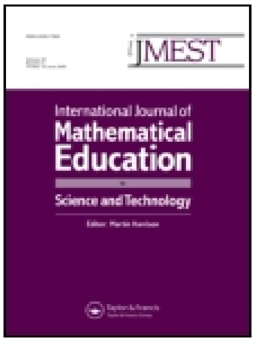

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A brief historical development of classical mathematics before the Renaissance

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'If you wish to foresee the future of mathematics our proper course is to study the history and present condition of the science.'

Henri Poincaré

'It is India that gave us the ingenious method of expressing all numbers by ten symbols, each symbol receiving a value of position, as well as an absolute value. We shall appreciate the grandeur of the achievement when we remember that it escaped the genius of Archimedes and Apollonius.'

P.S. Laplace

'The Greeks were the first mathematicians who are still 'real' to us today. Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. The Greek first spoke of a language which modern mathematicians can understand.'

G.H. Hardy

This article deals with a short history of mathematics and mathematical scientists during the ancient and medieval periods. Included are some major developments of the ancient, Indian, Arabic, Egyptian, Greek and medieval mathematics and their significant impact on the Renaissance mathematics. Special attention is given to many results, theorems, generalizations, and new discoveries of arithmetic, algebra, number theory, geometry and astronomy during the above periods. A number of exciting applications of the above areas is discussed in some detail. It also contains a wide variety of important material accessible to college and even high school students and teachers at all levels. Included also is mathematical information that puts the professionals and prospective mathematical scientists at the forefront of current research.

Keywords: the Hindu–Arabic numerals; Pythagorean theorem and its extension; history of ancient mathematics and Greek mathematicians

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1. Introduction

Mathematics has more than 5000 years of history and first came into existence from the ancient civilizations along the river valleys of Babylonia (or Mesopotamia), China, India and Egypt. About 3000 B.C., the Mesopotamias and the Egyptians civilizations first developed written alphabets and symbols to represent numbers. The Babylonians consisted of a large group of people who occupied an ancient region around and between the Tigris and Euphrates rivers in Mesopotamia which is now part of modern Iraq. A new numeral system with base 60 (sexagesimal system) was developed in Mesopotamia. In addition to elementary arithmetic operations with numbers, the Mesopotamian mathematicians were able to solve special problems in algebra, geometry and astronomy. While this great civilization was growing, it experienced many political changes with new cultural influences. However, its intellectual activity in mathematics and science remained almost unaffected.

Historically, mathematics originated from the fundamentals of counting in Arithmetic and is one of the most ancient, major, useful and fascinating disciplines of human knowledge. The creation of the fundamental idea of numbers including cardinal numbers based on the one-to-one correspondence principle, and the ordinal numbers based on the principle of ordered succession marked the beginnings of mathematics. In ancient civilizations of Mesopotamia, China, India and Egypt, numbers were used for counting, calculating, measuring, and keeping records of information which are still everyday occurrences in all of our lives, even among those with a high level of formal mathematics. From the very ancient civilizations, it has been realized that calculation and measurement are extremely useful for the study of mathematics, astronomy and other disciplines of human knowledge. During the past several millenniums, the ancient civilizations including China, India, Egypt, Babylonia, Greek and Arab kept mathematics fully alive and exciting through their numerous remarkable contributions to arithmetic, algebra, number theory, geometry, trigonometry, mechanics, astronomy and optics. In particular, the Babylonian mathematics provided an abundance of primary source material in the form of inscribed clay tablets, and a major collection of specific problems with solutions. However, they gave some insight of specific rules, physical evidence, trial and error, but no indication of the concept of proof, the idea of general rules and logical structure of mathematical principles. In marked contrast to Babylonians, the Chinese mathematics was more advanced in the sense that they formulated more general rules and logical principles, and often gave formal methods and proofs as well as solutions of particular mathematical problems in a systematic manner.

On the other hand, the Egyptian civilization remained almost unaffected by foreign influences, and it continued to develop arithmetic, algebra and geometry. The name geometry came from the Greek word that means *measurement of the earth*. Originally, it dealt with the measurement of land. The ancient Egyptians utilized geometry to determine the size of their farm lands and to redetermine the boundaries of their farms after the yearly flooding of the river Nile. So, geometry began as a practical tool and still has thousands of uses in every day lives. The Egyptian mathematicians also applied arithmetic and algebra to special problems involving points, lines and planes. It also deals with finding the lengths, areas and volumes of geometric figures of different shapes. Probably, the use of geometry dated back before the dawn of civilizations. The Egyptian architects and engineers designed and built huge temples and pyramids. The major sources of information regarding ancient Egyptian geometry are the Moscow and Rhind papyri, mathematical texts containing 25 and 85 problems, respectively, with approximate dates from 1850 B.C. and 1650 B.C. Among many mathematical formulas discovered by the Egyptian scholars, they formulated the approximate formula of the area of a circle of diameter d in the form $A = \left(\frac{8}{9}\,d\right)^2$ which led to a fairly accurate value for $\pi = 3.1605$. Although there is no documentary evidence that the Pythagoreans formula $c^2 = a^2 + b^2$ for any right-angled triangle was known to the ancient Egyptian mathematicians, however, the Egyptian surveyors were aware of the fact that a triangle with sides three, four, and five is a right-angled triangle. Recent historical study suggests that the ancient Egyptian mathematicians knew that the area of any triangle is equal to half of the product of base and attitude, and the volume of a right circular cylinder is the product of the area of the base and the length of the cylinder. Their remarkable discovery was the correct formula for the volume V of a truncated square pyramid as

$$V = \frac{h}{3}\left(a^2 + ab + b^2\right), \tag{1.1}$$

where a and b are sides of two parallel faces and h is the height. This formula constituted the so called *greatest Egyptian Pyramid*. It is equally remarkable that Egyptian mathematicians also developed geometrical designs and ideas to build the Great Pyramids which involved the golden number in the architecture of Pyramids that will be further discussed in the end of Section 5.

No doubt, the ancient Babylonians and Egyptian mathematicians discovered a great deal of simple and elementary results using empirical methods supplemented by extensive observations without developing any new or deep mathematical ideas, proofs and principles. Everyone is nevertheless struck by the extent and diversity of the mathematical problems successfully solved. However, mathematical achievements of the Babylonians and Egyptians provided a major significant role in the subsequent development of mathematical sciences. Indeed, their works marked the new beginnings of the golden age of Greek mathematics.

2. The classical Indian Hindu Mathematics

The most significant achievement of Indian mathematics during the European Dark Ages was the method of counting by symbols (numerals) for the numbers from one to nine. The first systematic algebra to use positive and negative numbers, zero and the decimal system was developed by Hindu mathematicians in India during the seventh century A.D. They used positive and negative numbers to handle all financial transactions involving credit and debit. All around the world numbers are commonly represented by ten digit symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 which led to *algorithms* for performing the basic operations of arithmetic including addition, subtraction, multiplication and division. Moreover, algorithms are developed for finding square and cube roots, the greatest common divisor (GCD) and least common multiple of two positive integers. This worldwide system is universally known as the *Hindu–Arabic numeral system* that was discovered by the Indian Hindu mathematicians, and then adopted and transmitted by the Arab mathematicians to Western Europe. Subsequently, mathematics has successfully and precisely been used to formulate laws of nature. One of the greatest mathematicians of the eighteenth century, P.S. Laplace (1749–1827) wrote, 'It is India that gave us the ingenious method of expressing all numbers by ten symbols, each symbol receiving a value of position, as well as an absolute value. We shall appreciate the grandeur of the achievement when we remember that it escaped the genius of Archimedes and Apollonius'. Although there was a lot of mathematical activity during the first millennium B.C. in India, there were no textbooks written earlier than the fifth century A.D. The Indian mathematician, Aryabhatta (476–550 A.D.) was the author of one of the oldest textbooks in mathematics. He became famous for the solution of the equation $by = ax + c$ in integers, development of algorithms for finding square and cube roots, discovery of the formula for sums of progressions involving squares and cubes, and computation of $\pi = 3.1416$ (based on the formula for the circumference, $C = 2\pi r$, of a circle of radius r) which was found to be accurate to four decimal places.

Most of the leading Indian mathematicians of the next many centuries, notably Brahmagupta (598–665 A.D.), Sridhara (8th century), Mahavira and Jayadeva (9th century), Bhaskara (1114–1185 A.D.), were mainly concerned with arithmetic, algebra, geometry and trigonometry. They developed formulas and computational methods needed in algebra and astronomy. Brahmagupta became very famous for his well known identity for products of sums of squares,

$$\left(a^2 + b^2\right)\left(c^2 + d^2\right) = (ad + bc)^2 + (ac - bd)^2. \tag{2.1}$$

When $c = d = 1$, this formula reduces to the form

$$2\left(a^2 + b^2\right) = (a + b)^2 + (a - b)^2. \tag{2.2}$$

Brahmagupta's formula for an area of a cyclic quadrilateral is

$$A = \sqrt{(s - a)(s - b)(s - c)(s - d)}, \tag{2.3}$$

where a, b, c, d are the sides of the quadrilateral with the semiperimeter, $s = \frac{1}{2}(a + b + c + d)$.

Sridhara proved a formula for the root of a general quadratic equation in one unknown as it is known today. Using a very simple and elegant method, he solved the quadratic equation

$$ax^2 + bx + c = 0 \tag{2.4}$$

by multiplying it by $4a$ so that it becomes

$$4a^2x^2 + 4abx + 4ac = (2ax + b)^2 - b^2 + 4ac = 0.$$

Or,

$$2ax + b = \pm\sqrt{b^2 - 4ac}. \tag{2.5}$$

This led Sridhara to discover the celebrated *quadratic formula* for the roots of the equation (2.4) as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.6}$$

Mahavira considered the solution of simultaneous equations with linear and quadratic expressions, and empirical rules for computing the areas and perimeter of a circle and an ellipse. Bhaskara first recognized the existence of irrational numbers and formulated rules for the sum of two irrational numbers, and developed a dissection proof for the Pythogorean formula for a right angled triangle. Subsequently, Madhava (1340–1425) and Nilkanta (1444–1545) made considerable progress in infinite series and mathematical analysis which provided some basis for the subsequent discovery of calculus. All of above Hindu Indian scholars paid considerable attention to solving equations, and discovering formulas with little or no rigorous proofs. In spite of their many remarkable achievements in ancient mathematics, they, however, provided no general methods or proofs or new major insight into any area of classical mathematics or astronomy. Subsequently, their work was later fully recognized by Greek mathematicians including Diophantus (250 A.D.) of Alexandria and Claudius Ptolemy (85–169 A.D.) of Alexandria.

3. The Golden Age of Greek mathematics

From his extensive visits in Babylonia and Egypt, the earliest Greek mathematical philosopher, Thales of Miletus (634–547 B.C.), learned mathematics and science and he was considered to be founder of Greek mathematics. It was the Greeks who first recognized the *natural* (or *positive integers* or *counting numbers*) *numbers* $1, 2, 3, \ldots, n, \ldots$, which represent an infinite sequence on which the basic arithmetic operations of addition and multiplication can be performed. Subsequently, real numbers consisting of positive and negative integers, zero, positive and negative rational and irrational numbers were discovered.

Most of our knowledge of classical arithmetic, algebra, geometry and number theory originally came from the writings of many notable mathematicians including Pythgoreas of Samas (580–500 B.C.), Euclid of Alexandria (330–275 B.C.), Archimedes of Syracuse (287–212 B.C.) and Appollonius of Perga (262–200 B.C.). After spending some time with Thales of Miletus, Pythagoras learned a great deal of mathematics and then developed a large body of knowledge in geometry and number theory and proved a large number of geometrical theorems. One of their most famous theorems known as the *Pythagorean Theorem*:

$$c^2 = a^2 + b^2, \tag{3.1}$$

for any right-angled triangle of the hypotenuse c, a and b other two sides. Similarly, the discovery of the classical geometrical proof of the irrationality of $\sqrt{2}$ often attributed to Pythagoras. However, the great Greek mathematical philosopher, Aristotle of Greece (384–322 B.C.) gave a first elegant proof of irrationality of $\sqrt{2}$ by contradiction (*reductio ad absurdum*). The great discoveries of theorem (3.1) and irrational numbers were the great achievements of the Pythagoras in the history of mathematics. The Pythagoreans were credited by Aristotle with the theory in mathematics that 'things themselves are numbers'. They provided mathematical formulation of numerical ratios of musical concords and harmony. Aristotle also said: 'They supposed the elements of number to be elements of all things, and the whole heavens to be a musical scale and number'. Above all, they had unique insights into geometrical structure as well as mathematical reality which was perhaps Pythagoras' true legacy. The great British mathematical philosopher, Bertrand Russell (1872–1970) praised Pythagoras by saying: 'intellectually one of the most important men that ever lived'. In 1971, Nicaragua (one of the countries of Central America) printed a series of commemorative stamps to pay tribute to the World's ten most celebrated mathematical formulas including the Pythagrorean formula, $c^2 = a^2 + b^2$. Each stamp features a particular formula accompanied by a short statement on its reverse side in Spanish language about the significance of the formula. These formulas have definitely contributed far more to human civilization than did many of the Kings, Queens, Presidents and Generals so often featured on commemorative stamps.

It is not out of place to mention works of Aristotle on physics, mechanics, logic, metaphysics and many other diverse fields. He made a serious attempt to formulate the basic principles of mathematics. One of his major achievements was the science of logic from mathematics. He first initiated the study of axioms and postulates. He discussed the fundamental problems of geometry (how points and lines are related) and arithmetic, that the theory of numbers is more accurate because numbers lead to abstraction more easily than the geometrical ideas. The fourth century of Greek mathematics and philosophy was dominated by two giants: Plato (427–347 B.C.) and Aristotle who was a student and colleague of Plato for a period of 20 years. Plato was a great mathematical philosopher of antiquity who provided the unique positive influence on the comprehensive development of arithmetic, geometry, astronomy and music. He firmly believed that mathematics is the key to understanding of eternal truth and reality. His quotation 'God eternally geometrized' revealed his general philosophy that the universe is mathematically designed and must be governed by mathematical laws. He founded the Academy of Athens which consisted of all the brilliant scholars of his time, and it became the leading mathematical centre of the Greek world. Among others, Eudoxus (408–355 B.C.), one of Plato's student at the Academy, was the most brilliant mathematician of the Greek classical period and was second to Archimedes in the whole of antiquity. For a period of three years (343–340 B.C.) Aristotle served as tutor of Alexander the Great (356–323 B.C.), who conquered Greece, Egypt and the Near East as far as India and built his Capital City of Alexandria in 332 B.C. on the Egyptian coast where Asia, Africa and Europe come together. This city was famous for its great Lighthouse, Museum and University, a true centre of advanced teaching, learning and research. It has the largest and most famous ancient library and became a world centre of science, commerce and culture. After being conquered by the Arabs in 642 A.D., the Arabian science and culture became dominant. Subsequently, Alexandria became a major intellectual and cultural centre of the world. Many great Greek mathematicians including Euclid, Appollonius, Archimedes, Hipparchus, Heron, Menelaus, Ptolemy, Diophantus and Pappus were called Alexandrian mathematicians as they continued to display the Greek genius for their works in mathematics, astronomy and science.

After $\sqrt{2}$ had been shown to be irrational, the celebrated Greek mathematician and philosopher, Plato reported that his teacher, Theodore of Cyrene in about 390 B.C. proved that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$, $\sqrt{15}$ and $\sqrt{17}$ are all irrational. Subsequently, many fundamental properties were proved by mathematicians of all Ages. The main distinction between rational and irrational numbers became more clear after the discovery of decimal fractions. It is found that any rational number can be written as either a *terminating* or a *nonterminating periodic* decimal fraction and conversely. For example, $\frac{5}{2} = 2.50$, $\frac{4}{3} = 1.3333\ldots$, $\frac{1}{7} = 0.142857142857\ldots$, and $\frac{47}{22} = 2.1363636\ldots$.

On the other hand, any irrational number can be represented by a nonterminating aperiodic decimal fraction, and conversely. For example, $\sqrt{2} = 1.41421356\ldots$, $\pi = 3.14159265\ldots$, and $e = 2.718281828\ldots$. Of many other properties, the distinction between the decimal representations of rational and irrational numbers became very useful in establishing other properties of these numbers.

Included are some major features of the irrational number, $\sqrt{2}$ to help solve other problems in subsequent sections of this article. The irrational number $x = \sqrt{2}$ can be expressed in terms of approximation of rational numbers

$$x = 1, 1.4, 1.41, 1.414, \ldots, \tag{3.2}$$

so that x^2 can also be written as

$$x^2 = 1, 1.96, 1.9881, 1.999396, \ldots. \tag{3.3}$$

This shows that the sequence of rational numbers tends to 2, and $x = \sqrt{2}$ is a root of the algebraic quadratic equation $x^2 - 2 = 0$. However, $\sqrt{2}$ can be expressed in other forms, such as $\sqrt{2} = |1 - i|$ where $i = \sqrt{-1}$ and $1 - i = 1 - \exp\left(\frac{i\pi}{2}\right)$.

Historically, continued fractions first appeared in ancient mathematics in connection with approximation of irrational numbers by a sequence of rational numbers as shown in the above example for $x = \sqrt{2}$. In general, the Euclid Algorithm also led to the unique continued fraction representation of a real number. More explicitly, every rational number can be represented by a unique *finite* continued fraction, and conversely. Similarly, every irrational number can be represented by an infinite continued fraction, and conversely. For example, $\sqrt{2}$ can be represented uniquely by an infinite continued fraction

$$\sqrt{2} = 1 + \frac{1}{n}, \quad \text{or} \quad n = 2 + \sqrt{2} - 1 = 2 + \frac{1}{n}, \tag{3.4}$$

so that

$$\sqrt{2} = 1 + \frac{1}{n} = 1 + \frac{1}{2 + \frac{1}{n}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{n}}}. \tag{3.5}$$

This process can be continued to generate the simple continued fraction for $\sqrt{2}$ written in compact form

$$\sqrt{2} = [1; 2, 2, \ldots], \tag{3.6}$$

with the first few partial fractions (or convergents)

$$\frac{p_n}{q_n} = \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots. \tag{3.7}$$

It follows that $\frac{17}{12}$ is the best rational approximation of $\sqrt{2}$ with a denominator ≤ 12 with the error estimate inequality

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}}, \tag{3.8}$$

where the error depends on q_n. In the present case,

$$\left| \sqrt{2} - \frac{17}{21} \right| = \left| \sqrt{2} - \frac{p_3}{q_3} \right| \leq \frac{1}{q_3 q_4} = \frac{1}{12 \times 29} < 3 \times 10^{-3}. \tag{3.9}$$

The continued fraction (3.5) is directly connected with the *Diophantine equation*

$$2x^2 - y^2 = \pm 1, \tag{3.10}$$

which is also known as the *Pell equation*. There are infinitely many pairs of integers (x, y) which satisfy (3.10) and the corresponding rational numbers (y/x) can be obtained by truncating the continued fraction (3.5) for $\sqrt{2}$. The first few solutions of (3.10) are (1, 1), (2, 3), (5, 7) and (12, 17) so that

$$\frac{1}{1} = 1, \quad \frac{3}{2} = 1.5, \quad \frac{7}{5} = 1.4, \quad \frac{17}{12} = 1.416, \ldots . \tag{3.11}$$

Replacing ± 1 in Equation (3.10) by zero yields $2x^2 - y^2 = 0$ so that $\left(\frac{y}{x}\right) = \sqrt{2}$. Thus, the sequence of rational numbers given by (3.11) which are alternately smaller and larger than $\sqrt{2} = 1.416$, converges to the limit $\sqrt{2} = 1.416$. The sequence (3.11) contains a list of rational numbers that best approximate $\sqrt{2}$. Finally, the conditionally convergent infinite series

$$\frac{1}{1} - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \cdots \pm \frac{1}{n} + \cdots = \frac{\log\left(1 + \sqrt{2}\right)}{\sqrt{2}}, \tag{3.12}$$

where n takes values of odd integers, and the sign of the term $\frac{1}{n}$ is *plus* or *minus* accordingly as n has a remainder of 1 or 7 when divided by 8; or n has a remainder of 3 or 5.

Most of our fundamental knowledge of the Greek mathematics of the Golden Age came from writings of three great mathematicians including Euclid, Appollonius and Archimedes who lived in the third century B.C. It was Euclid of Alexandria who made the first systematic development of Euclidean geometry in his most remarkable treatise, the *Elements* in 13 volumes. It contained perhaps all of the elementary mathematical knowledge of the Greek classical period. The *Euclid Elements* is probably the most influential mathematical work ever written in terms of his clarity, elegance and mathematical rigour. It contained a large body of mathematical results, theorems and problems covering plane geometry, solid geometry, number theory, arithmetic and algebra. It also covered a wide variety of results with proofs, such as Eudoxus' theory of proportion, Euxodus–Archimedes axiom, fundamental theorem of arithmetic that any integer $n > 1$ can be expressed uniquely as a product of primes, and elegant proof of the fact that the number of primes is infinite, generalization of the Pythagorean theorem known as the *laws of cosine*, a geometric derivation of the formulas for the sum of the first n terms of a geometric progression, many results involving prime and perfect numbers, and the *Euclidean Algorithm*. This was a very remarkable extensive work which primarily dealt with definitions, theorems, method of proofs and the axiomatic as well as deductive approach to mathematics. *Euclid's Elements* was the most organized superior kind of mathematical knowledge that eventually led to the remarkable discovery of non-Euclidean geometry which provided better models of physical space than Euclidean geometry. More than 1000 editions of the *Elements* have appeared since the first printed one in 1482. Its content, and organizational effort have made a tremendous positive impact on the development of both the subject matter and the logical foundation of mathematics. The *Elements* truly represents a permanent legacy of Euclid who was a great Greek mathematician of the Golden Age.

In several volumes of his *Elements*, Euclid described a simple process in elementary number theory which is universally referred to as the *Euclidean Algorithm* for finding the GCD of two or more integers. It can be illustrated by an elementary example. To find the GCD of 67 and 14, we first divide the larger number by the smaller one so that

$$\frac{67}{14} = 4 + \frac{11}{14} \quad \text{or} \quad 67 = \mathbf{4} \times 14 + 11$$

and repeat the process until the remainder is zero:

$$\begin{aligned}
\frac{14}{11} &= 1 + \frac{3}{11} \quad \text{or} \quad 14 = \mathbf{1} \times 11 + 3,\\
\frac{11}{3} &= 3 + \frac{2}{3} \quad \text{or} \quad 11 = \mathbf{3} \times 3 + 2,\\
\frac{3}{2} &= 1 + \frac{1}{2} \quad \text{or} \quad 3 = \mathbf{1} \times 2 + 1,\\
\frac{2}{1} &= 2 + 0 \quad \text{or} \quad 2 = \mathbf{2} \times 1 + 0.
\end{aligned}$$

The GCD is the last divisor, 1 in this example. Thus, $(67, 14) = 1$. The rational number (67/14) is uniquely represented by bold-faced four integers which can also be expressed in a simple *continued fraction* as [4; 1, 3, 1, 2]. Thus, the Euclidean algorithm can be expressed in a precise continued fraction as

$$\frac{67}{14} = 4 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{2}}}}. \tag{3.13}$$

On the other hand, Apollonius was known as the Great Greek Geometer for his outstanding work on geometry of conic sections. The *conics* in eight volumes represent his masterpiece of Greek mathematics. He first discovered that all four conic sections, circle, parabola, ellipse and hyperbola (names given by him) can be constructed by taking sections of a single circular cone by varying the inclination of the cutting plane. His conics contained over 450 theorems and propositions which have been proved by rigorous deductive methods. Included are also many theorems on asymptotes, tangents and axes, construction of conic sections which can intersect in the same plane under certain conditions. There are theorems in the conics that dealt with loci, that is, geometrical configurations consisting of all the points with certain properties. He became very famous for his thorough and systematic investigation of plane curves and their various properties. His conics completely superseded all earlier works on the subject.

One of the very greatest mathematicians of all time, and certainly the greatest of antiquity, was Archimedes. He was born in the Greek city of Syracuse in the island of Sicily. He was considered as the most creative and original mathematician of all ages. He is usually ranked with Sir Isaac Newton (1642–1727), Gottfried Wilhelm Leibniz (1646–1716) and Friedrich Gauss (1777–1855). Unlike his predecessors, Archimedes's work were not compilations or extensions of mathematical achievements, but are truly original creations. He provided unique insights into the ancient Greek mathematics and, at the same time, he made major discoveries in mathematics, mechanics and mathematical physics. His great works on geometry, mechanics and hydrostatics have been preserved as Greek text books. He was also called the *father of experimental science* as he often verified his mathematical findings by experimental observations. His great discoveries of plane and solid geometry can be found in *Quadrature of the Parabola*, *On the Sphere and Cylinder*, *On Conoids and Spheroids*, *On Spirals* and *The Measurement of the Circle*. He discovered the area of a circle, surface area of a sphere and volume of a sphere and of a cylinder. He first inaugurated the classical method of scientific computation for the number π with sufficient accuracy. He also proved the area of an ellipse, but was unable to determine the area of a segment of an ellipse or a hyperbola due to the lack of knowledge of elliptic integrals and elliptic functions. In his most favourite treatise '*On the Sphere and Cylinder*', Archimedes formulated a new geometrical postulate which has become known as the *Postulate of Archimedes: Given two unequal line segments, there is always some finite multiple of shorter one which is longer than the other*. Among many theorems, he proved that the volume of every sphere is two thirds of that of the cylinder circumscribing it. He was so proud of the figure of a sphere inscribed in a cylinder that he left an instruction to engrave this figure on his tombstone. In his treatise, *On Spirals*, he studied the properties of the Archimedian spiral curve with the polar equation, $r = a\theta$. Several works of Archimides have been preserved in Arabic translations, but some of his work was found to be lost. Some Arabian mathematician believed that Archimedes discovered the celebrated formula for the area $\triangle$ of a triangle

$$\triangle = \sqrt{s(s-a)(s-b)(s-b)}, \tag{3.14}$$

in terms of its three sides a, b, c and the semi-perimeter, $s = \frac{1}{2}(a+b+c)$. This formula was found in a later work of Heron of Alexandria (10–50 A.D.) in 60 A.D. Heron made some significant contribution to geometry and mechanics. His geometry was more concerned with measurement than the Euclid geometry. He was the first geometer who made the change of interest in the direction of measurement and numerical procedure in geometry. In marked contrast to Euclid's work, Heron first introduced the concepts of area and volume in Greek geometry and derived the area of a triangle in terms of its sides as stated in (3.14). In his books on *Metrica* and *Geometrica*, Heron developed theorems and rules for plane areas, surface areas and volumes of a large number of geometrical figures. He also first recognized the quantitative nature of Euclidean geometry in contrast to the works of Euclid and Apollonius. In his work *Dioptra*, a treatise on geodesy, Heron published the formula (3.14).

In addition to the above, Heron made numerous changes in *Euclid's Elements* and gave different proofs and added new theorems and their converses with special cases. He continued his work to expand the Egyptian science into quantitative analysis of measurements, and developed approximate formulas for square and cube roots of a number. For example, he wrote a given number $A = a^2 \pm b$, where a is a first approximation of the square root of A and b is the reminder, and first discovered his formula for the approximate square root of A in the form

$$\sqrt{A} = \sqrt{a^2 \pm b} \approx a \pm \frac{b}{2a}. \tag{3.15}$$

This has been further extended to give the lower and upper bounds as

$$a \pm \frac{b}{(2a \pm 1)} \leq \sqrt{A} = \sqrt{a^2 \pm b} \leq a \pm \frac{b}{2a}. \tag{3.16}$$

Indeed, Heron gave general formulation and solved algebraic problems by purely arithmetical procedures. His above example of finding the square root was related to the problem of finding good rational approximation of $\sqrt{A}$ involved in the equation

$$x^2 - Ay^2 = B, \tag{3.17}$$

where A is *not* a perfect square and B is any small number. This equation (3.17) has a long history in the ancient Hindu and Greek mathematics, and is now called the *Pell equation* when $B = 1$ which was known to the ancient mathematics for finding possible integral solutions for (x, y). Obviously, when x and y are large compared with B, then $\frac{x}{y}$ led to good approximation to $\sqrt{A}$. Pierre de Fermat (1601–1665) conjectured that the Pell equation, $x^2 - Ay^2 = 1$ has infinitely many integral solutions for (x, y). Leonhard Euler (1707–1783) found the solutions of the Pell equation for a wide range numerical values of A. However, Brahmagupta proved that if (x, y) is a solution of the Pell equation, then $(x^2 + Ay^2,\ 2xy)$ is also a solution because $(x^2 + Ay^2)^2 - A(2xy)^2 = (x^2 - Ay^2)^2 = 1$.

Historically, three mathematicians including Heron, Diophantus and Nichomachus of Gerasa made an attempt to treat arithmetic in the sense of number theory independent of geometry. In his two-volume famous book entitled *Introductio Arithmetica*, Nichomachus (who lived almost 100 A.D.) developed an independent treatment of arithmetic without geometry. Basically, *Introductio* was the major arithmetical work of the early Pythagoreans, and can be compared with *Euclid's Elements* for geometry. Among others, Nichomachus studied the properties of prime and composite numbers and parallelopipedal numbers of the form $n^2(n+1)$. He also examined odd and even, triangular, square and pentagonal numbers and their properties. The *nth triangular number* is denoted by $\triangle(n)$ and defined by

$$\triangle(n) = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1), \tag{3.18}$$

where $\triangle(1) = 1,\ \triangle(2) = 3,\ \triangle(3) = 6,\ \triangle(4) = 10, \ldots.$ Their first differences form a linear progressions $1, 2, 4, 5, \ldots.$

The *nth square number* $\square(n)$ is defined by

$$\square(n) = n^2, \tag{3.19}$$

where $\square(1) = 1,\ \square(2) = 4,\ \square(3) = 9,\ \square(4) = 16, \ldots.$

Obviously, the sum of two successive triangular numbers is a square number, that is,

$$\triangle(n-1) + \triangle(n) = \square(n). \tag{3.20}$$

The sum of consecutive odd positive integers is a square number, that is,

$$1 + 3 + 5 + \cdots + (2n-1) = n^2. \tag{3.21}$$

The *nth pentagonal number* $\omega(n)$ is defined by

$$\omega(n) = \frac{n}{2}(3n-1), \tag{3.22}$$

where $\omega(1) = 1$, $\omega(2) = 5$, $\omega(3) = 12$, $\omega(4) = 22, \ldots.$

The sum of the $(n-1)$th triangular number and the nth square number is equal to the nth pentagonal number, that is,

$$\triangle(n-1) + \square(n) = \omega(n). \tag{3.23}$$

The sum of n and the three times the $(n-1)$th triangular number is the nth pentagonal number, that is,

$$n + 3\triangle(n-1) = \omega(n). \tag{3.24}$$

In general, *the nth k-gonal number* is defined by

$$f_n(k) = \frac{n}{2}[(n-1)k - 2n + 4] \tag{3.25}$$

so that $f_n(3) = \triangle(n)$, $f_n(4) = \square(n)$, $f_n(5) = \omega(n)$, $f_n(6) = n(2n-1)$ (*hexagonal numbers*), $f_n(7) = \frac{n}{2}(5n-3)$ (*septagonal numbers*) and $f_n(8) = n(3n-2)$ (*octagonal numbers*).

It is easy to check that

$$\triangle(n-1) + f_n(k) = f_n(k+1). \tag{3.26}$$

It is interesting to point out that triangular numbers can be represented geometrically by the number of equidistant points in triangles of increasing size. These points form a triangular lattice. Similarly, square numbers can also be represented by the number of points in square lattices of increasing size [1]. Further, the nth triangular number, the nth square number, the nth pentagonal number and so on form an arithmetic progression with $(n-1)$th triangular number as the common difference.

A positive integer n is called a *perfect number* if the sum of its proper divisors (other than the number itself) is equal to n, that is, $s(n) = n$. One of the Pythagoreans used the Euclid method to generate perfect numbers which remains today the only method to generate perfect numbers. Nichomachus used this method to generate perfect numbers as follows. If

$$1 + 2 + 4 + \cdots + 2^{n-1} = \text{prime} = p(say), \tag{3.27}$$

then $p \times 2^{n-1}$ is a perfect number. For example, $1 + 2 = 3$ which is prime. Then $3 \times 2 = 6$ is perfect because the sum of its divisors 1, 2, 3 is 6. Similarly, $1 + 2 + 4 = 7$ which is also prime so that $7 \times 4 = 28$ which is the next perfect number. Then $1 + 2 + 4 + 8 + 16 = 31$ which is also prime so that $31 \times 16 = 496$ is a perfect number. The next perfect number is $8128 = (1 + 2 + 4 + 8 + 16 + 32 + 64) \times 64 = 127 \times 64$. Thus, the first four perfect numbers, 6, 28, 496, 8128 were discovered by Nichomachus. As of today, there are exactly 41 perfect numbers known.

If n is perfect, then the sum of the reciprocals of all divisors of n is always equal to 2. For example, 6 is perfect and has divisors 1, 2, 3, 6, and hence,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2. \tag{3.28}$$

The number 28 is also perfect with divisors 1, 2, 4, 7, 14, 28, and shares the same property. On the other hand, every perfect number is a sum of a consecutive odd cubes. For example,

$$28 = 1^3 + 3^3, \quad 496 = 1^3 + 3^3 + 5^3 + 7^3. \tag{3.29}$$

Marin Mersenne (1588–1648) had conjectured that there are only finitely many primes of the form $M_p = 2^p - 1$, where p is a prime. These are called *Mersenne primes*. Only 46 Mersenne primes are known today and the 45th Mersenne prime was discovered by Edison Smith in the United States on 23 August 2008 for $p = 43112609$ and the 46th Mersenne prime was discovered by Elvenrich in Germany on 6 September 2008.

In 1732, Euler discovered the 19-digit *eighth perfect number* $P = 2^{31-1}(2^{31} - 1)$ when $p = 31$. In book 1X of his *Elements*, Euclid proved in about 350–300 BC that if $2^p - 1$ is prime, the number

$$P = 2^{p-1}(2^p - 1) \tag{3.30}$$

is *perfect*. Two thousand years later, Euler showed that every *even perfect number* is of this type. Indeed, there are *no* known odd perfect numbers. It is conjectured that all perfect numbers are even. Although this has not yet been proved, some evidence has been found in favour of this conjecture. If an odd perfect number exists, it is known that it must be greater than 10^{300} and have at least nine distinct prime factors.

Nichomachus discovered different kinds of ratios and proportions that play important roles in music, and once he said that they are necessary for 'natural science, music, spherical trigonometry and planimetry and particularly for the study of the ancient mathematicians'.

In his *Arithmetica*, Nichomachus also stated the *sieve of Eratosthenes* which was a method for finding all the prime numbers less than a given number n. He also listed the odd numbers

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots \tag{3.31}$$

and then discovered that the first 1 is the cube of 1, the sum of the next two is the cube of 2, the sum of the next three is the cube of 3 and so on.

There are also two extant treatises of Archimedes on applied mathematics, *On the Equilibrium of Planes* and *On Floating Bodies*. In these works, Archimedes discovered physical postulates together with the axioms and postulates of geometry. It is also interesting to point out that in these works on mechanics and hydrostatics, Archimedes adopted the axiomatic – deductive method and experimental observations. One of the most signficant discoveries of modern times in the history of mathematical sciences, was Archimedes's long lost treatise 'The Method'. In this treatise, Archimedes exploited the 'method' only to discover new results, which he then proved rigorously by his great extension of the Eudoxian method of exhaustion. He also successfully employed the method to prove many results and theorems about the areas and volumes of geometrical figures. It is clear from the above discussion that Archimedes was a creative genius who revolutionized mathematics and mathematical physics with his remarkable contributions. Indeed, the Golden Age of Greek mathematics came to an end with his death in 212 B.C.

We next provide a summary of great works of a galaxy of eminent Greek mathematicians and mathematical astronomers of antiquity including Diophantus, Hipparchus, Menelaus, Ptolemy and Pappus. Diaphantus of Alexandria wrote a great treatise known as *Arithmetica* which contained a large collection of many new but different problems of arithmetic and algebra with his major works and their extensions. He first introduced symbolism in classical algebra and then developed a new analytical treatment in solving both determinate and indeterminate equations with an unlimited number of solutions. His extensive study of indeterminate equations led him to establish a new branch of algebra now known as *Diophantine Analysis* which dealt with the solutions of *Diophantine equations* in two or three unknowns in the form

$$y^2 = ax^2 + bx + c \quad \text{or} \quad y^2 = ax^3 + bx^2 + cx + d. \tag{3.32}$$

His analytical approach to solutions was entirely new and different from his predecessors. Apparently, Diophantus was familiar with Brahmagupta's identity (2.1) in a more general form to construct numbers as the sum of two squares in more than one way. The proof of (2.1) was available in Fibonacci's second book *Liber Quadratorum* of 1225. This book also contained more identities due to Diophantus. For a positive integer n, the properties of quadratic forms $x^2 \mp ny^2$ were studied using the generalized forms of (2.1) as

$$\left(a^2 + n\,b^2\right)\left(c^2 + n\,d^2\right) = (ac \pm n\,bd)^2 + n(ad \mp bc)^2, \tag{3.33}$$

$$\left(a^2 - n\,b^2\right)\left(c^2 - n\,d^2\right) = (ac \pm n\,bd)^2 - n(ad \pm bc)^2. \tag{3.34}$$

Diophantus was the first Greek mathematician who made the conjecture that every integer is a sum of two or three or four squares. He was interested in classifying problems and methods in algebra. In view of his major contributions to both arithmetic and algebra, Diophantus was then known as the 'father' of algebra.

Hipparchus made many notable contributions to spherical trigonometry and astronomy including a *table of chords*, and he was then considered the founder of spherical trigonometry. His original work on astronomy was the discovery of the precession of the equinoxes, construction of instruments for astronomical observations, and mathematical descriptions of the motions of all heavenly bodies including the Sun. Menelaus of Alexandria wrote a three-volume treatise, called *Sphaerica* which contained his significant work on spherical trigonometry and the Greek development of trigonometry. He also proved a chain of theorems of plane and spherical geometry and spherical trigonometry. In particular, he presented many results and theorems for spherical triangles analogous to what Euclid proved for plane triangles.

In the second century A.D., Claudius Ptolemy (85–169 A.D.) of Alexandria became very famous for his major contributions to geometry, astronomy, astrology, geography, optics, plane and spherical trignometry. He is author of the celebrated treatise best known to the world by its Greek–Arabic title of the *Almagest*, meaning the greatest. This great treatise played the same role for mathematical astronomy as *Euclid's Elements* did for geometry. Based on the work of Hipparchus, Ptolemy developed his complete geocentric theory of astronomy with the Earth at the centre of the Universe and also extended the work of Hipparchus by further improving the mathematical descriptions of the motions of all the heavenly bodies. However, Nicolaus Copernicus (1473–1543), the great Polish astronomer who boldly rejected the 1400 year old Ptolemy's geocentric theory of astronomy and discovered the revolutionary modern heliocentric picture of the Universe with the Sun at the centre and the Earth moving around the Sun. One of Ptolemy's many achievements was to construct a *table of chords* which is equivalent to a modern trigonometric sine table. He was able to prove many trigonometric identities for construction of his table. In constructing his table, he made the best use of a geometrical result known as

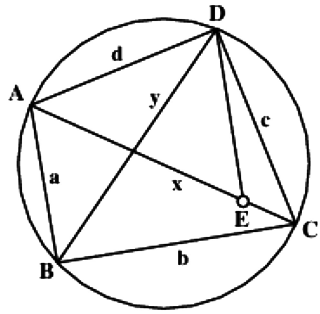

Figure 1. A cyclic quadrilateral $ABCD$.

Ptolemy's theorem which can be stated as follows: in a cyclic quadrilateral, the product of the two diagonals is equal to the sum of the products of the two pairs of opposite sides, and conversely. For its proof, see [2]. If $ABCD$ is a cyclic quadrilateral (Figure 1) with sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ and diagonals $AC = x$ and $BD = y$, then Ptolemy's theorem is $xy = ac + bd$. If a cyclic quadrilateral $ABCD$ is a cyclic rectangle $ABCD$, then $a = c$, $b = c$ and $x = y$, so Ptolemy's theorem reduces to the Pythagorean theorem $x^2 = a^2 + b^2$.

If t is the diameter of the above cyclic quadrilateral $ABCD$, and the angle between either diagonal and the perpendicular upon the other is θ, then the following Ptolemy's results hold:

$$xt\cos\theta = ad + bc, \quad yt\cos\theta = ab + cd, \tag{3.35}$$

$$\frac{x}{y} = \frac{ad + bc}{ab + cd}, \tag{3.36}$$

$$x^2 = \frac{(ac + bd)(ad + bc)}{(ab + cd)}, \quad y^2 = \frac{(ac + bd)(ab + cd)}{(ad + bc)}. \tag{3.37}$$

If, in the above quadrilateral, the diagonals intersect at right angles, then

$$t^2 = \frac{(ad + bc)(ab + cd)}{(ac + bd)}. \tag{3.38}$$

Then Ptolemy's theorem can be generalized for a convex quadrilateral $ABCD$ as follows:

$$ac + bd \geq xy, \tag{3.39}$$

where equality holds if and only if the quadrilateral is cyclic.

Ptolemy further extended his theorem for a convex quadrilateral $A_1A_2A_3A_4$ incribed in a circle C, where C_1, C_2, C_3, C_4 are four circles touching the circle C externally at A_1, A_2, A_3, A_4, respectively, so that

$$a_{12}a_{34} + a_{23}a_{41} = a_{13}a_{24}, \tag{3.40}$$

where a_{ij} is the length of a common external tangent to the circle C_i and C_j. However, Ptolemy's result (3.40) follows as a special case of a more general theorem proved by John Casey (1820–1891).

In addition to his remarkable achievements in geometry and astronomy, Ptolemy made some major contributions to geography, astrology and optics. In his eight-volume work on *Geographia*, Ptolemy first introduced the method of construction of geographical atlas (or maps) and charts, and then determined latitudes and longitudes of over 8000 places on the Earth. He also made a map of the Earth. Based on his own original idea of sterographic projection (i.e. the mapping of the x-y plane onto the surface of the three-dimensional unit sphere which was the x-y plane as its equatorial plane), Ptolemy discovered the *conical projection*, that is, the projection of a region on the surface of the Earth from the centre of the sphere onto a tangent plane.

Motivated by his work on Astronomy, Ptolemy wrote a famous book entitled *Quadripartite* or *Tetrabiblos*, or *Four Books Concerning the Influence of the Stars* in which he formulated rules and procedures for astrological predictions that have successfully been used for over a 1000 years.

Several Greek mathematical scientists initiated the study of optics which deals with the nature of light, vision and colour. They also examined the basic principles of optics including reflection, refraction, absorption and diffraction of light. They investigated the nature of light rays reflected from plane, convex and concave mirrors and the effect of this nature. Special attention was given to the phenomenon of refraction of light, that is, the bending of light rays as they pass through a medium, whose properties change from one point to another, or the sudden change in the direction of a light beam as it passes from one medium into another. Ptolemy also investigated the effects of refraction by the atmosphere as light rays are emitted from the Sun and Stars, and made an unsuccessful attempt to formulate the correct law of refraction of light when light passes from air to water or air to glass or vice versa.

In 320 A.D., Pappus of Alexandria published a comprehensive treatise entitled *Synagoge* (*Mathematical Collection*) which contained a collection of major works of several Greek mathematicians and a large number of theorems, new discoveries and extensions by Pappus himself. He proved several famous results on the complete quadrilateral (four sides and three diagonals) and on other geometrical problems which are still known as *Pappus theorem*. These theorems played the fundamental role for the development of the new subject of projective geometry by Girard Desargues (1591–1661) in the seventeenth century. Pappus made the most remarkable generalization of the Pythagorean theorem in two directions. First, he proved the celebrated law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{3.41}$$

for any arbitrary triangle ABC instead of the right-angled triangle with $C = \frac{\pi}{2}$ so that (3.41) reduces to the Pythagorean formula (3.1). Pappus replaced the right-angled triangle by any triangle ABC and $CADE$ and $CBGF$ are two parallelograms drawn externally on the sides CA and CB. If DE and GF meet externally at P (Figure 2), we draw two parallel and equal lines AM and BL which are also parallel to the extended line PC to meet ML at Q. Then the Pappus generalization of (3.1) is

$$\text{Area of } ABML = \text{Area of } CADE + \text{Area of } CBGF, \tag{3.42}$$

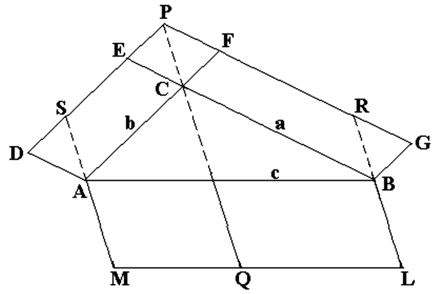

Figure 2. The extension of Pythagorean theorem.

where $ABML$ is a parallelogram drawn externally on the side AB. This is a an elegant generalization of the Pythagorean theorem (3.1), where the squares of the sides of a right-angled triangle are replaced by the areas of the parallelograms.

In 1783, J.P. de Gua de Malves (1712–1785) further generalized the Pythagorean theorem in three dimensions which is known as the *de Gua theorem* which can be stated as follows: the squares of the area of the base of a trirectangular tetrahedron is equal to the sum of the squares of the areas of its other three faces. (See also [3]).

Although Appolonius was universally remembered for his ground-breaking contributions to conic sections, Pappus also did some work on the classification of plane curves. He emphasized the importance of finding solutions by plane or solid figures so that solutions can be used for real world mathematical and physical problems. In his *Mathematical Collection*, he described considerable amount of works of the ancient and the Alexandrian Greeks from Euclid to Ptolemy with many theorems, lemmas, results, discoveries and extensions in many different directions. Pappus was considered the last important geometer in classical tradition.

The above short review reveals that there were several major elements that have probably been responsible for the unique contribution of Greek mathematicians. First, they strongly believed that all mathematical results must be proved by deductive methods, rather than the inductive argument, that is, from the particular to the general. They not only believed their ideas, but also used them in their research in algebra, number theory and geometry. Second, their major emphasis to geometry and geometrical methods in solving mathematical and physical problems provided the modern foundations of mathematical research. Third, they also thought that physical laws can be described by mathematical equations or formulas, and this basic idea is the key to unlock the secrets of nature. Fourth, Plato, one of the great Greek philosophers, became very famous for his first serious attempt to make mathematics abstract in a modern sense. G.H. Hardy (1877–1947), an eminent British mathematician, described eloquently that 'The Greeks were the first mathematicians who are still 'real' to us today. Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. The Greek first spoke of a language which modern mathematicians can understand.'

4. The classical mathematics of the Arab world

Many mathematicians of the Arab world made many contributions to arithmetic, algebra and to a lesser extent, geometry. It is generally true that they received a lot of intellectual nourishment from the works of Greek, Egypt and Indian mathematicians. In their mathematical contributions, the Arab (or Persian) mathematicians preserved the tradition of Greeks, Romans and Indians and translated many of the great works of Greek and Indian mathematicians. The ninth century was usually regarded as the Golden Age of Arab mathematics. At that time, the most notable Arab mathematician was Al-Khwarizmi (780–850 B.C.) who collected and improved the advances in arithmetic and algebra of earlier Indian mathematicians through Arabic translations. His major works were largely based on Arabic translations of Brahmagupta and other Indian mathematicians and provided a full account of the Hindu numerals and the Indian decimal system. Al-Khwarizmi wrote two books – one on arithmetic and the other on Algebra. His Arabic text on arithmetic was lost, but a twelfth century Latin translation entitled *Algorithmi de numero indorum* is extant. This book was mainly based on the Arabic translation of works of Indian scholars with a full description of the Hindu numerals and their basic properties. The Arabic word 'Algorithmi' in the above title subsequently became known as *Algorithm* in Europe. The present word algorithm means a formal mathematical procedure of calculation under certain rules. His second book entitled '*Hisab al-jabr w' almugabala*' (*the science of mathematics*) has given us the word *algebra*, from an English translation of the Arabic word *al-jabr* which means a quantity from one side of an equation to the other side. This famous book contained the translated works of classical Indian mathematics and his own work on algebra. More explicitly, it dealt with different types of linear and quadratic equations together with the basic algebraic rules for performing calculations.

The following example gives al-Khwarizmi's geometrical approach to solve a quadratic equation, $ax^2 + bx = c$ with $a = 1$, $b = 10$ and $c = 39$. He constructed a large square $ABCD$ (Figure 3) of side $(x + 5)$ around a central square of side x, 4 squares at the corners of the large square of each side of length $\frac{5}{2}$ so that the total area of the four corner squares is $4 \times \frac{25}{4} = 25$, and 4 equal rectangles each of area $\frac{5}{2}x$ so that the total area is $4 \times \frac{5}{2}x = 10x$. Thus, the total area of the large square is $(x + 5)^2 = x^2 + 10x + 25 = 39 + 25 = 64$ so that the side length of the central square is $x = 8 - 5 = 3$ which is a positive root of the quadratic equation $x^2 + 10x = 39$. This geometrical method of solution of a quadratic equation led to the development of algebraic methods that eventually turned into a major research topic of solution of equations of the third and fourth degrees in Algebra during the Renaissance period.

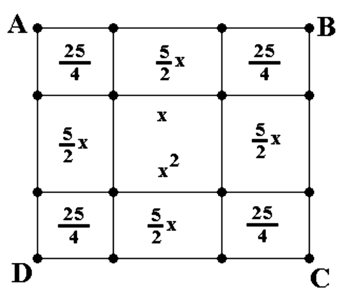

Figure 3. The geometrical approach of al-Khwarizmi to solve the quadratic equation $x^2 + 10x = 39$.

Subsequently, a Persian poet-mathematician and astronomer, Omar Khayyam's (1050–1123 A.D.) book on algebra contained the study of cubic and some biquadratic equations. This work was an expanded version of Al-Khwarizmi's book on algebra. It is important to point out the geometrical treatment of the general cubic equation, $x^3 + ax^2 + bx = c$ with positive coefficients a, b, c so that there is a single positive root x. The classical Delian problem led to the arithmetical formulation of finding the cube root of 2 or equivalently to the solution of the simple cubic equation $x^3 - 2 = 0$ that was solved geometrically by Archytas (428–347 B.C.) in about 400 B.C. His surprising method of solution involved the point of intersection of three surfaces in three-dimensional space: a cylinder, a cone and a torus generated by rotating a circle about one of its tangents. In the fourth century B.C., the Delian problem which dealt with the doubling of a cube was solved by Menaechmus (380–320 B.C.; one of the teachers of Alexander the Great) using his own discovery of the conic sections. The number x such that $x^3 = a^2b$ reduces to finding two numbers x and y such that $\frac{a}{x} = \frac{x}{y} = \frac{y}{b}$, where x and y are called *double proportional means* between two given numbers a and b. Clearly, $x^2 = ay$, $y^2 = bx$ and $xy = ab$ give $x^3 = axy = a^2b$. Geometrically, the point (x, y) is the point of intersection either of two parabolas ($x^2 = ay$ and $y^2 = bx$) or of a parabola and a hyperbola ($x^2 = ay$ and $xy = ab$). When $b = 2a$, x satisfies $x^3 = 2a^3$ and is, thus, the side of the cube with volume twice that of the cube of side a. If $a = 1$, $x^3 = 2$ or $x = 2^{\frac{1}{3}}$ represents the intersection of two conics. Menaechmus was an ancient Greek mathematician and geometer who became famous for his discovery of the conic sections and his solution of the Delian problem.

Khayyam's ingenious geometrical approach to the solution of a cubic equation laid the strong foundation for the Italian mathematicians who, almost 500 years later, finally developed a rigorous algebraic method to solve cubic and quartic equations. Their works led to the formulation of criteria for the possibility or impossibility of solving an algebraic equation by extraction of radicals. It was apparent from the works of Arab mathematicians that algebra and geometry were not essentially different in nature. These works can also be viewed as a nice blending between Greek and Indian mathematics, and provided some definite impact on further development of mathematics. In the course of time, all major mathematical works of Aristotle, Euclid, Archimedes, Diophantus, Heron, Ptolemy and of the Hindu mathematicians became accessible to Arabs in their own language. The Arab mathematicians translated *Euclid's Elements*, Ptolemy's *Mathematical Syntaxis* and *Tetrabiblos* or *Four Books Concerning the Influence of the Stars*, and other works into Arabic. Ptolemy's major contributions to mathematics, astronomy and astrology became well known to the Arabs as the *Almagest*.

On one hand, Arab mathematicians adopted the fully developed Hindu–Arabic numeral system along with all basic algebraic operations: addition, subtraction, multiplication and division including the extraction of square and cube roots. On the other hand, they collected, translated and preserved the classical Greek and Indian mathematics texts and then transmitted to the Western world through Latin translations and eventually to the whole world. By the year 1000, the new Hindu–Arabic numerals had become well known in the Western world. However, it took several years to replace other numeration systems by this new and universal Hindu–Arabic numeral system which finally became the foundation and fundamental of modern mathematics.

5. Progress in mathematics and science of the Medieval period

During the medieval period, the progress in mathematics and science was very slow in terms of new ideas and results. The first brilliant mathematician of this period was Leonardo of Pisa (1175–1250), better known as Fibonacci who was born in Pisa, Italy. He was well-educated in Africa, travelled extensively in different parts of the Mediterranean world from East to West. He began his study through early history of mathematical sciences in China, India, Greek and the Arabian countries. He also realized that there was hardly any progress in mathematics and science in Europe during the Middle Ages.

During his many extended visits to different countries of the world, Fibonacci made an extensive study of Greek, Babylonian, Indian and Arab mathematics and science. After his return to his home in Pisa at the age of thirty, he published his first book entitled, *Liber Abaci* (*Book of Calculations*) in 1202 and his second book, *Liber Quadratorum* (*Book of Squares*) in 1225. These were the earliest and influential text books on arithmetic and algebra ever written in Europe during the Middle Ages. His first book was essentially concerned with the new *Hindu–Arabic numerals* and their rules of computation. It also contained a large number of various problems of different kinds including his famous 800-year old rabbit population problem which led Fibonacci to discover in 1202 a new sequence $\{F_n\}_{n=1}^{\infty}$ of numbers, known as the *Fibonacci numbers*,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots, F_{n-2}, F_{n-1}, F_n, \ldots. \tag{5.1}$$

Thus, the Fibonacci numbers F_n are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 3 \tag{5.2}$$

with the initial values, $F_1 = 1$ and $F_2 = 1$.

Surprisingly, the Fibonacci numbers are found to occur in a wide variety of unexpected situations. Indeed, their occurrence is very common in nature including images of flowers and of fruits and vegetables. There are many examples of biological growth involving the Fibonacci numbers such as branches in trees, the reproduction of bees, the pattern of petals in many flowers and plants. They also form the number of leaves and seed grains of many plants. These numbers are found to arise in computer science and artificial languages. Another area in which the Fibonacci numbers have found useful applications is that of efficient sequential search algorithms for unimodal functions. The Fibonacci sequence has also been effectively used for the study of ladder networks of identical interacting cells, and of periodically loaded transmission lines. A unique and beautiful spiral pattern is observed in mature sunflowers which also display Fibonacci numbers. In particular, it is seen that the sunflower has $F_{10} = 55$ spirals in one direction and $F_{11} = 89$ spirals in the other direction representing fairly large Fibonacci numbers. The scale patterns on pineapples and pine cones provides excellent examples of Fibonacci numbers. The scales of pineapples are, indeed, hexagonal in shape. Fibonacci numbers are found to occur in music and musical instruments, particularly, the keyboard of a piano provides a beautiful visual example of the link between Fibonacci numbers and music. For more similar examples and applications, see [4]. Over the centuries, mathematical scientists became seriously interested in the study of the Fibonacci numbers and other related numbers with applications. It can be shown in different ways that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha, \tag{5.3}$$

where $\alpha = \frac{1}{2}(1 + \sqrt{5})$ is called the *golden ratio*.

One of the easiest proofs follows from definition (5.2), that is, $F_{n+1} = F_n + F_{n-1}$ and

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n}, \tag{5.4}$$

provided these limits exist so that

$$\alpha = 1 + \frac{1}{\alpha}, \quad \text{or} \quad \alpha^2 - \alpha - 1 = 0, \tag{5.5}$$

where the positive root of (5.5) is the golden ratio, α.

Historically, it is very fascinating that golden ratios and golden numbers have been frequently observed in ancient and modern Arts, Greek paintings and sculptures, Greek pottery, the ancient mosaic patterns in Syria, architectural design, furniture design and pictorial arts. The golden rectangles were also found in famous paintings of a French painter, Georges Seurat (1859–1891). The famous paintings of Leonardo da Vinci (1452–1519) contained the golden ratios and the golden rectangles which he inherited from the ancient Greek paintings and sculptures. The *Parthenon* was built at Athens, Greece in honour of Athena Parthenos, the patron goddess of Athens. It is a magnificent monument which contained the golden rectangles and the golden ratios. The golden rectangles were extensively used in the construction of the magnificent *Cathedral of Chartres* and the *Tower of Saint Jacques* in Paris, France. The beautiful statue of a seated Buddha (563–483 B.C.) also displayed the golden proportions and it fits exactly into a golden rectangle. In fact, many proportions of the human body exhibit the occurrence of the golden ratios and the golden rectangles.

Many modern artists and painters have extensively used golden ratios and golden rectangles in modern abstract art, architecture, music and in their dynamic paintings. They recognized that the golden ratios are not only aesthetically pleasing of all proportions, but also enhance beauty in art and paintings. It is now generally recognized that golden ratios and golden rectangles are fundamental elements in an art technique, known as *dynamical symmetry* in paintings and sculptures.

Some historical evidence revealed that the Great Pyramids of Egypt were built around 4750 B.C. With a typical pyramid of base side $2a$, height h and slant height l, these giant pyramids were so constructed that the area of one the inclined faces is equal to the square of the height so that $al = h^2$. Since $l^2 = h^2 + a^2$, it follows that $l(l - a) = a^2$ or $\frac{l}{a} = g = \alpha$, the golden number. With the slant angle θ, $\cos\theta = \frac{a}{l}$ and $\sec\theta = \frac{l}{a} = g$ so that $\theta = 51^\circ 51'$. It is a kind of pleasant surprise that the golden number was used in the architecture of all Pyramids of Egypt so that they have slant angles of about $51^\circ 10'$ and $53^\circ 10'$. From a mathematical point of view, this close agreement may be coincidental or may be highly significant information in the history of mathematics.

During the medieval period from 400 to 1100, the progress in both mathematics and physical science was rather slow. Some attempts were made to resolve the difficulties in Aristotle's work on mechanics and astronomy by medieval scholars including Nicole Oresme (1323–1382) and Jean Buridan (1300–1360). Perhaps their most significant contribution was to develop quantitative theory of mechanics to replace the qualitative arguments of Aristotle. The medieval scientists made more progress in the field of optics than that of other areas of physical sciences. They formulated basic laws of light including travelling of light in a straight line in a uniform medium, the laws of reflection and refraction of light. In addition, they made some progress in spherical aberration, the uses of lenses, function of the eye, atmospheric refraction, theory of rainbow and dispersion of light under refraction, that is, formation of colours from white light passing through a hexagonal crystal. Consequently, mathematics began to play a prominent role in order to study physical sciences.

6. Concluding remarks

The Renaissance began in the city of Florence in Italy around 1300 and then rapidly spread throughout Europe during the 1400s and 1500s. The revival of human learning and culture in the West took place in the Renaissance Period first with a wide range of creative activity in architecture, sculpture, paintings, arts and literature, and then with mathematics and science in the sixteenth century. During this period, the major progress was in algebra and the theory of equations. However, considerable attention had been given to the study of geometry, trigonometry, probability theory, mechanics and astronomy. Based on the Hindu–Arabic numeration system, the mathematical activity in the Western Europe continued to grow rapidly during the Renaissance period. The progress in mathematics was significantly influenced by the mathematical works of Arabs, Hindus, Greeks and others as these works, at that time, were available in Latin translations and increasingly in the form of printed books. In other words, the Renaissance mathematicians received tremendous inspiration from Arabic, Indian, Greek, ancient and medieval mathematics and science.

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