs given in the *Yuktibhāṣā* and the *Kriyākramakarī*. Highlighting the main difference between proof styles in India and the Greek counterparts, T. A. Sarasvati Amma observes:[25]

> *But one has to concede that there was an important difference between the Indian proofs and their Greek counterparts. The Indian's aim was not to build up an edifice of geometry on a few self-evident axioms but to convince the intelligent student of the validity of the theorem, so that visual demonstration was quite an accepted form of proof. This leads us to another characteristic of Indian mathematics which makes it differ profoundly from Greek mathematics. Knowledge for its own sake did not appeal to the Indian mind. Every discipline (Śāstra) must have a purpose.*

A proof for its own sake did not appeal to the Indian mind. Indian proofs were clearly purpose-oriented. Even for certain results that were attained by the power of human intellect through intuition, they often provided detailed arguments in order to convince others and make them accept and experience the truth of their inventions. Some of the main purposes in providing a proof may be stated as follows:

1. To check the validity of the results.
2. To demonstrate the truth and underlying concepts in the most convincing manner for various types of learners with different backgrounds.
3. To remove all kinds of doubts and misapprehensions, thereby enabling one to reject any wrong notion in understanding the results.
4. To convince all contemporary scholars in the field about the validity and authenticity of their results.
5. To enhance and stimulate the intellect of the learners by presenting several logical arguments and choosing the style of argumentation in such a way as to motivate them to make a further study or application of the results.

A prefatory note by Gaṇeśa, in his commentary *Buddhivilāsini*, on the *Līlāvatī* of *Bhāskarācārya* explicitly reveals these intentions. According to Gaṇeśa:[26]

> व्यक्ते वाऽव्यक्तसंज्ञे यदुदिततमखिलं नोपपत्तिं विना त-
> न्निर्भ्रान्तो (वा) ऋते तां सुगणकसदसि प्रौढतां नैति चायम्।
> प्रत्यक्षं दृश्यते सा करतल कलितादर्शवत्सुप्रसन्ना
> तस्मादग्र्योपपत्तिं निगदितुमखिलामुत्सहे बुद्धिवृद्ध्यै॥

[25] Amma, T. A. Sarasvati: *Geometry in Ancient and Medieval India* (p. 3).

[26] Apte, V. G.: *Līlāvatī* with *Buddhivilāsinī* of *Gaṇeśa Daivajña* and *Līlāvatī Vivaraṇa* of *Mahidhara*, Anandasramom series. No. 107, Poona (Part I)(p. 1, vs. 4) 1937.

vyakte vā'vyakta samjne yadutitamakhilam nopapattim vinā ta-
nnirbhrānto (vā) ṛte tām sugaṇakasadasi prauḍhatām naiti cāyam |
pratyakṣam dṛśyate sā karatala kalitādarśavatsuprasannā
tasmādagryopapattim nigaditumakhilāmutsahe buddhivṛddhyai ||

Whatever is stated in the *vyakta gaṇita* or *avyakta gaṇita* (arithmetic or algebra) without rationale may not be intelligible and not without confusion (*nirbhrānta*) and will not be acceptable to the assembly of great mathematicians. It should be crystal clear and perceivable as through a hand mirror. For enhancing the intellect of the learner, I shall describe the rationale of the enunciations in their fullness.

Gaṇeśa's remark not only reveals the main purpose of providing a proof argument but also throws light on the Indian mathematical tradition of accepting only well-established truths, which means that they were not used to accepting any result without valid proof.

4 The Role of Commentarial Literature in the Dissemination of Mathematical Knowledge

Most of the standard texts on Indian astronomy and mathematics contain only enunciations of established truths and not their proofs. Such precise texts are practically handbooks or reference manuals, and they contain a wealth of knowledge in a nutshell. The absence of proofs and other demonstrations in such precise texts, however, created a general feeling that Indian mathematicians were indifferent to the notion of proof, that they accepted any result without bothering to check its validity, and paid no serious attention to the methodology. A close look at the standard treatises along with their accompanying keys, called commentaries, reveals that this is baseless. Indian masters followed the oral tradition for imparting knowledge to their disciples. As such, they orally expounded the subject of study to their disciples and then summed up the quintessence of their discourses in the form of *sūtra*s in verse or prose of great precision, in such a way that one may easily recite, learn, and recall at times of need. The dearth of writing materials also prompted them to minimize the contents of their treatises by jotting down only very little in *sūtra* form for the sake of future reference. Such *sūtra*s would be sufficient and significant to those who knew the key to their meaning or to those who had attended the discourses on them. However, for others they would appear to be obscure and of little import. To throw light on the fund of information contained in the *sūtra*s and to preserve the elaborate proofs and other demonstrations acquired from their masters, some of the disciples who had drawn inspiration from the discourses endeavoured to take up the task of composing elaborate commentaries on works of their choice as accompanying keys after making a detailed study of the topic and gathering as much information as possible from all available sources. Such accompanying keys, which impart life and spirit to the highly precise *sūtra*s mentioned in the basic text, are called *paribhāṣā*s or *vyākhya*s. They form a perpetual gloss in which the information embedded in the *sūtra*s is brought to light, proved, elaborated, or amplified. The commentaries are also independent works in which the enunciations given in the basic treatise are expounded along with detailed proofs, illustrations, derivations, and various demonstrations including visual ones in geometrical forms, or whatever was necessary for a thorough understanding of the basic text as well as the subject. Quoting at length every verse of the basic treatise, the commentators expounded on them, often starting their discussions from a basic level. The commentators, being products of their own times, looked on the results from their point of view and gave their own interpretations. Quite often they committed themselves to several novel ideas. They blended their commentaries with current developments in the field and infused their composition with their own inventions and ideas, which naturally enhanced the quality and utility of such commentaries. Such commentarial literature plays an important role in disseminating mathematical knowledge gathered from over time.

5 Commentarial Literature: A Rich Source for the Study of Proof, Methodology, and Motivation

Kerala is rich in commentarial literature, which forms a rich source for study of detailed exposition, methodology and proof in Indian mathematics and astronomy. Some of the commentaries, such as the *Kriyākramakarī* and the *Yuktidīpika*, contain numerous *saṅgraha ślokas* (summary verses) toward the end of each discussion. On the one hand, these *saṅgraha ślokas* summarize the elaborate discussions given therein, enabling one to recapitulate the quintessence of the exposition. On the other hand, they provide several other important enunciations along with their elaborate proofs, derivations, and demonstrations in both algebraic and geometric contexts. Such *saṅgraha ślokas* are capable of throwing much light on the methodology adopted by Indian mathematicians in formulating various concepts and theories periodically in accordance with their needs. Some of the important works that are well known to contain detailed expositions of several mathematical or astronomical results include various commentaries on the *Āryabhaṭīya* of Āryabhaṭa I, such as those of Nīlakaṇṭha, Parameśvara, Sūryadeva, Someśvara, and Bhāskara I; commentaries on the *Līlāvatī* of Bhāskarācārya, such as those of Gaṇeśa Daivajña, Parameśvara, Śaṅkara, and Nārāyaṇa; the commentary on the *Mahābhāskarīya* of Bhāskara I by Govindasvāmin, including Parameśvara's supercommentary on it; Kṛṣṇa Daivajña's commentary on the *Bījagaṇita* of Bhāskarācārya; Bhāskarācārya's own commentary on the *Siddhānta Śiromaṇi*, and Nṛsimha Daivajña's commentary on it, commentaries on the *Siddhānta Śiromaṇi* by Gaṇeśa Daivajña and Munīśvara; *Śiṣyadhivṛddhida Tantra* of Lalla with commentary on it by Bhāskarācārya; *Tantrasaṅgraha* of Nilakaṇṭha

Somayāji with commentaries by Śaṅkara; *Yuktibhāṣā* of Jyeṣṭhadeva in two parts, the first part dealing with an exposition of mathematical enunciations and the second part with astronomy; *Gaṇitayuktayaḥ*, which is a compendium of arguments in Kerala astronomy; *Karaṇapaddhati* of Putumana Somayāji with commentaries on it; *Siddhānta-sarva-bhauma* of Munīśvara with his own commentary on it; *Siddhānta Tattva Viveka* of Kamalākara Bhaṭa and his own commentary on it, etc.

These and various other works containing detailed expositions are capable of throwing light on the nature of astronomical and mathematical contributions, the methodology adopted, motivations behind their contributions, notion of proof, etc. if proper attention is paid to the original and unbiased study of their contents. However, only a small part of the large mass of literature has been published so far, and from among those published only a few have been translated and studied. Extensive literature is still lying unexplored in various repositories. A monograph compiled by K. V. Sarma entitled *Science Texts in Sanskrit in the Manuscripts Repositories of Kerala and Tamil Nadu*[27] identifies from about 400 repositories in Kerala and Tamil Nadu over 3,473 science texts in Sanskrit and 12,244 science manuscripts, including astronomical and mathematical manuscripts. The knowledge embedded in these unexplored works handed down to us by the learned masters during the vigorous, youthful days of the land is waiting to see the light of day through the hands of the present and future generations. The present generation has unprecedented power in their hands with the tremendous advances in science and technology. With the advent of computers and internet facilities, modern ideas continue to pour in from all directions. Apart from all this, the products of the present age are capable of seeing much further than their predecessors because they can reverently climb on their shoulders with whatever has been gleaned so far from the vast store of writings lying in various repositories. A glimpse into the past will always help one to have a better understanding of the present. But while looking into the past, care has to be taken to give only due value and weight to the actual facts gathered from original sources. Explorative studies of an unbiased nature are capable of revealing the real flavor of the contributions, arousing the interest of all in the field, and revealing the insights and inspirations of the Indian mathematical mind. Unifying the multitude of facts gathered from such original sources, the old knowledge embedded in them can be recovered in their depth and fullness, and expounded in modern terms in a faithful manner, intelligible to the present generation. Leaving aside all outmoded methods and concepts, rejecting wrong notions if any, and accepting their ingenious ideas, one can restate the old knowledge in modern terms and

[27] Sarma, K. V. and Sastry, V. Kutumba: *Science Texts in Sanskrit in the Manuscripts Repositories of Kerala and Tamil Nadu*, Rashtriya Sanskrit Sansthan, Delhi (2002).

reconstruct the methodology in accordance with present needs, making full use of various ideas flowing from all directions. Viewing the content of the surviving materials in this manner through a modern perspective, one can construct bridges between the past and the present so that knowledge from the past can be understood by the present and handed down to the future generations.

The Algorithm of Extraction in Greek and Sino-Indian Mathematical Traditions

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1 Introduction

Theorem-proving, which originated in ancient Greece, has been the backbone of deductive tradition in the history of mathematics, while algorithm-creating flourished in ancient and medieval China and India, forming a strong trend in mathematics. However, in comparison with the deductive tradition, the functions of the algorithmic tradition have unfortunately been ignored.

Based on reviewing the case of the algorithm of extraction in Greece, China, and India, this paper shows that theorem-proving in ancient Greece and algorithm-creating in ancient and medieval China and India played indispensable roles in advancing the development of mathematics, highlighting the backgrounds of the different cultures and the disparate traditions in mathematics.

2 The Algorithm of Extraction in Ancient Greece

The fourth proposition – if a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments – from Book II of Euclid's *Elements* (around the third century B.C.E.). In this proposition, we found a figure for the presentation of the

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geometrical meaning of the formula, $(a+b)^2 = a^2 + 2ab + b^2$ and not for the algorithm of extraction. This figure was never used for extraction in Euclid's *Elements*.

The record of extraction comes from Heron and Theon of Alexandria.

2.1 Heron of Alexandria's Method

Heron of Alexandria [1, pp. 317–318] gives a very good and interesting example using this method in his treatise "Μετρικά," written probably in the first century C.E. His *Metrica* begins with the old legend of the practical traditional origin of geometry in Egypt and seems to be very rich in definite references to the discoveries of his predecessors. The names mentioned are Archimedes, Dionysodorus, Eudoxus, and Plato. The *Metrica* was first discovered in 1896 in a manuscript of the eleventh (or twelfth) century at Constantinople by R. Schöne and edited by his son, H. Schöne (*Herons Opera*, iii, Teubner, 1903). In *Metrica* he gives the theoretical basis of the formulas used, and this is not a mere application of rules to particular examples. It is also more akin to theory, in that it does not use concrete measures, but simple numbers or units which, may then in particular cases be taken to be feet, cubits, or any other unit of measurement. Up to 1896, the *Metrica* was known only by an allusion to it in Eutocius (on Archimedes' *Measurement of a Circle*), who states that the way to obtain an approximation to the square root of a nonsquare number is given by Heron, as well as by Pappus, Theon and others, who had commented on the *Syntaxis* (*Almagest*) of Ptolemy. The contents of the three books are as follows:

Book I of the *Metrica* contains the mensuration of squares, rectangles, and triangles (Chaps. 1–9), parallel trapezia, rhombi, rhomboids, and quadrilaterals with one right angle (Chaps. 10–16), regular polygons from the equilateral triangle to the regular dodecagon (Chaps. 17–25), a ring between two concentric circles (Chap. 26), segments of circles (Chaps. 27–33), an ellipse (Chap. 34), a parabolic segment (Chap. 35), the surfaces of a cylinder (Chap. 36), an isosceles cone (Chap. 37), a sphere (Chap. 38), and a segment of a sphere (Chap. 39).

Book II gives the mensuration of certain solids, the solid content of a cone (Chap. 1), a cylinder (Chap. 2), rectilinear solid figures, a parallelopiped, a prism, a pyramid and a frustum (Chaps. 3–8), a frustum of a cone (Chaps. 9 and 10), a sphere and a segment of a sphere (Chaps. 11 and 12), a spire or torus (Chap. 13), the section of a cylinder measured in Archimedes' *Method* (Chap. 14) and the solid formed by the intersection of two cylinders with axes at right angles inscribed in a cube, also measured in the *Method* (Chap. 15), the five regular solids (Chaps. 16–19).

Book III deals with the division of figures into parts having given ratios to one another; first plane figures (Chaps. 1–19), then solids, a pyramid, a cone and a frustum, a sphere (Chaps. 20–23). He gives this method in an exercise on the mensuration of the surface of a triangle with sides 7, 8, and 9 using the formula

$$E = \sqrt{\tau(\tau-\alpha)(\tau-\beta)(\tau-\gamma)},$$

where he obtains at $\tau = 12$, $\tau - \alpha = 5$, $\tau - \beta = 4$, $\tau - \gamma = 3$, so that $E = \sqrt{720}$, where τ is the semiperimeter. The text[3] [2, pp. 18–20] in liberal translation is as follows:

"Since," says Heron, "720 has not its side rational", we can obtain its side within a very small difference as follows.

> Since the next succeeding square number is 729, which has 27 for its side, divide 720 by 27. This gives $26\frac{2}{3}$. Add 27 to this, making $53\frac{2}{3}$, and take half of this or $26\frac{1}{2} + \frac{1}{3}$. The side of 720 will therefore be very nearly $26\frac{5}{6}$. In fact, if we multiply $26\frac{5}{6}$ by itself, the product is $720\frac{1}{36}$, so that the difference (in the square) is $\frac{1}{36}$. If we desire to make the difference smaller than $\frac{1}{36}$, we shall take $720\frac{1}{36}$ instead of 729 [or rather we should take $26\frac{5}{6}$ instead of 27] and by proceeding in the same way, we shall find that the resulting difference is much less than $\frac{1}{36}$. In modern terms [3, p. 3]:

$$\begin{aligned}
A &= 720, \quad \alpha_1 = 27 \ (\alpha_1^2 = 729 > 720),\\
\frac{A}{\alpha_1} &= \frac{720}{27} = 26\tfrac{2}{3},\\
\alpha_1 + \frac{A}{\alpha_1} &= 27 + 26\tfrac{2}{3} = 53\tfrac{2}{3},\\
\tfrac{1}{2}\left(\alpha_1 + \frac{A}{\alpha_1}\right) &= \tfrac{1}{2}\left(53\tfrac{2}{3}\right) = 26\tfrac{5}{6},\\
\sqrt{A} &\approx \tfrac{1}{2}\left(\alpha_1 + \frac{A}{\alpha_1}\right), \quad \text{or} \quad \sqrt{720} \approx 26\tfrac{5}{6}.
\end{aligned}$$

For a second approximation we take

$$\begin{aligned}
&\alpha_2 = 26\tfrac{5}{6} \ \left(\alpha_2^2 = 720\tfrac{1}{36} > 720\right)\\
&\frac{A}{\alpha_2},\ \alpha_2 + \frac{A}{\alpha_2},\ \tfrac{1}{2}\left(\alpha_2 + \frac{A}{\alpha_2}\right), \text{ and so on.}
\end{aligned}$$

The above seems to be the only extant classical rule to find second and further approximations to the value of a surd. But although Heron shows how to obtain a second approximation, he does not seem to make any direct use of this method himself, and consequently the question of how the approximations closer than the first that are to be found in his works were obtained still remains an open one.

[3] Έστι δε καθολική μέθοδος ώστε τριών πλευρών δοθεισών οιουδηποτούν τριγώνου το εμβαδόν ευρείν χωρίς καθέτου · οίον έστωσαν αι του τριγώνου πλευραί μονάδων ζ, η, θ · σύνθες τα ζ και τα η και τα θ · γίγνεται κδ · Τούτων λαβέ το ήμισυ · γίγνεται ιβ · Άφελε τας ζ μονάδας · λοιπαί ε · πάλιν ά φελε από των ιβ τας η · λοιπαί δ · και έτι τας θ · λοιπαί γ · ποίησον τα ιβ επί τα ε · γίγνονται ξ · ταύτα επί τον δ · γίγνονται σμ · ταύτα επί τον γ · γίγνεται ψκ · τούτων λαβέ πλευράν και έσται το εμβαδόν του τριγώνου · Επείοθν αι ψκ ρητήν την πλευράνοθκ έχουσι, ληψόμεθα μετά διαφόρου ελαχίστου την πλευράν ούτως · επεί ο συνεγγίζων τω ψκ τετράγωνός εστιν ο ψκθ και πλευράν έχει τον κζ, μέρισον τας ψκ εις τον κζ · γίγνεται κς και τρίτα δύο · πρόσθες τας κζ · γίγνεται νγ τρίτα δύο · Τούτων το ήμισυ · γίγνεται κςLγ́ · Έσται άρα του ψκ η πλευρά έγγιστα τα κςLγ́ · Τα γαρ κςLγ́ εφ εαυτά ψκλς́ · ώστε το διάφορον μονάδος εστί μόριον λς́ · Εάνδε βουλώμεθα εν ελάσσονι μορίω του λς́ την διαφοράν γίγνεσθαι, αντί του ψκθ τάξομεν τα νυν ευρεθέντα ψκ και λς́, και ταυτά ποιήσαντες ευρήσομεν πολλώ ελάττονα < του > λς́ την διαφοράν γιγνομένην · .

2.2 Theon of Alexandria's Method

Theon of Alexandria [1, p. 526] lived toward the end of the fourth century C.E. Suidas places him in the reign of Theodosius I (379–395); he tells us himself that he observed a solar eclipse at Alexandria in the year 365, and his notes on the chronological tables of Ptolemy extend to 372.

He was the author of a commentary on Ptolemy's *Syntaxis*(*Almagest*)in eleven books. We are indebted to it for a useful account of the Greek method of operating with sexagesimal fractions, which is illustrated by examples of multiplication, division and the extraction of the square root of a nonsquare number by way of approximation. Theon of Alexandria has given a geometrical explanation [3, pp. 5–6] of the algorithm of the square roots in his *Commentary* on Ptolemy's *Almagest*. Ptolemy, while constructing for astronomical use a table of arc lengths, needed to calculate the chord of an arc of 36°. This chord, which is the side of an inscribed regular decagon, is $R(\sqrt{5}-1)/2$, where R is the chord of the circle.

Ptolemy used a circle with $R = 60$, so the above relation is $30(\sqrt{5}-1)$, or $\sqrt{4{,}500}-30$. For $\sqrt{4{,}500}$ Ptolemy gives (in the sexagesimal number system) the approximation $67°4'55''$, without explanation of the manner by which he arrived at this result. The same procedure is illustrated by Theon's explanation of Ptolemy's method of extracting square roots according to the sexagesimal system of fractions. The problem is to find approximately the square root of 4,500°, and a geometrical figure is used, which essentially shows the Euclidean basis of the method.

Theon in his commentaries gives the following explanation:

First he takes the square $AB\Gamma\Delta$ with surface 4,500°, so the calculation of $\sqrt{4{,}500}$ is reduced to the calculation of the side $B\Gamma$ of the square. The nearest square to 4,500 is 4,489 (=67^2). He constructs the square $AEZH$ with area 4,489°, the side of which is $EZ = 67°$ and is the first approximation of $B\Gamma$. Hence, from Fig. 1 we have

$$(AB\Gamma\Delta) = (AEZH) + (EBMZ) + (HZN\Delta) + (ZM\Gamma N).$$

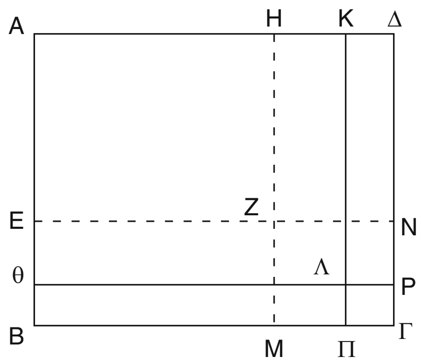

Fig. 1. Figure for Heron's of Alexandria method of extraction

Since the rectangles $EBMZ, HZN\varDelta$ are equal and the square $ZM\varGamma N$ is very small, we have the following:

$$(AB\varGamma\varDelta) \;=\; (AEZH) \;+\; 2(EBMZ),$$

or

$$4{,}500^\circ \;=\; 4{,}489^\circ \;+\; 2 \times 67^\circ \cdot ZM,$$

or

$$ZM \;=\; \frac{4{,}500^\circ - 4{,}489^\circ}{2 \times 67^\circ} \;=\; \frac{11^\circ}{134^\circ}.$$

By this relation we have that the side of the square $ZM\varGamma N$, in the sexagesimal system, approximately equals $4'$. We can count as a second approximation of the side $B\varGamma$ $(=\sqrt{4{,}500})$,

$$B\varGamma \;=\; BM + M\varGamma \;=\; 67^\circ + 4' \;=\; 67^\circ 4'.$$

Theon continues with the same procedure:

He constructs the square $A\varTheta\varLambda K$ with side $\varTheta\varLambda \;=\; 67^\circ 4'$ and surface

$$(A\varTheta\varLambda K) \;=\; (67^\circ 4')^2 \;=\; 4{,}497^\circ 56' 16'' \;=\; 4{,}489^\circ + 268' + 268' + 16''.$$

For the surfaces in the figure we have

$$(AB\varGamma\varDelta) \;=\; (A\varTheta\varLambda K) + (\varTheta B\varPi\varLambda) + (K\varLambda P\varDelta) + (\varLambda\varPi\varGamma P).$$

And we have as an approximation the following:

$$4{,}500^\circ \;=\; 4{,}497^\circ 56' 16'' + 2(67^\circ 4')\varLambda\varPi,$$

or

$$\varLambda\varPi \;=\; \frac{4{,}500^\circ - 4{,}497^\circ 56' 16''}{2(67^\circ 4')} \;=\; \frac{2^\circ 3' 44''}{134^\circ 8'}.$$

From the last relation we have that the side of the square $\varLambda\varPi\varGamma P$ is approximately equal to $55''$. A third approximation of the side $B\varGamma$ is

$$B\varPi + \varPi\varGamma \;=\; 67^\circ 4' + 55'' \;=\; 67^\circ 4' 55'',$$

the value given by Ptolemy for $\sqrt{4{,}500}$ in his *Syntaxis* (*Almagest*). After this explanation, Theon summarizes the procedure into a general rule, or algorithm:

> When we seek a square root, we take first the root of the nearest square number. We then double this and divide by it the remainder reduced to minutes, and subtract the square of the quotient; then we reduce the remainder to seconds and divide by twice the degrees and minutes [of the whole quotient]. We thus obtain nearly the root of the quadratic.[4]

[4] "εάν ζητώμεν αρτθμού τινός την τετραγωνικήν πλευράν, λαμβάνομεν πρώτον του σύνεγγυς τετραγώνου αριθμού την πλευράν, είτα ταύτην διπλάσιάζοντες και περί τον γενόμενον αριθμόν μερίζοντες τον λοιπόν αριθμόν, αναλυθέντα εις πρώτα εξηκοστά, από του εκ παραβολής γινομένου αφαιρούμεν τετραγωνον και αναλύοντες πάλιν τα υπολειπόμενα εις δεύτερα εξηκοστά και μερίζοντες παρά τον διπλασίονα των μοιρών και εξηκοστών, έξομεν έγγιστα τον επιζητούμενον της πλευράς του τετραγωνικού χωρίου αριθμόν·."

2.3 The Influence and Evolution of the Algorithm of Extraction in Western Europe[5]

Early printed sources on arithmetic generally use an arrangement of figures similar to that found in the galley method of division. Pacioli (1494) gives what follows in Fig. 2, and gradually, in the sixteenth century, the galley method gave way to our modern arrangement, although it was occasionally used until the eighteenth century. Among the early writers to take an important step towards our present method was Cataneo (1546), who arranged the work substantially as follows.[6]

Among the first of the well-known writers to use our method in its entirety was Cataldi, in his *Trattato* of 1613. Most early writers gave directions for "pointing off" in periods of two figures each, some placing dots above, some placing dots below, some using lines, some using colons, and some using vertical bars. Many writers, however, did not separate the figures into groups.

Pacioli's Method	**Cataneo's Method**
Extractio radicu	54756 (234
00	4 primo duplata 4
~~018~~	14 secondo
~~1270~~	
~~20880~~	46
0~~996980~~	12
~~18778980~~	27
~~99980001~~	9
9999	185
....	184
~~9898989~~	16
11999	
1	16
that is, $(99.980.001)^{1/2} = 9999$	0

Fig. 2. Pacioli and Cataneo's method of extractions

[5] [4, pp. 146–147].

[6] It is like the method from India, probably influenced by it.

In finding the square root, most of the early writers gave the rules without explanation, or at most with merely a reference to the fact $(a+b)^2 = a^2 + 2ab + b^2$.

A belief of the value of showing the reasoning behind the result led various writers in the sixteenth century to give clear explanations based on the geometric diagram.

3 The Algorithm of Extraction in Ancient China

3.1 The Pre-Method of the Algorithm of Extraction

"Squaring a field" in *Suanshu shu* [5], [6]

Question: There is a field of one *mŭ*: how many bu is it square?
Reply: It is square 15 bu and $\frac{15}{31}$ bu.
Method: If it is square 15 bu it is in deficit by 15 bu; if it squares, 16 bu, there is a remainder of 16 bu in which "bu" is a length unit in ancient China, and it is the English translation of a Chinese original text. Combine the excess and the deficit to make the divisor; Let the numerator of the deficit be multiplied by the denominator of the excess, and the numerator of the excess be multiplied by the denominator of the deficit; combine to make the dividend. Reverse this as in the method of revealing the width

$$\frac{15 \times 16 + 16 \times 15}{15 + 16} = 15\frac{15}{31}.$$

It is intriguing to find the "false position" method used here to find a good approximation to a square root, therefore perhaps suggesting the possibility that at the time of writing the algorithm for finding square roots had not been discovered. Nowadays we know that the method of "false position" for extraction in *Suanshu shu*, of course, works exactly only when linear functions are involved, but in fact,

$$\left(15 + \frac{15}{31}\right)^2 = 239\frac{721}{961},$$

which is very close to the desired value of 240. The method works well for finding square roots if the two trial values are close enough to the solution so that a straight line is a good approximation to the second-degree curve on which the solution lies.

In ancient times the Chinese were familiar with the "false position" method. There were 20 problems in the fourth chapter (Yíng Bù Zú) of the *Nine Chapters*, in which only three are nonlinear.

3.2 The Method of the Algorithm of the Extraction in the Nine Chapters and Thereafter

The method of the algorithm of extraction is first found in *Zhou Bi Suanjing* (around 100 B.C.E.), but only the name of the method. The method mentions that a, b, and c are sides of a right triangle with c as the diagonal, and if a and b are given, the general rule for finding c is equal to $\sqrt{a^2 + b^2}$. It is known as the Pythagorean theorem, and it is important to mention that the procedure of calculation is not given. However, in the *Nine Chapters*, a set of procedures for calculating the extraction fortunately was given, and it is a beautiful program with the probability to become a perfect program – the same as Horner's method – in China in the eleventh century. Here we must note that the procedure is totally algebraic, something like a computer procedure written in FORTRAN [7]. The procedure in the *Nine Chapters* is different from the notes of the Chinese mathematician Liu Hui. He has changed the division in one step of the procedure in the *Nine Chapters* into subtraction. Let us take a look at an example of the procedure enunciated by Liu Hui:

Extraction of the square root

		2	2	2	2	23	23	23
55,225	55,225	15,225	15,225	15,225	15,225	15,225	2,325	2,325
1	1	2	4	4	4	43	43	46
					1	1	1	
					3	3	3	
a	b	c	d	f	g	h	i	j

a. Put the number as dividend.
b. Make use of a rod; shift it, jumping one column.
c. When you have got the quotient, with it, multiply the used rod once. That gives you the divisor; and then with this, eliminate (making the division, not subtracting from dividend directly).[7]
d. After having eliminated, double the divisor, that gives you the fixed divisor.
e. If again one eliminates, reduce the divisor, moving it backwards.
f. Again use a rod; shift it as you did at the beginning.
g. With the new quotient, multiply it once.
h. What you get, as auxiliary, is added to the fixed divisor; with this eliminate.
i. What has been got auxiliary joins the fixed divisor; if again one eliminates, moving backwards, reduce as before; If, by extraction, you do not use up your number that means that one cannot take its root. You must then put the side as its denominator.

[7] It is not method of *Liu Hui*, but in the *Nine Chapters.*

The flow chart of the procedure is as followed in the Chemla [7, p. 316].

Compared with the Greek method, we can say that a similar method was used in the *Nine Chapters* compiled in 100 B.C.E. or so, a century before Heron of Alexandria's Method. If a is an integral value, its square is approximately close to A. They took

$$a + \frac{A - a^2}{2a + 1} \quad \text{or} \quad a + \frac{A - a^2}{2a}$$

as the approximate value for $\sqrt{A}$. That is, they took the inequality

$$a + \frac{A - a^2}{2a + 1} < \sqrt{A} < a + \frac{A - a^2}{2a}$$

for granted. Here it can be proved easily that

$$\sqrt{A} \approx \frac{1}{2}\left(a + \frac{A}{a}\right) = \frac{1}{2}\left(a + \frac{A - a^2 + a^2}{a}\right) = a + \frac{1}{2}\frac{A - a^2}{2a}.$$

Consider the statement "if, by extraction, you do not use up your number, that means that one cannot take its root." Unlike his predecessors, Liu Hui does it with "put the side as its denominator." As a matter of fact, he does not think that

$$a + \frac{A - a^2}{2a + 1} \quad \text{or} \quad a + \frac{A - a^2}{2a}$$

is exact; therefore, he defines and gives a method to calculate the irrational number as a decimal fraction.

Liu Hui's "put the side as its denominator" means that it is correct to express $\sqrt{A}$ as a square root of the radicand, whether it is a perfect square or not. As for the unperfected square number, whose square root an irrational number, Liu Hui gave the method named "qiu weishu" not the method "put the side as its denominator," which is correct. This is an approximation of $\sqrt{A}$ in the decimal number system.

3.3 Liu Hui's Geometrical Explanation of the Algorithm of Extraction

Liu Hui not only improves the algorithm of extraction, but also gives an explanation for extracting the square root. It is very interesting that his explanation is exactly the same as Theon's method. Although Liu Hui lived about 100 years before Theon, we have no evidence to show that the two methods affected each other.

Let us look at *Liu Hui's* geometrical explanation of the algorithm of extraction. For example, $\sqrt{55{,}225} = 235$. Depending on Fig. 3, he gives the explanation for the algorithm of the square root, and then a solid for the cube root. Although the figure used by Liu Hui is similar to Theon's, and both

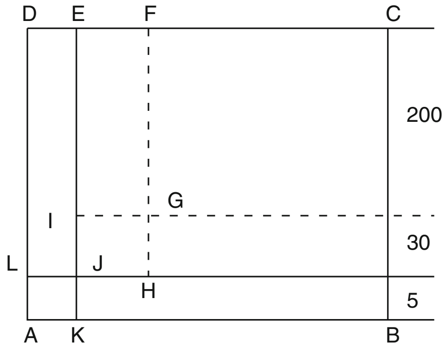

Fig. 3. Figure for Liu Hui's method of extraction

explain the algorithm of extraction. The obvious difference is that Theon's $67°$, $67°4$, $67°455''$ is the integral part in the sexagesimal number system while calculating, but Liu Hui's 200, 30 and 5 is the square root, one by one, in the decimal number system.

Another improved step made by Liu Hui is to change the division – the last step of every circle in the whole procedure – into subtraction, as mentioned above.

After this, in *Zhang Qiujian's Treatise* (end of the fourth century C.E.) and *Sun Zi's Treatise* (fifth century C.E.), the procedure of the algorithm of extraction is gradually improved, whereby the "borrowed rod" eliminated and borrowed again moves like the divisor or fixed divisor.

Thus, the procedure is more orderly and continuous. It more resembles a computer algorithm. In fact, It evolves to be a more mechanical and programming algorithm of extraction named Zengcheng method in eleventh century C.E. in China, which called P. Ruffiini–W.G. Horner's method (1804, 1819) in the West.

3.4 The Influence and Evolution of the Algorithm of Extraction in China

The algorithm of extraction in China was formed before the age of the *Nine Chapters*, and the procedure of the algorithm of extraction is actually used to solve a special kind of equation of a higher degree, namely $x^n = A$.

Based in the achievements in the *Nine Chapters*, even that of Liu Hui', *Sun Zi's Treatise* and *Zhang Qiujian's Treatise*, Jia Xian creates the figure "Jia Xian-triangle," which is called the Pascal's-Triangle in the West, created by Pascal 400 years later.

Being familiar with the program of extraction and "Jia Xian-Triangle," Jia Xian eventually created the Zengcheng method for extraction, enlightened by the "Zengcheng Qiulian Cao" obtained the numbers in "Jia Xian-Triangle." Getting the 7th root of a number, Jia Xian needs to know the coefficients (1-7-21-35-21-7-1) in the triangle (Fig. 4).

1 – 7 – 21 – 35 – 35 – 21 – 7 – 1

1	7					
1	6	21				
1	5	15	35			
1	4	10	20	35		
1	3	6	10	15	21	
1	2	3	4	5	6	7
1	1	1	1	1	1	1

Fig. 4. The table of the method of Zengcheng

Jia Xian does the algorithm with his triangle, the method is called "Lichneg Shisuo." Moreover, based on the "Zengcheng Qiulian Cao," Jia Xian creates the Zengcheng method for extraction (being the same as Horner's method), which is one of the greatest achievements of mathematics in ancient China. In addition, another great achievements called "Tianyuan shu" and "Siyuan shu" are also used for solving the higher degree equations. The theory of equations and the expansion of rational numbers are motivated by the algorithm of extraction and the solution of equations of higher degree.

4 The Algorithm of Extraction in Ancient India

The algorithm of extraction is found in *Lilāvati*, written in the twelfth century in India. The method is very similar to the Chinese method in the *Nine Chapters*, the only little difference concerning the course of calculating [8]:

> Starting from the units place, mark alternately vertical and horizontal bars above the digits so that the given number is divided into groups of two digits each with the possible exception of the extreme left group. The extreme left group will contain either one digit or two digits and will have a vertical bar on its top on the right digit respectively (Fig. 5).

From the group on the extreme left, deduce the highest possible square of a_1 (say). Then write $2a_1$ in the neighbouring column; this is called *pankti* (row). To the right of the number obtained from the above subtraction, write the digit from the next group with a horizontal bar. Now divide the number obtained by $2a_1$; this quotient a_2 should not be more than 9. Now write $2a_2$

88209 (2 4 4) 48 (9	Root	*pankti*	
36 122	2 9	4 18	1st
81 58) 410 (7	7	58 14	2nd
406 049 49		594 ÷2	3rd
00	297	297	

Fig. 5. The table of extraction in *Lilāvatī*

below $2a_1$ after shifting it one place to the right and add. The result is the second *pankti*. Write the next digit to the right of the remainder so obtained and from that subtract the square of the second quotient a_2. Now to the right of the remainder obtained, write the next digit and divide this by the second *pankti*. This gives the third digit of the required square root. Now double the third digit of the square root that should be added to the second *pankti* after shifting it by one place to the right. The result is the third *pankti*. Then write the next digit of the given number to the right of the remainder and subtract from it the square of the third digit of the square root. Repeat this process. The result is the required square root. For example, $\sqrt{88{,}209} = ?$

We'll find the square root of $\sqrt{88{,}209}$ by *Bhāskarācārya's* method. The first thing is to make horizontal and vertical bars: $\acute{8}\bar{8}\acute{2}\bar{0}\acute{9}$.

From the first group, 8, subtract the highest possible square, which is 4. We get the first reminder $4 = \acute{8} - 4$. Now write 8 (from the given number) to the right of the remainder 4 to get 48. That $2 \times 2 = 4$ is the first *pankti*. Divide 48 by 4, and see that the highest one digit quotient does not exceed 9. Here, the quotient is 9. Write this 9 below 2 in the root column. In the same horizontal line, write $2 \times 9 = 18$ with 1 below 4. Add two to get, 58, which is the second *pankti*. Then subtract from 48 to get 12. To its right write the next digit 2 and we get 122. From this subtract the square of 9 to get 41. To the right of 41 write 0, the next digit from the given number. Divide 410 by the second *pankti*, 58, and get 7 as the quotient and 4 as remainder. Next we write this number 7 in the root column and to its right with 1 below 8. Add two to get 594, which is the third *pankti*. Write the last digit 9 of the given number to the right of the number 4 to get 49. From this subtract $7^2 = 49$

to get the remainder 0. The required square root is the number obtained by writing the digit from the root column in the order in which we divided them. Thus it is 297. We can get the same number as half of the third *pankti*.

5 A Brief Comparison and Conclusions

5.1 The Accuracy in the Algorithm, Approximation in the Theorem-Proving System

In ancient Greece, according to the tradition of Euclid's *Elements*, accuracy was always emphasized, that is to say, the Greeks preferred theorem-proving as the backbone of the deductive tradition to algorithmic methods. Therefore their algorithm of extraction is approximately that of China and India.

5.2 The Minor Difference

There is a minor difference in the algorithm of extraction of the square root, though both of them belong to the same mathematical tradition of algorithm-construction. One point should be noted that the algorithm of extraction in China is more that of India, and therefore it is the main reason that the algorithm of extraction in China, first, developed to be the Zengcheng method of extraction being the same as Horner's method.

5.3 Brief Conclusions

Theorem-proving, as originated in ancient Greece, is the backbone of the deductive tradition in the history of mathematics, while the algorithm-construction, which flourished in ancient and medieval China and India, led to a strong algorithmic trend in mathematics. However, in comparison to the deductive tradition, the functions of the algorithmic tradition have unfortunately been largely ignored.

Both theorem-proving in ancient Greece and algorithm-construction in ancient and medieval China and India played indispensable roles in advancing the development of mathematics.

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Brahmagupta: The Ancient Indian Mathematician

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The five centuries extending from 500 to 1000 C.E. saw tremendous development of immense depth and complexity in the mathematics of Eastern countries. The West was intellectually dormant in this period. Brahmagupta, one of the most celebrated mathematicians of the East, indeed of the world, was born in the year 598 C.E., in the town of Bhillamala during the reign of King Vyaghramukh of the Chapa Dynasty. Bhillamala, referred to as Pi-lo-mo-lo by Hiuen-Tsang, belonged to Sind, of undivided India. Brahmagupta was the son of Jishnugupta and carried on his activities in Ujjain, the centre of ancient Indian science. He made original contributions to mathematics and astronomy that were embodied in the highly acclaimed treatises, *Brahma-sphuta-siddhānta* and *Khaṇḍa-khādyaka*. The former was composed, in 628 C.E., in his 30th year under the patronage of King Vyaghramukh, and the latter in 665 C.E., at the mature age of 67. Aryabhata I (476 C.E.), Varahamihira (505 C.E.), and Bhaskara I (522 C.E.) were his illustrious predecessors, while the giant of ancient Indian mathematics, Bhaskara II (1114–1185 C.E.), appeared several hundred years after him [3, 4]. Brahmagupta probably lived a long life beyond 665 C.E., and died in Ujjain. He belonged to the Shiva system of religion. The title "Ganak Chakra Chudamoni" (the gem of the circle of mathematicians) was ascribed to him by Bhaskara II as a mark of recognition of his talents in mathematics and astronomy.

The *Brahma-sphuta-siddhānta* and *Khaṇḍa-khādyaka* were composed in Sanskrit verse, as was the custom of the day. Prthudaksvami (864 C.E.), the noted Sanskrit scholar, rendered the difficult slokas into simpler language with interpretations, and adding illustrations some of which were his own. Lalla and Bhattotpala, who appeared in 748 C.E., and 866 C.E., respectively, were two other renowned commentators. The *Brahma-sphuta-siddhānta*

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consists of more than 24 chapters. The major portion of the treatise deals with astronomy, arithmetic, geometry, while kuttaka, or algebra, is discussed in the remaining chapters. In astronomy, Brahmagupta discussed the average and real motions of the planets, the problems of place-time-distance concerning the earth, sun, and planets, planetary conjunctions, and the rising and setting of celestial objects. He correctly described the phenomena of solar and lunar eclipses as being caused by the moon and earth casting shadows, on which be based his calculations. One chapter is devoted to the description and use of various astronomical instruments. In the chapters on mathematics, Brahmagupta discussed and established concrete rules and procedures for various operations, all in verse as usual. He discussed some of the important results obtained by his predecessors. He played a pioneering role in framing the direct and inverse rule of three. The earliest treatment of zero in algebra is to be found in *Brahma-sphuta-siddhānta*. A few rules and results in algebra and geometry may be cited by way of illustration [8, 12, 15]:

(1) To evaluate $\frac{a}{b}$ Brahmagupta gave the rule:

$$\frac{a}{b} = \frac{a}{b+h} + \frac{a}{b+h}\,\frac{h}{b}.$$

Example: Evaluate $\frac{9{,}999}{97}$ by employing Brahmagupta's rule. Choose $h = 2$. Then

$$\frac{9{,}999}{97} = \frac{9{,}999}{97+2} + \frac{9{,}999}{97+2}\,\frac{2}{97} = 101 + \frac{202}{97}.$$

Employing the rule again,

$$\frac{202}{97} = \frac{202}{97+4} + \frac{202}{97+4}\,\frac{4}{97} = 2 + \frac{8}{97}.$$

Therefore

$$\frac{9{,}999}{97} = 101 + 2 + \frac{8}{97} = 103\frac{8}{97}.$$

Similarly,

$$\begin{aligned}\frac{505}{83} &= \frac{505}{83+18} + \frac{505}{83+18}\,\frac{18}{83}\\ &= 5 + \frac{90}{83} = 5 + \frac{90}{83+7} + \frac{90}{83+7}\,\frac{7}{83}\\ &= 5 + 1 + \frac{7}{83} = 6\frac{7}{83}.\end{aligned}$$

(2) To find the square of an integer the rule is:

$$x^2 = (x-y)(x+y) + y^2.$$

(3) Given a side a of a right-angled triangle, the other sides are:

$$\frac{1}{2}\left(\frac{a^2}{m} - m\right), \quad \frac{1}{2}\left(\frac{a^2}{m} + m\right),$$

where m is any rational number.

(4) The circumradius of a triangle two of whose sides are b, c, with p the altitude at their point of intersection, is bc/p.

(5) The area of a cyclic quadrilateral with sides a, b, c, d is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $2s = a + b + c + d$. The diagonals are

$$x = \left\{\frac{(ab+cd)(ac+bd)}{ad+bc}\right\}^{1/2} \quad \text{and} \quad y = \left\{\frac{(ad+bc)(ac+bd)}{ab+cd}\right\}^{1/2}.$$

Besides this basic problem, Brahmagupta discussed several other interesting problems on quadrilaterals.

However, Brahmagupta's most important mathematical contribution lies in attempting to solve three types of algebraic equations:

1. Equations in one unknown (linear and quadratic)
2. Equations in several unknowns
3. Equations involving products of unknowns.

He found:

$$x = \frac{(4ac + b^2)^{1/2} - b}{2a}$$

as a root of the quadratic equation $ax^2 + bx + c = 0$, although from his discussion of some other problems it appears that he was aware of the existence of the other root as well. He found the solution to the indeterminate quadratic equation involving two unknowns, $nx^2 + 1 = y^2$, in integers, provided one could obtain a solution (α, β) for the equation $nx^2 + k = y^2$ for $k = \pm 1, \pm 2, \text{or} \pm 4$. The splendid ingenuity with which Brahmagupta solved this equation for the first time in the world is highly acclaimed by modern mathematicians. With the help of the solution so obtained, one can now obtain an infinite number of solutions by repeated application of the process known as "*samāsa*." Considering that this scholastic feat was demonstrated as far back as the beginning of the seventh century, Brahmagupta has been described by many modern historians as one of the most gifted mathematicians in the world. Florian Cajori [5], the noted historian, summed up the matter in an extraordinarily suggestive manner: "The perversity of fate has willed it that the equation $y^2 = nx^2 + 1$ should now be called Pell's Problem, while in recognition of Brahmin scholarship it ought to be called the 'Hindu Problem.' It is a problem that has exercised the highest faculties of some of our greatest modern analysts." Indian mathematical historians would like to call it the Brahmagupta–Bhaskara problem, keeping in mind that Bhaskara perfected Brahmagupta's method of solution in the twelfth century; Bhaskara used "Chakra-vala," or a cyclic process, to improve Brahmagupta's method by doing away with the necessity of finding a trial solution.

The *Khaṇḍa-khādyaka* was a systematic and complete work on Hindu astronomy. It was composed by Brahmagupta in the later years of his life. Of the 11 chapters, *Khaṇḍa-khādyaka* proper consists of eight, together with the *Uttara* part of three chapters. In the first, the author discusses methods and topics in astronomy, and the latter part gives corrections to the former. In *Uttara Khaṇḍa-khādyaka*, Brahmagupta taught for the first time in the history of mathematics the improved rules for interpolation using second differences and he used it to compute the sine of an angle between the angles given in Aryabhata's sine table. The rule employed is equivalent to the modern Newton–Stirling interpolation equation up to second-order differences. The modern mathematical world has yet to pay due recognition to Brahmagupta's mathematical genius on this score. The researcher, Kim Plofker [10], has described in detail the method enunciated by Brahmagupta in *Brahma-sphuta-siddhānta* to calculate the mean longitude of the sun if its true longitude is known.

Brahmagupta's contributions influenced the mathematicians Sridhara, Mahavira, Bhaskara II, and others of later periods in his own country. In a sense, the great mathematical tradition that prevailed in the country from the days of the Vedas, the *Sulbas*, and the *Bakshali* manuscripts of ancient ages found its culmination in the appearance of Brahmagupta. Bhaskara II was his most able and celebrated successor, upholding this great tradition of the land. Moreover it was through the *Brahma-sphuta-siddhānta* and *Khaṇḍa-khādyaka* that the Arabs became conversant with Indian astronomy, arithmetic, and algebra. These treatises reached Arabia during the reign of Khalif Al Mansur (753–774 C.E.) of Baghdad and were translated by Al Fazari and Yakub ibn Tarik into Arabic at Khalif's insistence. The famous Sindhind of Arabian scientific literature was a translation of the *Brahma-sphuta-siddhānta*, while the only other contemporary astronomical treatise in Arabic, called Alarkand, was supposedly the translation of *Kanda-khādyaka*. Both works were widely used for a long time and exercised great influence on Arab mathematics and astronomy. Musa Al Khowarizmi (825 C.E.), the great Arab mathematician, based his astronomical tables on these translations. Arabs transmitted some of these results to Europe centuries later. Albiruni resided in India during the period 1017–1030 C.E., and learned Indian astronomy chiefly by studying the *Khaṇḍa-khādyaka*.

The 12th and 18th chapters of *Brahma-sphuta-siddhānta*, dealing with arithmetic and algebra, were translated into English by Colebrooke [6]. The *Khaṇḍa-khādyaka* was translated into English with notes and comments by Sengupta [14].

"Brahmagupta holds a remarkable place in the history of eastern civilization." His masterly treatises influenced Arab scholars for nearly 350 years when Islamic civilization was at its peak at Baghdad and elsewhere till about 1100 C.E. The influence spread to the rising Christian Europe through the Arabs, via Spain, Italy, and other channels [9, 11]. Hindu creativity in mathematics and astronomy, however, gradually came to a halt at the advent of barbaric invasions of the country by foreign invaders, while the Arab enlightenment in mathematics and astronomy also crumbled owing to their inherent historical anti-science contradictions at about the same time [2]. Europe, having remained dormant for centuries before this period, came forward and swiftly and successfully occupied centre stage in cultivating mathematics and science [1, 13]. Their march forward culminated in the industrial revolution, and their march forward continues through modern times!

It has been accepted by all historians of science and mathematics that these subjects can be creatively pursued only in an environment of social, political, and economic stability. Prior to the Muslim invasions, India did have a history of stability: the emergence of Aryabhata, Varahamihira, Bhaskara I, Brahmagupta, and others indeed occurred in a golden phase of Indian history. But with the raids of Mahmud of Ghazni in the early part of the eleventh century C.E., the disintegration of the country began, expansion ended and cultural and scientific advancement stopped. "The period of resistance began" [7, 16, 17].

Brahmagupta is uniquely placed in the history of these great transitions of civilization across geographical and temporal boundaries, across different cultures, and across phases of human enlightenment and scholarship in mathematics and astronomy.

Brahmagupta was truly a man of immense genius, a genuine creative writer with incomparable suggestions on many difficult mathematical problems (e.g., use of second differences, solution of indeterminate equations). His concept of zero as a fundamental mathematical notion opened up a new vista. He made far-reaching contributions to the coordinated structure of mathematics. Historians of mathematics are to this day engaged in painstaking research in measuring the Stature of Brahmagupta in the perspective of world mathematics.

Mathematics and science are considered to be a shared heritage of mankind. Brahmagupta's contributions are part of this heritage.

Appendix: Solution of $Nx^2 + 1 = y^2$, Where N Is an Integer

Let (α, β) and (α', β') be integral solutions of the equations

$$Nx^2 + k = y^2 \quad \text{and} \quad Nx^2 + k' = y^2$$

for some chosen values of k and k'. It can be easily shown that

$$x = \alpha\beta' \pm \alpha'\beta, \quad y = \beta\beta' \pm N\alpha\alpha'$$

are solutions of $Nx^2 + kk' = y^2$.

This is known as Brahmagupta's lemma. It was described in *Brahmasphuta-siddhānta* (628 C.E.). This is known as **the principle of composition** or ***samāsa***. Euler rediscovered it in 1764. In particular, set $k = k'$. If $N\alpha^2 + k = \beta^2$, then $x = 2\alpha\beta, \quad y = \beta^2 + N\alpha^2$ is a solution of the equation $Nx^2 + k^2 = y^2$. Hence one get:

$$N\left(\frac{2\alpha\beta}{k}\right)^2 + 1 = \left(\frac{\beta^2 + N\alpha^2}{k}\right)^2.$$

Therefore

$$x = \frac{2\alpha\beta}{k}, \quad y = \frac{\beta^2 + N\alpha^2}{k}$$

is a solution of $Nx^2 + 1 = y^2$.

We require, however, that the above solution be an integral solution. This can happen under the following cases:

1. Suppose $k = \pm 1$; then the solution is integral.
2. Suppose $k = \pm 2$; again an integral solution $x = \alpha\beta, \ y = \beta^2 - 1$ is obtained (with positive sign).
3. Suppose $k = 4$; then $x = \alpha\beta/2, \quad y = (\beta^2 - 2)/2$. If α is even, then since $N\alpha^2 + 4 = \beta^2$, clearly β is also even. Hence an integral solution is obtained. If, however, α is odd, then the *samāsa* operation is applied between

   $$\begin{array}{lll} \frac{1}{2}\alpha\beta & \frac{1}{2}(\beta^2 - 2) & 1 \\ \frac{1}{2}\alpha & \frac{1}{2}\beta & 1 \end{array}$$

   Thus Brahmagupta obtained

   $$x = \frac{1}{2}\alpha(\beta^2 - 1), \quad y = \frac{1}{2}\beta(\beta^2 - 3),$$

   which are both integers when β is odd. When β is even, the earlier values $x = \alpha\beta/2, \quad y = (\beta^2 - 2)/2$ are integers.
4. Suppose $k = -4$. Then,

   $$N\left(\frac{1}{2}\alpha\beta\right)^2 + 1 = \left(\frac{1}{2}(\beta^2 + 2)\right)^2.$$

   Applying ***samāsa*** to

   $$\begin{array}{lll} \frac{1}{2}\alpha\beta & \frac{1}{2}(\beta^2 + 2) & 1 \end{array}$$

   with itself and eliminating N, Brahmagupta obtained

$$x = \frac{1}{2}\alpha\beta(\beta^2+2), \quad y = \frac{1}{2}(\beta^4+\beta^2+2)$$

as the solution of $Nx^2+1=y^2$. Similarly applying *samāsa* between

$$\begin{array}{ccc} \frac{1}{2}\alpha\beta & \frac{1}{2}(\beta^2+2) & 1 \\ \frac{1}{2}\alpha\beta(\beta^2+2) & \frac{1}{2}(\beta^4+\beta^2+2) & 1 \end{array}$$

the following integral solution is obtained when β is odd or even:

$$x = \frac{1}{2}\alpha\beta(\beta^2+1)(\beta^2+3), \quad y = (\beta^2+2)\left[\frac{1}{2}(\beta^2+1)(\beta^2+3)-1\right].$$

Examples: Starting with $92\times1^2+8=100$, a solution $x=120,\ y=1{,}151$ for the equation $92x^2+1=y^2$ is obtained. Similarly starting with $83\times1^2-2=9^2$, a solution $x=9,\ y=82$ for the equation $83x^2+1=y^2$ is obtained.

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Mainland Southeast Asia as a Crossroads of Chinese Astronomy and Indian Astronomy

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1 Introduction

Southeast Asia is divided into two parts, namely, Mainland Southeast Asia and Insular Southeast Asia (also called the Indo-Malay Archipelago). Mainland Southeast Asia is further divided into two parts; one is Vietnam, where Chinese influence is larger than Indian influence, and the other includes Burma (Myanmar), Cambodia, Laos, and Thailand, where Indian influence is greater. The Malay Peninsula is a part of Mainland Southeast Asia geographically, but is culturally closer to Insular Southeast Asia.

Mainland Southeast Asia is a kind of crossroads of Chinese and Indian culture, and the traditional astronomies there were also influenced by Chinese and Indian astronomy.

I discussed Mainland Southeast Asian astronomy at a conference held in Singapore in 1999 [11] and a conference held in Shanghai in 2002 [13]. The present paper is a continuation of these papers, and is a revised version of a paper presented at the First International Conference on History of Exact Sciences along the Silk Road, held in Xi'an, China from July 31 to August 3, 2005.

2 Vietnamese Calendrical Astronomy

In Vietnam, the Chinese calendar was introduced first, and modified later. I discussed the history of Vietnamese mathematics and astronomy at a conference held in Tianjing in 2002 [12]. A rough history of the Vietnamese calendar is as follows

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(1) Acceptance of Chinese calendars, notably the *Shoushi* calendar.

The *Yuanshi* (official history of the Yuan Dynasty) says that a Chinese calendar was given to the Vietnamese king (Tran Dynasty) in 1265. At that time, the famous *Shoushi* calendar, which is one of the best traditional calendars in China, was not developed, and the *Daming* calendar of the previous Jin Dynasty was still used in China.

The *Dai-Viet su-ky toan-thu* (official dynastic history of Vietnam) says that the *Shoushi* calendar was given to the Vietnamese king (Tran Dynasty) by the Chinese emperor in 1324.

From the above sources, we can presume that the Chinese calendars were accepted in Vietnam by the early fourteenth century.

(2) *Hiep-ky* calendar

The *Dai-Viet su-ky toan-thu* says that the *Shoushi* calendar was converted into the *Hiep-ky* calendar in 1339 (Tran Dynasty). This record possibly means that the name of the calendar was changed, and does not necessarily mean that the method of calculation was amended.

The *Mingshi* (official history of the Ming Dynasty) reports that the *Datong* calendar of China was given to the Vietnamese king (Tran Dynasty) in 1369, the year after the establishment of the Ming Dynasty.

It is possible that the *Datong* calendar, which is almost the same as the *Shoushi* calendar, was harmoniously accepted in Vietnam at that time.

(3) *Thuan-thien* calendar

The *Dai-Viet su-ky toan-thu* mentions that the *Hiep-ky* calendar was abolished, and the *Thuan-thien* calendar was adopted in 1401 (Ho Dynasty). The difference between these two calendars is not recorded.

(4) Acceptance of the Chinese *Datong* calendar

Vietnam was directly ruled by the Ming Dynasty of China from 1413 to 1428, and the *Datong* calendar must have been used. The Le Dynasty was founded in 1428 in Vietnam, but there is no record that the calendar was changed.

The *Mingshi* reports that the *Datong* calendar was given to the Mac, who ruled Vietnam for a certain period, in 1540.

According to Chang [2], Nguyen Huu-than wrote in his *Y-trai toan-phap nhat-dac-luc* in 1829 that the *Datong* calendar had been used until the *Hiep-ky* calendar (= *Shixian* calendar, which is the last traditional Chinese calendar, made under Western (Jesuit) influence) was adopted in 1813.

(5) *Hiep-ky* calendar (= *Shixian* calendar)

According to Nguyen Huu-than, as we have seen above, the Chinese *Shixian* calendar was adopted in Vietnam as the *Hiep-ky* calendar in 1813 (Nguyen Dynasty). This *Hiep-ky* calendar should not be confused with its previous namesake.

Chang [2] compared Vietnamese chronological tables with the Chinese calendar, and pointed out that the *Shixian* calendar was actually used in Vietnam from 1813 to 1840.

(6) Consideration of the longitudinal difference

Chang [2] pointed out that the Vietnamese *Hiep-ky* calendar has differed from the Chinese *Shixian* calendar from 1841.
This must be due to the consideration of the longitudinal difference between Vietnam and China.

3 Mainland Southeast Asian Astronomy (Except for Vietnam)

In the traditional calendars of Mainland Southeast Asia (except for Vietnam) [5,6], namely Burma [8,9,18], Cambodia [7], Laos [4,17] and Thailand [1,10], and also the Tai (Dai) people in Sipsong-panna in Yunnan province in South China [20], the sidereal year is usually used instead of the tropical year. The length of a year and that of a month used there are as follows:

Some of the Southeast Asian calendars that appear to have followed the Ārdharātrika school (one of the schools of Hindu astronomy [16]) are as follows:

	Length of a year (days)	Length of a month (days)
Souriat (Ayutthaya dynastsy, Thailand) [3,10]	365.25875	29.530583
Suding and *Suliya* (Sipsong-panna, Yunnan, China) [20]	365.25875	29.530583
Makaranta (Burma) [9]	365.25875	29.530583

Some of the Southeast Asian calendars that appear to have followed the Saura school (one of the schools of Hindu astronomy) are as follows:

	Length of a year (days)	Length of a month (days)
Xitan (Sipsong-panna) [20]	365.258756481	29.530588
Thandeikta (Burma) [9]	365.258756477	29.530588

Since these calendars are lunisolar, a certain cycle of intercalation is needed, and the 19-year cycle is usually used. This cycle has 7 intercalary months in 19 years. It is harmonious with a tropical year, but not with a sidereal year. This fact obliges us to suspect that the origins of the sidereal year and the 19-year cycle are different.

The length of a year used in Mainland Southeast Asia as well as in Hindu astronomy is slightly longer than the length of the actual sidereal year, while the length of a synodical month used there is almost exact. There is almost no room to doubt the Indian origin of the sidereal year. However, the origin of the 19-year cycle is controversial.

4 Mainland Southeast Asian 19-Year Cycle

I discussed the origin of the 19-year cycle in the Mainland Southeast Asian calendar at a conferences held at Chengdu in 2004 [14] and at Chiang Mai in 2004 [15]. I would like to continue my discussion of this topic in this paper.

In the Mainland Southeast Asian calendars, there are 7 intercalary months every 19 years. Further, the 29-day month and the 30-day month are arranged alternately. An intercalary month has 30 days, and 11 extra days are distributed in every 57 years. Therefore, a cycle of 57 years (3×19 years) has 705 months, and has 20,819 days. This method was widely used in Mainland Southeast Asia [6, 20]. The 19-year cycle of intercalation was already mentioned in the *"Souriat"* of the Ayutthaya Dynasty, Thailand [10].

This method is not harmonious with the proper length of a year and that of a month in the Mainland Southeast Asian calendars, and may be considered to be a kind of simplified method. According to this method, a year becomes 365.2456 (=20,819/57) days, and a month becomes 29.530496 (=20,819/705) days. Both lengths are different from their correct lengths in the Mainland Southeast Asian calendars. It is likely that this difference is due to the combination of two different traditions.

In Fig. 1, the relationship between the length of a year and that of a month is shown. If the 19-year cycle is used, a combination of a year and a month is on a horizontal line in the figure. If the 19-year cycle should be used, it is clear at first sight that a synodical month can be combined with a tropical year with some inaccuracy, but it is almost impossible to combine a synodical month with a sidereal year. It is impossible to suppose that the 57-year cycle was created in Southeast Asia, where the sidereal year is used instead of the tropical year. If a synodical month was combined with a sidereal year, the well-known 76-year cycle ($76 = 4 \times 19$) would have been still better. The 57-year cycle evidently presupposes a certain knowledge of a tropical year, because it is a very good harmonization of a synodical month and a tropical year.

First, I shall show that this 19-year cycle is not of Indian origin. Usually, Hindu calendars use the sidereal year, and the 19-year cycle is not harmonious with the sidereal year and can be used with the tropical year only. Almost the only Sanskrit astronomical work that mentions the 19-year cycle is

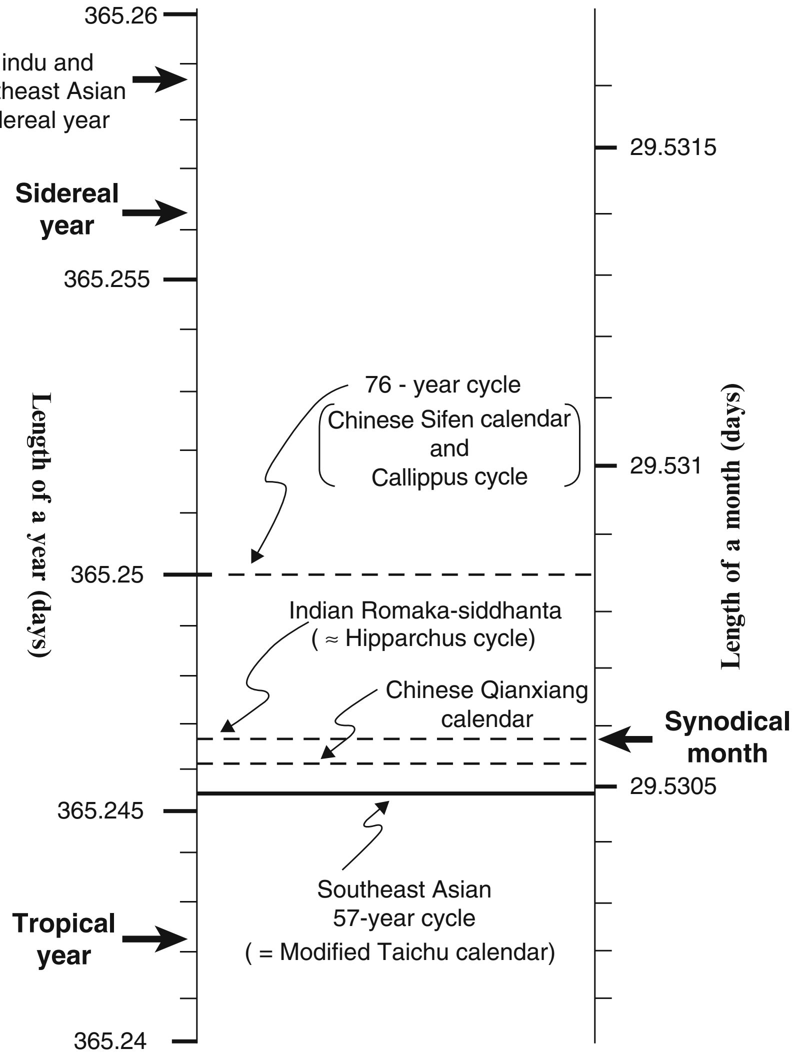

Fig. 1. Relationship between the length of a year and a month

the *Romaka-siddhānta*, which has been summarized in the *Pañca-siddhāntikā* of Varāhamihira (sixth century C.E.). The *Pañca-siddhāntikā* (I.15) states that there are 1050 intercalary months and 16,547 elided days in 2,850 years in the *Romaka-siddhānta*. In other words, 2,850 (=19 × 150) years have 35,250 (=2,850 × 12 + 1,050) months, including 1,050 (=7 × 150) intercalary months, and 1.040,953 (=35,250 × 30 − 16,547) days. Accordingly, 1 year becomes 365.2467 days (tropical year) and 1 month becomes 29.530582 days. (The use of the tropical year is exceptional in classical Sanskrit texts.) This value is very close to that of Hipparchus of ancient Greece. (The value in the *Romaka-siddhānta* may have been influenced by the Greek value.) Both lengths used in the *Romaka-siddhānta* (and also Hipparchus) are different from the value used in Mainland Southeast Asian calendars. Therefore, the Indian 19-year cycle is not the origin of that of Mainland Southeast Asia.

It is well known that the 19-year cycle of intercalation was widely used in ancient China. Let us see the calendars that were used or produced during the Western Han (206 B.C.E.–23 C.E.) and Eastern Han (25–220 C.E.) Dynasties. The *Sifen* calendar (used until the beginning of the Western Han Dynasty, and again used in the Eastern Han Dynasty), the *Taichu* calendar (produced in the Western Han Dynasty and used until the beginning of the Eastern Han Dynasty), the *Qianxiang* calendar (produced at the end of the Eastern Han Dynasty, and used in the Three Kingdoms period), etc., used the 19-year cycle. The length of a year and that of a month are as follows. (The value used in the *Sifen* calendar is the same as that of Callippus in ancient Greece. The Chinese cycle and the Callippus cycle are probably independent.)

Chinese calendars [19]	Length of a year (days)	Length of a month (days)
Xitan The *Sifen* calendar	365.2500	29.53085
The *Taichu* calendar	365.2502	29.53086
The *Qianxiang* calendar	365.2462	29.53054

It is clear that all of them are different from the value used in Mainland Southeast Asian calendars.

There was also the "modified *Taichu* calendar" in the Eastern Han Dynasty. It was proposed by the followers of the *Taichu* calendar, but was not officially used. The proposal is recorded in the "Treatise of Tuning System and Calendar" of the *Xuhanshu* (continuation of the history of the Han Dynasty), which is included in the *Houhan-shu* (official history of the Later (Eastern) Han Dynasty). In the original *Taichu* calendar, 171 years have $62{,}457\,{}^{63}\!/_{81}$ ($=62{,}457\,{}^{1{,}197}\!/_{1{,}539}$) days, but the "modified *Taichu* calendar" proposed to omit the fraction ($63/81 = 1{,}197/1{,}539$) every 171 years. Then, $171(=9 \times 19)$ years become 62,457 days, and $57(=3 \times 19)$ years become 20,819 days.

This value (57 years = 20,819 days) is exactly the same as that of the Mainland Southeast Asian calendars. Therefore, I presume that the origin of the 19-year cycle and the 57-year cycle of the Mainland Southeast Asian calendars is the "modified *Taichu* calendar," which was proposed in the Eastern Han Dynasty. We can also note that the 60-year cycle seems to have been introduced to the Tai people from Central China during the Han Dynasty or so, and accordingly, the supposition that the "modified *Taichu* calendar" was also introduced to the Tai people is also possible.

5 Conclusion

Vietnamese astronomy was originally based on Chinese astronomy, and developed into Vietnamese traditional astronomy suitable to the Vietnamese longitude.

Other Mainland Southeastern astronomies were influenced by both Chinese and Indian astronomy, and developed into their traditional astronomies, which are still used to determine traditional festivals, etc.

I presume that the Tai people in South China received the influence of the Chinese calendar, such as the use of the 60-year cycle, at the time of the Han Dynasty (206 B.C.E.–220 C.E.) or so, and the "modified *Taichu* calendar" (Eastern Han Dynasty (25–220 C.E.)) may have been the origin of the 19-year and 57-year cycles used in the Tai calendar and other Mainland Southeast Asian calendars. Later, the Indian calendar was introduced into Southeast Asia, and is the origin of the use of the sidereal year, etc. These Chinese and Indian traditions were combined in mainland Southeast Asia, and developed into the Mainland Southeast Asian calendars, which have the unique characteristics of this region.

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Mathematical Literature in the Regional Languages of India*

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Ever since the great Henry Thomas Colebrooke (1765–1837) translated the *Līlāvatī* into English[2] at the beginning of the nineteenth century, the concern of the historians of Indian mathematics has been the exploration of primary sources in Sanskrit. This emphasis on the Sanskrit texts is unexceptionable because Sanskrit has been the chief medium of intellectual discourse in India and the major vehicle of pan-Indian dissemination of ideas.

However, there have also been other parallel streams of intellectual communication in India: the Middle Indo-Aryan variants called the Prakrits, the succeeding New Indo-Aryan languages, the Dravidian languages of the South, and Persian. All these languages possess rich and varied literature, which may contain works on mathematics as well.

The extent and the nature of the exchanges between the pan-Indian Sanskrit on the one hand, and these regional languages on the other, have yet to be properly mapped. We may, however, postulate certain hypotheses on the nature of the exchanges. It is certain that these exchanges were never one-sided, i.e., from the "Great Tradition" of Sanskrit to the "Little Traditions" of regional languages. The two traditions were mutually complementary. While mathematical ideas and processes were systematized in Sanskrit manuals, the

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[2] Henry Thomas Colebrooke (tr), *Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brehmegupta and Bháscara*, London 1817, First Indian Reprint: *Classics of Indian Mathematics: Algebra with Arithmetic and Mensuration, From the Sanskrit of Brahmagupta and Bhāskara*, with a foreword by S. R. Sarma, Sharda Publishing House, Delhi (2005).

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broader dissemination of these ideas took place in the regional languages. Conversely, Sanskrit has also absorbed much from the local traditions. Anthropologists recognize today that the so-called "Little Traditions" played a significant role in shaping the "Great Tradition."[3]

As mentioned earlier, the process of give-and-take is yet to be mapped, and this is especially true of mathematical literature. Without exploring the literature in regional languages, a full picture will not emerge on how mathematical ideas were developed and systematized in Sanskrit manuals and how they were disseminated and popularized in the regional languages.

The mathematical literature in Sanskrit has been surveyed and studied to a large extent.[4] But no attempts have been made so far to even survey the mathematical literature available in the regional languages. There have been one or two exercises to compile bibliographies of source materials in the regional languages, but none have come to fruition.[5] In the late 1950s, K. R. Rajagopalan published brief surveys of mathematical literature in the four states of South India.[6] This includes works composed in Tamil,

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[3] On this, see, *inter alia*, Swami Agehananda Bharati, *Great Tradition and Little Traditions: Indological Investigations in Cultural Anthropology*, Chowkhamba Sanskrit Studies, Vol. XCVI, Chowkhamba Sanskrit Series Office, Varanasi (1978).

[4] To mention the most prominent works: Bibhutibhusan Datta and Avadhesh Narayan Singh, *History of Hindu Mathematics: A Source Book*, 2 parts, 1935, 1938; single volume edition: Asia Publishing House, Bombay etc., 1962; A. K. Bag, *Mathematics in Ancient and Medieval India*, Chaukhamba Orientalia, Varanasi-Delhi, 1979; T. A. Saraswati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarsidass, Delhi-Varanasi-Patna, 1979; David Pingree, *Census of the Exact Sciences in Sanskrit*, Series A, Volumes 1–5, American Philosophical Society, Philadelphia, 1970–1994 (in progress); also the large number of papers by R. C. Gupta listed in: Takao Hayashi, A Bibliography (1958–1995) of Radha Charan Gupta, Historian of Indian Mathematics, *Historia Scentiarum*, 6.1 (1996) 43–53.

[5] K. V. Sarma, *A Bibliography of Kerala and Kerala-based Astronomy and Astrology*, Vishveshvaranand Institute, Hoshiarpur, 1972, though primarily devoted to works in Sanskrit, contains several works on mathematics composed in Malayalam as well. The Government Oriental manuscripts Library, Madras, brought out a Malayalam work on Mathematics, *Kaṇakkusāram*, ed. D. Achyutha Menon, Madras, 1950. But, as far as I know, no study of this work has appeared to date.

[6] K. R. Rajagopalan, Mathematics in Karnataka, *Bhavan's Journal*, 5.6 (October 1958) 52–56; Mathematics in Tamil Nadu, ibid. 5.20 (May 1959) 39, 42–44; Mathematics in Andhra, ibid. 6.8 (November 1959) 47–49; Mathematics in Kerala, ibid. 6.10 (December 1959) 61–64. See also R. C. Gupta, Some Telugu Authors and Works on Ancient Indian Mathematics, *Souvenir of the 44th Conference of the Indian Mathematical Society*, Hyderabad, pp. 25–28 (1978).

Malayalam, Kannada, and Telugu. In a recent article, K. K. Bishoi mentions the names of several scholars who composed mathematical works in Oriya.[7] One wishes to know more about them.

Popularization of mathematics or any other science in India is not necessarily coterminous with vernacularization. Within the Sanskrit tradition itself there were attempts to compile popular handbooks of mathematics. In the eighth century, Śrīdhara abridged his own voluminous *Pāṭīgaṇita* and prepared the *Triśatikā* in 300 verses. In his admirable analysis of *The Bakhshālī Manuscript*, Takao Hayashi has shown that it was not an independent manual but a compilation made from diverse sources for practical application.[8] Hayashi also brought to light two other compilations of popular nature, namely the anonymous *Pañcaviṃśatikā*[9] and the *Caturacintāmaṇi* of Giridharabhaṭṭa.[10]

But these attempts at popularization received sharper focus in Prakrit and other regional languages. The mathematical works composed in these languages, though largely modeled on Sanskrit manuals, contain much information of contemporary relevance. The Śrīmāla Jainas in the West, the Kāyasthas in the North, the Karaṇams and other village accountants in the South, and the merchants in all parts of India were the numerate professionals who used mathematics in their daily transactions. The experience these classes of people gained in the application of arithmetical computations in their professions may be available from the sources in regional languages. To put it differently, while the theoreticians of mathematics wrote in Sanskrit, the practitioners of mathematics wrote in the regional languages. It is in the writings of these professionals that we come across shortcuts in computational

[7] K. K. Bishoi, Palm-Leaf Manuscripts in Orissa, in: A. Pandurangan and P. Maruthanayagam (ed), *Palm-Leaf and Other Manuscripts in Indian Languages*, Institute of Asian Studies, Madras, 1996 pp. 46–56; esp. pp. 52–53: "Orissa... has a rich heritage of mathematical treatises. Proficiency in mathematics is exemplified in the manuscripts. The authors of Orissan Mathematical manuscripts are Anirdha, Artta Dasa, Krushna Padhiari, Ucchavananda, Kunjabana Pattnayaka Krupasindhu, Gangadhara, Nimai Charana, Radha Charana, Brajabhusana, Vamadeva, Shiva Mohanty, Sarangadhara, Hari Nayaka and Srinatha of the Ganita Sāstras. The *Līlāvatī Sūtra* is very popular in Orissa. The manuscript is available in all parts of the state. It provides scope for all age groups to study mathematics through works of addition, subtraction, multiplication, division, mensuration, trigonometry, algebra, etc."

[8] Takao Hayashi, *The Bakhshālī Manuscript: An Ancient Indian Mathematical Treatise*, Groningen Oriental Studies, Vol. XI, Egbert Forsten, Groningen, 1995.

[9] Takao Hayashi, *The Pañcaviṃśatikā* in its Two Recensions: A Study in the Reformation of a Medieval Sanskrit Mathematical Textbook, *Indian Journal of History of Science*, 26, 393–448 (1991).

[10] Takao Hayashi, The *Caturacintāmaṇi* of Giridharabhaṭṭa: A Sixteenth Century Sanskrit Mathematical Treatise, *SCIAMVS: Sources and Commentaries in Exact Sciences*, 1, 133–209 (2000).

processes, verifying results, e.g., by casting off nines, conversion of one set of monetary units into another, and so on.

For example, a Telugu manuscript of uncertain date contains an elaborate classification of the variations of the Rule of Three, and also a simpler method of solving the problems of the Rule of Five, etc. Thus, in the case of the Rule of Five, the product of the last three terms is divided by the product of the first two terms; or in the case of the Rule of Seven, the product of the last four terms is divided by the product of the first three terms. This, in effect, is what Bhāskara I seemed to suggest before Brahmagupta proposed the arrangement of all the terms in two vertical columns. The Telugu solution is the ultimate stage of a mechanical solution.[11]

Furthermore, the Sanskrit texts on arithmetic employ in their sums the so-called *Māgadhamāna*, i.e., units of measurement, weight, and coinage, which are said to have been prevalent in Magadha in ancient times (probably when Āryabhaṭa was writing at Kusumapura) and not the contemporary units. Thus the Sanskrit texts – be it the *Āryabhaṭīya* composed in the fifth century in Kusumapura or the *Gaṇitalatā* by Vallabha Gaṇaka of Jayanagara of the mid-nineteenth century[12] – are neutral in relation to space and time. Not so the texts in the regional languages. Even when they are directly translated from Sanskrit, these texts employ in their sums the contemporary metrological units. This is of great value not only to metrology, but also for the economic history of the region. I shall illustrate these features through the example of an Apabhraṃśa text composed by a Śrīmāla Jaina, called Pherū.

In the first quarter of the fourteenth century, Ṭhakkura Pherū,[13] a learned Jaina employed as the assay-master at the court of the Khalji sultans of Delhi, wrote six scientific texts of a popular nature in Apabhraṃśa.[14] One of the six works composed by him is on arithmetic and geometry. It is variously called *Gaṇitasārapāṭīkaumudī*, *Gaṇitasārakaumudī*, or just *Gaṇitasāra*.[15] This

[11] Sreeramula Rajeswara Sarma, Rule of Three and Its Variations In India: Yvonne Dold-Samplonius et al. (eds.), From *China to Paris: 2000 Years Transmission of Mathematical Ideas*, Steiner Verlag, Stuttgart, pp. 133–156, especially 149 (2002).

[12] This is perhaps the last work on mathematics to be composed in Sanskrit in the traditional style, and is yet to be published. The Department of Sanskrit, Aligarh Muslim University, possesses three manuscripts of this work.

[13] On his life and works, see Sreeramula Rajeswara Sarma, *Ṭhakkura Pherū's Rayaṇaparikkhā*, Viveka Publications, Aligarh, 1984, Introduction.

[14] These six scientific works (and a seventh of a religious nature) were edited and published by the Jaina savant Jinavijaya Muni under the title *Ṭhakkura-Pherū-viracita-Ratnaparīkṣādi-sapta-granthasaṃgraha*, Rajasthan Oriental Research Institute, Jodhpur, 1961.

[15] *Gaṇitasārakaumudī: The Moonlight of the Essence of Mathematics by Ṭhakkura Pherū*, edited with Introduction, Translation and Mathematical Commentary by SaKHYa (Sreeramula Rajeswara Sarma, Takanaori Kusuba, Takao Hayashi and Michio Yano), Manohar, New Delhi (2009).

work is largely based on Śrīdhara's *Pāṭīgaṇita*; several verses are phonetically converted from Śrīdhara's Sanskrit.

But there is a considerable amount of input by Pherū himself, which is related to contemporary society. The units of measurement and the illustrative examples given by Pherū reflect their wide applications in different professions of that period, such as that of traders, carpenters and masons. The section on solid geometry contains rules for calculating the volumes of bridges (*pulabaṃdha*), niches (*tāka*), staircases (*sopāna*), domes (*goṃmaṭa*), square and circular towers with a spiral stairway in the middle (*pāyaseva*), and so on. Some of these are new architectural features that were being introduced by the sultans in the fourteenth century. Consider the following definition: "The *munārayās* are like circular towers with a spiral stairway in the middle, as far as the inside is concerned. But outside there is this difference. The outer wall consists of half triangles and half circles." The meaning of the cryptic last sentence is this: the horizontal cross-section of the outer circumference consists of alternate triangles and semicircles. It should be remembered that about one hundred years prior to this, Qutbuddin Aibak built the Qutb Minar in Delhi. The lower storey of the Qutb Minar has alternately circular and angular columns, the second storey has circular columns, and the third has angular columns. I believe that Pherū is referring here to such a tower with fluted columns.

Pherū also touches upon various aspects of contemporary life that are quantifiable, from the average yield of different crops per *bīghā* to the quantity of ghee that can be extracted from cow's milk and buffalo's milk. He is perhaps the first mathematician to devise rules for converting the dates of the Vikrama era into those of the Hijrī era and vice versa.[16] Finally, he teaches us how to construct magic squares for even (*sama*), odd (*viṣama*), and oddly even (*sama-viṣama*) orders. This is the first systematic treatment of magic squares in India; and it precedes the most elaborate discussion by Nārāyaṇa in his *Gaṇitakaumudī* by about 40 years.[17]

[16] Cf. S. R. Sarma, Conversion of Vikrama-Saṃvat to Hijrī in: B. V. Subbarayappa and K. V. Sarma (eds.), *Indian Astronomy: A Source-Book*, Bombay, pp. 59–60 (1985); idem, Islamic Calendar and Indian Eras in: G. Kuppuram and K. Kumudamani (eds.), *History of Science and Technology in India*, Delhi, Vol. 2, pp. 433–441 (1990).

[17] On Nārāyaṇa's treatment of magic squares, see Schuyler Cammann, Islamic and Indian Magic Squares, *History of Religions*, 8.3–4, 181–209, 271–299 (1969); Parmanand Singh, Total Number of Perfect Magic Squares: Nārāyaṇa's Rule, *The Mathematics Education*, 16.2 (June 1982) 32–37; idem, Nārāyaṇa's Treatment of Magic Squares, *Indian Journal of History of Science*, 21.2, 123–130 (1986); idem, The Gaṇitakaumudī of Nārāyaṇa Paṇḍita, Chapter XIV, English Translation with Notes, *Gaṇita-Bhāratī*, 24, 35–98 (2002); Takanori Kusuba, Combinatorics and Magic Squares in India: A Study of Nārāyaṇa Paṇḍita's Gaṇitakaumudī, Chapters 13–14, PhD Thesis, Brown University 1993.

About the mathematical activity of Kāyasthas in North India, information is available only from Assam, where they were known as Kāiths who kept the land records. They developed a professional variety of arithmetic called *Kāithelī Aṃka*, which was in verse form. A certain Daṇḍirām Datta meticulously collected these *Kāithelī* sums and puzzles, and published them in a book entitled *Kautuk Āru Kāithelī Aṃka.*[18] In the sixteenth century, the *Līlāvatī* of Bhāskara II was brought to Assam from Bengal by Durgāracan Barkāith, and several translations were made into Assamese by different mathematicians.[19]

But the earliest of such translations from Sanskrit into a regional language is the *Pāvulūrigaṇitamu* in Telugu.[20] That under the caliphate of al-Mansūr at Baghdad Sanskrit astronomical-cum-mathematical texts were translated or adapted into Arabic is well known, but not so well known is the fact that Mahāvīra's *Gaṇitasārasaṃgraha* was translated into Telugu by Pāvulūri Mallana in the eleventh century. Indeed it is the second extant work in the Telugu language. Yet, only a small fragment of this text was published.[21]

From the small fragment published so far, we can see that Mallana was a superb translator. The lucidity with which he rendered the terse Sanskrit of Mahāvīra is worth emulating by every modern translator of scientific texts. His way of handling mathematical rules or examples containing large numbers – some examples have as many as 36 digits – is unrivaled even in Sanskrit. He abridged the material of the Sanskrit original at certain places and expanded at others. Thus, while the Sanskrit *Gaṇitasārasaṃgraha* contains five methods of squaring and seven of cubing, the Telugu version has only one each and avoids all algebraic methods. Mallana also employs units of measure that were prevalent in the Andhra region of his time. Another innovation or addition in the Telugu version pertains to mathematics proper. There are 45 additional examples under multiplication and 21 under division, which are not found in Sanskrit. All these examples have one common feature: to produce numbers containing a symmetric arrangement of digits. The Sanskrit original has only a few, and Mahāvīra calls them "necklace numbers" (*kaṇṭhikā*) because the

[18] Dilip Kumar Sarma, Kautuk Āru Kāithelī Aṃka: A Study, *Summaries of Papers*, All-India Oriental Conference, 40th Session, Chennai, 2000. p. 505 (TS & FA-32).

[19] Dilip Kumar Sarma, A Peep into the Study of Development of Mathematics in Assam from Ancient to Modern Times, *Summaries of Papers*, All-India Oriental Conference, 39th Session, Vadodara, 1998, pp. 437–438 (TS & FA-11). See also Ganganand Singh Jha, Asam kī gaṇitīya den, *Pūrvānchal Praharī*, Guahati, 3 May 2000, p. 5; cited in : Hiteshwar Singh, Dr. G. S. Jha: A Broad-Based Historian of Mathematics, *Gaṇita Bhāratī*, 25, 150–153 (2003).

[20] Sreeramula Rajeswara Sarma, The Pāvulūrigaṇitamu: the First Telugu Work on Mathematics, *Studien zur Indologie und Iranistik*, Hamburg, 13–14, pp. 163–176 (1987).

[21] *Sārasaṃgrahagaṇitamu, Pāvulūri Mallana (Mallikārjuna) praṇītamu,* ed, Veṭūri Prabhākara Śastri, Part 1, Tirupati, 1952. This edition contains only a small part of the text, corresponding to Sanskrit *Gaṇtasārasaṃgraha* 1.1–3.53.

symmetric arrangement of digits is like the symmetric arrangement of beads in a necklace. The Telugu version abounds in necklace numbers of diverse patterns. For example, necklaces made up of just unities:

$$\begin{aligned} 37 \times 3 &= 111, \\ 101 \times 11 &= 1111, \\ 271 \times 41 &= 11111, \\ 37 \times 3003003 &= 111111111, \\ 37 \times 3003003003003 &= 111111111111111, \end{aligned}$$

and finally,

$$1\,763\,668\,430\,335\,097\,001\,763\,668\,430\,335\,097 \times 63 = \\ 111\,111\,111\,111\,111\,111\,111\,111\,111\,111\,111\,111\ .$$

Or necklaces containing unities intermingled with pearl-like zeros:

$$\begin{aligned} 14287143 \times 7 &= 100010001, \\ 157158573 \times 7 &= 1100110011, \\ 142857143 \times 7 &= 1000000001, \\ 777000777 \times 13 &= 10101010101. \end{aligned}$$

And here is the largest pearl necklace:

$$20\,408\,163\,265\,306\,122\,449 \times 49 = 10\,000\,000\,000\,000\,000\,000\,01.$$

Mallana introduces a new pattern and calls it a "moon-like" number because here the digits increase from 1 to n and then decrease to 1 just as the phases of the moon gradually increase up to the full moon and then decrease in an *amānta* lunar month, e.g., $111111 \times 111111 = 12345654321$.

There are also reverse "moon numbers" in which the digits first decrease from n to 1 and then increase up to n, like the phases of the moon in a *pūrṇimānta* lunar month, e.g., $146053847 \times 448 = 65432123456$.

I should also add that often several sets of factors are given for one product.

It is indeed likely that problems such as these which produce startling results attracted the attention, not just of serious mathematicians who invented more problems like these, but also of laymen who posed these problems as puzzles or riddles under the village tree. Thus we come to the realm of recreational mathematics.[22] A large corpus of such mathematical riddles exists as oral literature, now styled ethno-mathematics.[23] This oral literature has not

[22] David Singmaster, South Bank University, London, is compiling the *Sources in Recreational Mathematics*: *An Annotated Bibliography*. The seventh preliminary edition was released in January 2002.

[23] Cf. D. K. Sinha, Ethno-mathematics: A Philosophical and Historical Critique, in: D. P. Chattopadhyaya and Ravinder Kumar, *Mathematics, Astronomy and*

yet been recorded in a systematic manner. It consists of mnemonic tables of multiplication and the like and also recreational mathematics.

In the seventeenth century, European travelers were much impressed by the Indian merchant's ability to perform mental calculations with great speed. Thus, the French jeweller Jean-Baptiste Tavernier wrote in 1665 that the Indian merchants learned arithmetic "perfectly, using for it neither pen nor counters, but the memory alone, so that in a moment they will do a sum, however difficult it might be."[24] The secret lies naturally in the number of multiplication and other tables the merchant had committed to memory in childhood. Hemādri, the chancellor of the exchequer (*mahākaraṇādhipa*) under the last Yādavas of Devagiri in the second half of the thirteenth century, was described as the outstanding computer (*gaṇakāgraṇi*). D. D. Kosambi writes that a few tables for quick assessment survive in Hemādri's name.[25] Writing in the first quarter of the nineteenth century, John Taylor records that "in the Mahratta schools, this table [of multiplication] consists in multiplying ten numbers as far as 30, and in Gujarati schools, in multiplying ten numbers as far as one hundred."[26] At the beginning of the twentieth century, the *Gazetteer of the Bombay Presidency* reports that merchant boys memorized no fewer than 20 types of tables: multiplication tables of whole numbers and of fractions, tables of squares, tables of interest, and so on.[27]

In Bengal, Śubhaṅkara is a household name as a repository of mathematical or computational expertise,[28] but nobody seems to have collected the

Biology in Indian Tradition: Some Conceptual Preliminaries, PHISPC Monograph Series on History of Philosophy, Science and Culture in India, No. 3, Project of History of Indian Science, Philosophy and Culture, New Delhi, pp. 94–119 (1995).

[24] Jean-Baptiste Tavernier, *Travels in India,* tr. V. Ball, second edition, edited by William Crooke, London, Vol. 2, p. 144 (1925).

[25] Damodar Dharmanand Kosambi, Social and Economic Aspects of the *Bhagavad-Gītā*, in: idem, *Myth and Reality: Studies in the Formation of Indian Culture*, Popular Prakashan, Bombay, 1962, pp. 12–41, especially 32. I have not been able to find any information on these surviving tables.

[26] John Taylor, *Līlāwatī*: *or a Treatise on Arithmetic and Geometry by Bhascara Acharya*, translated from the Original Sanskrit by John Taylor, M. D. of the Hon'ble East India Company's Bombay Medical Establishment, Bombay, 1816, p. 145. The quotation is from a highly interesting "Short Account of the Present Mode of Teaching Arithmetic in Hindu Schools" (pp. 143–161) which he appended to his introduction.

[27] *Gazetteer of the Bombay Presidency*, Volume IX, Part 1: Gujarat, Population, Hindus, Bombay, 1901; reprinted as *Hindu Castes and Tribes of Gujarat*, compiled by Bhimbhai Kriparam, ed. James M. Campbell, Gurgaon, 1988, Vol. 1, p. 80.

[28] The Rev. Lál Behári Day, *Govinda Sámanta or the History of a Bengal Ráiyat*, London, 1874; new edition under the title *Bengal Peasant Life,* London, 1878; reprint: Macmillan and Co., Limited, London, 1920, p. 75: "He (the village school master) was the first mathematician of the village. He had not only Subhankara,

texts or sayings attributed to him.[29] Surely children must have memorized multiplication tables throughout the ages in India, but we do not know how these were formulated or under what name they were known. In a Telugu commentary on the *Pāvulūrigaṇitamu*, I came across fragments of tables of multiplication, of squares and square roots, and of cubes and cube roots. These tables are in Prakrit and must have been in use in the Andhra region at some time.[30]

But what is so special about these tables in the regional languages, when the results can be obtained from any pocket calculator today? These mnemonic tables are couched not in the modern form of the regional languages, but in earlier forms of languages. Thus in Uttar Pradesh, elderly people tell me that they had memorized several multiplication tables of whole numbers and fractions in Vrajbhāsā or in Avadhī. Therefore the importance of the tables is more cultural than mathematical. These tables tell us about the milieu in which they were formulated; the variety and the extent of the tables tell us about the nature of mathematical education. Therefore these tables are important and deserve to be recorded.

I mentioned earlier that much of the recreational mathematics is oral and unrecorded. Allow me to present one case in which I luckily found a written record as well as an oral version. In one of my visits to my native Andhra Pradesh, a friend of my father gave me several copybooks in which his own

the Indian Cocker, at his finger tips, but was acquainted with the elements of Víjaganita or Algebra."

See also W. Adam, *State of Education in Bengal*, 1835–1838 (Extracts reprinted in: Dharampal, *The Beautiful Tree: Indigenous Indian Education in the Eighteenth Century,* Biblio Impex Private Limited, New Delhi 1983, pp. 269–270): "The only other written composition used in these schools, and that only in the way of oral dictation by the master, consists of a few of the rhyming arithmetical rules of *Subhankar*, a writer whose name is as familiar in Bengal as that of Cocker in England, without anyone knowing who or what he was or when he lived. It may be inferred that he lived, or, if not a real personage, that the rhymes bearing that name were composed, before the establishment of the British rule in this country, and during the existence of the Muslim power, for they are full of Hindustani or Persian terms, and contain references to Muslim usages without the remotest allusion to English practices or modes of calculation."

Edward Cocker (1631–1675) was an English pedagogue whose posthumous publications *Arithmetick, Being a Plain and Easy Method* (1678) and *Algebraical Arithmetic or Equations* (1684) were so popular that "according to Cocker" has become a proverbial expression to mean "very reliable." An analogous expression in German "nach Adam Riese" perpetuates the memory of Adam Riese (1492–1559), who wrote the earliest mathematical primers in German.

[29] D. K. Sinha, Ethnomathematics: A Philosophical and Historical Critique, op. cit., discusses on pp. 99–102 some old Bengali rhymes, which may be of Śubhaṅkara.

[30] Sreeramula Rajeswara Sarma, Some Medieval Arithmetical Tables, *Indian Journal of History of Science*, 32, 191–198 (1997).

father had collected various items of mathematical interest.[31] Here I found a Telugu version of the so-called Josephus problem.[32] The solution to this problem consists in arranging in a circle two groups of an equal number of persons or objects in such a manner that each nth person or object belongs to the same group. Though named after the Jewish historian, Flavius Josephus (37–100 C.E.), this problem was not known in Europe before the tenth century. There the problem runs as follows:

Fifteen Jews and fifteen Christians were traveling in a boat when the boat developed a leak. So the Christian captain arranged all the thirty persons in a circle and kicked out each ninth person and thus got rid of all the Jews.

Japan is the only other place where this problem was known, and there it became popular some time after the twelfth century. In the Japanese version, a man had 15 sons by his first wife. After her death, he married another woman who already had 15 sons of her own. The second wife arranged all the 30 sons and stepsons in a circle, explaining that she would count and take out each tenth one from the circle and that the last one in the circle would inherit the father's property. After she had thus eliminated 14 stepsons one after the other, the 15th stepson realized the trick and insisted that the counting should begin from him. She agreed to do so, but the consequence was that all her 15 sons were eliminated. The last one to remain in the circle and thus to inherit the patrimony was the 15th stepson, who cleverly saw through the stepmother's game.[33]

The Telugu version that I discovered runs as follows. Fifteen Brahmins and 15 thieves had to spend a dark night in an isolated temple of Durgā. The goddess appeared in person at midnight and wanted to devour exactly 15 persons, since she was hungry. The thieves naturally suggested that she should consume the 15 plump Brahmins. But the clever Brahmins proposed that all the 30 would stand in a circle and that Durgā should eat each ninth person. The proposal was accepted by Durgā and the thieves. So the Brahmins arranged themselves and the thieves in a circle, telling each one where to stand. Durgā then counted out each ninth person and devoured him. When the 15 were eaten, she was satiated and disappeared, and only Brahmins remained in

[31] My father's friend and his father were hereditary Karaṇams who maintained the village records. The copybooks are datable to the 1930s, but the material collected therein is much older.

[32] On the Josephus problem, see David Eugene Smith, *History of Mathematics*, New York, Vol. II, pp. 541–544 (1925).

[33] Osamu Takenouchi et al. (eds. and trs.), *Jinko-ki*, Wasan Institute, Tokyo 2000, pp. 139–140. "Wasan" is the indigenous mathematics developed in Japan during the Edo Period (1603–1867). The *Jinko-ki*, which was published in 1627, is one of the earliest texts of this genre. The present edition contains an English translation, together with the facsimile reproduction of the original Japanese illustrated woodblock edition of 1627.

the circle. The problem is, how did the Brahmins arrange themselves and the thieves in the circle? The answer is composed in a Telugu verse of a classical meter.

The copybooks contained another variant of the problem, namely, to arrange 30 Brahmins and 30 thieves in a circle in such a way that each 12th person would be a thief. The solution to this also is given in a classical meter. Since I was pleased with this discovery, a farmer in my village posed the same problem to me. His solution is the same, but it is couched in free verse.[34]

Two versions of the solution to the same riddle in the same geographic area does indeed demonstrate the wide popularity of mathematical riddles in Andhra Pradesh. Whether this is an offshoot of the popularity of mathematical literature, or whether riddles – mathematical or otherwise – are transmitted in a different process independent of literature, is a question I am not competent to answer. But a collection of such mathematical riddles would certainly enrich the history of our mathematics.

There is yet another area in which regional languages provide valuable source material, viz. the dissemination of modern mathematics in the nineteenth century through mathematical textbooks in regional languages. As far as I know, Dhruv Raina and S. Irfan Habib are the only scholars who have studied this aspect, in connection with the Urdu textbooks and other popular writings on mathematics by Master Ramchandra (b. 1821).[35]

I conclude this presentation with a plea that organized efforts be made to save this mathematical heritage in the regional languages, both of the recorded and of the oral varieties.

A final poser: Watching TV on a visit to Tamil Nadu, I discovered that zero is called *pūjyam* in Tamil, "worthy of worship." I would worship any person who can explain why zero has such an exalted name in the Tamil language.[36]

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[34] Sreeramula Rajeswara Sarma, Mathematical Literature in Telugu: An Overview, *Sri Venkateswara University Oriental Journal*, 28, 7790 (1985).

[35] S. Irfan Habib and Dhruv Raina, The Introduction of Scientific Rationality into India: A Study of Master Ramchandra, Urdu Journalist, Mathematician and Educationist, *Annals of Science,* 46.6 (November 1989), pp. 597–610; Dhruv Raina and S. Irfan Habib, Ramchandra's Treatise through the "Haze of the Golden Sunset": An Aborted Pedagogy, *Social Studies of Science,* 20.3 (1990), pp. 455–472; Dhruv Raina, Mathematical Foundations of a Cultural Project: Ramchandra's Treatise through the "Unsentimentalized Light of Mathematics," *Historia Mathematica*, 19, pp. 371–384 (1992).

[36] In the discussion following my lecture, I learned that zero is called *pūjyam* in Malayalam and Marathi also. It would be interesting to know when this designation came into vogue and in what context. I also learned a Tamil proverb, which declares, "Inside the *pūjyam* (zero), there exists a *rājyam* (kingdom)."

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