Ancient Indian Square Roots: An Exercise in Forensic Paleo-Mathematics

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Ancient Indian Square Roots: An Exercise in Forensic Paleo-Mathematics

David H. Bailey and Jonathan M. Borwein

Abstract. This article examines the computation of square roots in ancient India in the context of the discovery of positional decimal arithmetic.

1. INTRODUCTION. Our modern system of positional decimal notation with zero, together with efficient algorithms for computation, which were discovered in India some time prior to 500 CE, certainly must rank among the most significant achievements of all time. As Pierre-Simon Laplace explained:

> It is [in India] that the ingenious manner of expressing all numbers in ten characters originated, by assigning to them at once an absolute and a local [positional] value, a subtle and important conception, of which the simplicity is such that we can [only] with difficulty, appreciate its merit. But this very simplicity and the great facility with which we are enabled to perform our arithmetical computations place it in the very first rank of useful inventions; the difficulty of inventing it will be better appreciated if we consider that it escaped the genius of Archimedes and Apollonius, two of the greatest men of antiquity. [**16**, pp. 222–223]

In a similar vein, Tobias Dantzig (father of George Dantzig of simplex fame), adds the following:

> When viewed in this light, the achievement of the unknown Hindu who some time in the first centuries of our era discovered the *principle of position* assumes the proportions of a world-event. Not only did this principle constitute a radical departure in method, but we know now that without it no progress in arithmetic was possible. [**5**, pp. 29–30]

The Mayans came close, with a system that featured positional notation with zero. However, in their system successive positions represented the mixed sequence $(1, 20, 360, 7200, 144000, \ldots)$, i.e., $18 \cdot 20^{n-2}$ for $n \geq 3$, rather than the purely vigesimal (base-20) sequence $(1, 20, 400, 8000, 160000, \ldots)$, i.e., 20^{n-1} for $n \geq 1$. This choice precluded any possibility that their numerals could be used as part of a highly efficient arithmetic system [**13**, p. 311].

2. THE DISCOVERY OF POSITIONAL ARITHMETIC. So who exactly discovered the Indian system? Sadly, there is no record of this individual or individuals, who would surely rank among the greatest mathematicians of all time.

The earliest known piece of physical evidence of positional decimal notation using single-character Brahmi numerals (which are the ancestors of our modern digits) is an inscription of the date 346 on a copper plate, which corresponds to 595 CE. No physical

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artifacts are known earlier than this date [**3**, p. 196]. But there are numerous passages of more ancient texts that strongly suggest that both the concept and the practice of positional decimal numeration was known much earlier [**19**, p. 122].

For example, a fifth century text includes the passage "Just as a line in the hundreds place [means] a hundred, in the tens place ten, and one in the ones place, so one and the same woman is called mother, daughter, and sister [by different people]" [**19**, p. 46]. Similarly, in 499 CE the Indian mathematician Aryabhata wrote, "The numbers one, ten, hundred, thousand, ten thousand, hundred thousand, million, ten million, hundred million, and billion are from place to place each ten times the preceding" [**4**, p. 21].

These early texts did not use Brahmi numerals, but instead used the Sanskrit words for the digits one through nine and zero, or, when needed to match the meter of the verse, used one of a set of literary words (known as "word-symbols") associated with digits. For example, the medieval Indian manuscript *Surya Siddhanta* included the verse, "The apsids of the moon in a cosmic cycle are: fire; vacuum; horsemen; vast; serpent; ocean." Here the last six words are word-symbols for 3, 0, 2, 8, 8, 4, respectively (meaning the decimal number 488,203, since the order is reversed) [**13**, p. 411].

One issue some have raised is that most ancient Indian documents are more recent copies, so that we cannot be absolutely certain of their ancient authenticity. But one manuscript whose ancient authenticity cannot be denied is the *Lokavibhaga* ("Parts of the Universe") [**17**]. This has numerous large numbers in positional decimal notation (using Sanskrit names or word-symbols for the digits), such as 14236713, 13107200000, and 70500000000000000, and detailed calculations [**17**, pp. 70, 79, 131]. Near the end of the *Lokavibhaga*, the author provides some astronomical observations that enable modern scholars to determine, in two independent ways, that this text was written on 25 August 458 CE (Julian calendar). The text also mentions that it was written in the 22nd year of the reign of Simhavarman, which also confirms the 458 CE date. Thus its ancient date is beyond question [**13**, p. 417].

One even earlier source of positional word-symbols is the mid-third-century CE text *Yavana-jataka*. Its final verse reads, "There was a wise king named Sphujidhvaja who made this [work] with four thousand [verses] in the Indravajra meter, appearing in the year Visnu; hook-sign; moon." The three word-symbols, "Visnu," "hook-sign," and "moon," mean 1, 9, and 1, signifying year 191 of the Saka era, which corresponds to 270 CE [**19**, p. 47].

The earliest record of zero may be in the *Chandah-sutra*, dated to the second or third century BCE. Here we see the solution to a mathematical problem relating to the set of all possible meters for multi-syllable verse, which involves the expression of integers using a form of binary notation [**19**, p. 55]. The very earliest origin of the notion of positional decimal notation and arithmetic, however, is still obscure; it may be connected to the ancient Chinese "rod calculus" [**19**, p. 48].

Additional details on the origin, proliferation and significance of positional decimal arithmetic are given in [**1**].

3. ARYABHATA'S SQUARE ROOT AND CUBE ROOT. One person who deserves at least some credit for the proliferation of decimal arithmetic calculation is the Indian mathematician Aryabhata, mentioned above (see Figure 1). His ingenious digit-by-digit algorithms for computing square roots and cube roots, based on terse statements in his 499 CE work *Aryabhatiya* [**4**, pp. 24–26]), are illustrated by examples (due to the present authors) shown in Figure 2. These schemes were used, with only minor variations, by Indian mathematicians such as Siddhasena Gani ($\sim$550 CE), Bhaskara I ($\sim$600 CE), Sridhara ($\sim$750 CE), and Bhaskara II ($\sim$1150 CE), as well as by numerous later Arabic and European mathematicians [**7**, vol. I, pp. 170–175].

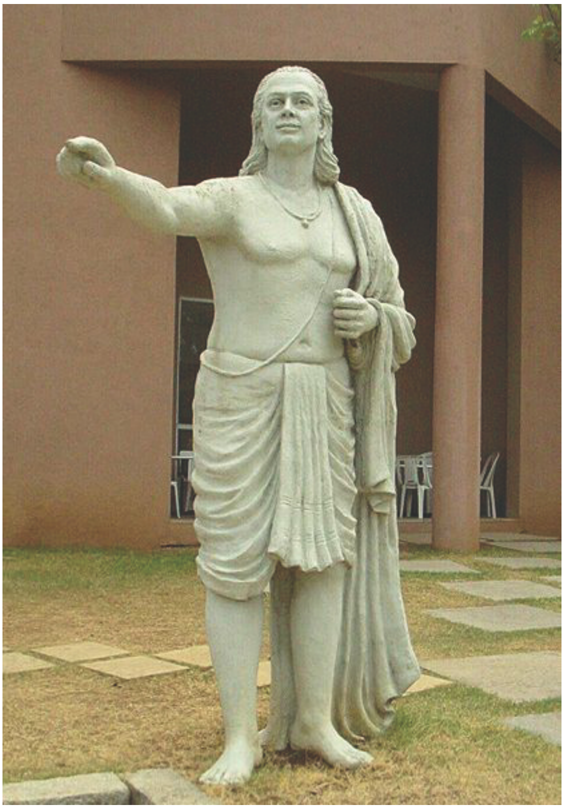

Figure 1. Statue of Aryabhata on the grounds of IUCAA, Pune, India (no one knows what Aryabhata actually looked like) [courtesy Wikimedia]

4. THE BAKHSHALI MANUSCRIPT. Another ancient source that clearly exhibits considerable familiarity with decimal arithmetic in general and square roots in particular is the Bakhshali manuscript. This document, an ancient mathematical treatise, was found in 1881 in the village of Bakhshali, approximately 80 kilometers northeast of Peshawar (then in India, now in Pakistan). Among the topics covered in this document, at least in the fragments that have been recovered, are solutions of systems of linear equations, indeterminate (Diophantine) equations of the second degree, arithmetic progressions of various types, and rational approximations of square roots.

The manuscript appears to be a copy of an even earlier work. As Japanese scholar Takao Hayashi has noted, the manuscript includes the statement "sutra bhrantim asti" ("there is a corruption in the numbering of this sutra"), indicating that the work is a commentary on an earlier work [**11**, pp. 86, 148].

Ever since its discovery in 1881, scholars have debated its age. Some, like British scholar G. R. Kaye, assigned the manuscript to the 12th century, in part because he believed that its mathematical content was derivative from Greek sources. In contrast, Rudolf Hoernle assigned the underlying manuscript to the "3rd or 4th century CE" [**12**, p. 9]. Similarly, Bibhutibhusan Datta concluded that the older document was dated "towards the beginning of the Christian era" [**6**]. Gurjar placed it between the second century BCE and the second century CE [**9**].

In the most recent and arguably the most thorough analysis of the Bakhshali manuscript, Japanese scholar Takao Hayashi assigned the commentary to the seventh century, with the underlying original not much older [**11**, p. 149].

Tableau								Result				Notes
4	5	4	6	8	0	4	9	6				$\lfloor\sqrt{45}\rfloor = 6$
3	6											$6^2 = 36$
	9	4						6	7			$\lfloor 94/(2\cdot 6)\rfloor = 7$
	8	4										$7\cdot(2\cdot 6) = 84$
	1	0	6									
		4	9									$7^2 = 49$
		5	7	8				6	7	4		$\lfloor 578/(2\cdot 67)\rfloor = 4$
		5	3	6								$4\cdot(2\cdot 67) = 536$
			4	2	0							
				1	6							$4^2 = 16$
			4	0	4	4		6	7	4	3	$\lfloor 4044/(2\cdot 674)\rfloor = 3$
			4	0	4	4						$3\cdot(2\cdot 674) = 4044$
						0	9					
							9					$3^2 = 9$
							0					Finished; result $= 6743$

Tableau								Result				Notes
7	7	8	5	4	4	8	3	4				$\lfloor\sqrt[3]{77}\rfloor = 4$
6	4											$4^3 = 64$
1	3	8						4	2			$\lfloor 138/(3\cdot 4^2)\rfloor = 2$
	9	6										$2\cdot(3\cdot 4^2) = 96$
	4	2	5									
		4	8									$3\cdot 2^2\cdot 4 = 48$
	3	7	7	4								
				8								$2^3 = 8$
	3	7	6	6	4			4	2	7		$\lfloor 37664/(3\cdot 42^2)\rfloor = 7$
	3	7	0	4	4							$7\cdot(3\cdot 42^2) = 37044$
			6	2	0	8						
			6	1	7	4						$3\cdot 7^2\cdot 42 = 6174$
					3	4	3					
					3	4	3					$7^3 = 343$
							0					Finished; result $= 427$

Figure 2. Illustration of Aryabhata's digit-by-digit algorithms for computing $\sqrt{45468049} = 6743$ (top) and $\sqrt[3]{77854483} = 427$ (bottom).

4.1. The Bakhshali square root.

One particularly intriguing item in the Bakhshali manuscript is the following algorithm for computing square roots:

> [1:] In the case of a non-square [number], subtract the nearest square number; divide the remainder by twice [the root of that number]. [2:] Half the square of that [that is, the fraction just obtained] is divided by the sum of the root and the fraction and subtract [from the sum]. [3:] [The non-square number is] less [than

the square of the approximation] by the square [of the last term]. (Translation is due to B. Datta [**6**], except last sentence is due to Hayashi [**11**, p. 431]. Sentence numbering is by present authors.)

4.2. The Bakhshali square root in modern notation. In modern notation, this algorithm is as follows. To obtain the square root of a number q, start with an approximation x_0 and then calculate, for $n \geq 0$,

$$a_n = \frac{q - x_n^2}{2x_n} \qquad \text{(sentence \#1 above)}$$

$$x_{n+1} = x_n + a_n - \frac{a_n^2}{2\,(x_n + a_n)} \qquad \text{(sentence \#2 above)}$$

$$q = x_{n+1}^2 - \left[\frac{a_n^2}{2\,(x_n + a_n)}\right]^2. \qquad \text{(sentence \#3 above)}$$

The last line is merely a check; it is not an essential part of the calculation. In the examples presented in the Bakhshali manuscript, this algorithm is used to obtain rational approximations to square roots only for integer arguments q, only for integer-valued starting values x_0, and is only applied once in each case (i.e., it is not iterated). But from a modern perspective, the scheme clearly can be repeated, and in fact converges very rapidly to $\sqrt{q}$, as we shall see in the next section.

Here is one application in the Bakhshali manuscript [**11**, pp. 232–233].

Problem 1. *Find an accurate rational approximation to the solution of*

$$3x^2/4 + 3x/4 = 7000 \tag{1}$$

(which arises from the manuscript's analysis of some additive series).

Answer. $x = (\sqrt{336009} - 3)/6$. To calculate an accurate value for $\sqrt{336009}$, start with the approximation $x_0 = 579$. Note that $q = 336009 = 579^2 + 768$. Then calculate as follows (using modern notation):

$$a_0 = \frac{q - x_0^2}{2x_0} = \frac{768}{1158}, \quad x_0 + a_0 = 579 + \frac{768}{1158}, \quad \frac{a_0^2}{2(x_0 + a_0)} = \frac{294912}{777307500}. \tag{2}$$

Thus we obtain the refined root

$$x_1 = x_0 + a_0 - \frac{a_0^2}{2\,(x_0 + a_0)} = 579 + \frac{515225088}{777307500} = \frac{450576267588}{777307500} \tag{3}$$

(note: This is $579.66283303325903841\ldots$, which agrees with

$$\sqrt{336009} = 579.66283303313487498\ldots$$

to 12-significant-digit accuracy).

The manuscript then performs a calculation to check that the original quadratic equation is satisfied. It obtains, for the left-hand side of (1),

$$\frac{50753383762746743271936}{7250483394675000000}, \tag{4}$$

which, after subtracting the correction

$$\frac{21743271936}{7250483394675000000}, \tag{5}$$

gives,

$$\frac{50753383762725000000000}{7250483394675000000} = 7000. \tag{6}$$

Each of the integers and fractions shown in the above calculation (except the denominator of (5), which is implied) actually appears in the Bakhshali manuscript, although some of the individual digits are missing at the edges—see Figure 3. The digits are written left-to-right, and fractions are written as one integer directly over another (although there is no division bar). Zeroes are denoted by large dots. Other digits may be recognized by those familiar with ancient Indian languages.

It is thrilling to see, in a very ancient document such as this, a sophisticated calculation of this scope recorded digit by digit. And we are not aware, in the Western tradition, of a fourth-order convergent formula being used until well after the Enlightenment of the 1700s.

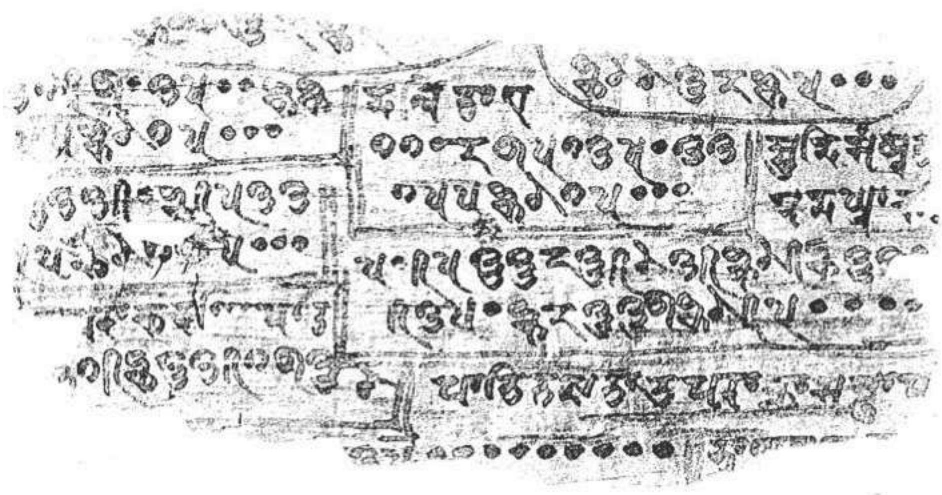

Figure 3. Fragment of Bakhshali manuscript with a portion of the square root calculation mentioned in Problem 1. For example, the large right-middle section corresponds to the fraction $\frac{50753383762746743271936}{7250483394675000000}$ in Formula (4). Graphic from [**11**, p. 574].

5. CONVERGENCE OF THE BAKHSHALI SQUARE ROOT. Note, in the above example, that starting with the 3-digit approximation 579, one obtains, after a single application of (4.2), a value for $\sqrt{336009}$ that is correct to 12 significant digits. From a modern perspective, this happens because the Bakhshali square root algorithm is *quartically convergent*—each iteration approximately quadruples the number of correct digits in the result, provided that either exact rational arithmetic or sufficiently high precision floating-point arithmetic is used (although, as noted above, there is no indication of the algorithm being iterated more than once in the manuscript itself). For example, with $q = 336009$ and $x_0 = 579$, successive iterations are as shown in Table 1.

Table 1. Successive iterations of the Bakhshali square root scheme for $q = 336009$ and $x_0 = 579$.

Iteration	Value	Relative error
0	579.00000000000000000000000000000...	1.143×10^{-3}
1	579.662833033259038411439488233421...	2.142×10^{-13}
2	579.662833033134874975589542464552...	2.631×10^{-52}
3	579.662833033134874975589542464552...	5.993×10^{-208}
4	579.662833033134874975589542464552...	1.612×10^{-830}
5	579.662833033134874975589542464552...	8.449×10^{-3321}
6	579.662833033134874975589542464552...	6.371×10^{-13282}
7	579.662833033134874975589542464552...	2.060×10^{-53126}

The proof that iterates of the Bakhshali square root formula are quartically convergent is relatively straightforward.

Theorem 1. *The Bakhshali square root algorithm, as defined above in* (4.2)*, is quartically convergent.*

Proof. It suffices to demonstrate that the scheme is mathematically equivalent to performing *two* consecutive iterations of the Newton-Raphson iteration [**2**, pp. 226–229] for finding the root of $f(x) = x^2 - q = 0$, which are

$$x_{n+1} = x_n + \frac{q - x_n^2}{2x_n} \quad \text{and} \tag{7}$$

$$x_{n+2} = x_{n+1} + \frac{q - x_{n+1}^2}{2x_{n+1}}. \tag{8}$$

Expanding the expression for x_{n+2}, one obtains

$$x_{n+2} = x_n + \frac{q - x_n^2}{2x_n} + \frac{q - \left(x_n + \frac{q-x_n^2}{2x_n}\right)^2}{2\left(x_n + \frac{q-x_n^2}{2x_n}\right)} \tag{9}$$

$$= x_n + \frac{q - x_n^2}{2x_n} + \frac{q - x_n^2 - 2x_n\left(\frac{q-x_n^2}{2x_n}\right) - \left(\frac{q-x_n^2}{2x_n}\right)^2}{2\left(x_n + \frac{q-x_n^2}{2x_n}\right)} \tag{10}$$

$$= x_n + \frac{q - x_n^2}{2x_n} - \frac{\left(\frac{q-x_n^2}{2x_n}\right)^2}{2\left(x_n + \frac{q-x_n^2}{2x_n}\right)}, \tag{11}$$

which is the form of a single Bakhshali square root iteration. Since a single Newton-Raphson iteration (7) for the square root (which is often referred to as the Heron formula, after Heron of Alexandria ~70 CE), is well-known to be quadratically convergent, two consecutive iterations (and thus a single Bakhshali iteration) are quartically convergent.

For completeness, we include a proof that the Newton-Raphson-Heron iteration, which is equivalently written $x_{n+1} = (x_n + q/x_n)/2$, is quadratically convergent. Note that

$$x_{n+1} - x_n = \frac{q/x_n - x_n}{2} = \frac{q - x_n^2}{2x_n}, \quad \text{and} \tag{12}$$

$$x_{n+1}^2 - q = \left(\frac{x_n + q/x_n}{2}\right)^2 - q = \left(\frac{x_n^2 - q}{2x_n}\right)^2. \tag{13}$$

By (13), $x_n \geq \sqrt{q}$ for all $n \geq 1$, and by (12), x_n is monotonically decreasing for $n \geq 1$. Then x_n must converge to some limit r, and again by (12), $r = \sqrt{q}$. Finally, (13) implies that

$$\left|x_{n+1}^2 - q\right| \leq \frac{\left|x_n^2 - q\right|^2}{4q}, \tag{14}$$

which establishes quadratic convergence, so that once the right-hand side is sufficiently small, the number of correct digits approximately doubles with each iteration.

■

The fact that the Bakhshali square root scheme is quartically convergent when iterated has not been clearly recognized in the literature, to our knowledge. G. R. Kaye, for instance, evidently presumed that the Bakhshali square root is equivalent to and derived from Heron's formula. He also claimed that the Bakhshali square root scheme was extended to "second approximations" in some instances, but this is not true—it was always implemented as stated in the manuscript (see translation above). Also, Kaye erred in his arithmetic, since the numerical value he gave for the $\sqrt{336009}$ result is only correct to four digits instead of 12 digits [**15**, pp. 30–31]. Similarly, Srinivasiengar stated that the Bakhshali square root is "identical" to Heron's formula, even though he presented a mathematically correct statement of the Bakhshali formula that is clearly distinct from Heron [**20**, p. 35]. Hayashi and Plofker correctly observed that the Bakhshali scheme can be mathematically derived by twice iterating the Newton-Raphson-Heron formula, although neither of them discussed convergence rates when the scheme is iterated [**11**, p. 431] [**18**, p. 440].

We might add that Heron's formula was known to the Babylonians [**8**], although, as with the Bakhshali formula, it is not clear that the Babylonians ever iterated the process. As to the source of the Bakhshali scheme, Hayashi argues that it may be based either on the Aryabhata square root scheme or on an ancient Heron-like geometric scheme described in the *Sulba-sutras* (between 600 BCE and 200 CE) [**11**, pp. 105–106].

6. AN EVEN MORE ANCIENT SQUARE ROOT. There are instances of highly accurate square roots in Indian sources that are even more ancient than the Bakhshali manuscript. For example, Srinivasiengar noted that the ancient Jain work *Jambudvipa-prajnapti* ($\sim$300 BCE), after erroneously assuming that $\pi = \sqrt{10}$, asserts that the "circumference" of a circle of diameter 100,000 yojana is 316227 yojana + 3 gavyuti + 128 dhanu + $13\frac{1}{2}$ angula, "and a little over" [**20**, pp. 21–22]. Datta added that this statement is also seen in the *Jibahigama-sutra* ($\sim$200 BCE) [**6**, p. 43]. Joseph noted that it also seen in the *Anuyogadvara-sutra* ($\sim$0 CE) and the *Triloko-sara* ($\sim$0 CE) [**14**, p. 356].

According to one commonly used ancient convention these units are: 1 yojana = 14 kilometers (approximately); 4 gavyuti = 1 yojana; 2000 dhanu = 1 gavyuti; and 96 angula = 1 dhanu [**14**, p. 356]. Converting these units to yojana, we conclude that the "circumference" is 316227.766017578125 ... yojana. This agrees with $100000\sqrt{10} = 316227.766016837933\ldots$ to 12-significant-digit accuracy!

What algorithm did these ancient scholars employ to compute square roots? Here we offer some analysis, which might be termed an exercise in "forensic paleo-mathematics":

First, note that the exact value of $100000\sqrt{10}$ is actually slightly *less* than the above, even though the ancient writers added the phrase "and a little over" to the listed value that ends in $13\frac{1}{2}$. Also, we can justifiably infer that the underlying target value (most likely a fraction) was less than the given value with $13\frac{3}{4}$ at the end, or presumably it would be listed with 14 instead of $13\frac{1}{2}$. Thus, a reasonable assumption is to take a slightly larger value, say $13\frac{5}{8}$ (the average of $13\frac{1}{2}$ and $13\frac{3}{4}$) at the end, as a closer approximation of the underlying fractional value. Now let us compare the corresponding decimal values, together with the results of a Newton-Raphson-Heron iteration (starting with 316227), a Bakhshali iteration (starting with 316227), and the exact result:

Manuscript value, with $13\frac{1}{2}$	316227.76601757812500 ...
Manuscript value, except with $13\frac{5}{8}$	316227.76601774088541 ...
One Heron iteration (316227)	316227.76601776571892 ...
One Bakhshali iteration (316227)	316227.76601683793319 ...
Exact value of $100000\sqrt{10}$	316227.76601683793319 ....

Comparing these values, it is clear that the manuscript value, with $13\frac{5}{8}$ at the end, is very close to the result of one Newton-Raphson-Heron iteration, but is 36 times more distant from the result of either Bakhshali iteration or the exact value (note that the value of the Bakhshali iteration, starting with 316227, is identical to the exact result, to 20-significant-digit accuracy). Thus, the most reasonable conclusion is that the Indian mathematician(s) did some preliminary computation to obtain the approximation 316227, then used one Newton-Raphson-Heron iteration to compute an approximate fractional value, and then converted the final result to the length units above. Evidently the Bakhshali formula had not yet been developed.

Along this line, R. C. Gupta analyzed the *Triloya-pannatti*, an Indian document dating to the between the fifth and tenth century CE, which gives the "circumference" above expressed in even finer units. Gupta concluded, as we did, that the result was based on a calculation using the Newton-Raphson-Heron formula [**10**].

Note that just to perform one Newton-Raphson-Heron iteration, with starting value 316227, one would need to perform at least the following rather demanding calculation:

$$\begin{aligned} x_1 &= \frac{1}{2}\left(x_0 + \frac{q}{x_0}\right) = \frac{1}{2}\left(316227 + \frac{100000000000}{316227}\right) \\ &= \frac{1}{2}\left(\frac{316227^2 + 100000000000}{316227}\right) = \frac{99999515529 + 100000000000}{2 \cdot 316227} \\ &= \frac{199999515529}{632454} = 316227 + \frac{484471}{632454}, \end{aligned} \tag{15}$$

followed by several additional steps to convert the result to the given units. By any reasonable standard, this is a rather impressive computation, which we were surprised to find evidence for in manuscripts of this ancient vintage (200–300 BCE). Numerous other examples of prodigious computations in various ancient Indian sources are mentioned by Datta [**7**], Joseph [**14**], Plofker [**19**], and Srinivasiengar [**20**]. Although some impressive calculations are also seen in ancient Mesopotamia, Greece, and China, as far as we are aware there are more of these prodigious calculations in ancient Indian literature than in other ancient sources.

In any event, all of this analysis leads to the inescapable conclusion that ancient Indian mathematicians, roughly contemporaneous with Greeks such as Euclid and Archimedes, had command of a rather powerful system of arithmetic, possibly some variation of the Chinese "rod calculus," or perhaps even some primitive version of decimal arithmetic. We can only hope that further study of ancient Indian mathematics will shed light on this intriguing question.

7. CONCLUSION. We entirely agree with Laplace, Tobias Dantzig, Georges Ifrah, and others that the discovery of positional decimal arithmetic with zero, together with efficient algorithms for computation, by Indian mathematicians (who likely will never be identified), certainly by 500 CE and probably several centuries earlier, is a mathematical development of the first magnitude. The fact that the system is now taught and mastered in grade schools worldwide, and is implemented (in binary) in every computer chip ever manufactured, should only enhance its historical significance. Indeed, these facts emphasize the enormous advance that this system represents, both in simplicity and efficiency, as well as the huge importance of this discovery in modern civilization.

It should be noted that these ancient Indian mathematicians missed some key points. For one thing, the notion of decimal fraction notation eluded them and everyone else until the tenth century, when a rudimentary form was seen in the writings of the Arabic mathematician al-Uqlidisi, and the twelfth century, when al-Samawal illustrated its use in division and root extraction [**14**, p. 468]. Also, as mentioned above, there is no indication that Indian mathematicians iterated algorithms for finding roots.

Aside from historical interest, does any of this matter? As historian Kim Plofker notes, in ancient Indian mathematics, "True perception, reasoning, and authority were expected to harmonize with one another, and each had a part in supporting the truth of mathematics." [**19**, p. 12]. As she neatly puts it, mathematics was not "an epistemologically privileged subject." Similarly, mathematical historian George G. Joseph writes:

> A Eurocentric approach to the history of mathematics is intimately connected with the dominant view of mathematics, both as a sociohistorical practice and as an intellectual activity. Despite evidence to the contrary, a number of earlier histories viewed mathematics as a deductive system, ideally proceeding from axiomatic foundations and revealing, by the necessary unfolding of its pure abstract forms, the eternal/universal laws of the "mind."
>
> The concept of mathematics found outside the Graeco-European praxis was very different. The aim was not to build an imposing edifice on a few self-evident axioms but to validate a result by any suitable method. Some of the most impressive work in Indian and Chinese mathematics..., such as the summations of mathematical series, or the use of Pascal's triangle in solving higher-order numerical equations or the derivations of infinite series, or "proofs" of the so-called Pythagorean theorem, involve computations and visual demonstrations that were not formulated with reference to any formal deductive system. [**14**, p. xiii]

So this is why it matters. The Greek heritage that underlies much of Western mathematics, as valuable as it is, may have unduly predisposed many of us against experimental approaches that are now facilitated by the availability of powerful computer technology [2]. In addition, more and more documents are now accessible for careful study—from Chinese, Babylonian, Mayan, and other sources as well. Thus a renewed exposure to non-Western traditions may lead to new insights and results, and may clarify the age-old issue of the relationship between mathematics as a language of science and technology, and mathematics as a supreme human intellectual discipline.

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A Sixth Proof of an Inequality

The following theorem was proved in [**1**], and in turn implies the result proved in three ways in [**2**].

Theorem. *Let* $u_n = (1 + \frac{1}{n})^{n+1/2}$. *Then* (u_n) *is decreasing, and consequently greater than its limit, e.*

Another proof. We show that the function $(1 + \frac{1}{x})^{x+1/2}$, or equivalently $(x + \frac{1}{2}) \log(1 + \frac{1}{x})$, is decreasing for $x > 0$. Substitute $y = 1/x$: we have to show that $F(y) = \left(\frac{1}{y} + \frac{1}{2}\right) \log(1 + y)$ is increasing for $y > 0$. We have

$$F'(y) = \frac{1/y + 1/2}{1 + y} - \frac{1}{y^2} \log(1 + y),$$

so

$$y^2 F'(y) = \frac{y(2 + y)}{2(1 + y)} - \log(1 + y).$$

Now $\log(1 + y) = \int_1^{1+y} (1/t)\, dt$ and the function $1/t$ is convex. A convex function lies below its straight-line interpolation between two points, so its integral is overestimated by the trapezium rule. Hence, for $y > 0$,

$$\log(1 + y) \le \frac{y}{2} \left(1 + \frac{1}{1 + y}\right) = \frac{y(2 + y)}{2(1 + y)},$$

so $F'(y) \ge 0$, as required. ■

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