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The logic of non-Western science: mathematical discoveries in medieval India

One of the most significant things one learns from the study of the exact sciences as practiced in a number of ancient and medieval societies is that, while science has always traveled from one culture to another, each culture before the modern period approached the sciences it received in its own unique way and transformed them into forms compatible with its own modes of thought. Science is a product of culture; it is not a single, unified entity. Therefore, a historian of premodern scientific texts – whether they be written in Akkadian, Arabic, Chinese, Egyptian, Greek, Hebrew, Latin, Persian, Sanskrit, or any other linguistic bearer of a distinct culture – must avoid the temptation to conceive of these sciences as more or less clumsy attempts to express modern scientific ideas. They must be understood and appreciated as what their practitioners believed them to be. The historian is interested in the truthfulness of his own understanding of the various sciences, not in the truth or falsehood of the science itself.

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In order to illustrate the individuality of the sciences as practiced in the older non-Western societies, and their differences from early modern Western science (for contemporary science is, in general, interested in explaining quite different phenomena than those that attracted the attention of earlier scientists), I propose to describe briefly some of the characteristics of the medieval Indian *śāstra* of *jyotiṣa*. This discipline concerned matters included in such Western areas of inquiry as astronomy, mathematics, divination, and astrology. In fact, the *jyotiṣīs*, the Indian experts in *jyotiṣa*, produced more literature in these areas – and made more mathematical discoveries – than scholars in any other culture prior to the advent of printing. In order to explain how they managed to make such discoveries – and why their discoveries remain largely unknown – I will also need to describe briefly the general social and economic position of the *jyotiṣīs*.

'*Śāstra*' ('teaching') is the word in Sanskrit closest in meaning to the Greek 'ἐπιστήμη' and the Latin 'scientia.' The teachings are often attributed to gods or considered to have been composed by divine *ṛṣis*; but since there were many of both kinds of superhuman beings, there were many competing varieties of each *śāstra*. Sometimes, however, a school within a *śāstra* was founded by a human; scientists were free to modify their *śāstras* as they saw fit. No one was constrained to follow a system taught by a god.

Jyotiḥ is a Sanskrit word meaning 'light,' and then 'star'; so that *jyotiḥśāstra* means 'teaching about the stars.' This *śāstra* was conventionally divided into three subteachings: *gaṇita* (mathematical astronomy and mathematics itself), *saṃhitā* (divination, including by means of celestial omens), and *horā* (astrology). A number of *jyotiṣīs* (students of the stars) followed all three branches, a larger number just two (usually *saṃhitā* and *horā*), and the largest number just one (*horā*).

The principal writings in *jyotiḥśāstra*, as in all Indian *śāstras*, were normally in verse, though the numerous commentaries on them were almost always in prose. The verse form with its metrical demands, while it aided memorization, led to greater obscurity of expression than prose composition would have entailed. The demands of the poetic meter meant that there could be no stable technical vocabulary; many words with different metrical patterns had to be devised to express the same mathematical procedure or geometrical concept, and mathematical formulae had frequently to be left partially incomplete. Moreover, numbers had to be expressible in metrical forms (the two major systems used for numbers, the *bhūtasaṅkhyā* and the *kaṭapayādi*, will be explained and exemplified below), and the consequent ambiguity of these expressions encouraged the natural inclination of Sanskrit *paṇḍits* to test playfully their readers' acumen. It takes some practice to achieve sureness in discerning the technical meanings of such texts.

But in this opaque style the *jyotiṣīs* produced an abundant literature. It is estimated that about three million manuscripts on these subjects in Sanskrit and in other Indian languages still exist. Regrettably, only a relatively small number of these has been subjected to modern analysis, and virtually the whole ensemble is rapidly decaying. And because there is only a small number of scholars trained to read and understand these texts, most of them will have disappeared before anyone will be able to describe correctly their contents.

In order to make my argument clearer, I will restrict my remarks to the first branch of *jyotiḥśāstra* – *gaṇita*. Geometry, and its branch trigonometry, was the mathematics Indian astronomers used most frequently. In fact, the Indian astronomers in the third or fourth century, using a pre-Ptolemaic Greek table of chords,[1] produced tables of sines and versines, from which it was trivial to derive cosines. This new system of trigonometry, produced in India, was transmitted to the Arabs in the late eighth century and by them, in an expanded form, to the Latin West and the Byzantine East in the twelfth century. But, despite this sort of practical innovation, the Indians practiced geometry without the type of proofs taught by Euclid, in

1 For a description of the table of chords, cyclic quadrilaterals, two-point iteration, fixed-point iteration, and several other mathematical terms mentioned in this essay, please see Victor J. Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993).

which all solutions to geometrical problems are derived from a small body of arbitrary axioms. The Indians provided demonstrations that showed that their solutions were consistent with certain assumptions (such as the equivalence of the angles in a pair of similar triangles or the Pythagorean theorem) and whose validity they based on the measurement of several examples. In their less rigorous approach they were quite willing to be satisfied with approximations, such as the substitution of a sine wave for almost any curve connecting two points. Some of their approximations, like those devised by Āryabhaṭa in about 500 for the volumes of a sphere and a pyramid, were simply wrong. But many were surprisingly useful.

Not having a set of axioms from which to derive abstract geometrical relationships, the Indians in general restricted their geometry to the solution of practical problems. However, Brahmagupta in 628 presented formulae for solving a dozen problems involving cyclic quadrilaterals that were not solved in the West before the Renaissance. He provides no rationales and does not even bother to inform his readers that these solutions only work if the quadrilaterals are circumscribed by a circle (his commentator, Pṛthūdakasvāmin, writing in about 864, follows him on both counts). In this case, and clearly in many others, there was no written or oral tradition that preserved the author's reasoning for later generations of students. Such disdain for revealing the methodology by which mathematics could advance made it difficult for all but the most talented students to create new mathematics. It is amazing to see, given this situation, how many Indian mathematicians did advance their field.

I will at this point mention as examples only the solution of indeterminate equations of the first degree, described already by Āryabhaṭa; the partial solution of indeterminate equations of the second degree, due to Brahmagupta; and the cyclic solution of the latter type of indeterminate equations, achieved by Jayadeva and described by Udayadivākara in 1073 (the cyclic solution was rediscovered in the West by Bell and Fermat in the seventeenth century). Interpolation into tables using second-order differences was introduced by Brahmagupta in his *Khaṇḍakhādyaka* of 665. The use of two-point iteration occurs first in the *Pañcasiddhāntikā* composed by Varāhamihira in the middle of the sixth century, and fixed-point iteration in the commentary on the *Mahābhāskarīya* written by Govindasvāmin in the middle of the ninth century. The study of combinatorics, including the so-called Pascal's triangle, began in India near the beginning of the current era in the *Chandaḥsūtras*, a work on prosody composed by Piṅgala, and culminated in chapter 13 of the *Gaṇitakaumudī* completed by Nārāyaṇa Paṇḍita in 1350. The fourteenth and final chapter of Nārāyaṇa's work is an exhaustive mathematical treatment of magic squares, whose study in India can be traced back to the *Bṛhatsaṃhitā* of Varāhamihira.

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In short, it is clear that Indian mathematicians were not at all hindered in solving significant problems of many sorts by what might appear to a non-Indian to be formidable obstacles in the conception and expression of mathematical ideas.

Nor were they hindered by the restrictions of 'caste,' by the lack of societal support, or by the general absence of monetary rewards. It is true that the overwhelming majority of the Indian mathematicians whose works we know were Brāhmaṇas, but there are exceptions (e.g., among Jainas, non-Brahmānical scribes, and craftsmen). Indian society was far from open, but it was not absolutely rigid; and talented mathematicians, whatever their origins, were not ignored by their colleagues. However, astrologers (who frequently were not Brāhmaṇas) and the makers of calendars were the only *jyotiṣīs* normally valued by the societies in which they lived. The attraction of the former group is easily understood, and their enormous popularity continues today. The calendar-makers were important because their job was to indicate the times at which rituals could or must be performed. The Indian calendar is itself intricate; for instance, the day begins at local sunrise and is numbered after the tithi that is then current, with the tithis being bounded by the moments, beginning from the last previous true conjunction of the Sun and the Moon, at which the elongation between the two luminaries had increased by twelve degrees. Essentially, each village needed its own calendar to determine the times for performing public and private religious rites of all kinds in its locality.

By contrast, those who worked in the various forms of *gaṇita* usually enjoyed no public patronage – even though they provided the mathematics used by architects, musicians, poets, surveyors, and merchants, as well as the astronomical theories and tables employed by astrologers and calendar-makers. Sometimes a lucky mathematical astronomer was supported by a Mahārāja whom he served as a royal astrologer and in whose name his work would have been published. For example, the popular *Rājamṛgāṅka* is attributed, along with dozens of other works in many *śāstras*, to Bhojadeva, the Mahārāja of Dhārā in the first half of the eleventh century. Other *jyotiṣīs* substituted the names of divinities or ancient holy men for their own as authors of their treatises. Authorship often brought no rewards; one's ideas were often more widely accepted if they were presented as those of a divine being, a category that in many men's minds included kings.

One way in which a *jyotiṣī* could make a living was by teaching mathematics, astronomy, or astrology to others. Most frequently this instruction took place in the family home, and, because of the caste system, the male members of a *jyotiṣī*'s family were all expected to follow the same profession. A senior *jyotiṣī*, therefore, would train his sons and often his nephews in their ancestral craft. For this the family maintained a library of appropriate texts that included the compositions of family members, which were copied as desired by the younger members. In this way a text might be preserved within a family over many generations without ever being seen by persons outside the family. In some cases, however, an expert became well enough known that aspirants came from far and wide to his house to study. In such cases these students would carry off copies of the manuscripts in the teacher's collection to other family libraries in other locales.

The teaching of *jyotiḥśāstra* also occurred in some Hindu, Jaina, and Buddhist monasteries, as well as in local schools. In these situations certain standard texts were normally taught, and the status of these texts can be established by the number of copies that still exist, by their geographical distribution, and by the number of commentaries that were written on them.

Thus, in *gaṇita* the principal texts used in teaching mathematics in schools were clearly the *Līlavatī* on arithmetic and the *Bījagaṇita* on algebra, both written by Bhāskara in around 1150, and, among Jainas, the *Gaṇitasārasaṅgraha* composed in about 850 by their coreligionist, Mahāvīra. In astronomy there came to be five *pakṣas* (schools): the *Brāhmapakṣa*, whose principal text was the *Siddhānta-*

śiromaṇi of the Bhāskara mentioned above; the *Āryapakṣa*, based on the *Āryabhaṭīya* written by Āryabhaṭa in about 500; the *Ārdharātrikapakṣa*, whose principal text was the *Khaṇḍakhādyaka* completed by Brahmagupta in 665; the *Saurapakṣa*, based on the *Sūryasiddhānta* composed by an unknown author in about 800; and the *Gaṇeśapakṣa*, whose principal text was the *Grahalāghava* authored by Gaṇeśa in 1520. Each region of India favored one of these *pakṣas*, though the principal texts of all of them enjoyed national circulation. The commentaries on these often contain the most innovative advances in mathematics and mathematical astronomy found in Sanskrit literature. By far the most popular authority, however, was Bhāskara; a special college for the study of his numerous works was established in 1222 by the grandson of his younger brother. No other Indian *jyotiṣī* was ever so honored.

Occasionally, indeed, an informal school inspired by one man's work would spring up. The most noteworthy, composed of followers of Mādhava of Saṅgamagrāma in Kerala in the extreme south of India, lasted for over four hundred years without any formal structure – simply a long succession of enthusiasts who enjoyed and sometimes expanded on the marvelous discoveries of Mādhava.

Mādhava (c. 1360 – 1420), an Emprāntiri Brāhmaṇa, apparently lived all his life on his family's estate, Ilaññipaḷḷi, in Saṅgamagrāma (Irinjālakhuḍa) near Cochin. His most momentous achievement was the creation of methods to compute accurate values for trigonometric functions by generating infinite series. In order to demonstrate the character of his solutions and expressions of them, I will translate a few of his verses and quote some Sanskrit.

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He began by considering an octant of a circle inscribed in a square, and, after some calculation, gave the rule (I translate quite literally two verses):

> Multiply the diameter (of the circle) by 4 and divide by 1. Then apply to this separately with negative and positive signs alternately the product of the diameter and 4 divided by the odd numbers 3, 5, and so on .... The result is the accurate circumference; it is extremely accurate if the division is carried out many times.

This describes the infinite series:

$$C = \frac{4D}{1} - \frac{4D}{3} + \frac{4D}{5} - \frac{4D}{7} + \frac{4D}{9} \ldots.$$

That in turn is equivalent to the infinite series for π that we attribute to Leibniz:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots.$$

Mādhava expressed the results of this formula in a verse employing the *bhūtasaṅkhyā* system, in which numbers are represented by words denoting objects that conventionally occur in the world in fixed quantities:

> vibudhanetragajāhihutāśanatriguṇavedabhavāraṇabāhavaḥ |
> navanikharvamite vṛtivistare
> paridhimānam idaṃ jagadur budhāḥ ||

A literal translation is:

> Gods [33], eyes [2], elephants [8], snakes [8], fires [3], three [3], qualities [3], Vedas [4], nakṣatras [27], elephants [8], and arms [2] – the wise say that this is the measure of the circumference when the diameter of a circle is nine hundred billion.

The *bhūtasaṅkhyā* numbers are taken in reverse order, so that the formula is:

$$\pi = \frac{2827433388233}{900000000000}$$

(= 3.14159265359, which is correct to the eleventh decimal place).

Another extraordinary verse written by Mādhava employs the *kaṭapayādi* system in which the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 are represented by the consonants that are immediately followed by a vowel; this allows the mathematician to create a verse with both a transparent meaning due to the words and an unrelated numerical meaning due to the consonants in those words. Mādhava's verse is:

> vidvāṃs tunnabalaḥ kavīśanicayaḥ sarvārthaśīlasthiro
> nirviddhāṅganarendraruṅ

The verbal meaning is: "The ruler whose army has been struck down gathers together the best of advisors and remains firm in his conduct in all matters; then he shatters the (rival) king whose army has not been destroyed."

The numerical meaning is five sexagesimal numbers:

0;0,44
0;33,6
16;5,41
273;57,47
2220;39,40.

These five numbers equal, with R = 3437;44,48 (where R is the radius):

$$\frac{5400^{11}}{R^{10}11!}$$

$$\frac{5400^{9}}{R^{8}9!}$$

$$\frac{5400^{7}}{R^{6}7!}$$

$$\frac{5400^{5}}{R^{4}5!}$$

$$\frac{5400^{3}}{R^{2}3!}$$

These numbers are to be employed in the formula:

$$sin\theta = \theta - \left(\frac{\theta}{5400}\right)^3\left[\frac{5400^3}{R^2 3!} - \left(\frac{\theta}{5400}\right)^2\left[\frac{5400^5}{R^4 5!} - \left(\frac{\theta}{5400}\right)^2\left[\frac{5400^7}{R^6 7!} - \left(\frac{\theta}{5400}\right)^2\left[\frac{5400^9}{R^8 9!} - \left(\frac{\theta}{5400}\right)^2\left[\frac{5400^{11}}{R^{10} 11!}\right]\right]\right]\right]\right]$$

and this formula is a simple transformation of the first six terms in the infinite power series for $sin\theta$ found independently by Newton in 1660:

$$sin\theta = \theta - \frac{\theta^3}{R^2 3!} + \frac{\theta^5}{R^4 5!} - \frac{\theta^7}{R^6 7!} + \frac{\theta^9}{R^8 9!} - \frac{\theta^{11}}{R^{10} 11!}$$

Not surprisingly, Mādhava also discovered the infinite power series for the cosine and the tangent that we usually attribute to Gregory.

The European mathematicians of the seventeenth century derived their trigonometrical series from the application of the calculus; Mādhava in about 1400 relied on a clever combination of geometry, algebra, and a feeling for mathematical possibilities. I cannot here go through his whole argument, which has fortunately been preserved by several of his successors; but I should mention some of his techniques.

He invented an algebraic expansion formula that keeps pushing an unknown quantity to successive terms that are alternately positive and negative; the series must be expanded to infinity to get rid of this unknown quantity. Also, because of the multiplications, as the terms increase, the powers of the individual factors also increase. One of these factors in the octant is one of a series of integers beginning with 1 and ending with 3438 – the number of parts in the radius of the circle that is also the tangent of 45°, the angle of the octant; this means that there are 3438 infinite series that must be summed to yield the final infinite series of the trigonometrical function.

It had long been known in India that the sum of a series of integers beginning with 1 and ending with n is: $n\left(\frac{(n-1)}{2}+1\right)$, that is, $\frac{n^2}{2}-\frac{n}{2}+n$. Since n here equals 3438, Mādhava decided that $n-\frac{n}{2}$, which equals $\frac{3438}{2}$, is negligible with respect to $\frac{3438^2}{2}$. Therefore, an approximation to the sum of the series of n integers is $\frac{n^2}{2}$. Similarly, the sums of the squares of a series of n integers beginning with 1 was known to be $\frac{\frac{n}{2}[2(n+1)^2-(n+1)]}{3}$. If n is large, this is approximately equal to $\frac{n(n+1)^2}{3}$ since $\frac{-(n+1)}{6}$ is negligible. But, with $n = 3438$, $\frac{3438 \times 3439^2}{3}$ is little different from $\frac{3438^3}{3}$. Therefore, as an approximation, the sum of the series of the squares of 3438 integers beginning with 1 is $\frac{n^3}{3}$. Finally, it was known that the sum of the cubes of a series of n numbers beginning with 1 is: $\left(\frac{n}{2}\right)^2(n+1)^2$ or $\frac{n^2(n+1)^2}{4}$. If n is 3438, there is little difference between $\frac{3438^2 \times 3439^2}{4}$ and $\frac{3438^4}{4}$. Therefore, the expression $\frac{n^4}{4}$ is a close approximation to the sum of the cubes of a series of n numbers beginning with 1. From these three examples Mādhava guessed at the general rule that the sum of n numbers in an arithmetical series beginning with 1 all raised to the same power, p, is approximately equal to $\frac{n^{p+1}}{p+1}$.

It had also been realized in India since the fifth century – from examining the sine table in which the radius of the circle, R, is $\frac{21{,}600}{2\pi}$ (which was approximated by 3438) and in which there are 24 sines in a quadrant of 90°, so that the length of each arc whose sine is tabulated is 225′ – that the sine of any tabulated angle θ is equal to θ minus the sum of the sums of the second differences of the sines of the preceding tabulated angles. Mādhava discovered, by some very clever geometry, that the sum of the sums of the second differences approximately equals $\frac{\theta^3}{R^2 3!}$ and that the versine of θ is approximately equal to $\frac{\theta^2}{2R}$. Since $sin^2\theta = R^2 - cos^2\theta$ and $vers\theta = R - cos\theta$, Mādhava could correct the approximation to the versine by the approximation to the sum of the sums of the second differences of tabulated sines; then he could correct the approximation to the sum of the sums of the second differences by the corrected approximation to the versine; and he could continue building up the two parallel series by applying alternating corrections to them. He finally arrives at two infinite power series, equivalent, if $R = 1$, to:

$$sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} \ldots,$$

and

$$cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \ldots.$$

Subsequent members of the 'school' of Mādhava did remarkable work as well, in both geometry (including trigonometry) and astronomy. This is not the occasion to recite their accomplishments, but I should remark here that, among these members, Indian astronomers attempted especially to use observations to correct astronomical models and their parameters.

This began with Mādhava's principal pupil, a Nampūtiri Brāhmaṇa named Parameśvara, whose family's illam was Vaṭaśreṇi in Aśvatthagrāma, a village about thirty-five miles northeast of Saṅgamagrāma. He observed eighteen lunar and solar eclipses between 1393 and 1432 in an attempt to correct traditional Indian eclipse theory. One pupil of Parameśvara's son, Dāmodara, was

Nīlakaṇṭha – another Nampūtiri Brāhmaṇa who was born in 1444 in the Kelallūr illam located at Kuṇḍapura, which is about fifty miles northwest of Aśvatthagrāma.

Nīlakaṇṭha made a number of observations of planetary and lunar positions and of eclipses between 1467 and 1517. Nīlakaṇṭha presented several different sets of planetary parameters and significantly different planetary models, which, however, remained geocentric. He never indicates how he arrived at these new parameters and models, but he appears to have based them at least in large part on his own observations. For he proclaims in his *Jyotirmīmāṃsā* – contrary to the frequent assertion made by Indian astronomers that the fundamental *siddhāntas* expressing the eternal rules of *jyotiḥśāstra* are those alleged to have been composed by deities such as Sūrya – that astronomers must continually make observations so that the computed phenomena may agree as closely as possible with contemporary observations. Nīlakaṇṭha says that this may be a continuous necessity because models and parameters are not fixed, because longer periods of observation lead to more accurate models and parameters, and because improved techniques of observing and interpreting results may lead to superior solutions. This affirmation is almost unique in the history of Indian *jyotiṣa*; *jyotiṣīs* generally seem to have merely corrected the parameters of one *pakṣa* to make them closely corresponded to those of another.

The discoveries of the successive generations of Mādhava's 'school' continued to be studied in Kerala within a small geographical area centered on Saṅgamagrāma. The manuscripts of the school's Sanskrit and Malayālam treatises, all copied in the Malayālam script, never traveled to another region of India; the furthest they got was Kaṭattanāt in northern Kerala, about one hundred miles north of Saṅgamagrāma, where the Rājakumāra Śaṅkara Varman repeated Mādhava's trigonometrical series in a work entitled *Sadratnamālā* in 1823. This was soon picked up by a British civil servant, Charles M. Whish, who published an article entitled "On the Hindú Quadrature of the Circle and the Infinite Series of the Proportion of the Circumference to the Diameter in the Four Sástras, the Tantra Sangraham, Yocti Bháshá, Carana Paddhati and Sadratnamála" in *Transactions of the Royal Asiatic Society* in 1830.[2] While Whish was convinced that the Indians (he did not know of Mādhava) had discovered calculus – a conclusion that is not true even though they successfully found the infinite series for trigonometrical functions whose derivation was closely linked with the discovery of calculus in Europe in the seventeenth century – other Europeans scoffed at the notion that the Indians could have achieved such a startling success. The proper assessment of Mādhava's work began only with K. Mukunda Marar and C. T. Rajagopal's "On the Hindu Quadrature of the Circle," published in the *Journal of the Bombay Branch of the Royal Asiatic Society* in 1944.

So while the discoveries of Newton, Leibniz, and Gregory revolutionized European mathematics immediately upon their publication, those of Mādhava, Parameśvara, and Nīlakaṇṭha, made between the late fourteenth and early sixteenth centuries, became known to a handful of scholars outside of Kerala in

2 Note that the *Tantrasaṅgraha* was written by the Nīlakaṇṭha whom we have already mentioned, the *Yuktibhāṣa* by his colleague and fellow pupil of Dāmodara, Jyeṣṭhadeva, and the *Karaṇapaddhati* by a resident of the Putumana illam in Śivapura in 1723.

India, Europe, America, and Japan only in the latter half of the twentieth century. This was not due to the inability of Indian *jyotiṣīs* to understand the mathematics, but to the social, economic, and intellectual milieux in which they worked. The isolation of brilliant minds was not uncommon in premodern India. The exploration of the millions of surviving Sanskrit and vernacular manuscripts copied in a dozen different scripts would probably reveal a number of other Mādhavas whose work deserves the attention of historians and philosophers of science. Unfortunately, few scholars have been trained to undertake the task, and the majority of the manuscripts will have crumbled in just another century or two, before those few can rescue them from oblivion.