Indian Mathematics

I

1. The orientalists who exploited Indian history and literature about a century ago were not always perfect in their methods of investigation and consequently promulgated many errors. Gradually, however, sounder methods have obtained and we are now enabled to see the facts in more correct perspective. In particular the early chronology has been largely revised and the revision in some instances has important bearings on the history of mathematics and allied subjects. According to orthodox Hindu tradition the "Sûrya Siddhânta", the most important Indian astronomical work, was composed over two million years ago! BAILLY, towards the end of the eighteenth century, considered that Indian astronomy had been founded on accurate observations made thousands of years before the Christian era. LAPLACE, basing his arguments on figures given by BAILLY considered that some 3,000 years B. C. the Indian astronomers had recorded actual observations of the planets *correct to one second*; PLAYFAIR eloquently supported BAILLY's views; Sir WILLIAM JONES argued that correct observations must have been made at least as early as 1181 B. C., and so on; but with the researches of COLEBROOKE, WHITNEY, WEBER, THIBAUT, and others more correct views were introduced and it was proved that the records used by BAILLY were quite modern and that the actual period of the composition of the "Sûrya Siddhânta" was not earlier than 400 A. D.

It may, indeed, be generally stated that the tendency of the early orientalists was towards antedating and this tendency is exhibited in discussions connected with two notable works, the "Sulvasûtras" and the "Bakhshâli" arithmetic the dates of which are not even yet definitely fixed.

2. In the 16th century A. D. Hindu tradition ascribed the invention

of the "nine figures with the device of places to make them suffice for all numbers" to "the beneficent Creator of the universe"; and this was accepted as evidence of the very great antiquity of the system! This is a particular illustration of an attitude that was quite general, for early Indian works claim either to be directly revealed or of divine origin. One consequence of this attitude is that we find absolutely no references to foreign origins or foreign influence (1). We have, however, a great deal of direct evidence that proves conclusively that foreign influence was very real indeed — Greek and Roman coins, coins with Greek and Indian inscriptions, Greek technical terms, etc.; and the implication of considerable foreign influence occurs in certain classes of literature and also in the archaeological remains of the north-west of India. One of the few references to foreigners is given by Varâha Mihira who acknowledged that the Greeks knew something of astrology; but although he gives accounts of the "Romaka" and the "Pauliśa siddhântas" he never makes any direct acknowledgement of western influence.

II

3. For the purpose of discussion three periods in the history of Hindu mathematics may be considered:

(I) The "Śulvasûtra" period with upper limit c. 200 A. D.;

(II) The astronomical period c. 400-600 A. D.;

(III) The Hindu mathematical period proper 600-1200 A. D.

Such a division into periods does not, of course, perfectly represent the facts, but it is a useful division and serves the purposes of exposition with sufficient accuracy. We might have prefixed an earlier, or Vedic, period but the literature of the Vedic age does not exhibit anything of a mathematical nature beyond a few measures and numbers used quite informally. It is a remarkable fact that the second and third of our periods have no connection whatever with the first or "Śulvasûtra" period. The later Indian mathematicians completely ignored the mathematical contents of the "Śulvasûtras". They not only never refer to them but do not even utilise the results given therein. We can go even further and state that no Indian writer earlier than the nineteenth century A. D. is known to have referred to

(1) It may be noted that, beyond the vague pseudo-prophetic references to the Javanas in the "Purâṇas", no early Indian writer mentions the invasion of Alexander the Great.

the "Śulvasûtras" as containing anything of mathematical value. This disconnection will be illustrated as we proceed and it will be seen that the works of the third period may be considered as a direct development from those of the second.

4. *The Śulvasûtra period.* — The term "Śulvasûtra" means "the rules of the cord" and is the name given to the supplements of the "Kalpasûtras" which treat of the construction of sacrificial altars The period in which the "Śulvasûtras" was composed has been variously fixed by various authors. MAX MÜLLER gave the period as lying between 500 and 200 B. C.; R. C. DUTT gave 800 B. C.; BÜHLER places the origin of the Āpastamba school as probably somewhere within the last four centuries before Christian era, and Baudhâyana somewhat earlier; MACDONNELL gives the limits as 500 B. C. and 200 A. D. and so on. As a matter of fact the dates are not known and those suggested by different authorities must be used with the greatest circumspection. It must also be borne in mind that the contents of the "Śulvasûtras", as known to us, are taken from quite modern manuscripts; and that in matters of detail they have probably been extensively edited. The editions of Āpastamba, Baudhâyana and the Kâtyâyana which have been used for the following notes, indeed, differ from each other to a very considerable extent.

The "Śulvasûtras" are not primarily mathematical but are rules ancillary to religious ritual — they have not a mathematical but a religious aim. No proofs or demonstrations are given and indeed in the presentation there is nothing mathematical beyond the bare facts. Those of the rules that contain mathematical notions relate to (1) the construction of squares and rectangles, (2) the relation of the diagonal o the sides, (3) equivalent rectangles and squares, (4) equivalent circles and squares.

5. In connection with (1) and (2) the PYTHAGOREAN theorem is stated quite generally. It is illustrated by a number of examples which may be summarised thus :

ĀPASTAMBA.	BAUDHĀYANA.
$3^2 + 4^2 = 5^2$	$3^2 + 4^2 = 5^2$
$12^2 + 16^2 = 20^2$	$5^2 + 12^2 = 13^2$
$15^2 + 20^2 = 25^2$	$8^2 + 15^2 = 17^2$
$5^2 + 12^2 = 13^2$	$7^2 + 24^2 = 25^2$
$15^2 + 36^2 = 39^2$	$12^2 + 35^2 = 37^2$
$8^2 + 15^2 = 17^2$	$15^2 + 36^2 = 39^2$
$12^2 + 35^2 = 37^2$	

Kâtâyyana gives no such rational examples but gives (with Āpastamba and Baudhāyana) the hypotenuse corresponding to sides equal to the side and diagonal of a square *i. e.* the triangle a, $a\sqrt{2}$, $a\sqrt{3}$, and he alone gives $1^2 + 3^2 = 10$, and $2^2 + 6^2 = 40$. [It will be noted that Bauddhâyana's list is more general than that given by Āpastamba although the former is supposed to be earlier.] There is no indication that the "Śulvasûtra" rational examples were obtained from any general rule.

Incidentally is given an arithmetical value of the diagonal of a square which may be represented by

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.\ 4} - \frac{1}{3.\ 4.\ 34.}$$

This has been much commented upon but, given a scale of measures based upon the change ratios 3, 4, and 34 (and Baudhâyana actually gives such a scale) the result is only an expression of a direct measurement; and for a side of 6 feet it is accurate to about $^1/_7$ of an inch; or it is possible that the result was obtained by the approximation $\sqrt{a^2 + b} = a + \frac{b}{2a}$ by TANNERY's R- process, but it is quite certain that no such process was known to the authors of the "Śulvasûtras". The only noteworthy character of the fraction is the form with its unit numerators. Neither the value itself nor this form of fraction occurs in any later Indian work.

There is one other point connected with the PYTHAGOREAN theorem, viz. the occurrence of an indication of the formation of a square by the successive addition of gnomons. The text relating to this is as follows :

" Two hundred and twenty-five of these bricks constitute the seven-
" fold "agni" with "aratni" and "prâdeśa". To these sixty-four
" more are to be added. With these bricks a square is formed. The
" side of the square consists of sixteen bricks. Thirty-three bricks
" still remain and these are placed on all (*sic*) sides round the
" borders. "

This subject is never again referred to in Indian mathematical works.

The questions (*a*) whether the Indians of this period had completely realised the generality of the PYTHAGOREAN theorem, and (*b*) whether they had a sound notion of the irrational have been much discussed; but the ritualists who composed the "Śulvasûtras" were not interested in the PYTHAGOREAN theorem beyond their own actual

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wants, and it is quite certain that even as late as the 12^{th} century A. D. no Indian mathematician gives evidence of a complete understanding of the irrational. Further at no period did the Indians develop any real theory of geometry, and a comparatively modern Indian work denied the possibility of any proof of the PYTHAGOREAN theorem other than experience. The fanciful suggestion of BÜRK that possibly PYTHAGORAS obtained his geometrical knowledge from India is not supported by any actual evidence. The Chinese had acquaintance with the theorem over a thousand years B. C., and the Egyptians as early as 2000 B. C.

6. Problems relating to equivalent squares and rectangles are involved in the prescribed altar constructions and consequently the "Śulvasûtras" give constructions, by means of the PYTHAGOREAN theorem, of

(1) a square equal to the sum of two squares;

(2) a square equal to the difference of two squares;

(3) a rectangle equal to a given square;

(4) a square equal to a given rectangle;

(5) the decrease of a square into a smaller square.

These are from Baudhâyana's rules : Āpastamba differs slightly.

Again we have to remark the significant fact that none of these geometrical constructions occur in any later Indian work. The first two are direct geometrical applications of the rule $c^2 = a^2 + b^2$; the third gives in a geometrical form the sides of the rectangle as $a\sqrt{2}$ and $\frac{a\sqrt{2}}{2}$; the fourth rules gives a geometrical construction for $ab = \left(b + \frac{a-b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$ and corresponds to EUCLID, II, 5; the fifth is not perfectly clear but evidently corresponds to EUCLID, II, 4.

7. *The circle.* — According to the altar building ritual of the period it was, under certain circumstances, necessary, to "square the circle", and consequently we have recorded in the "Śulvasûtras" attempts at the solution of this problem, and its connection with altar ritual reminds us of the celebrated DELIAN problem. The solutions offered are very crude although in one case there is pretence of accuracy. Denoting by a the side of the square and by d the diameter of the circle whose area is supposed to be a^2 the rules given may be expressed by

$$(\alpha) \quad d = a + \frac{1}{3}(a\sqrt{2} - a)$$

$$(\beta)\quad a = d - \frac{2}{15}d$$

$$(\gamma)\quad a = d\left(1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}\right)$$

Neither of the first two rules, which are given by both Āpastamba and Baudhâyana, is of particular value or interest. The third is given by Baudhâyana *only* and is evidently obtained from (α) by utilising the value for $\sqrt{2}$ given in paragraph 5 above. We thus have

$$\frac{a}{d} = \frac{3}{2+\sqrt{2}} = \frac{3}{2+\frac{577}{408}} = \frac{1224}{1393}$$

$$= 1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8} - \frac{41}{8.29.6.8.1393}$$

which, neglecting the last term, is the value given in rule (γ). This implies a knowledge of the process of converting a fraction into partial fractions with unit numerators, a knowledge most certainly not possessed by the composers of the "Śulvasûtras"; for as THIBAUT says there is nothing in these rules which would justify the assumption that they were expert in long calculations; and there is no indication in any other work that the Indians were ever acquainted with the process and in no later works are fractions expressed in this manner.

It is worthy of note that later Indian mathematicians record no attempts at the solution of the problem of squaring the circle and never refer to those recorded in the "Śulvasûtras".

III

400 to 600 A. D.

8. There appears to be no connecting link between the "Śulvasûtra" mathematics and later Indian developments of the subject. Subsequent to the "Śulvasûtras" nothing further is recorded until the introduction into India of western astronomical ideas (1). In the

(1) This has a somewhat important bearing on the date of the « Śulvasûtras ». If for example the date of their composition were accepted as 500 B. C. a period of nearly 1000 years, absolutely blank as far as mathematical notions are concerned, would have to be accounted for.

sixth century A. D., VARĀHA MIHIRA wrote his "Pañcha siddhântikâ" which gives a summary account of the five most important astronomical works then in use. Of these the "Sûrya Siddhânta" which was probably composed in its original form not earlier than 400 A. D. afterwards became the standard work. VARĀHA MIHIRA's collection is the earliest and most authentic account we have of what may be termed the scientific treatment of astronomy in India. "Although, " writes THIBAUT, not directly stating that the Hindus learned from the " Greeks, he at any rate mentions certain facts and points of doctrine " which suggest the dependence of Indian astronomy on the science " of Alexandria; and, as we know already from his astrological " writings, he freely employs terms of undoubted Greek origin."

VARĀHA MIHIRA writes : "There are the following Siddhântas — the " Pauliśa, the Romaka, the Vâsishtha, the Saura and the Paitâmaha... " The Siddhânta made by Pauliśa is accurate, near to it stands the " Siddhânta proclaimed by Romaka, more accurate is the Sâvitra " (Sûrya). The two remaining ones are far from the truth".

9. The "Pañcha Siddhântikâ" contains material of considerable mathematical interest and from the historical point of view of a value not surpassed by that of any later Indian works. The mathematical section of the "Pauliśa siddhânta" is perhaps of the most interest and may be considered to contain the whole essence of Indian trigonometry. It is as follows :

" (1) The square-root of the tenth part of the square of the circumference, which comprises 360 parts, is the diameter. Having " assumed the four parts of a circle the sine of the eighth part of a " sign [is to be found];

" (2) Take the square of the radius and call it the constant. The " fourth part of it is [the square of] Aries. The constant square is to " be lessened by the square of Aries. The square-roots of the two " quantities are the sines;

" (3) In order to find the rest take the double of the are, deduct " from it the quarter, diminish the radius by the sine of the remainder " and add to the square of half of that the square of half the sine of " double the arc. The square-root of the sum is the desired sine".

[The eighth part of a "sign" (=30°) is 3°45′ and by "Aries" is indicated the first "sign" of 30°]. The rules given may be expressed in our notation (for unit radius) as

$$(1)\quad \pi = \sqrt{10}$$

(2) $\sin 30^\circ = \frac{1}{2}$, $\sin 60^\circ = \sqrt{1 - \frac{1}{4}}$

(3) $\sin^2\gamma = \left(\frac{\sin 2\gamma}{2}\right)^2 + \left(\frac{1 - \sin(90 - 2\gamma)}{2}\right)^2$

They are followed by a table of 24 sines progressing by intervals of $3^\circ 45'$ obviously taken from Ptolemy's table of chords. Instead, however, of dividing the radius into 60 parts, as did Ptolemy, Pauliśa divides it into 120 parts; for as $\sin\frac{\alpha}{2} = \frac{\text{chord } \alpha}{2}$ this division of the radius enabled him to convert the table of chords into sines without numerical change. The "Sûrya Siddhânta" (and Āryabhata) gives another measure for the radius (3438′) which enabled the sines to be expressed in a sort of circular measure.

We thus have three distinct stages :

(*a*) The "chords" of Ptolemy or ch'd α, with $r = 60$

(*b*) The Paulisâ "sine" or $\sin\frac{\alpha}{2} = \frac{\text{ch'd } \alpha}{2}$, with $r = 120$

(*c*) The Sûrya Siddhanta "sine" or $\sin\frac{\alpha}{2} = \frac{\pi}{3}\,\frac{\text{ch'd } \alpha}{2}$, with $r = 3448'$

To obtain (*c*) the value of π actually used was $\frac{600}{191}$ (=3·14 136..)

Thus the earliest known record of the use of a sine function occurs in the Indian astronomical works of this period. At one time the invention of this function was attributed to el-Battânî [877-919 A. D.] and although we now know this to be incorrect we must acknowledge that the Arabs utilised the invention to a much more scientific end than did the Indians.

In some of the Indian works of this period an interpolation formula for the construction of the table of sines is given. It may be represented by

$$\Delta_{n+1} = \Delta_n - \frac{\sin n.\alpha}{\sin \alpha} \qquad \text{where } \Delta_n = \sin n.\alpha - \sin (n-1)\,\alpha.$$

This is given ostensibly for the formation of the table, but the table actually given cannot be obtained from the formula.

10. Āryabhaṭa. — Tradition places Āryabhaṭa (born 476 A. D.) at the head of the Indian mathematicians and indeed he was the first to

write formally on the subject. He was renowned as an astronomer and as such tried to introduce sounder views of that science but was bitterly opposed by the orthodox. The mathematical work attributed to him consists of thirty-three couplets into which is condensed a good deal of matter. Starting with the orders of numerals he proceeds to evolution and involution, and areas and volumes. Next comes a semi-astronomical section in which he deals with the circle, shadow problems, etc.; then a set of propositions on progressions followed by some simple algebraic identities. The remaining rules may be termed practical applications with the exception of the very last which relates to indeterminate equations of the first degree. Neither demonstrations nor examples are given, the whole text consisting of sixty-six lines of bare rules so condensed that it is often difficult to interpret their meaning. As a mathematical treatise it is of interest chiefly because it is some record of the state of knowledge at a critical period in the intellectual history of the civilised world; because, as far as we know, it is the earliest Hindu work on pure mathematics; and because it forms a sort of introduction to the school of Indian mathematicians that flourished in succeeding centuries.

Āryabhaṭa's work contains one of the earliest records known to us of an attempt at a general solution of indeterminates of the first degree by the continued fraction process. The rule, as given in the text, is hardly coherent but there is no doubt as to its general aim. It may be considered as forming an introduction to the somewhat marvellous development of this branch of mathematics that we find recorded in the works of Brahmagupta and Bhāskara. Another noteworthy rule given by Āryabhaṭa is the one which contains an extremely accurate value of the ratio of the circumference of a circle to the diameter, viz. $\pi = 3\frac{177}{1250}$ $(= 3{\cdot}1416)$; but it is rather extraordinary that Āryabhata himself never utilised this value, that it was not used by any other Indian mathematician before the 12^{th} century A. D., and that no Indian writer quotes Āryabhaṭa as recording this value. Other noteworthy points are the rules relating to volumes of solids which contain some remarkable inaccuracies, *e. g.* the volume of a pyramid is given as *half* the product of the height and the base; the volume of a sphere is stated to be the product of the area of a circle (of the same radius as the sphere) and the root of this area, or $\pi^{\frac{3}{2}}r^3$. Similar errors were not uncommon in later Indian works. The rule known as the *epanthem* occurs in Aryabhaṭa's work and there is a type of definition that occurs in no other Indian work *e. g.* "The product of three " equal numbers is a cube and it also has twelve edges."

IV

A. D. 600-1200.

11. Āryabhaṭa appears to have given a definite bias to Indian mathematics, for following him we have a series of works dealing with the same topics. Of the writers themselves we know very little indeed beyond the mere names but some if not all the works of the following authors have been preserved.

Brahmagupta	born A. D. 598.
Mahāvīra	? IX[th] century A. D.
Śrīdhara	born A. D. 991.
Bhâskara	born A. D. 1114.

Bhāskara is the most renowned of this school, probably undeservedly so, for Brahmagupta's work is possibly sounder mathematically and is of much more importance historically. Generally these writers treat of the same topics — with a difference — and Brahmagupta's work appears to have been used by all the others.

Bhāskara mentions another mathematician, Padmanābha, but omits from his list Mahāvīra.

One of the chief points of difference is in the treatment of geometry. Brahmagupta deals fairly completely with cyclic quadrilaterals while the later writers gradually drop this subject until by the time of Bhāskara it has ceased to be understood.

The most interesting characteristics of the works of this period are the treatment of :

(I) indeterminate equations;
(II) the rational right-angled triangle;
and (III) the perfunctory treatment of pure geometry.

Of these topics it will be noted that the second was dealt with to some extent in the " Sulvasûtras "; but a close examination seems to show that there is no real connection and that the writers of the third period were actually ignorant of the results achieved by Baudhâyana and Āpastamba.

12. *Indeterminate equations.* — The interesting names and dates connected with the early history of indeterminates in India are :

		cir. A. D.
Diophantus		360
Hypatia		400
Āryabhata	born	476
Brahmagupta	—	598
Bhāskara	—	1114

That we cannot fill up the gap between Diophantus and Āryabhata with more than the mere name of Hypatia is probably due to the fanatic ignorance and cruelty of the early Alexandrian Christians rather than the supposed destruction of the Alexandrian library by the Muhammadans. It would be pleasant to conceive that in the Indian works we have some record of the advances made by Hypatia, or of the contents of the lost books of Diophantus — but we are not justified in indulging in more than the mere fancy. The period is one of fascinating interest. The murder of Hypatia (A. D. 415), the imprisonment and execution of Boethius (A. D. 524), the closing of the Athenean schools in A. D. 530 and the fall of Alexandria in 640 are events full of suggestions to the historian of mathematics. It was during this period also that Isidore, Damascius, Simplicius (mathematicians of some repute) and others of the schools of Athens, having heard that PLATO's ideal form of government was actually realised under Chosroes I in Persia, emigrated thither (c. A. D. 532). They were naturally disappointed but the effect of their visit may have been far greater than historical records show.

13. The state of knowledge regarding indeterminate equations in the west at this period is not definitely known. Some of the works of Diophantus and all those of Hypatia are lost to us; but the extant records show that they had explored the field of this analysis so far as to achieve rational solutions (not necessarily integral) of equations of the first and second degree and certain cases of the third degree. The Indian works record distinct advances on what is left of the Greek analysis. For example they give rational *integral* solutions of

$$\text{(A)} \quad ax \pm by = c$$

$$\text{(B)} \quad Du^2 + 1 = t^2$$

The solution of (A) is only roughly indicated by Āryabhata but Brahmagupta's solution (for the positive sign) is practically the same as

$$x = \pm cq - bt, \quad y = \mp cp + at$$

where t is zero or any integer and p/q is the penultimate convergent of a/b.

To solve $$Du^2 + 1 = t^2$$

the Indian methods may be summarised as follows :

If $$Dq^2 + s = p^2 \text{ and } Df^2 + g = e^2$$ then will

$$(a) \quad D(pf \pm eq)^2 \pm sg = (pe \pm qfD)^2$$

$$(b) \quad D\left(\frac{p + qr}{s}\right)^2 \pm \frac{r^2 - D}{s} = \left(\frac{Dq + pr}{s}\right)^2$$

r being any suitable integer.

Also

$$(c) \quad D\left(\frac{2n}{D \sim n^2}\right)^2 + 1 = \left(\frac{D + n^2}{D \sim n^2}\right)^2$$

n being any assumed number.

The complete integral solution is given by a combination of (a) and (b) of which the former only is given by Brahmagupta, while both are given by Bhāskara (five centuries later !). The latter designates (a) the " method by composition " and (b) the " cyclic method ". These solutions are alone sufficient to give to the Indian works an important place in the history of mathematics. Of the " cyclic method " (*i. e.* the combination of [a] and [b]) HANKEL says " It is beyond all praise : it is " certainly the finest thing achieved in the theory of numbers before " Lagrange ". He attributed its invention to the Indian mathematicians, but the opinions of the best modern authorities (*e. g.* TANNERY, CANTOR, HEATH) are rather in favour of the hypothesis of ultimate Greek origin.

The following conspectus of the indeterminate problems dealt with by the Indians will give some idea of their work in this direction; and although very few of the cases actually occur in Greek works now known to us the conspectus significantly illustrates a general similarity of treatment.

(1)	$ax \pm by = c$	[1] B, 8 f; M, VI, 115 f; L, 248 f; V, 65 f.
(2)	$ax + by + cz + \ldots = l$	B, 55 f; V, 157 f.
(3)	$x \equiv a_1 \text{ mod. } b_1 \equiv \ldots \equiv a_4 \text{ Mod. } b_4$	B, 7; M, VI, 123 f; V, 160.
(4)	$Ax + By + Cxy = D$	B, 62; V, 209 f.
(5)	$Du^2 + 1 = t^2$	B, 67; V, 82, 87, 94.
(6)	$Du^2 - 1 = t^2$	V, 90.
(7)	$Du^2 \pm s = t^2$	B, 70, 72, 73, 76.
(8)	$D^2u^2 \pm s = t^2$	B, 73, 76; V, 96.
(9)	$u^2 + s = at^2$	V, 184.
(10)	$Du^2 \pm au = t^2$	V, 177, 181.
(11)	$s - Du^2 = t^2$	V, 98.
(12)	$Du + s = t^2$	B, 86.
(13)	$x \pm a = s^2,\ x \pm b = t^2$	B, 83, 85; M, 275-280; Bk. 50.
(14)	$ax + 1 = s^2,\ bx + 1 = t^2$	B, 79; V, 197.
(15)	$ax + b = s^2,\ bx + a = t^2$	V, 199.
(16)	$ax^2 + by^2 = s^2,\ ax^2 - by^2 + 1 = t^2$	V, 187.
(17)	$x^2 + y^2 \pm 1 = s^2,\ x^2 - y^2 \pm 1 = t^2$	V, 194.
(18)	$x^2 - a \equiv x^2 - b \equiv o \text{ Mod. } c$	V, 202.
(19)	$ax^2 + b \equiv o \text{ Mod. } c$	V, 207.
(20)	$x + y = s^2,\ x - y = t^2,\ xy = u^3$	V, 201.
(21)	$x^3 + y^3 = s^2,\ x^2 + y^2 = t^3$	V, 122.
(22)	$x - y = s^3,\ x^2 + y^2 = t^3$	V, 182.
(23)	$x + y = s^2,\ x^3 + y^2 = t^2$	V, 188.
(24)	$x^3 + y^2 + xy = s^2,\ (x + y)s + 1 = t^2$	V, 189.
(25)	$ax + 1 = s^3,\ as^2 + 1 = t^2$	V, 198.
(26)	$wxyz = a(w + x + y + z)$	V, 210.
(27)	$x^3 - a \equiv o \text{ Mod. } b$	V, 206.

14. *Rational right-angled triangles.* — The Indian mathematicians of this period seem to have been particularly attracted by the problem of the rational right-angled triangle and give a number of rules for obtaining integral solutions. The following summary of the various rules relating to this problem shows the position of the Indians fairly well.

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(1) The references are to Brahmagupta (indicated by B); Mahāvīra (M); Bhāskara's *Līlāvatī* (L) and *Vija Gaṇita* (V); the Bakhshāli Ms. (Bk.) It should be remembered that Brahmagupta flourished five centuries before Bhāskara.

	A	B	$\sqrt{A^2+B^2}$	Authorities
(I)	n	$\frac{n^2-1}{2}$	$\frac{n^2+1}{2}$	Pythagoras (according to Proclus)
(II)	$\sqrt{mn}$	$\frac{m-n}{2}$	$\frac{m+n}{2}$	Plato (according to Proclus)
(III)	$2mn$	m^2-n^2	m^2+n^2	Euclid & Diophantus
(IV)	$\frac{m(n^2-1)}{n^2+1}$	$\frac{2mn}{n^2+1}$	m	Diophantus
(V)	$2mn$	m^2-n^2	m^2+n^2	BRAHMAGUPTA & MAHĀVĪRA
(VI)	$\sqrt{m}$	$\frac{1}{2}\left(\frac{m}{n}-n\right)$	$\frac{1}{2}\left(\frac{m}{n}+n\right)$	BRAHMAGUPTA, MAHĀVĪRA & BHĀSKARA
(VII)	m	$\frac{2mn}{n^2-1}$	$m\frac{(n^2+1)}{n^2-1}$	BHĀSKARA
(VIII)	$\frac{m(n^2-1)}{n^2+1}$	$\frac{2mn}{n^2-1}$	m	BHĀSKARA
(IX)	$2lmn$	$l(m^2-n^2)$	$l(m^2+n^2)$	General formula

Mahâvīra gives many examples in which he employs formula (v) of which he terms m and n the " elements ". From given elements he constructs triangles and from given triangles he finds the elements *e. g.* " What are the ' elements ' of the right-angled triangle (48, 55, 73)? Answer : 3, 8."

Other problems connected with the rational right-angled triangle given by Bhāskara are of some historical interest: *e. g.* (1) The sum of the sides is 40 and the area 60, (2) The sum of the sides is 56 and their product 7×600, (3) The area is numerically equal to the hypotenuse, (4) See also 17 (IV) below.

15. The geometry of this period is characterised by :

(1) Lack of definitions, etc.;

(2) Angles are not dealt with at all;

(3) There is no mention of parallels and no theory of proportion;

(4) Traditional inaccuracies are common;

(5) A gradual decline in geometrical knowledge is noticeable.

On the other hand we have the following noteworthy rules relating to cyclic quadrilaterals

$$\text{(I)} \quad Q = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

$$\text{(II)} \quad x^2 = (ad+bc)\,ac+bd)/(ab+cd)$$

$$y^2 = (ab+cd)\,(ac+bd)/ad+bc)$$

where x and y are the diagonals of the cyclic quadrilateral (a, b, c, d). This (II) is sometimes designated " Brahmagupta's theorem ".

16. The absence of definitions and indifference to logical order sufficiently differentiate the Indian geometry from that of the early Greeks; but the absence of what may be termed a theory of geometry hardly accounts for the complete absence of any reference to parallels and angles. Whereas, on the one hand the Indians have been credited with the invention of the sine function, on the other there is no evidence to show that they were acquainted with even the most elementary theorems (as such) relating to angles.

The presence of a number of incorrect rules side by side with correct ones is significant. The one relating to the area of triangles and quadrilaterals, viz. the area is equal to the product of half the sums of pairs of opposite sides, strangely enough occurs in a Chinese work of the 6th century A. D. as well as in the works of Ahmes, Brahmagupta, Mahâvīra, Boethius and Bede. By Mahâvīra, the idea on which it is based — that the area is a function of the perimeter — is further emphasized. Āryabhata gives an incorrect rule for the volume of a pyramid; incorrect rules for the volume of a sphere are common to Āryabhata, Śrīdhara and Mahâvira. For cones all the rules assume that $\pi = 3$. Mahâvira gives incorrect rules for the circumference and area of an ellipse, and so on.

17. Brahmagupta gives a fairly complete set of rules dealing with the cyclic quadrilateral and either he or the mathematician from whom he obtained his material had a definite end in view — the construction of a cyclic quadrilateral with rational elements.

The commentators did not fully appreciate the theorems, some of which are given in the works of Mahâvira and Śrīdhara; but by the time of Bhâskara they had ceased to be understood. Bhâskara indeed condemns them outright as unsound. " How can a person " he says " neither specifying one of the perpendiculars, nor either of the " diagonals, ask the rest. Such a questioner is a blundering devil and " still more so is he who answers the question. "

Besides the two rules (I) and (II) already given in paragraph 15, Brahmagupta gives rules corresponding to the formula

$$\text{(III)} \quad 2r = \frac{a}{\sin A} \text{etc:, and}$$

(IV) If $a^2 + b^2 = c^2$ and $\alpha^2 + \beta^2 = \gamma^2$ then the quadrilateral ($a\gamma$, $c\beta$, $b\gamma$, $c\alpha$) is cyclic and has its diagonals at right angles. This figure is sometimes termed " Brahmagupta's trapezium ". From the triangles (3, 4, 5) and (5, 12, 13) a commentator obtains the quadrilateral (39, 60, 52, 25) with diagonals 63 and 56, etc. He also introduces a proof of Ptolemy's theorem and in doing this follows Diophantus (III, 19) in constructing from triangles (a, b, c) and (α, β, γ) new triangles $(a\gamma, b\gamma, c\gamma)$ and $(\alpha c, \beta c, \gamma c)$ and uses the actual arithmetical examples given by Diophantus, namely (39, 52, 65) and (25, 60, 65).

18. An examination of the Greek mathematics of the period immediately anterior to the Indian period with which we are now dealing shows that geometrical knowledge was in a state of decay. After Pappus (c. A. D. 300) no geometrical work of much value was done. His successors were, apparently, not interested in the great achievments of the earlier Greeks and it is certain that they were often not even acquainted with many of their works. The high standard of the earlier treatises had ceased to attract, errors crept in, the style of exposition deteriorated and practical purposes predominated. The geometrical work of Brahmagupta is almost what one might expect to find in the period of decay in Alexandria. It contains one or two gems but it is not a scientific exposition of the subject and the material is obviously taken from western works.

V.

19. We have, in the above notes, given in outline the historically important matters relating to Indian mathematics. For points of detail the works mentioned in the annexed bibliography should be consulted; but we here briefly indicate the other contents of the Indian works, and in the following sections we shall refer to certain topics that have achieved a somewhat fictitious importance, to the personalities of the Indian mathematicians and to the relations between the mathematics of the Chinese, the Arabs and the Indians.

20. Besides the subjects already mentioned BRAHMAGUPTA deals very briefly with the ordinary arithmetical operations, square and cube-root, rule of three, etc.; interest, mixtures of metals, arithmetical progressions, sums of the squares of natural numbers; geometry as already described but also including elementary notions of the circle, elementary mensuration of solids, shadow problems; negative and positive qualities, cipher, surds, simple algebraic identities, indeterminate equations of the first and second degree which occupy the greater portion of the work while simple equations of the first and second degree receive comparatively but little attention.

MAHĀVIRA's work is fuller but more elementary on the whole. The ordinary operations are treated with more completeness and geometrical progressions are introduced; many problems on indeterminates are given but no mention is made of the "cyclic method" and it contains no formal algebra. It is the only Indian work that deals with ellipses (inaccurately).

The only extant work by Śṛidhara is like Mahâvira's but shorter; but he is quoted as having dealt with quadratic equations, etc.

Bhâskara's "Līlâvatī" is based on Śrīdhara's work and, besides the topics already mentioned, deals with combinations, while his "Vīja-gaṇita", being a more systematic exposition of the algebraical topics dealt with by Brahmagupta, is the most complete of the Indian works.

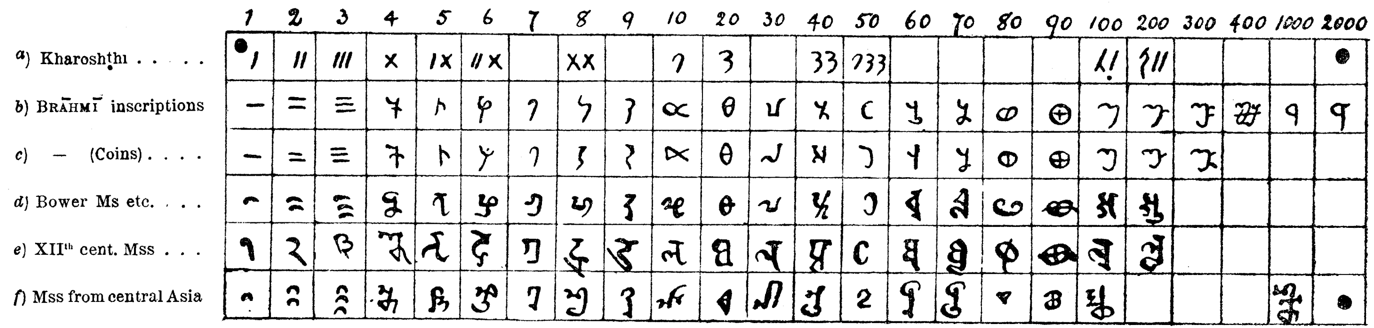
After the time of Bhâskara (born A. D. 1114) no Indian mathematical work of historical value or interest is known. Even before his time deterioration had set in and although a "college" was founded to perpetuate the teaching of Bhâskara it, apparently, took an astrological bias.

VI

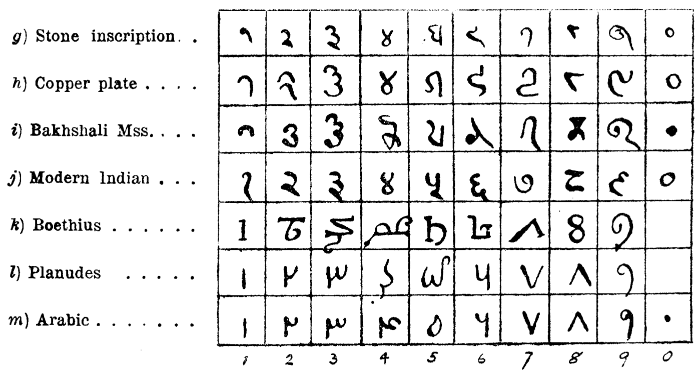
21. According to the Hindus the modern place-value system of arithmetical notation is of divine origin. This led the early orientalists to believe that, at any rate, the system had been in use in India from time immemorial; but an examination of the real facts shows that the early notations in use were not place-value ones and that the modern place-value system was not introduced until comparatively modern times. The early systems employed may be conveniently termed (*a*) the Kharoshṭhi (*b*) the Brâhmi (*c*) Āryabhaṭa's alphabetic notation (*d*) the word-symbol notation.

(*a*) The "Kharoshṭhi" script is written from right to left and was in use in the north-west of India, Afghanistan and Central Asia at the beginning of the Christian era. The notation is shown in the accompanying table. It was apparently derived from the Aramaic system and has little direct connection with the other Indian notations. The smaller elements are written on the left.

(*b*) The "Brâhmī notation" is the most important of the old notations of India. It might appropriately be termed *the* Indian notation for it occurs in early inscriptions and was in fairly common use throughout India for many centuries, and even to the present day is occasionally used. The symbols employed varied somewhat in form according to time and place but on the whole the consistency of form exhibited is remarkable. They are written from left to right with the smaller elements on the right. Several false theories as to the origin



PLACE VALUE NOTATIONS.



REFERENCES.

a) After Bühler c. 100 A. D.

b) Inscriptions c. 100-500 A. D.

c) After Rapson c. 200-400 A. D.

d) Bower Mss. & Nepal Mss. c. 500-900 A. D.

e) After Bendall XIth century.

f) Eastern Turkestan Mss. (Stein collection) J. R. A. S. 1911, 455.

g) Gwâlior stone insep. IXth cent. *Arch. Surv.*, 1903-4. Pl. LXXII.

h) Surat grant of 1050 A. D. *Ind. Ant.* XII, 202.

XI-82.

NUMERICAL NOTATIONS

of these symbols have been published some of which still continue to be recorded. The earliest orientalists gave them place-value, but this error soon disproved itself; it was then suggested that they were initial letters of numerical words; then it was propounded that the symbols were *aksharas* or syllables; then it was again claimed that the symbols were initial letters (this time "kharoshṭhî") of the corresponding numerals. These theories have been severally disproved.

The notation was possibly developed on different principles at different times. The first three symbols are natural and only differ from those of many other systems in consisting of horizontal instead of vertical strokes. No principle of formation of the symbols for "four" to "thirty" is now evident but possibly the "forty" was formed from the thirty by the addition of a stroke and the "sixty" and "seventy" and "eighty" and "ninety" appear to be connected in this way. The hundreds are (to a limited extent) evidently built upon such a plan, which, as BAYLEY pointed out, is the same as that employed in the Egyptian hieratic forms; but after the "three hundred" the Indian system forms the "four hundred" from the elements of "a hundred" and "four", and so on. The notation is exhibited in the table annexed.

(*c*) Āryabhaṭa's alphabetic notation also had no place-value and differed from the Brâhmî notation in having the smaller elements on the left. It was, of course, written and read from left to right. It may be exhibited thus :

Letters.	*k*	*kh*	*g*	*gh*	*ṅ*	*c*	*ch*	*j*	*jh*	*ñ*.
Values.	1	2	3	4	5	6	7	8	9	10.
Letters.	*ṭ*	*ṭh*	*ḍ*	*ḍh*	*ṇ*	*t*	*th*	*d*	*dh*	*n*.
Values.	11	12	13	14	15	16	17	18	19	20.

Letters.	*p*.	*ph*	*b*	*bh*	*m*	*y*	*r*	*l*	*v*	'	*sh*	*s*	*h*
Values.	21	22	23	24	25	30	40	50	60	70	80	90	100.

The vowels indicate multiplication by powers of one hundred. The first vowel *a* may be considered as equivalent to 100^{0}, the second vowel $i = 100^{1}$ and so on. The values of the vowels may therefore be shown thus :

Vowels.	*a*	*i*	*u*	*ṛi*	*ḷi*	*e*	*ai*	*o*	*au*.
Values.	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}	10^{12}	10^{14}	10^{16}.

The following examples taken from Āryabhata's *Gîtikâ* illustrate the application of the system :

$$khyughṛi = (2 + 30).10^4 + 4.10^6 = 4{,}320{,}000.$$

$$cayagiyiṅuśulchḷi = 6 + 30 + 3.10^2 + 30.10^2 + 5.10^4 + 70.10^4 + (50 + 7).10^8 = 57{,}753{,}336.$$

The notation could thus be used for expressing large numbers in a sort of mnemonic form. The table of sines referred to in paragraph 9 above was expressed by Āryabhaṭa in this notation which, by the way, he uses only for astronomical purposes. It did not come into ordinary use in India, but some centuries later it appears occasionally in a form modified by the place-value idea with the following values :

1	2	3	4	5	6	7	8	9	0
k	*kh*	*g*	*gh*	*ṅ*	*c*	*ch*	*j*	*jh*	*ñ*
ṭ	*ṭh*	*ḍ*	*ḍh*	*ṇ*	*t*	*th*	*d*	*dh*	*n*
p	*ph*	*b*	*bh*	*m*					
y	*r*	*l*	*v*	*ś*	*sh*	*s*	*h*	*l*	

Initial vowels are sometimes used as ciphers also. The earliest example of this modified system is of the twelfth century A. D. Slight variations occur.

(*d*) *The word-symbol notation.* — A notation that became extraordinarily popular in India and is still in use was introduced about the ninth century A. D., possibly from the East. In this notation any word that connotes the idea of a number may be used to denote that number *e. g.* Two may be expressed by *nayana*, the eyes, or *karṇa*, the ears, etc. ; seven by *aśva*, the horses (of the sun) ; fifteen by *tithi*, the lunar days (of the half month) ; twenty by *nakha*, the nails (of the hands and feet) ; twenty-seven by *nakshatra* the lunar mansions ; thirty-two by *danta*, the teeth ; etc.

(*e*) *The modern place-value notation.* — The orthodox view is that the modern place-value notation that is now universal was invented in India and until recently it was thought to have been in use in India at a very early date. Hindu tradition ascribes the invention to God! According to Maçoudi a congress of sages, gathered together by order of king Brahma (who reigned 366 years), invented the nine figures! Patañjali and other early writers are supposed to make references to the place-value system. An inscription of A. D. 595 is supposed to contain a genuine example of the system. According to M. NAU the "Indian figures" were known in Syria in 662 A. D.; but his authority makes such erroneous statements about Indian astronomy that we have no faith in what he says about other Indian matters. Certain other mediæval works refer to "Indian numbers" and so on.

On the other hand it is held that there is no sound evidence of the employment in India of a place-value system earlier than about the ninth century A. D. The suggestion of "divine origin" indicates nothing but historical ignorance; Maçoudi is obviously wildly erratic; the inscription of A. D. 595 is not above suspicion [1] and the next inscription with an example of the place-value system is nearly three centuries later while there are hundreds intervening with examples of the old non-place-value system; the modern texts of Patañjali and others are well known to have been freely edited and in no direction is editing so free as when dealing with numbers. The references in mediæval works to India do not necessarily indicate India proper but often simply refer to "the East" and the use of the term with regard to numbers has been further confused by the misreading by Woepcke and others of the Arabic term *hindasi* (geometrical, having to do with numeration, etc.) which has nothing to do with India. Again, it has been assumed that the use of the abacus "has been universal in India from time immemorial" but this assumption is not based upon fact, there being actually no evidence of its use in India until quite modern times. Further, there is evidence that indicates that the notation was introduced into India, as it was into Europe, from a right-to-left script.

22. In paragraph 7 above certain attempts at squaring the circle are briefly described and it has been pointed out (in § 10) that Āryabhaṭa gives an extremely accurate value of π. The topic is perhaps of sufficient interest to deserve some special mention. The Indian values given and used are not altogether consistent and the subject is wrapped in some mystery. Briefly put — the Indians record an extremely accurate value at a very early date but seldom or never actually use it. The following table roughly exhibits how the matter stands.

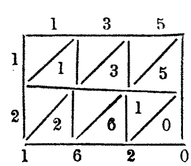
---

(1) The figures were obviously added at a later date.

DATE CIRCA	AUTHORITY	VALUE OF π	
B.C. 1700	Ahmes the Egyptian	$\left(\frac{16}{9}\right)^2$	$= 3,160..$
» 250	Archimedes	$< 3\,\frac{1}{7}$ and $> 3\,\frac{10}{71}$	
» ?	The ŚULVASŪTRAS	$\left(\frac{26}{15}\right)^2$	$= 3,004..$
	»	$\left(\frac{9785}{5568}\right)^2$	$= 3,097..$
» 230	Apollonius	$3\,\frac{17}{120}$	$= 3,14166..$
» 120	Heron	3	$= 3$
	»	$3\,\frac{1}{7}$	$= 3,14286..$
A. D. 150	Ptolemy	$3\,\frac{17}{120}$	$= 3,14166..$
» 263	Liu Hiu (Chinese)	$3\,\frac{7}{50}$	$= 3,14$
» ?	PULIŚA (According to Alberuni)	$3\,\frac{177}{1250}$	$= 3,1416$
» 450	Tsu Ch'ung-chi (Chinese)	$3\,\frac{1}{7}$	$= 3,14286..$
» 500	ĀRYABHATA	3	$= 3$
		$\frac{62832}{20000}$	$= 3,1416..$
	» (According to Alberuni)	$\frac{3393}{1080}$	$= 3,14166.$
» 628	BRAHMAGUPTA	3	$= 3$
		$\sqrt{10} = 3\,\frac{1}{7}$	$= 3,14286.$
	» (According to Alberuni)	$\sqrt{10} = \frac{721}{228}$	$= 3,16228.$
» 800	M. ibn Musa	$\overline{10}$	$= ?$

DATE CIRCA	AUTHORITY	VALUE OF π	
» 800	M. ibn Musa	$\frac{62832}{20000}$	$= 3,1416$
»	MAHAVIRA	3	$= 3$
	»	$\sqrt{10}$	$= ?$
» 1020	ŚĪRDHARA	3	$= 3$
	»	$\sqrt{10}$	$= ?$
	»	$3\frac{1}{6}$	$= 3,1666..$
» 1150	BHĀSKARA	3	$= 3$
	»	$3\frac{1}{7}$	$= 3,14286..$
	»	$3\frac{17}{120}$	$= 3,14166$
	»	$3\frac{177}{1250}$	$= 3,1416..$
		Approximately correct value	3,14159..

23. The mistakes made by the early orientalists have naturally misled the historians of mathematics, and the opinions of CHASLES, WOEPCKE, HANKEL and others founded upon such mistakes are now no longer authoritative. In spite, however, of the progress made in historical research there are still many errors current, of which, besides those already touched upon, the following may be cited as examples : *a*) The proof by " casting out nines " is not of Indian origin and occurs in no Indian work before the twelfth century; *b*) The scheme of multiplication, of which the following is an Indian example of the 16th century, was known much earlier to the Arabs and there is no evidence that it is of Indian origin; *c*) The *Regula*



duorum falsorum occurs in no Indian work; *d*) The Indians were not the first to give negative and positive solutions of quadratic equations, etc.

VII

24. Of the personalities of the Indian mathematicians we know very little indeed but Alberuni has handed down Brahmagupta's opinion of Āryabhaṭa and Pauliśa (1) and his own opinions are worth repeating. We have also Bhâskara's inscription. The following notes contain, perhaps, all that is worth recording.

Alberuni writes (I,376) : " Now it is evident that that which Brahmagupta relates on his own authority, and with which he himself agrees, is entirely unfounded; but he is blind to this from sheer hatred of Āryabhaṭa, whom he abuses excessively. And in this respect Āryabhaṭa and Puliśa are the same to him. I take for witness the passage of Brahmagupta where he says that Āryabhaṭa has subtracted something from the cycles of *Caput Draconis* and of the *apsis* of the moon and thereby confused the computation of the eclipse. He is rude enough to compare Āryabhaṭa to a worm which, eating the wood, by chance describes certain characters in it, without intending to draw them. " He, however, who knows these things thoroughly stands opposite to Āryabhaṭa, Śrîshena and Vishnucandra like the lion against gazelles. They are not capable of letting him see their faces." In such offensive terms he attacks Āryabhaṭa and maltreats him? Again : " Aryabhaṭa... differs from the doctrine of the book "Smriti", just mentioned, and he who differs from us is an opponent." On the other hand, Brahmagupta praises Puliśa for what he does, since he does not differ from the book " Smṛiti." Again, speaking of Varâhamihira, Srishena, Āryabhaṭa and Vishnucandra, Brahmagupta says : " If a man declares these things illusory he stands outside the generally acknowledged dogma, and that is not allowed. "

Of Varâhamihira Alberuni writes : " In former times, the Hindus used to acknowledge that the progress of science due to the Greeks is much more important than that which is due to themselves. But from this passage of Varâhamihira alone (see paragraph 2 above) you see what a self-lauding man he is, whilst he gives himself airs as doing justice to others"; but in another place (II, 110) he says : " On the whole his foot stands firmly on the basis of truth and he clearly speaks out the truth... Would to God all distinguished men followed his example. "

Of Brahmagupta, Alberuni writes (II, 110) : " But look, for instance, at Brahmagupta, who is certainly the most distinguished of their astronomers... he shirks the truth and lends his support to imposture...

(1) According to Alberuni Puliśa was an Indian and Pauliśa a Greek.

under the compulsion of some mental derangement, like a man whom death is about to rob of his consciousness... If Brahmagupta... is one of those of whom God says, " They have denied our signs, although their hearts knew them clearly, from wickedness and haughtiness," we shall not argue with him, but only whisper into his ear. " If people must under circumstances give up opposing the religious codes (as seems to be your case), why then do you order people to be pious if you forget to be so yourself. ." I for my part, am inclined to the belief that that which made Brahmagupta speak the above mentioned words (which involve a sin against conscience) was something of a calamitous fate, like that of Socrates, which had befallen him, notwithstanding the abundance of his knowledge and the sharpness of his intellect, and notwithstanding his extreme youth at the time. For he wrote the " Brahmasiddhânta " when he was only thirty years of age. If this indeed is his excuse we accept it and drop the matter."

An inscription found in a ruined temple at Pâṭṇâ, a deserted village of Khandesh in the Bombay Presidency refers to BHĀSKARA in the following terms : " Triumphant is the illustrious Bhâskarâcârya whose feet are revered by the wise, eminently learned... who laid down the law in metrics, was deeply versed in the Vaiśeshika system... was in poetics a poet, like unto the three-eyed in the three branches, the multifarious arithmetic and the rest... Bhâskara, the learned, endowed with good fame and religious merit, the root of the creeper — true knowledge of the Veda, an omniscient seat of learning; whose feet were revered by crowds of poets, etc."

The inscription goes on to tell us of Bhâskara's grandson " Chaṅga-deva, chief astrologer of King Siṁghaṇa, who, to spread the doctrines promulgated by the illustrious Bhâskarâcârya, founds a college, that in his college the " Siddhântaśiromani " and other works composed by Bhâskara, as well as other works by members of his family, shall be necessarily expounded ".

Bhâskara's most popular work is entitled the " Lîlâvati" which means " charming." He uses the phrase " Dear intelligent " Lîlâ-vati," etc. : and thus have arisen certain legends as to a daughter he is supposed to be addressing. The legends have no historical basis.

Bhâskara at the end of his " Vîja gaṇita " refers to the treatises on algebra of Brahmagupta, Srîdhara and Padmanâbha as too diffusive and states that he has compressed the substance of them in a well reasoned compendium, for the gratification of learners.

VIII

25. *Chinese Mathematics.* — There is abundant evidence of an intimate connection between Indian and Chinese mathematics. A number of Indian embassies to China and Chinese visits to India

are recorded in the fourth and succeeding centuries. The records of these visits are not generally found in Indian works and our knowledge of them in most cases comes from Chinese authorities, and there is no record in Indian works that would lead us to suppose that the Hindus were in any way indebted to China for mathematical knowledge. But, as pointed out before, this silence on the part of the Hindus is characteristic, and must on no account be taken as an indication of lack of influence. We have now before us a fairly complete account of Chinese mathematics (1) which conclusively proves a very close connection between the two countries. This connection is briefly illustrated in the following notes.

The earliest Chinese work that deals with mathematical questions is said to be of the 12th century B. C. and it records an acquaintance with the PYTHAGOREAN theorem. Perhaps the most celebrated Chinese mathematical work is the *Chiu-chang Suan-shû* or "Arithmetic in Nine Sections" which was composed at least as early as the second century B. C. while Chang T'sang's commentary on it is known to have been written in A. D. 263. The "Nine Sections" is far more complete than any Indian work prior to Brahmagupta (628 A. D.) and in some respects is in advance of that writer. It treats of fractions, percentage, partnership, extraction of square and cube-roots, mensuration of plane figures and solids, problems involving equations of the first and second degree. Of particular interest to us are the following: The area of a segment of a circle $= \frac{1}{2}(c + a)a$, where "c" is the chord and "a" the perpendicular, which actually occurs in Mahâvira's work; in the problems dealing with the evaluation of roots, partial fractions with unit numerators are used (cf. paragraphs 5 and 7 above); the diameter of a sphere $= \sqrt[3]{\frac{16}{9} \times \text{volume}}$, which possibly accounts for Āryabhaṭa's strange rule; the volume of the cone $= \left(\frac{circumference}{6}\right)^2$ which is given by all the Indians and the correct volume for a truncated square pyramid which is reproduced by Brahmagupta and Śrîdhara. One section deals with right-angled triangles and gives a number of problems like the following: "There is a bamboo 10 feet " high, the upper end of which being broken reaches to the ground " 3 feet from the stem. What is the height of the break?". This

(1) By Yoshio Mikami.

occurs in every Indian work after the 6th century A. D. The problem about two travellers meeting on the hypotenuse of a right-angled triangle occurs some ten centuries later in exactly the same form in Mahâvira's work. *The Sun-Tsu Suan-ching* is an arithmetical treatise of about the first century A. D. It indulges in big numbers and elaborate tables like that contained in Mahavira's work; it gives a clear explanation of square-root and it contains examples of indeterminate equations of the first degree. The example : "There are " certain things whose number is unknown. Repeatedly divided by " 3 the remainder is 2; by 5 the remainder is 3, and by 7 the " remainder is 2. What will be the number?" reappears in Indian works of the seventh and ninth centuries. The earliest Indian example is given by Brahmagupta and is : "What number divided by " 6 has a remainder 5, and divided by 5 has a remainder of 4, and by " 4 a remainder of 3, and by 3 a remainder of 2?" Mahâvîra has similar examples.

In the third century the *Sea island Arithmetical Classic* was written. Its distinctive problems concern the measurement of the distance of an island from the shore, and the solution given occurs in Āryabhaṭa's *Gaṇita* some two centuries later. The *Wu-t'sao* written before the sixth century appears to indicate some deterioration. It contains the erroneous rule for areas given by Brahmagupta and Mahavira. The arithmetic of CHANG-CH'IU-CHIEN written in the sixth century contains a great deal of matter that may have been the basis of the later Indian works. Indeed the later Indian works seem to bear a much closer resemblance to Chang's arithmetic than they do to any earlier Indian work.

The problem of "the hundred hens" is of considerable interest. Chang gives the following example : "A cock costs 5 pieces of money, " a hen 3 pieces and 3 chickens 1 piece. If then we buy with " 100 pieces 100 of them what will be their respective numbers?"

No mention of this problem is made by Brahmagupta but it occurs in Mahâvira and Bhâskara in the following form : "Five doves are to " be had for 3 *drammas*, "7 cranes for 5, 9 geese for 7 and 3 peacocks " for 9. Bring 100 of these birds for 100 *drammas* for the prince's " gratification?" It is noteworthy that this problem was also very fully treated by Abû Kâmil (Soğâ) in the 9th century, and in Europe in the middle ages it acquired considerable celebrity.

Enough has been said to show that there existed a very considerable intimacy between the mathematics of the Indians and Chinese; and assuming that the chronology is roughly correct (1), the distinct priority of the Chinese mathematics is fully established. On the other hand Brahmagupta gives more advanced developments of indeterminate equations than occurs in the Chinese works of his period, and it is not until after Bhâskara that Ch'in Chu-sheo recorded (in A. D. 1247) the celebrated *t'ai-yen ch'iu-yi-shu* or process of indeterminate analysis, which is, however, attributed to I'-hsing nearly six centuries earlier. The Chinese had maintained intellectual intercourse with India since the first century A. D. and had translated many Indian (Buddhistic) works. They (unlike their Indian friends) generally give the source of their information and acknowledge their indebtedness with becoming courtesy. From the 7th century Indian scholars were occasionally employed on the Chinese Astronomical Board. Mr. Yoshio Mikami states that there is no evidence of Indian influence on Chinese mathematics. On the other hand he says "the " discoveries made in China may have touched the eyes of Hindoo " scholars."

26. *Arabic Mathematics.* — It has often been assumed, with very little justification, that the Arabs owed their knowledge of mathematics to the Hindus. Muhammad b. Mûsâ el- Chowârezmi (A. D. 782) is the earliest Arabic writer on mathematics of note and his best known work is the *Algebra*. The early orientalists appear to have been somewhat prejudiced against Arabic scholarship for, apparently without examination, they ascribed an Indian origin to M. b. Mûsâ's work. The argument used was as follows : "There is nothing in history" wrote Cossali, and Colebrooke repeated it, "respecting " Muhammad ben Musa individually, which favours the opinion that " he took from the Greeks, the algebra which he taught to the Muham- " madans. History presents him in no other light than a mathe- " matician of a country most distant from Greece and contiguous to " India... 'Not having taken algebra from the Greeks, he must either " have invented it himself or taken it from the Indians." As a matter of fact his algebra shows, as pointed out by Rodet, no sign of Indian influence and is practically wholly based upon Greek knowledge; and it is now well known that the development of mathematics among the Arabs was largely, if not wholly, independent of Indian

(1) My information about Chinese mathematics is second-hand and the correctness of the information given in this section is largely dependent on Mr. Yoshio Mikami's interesting book.

influence and that, on the other hand, Indian writers on mathematics later than Brahmagupta were possibly influenced considerably by the Arabs. Alberuni early in the eleventh century wrote : " You mostly " find that even the so-called scientific theorems of the Hindus are in " a state of utter confusion, devoid of any logical order... since they " cannot raise themselves to the methods of strictly scientific deduc- " tion... I began to show them the elements on which this science " rests, to point out to them some rules of logical deduction and the " scientific method of all mathematics, etc. "

The fact is that in the time of el-Mâmûn (c. 772 A. D.) a certain Indian astronomical work (or certain works) was translated into Arabic. On this basis it was *assumed* that the Arabic astronomy and mathematics was wholly of Indian origin, while the fact that Indian works were translated is really only evidence of the intellectual spirit then prevailing in Baghdad. No one can deny that Āryabhata and Brahmagupta preceded M. b. Mûsâ (1) but the fact remains that there is not the slightest resemblance between the previous Indian works and those of M. b. Mûsa. The point was somewhat obscured by the publication in Europe of an arithmetical treatise by M. b. Mûsâ under the title *Algoritmi de Numero Indorum*. As is well known the term India did not in mediæval times necessarily denote the India of to-day and despite the title there is nothing really Indian in the work. Indeed its contents prove conclusively that it is not of Indian origin. The same remarks apply to several other mediæval works.

27. From the time of M. b. Mûsâ (c. 820 A. D.) onwards the Muhammadan mathematicians made remarkable progress. To illustrate this fact we need only mention a few of their distinguished writers and their works on mathematics. 'Tâbit b. Qorra b. Merwân (826-901) wrote on Euclid, the Almagest, the arithmetic of Nicomachus, the right-angled triangle, the parabola, magic squares, amicable numbers, etc. Qosṭâ b. Lûkâ el-Ba'albekî (died c. 912 A. D.) translated Diophantus and wrote on the sphere and cylinder, the rule of two errors, etc. El-Battânî (M. b. Ǧâbir b. Sinan, 877-919 A. D.) wrote a commentary on Ptolemy and made notable advances in trigonometry. Abû Kâmil Shoǧâ b. Aslam (c. 850-930 A. D.) wrote on algebra and geometry, the pentagon and decagon, the rule of two errors, etc.

(1) It should not be forgotten, however, that Nicomachus (c. 100 A. D.) was an Arabian, while Jamblichus, Damascius, and Eutocius were natives of Syria.

Abû'l-Wefâ el Bûzğâni, born in A. D. 940, wrote commentaries on Euclid, Diophantus, Hipparchus, and M. b. Mûsâ, works on arithmetic, on the circle and sphere etc. ABû SA ID, EL SIGZI (Ahmed b. M. b. Abdelgalil, 951-1024 A. D.) wrote on the trisection of an angle, the sphere, the intersection of the parabola and hyperbola, the Lemmata of Archimedes, conic sections, the hypebola and its asymptotes, etc. ABû BEKR, EL-KARCHÎ (M. b. el-Haṣan, c. 1016 A. D.) wrote on arithmetic and indeterminate equations after Diophantus. ALBERUNI (M. b. Aḥmed, Abu'l-Rihan el-Biruni) was born in 973 A. D. and besides works on history, geography, chronology and astronomy wrote on mathematics generally, and in particular on tangents, the chords of the circle. etc. OMAR B. IBRAHIM EL-CHAIJAMI, the celebrated poet, was born about 1046 A. D. and died in 1123 A. D. a few years after Bhâskara was born. He wrote an algebra in which he deals with cubic equations, a commentary on the difficulties in the postulates of Euclid; on mixtures of metals, and on arithmetical difficulties.

This very brief and incomplete resumé of Arabic mathematical works written during the period intervening between the time of Brahmagupta and Bhâskara indicates at least considerable intellectual activity and a great advance on the Indian works of the period in all branches of mathematics except, perhaps, indeterminate equations.

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Note. — J. B. A. S. = *Journal of the Bengal Asiatic Society;*
J. R. A. S. = *Journal of the Royal Asiatic Society;*
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