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Figuring Out

The Spread of Hindu-Arabic Numerals in the European Tradition of Practical Mathematics (13th–16th Centuries)

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Abstract

The paper focusses on the spread of Hindu-Arabic arithmetic among European practitioners. The analysis is based on an original database recording detailed information on over 1200 practical arithmetic manuals, both manuscript and printed. This database provides the most detailed reconstruction available of the European tradition of practical arithmetic from the late 13th to the end of the 16th century. The paper argues that studying this spread makes it possible to open a perspective on a progressive transmission of 'useful knowledge' from the 'commercial revolution' to the 'little divergence'. Focussing on the transmission of practical arithmetic allows to stress the role of skills and human capital in pre-modern European economic development. Moreover, it allows to reconstruct a progressive transmission, from the Mediterranean to the Atlantic, of a 'practical knowledge' which eventually contributed to major developments in European 'theoretical knowledge'.

Keywords

human capital – practical knowledge – Renaissance

1 Introduction

Given their almost universal diffusion, Hindu-Arabic numerals can easily be taken for granted. However, for a long stretch of European history, numbers have been represented using Roman numerals, as the positional numeral system reached western Europe at a relatively late stage. This paper provides new

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evidence to reconstruct the transition of European practitioners from Roman to Hindu-Arabic numerals. Moreover, since Hindu-Arabic arithmetic provided a key mathematical underpinning of such innovations, this paper argues that studying the diffusion of Hindu-Arabic numerals among European practitioners provides a proxy to study the European diffusion of the financial and organisational innovations first developed with the 'commercial revolution' of the 13th century.[1]

The origins and spread of Hindu-Arabic numerals from 5th-century India have been the object of a considerable amount of research.[2] Scholars have also focused on the complex introduction of Hindu-Arabic numerals in Europe from the 10th to the 13th century.[3] Yet, the spread of Hindu-Arabic numerals within Europe has received limited scholarly attention, with only one relatively recent work touching on the subject.[4] This paper contributes to this field by providing a detailed reconstruction of the continuous and European-wide tradition of practical arithmetic from the late 13th century to 1600, based on an original database recording 1280 texts written by over 340 authors, who mainly wrote in European vernaculars. Through this tradition, the use of Hindu-Arabic numerals spread among merchants and other practitioners, who needed such mathematics for their crafts and commercial practices. Hindu-Arabic mathematics was adopted by the first sedentary merchants, who in the 13th century were increasingly involved in international trade. As these merchants were facing a growing need to calculate conversions and exchange rates, they adopted

1 I understand 'commercial revolution' in its stricter sense, as originally formulated by De Roover, that is to say a "a complete or drastic change in the methods of doing business or in the organization of business enterprise" which occurred in the 13th century. This 'drastic' change led to the emergence of the first sedentary merchants and to the introduction of key innovations for European business history, such as the bill of exchange, permanent partnership contracts, and double entry bookkeeping. See Raymond De Roover, "The Commercial Revolution of the Thirteenth Century," in *Enterprise and Secular Change: Readings in Economic History*, edited by Frederic Chapin Lane, Jelle C. Riemersma (London: George Allen and Unwin, 1953), pp. 80–85.

2 David Eugene Smith, Louis Charles Karpinski, *The Hindu-Arabic Numerals* (Boston; London: Ginn and Company, 1911). George Francis Hill, *The Development of Arabic Numerals in Europe: Exhibited in Sixty-Four Tables* (Oxford: Clarendon Press, 1915). Georges Ifrah, *The Universal History of Numbers*, translated by David Bellos. 3 vols. (London: Harvill Press, 2000). Stephen Chrisomalis, *Numerical Notation: A Comparative History* (Cambridge: Cambridge University Press, 2010).

3 As summarised in Charles Burnett, *Numerals and Arithmetic in the Middle Ages* (Farnham: Ashgate Variorum, 2010).

4 Alain Schärlig, *Du zéro à la virgule: les chiffres arabes à la conquête de l'Europe: 1143–1585* (Lausanne: Presses polytechniques et universitaires romandes, 2010).

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a mathematical symbolism which provided the best tools to deal with multiplications and divisions. Moreover, the symbolism for fractions allowed these merchants to effectively handle rational numbers, a key feature for merchants operating on the slightest variations of exchange rates. The bill of exchange, which revolutionised the European international monetary market and was based on a double operation of exchange, was in fact developed by the same mercantile class that adopted Hindu-Arabic numerals.

The paper also contributes to the literature focusing on the role of ideas, practices and human capital in pre-modern European economic history. Qualitative and quantitative studies have documented the role of human capital in the industrial revolution.[5] Drawing from a tradition of classic studies,[6] a recent literature is investigating the role of human capital in the pre-modern period as well. Providing estimates of European book production Van Zanden and Baten argue that it is possible to observe an emerging "knowledge economy" starting from the late middle ages.[7] This paper elaborates on this thesis providing evidence grounded on clearly identifiable sources and focussing on a particularly significant cross-section of overall book production.

Mokyr stressed the importance for economic development of the growing diversity and permeability of the European "market for ideas" from the beginning of the early modern period.[8] Dittmar has documented the significant role of such ideas identifying a causal relationship between the spread of the printing press and urban growth.[9] More recently, Dittmar and Seabold have narrowed their focus by isolating the impact of the publication of printed business manuals – which consist mainly of practical arithmetics – on economic

5 Joel Mokyr, "The Intellectual Origins of Modern Economic Growth," *The Journal of Economic History*, 2005, *65*/2:285–351. Joel Mokyr, *The Enlightened Economy: Britain and the Industrial Revolution 1700–1850* (London: Penguin, 2011). Mara P. Squicciarini, Nico Voigtländer, "Human Capital and Industrialization: Evidence from the Age of Enlightenment," *The Quarterly Journal of Economics*, 2015, *130*/4:1825–1883.

6 Werner Sombart, *Der moderne Kapitalismus: historisch-systematische Darstellung des gesamteuropäischen Wirtschaftslebens von seinen Anfängen bis zur Gegenwart* (München: Duncker & Humblot, 1928). Werner Sombart, "Medieval and Modern Commercial Enterprise," in *Enterprise and Secular Change: Readings in Economic History*, edited by Frederic Chapin Lane, Jelle C. Riemersma (London: Allen & Unwin, 1953), pp. 25–40.

7 Joerg Baten, Jan Luiten van Zanden, "Book Production and the Onset of Modern Economic Growth," *Journal of Economic Growth*, 2008, *13*/3:217–235. Jan Luiten van Zanden, *The Long Road to the Industrial Revolution: The European Economy in a Global Perspective, 1000–1800* (Leiden: Brill, 2009).

8 Joel Mokyr, *A Culture of Growth: The Origins of the Modern Economy* (New Jersey: Princeton University Press, 2017).

9 Jeremiah E. Dittmar, "Information Technology and Economic Change: The Impact of The Printing Press," *The Quarterly Journal of Economics*, 2011, *126*/3:1133–1172.

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development.[10] Their econometric analyses identified a causal effect between the spread of printed commercial manuals and urban growth, documenting the economic impact of the spread of these skills in print in the 16th century.[11]

The analysis of this paper contributes to this debate by taking into consideration also manuscript sources and by extending the observation period. While the impact of printing clearly played a major role in lowering access costs and in accelerating the diffusion of ideas, we should not forget that those very ideas had already been circulating and influencing social and economic phenomena for a long time. The main consequence of analysing both manuscript and printed sources is that the picture of the diffusion of the innovations of the commercial revolution – understood as specified above – changes considerably. The data show a transmission of practical arithmetic that followed a marked south-to-north axis. This pattern, which is different from that of the diffusion of the printing press, documents a progressive transmission of knowledge from the first centres of the commercial revolution to the rest of Europe.

England, where the publication of practical arithmetics took off in the second half of the 16th century, and where Hindu-Arabic numerals still had a limited circulation at the end of the century, stands out as the latecomer in this cycle. Following the Mokyr's own suggestion that the origins of his framework are to be traced back to the renaissance, I employ the concept of 'useful knowledge' to capture the market- and practice-oriented nature of the diffusion of practical arithmetic texts.[12]

Additionally, the data suggest that the spread of practical arithmetic did not follow the network of European maritime trade, but occurred through inland networks of proximity. These networks of proximity enabled intellectual exchanges across linguistic and economic communities. This finding adds to previous research on the role of human capital formation in the transition of the core of European trade from the Mediterranean to the Atlantic.[13] Inves-

10 Jeremiah Dittmar, Skipper Seabold, "New Media and Competition: Printing and Europe's Transformation After Gutenberg," *Journal of Political Economy*, 2020, forthcoming, http://www.jeremiahdittmar.com/research (accessed 29 Oct. 2020).

11 More precisely, Dittmar and Seabold identified no robust relationship between non-commercial publications and city growth, a causal effect of the spread of printed business manuals on urban growth, and a strong correlation with individual achievement in bourgeois occupations between 1500 and 1600.

12 Mokyr, *The Intellectual Origins of Modern Economic Growth* (cit. note 5).

13 Alexandra M. de Pleijt, Jan Luiten van Zanden, "Accounting for the 'Little Divergence': What Drove Economic Growth in Pre-Industrial Europe, 1300–1800?," *European Review of Economic History*, 2016, *20*/4:387–409.

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tigating the origins and transmission of this human capital, this paper argues that a relevant contribution to the transition of the core of European trade from the Mediterranean to the Atlantic – the premise of the so-called 'little divergence' – came not only from maritime trade, but also from intellectual exchanges, i.e. from the spread of useful knowledge and skills.

The paper is organised in four sections. The first section is dedicated to explaining the adoption of Hindu-Arabic numerals during the commercial revolution. The second section describes the European tradition of practical arithmetic, providing both qualitative evidence and an analysis of the dataset. Section three shows that the spread of practical arithmetic correlated with the adoption of Hindu-Arabic numerals in commercial documentation. Section four wraps up and concludes.

2 The Adoption of Hindu-Arabic Numerals in European Commercial Practice

Like other techniques imported from the East, such as the manufacturing of paper, glass and silk,[14] Hindu-Arabic numerals were first developed in the East – probably in 5th century India – and were brought to Europe across the Mediterranean during the late medieval period.[15] Like other technologies imported from the Levant, Hindu-Arabic mathematics was re-contextualised and adapted to European needs. In 1202 Pisa, the merchant-mathematician Leonardo Pisano (also known as Fibonacci) completed his *Liber Abaci*.[16] Fibonacci presented his work as the result of the studies he had carried out while travelling for business reasons to Maghreb and other Mediterranean ports. It was a ground-breaking work not so much for its original contributions, but rather for its key role in offering a comprehensive *summa* in the Latin language of Hindu-Arabic mathematics, making it accessible to a European read-

14 David Abulafia, "The Impact of the Orient: Economic Interactions between East and West in the Medieval Mediterranean," in *Across the Mediterranean Frontiers: Trade, Politics and Religion, 650–1450*, edited by Dionisius A. Agius, Ian Richard Netton (Turnhout: Brepols Publishers, 1997), pp. 1–40.

15 Ifrah, *The Universal History of Numbers* (cit. note 2). Paul Kunitzsch, "The Transmission of Hindu-Arabic Numerals Reconsidered," in *The Enterprise of Science in Islam: New Perspectives*, edited by Jan P. Hogendijk, Abdelhamid I. Sabra (Cambridge; London: MIT Press, 2003), pp. 3–23. Roshdi Rashed, *D'Al-Khwārizmī à Descartes: étude sur l'histoire des mathématiques classiques* (Paris: Hermann, 2011).

16 Raffaella Franci, "Il Liber Abaci di Leonardo Fibonacci 1202–2002," *Bollettino dell'Unione Matematica Italiana*, 2002, 5/2:293–328.

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ership. The *Liber Abaci* showed the functioning of the positional numeral system, presented a thorough summary of Hindu-Arabic mathematics of the time and included several sections on how such mathematics could have been used to solve practical and commercial problems.[17] The *Liber Abaci* is a sophisticated mathematical work which cannot be reduced to a commercial arithmetic, but its practical sections had a fundamental influence on successive developments in the applications of mathematics.

The 13th century was the heyday of Mediterranean trade and of maritime republics.[18] The financial and organizational innovations of the commercial revolution, such as permanent partnerships (the *compagnia* and the *accomandita*), the European bill of exchange and double-entry bookkeeping were developed in Italian city-states by the first European sedentary merchants.[19] Together with the development of these innovations, merchant-bankers were the first European practitioners to use – alongside the previous Roman notation – Hindu-Arabic numerals.

The positional numeral system presents advantages over the Roman one. It makes it possible to handle large numbers. It does not require a separate reckoning tool, such as a reckoning table or other counting devices. It allows to check a calculation without repeating the entire procedure. Hindu-Arabic mathematics also provided the first step in what would have become the modern European mathematical symbolism. It introduced the symbolism we still employ to represent fractions, and the positional notation made it possible to handle proportions, multiplications and divisions effectively. Furthermore, the positional numeral system also provided a standardised symbolism to represent (and therefore to operate with) rational numbers, a feature which the previous Roman system did not provide.[20]

17 Enrico Giusti, Raffaella Petti (eds.), *Un ponte sul Mediterraneo: Leonardo Pisano, la scienza araba e la rinascita della matematica in Occidente* (Firenze: Polistampa, 2002).

18 Frederic Chapin Lane, *Venice: A Maritime Republic* (Baltimore; London: Johns Hopkins University Press, 1973). Ronald Findlay, Kevin H. O'Rourke, *Power and Plenty: Trade, War, and the World Economy in the Second Millennium* (Princeton: Princeton University Press, 2009).

19 Federigo Melis, *Storia della ragioneria: contributo alla conoscenza e interpretazione delle fonti più significative della storia economica, illustrato con 54 tavole fuori testo e 55 figure* (Bologna: Zuffi, 1950). De Roover, *The Commercial Revolution of the Thirteenth Century* (cit. note 1). Raymond De Roover, *L'évolution de la lettre de change, xive-xviiie siècles* (Paris: A. Colin, 1953). Sushil Chaudhury, Markus A. Denzel (eds.), *Cashless Payments and Transactions from the Antiquity to 1914* (Stuttgart: Franz Steiner Verlag, 2008). Richard A. Goldthwaite, *The Economy of Renaissance Florence* (Baltimore: Johns Hopkins University Press, 2009).

20 For examples of the Roman-medieval symbolism for fractions, see Florence A. Yeldham,

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The written character of Arabic mathematics calls for a community of literate users, a rather high requirement for late medieval and early modern practitioners.[21] Since they had to rely on written communication in order to coordinate with their agents abroad, sedentary merchants became literate during the 13th century.[22] The growing scale of their international business brought these merchants to progressively handle higher numbers and to increasingly deal with conversions between currencies, weights and measures. The bill of exchange in its fully-fledged form was probably developed by the end of the 13th century and it became the main instrument to carry out international payments by the first half of the 14th century.[23] The net quantity of precious metal transported between the main European trading centres did not decrease with the introduction of the bill of exchange, but "the amount of business that it represented was increased out of all proportion," as it "enormously multiplied the supply of money available for international transactions."[24]

Together with providing the main tool to carry out cashless international payments, the bill of exchange was also used as an instrument of credit, as it made it possible to conceal a credit at interest under a double operation of exchange, bypassing the church's ban on interest-bearing loans.[25] This is the key feature that separates the European bill of exchange from its Islamic homologue, as it seems that the 'Islamic bill of exchange' was not used as an instrument of credit.[26] Since the actual interest rate of the loan depended on the relative movement of exchange rates across two markets, lenders needed

"Notation of Fractions in the Earlier Middle Ages," *Archeion. Archivio di Storia della Scienza*, 1927, *8*:313–329.

21 Jessica Otis, "'Set Them to the Cyphering Schoole': Reading, Writing, and Arithmetical Education, circa 1540–1700," *Journal of British Studies*, 2017, *56*/3:453–482.

22 As early as 1201, Boncompagno da Signa, professor of *ars dictaminis* in Bologna, noticed that these merchants continuously "wrote and rewrote" letters among themselves in either a "corrupted Latin" or directly in their own vernaculars. See Gianfranco Folena, *Culture e lingue nel Veneto medievale* (Padova: Programma, 1990), p. 229.

23 The earliest bill of exchange surviving in its original format dates to the first decade of the 14th century. See Roberta Cella, *La documentazione Gallerani-Fini nell'Archivio di Stato di Gent (1304–1309)* (Tavarnuzze, Impruneta: SISMEL-edizioni del Galluzzo, 2009). On its use, see Peter Spufford, *Money and Its Use in Medieval Europe* (Cambridge; New York: Cambridge University Press, 1988). Chaudhury, Denzel, *Cashless Payments and Transactions from the Antiquity to 1914* (cit. note 19).

24 Spufford, *Money and Its Use in Medieval Europe* (cit. note 23), pp. 254–255.

25 De Roover, *L'évolution de la lettre de change, xive-xviiie siècles* (cit. note 19).

26 Jared Rubin, "Bills of Exchange, Interest Bans, and Impersonal Exchange in Islam and Christianity," *Explorations in Economic History*, 2010, *47*/2:213–227.

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to anticipate exchange rates trends. The Venetian monetary market offered a suitable context to carry out such operations, as the seasonal departure of the state-owned galley fleets determined regular trends in the demand for liquidity and therefore foreseeable peaks and troughs in the quotation of money.[27] While the first experimentations of the credit function of the bill of exchange could have already occurred in the middle of the 13th century, they certainly came into use by the early 14th, as in 1301 the Venetian government instructed the consuls of the merchants "not to register any except licit exchanges."[28] Florentine merchant-bankers exploited their strategic position and specialised in these operations, to the point that they held a "virtual monopoly of exchange" in the Rialto from the first half of the 14th century.[29]

These developments corresponded with the adoption of Hindu-Arabic numerals by Tuscan merchant-bankers.[30] The growing scale of international trade, the complexity of monetary and accounting units, and the central role given to exchange by the development of the bill of exchange led to an increasing need to handle rational numbers and to calculate multiplication and divisions. This emerging need made the gains associated with adopting the positional numeral system outweigh its costs, as Hindu-Arabic numerals provided a more effective mathematical notation to satisfy these needs. The changing nature of their business made Tuscan merchant-bankers adopt in their practice the mathematical instruments which had been introduced by Fibonacci in the beginning of the century.[31] During this period of financial and organisational experimentation, the use of a technology imported from the Levant was recontextualised as part of a new framework. In the case of the bill of exchange,

27 Reinhold C. Mueller, *The Venetian Money Market: Banks, Panics, and the Public Debt, 1200–1500* (Baltimore; London: Johns Hopkins University Press, 1997), ch. 8, Appendix C. Adrian R. Bell, Chris Brooks, Tony K. Moore, "Cambium Non Est Mutuum: Exchange and Interest Rates in Medieval Europe," *The Economic History Review*, 2017, *70*/2:373–396. Ling-Fan Li, Arbitrage, "Communication, and Market Integration at the Time of Datini," *European Review of Economic History*, 2017, *21*/4:414–433.

28 Peter Spufford, *Handbook of Medieval Exchange; with the Assistance of Wendy Wilkinson and Sarah Tolley* (London: Royal Historical Society, 1986), p. xliv.

29 Mueller, *The Venetian Money Market* (cit. note 27), p. 256.

30 Warren Van Egmond, *The Commercial Revolution and the Beginnings of Western Mathematics in Renaissance Florence, 1300–1500* (PhD Diss., Indiana University, 1976).

31 The argument here is not that Hindu-Arabic numerals replaced previous mathematic techniques, but rather that the adoption of Hindu-Arabic numerals opened possibilities that were hampered by previous systems. Italian late medieval merchants did not stop using Roman numerals and other reckoning tools such as the reckoning table. They had different reckoning technologies at hand and probably switched from one to the other depending on convenience.

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Hindu-Arabic mathematics provided the mathematical underpinning of a key innovation in international finance.

As proficiency in basic arithmetic and written calculation requires training, starting from the late 13th century Italian commercial centres developed a tradition of practical arithmetic schools and of practical arithmetic manuals. Following the terminology of primary sources, this tradition of Italian practical arithmetic is known as 'abacus mathematics'. This is a misleading label, as abacus treatises show almost exclusively Hindu-Arabic numerals and never illustrate how to make calculations with a reckoning table. Like the other mercantile sources of the period, abacus treatises are mainly written in Italian vernaculars.

Abacus mathematics developed within the *ad hoc* institutional framework of 'abacus schools', i.e. both private and public schools, independent from universities, where lay 'abacus masters' taught the fundamentals of Hindu-Arabic mathematics to children usually aged between 11 and 13. We have archival evidence of a considerable diffusion of abacus schools in central-northern Italian cities. Florence, the capital of international commercial-banking, was not surprisingly also the most important centre for abacus mathematics, with documented evidence of more than 70 abacus masters active in over 20 abacus schools.[32]

Abacus masters competed among each other to secure pupils and to solve mathematical problems of a growing complexity, acting also as consultants to merchant-bankers.[33] Already in 14th-century Florence almost every practitioner had attended an abacus school, and in 15th century Italian commercial cities the diffusion of this mathematics was widespread among economic agents.[34] Abacus schools were attended not only by would-be merchant bankers, but also by other practitioners, such as engineers, architects, artisans and artists.[35]

32 By the 15th century, abacus schools were founded in Verona, Venice, Florence, Bologna, Siena, Perugia, Palermo, Arezzo, Pisa, Volterra, Colle Val d'Elsa, Lucca, Milan, Pistoia, Prato, Fucecchio, Genoa, Savona, and Città di Castello Elisabetta Ulivi, "Scuole d'abaco e Insegnamento Della Matematica," in *Il Rinascimento Italiano e l'Europa*, edited by Luca Molà, Giovanni Luigi Fontana. 12 vols. (Treviso: Angelo Colla Editore, 2008), Vol. 5, pp. 403–420.

33 Elisabetta Ulivi, "Masters, Questions and Challenges in the Abacus Schools," *Archive for History of Exact Sciences*, 2015, *69*/6:651–670.

34 R.A. Goldthwaite, "Schools and Teachers of Commercial Arithmetic in Renaissance Florence," *Journal of European Economic History*, 1972, *1*/2:418–433. Paul F. Grendler, *Schooling in Renaissance Italy: Literacy and Learning, 1300–1600* (Baltimore: Johns Hopkins University Press, 1989). Goldthwaite, *The Economy of Renaissance Florence* (cit. note 19), pp. 341–407.

35 For example, we know that Filippo Brunelleschi, Leonardo da Vinci, Leon Battista Alberti,

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Abacus masters were also the main authors of abacus manuals, which circulated first of all within abacus schools. Contrary to what is commonly assumed about manuscript sources, abacus manuals had a particularly wide public circulation. These texts were in fact used by abacus masters as teaching aids while lecturing, and we have evidence showing that manuscript abacus manuals were passed down across generations of masters.[36] This implies that manuscript manuals which were used as teaching aids had a readership of generations of masters, and an audience of a number of generations of students.

With regards to the mathematical contents of these texts, though with a rich degree of variation, abacus manuals spread a standard package of mathematical knowledge. Although a degree of experimentation in mathematical theory was achieved along its history, the main contribution of this tradition was to spread in vernacular contexts the mathematics that had been introduced in the time of Fibonacci. Together with the introduction of the positional numeral system and of fractions, these texts were founded on the so-called "rule of three," i.e. the calculation of the unknown term of a proportion where three terms are known. The first lines of the earliest abacus manual (c. 1290) directly introduce the rule of three:

> Quisto è-ne lo livero de l'abbecho secondo la oppenione de
> maiestro Leonardo de la chasa degl figluogle Bonaçie da
> Pisa. Lo p(r)imo chapitolo è-ne de le reg(o)le d(e) le tre chose.
> Se ce fosse dicta alchuna ragione e · lla q(ua)-
> le se proponesse tre chose, sì devemo m(ultip)licare
> quilla chosa che noie volemo sap(er)e (contra) q(ui)lla
> che non è de quilla medessme, a pa(r)ti(r)e nell'a-
> ltra.[37]

Piero della Francesca and Michelangelo attended an abacus school. Abacus mathematics was fundamental for the development of linear perspective. Piero della Francesca was so versed in abacus mathematics that he was also the author of an advanced abacus manual Piero della Francesca, *Trattato d'abaco: dal Codice Ashburnhamiano 280, 359*–291*, della Biblioteca medicea laurenziana di Firenze*, edited by Gino Arrighi (Pisa: Domus Galilaeana, 1970), Piero della Francesca, *Trattato d'abaco*, edited by Marisa Dalai Emiliani et al. 3 vols. (Roma: Istituto Poligrafico e Zecca dello Stato, 2012).

36 See the introduction to Andrea Bocchi, *Lo livero de l'abbecho* (Pisa: ETS, 2017). Raffaele Danna, "Una scienza per la rinascita. Note su Paolo dell'Abaco e la matematica abacistica fiorentina," *Rinascimento*, 2019, *59*:245–269.

37 This is the *Livero de l'abbecho* [the book of the abacus] according to the opinion of master Leonardo of the house of the sons of Bonaccio [*filii Bonaccii*] of Pisa. The first chapter is (about) the rule of three. / If we were told any question where three things are given, we have to multiply the thing we want to know by the thing that is not of the same kind, and

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As shown in this passage, the main focus of these texts is not on mathematical theory, but on the pragmatic relevance of mathematics. Theoretical explanations, when present at all, are given a very limited space. The relevant mathematical rule is generally stated without providing a demonstration and is followed by a list of practical examples where the rule is applied. Most of these texts consist of lists of solved practical problems arranged according to their context of application, such as exchange, conversion, interest rates, divisions of shares, and alloying. This is a typical example, taken from the manual of the most famous abacus master of 14th century Florence, Paolo dell'Abaco's *Trattato di tutta l'arte dell'abacho* (c. 1339):

> Quest'è una regola la quale mostra gieneralmente per tutte terre e da tutti pesi e da tutte monete. Quando noi comparassimo argento a biglone o d'oro, o sseta o spezierie o grascia o qualunque altro avere di peso fosse e noi lo portassimo da una terra ad un'altra, di sapere quanto mi chonviene vendere lo marcho o l'oncia o la libra o 'l quintale o la charica o qualunque altro peso fosse a salvarmi della vendita e riavere mio chapitale al peso ed alla moneta. Fa chosie: sempre multiplicha lo peso della terra onde tu lo trai chontra la valuta di quello luogho. E parti per la quantitade del peso oue tu lo porti sieno oncie o libre o chosse fosse.
>
> Verbi grazia lo marcho dell'argento di chorte lo qual è 8 oncie mi chosta 75 soldi della moneta. Portolo a gienova e questo marcho mi torna 8 oncie e quarta. Dimmi per quanto posso dare lo marcho in gienova e ssalvarmi. Dovemo multiplicare 8 via 75 soldi fanno 600 soldi et partire per 8 et $\frac{1}{4}$ sechondo la reghola che ne viene 72 soldi e 8 denari e $\frac{8}{11}$. Et per tanto possiamo dare lo marcho a gienova della moneta di vignone cioè 72 soldi 8 denari $\frac{8}{11}$.[38]

divide into the other. Florence, Biblioteca Riccardiana, ms. 2404, f. 1r, 1–8, edited in Bocchi, *Lo livero de l'abbecho* (cit. note 36), p. 163. English translation by the author.

38 This is a rule which can be applied [literally: is shown] to every city, every weigh and every currency. When we compare silver or bullion or gold, or silk or spices or provisions or any other kind of goods sold by weigh, and we carry such goods from one city to another, [we want to] know at what price it is convenient to sell by the mark or ounce or pound or quintal or *charica* or any other weight in order to be refunded [literally: be saved] from the selling and regain our capital. Do as follows: always multiply the weight of the city you are shipping from by the currency of that city. And divide into the weight of the city you are shipping to, whatever this may be. For example, the mark of silver of the court [the papal court then in Avignon], which is 8 ounces, costs 75 soldi of money of Avignon. I take it to Genoa, where the mark of silver weights 8 ounces and a quarter. Tell me at what price I can sell the mark in Genoa in order to be refunded. We have to multiply 8 by 75

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As shown in this passage, this arithmetic was fundamental to solve problems of exchange. It is interesting to note the dialogical form of the text, where the writer directly addresses the reader ('do as follows', 'tell me'), which mimics the teaching practice of abacus schools and the style of other practical texts such as books of recipes.[39] Although they do not directly explain the functioning of the bill of exchange – whose details, in the same way as bookkeeping, were probably learnt during subsequent apprenticeships – abacus manuals provided the mathematical tools necessary to handle them. For example, the following passage, taken from an early 14th-century Pisan manual, asks to compare exchange rates in order to know which exchange rate would be more convenient to transfer money across markets:

> Pagamento da una città a un'altra
>
> Uno ha fare uno pagamento a bologna di lb 500 di bolognini e truova che il bolognino grosso vale in bologna d 12 vale in firenze d 20. E llo grosso guelfo che vale in firenze d 48 vale in bologna d 30 e l'ancontano grosso che vale in firenze d 40 vale in bologna d 25. Adomando quale di queste 3 monete sarà milliore a portare e quanto.[40]

Italian abacus mathematics was a forerunner in Europe, as the production of practical arithmetic texts employing Hindu-Arabic numerals outside Italy consolidated only in the 15th century. Abacus schools predated the development of vernacular schools teaching Hindu-Arabic arithmetic in the rest of Europe. Like in Italy, the European tradition of practical arithmetic was not dominated by the Latin language and was accessible to a relatively wide readership. As

soldi, which is 600 soldi, and divide 600 into 8 and $\frac{1}{4}$, according to the rule, which results in 72 soldi, 8 denari and $\frac{8}{11}$. And this is the price we can sell in Geona the mark of Avignon, i.e. 72 soldi, 8 denari and $\frac{8}{11}$. Paolo dell'Abaco, *Trattato di tutta l'arte dell'abacho*, Florence, Biblioteca Nazionale Centrale, ms. Fondo Nazionale II. IX. 57, fol. 53 r. English translation by the author. For a discussion of this manuscript within its socio-economic context, see Danna, "Una scienza per la rinascita" (cit. note 36). For the extant sources on the author, see Warren Van Egmond, "New Light on Paolo dell'Abbaco," *Annali dell'Istituto e Museo di Storia della Scienza di Firenze*, 1977, 2/2:3–21.

39 Robert Recorde's *The Ground of Artes* (London, 1543), the most important practical arithmetic manual of 16th century England, reproduces this dialogical structure.

40 Payment from one city to the other. / One has to make a payment in Bologna of 500 pounds of bolognini, and they find that the bolognino grosso is worth 12 denari in Bologna and 20 denari in Florence. And the Guelf grosso, which is worth 48 denari in Florence, is worth 30 denari in Bologna. And the grosso of Ancona, which is worth 40 denari in Florence, is worth 25 denari in Bologna. I ask which of these three currencies will be best [to make the payment] and by how much. Florence, Biblioteca Riccardiana, ms. 2252, fol. 16 r.

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with abacus mathematics, also in the European tradition the focus was not on mathematical theory, but rather on its application to the solution of practical problems mainly through applications of the rule of three. European practical arithmetic was a continuous tradition largely stemming from abacus mathematics. To the best of my knowledge, this tradition has never been studied in a comprehensive manner and has never been reconstructed as a continuous and European-wide phenomenon.

3 The Spread of Practical Arithmetic Manuals in Europe

The analysis presented in this section is based on an original database reconstructing the European tradition of practical arithmetic. The most complete catalogue available on the tradition of abacus mathematics identifies 288 manuscripts and 153 printed works composed from the 13th century to 1600: a tradition of remarkable size in late-medieval terms.[41] The database presented in this paper expands this approach to the main areas of western Europe relying both on secondary sources and on extensive archival research. I have recorded detailed information on 1280 practical arithmetic texts written from the *Liber Abaci* (1202) to 1600. Of these texts, 342 are manuscripts and 938 are printed. Following Van Egmond's methodology, every manuscript document has been considered as an independent text. Following the main catalogues on the period, I have considered printed texts at the book-edition level.[42] The database records over 340 authors who wrote in more than 130 cities in most western-European languages.[43] All the recorded texts employ Hindu-Arabic numerals, and most of them use arithmetic as a commercial tool. The introduction of the printing press imposed a higher degree of standardisation on these manuals, which became the ancestors of modern mathematical primers. The dataset is described in further detail in the Data Appendix.

The tradition of practical arithmetic is particularly suitable for a long-run analysis because its size is considerably bigger and its preservation is less scattered than other commercial sources, such as mercantile handbooks, account-

41 Warren Van Egmond, *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600* (Firenze: Giunti Barbera, 1980).

42 Incunabula Short Title Catalogue (ISTC), www.bl.uk/catalogues/istc/. Universal Short Title Catalogue (USTC), www.ustc.ac.uk/. Jochen Hoock, Pierre Jeannin, Wolfgang Kaiser, *Ars Mercatoria: Handbücher Und Traktate Für Den Gebrauch Des Kaufmanns, 1470–1820: Eine Analytische Bibliographie in 6 Bänden* (Paderborn: Schöningh, 1991).

43 Several texts are written by anonymous authors, hence the estimate on their number.

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ing accounting treatises, and account books. Lists of late-medieval 'merchant notebooks' do not go beyond 25 surviving manuscripts, while only the Italian abacus tradition lists more than 280 manuscript texts.[44] In comparison with bookkeeping manuals, practical arithmetic constitutes their premise for two reasons: because basic arithmetic is necessary to keep accounts and because, as a consequence, its spread occurred earlier than that of bookkeeping treatises. Moreover, the tradition of bookkeeping manuals is not comparable to that of practical arithmetic in terms of size, as, despite having been the focus of important scholarly efforts, the most comprehensive edition of early bookkeeping manuals includes only 31 texts.[45] In comparison with the direct study of accounting sources (i.e. of account books), the tradition of European practical mathematics provides a source of evidence whose preservation seems to be less scattered. For example, while thousands of account books from Florence survive, no bankers' account book written in Venice before 1500 survived in its entirety.[46] Besides, this tradition follows quite closely the path of the diffusion of modern accounting as investigated by accounting historians.

The size of the sample makes it possible to identify some interesting trends. From a general point of view, it is possible to observe an exponential increase in the overall production of practical arithmetic manuals across the observation period, with a marked acceleration occurring in the 16th century, as a consequence of the introduction of the printing press and of the diffusion of the tradition across Europe. In Fig. 1, which reports both the overall data and the regional contribution for each interval, it is possible to notice the leading role played by the Italian abacus tradition until the first half of the 15th century. Starting from this date, there was a marked expansion in the production of texts across all other European areas, while the Italian production levelled off.

Fig. 2 shows the trends in areas of publication using 40-years moving averages. The graph represents absolute (i.e. not weighted) values, which makes trends more significant than comparative numbers. Again, the main feature of the first half of the observation period is the leading Italian production. Interestingly, before the 16th century abacus mathematics shows two periods

44 Peter Spufford, "Late Medieval Merchants Notebooks: A Project. Their Potential for the History of Banking," in *Kaufmannsbücher und Handelspraktiken vom Spätmittelalter bis zum beginnenden 20. Jahrhundert*, edited by Markus A. Denzel, Jean Claude Hocquet, Harald Witthöft (Stuttgart: Franz Steiner, 2002), pp. 47–61.

45 Basil Yamey, Michael F. Bywater (eds.), *Historic Accounting Literature: A Companion Guide* (London: Scolar Press, 1982).

46 Mueller, *The Venetian Money Market* (cit. note 27), p. 81.

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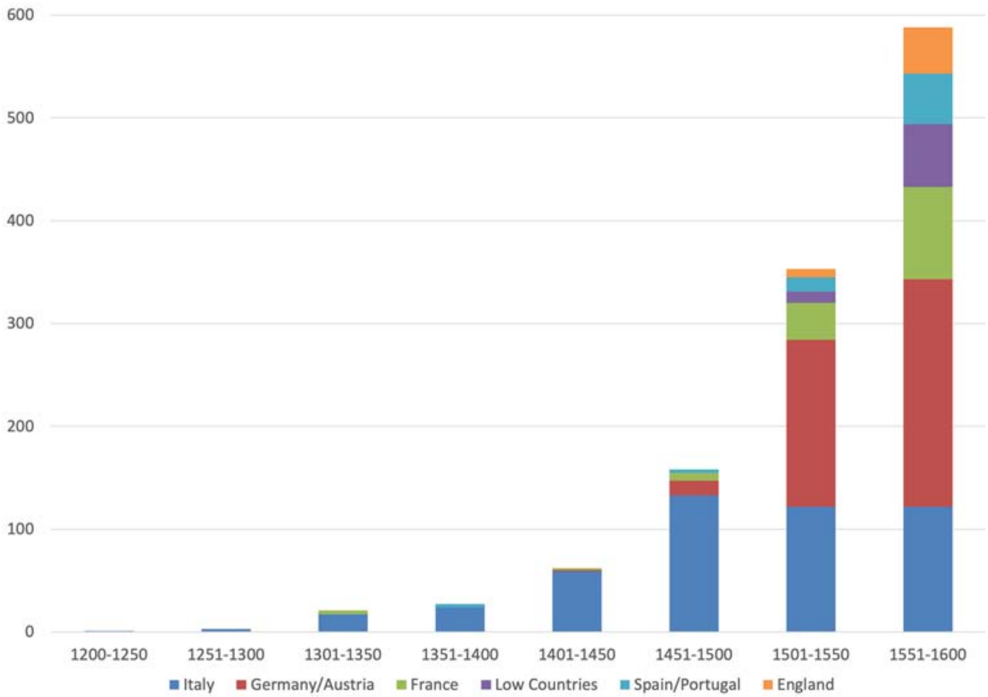

FIGURE 1 Overall output of practical arithmetic manuals, for areas of publication

of growth, namely the period until the middle of the 14th century and the entire 15th century. The second half of the 14th century, instead, is characterized by a slight decline in text production. This could be due to a heaping effect in 1350 (several manuscripts do not report a date, and can therefore only be approximately dated, which increases the probability of text heaping in round dates), or to a decline in text demand.[47] This second option could be due to the demographic collapse in the aftermath of the Black Death.

Fig. 2 shows that the 15th century was the golden age of Italian abacus mathematics and suggests that by the end of the century abacus mathematics saturated its diffusion within Italy. This hypothesis is strengthened by the fact that the introduction of the printing press did not determine a marked increase in the production of abacus manuals. After 1478 (the year in which the first printed abacus manual was published), the Italian production levels off around an average of 12 texts published every five years. Given the higher amount of

47 I am borrowing the concept of 'heaping effect' from Brian A'Hearn, Jörg Baten, Dorothee Crayen, "Quantifying Quantitative Literacy: Age Heaping and the History of Human Capital," *The Journal of Economic History*, 2009, *69*/3:783–808.

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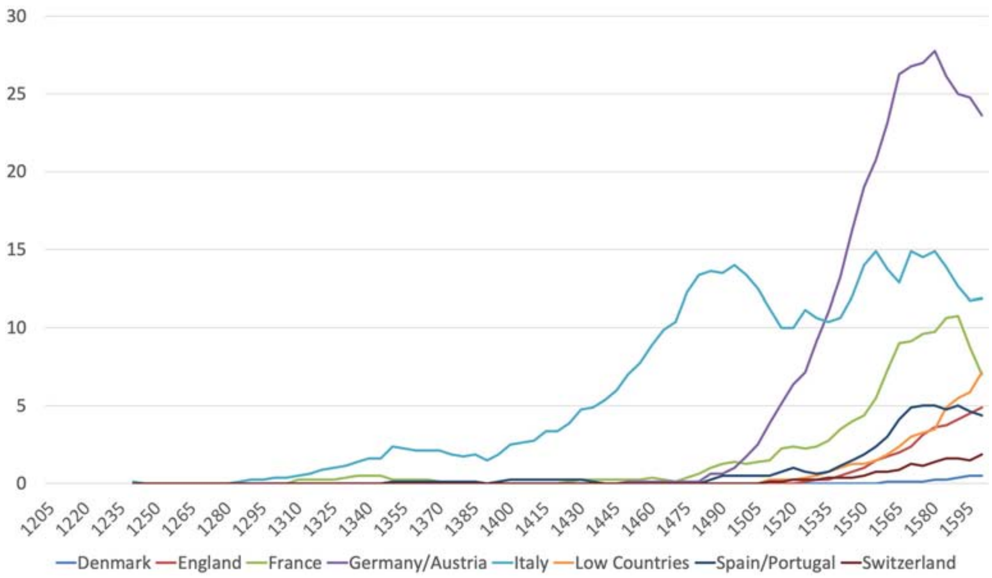

FIGURE 2 Areas of publication (40 years moving average)

texts produced in Italy in this period (307 texts between 1478 and 1600), the still high component of manuscript texts (148) and of texts without precise dating (103, both manuscript and printed), the text-heaping effect in this period is stronger. This could explain the two apexes in the beginning as well as in the middle of the 16th century, but does not explain the drop at the very end of the period. In other words, the data suggest a levelling off of the Italian production of practical arithmetic texts in the 16th century, with a decline at the end of the period. An alternative explanation for the decline in the Italian production in this period could again rely on external factors, such as the aftermath of the so-called Italian wars of the 16th century. Overall, the Italian production seems to follow the typical s-shaped curve characterising the diffusion of innovations.[48]

All the other areas show a marked increase in production of practical arithmetic texts, with the printing press as a fundamental driving factor. Germany, the cradle of European printing, stands out as the most important producer of these texts. The so-called *Rechenbücher* were a particularly successful genre in 16th century Germany, as shown by the steep increase in the production of these texts. Interestingly, Germany shows a decline in production at the end of

48 As discussed in Everett M. Rogers, *Diffusion of Innovations* (New York: Free Press of Glencoe, 1962).

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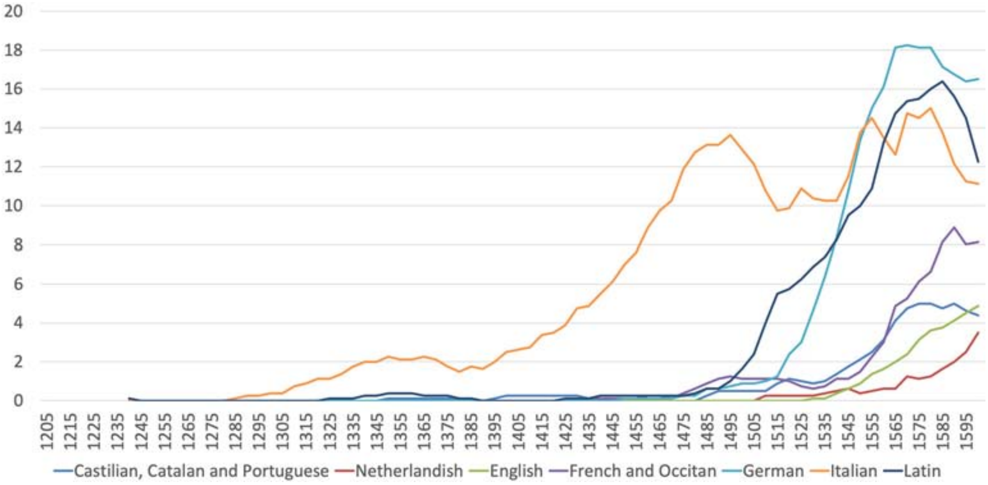

FIGURE 3 Languages (40 years moving average)

the 16th century as well. Very similar trends apply to France, and, with a less marked decline at the end of the period, to Spain. A different trend is instead shown by the latecomers. Among these, the Low Countries and England show a growing production of texts in the end of the 16th century, a period when both areas were laying the foundations for the expansion of their Atlantic trade, and of the so-called "little divergence."[49]

Fig. 3 shows language trends, again using 40-years moving averages. The overall picture is similar to the one given in Fig. 2. What is interesting to notice in this graph is the curve of Latin texts, which cannot be captured by an analysis based on areas. Latin texts tend to be more élite publications, even though in this period, which is also marked by the expansion of humanism north of the Alps, Latin started to circulate also within urban societies. Latin texts are more mobile in terms of geographic span than vernacular ones, as they were published in most of the areas covered by the database. It is interesting to notice that the production of Latin texts was remarkable in size (in absolute terms, it is the second language after German) and that it followed the trend of most continental areas, showing a decline in production at the end of the 16th century. The expanding Netherlandish and English productions at the end of the period are, in fact, mainly in the vernacular.

It is possible to visualise the spread of practical arithmetic on intertemporal maps.[50] The data have been divided into 50-year intervals until 1450, and in 25-

49 de Pleijt and van Zanden, "Accounting for the 'Little Divergence'" (cit. note 13).
50 138 recorded texts could not be located with precision, and as a consequence these have

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year intervals until 1600. The maps record cumulative data, i.e. they record all the texts published up to the map date. Fig. 4 shows the situation in 1350. We can see the early production of abacus manuscripts in Pisa starting from the late 13th century, together with the earliest text written in Perugia.[51] Close to Pisa it is possible to spot Lucca, and moving eastwards we find Florence, which would grow to become the capital of the manuscript phase of abacus mathematics. Further north, we can see the first abacus text appearing in Venice. We also find abacus manuals written in Avignon as well as in Montpellier. These texts belong to the Italian abacus tradition, as they were written by Italian (probably all Florentine) authors who were active in the network of Italian merchants trading with the Papal court, based in Avignon and supplied by the port of Montpellier.

Fig. 5 portrays the spread in 1400. It is possible to notice the growing role of Florence, whose production overtook the Pisan one. In the north of Italy, we see appearing other commercial centres, such as Genoa, Bologna and Milan. With Fig. 6 we move the analysis to the European scale in 1450. It is possible to notice the emergence of two main hubs for the diffusion of abacus texts in Italy (Florence and Venice) as well as the appearance of new centres such as Como, Padua, other Tuscan centres, and Rome.

Outside of Italy, the earliest examples of non-Italian practical arithmetic texts employing Hindu-Arabic numerals come from the Mediterranean context. The earliest extant manuscripts are from Spain and can be dated between the end of the 14th and the beginning of the 15th century.[52] These texts

been excluded from GIS analysis. Since it is more difficult to have precise information on early manuscript material, relatively more documents without a specific location are concentrated in the first phases of the observation period. More specifically, the dataset records 1 unlocated text in the 13th century (out of a total of 4 texts: 25 % of total), 19 unlocated texts in the 14th century (out of 41: c. 46 % of total), 72 in the 15th century (out of 225: 32 % of total) and 47 in the 16th century (out of 1009: c. 4.6 % of total).

51 Raffaella Franci, "Leonardo Pisano e la trattatistica dell'abaco in Italia nei secoli XIV e XV," *Bollettino di Storia delle Scienze Matematiche*, 2003, *23*/2:33–54. Ead., "Un trattato d'abaco pisano della fine del XIII secolo," *Bollettino di Storia delle Scienze Matematiche*, 2015, *35*/1:9–96. Bocchi, *Lo livero de l'abbecho* (cit. note 36).

52 These are the *Libro de arismética que es dicho alguarismo* (c. 1395), ms. 46 de la Real Colegiata de San Isidoro de León; the ms. 155, ff. 144r–164r of the Real Academia Española (beginning of 15th c.); the ms. 10106 of the Biblioteca Nacional in Madrid (beginning of 15th c.). The earliest of these texts was probably written in Seville. See Javier Docampo Rey, "A New Source for Medieval Mathematics in the Iberian Peninsula: The Commercial Arithmetic in Ms 10106 (Biblioteca Nacional, Madrid)," *Revue d'histoire des mathématiques*, 2009, *15*/1:123–177. Betsabé Caunedo del Potro, Ricardo Cordoba de la Llave (eds.), *El Arte del alguarismo: un libro castellano de aritmética comercial y de ensayo de moneda del siglo XIV: Ms. 46 de la Real Colegiata de San Isidoro de León* (Junta de Castilla y León, Consejería de Educacíon y Cultura, 2000).

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appeared at the convergence of the vernacular tradition of practical arithmetic and of the Latin tradition which originated in 12th-century al-Andalus. The relative contribution of these two traditions is a subject which is currently debated among specialists. In the south of France we find the so-called *Manuscrit de Pamier*, written around 1420–1430 in Pamier, a town on the main road between Toulouse and Barcelona.[53] This manuscript preserves an anonymous text written in Provençal which presents the typical contents and structure of an abacus treatise, as it shows how to make operations with integers and fractions, followed by applied problems.

The earliest practical arithmetic text employing Hindu-Arabic numerals north of the Alps is the so-called *Algorismus Ratisbonensis*. This is an interesting text, as it could represent an intersection of the Latin tradition of the *algorismi* and that of abacus mathematics. Written by a Benedictine monk at the monastery of Sankt Emmeran in Regensburg – a free imperial city on the trade routes to Italy – in the middle of the 15th century, the *Algorimsus Ratisbonensis* played a similar role in Germany as Fibonacci's *Liber Abaci* had done in Italy.[54]

Venice played a central role for the transmission towards north of practical knowledge in general, and of practical arithmetic in particular. The *Fondaco dei Tedeschi* was a centre where German and Italian commercial communities encountered each other. Starting from the 15th century, we find a growing number of German prospective merchants moving to Italian commercial centres (and mostly to Venice) to gain an education in commerce or in other practical arts. The international merchant-banker Jakob Fugger, his accountant Matthaüs Schwarz, Lucas Rem and Albrecht Dürer are just the best-known examples of a wider phenomenon.[55]

With Fig. 7 we can observe the situation in 1475. This is still a world of only manuscript sources. We can see more texts appearing in the south of France as well as the first text appearing in Paris. This is Jehan Adam's *L'arithmétique aux jetons* which, despite being mainly dedicated to calculations on the reckoning

53 Paris, Bibliothèque Nationale, nouv. Acq. Fr. 4140 (NAF 4140). See Jacques Sesiano, "On an Algorithm for the Approximation of Surds from a Provençal Treatise," in *Mathematics from Manuscript to Print, 1300–1600*, edited by Cynthia Hay (Oxford: Clarendon, 1988), pp. 30–55.

54 Kurt Vogel, *Die Practica des Algorismus Ratisbonensis: ein Rechenbuch des Benediktinerklosters St. Emmeram aus der Mitte des 15. Jahrhunderts nach den Handschriften der Münchner Staatsbibliothek und der Stiftsbibliothek St. Florian* (München: Beck, 1954).

55 Philippe Braunstein, *Les Allemands à Venise: (1380–1520)* (Rome: École française de Rome, 2016), p. 435.

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table, shows that its author was also acquainted with Hindu-Arabic numerals.[56] In Italy, it is possible to notice the rapid expansion of Florence as the leading centre in 15th century abacus mathematics as well as the diffusion of the production of these texts in several other central-northern cities. This is probably the apex of Florence both as financial hub and as capital of abacus mathematics. The most successful abacus treatise of the 15th century – Benedetto da Firenze's *Trattato d'abacho* – is attested for the first time during this period.[57]

With Fig. 8 we move to the end of the 15th century, and we start to see the acceleration in text production triggered by the printing press. In Italy, Florence still appears as the leading centre, followed by Venice, and the production of abacus treatises is increasingly widespread, with texts published also in Naples, Nola and in the region of Calabria. The earliest printed practical arithmetic was published in this period. This is the so-called *Aritmetica di Treviso*, published in 1478.[58] The second-oldest practical arithmetic manual printed in Europe was published in Barcelona. This is the *Suma de la art de arismetica* by Francesc de Santcliment, written in Catalan and printed in 1482, which was probably influenced by the model of abacus manuals.[59]

In France, we see an expansion in Paris and the appearing of Lyon, following the royal protection to local fairs granted by Louis XI in 1463. This is the city in which Nicolas Chuquet completed in 1484 a manuscript which includes a commercial arithmetic, a collection of problems, a geometry and the *Triparty en la science des nombres*.[60] Chuquet was a sophisticated mathematician, and his *Triparty* has drawn the attention of historians of mathematics because

56 Lynn Thorndike, "The Arithmetic of Jehan Adam, 1475 A.D.," *The American Mathematical Monthly*, 1926, *33*/1:24–28.

57 Elisabetta Ulivi, "Benedetto Da Firenze (1429–1479) un maestro d'abaco del XV secolo, con documenti inediti e con un'appendice su abacisti e scuole d'abaco a Firenze nei secoli XIII–XVI," *Bollettino di Storia delle Scienze Matematiche*, 2002, *22*/1.

58 Frank J. Swetz, *Capitalism and Arithmetic: The New Math of the 15th Century, Including the Full Text of the Treviso Arithmetic of 1478, Translated by David Eugene Smith* (La Salle: Open Court Publishing, 1987).

59 In Catalunia, practical arithmetic employing Hindu-Arabic numerals is often identified using the term 'abba', which clearly echoes the term 'abaco' employed in the Italian tradition. In Santcliment's text, the formulation of the rule of three follows the standard definition of abacus manuals. The text is available online: https://mdc.csuc.cat/digital/collection/incunableBC/id/25441 (accessed 29 Oct. 2020).

60 Bibliothèque Nationale, Français 1346. The *Triparty* is the first section of the ms. (fols. 1–147), followed by a collection of problems (fols. 148–210), a geometry (fols. 211–262) and a commercial arithmetic (fols. 264–321). The manuscript is available online at http://gallica.bnf.fr/ark:/12148/btv1b90588845h (accessed 29 Oct. 2020).

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of its original mathematical contributions.[61] However, Chuquet's commercial arithmetic includes the typical contents of abacus mathematics, and is possibly based on Italian sources.[62] Another French source of this early phase is the *Kadran aux marchans* by Jehan Certain, probably completed in 1485 Marseilles.[63] This is a purely commercial texts, the subjects covered are those of abacus mathematics, and the sources and influences are from the Italian context.[64]

Moving towards the German-speaking world, the easternmost point represents Vienna, where the first printed edition of Georg von Peuerbach's *Elementa Arithmetices* was published in 1495.[65] This was a very synthetic manual, covering the main subjects of arithmetic in the turn of 8 folios. We can also see appearing several cities in the central region of Germany, among which are Cologne, Strasbourg, Nuremberg, Bamberg, and Leipzig.[66] Leipzig had a consolidated trading history, as its fair, founded in 1190, was one of the oldest in Europe. Starting from the late 15th century, Francone and Swabian commercial centres such as Augsburg and Nuremberg founded schools which also offered mathematical teaching. The masters active in these schools (*Rechenmeisters*) produced a remarkable tradition of practical arithmetic texts (*Rechenbücher*) during the 16th century.

Just to quote one example, the most successful of these manuals is Adam Ries' *Rechenung auff der linihen und federn*, first published in 1522 Erfurt. This was Ries' second *Rechenbuch*,[67] and it covered calculation with both the reckoning table and with Hindu-Arabic numerals, together with a thorough illustration of their commercial applications.[68] This text circulated widely, as we have

61 The most interesting of which is a non-geometrical symbolism for exponents.

62 Graham Flegg, Cynthia Hay, Barbara Moss (eds.), *Nicolas Chuquet, Renaissance Mathematician: A Study with Extensive Translation of Chuquet's Mathematical Manuscript Completed in 1484* (Dordrecht; Boston: D. Reidel Pub. Co., 1985). Paul Benoit, "The Commercial Arithmetic of Nicolas Chuquet," in *Mathematics from Manuscript to Print* (cit. note 53), pp. 96–116.

63 Paris, Bibliothèque de l'Arsenal, ms. 2904.

64 Paul Benoît, "La formation mathématique des marchands français à la fin du Moyen Âge: l'exemple du Kadran aux marchans (1485)," *Actes des congrès de la Société des historiens médiévistes de l'enseignement supérieur public*, 1981, *12*/1:209–224.

65 Before returning to Vienna, Peuerbach studied in Italy, where he probably came in contact with abacus mathematics.

66 Of these, Cologne, Strasbourg, Nuremberg and Augsburg were free imperial cities.

67 Adam Ries' first *Rechenbuch* showed only calculations with the reckoning table and was never reprinted.

68 The popularity of Adam Ries as a *Rechenmeister* has survived until the present day in the German saying "nach Adam Ries," a common idiomatic expression employed when deal-

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evidence of 113 editions of it printed until 1656 throughout German-speaking lands, with several revisions and updates.[69] The spread of these schools, masters and publications in Germany seems to have strong similarities with that of abacus mathematics in Italy.

With Fig. 9 we enter the 16th century. In the Iberian peninsula we see for the first time Valencia and Lisbon. The first practical arithmetic text published in Portugal was the *Tratado da pratica darismetyca* by Gaspar Nicolas, printed in 1519.[70] This was a rather successful text, receiving at least 11 editions stretching until the 18th century. In Italy as well as in France we see a consolidation of previous trends. As far as Germany is concerned, we see the consolidation of the first publication centres as well as the introduction of new ones (Nuremberg, Erfurt, Frankfurt an der Oder). Moving further north, vernacular practical arithmetic reached the Low Countries with texts such as the anonymous *Die maniere om te leeren cyffren*, printed by Thomas van der Noot in Brussels in 1508. This is one of the first practical arithmetic texts published in the north of Europe, and the first one known written Netherlandish. A second edition was printed in 1510 in Antwerp by Willem Vorsterman, followed by a French translation published by the same printer in 1529 with the title *La maniere pour apprendre a cyfrer*.[71]

Fig. 10 takes us to 1550. In Spain, Madrid, Valladolid and Seville appear for the first time. In France, production increases in the two main centres (Paris and Lyon) and we find a publication in Poitier as well. In Italy, it is possible

ing with basic arithmetic. A modern edition of Adam Ries' *Rechenbuch* is published in Stefan Deschauer (ed.), *Das zweite Rechenbuch Von Adam Ries: Eine moderne Textfassung mit Kommentar und metrologischem Anhang und einer Einführung in Leben und Werk des Rechenmeisters* (Braunschweig; Wiesbaden: Springer-Verlag, 2013).

69 For example, in 1533 the printer Egenolff in Frankfurt added the Visierbüchlein by Erhart Helm to cover a subject (gauging) which was not dealt with in Ries' original text. For the details on the editions of this work, see Rainer Gebhardt, *Die gedruckten Bücher von Adam und Isaak Ries: Verzeichnis und Beschreibung aller bekannten Exemplare* (Annaberg-Buchholz: Adam-Ries-Bund, 2017). Rainer Gebhardt, Peter Rochhaus, Pia Meyer (eds.), *Verzeichnis Der Adam Ries Drucke: Katalog Der Gedruckten Rechenbücher Und Rechenhilfsmittel Des Rechenmeisters Adam Ries* (Annaberg-Buchholz: Adam-Ries-Bund, 1997). In 1550, Ries published a third *Rechenbuch* which is usually referred to as *Die Practica*. However, probably because this text was published with a printing privilege, it was never reprinted.

70 Joaquim Barradas de Carvalho, "Sur l'introduction et la diffusion des chiffres arabes au Portugal," *Bulletin des Études Portugaises*, 1958, *20*:110–151.

71 Donald J. Harreld, "An Education in Commerce: Transmitting Business Information in Early Modern Europe," in *Information Flows: New Approaches in the Historical Study of Business Information*, edited by Jari Ojala, Leos Müller (Helsinki: Suomalaisen Kirjallisuuden Seura, 2007), pp. 21–25.

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to notice the acceleration of production in Venice, which eventually overtook Florence thanks to its flourishing printing industry. We observe a consolidation of text production in the central part of Germany, with some relevant accelerations. For example, in Frankfurt am Main, another key hub of German trade, at least 19 *Rechenbücher* were printed in the span of these 25 years. We also see the first publications in centres in northern-German lands. These are Gdansk (1538, 1540), Lübeck (1547) and Hamburg (1549). This lag of northern German cities – which were, nevertheless, relevant centres of commerce – is an interesting feature which will be discussed in more detail below.

Further East, we see for the first time Prague, Wrocław and Krakow, which were all on a main trade route. In the Low Countries, we can see a growing production in Antwerp, where in 1540 appeared the first edition of the *Arithmeticae practicae methodus facilis*. This is a compact compendium of practical mathematics written by the Dutch mathematician, cartographer and physician Rainer Gemma Frisius. This was another successful work which received over 100 reprints, with translations in French, Italian and Hungarian.[72]

This is also the period in which English production took off. The oldest completely surviving practical arithmetic text published in England is the anonymous *An Introduction for to lerne to reckon with the Pen and with the Counters*, published in St. Albans in 1536/37.[73] Early English texts have continental sources, i.e. both French and Dutch practical arithmetic works.[74] The turning point for the diffusion of this kind of publications in England arrived in 1543, with the publication of Robert Recorde's *The ground of artes teachyng the worke and practise of arithmetike*, printed in London by Reyner Wolfe.[75] With at least 37 reprints, this is the work that triggered the diffusion of arithmetic primers in England.

With Fig. 11 we approach a mature stage of the European diffusion of practical arithmetic. Moving from west to east, we see the first work published in Porto. This is the 1555 *Tratado da arte de arismetica*, the first work published in Portugal to deal with algebra. In Spain we see a growing production in Barcelona, Valencia, Zaragoza, and Seville. We also see the first publications in Burgos, Salamanca, Granada and Alcalá de Henares. In France, it is possible

72 Ulrich Reich, "Über hundert Auflagen: Das Arithmetikbuch des Gemma Frisius (1508–1555)," in *Arithmetische und algebraische Schriften der frühen Neuzeit*, edited by Rainer Gebhardt (Annaberg-Buchholz: Adam-Ries-Bund, 2005), pp. 323–340.

73 Travis D. Williams, "The Earliest English Printed Arithmetic Books," *The Library*, 2012, *13*/2:164–184.

74 A.W. Richeson, "The First Arithmetic Printed in English," *Isis*, 1947, *37*:47–56. P. Bockstaele, "Notes on the First Arithmetics Printed in Dutch and English," *Isis*, 1960, *51*/3:315–321.

75 A digitized reproduction of the work is available on eebo.chadwyck.com.

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to observe again a growing production in Paris as well as in Lyon. The publication of practical arithmetic texts is by 1575 consolidated in Switzerland, with the first publication in Zurich (a 1565 reprint of Adam Ries' *Rechenbuch*) and 11 texts published in Basel.

Germany presents a consolidation in production in its main centres as well (Cologne, Frankfurt am Main, Nuremberg, Augsburg, Leipzig, Wittemberg), some of which show a marked growth in text production. This is again the case of Frankfurt am Main, where in 50 years were published over 55 practical arithmetic texts. We can also note the beginning of a spread in the north of Germany, with Rostock, Szczecin and Königsberg appearing for the first time. In 1560 we also find the first publication of a practical arithmetic text in Denmark. This is the *En ny konstig regne Bog, udi Tal maader oc Vecter, paa lynnerne och met ziffre* by Anders Olsen, which had at least 4 other reprints in Copenhagen. Starting from this period, the already mentioned steep growth in text production in England as well as the Low Countries can be observed. Practical arithmetic manuals published in London increased from 8 editions in 1550 to over 25 in 1575. In the Low Countries we can see the leading role in the diffusion played by Antwerp, as well as the spread to new cities (Bruges, Gent, Louvain, Amsterdam). In Italy it is possible to note, together with a well-established diffusion among most northern and central centres, the overtake of Venice on Florence.

Fig. 12 brings us to the end of the observation period. The overall picture is that of an incremental growth of already established trends. This applies to most of western Europe (Portugal, Spain, France, Germany, Italy) where production shrank and only a few new centres appeared in this 25-years period. England and the Low Countries stand out of this trend. London is characterised by a marked growth in text production, going from 26 texts in 1575 to over 50 texts in 1600. A similar pattern is followed by the Low Countries, where overall production jumped from more than 30 texts in 1575 to over 70 texts in 1600 and where text production spread to several new centres, especially in Netherlandish-speaking territories. Among these new centres were Middelburg, Dordrecht, Rotterdam, Delft, Leiden, Haarlem, and Alkmaar. Further east, the diffusion of practical arithmetic reached for the first time Hungarian-speaking lands, with publications in Debrecen and Cluj.

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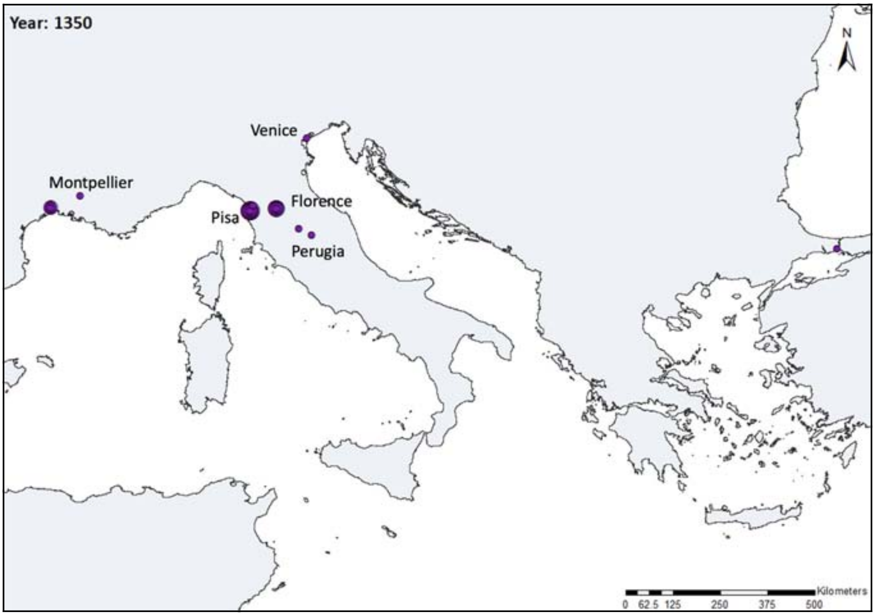

FIGURE 4 The spread of practical arithmetic manuals up to 1350, cumulative data

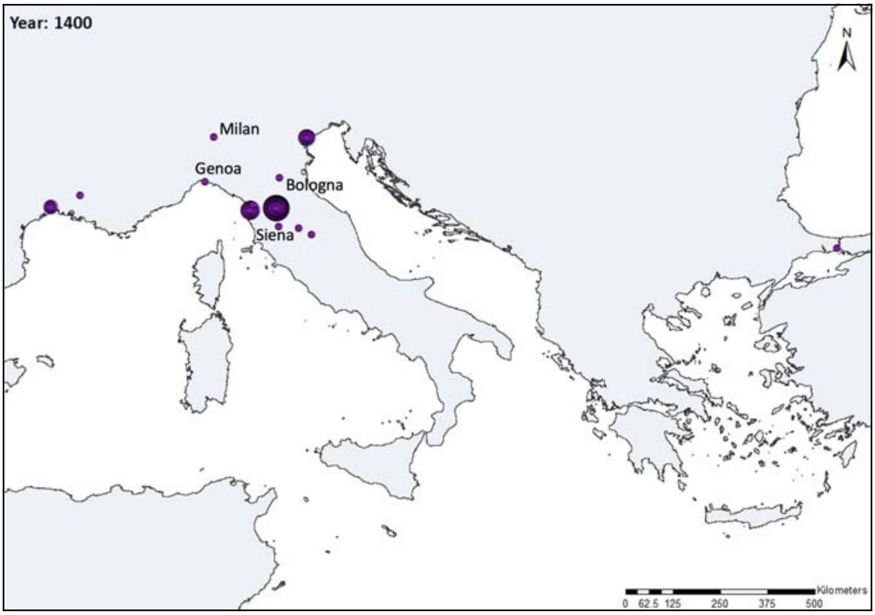

FIGURE 5 The spread of practical arithmetic manuals up to 1400, cumulative data

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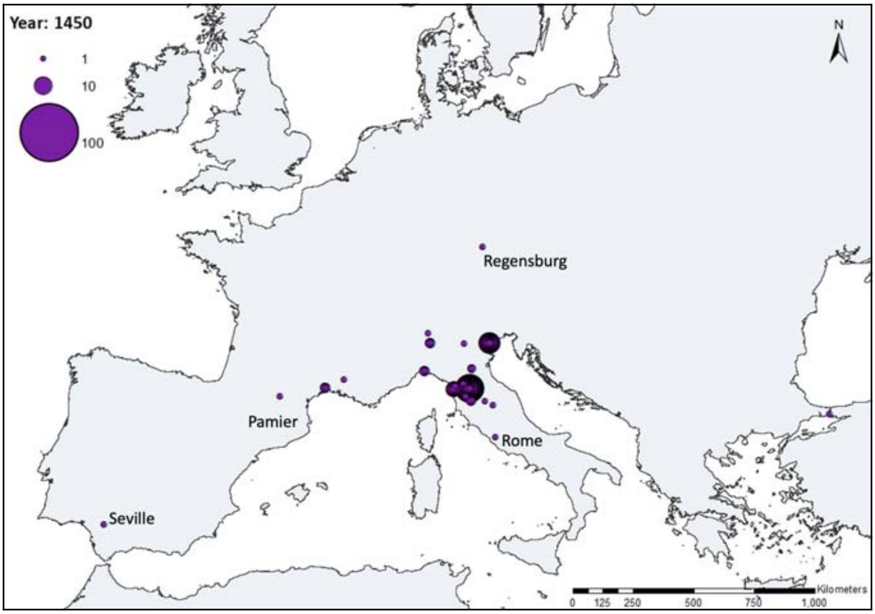

FIGURE 6 The spread of practical arithmetic manuals up to 1450, cumulative data

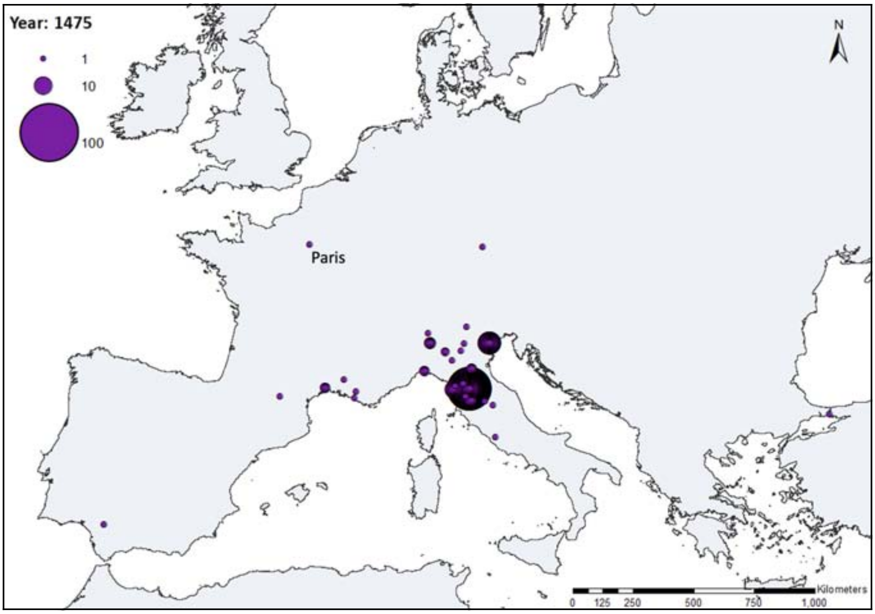

FIGURE 7 The spread of practical arithmetic manuals up to 1475, cumulative data

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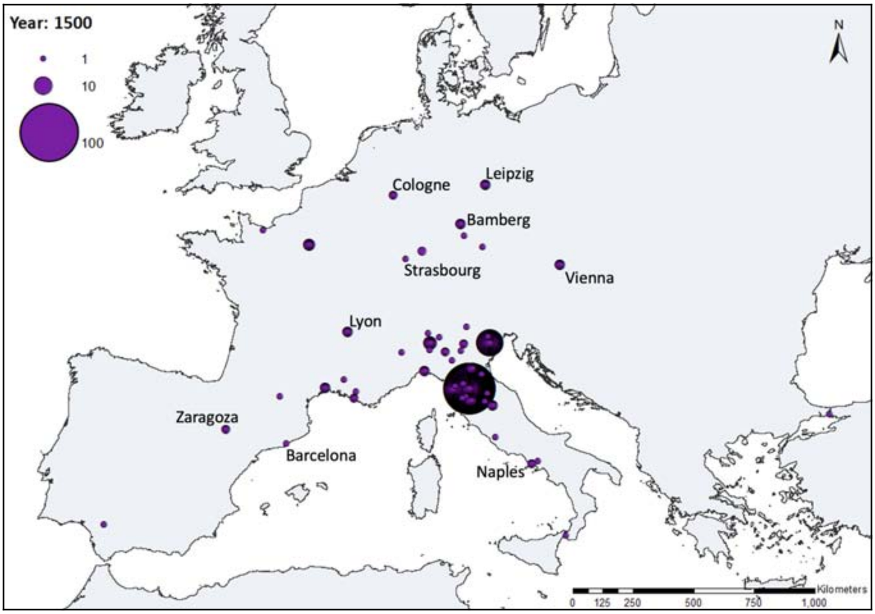

FIGURE 8 The spread of practical arithmetic manuals up to 1500, cumulative data

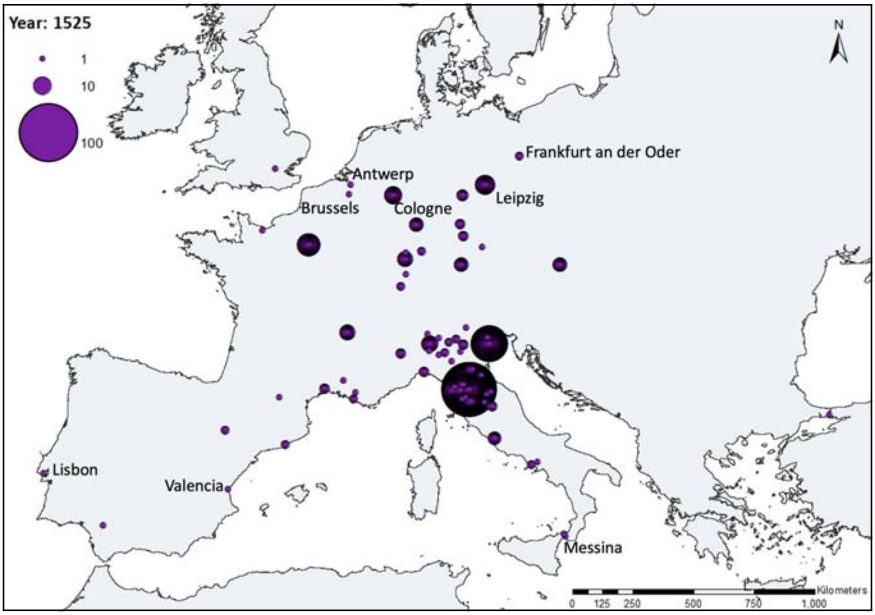

FIGURE 9 The spread of practical arithmetic manuals up to 1525, cumulative data

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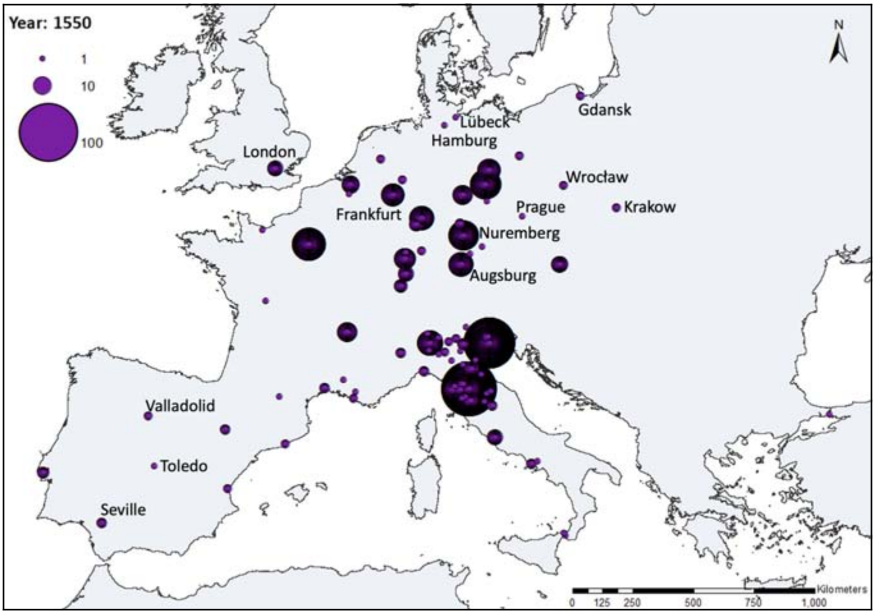

FIGURE 10 The spread of practical arithmetic manuals up to 1550, cumulative data

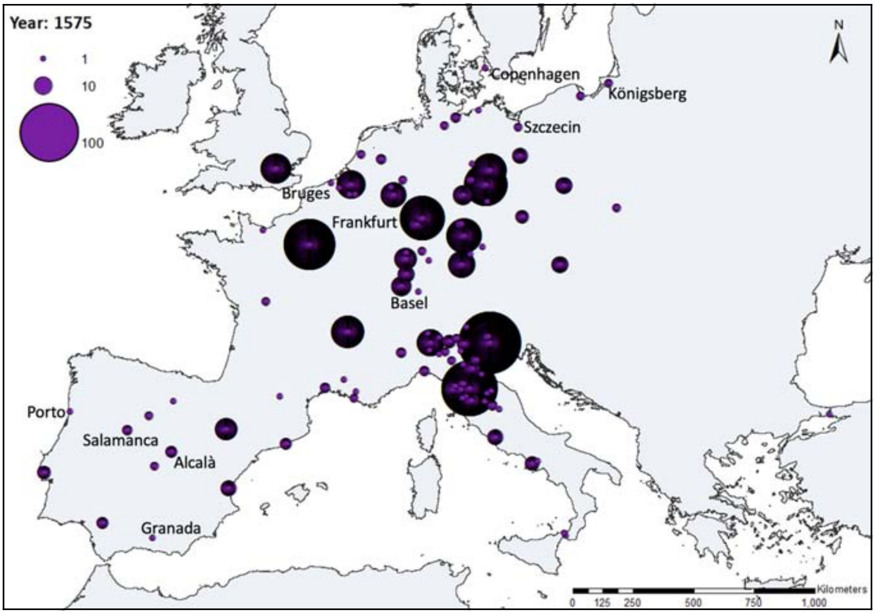

FIGURE 11 The spread of practical arithmetic manuals up to 1575, cumulative data

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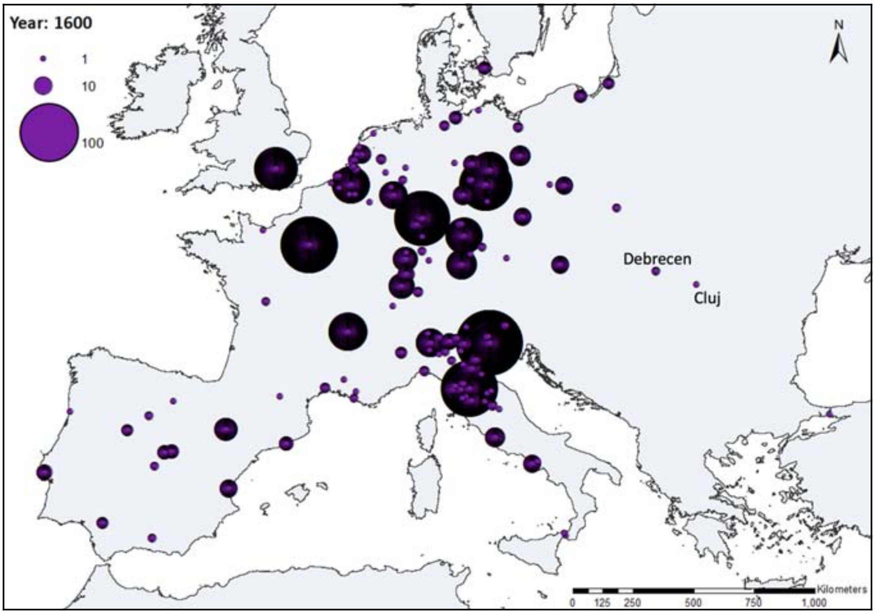

FIGURE 12 The spread of practical arithmetic manuals up to 1600, cumulative data

4 General Features of the Spread of Practical Arithmetic

The previous analysis gives a picture of a continuous tradition addressed to practitioners which spread a set of mathematical tools from the south to the north of Europe. This spread can roughly be divided into three phases. The first phase spans from the 13th to the middle of the 15th century. This is the period in which merchants faced a growing need for advanced mathematical tools, which led them to adopt Hindu-Arabic numerals. This is also the period in which Hindu-Arabic numerals appeared in Italian commercial documentation. The facsimile reproductions of Italian commercial documents published by Melis show that, starting from the 14th century, Hindu-Arabic numerals became increasingly frequent in Italian merchants' accounting practice.[76]

76 The transcriptions provided in Federigo Melis, *Documenti per la storia economica dei secoli XIII–XVI* (Firenze: L.S. Olschki, 1972). do not follow the use of the different numerical notations of the sources, so it is necessary to refer to the facsimile reproductions. Particularly interesting are documents 123–129, 136, 151–152 (reproducing both personal and companies' accounts, where Hindu-Arabic numerals are used to make calculations or to number pages); 132–133, 142 (where Roman and Arabic numerals are used to distinguish between currencies), 155 (where the entries in the *Libro maestro* are recorded in Roman numer-

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Hindu-Arabic numerals also appeared in Italian public accounts from the 14th century. Among the archives of the *Estimo* of Florence, it is possible to find Hindu-Arabic numerals within the text of single entries from the middle of the century.[77] The ten digits can be found in the registers of the *Prestanze* from the 1380s in the foliation and in draft calculations carried out on spare spaces (such as on the back cover and on the endpapers).[78] Moving to the following century, the most remarkable public accounting enterprise was the 1427 *Catasto*.[79] The officials who compiled the so-called *campioni* – i.e. the summary reports assessing the wealth of each Florentine household – resorted extensively to Hindu-Arabic numerals. What is more, in the *campioni* the use of the two numeral systems is reversed in comparison with previous examples, with Hindu-Arabic numerals appearing as the main numerical notation.[80]

Between the late 14th and the 15th century we see the first texts published outside of Italy, but never with the continuity or scale of abacus mathematics. The evidence tentatively suggests that the use of Hindu-Arabic numerals in commercial practice remained a virtual Italian monopoly until the late 14th century. The role of the printing press in the diffusion of practical arithmetic outside of Italy is clear. It may be that a considerable amount of early manuscript material outside of Italy has been lost and that, if it were preserved, one would observe a smoother acceleration in production outside of Italy. At the same time, though, we have to consider that there was also a linguistic factor at play until well into the 15th century. Since the Italian abacus tradition is overwhelmingly written in the Italian vernaculars and circulated either in abacus schools or within private collections, it was difficult for non-Italian practitioners to have access to it.

The diffusion of practical arithmetic manuals and schools in central and northern Europe followed a generation of merchants and other practitioners who moved to Italy to gain an education in the arts or in commerce, who there-

als, while the corresponding cheques report values in Hindu-Arabic numerals); 159–160 (where the entries in the *Quaderno dei cambi*, recorded with Hindu-Arabic numerals, are reported in the *Mastro* in Roman numerals).

77 Florence, Archivio di Stato, Estimo Nr. 6: Registro dei pagamenti degli allibrati del quartiere di S. Spirito (1351–1352); Nr. 8: Libro d'entrata dell'estimo Bue Nero, Leon Nero e Ruote (1351–1352).

78 For example: Florence, Archivio di Stato, Prestanze Nr. 945: Registro della 19° prestanza di fiorini pel gonfalone scala (1385).

79 D. Herlihy, C. Klapisch-Zuber, *Tuscans and Their Families: A Study of the Florentine Catasto of 1427* (New Haven: New Haven, 1985).

80 Florence, Archivio di Stato, Catasto 1427. Such widespread resorting to Hindu-Arabic numerals is common to the *campioni* of every *quartiere* as well as *gonfalone*.

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fore were trained in Italy and were familiar with abacus mathematics. The booming central European diffusion of practical arithmetic texts could have occurred at the convergence of these two unrelated phenomena, i.e. the presence of central European practitioners trained in Italy and the introduction of the printing press. Previous research also found that the late 15th century is the period in which the account books of Augsburg started to adopt Hindu-Arabic numerals.[81]

The second stage of the spread spans roughly from 1475 to 1525. This is the phase of the first introduction of the printing press, and of the centres of early adoption of practical arithmetic employing Hindu-Arabic numerals outside of Italy. These early adopters are mainly in southern and central Europe: Lisbon, Zaragoza, Valencia, Barcelona, Paris, Lyon, Cologne, Strasbourg, Oppenheim, Augsburg, Leipzig, Vienna. The third phase spans approximately from 1525 to 1600, when we can observe the introduction and consolidation of the spread in northern Europe. By 1550, it is possible to see the early adopters in this region: Antwerp, Brussels, Hamburg, Lübeck, and the initial production in London. In accordance with this trend, in Antwerp Hindu-Arabic numerals started to appear in private and public accounts starting from the middle of the 16th century, and their use became consolidated from the 1580s.[82]

The lag between the north and the centre of Europe is particularly evident in Germany and the Low Countries. Both areas are divided between a southern group of centres adopting Hindu-Arabic arithmetic by 1525, and a northern group of centres adopting it only from the second half of the 16th century. This lag of the northern part of both Germany and the Low Countries (which is apparent also from a linguistic point of view) is an unexpected finding, given the fact that this was a developed commercial region.[83] It is possible to formulate hypotheses on the reasons why this lag took place. The distinction between

81 Alfred W. Crosby, *The Measure of Reality: Quantification and Western Society, 1250–1600* (Cambridge: Cambridge University Press, 1997), p. 115.

82 Ad Meskens, *Practical Mathematics in a Commercial Metropolis: Mathematical Life in Late 16th Century Antwerp* (Dordrecht: Springer, 2013).

83 As far as the Low Countries are is concerned, whereas in Antwerp (an early adopter) the majority of texts were published in French, all of the 16 texts published in Amsterdam (a late adopter) were in Netherlandish. In Germany, all of the texts published in Hamburg, Lübeck, Szczecin, Gdansk and Königsberg (all late adopters) were published in German. In early adopting centres, instead, there is a good mixture of German as well as Latin texts. Of the 22 texts published in Cologne, 20 are in Latin and only 2 in German. In Strasbourg at least 17 works were published, of which 10 were in Latin and 7 in German. In Leipzig it is possible to find 41 German texts and 37 Latin ones, for a total of 78 publications before 1600. Augsburg, however, does not follow this trend: all of its 26 texts were published in German.

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Catholic and Protestant cities does not correlate neatly with this lag. While it is true that quite a few late adopters (Wittenberg, Szczecin, Hamburg) were cities which turned quickly to the Protestant faith, it is also true that a few early adopters did so as well, such as Leipzig and Strasbourg.

On the institutional level, the most important divide was between Hanseatic and non-Hanseatic cities. Among the adopters in 1500, only Cologne was a Hanseatic city (and a peripherical one), whereas all of the others (Strasbourg, Nuremberg, Bamberg, and Leipzig) had never been members of the League. In 1525 the only Hanseatic city in which practical arithmetic manuals were published was Erfurt (again, a peripherical centre), whereas all of the other centres were not part of the League. Similar considerations apply to the Low Countries, where up to 1525 none of the recorded cities (Brussels and Antwerp) was part of the Hansa. Only in 1550 can we see the first publications in Hanseatic centres (Hamburg, Lübeck, Gdansk), which consolidated in the following years, with a substantial lag on other non-Hanseatic cities of the centre of Germany. For the Low Countries, such a lag is even more evident, with the first publication in the Hanseatic territory appearing in 1563 in Amsterdam.

This lag could be a result of the focus of Hanseatic trade on the North and the Baltic sea, rather than on trade with the south, and it is consistent with research on the history of accounting, according to which the development of accounting techniques was slower in Hanseatic cities.[84] This finding is interesting because it opens the hypotheses of a reluctance among Hanseatic merchants to adopt techniques that were coming from abroad or, alternatively, the hypothesis that southern merchants adopted practices of secrecy, in order to safeguard their commercial techniques.[85] This in turn opens up the question as to what extent commercial communities were aware of their identity and protective of their economic and commercial competences.

There is anecdotal evidence that goes in this direction. In a similar fashion to double-entry bookkeeping, which was often labelled as 'Italian' in bookkeeping manuals, also practical mathematics employing Hindu-Arabic numerals was perceived as a science coming from the south. It is not rare, among practical arithmetic texts written in German, to define Hindu-Arabic numerals as 'welsch' – a blurred term, probably deriving from 'Guelf', and generally mean-

84 Braunstein, *Les Allemands à Venise* (cit. note 55), pp. 444–448. Raymond De Roover, *Business, Banking, and Economic Thought in Late Medieval and Early Modern Europe*, edited by Julius Kirshner (Chicago: University of Chicago Press, 1974), p. 170.

85 Tognetti has recently written of a *conventio ad excludendum* of Tuscan merchant-bankers against their competitors. See Sergio Tognetti, "Le compagnie mercantili-bancarie toscane e i mercati finanziari europei tra metà XIII e metà XVI secolo," *Archivio Storico Italiano*, 2015, *173*/4:687–718.

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ing the territories of the ancient Roman world south of the Alps.[86] I am only touching on this complex theme here, because this evidence might contribute to addressing the question. Moreover, more research is needed to tell whether this gap occurred also in the adoption across commercial communities of new financial instruments, such as the bill of exchange.

England stands out as the latecomer in the publication of practical arithmetics with Hindu-Arabi numerals. This is reflected in a remarkably late adoption of Hindu-Arabic numerals in mathematical practice. For example, as the English university curriculum did not cover practical arithmetic, Samuel Pepys decided to take private lessons of basic arithmetic after his appointment as Clerk of the Acts to the Navy Board.[87] The ten figures started to appear in private English accounts between the last decade of the 16th and the middle of the following century, while public accounts were more reluctant in adopting the new numeral system.[88]

5 Conclusions and Open Questions

The evidence provided in this paper shows that the transition of European practitioners from Roman to Hindu-Arabic numerals was an innovation cycle which started from the Mediterranean in the late 13th century and took over two centuries to complete. Primary and secondary sources suggest that the spread of practical arithmetics correlated with the adoption of Hindu-Arabic numerals in commercial documentation. Even though further research is needed to document more in detail when and where Hindu-Arabic numerals were adopted in commercial practice, this correlation suggests that the tradition of practical arithmetic was a relevant driver for the definite adoption of the positional numeral system in European vernacular contexts.[89]

86 An example is the 1518 *Ayn new kunstlich buech* by Heinrich Ludwig Schreiber (Henricus Grammaticus).

87 Robert Latham, William Matthews (eds.), *Pepys's Diary, Selected and Edited by Robert Latham from the Diary of Samuel Pepys, a New and Complete Transcription* (London: Folio Society, 1996). A. Geoffrey Howson, *A History of Mathematics Education in England* (Cambridge: Cambridge University Press, 1982), p. 12.

88 Hilary Jenkinson, "The Use of Arabic and Roman Numerals in English Archives," *The Antiquaries Journal*, 1926, *6*/3:263–275. Peter Wardley, Pauline White, "The Arithmeticke Project: A Collaborative Research Study of the Diffusion of Hindu-Arabic Numerals," *Family & Community History*, 2003, *6*/1:5–17. Amy Froide, "Learning to Invest: Women's Education in Arithmetic and Accounting in Early Modern England," *Early Modern Women*, 2015, *10*/1:3–26. Otis, "Set Them to the Cyphering Schoole" (cit. note 21).

89 The study of building accounts, which are probably among the best surviving accounting sources, could be an interesting direction to pursue such research.

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It is important to point out that the spread of practical arithmetic manuals does not represent a story of forerunners. Italian merchant-bankers had been active in northern Europe using Hindu-Arabic numerals, bills of exchange and double-entry bookkeeping for at least two centuries when the first practical arithmetic and bookkeeping manuals appeared there in conjunction with the activity of practical arithmetic masters and schools, i.e. when northern European merchants started to adopt these commercial techniques. The spread of the tradition of practical arithmetic manuals is therefore a conservative estimate for the diffusion of a set of skills and practices associated with the commercial revolution, and should be understood as a proxy to observe their consolidated transmission across commercial and linguistic communities.

Moreover, practical arithmetic was arguably the premise for the adoption of other innovations of the commercial revolution, as a basic understanding of arithmetic and the use of an effective mathematical symbolism to handle rational numbers was necessary to use tools and techniques such as the bill of exchange and double-entry bookkeeping. In this sense, the spread of practical arithmetic testifies of the diffusion of a wider 'tacit knowledge', of a set of skills which were not necessarily expressed in the manuals themselves, but which found in practical arithmetic their necessary mathematical premise.[90] The evidence provided suggests that Hindu-Arabic numerals, and the commercial techniques associated with them, remained a virtual Italian monopoly until the 15th century. This is in accordance with the fact that the first non-Italian international merchant-bankers, such as Jakob Fugger, emerged in this period, as well as with anecdotal evidence. However, further research is needed to document this in detail.

This paper shows that it is possible to trace a continuous transmission of practical arithmetic from the late 13th to the end of the 16th century. Econometric research found that the spread of commercial and arithmetic manuals in print had a causal impact on economic development in 16th-century Europe.[91] The evidence provided with this paper suggests the possibility of extending these findings to the origins of the commercial revolution, as it documents a continuous transmission of practical arithmetic texts from the late 13th cen-

90 I am borrowing the concept of 'tacit knowledge' from the social sciences. First formulated by Michael Polanyi, I use the term to identify a knowledge which was not formulated in explicit form but was undoubtedly present. See Michael Polanyi, *The Tacit Dimension* (London: Routledge & Kegan Paul, 1966). Harry Collins, *Tacit and Explicit Knowledge* (Chicago: University of Chicago Press, 2010).

91 Dittmar, Seabold, "New Media and Competition: Printing and Europe's Transformation After Gutenberg" (cit. note 10).

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tury. Even if they were written in the manuscript medium, these texts had a public circulation, as they were produced in remarkable numbers, reached generations of students who attended *ad hoc* vernacular schools, and were written by masters who operated in a competitive environment. This, in turn, suggests the possibility of extending Mokyr's conception of the role of 'useful knowledge' in economic development to the onset of the commercial revolution of the 13th century.[92]

This is also suggested by the characteristics of the spread of practical arithmetic. The analysis provided in this paper shows that practical arithmetic manuals did not spread through maritime, but rather through inland routes. If it is true that the adoption of practical arithmetic is a proxy to capture the spread of practical knowledge and skills that had a significant impact on market development, this form of knowledge did not spread following the hubs of maritime trade. On the contrary, practical arithmetic spread through an inland, incremental, and proximity-based network, that is to say a network which enabled the transfer of knowledge. This is an important finding, which raises the question as to whether a relevant contribution to the shift of the core of European trade from the Mediterranean to the Atlantic came not only from international trade, but also from intellectual exchanges, i.e. from the transmission of practical knowledge and skills. In other words, this finding raises the hypothesis that, together with the expansion of market access, the 'little divergence' was also underpinned by a continuous and tacit transmission of the useful knowledge first developed with the commercial revolution. This hypothesis of an intellectual perspective on Europe's pre-modern economic development is an open question which goes beyond the scope of the present paper, but it suggests a possible direction for further research.

Finally, the reconstruction provided in this paper also allows to appreciate the wider socio-economic context in which major developments in the history of science occurred as a result of the influence and codification of practical knowledge, a subject which has recently attracted the attention of scholars.[93] For example, the first major breakthrough in European algebra since antiquity – the identification of the general solution to cubic equations – was achieved in 16th-century Italy. The protagonists of this discovery – such as Scipione Dal Ferro, Niccolò Tartaglia, and Girolamo Cardano – were middling figures between the world of abacus mathematics and that of the universi-

92 Mokyr himself suggested this possibility, arguing that the economy of the enlightenment "had roots in the commercial capitalism of the later middle ages and the sixteenth century," see Mokyr, *The Intellectual Origins of Modern Economic Growth* (cit. note 5), p. 339.

93 Matteo Valleriani (ed.), *The Structures of Practical Knowledge* (Cham: Springer, 2017).

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ties. These middling figures appeared as a result of the increasingly wide social circulation of abacus mathematics in 16th-century Italy and were able, as a consequence of this intermediate position, to hybridize the practical and the theoretical mathematics of their time.[94]

Moreover, the first developments of what would become European modern algebraic notation were also achieved within the tradition of practical arithmetic.[95] For example, in the late 15th-century Rechenmeister Johannes Widmann, who was the first to give a lecture about algebra in Germany, was also the first to use in print the signs + and –.[96] In the middle of the 16th century, Robert Recorde was the first to use the symbol = as the equals sign.[97] These developments, which occurred as a result of a bottom-up trial and error process which led to a 'plurality' of algebras,[98] were subsequently unified and systematised by figures such as François Viète and John Wallis. From this point of view, the reconstruction of the European tradition of practical arithmetic makes it possible to identify a continuous transmission of practical knowledge which linked Fibonacci to Italian abacus masters, German *Rechenmeisters*, Netherlandish *rekenmeesters*, and English reckoning masters. The latter were, in turn, the first of the so-called 'mathematicall practitioners' of Elizabethan England, who gave an essential contribution to the turn towards quantification and experimentation in 17th-century English natural philosophy,[99] in a hybridization

94 Ettore Bortolotti, *La storia della matematica nella università di Bologna* (Bologna: Nicola Zanichelli, 1947). Veronica Gavagna, "Alcune osservazioni sulla *Practica Arithmetice* di Cardano e la tradizione abachistica quattrocentesca," in *Girolamo Cardano. Le opere, le fonti, la vita*, edited by Marialuisa Baldi, Guido Canziani (Milano: Franco Angeli, 1999), pp. 273–312. Ead., "Radices Sophisticae, Racines Imaginaires: The Origins of Complex Numbers in the Late Renaissance," in *The Art of Science*, edited by Annarita Angelini, Rossella Lupacchini (Cham: Springer, 2014), pp. 165–190.

95 As summarised in Florian Cajori, *A History of Mathematical Notations* (New York: Dover, 1993).

96 Johannes Widmann, *Behende und hübsche Rechenung auff allen kauffmanschaft* (Leipzig: Konrad Kachelofen, 1489). Barbara Gärtner, *Johannes Widmanns 'Behende vnd hubsche Rechenung': die Textsorte 'Rechenbuch' in der Frühen Neuzeit* (Tübingen: Niemeyer, 2000).

97 Robert Recorde, *The whetstone of witte* (London: John Kingston, 1557). See Gordon Roberts, *Robert Recorde: Tudor Scholar and Mathematician* (Cardiff: University of Wales Press, 2016).

98 Karen Hunger Parshall, "A Plurality of Algebras, 1200–1600: Algebraic Europe from Fibonacci to Clavius," *BSHM Bulletin: Journal of the British Society for the History of Mathematics*, 2017, *32*/1:2–16. Maryvonne Spiesser, Sabine Rommevaux, Maria Rosa Massa Esteve (eds.), *Pluralité de l'algèbre à la Renaissance* (Paris: Honore Champion, 2012).

99 Stephen Johnston, "Mathematical Practitioners and Instruments in Elizabethan England," *Annals of Science*, 1991, *48*/4:319–344, p. 341. Jim A. Bennett, "The Challenge of Practical Mathematics," in *Science, Culture and Popular Belief in Renaissance Europe*, edited by

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of practical and theoretical knowledge which echoed the algebraic developments of 16th-century Italy. Also in this case, these developments need further research to be addressed more in detail, but they show some possible continuities which can be identified by following the dissemination of Hindu-Arabic numerals in European vernacular contexts.

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Data appendix

The database records 1280 texts written from the *Liber Abaci* (1202) and 1600, by more than 340 authors active in over 130 cities.[100] Of these texts, 342 are manuscripts and 938 are printed documents. The recorded languages are Castilian, Catalan, Czech, Danish, English, French, German (when possible distinguishing for Low German), Greek, Hebrew, Hungarian, Italian (with several regional variations), Latin, Netherlandish, Polish, Portuguese, and Provençal. Other European languages, like Icelandic, have been excluded because the first practical arithmetic texts in such languages were written after 1600.[101] The

Stephen Pumfrey, Paolo L. Rossi, Maurice Slawinski (Manchester: Manchester University Press, 1991), pp. 176–190: 189.

100 Several texts are written by anonymous authors, hence the estimate on their number.

101 For example, the first Icelandic practical arithmetic was the so-called *Arithmetica Islandica*, written in 1716. The text is preserved in the Iceland National Library, manuscript Lbs. 1694 8vo, fols. 37^r–109^v. See Kristín Bjarnadóttir, "17th and 18th Century European Arithmetic in an 18th Century Icelandic Manuscript," in *History and Epistemology in Math-*

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areas covered by the dataset are Austria, Bohemia, Denmark, England, France, Germany, Greece, Hungary, Italy, the Low Countries, Poland, Portugal, Russia, Spain and Switzerland. As the focus of this paper is on practical arithmetic, works on geometry have not been included in the database.

The catalogue is based both on secondary and on primary sources. The main secondary sources are catalogues. As far as the Italian abacus tradition is concerned, I have included all the evidence recorded in the catalogue edited by Van Egmond, which is still the most comprehensive catalogue on this tradition.[102] However, this catalogue – which is particularly thorough on manuscript sources – does not provide summaries of contents for printed books. Every time it was possible, I have therefore consulted such sources (either by direct inspection or through digitised copies) and I have recorded their contents. I have also integrated the evidence provided by Van Egmond with the several findings of successive research in the field.[103]

As far as the European tradition is concerned, I relied on a variety of sources. The study by Smith is the foundation of the data on the European scale.[104] This monumental work, which is based on a remarkable effort of archival research, had to be addressed with precise criteria as it includes texts which cannot be considered part of a European tradition of practical arithmetic. The first criterium has been excluding works not presenting Hindu-Arabic numerals. Secondly, among the works employing Hindu-Arabic numerals, only the ones which could be considered as belonging to the tradition of practical arithmetic have been included. This means that the early modern reprints of classical as well as early medieval sources have not been included.

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ematics Education: Proceedings of the Sixth European Summer University, edited by Evelyne Barbin, Manfred Kronfellner, Constantinos Tzanakis (Vienna: Holzhausen, 2011), pp. 605–624. Jens Ulff-Møller, "Stefán Einarssons Isländisches Rechenbuch von 1736," in *Visier- und Rechenbücher der frühen Neuzeit*, edited by Rainer Gebhardt (Annaberg-Buchholz: Adam-Ries-Bund, 2008), pp. 215–233.

102 Van Egmond, *Practical Mathematics in the Italian Renaissance* (cit. note 41).

103 Such as Bocchi, *Lo livero de l'abbecho* (cit. note 36). Franci, *Un trattato d'abaco pisano della fine del XIII secolo* (cit. note 51). Ulivi, *Masters, Questions and Challenges in the Abacus Schools* (cit. note 33). Ead., "Su Leonardo Fibonacci e sui maestri d'abaco pisani dei secoli XIII–XV," *Bollettino di Storia delle Scienze Matematiche*, 2011, *31*/2:247–288. Franci, *Leonardo Pisano e la trattatistica dell'abaco in Italia nei secoli XIV e XV* (cit. note 51). Ulivi, *Benedetto Da Firenze* (cit. note 57). Pamela O. Long, David McGee, Alan M. Stahl (eds.), *The Book of Michael of Rhodes: A Fifteenth-Century Maritime Manuscript* (Cambridge; London: MIT Press, 2009).

104 David Eugene Smith, *Rara Arithmetica: A Catalogue of the Arithmetics Written before the Year MDCI, with Description of Those in the Library of George Arthur Plimpton, of New York* (Boston; London: Ginn, 1908).

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This criterium ruled out all the reprints of the *Arithmetica* by Boethius and all the early modern works belonging to such a tradition, together with texts dealing with numerology, and the *computi* for the calculation of the calendar.[105] Thirdly, only works which could have been relevant to practitioners have been included. This means that only texts showing practical applications of mathematics, or that are explicitly addressed to practitioners, or that could have been used as part of a commercial training, have been included. This, in turn, means that purely theoretical works developed within universities, as most of the tradition of the *algorismi*, have not been included. Algebraic and geometric texts have been included only if they presented practical applications (i.e. purely theoretical texts in these fields have been excluded). However, it is necessary to point out that, as the primary material is not clearly divisible according to strict categories, these criteria were assessed against the singular characteristics of the individual texts, which led to the inclusion of some theoretical works and the exclusion of some practical ones.[106]

The information provided by Smith has been checked, depending on the kind of text, on the relevant online repositories. The main ones I have consulted are the Incunabula Short Title Catalogue (ISTC), the Universal Short Title Catalogue (USTC), the English Short Title Catalogue (ESTC), the online catalogue of the Bibliothèque Nationale de France and their digitalisation project (Gallica), the Münchener DigitalisierungsZentrum of the Bayerische Staatsbibliothek, and the Biblioteca Virtual Miguel de Cervantes (Cervantes Virtual). Such research tools, which were not available neither to Smith nor to Van Egmond, allowed to expand considerably the available information on every text, to identify new texts by the same author as well as to identify new authors, to record editions that were missing in previous catalogues, holding institutions

105 The *computi* were texts showing how to calculate the calendar and were widely employed, especially within the Church, for the calculation of specific dates, such as Easter. Hindu-Arabic numerals spread also through this tradition, which in some cases considerably preceded the spread of practical arithmetic. See C. Philipp E. Nothaft, "The Reception and Application of Arabic Science in Twelfth-Century Computistics: New Evidence from Bavaria," *Journal for the History of Astronomy*, 2014, *45*/1:35–60.

106 Just to give a few examples, the already mentioned *De arte supputandi libri quattuor* by Cuthbert Tonstall (London, 1522) has been included despite its primarily theoretical focus, given its importance as the first work entirely dedicated to arithmetic published in England. In a similar way, Petrus Ramus' *Arithmeticae libri tres* (1st ed., Paris, 1555) has been included because of the importance given by Ramus to applied mathematics. See Annarita Angelini, *Metodo ed Enciclopedia nel Cinquecento francese* (Firenze: L.S. Olschki, 2008). Examples of texts employing Hindu-Arabic numerals which, despite being of evident practical relevance, have not been included are the manuals dealing with finger reckoning.

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and classmarks of original copies, and to report every digital reproduction identifiable. When referring to these tools, I have tried to avoid circularity as much as possible, i.e. to include their evidence only if based on information richer or independent from the sources I was already consulting.

This approach (i.e. the stated criteria followed by the consultation of online repositories) has been adopted when dealing with all consulted catalogues. The works quoted in Smith's *Addenda* have been included only if they had clearly been consulted by Smith himself or if the USTC provided more information on them.[107] The *Bibliographia physico-mathematica hispanica* provided precious information on Iberian sources, while the catalogue *Ars Mercatoria* allowed to consolidate the evidence for central and northern Europe.[108] The information provided by these main catalogues has been integrated resorting to an extensive list of specialistic papers and local studies.

Furthermore, I have paid visits to Italian archives in Florence and in Bologna, as well as to archives in the British Library, the University Library of Cambridge and the Bibliothèque nationale de France in Paris, in order to gather first-hand evidence and documentation on early and particularly interesting primary sources of Italian and European practical arithmetic traditions. Every time it was possible, I have checked the contents of the texts. At the present stage of development, I have recorded the contents of 1051 texts (82 % of total), collected either through direct inspection of original or digitised copies, or relying on secondary studies.

As far as the representativeness of the sample is concerned, I can only give tentative considerations. The main issue to estimate the representativeness is that the size of the population (of which the database is a sample) is unknown. In other words, it is difficult to estimate the representativeness of the database because we do not know how many practical arithmetic manuals were actually written in Europe from the 13th to the end of the 16th century. What is possible to say is that, for the early stages of the spread, the Italian abacus tradition surely is the best documented case. The same does not hold for Spain and the south of France, where the number of known manuscript documents is in far smaller quantity than the Italian case. As we have seen, it is possible that several sources from the early stages of Mediterranean-wide mathematical exchanges

107 David Eugene Smith, *Addenda to Rara Arithmetica Which Described in 1908 Such European Arithmetics Printed before 1601 as Were Then in the Library of the Late George Arthur Plimpton* (Boston: Ginn, 1939).

108 Víctor Navarro Brotons, *Bibliographia Physico-Mathematica Hispanica* (*1475–1900*). Vol. 1, *Libros y Folletos,* (*1475–1600*) (Valencia: Universitat de Valencia-C.S.I.C., 2000). Hoock, Jeannin, Kaiser, *Ars Mercatoria* (cit. note 42).

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did not survive to the present day. In other words, the manuscript material of the first stages of the tradition of European practical mathematics may be lost, the best documented case being the Italian one. Among the known surviving manuscript sources of the pre-printing age, the database offers a good coverage.

Entering the age of the printing press, the quantity and quality of available sources sensibly increases. Because of the overwhelming amount of the material produced, I may have missed some texts, but I have resorted to the most reliable repositories currently available and I have tried, every time it was possible, to double-check the recorded information.[109] The sources on which the database is arguably weakest are printing-press-era manuscripts from outside of Italy, as these sources have not been comprehensively studied yet. Nevertheless, as far as European printed texts are concerned, known sources can probably be considered as a representative sample of their respective traditions, and the database provides a good coverage of them. As far as Italy is concerned, the data can arguably be considered representative of both the manuscript and printed traditions.

In conclusion, as far as the Italian tradition is concerned, both manuscript and printed traditions have been studied and are quite well preserved. The database offers a good coverage of both. Moving to the European level, limited early manuscript material has survived, but it has been often studied in detailed and has been recorded. Printing-press-era manuscript material, on the contrary, has not been thoroughly studied and, as a consequence, is possibly not fully recorded in the catalogue. On the other hand, printed European material is abundant, can be considered as a representative sample of its tradition and has been recorded in the database as thoroughly as possible. For all of these reasons, I believe the database can be considered as a representative sample of known sources. In the first analysis of this paper I focus on the periods and areas for which the evidence is stronger.

Following Van Egmond's model, every manuscript document has been considered as an independent text.[110] Following the methodology of *Ars Mercatoria*, printed texts have been recorded at the book-edition level, which means that every reprint has been considered as an independent text.[111] Among

109 Even if the resources used to compile the database cover most of the European area, I believe that the collected data is most solid for German, Italian and English printed texts. This is due both to the quantity of studies published on these areas (Italy) and to the quality of the work of cataloguing and digitisation of primary sources (Germany and England).

110 Van Egmond, *Practical Mathematics in the Italian Renaissance* (cit. note 41).

111 Hoock, Jeannin, and Kaiser, *Ars Mercatoria* (cit. note 42).

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printed documents, the first editions, without counting translations, are at least 310.[112] For every text, I have recorded the following details:

- Year of publication (dated vs estimated)
- Area (ex: Italy, France, England)
- City
- Author
- Title
- Language
- Manuscript vs printed document
- (number of ms. copies)
- (number of reprints)
- (printer)
- Source of the recorded information (both secondary and primary)
- (link to digitized document)
- Holding institution, (classmark)
- Table of contents
- Notes

The main focus of the tradition of practical arithmetic was not on mathematical theory, but rather on its practical applications. As a consequence, practical arithmetic manuals do not follow the deductive structure of the Euclidean model, and are mostly organised into thematic sections comprising lists of solved problems. Theoretical explanations, when present, are given a limited space. Mathematical rules, when present, are generally given without a demonstration at the beginning of their section, and are followed by a list of solved problems in which the rule is applied. All the recorded texts employ Hindu-Arabic numerals and rely extensively on the "rule of three", i.e. the calculation of the unknown in a proportion where three terms are known. The rule of three is used in a wide variety of commercial problems, such as exchange, conversion, interest rates, divisions of shares, alloying, barter, etc.

As argued in this paper, the fundamental role of practical arithmetic was to drive the adoption of Hindu-Arabic numerals in Europe. From this point of view, European practical arithmetic can be thought of as a long process of vulgarisation of the mathematics introduced in the times of Fibonacci. As such, this tradition spread a package of mathematical theory and mathematical applications which can be summarised in the following subjects:[113]

112 Sometimes it is not possible to tell whether a document is a first edition. 310 is the number of the documents clearly printed for the first time.

113 Franci, *Leonardo Pisano e la trattatistica dell'abaco in Italia nei secoli XIV e XV* (cit. note 51).

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TABLE 1 Distribution of manuals' contents, divided for typology (manuscript vs print) and for periodisation

	All documents		Manuscripts		Printed books		Pre-1500		Post-1500	
Total observations	1051	100%	283	100%	768	100%	222	100%	829	100%
Numeration	644	61.27%	27	9.54%	617	80.34%	34	15.32%	610	73.58%
Operations	792	75.36%	112	39.58%	680	88.54%	94	42.34%	698	84.20%
Rule of three	694	66.03%	69	24.38%	625	81.38%	63	28.38%	631	76.12%
Fractions	709	67.46%	145	51.24%	564	73.44%	126	56.76%	583	70.33%
Commercial problems	665	63.27%	244	86.22%	420	54.69%	188	84.68%	477	57.54%
Alloying	293	27.88%	29	10.25%	264	34.38%	37	16.67%	256	30.88%
Algebra	220	20.93%	90	31.80%	130	16.93%	77	34.68%	143	17.25%
Geometry	195	18.55%	127	44.88%	68	8.85%	111	50.00%	84	10.13%
Recreational problems	167	15.89%	79	27.92%	88	11.46%	67	30.18%	100	12.06%
Measures/weights	167	15.89%	64	22.61%	103	13.41%	45	20.27%	122	14.72%
Calendar/Astronomy	69	6.57%	47	16.61%	23	2.99%	43	19.37%	26	3.14%

'Numeration', 'operations', 'rule of three' and 'fractions' stand for the presence a section specifically dedicated to the illustration of such topics.

- Introduction of the positional numeral system
- Operations with integers
- Operations with fractions
- Rule of three (proportion with one unknown term)
- Rule of false position (a method to solve equations with one unknown)
- Mercantile problems (monetary systems, money exchange, weights and measures, division of profit/losses, barter, interest rates, alloying)
- Practical geometry
- Recreational problems
- Algebra

Even if these are the main topics covered by this tradition, not all texts present all of them. There is a high degree of variation among the recorded texts: from collections of solved problems preserved in personal manuscript notebooks to systematic printed treatises. Table 1 summarises the distribution of contents in the recorded texts, distinguishing for typology (manuscript vs print) and for periodisation (pre- and post-1500). These statistics show the strong effect of the technology of the printing press in the specialisation and standardisation of the tradition. Since the manuscript text is a more flexible medium, manuscript material is more heterogeneous and less systematic. The vast majority of manuscript texts deal with the general subject of 'commercial problems'. In comparison with printed texts, manuscript sources are more likely to show a variety of subjects close but not strictly pertinent to practical arith-

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metic, such as weights and measures, algebra, geometry, recreational problems and astronomy.

Printed manuals, on the contrary, consistently show higher percentages of more specific contents, such as sections dedicated to illustrating the positional numeral system, how to carry out the fundamental operations, or to a separate treatment of the rule of three. This growing standardisation is at the origin of modern mathematical primers. Subjects such as weights and measures, algebra, practical geometry, and gauging evolved into specific printed traditions independent of practical mathematics.

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