An Early Reference to Division by Zero

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tinue most indefinitely giving examples and properties of lattices. But I hope that the preceding examples will give some idea of the generality and simplicity of the notion of a lattice. In order to round out the picture, I shall conclude by mentioning two other lattices which play basic roles in function theory and logic.

The continuous real functions $f(x)$ of a real variable form a lattice, if one defines $f \leqq g$ to mean $f(x) \leqq g(x)$ for all x. Here $f \cap g$ is the function $h(x)$ which, for all x, is the lesser of $f(x)$ and $g(x)$; we define $f \cup g$ dually. Thus $f \cup -f$ is the *absolute* $|f(x)|$ of $f(x)$.

Finally, in mathematical logic, attributes (good, rich, female, *etc.*) form a lattice. Here $p \leqq q$ is interpreted to mean p is *implied* by q, $p \cap q$ to mean p *or* q, $p \cup q$ to mean p *and* q. In this lattice (often called a *Boolean algebra*), a fundamental role is also played by the operation p' (meaning *not* p). It satisfies

L7. $(p')' = p$, $p \cap p' = 0$, $p \cup p' = I$,

$$(p \cap q)' = p' \cup q', \quad \text{and} \quad (p \cup q)' = p' \cap q'.$$

It is noteworthy that the same laws are satisfied by sets, if one lets X' denote the *complement* of X (set of points not in X); further, they are satisfied in projective geometry if one lets x' denote the *polar* of x. In fact, it can be proved that the chief difference between projective geometry and Boolean algebra (or the algebra of logic) is that the distributive laws L6′–L6″ of Boolean algebra must be replaced in projective geometry by the weaker *modular* law.

L5. If $x \leqq z$, then $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.

Since $x \leqq z$, clearly $x \cup z = z$, so that the conclusion of L5 assumes the *self-dual* form $x \cup (y \cap z) = (x \cup y) \cap z$; thus it is also equivalent to $(x \cup y) \cap z = (x \cap z) \cup (y \cap z)$.

The self-dual modular law L5 is important for another reason. It is satisfied by the normal subgroups of any group, the ideals of any ring, *etc.*, and may be made the basis of most of the known decomposition theorems of modern algebra. But this is a long story!

AN EARLY REFERENCE TO DIVISION BY ZERO

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1. The Hindu legend. In this MONTHLY for 1924 we read* that Brahmagupta, writing in 628 A.D., was the first person to speak of division by cipher, although he gave no quotient. We read further that Bhaskara in 1152 termed such a quotient infinite. This is a reiteration of statements which one can find in almost any history or reference work in mathematics.† It is but one aspect of the impression generally prevailing that "the arithmetic of zero is entirely the Hindu contribution to the development of mathematical science. With no

* H. G. Romig, "Early history of division by zero," this MONTHLY, 31, 1924, 387–389.

† See Encyclopédie des sciences mathématiques, I, 1[1], 1904, p. 33; Moritz Cantor, Vorlesungen über Geschichte der Mathematik, I, 2nd ed., Leipzig, 1894, p. 576; H. T. Colebrooke, Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brahmagupta and Bhaskara, London, 1817, pp. 137–138, 339–340; B. Datta and A. N. Singh, History of Hindu Mathematics. A Source Book. Part I. Numeral Notation and Arithmetic, Lahore, 1935, pp. 238–243.

other early nations do we find any treatment of zero."* Such categorical assertions are serious misrepresentations of the true situation.

It is now well known that symbols for empty places in the positional representation of numbers—corresponding in this respect to our zero—were used by the Babylonians, the Greeks, and the Mayas. Such use antedates historical evidence for the Hindu zero by some thousand years. Nevertheless, such symbols did not appear alone, or independently of other ciphers or digits, to represent zero as itself a number; they were simply part of the representations of positive integers and fractions. Hence the adoption of such symbolisms does not necessarily refute the Hindu claims of priority to the use of zero in this independent sense. Nevertheless, the numerational concept of zero—as distinct from a symbol for empty places—was also familiar to the ancient Greeks. This idea frequently is attributed to Plato,† but such ascription rests largely on gratuitous inferences derived from his sybilline phraseology. In the works of his pupil Aristotle, however, there appears a clear-cut reference not only to the numerical notion of zero but also to the result of division by zero. Historians of mathematics seem generally to have overlooked‡ this significant passage in the *Physics* which anticipates the earliest extant Hindu evidence in this connection by almost a millenium.

2. Aristotle's law of motion. A study of moving bodies had led Aristotle to conclude§ that for a given force or impulse the velocity was in all cases inversely

* Bibhutibuhusan Datta, "Early history of the arithmetic of zero and infinity in India," Bulletin of the Calcutta Mathematical Society, XVIII, 1927, 165–176. See articles by the same author in this MONTHLY for 1926 and 1931.

† See John Burnet, Greek Philosophy. Thales to Plato, London, 1932, pp. 320–322, 330; A. E. Taylor, Plato. The Man and his Work, new ed., New York, 1936, pp. 505–506.

‡ The passage is noted quite incidentally in a philosophical treatment by Albert Görland, Aristoteles und die Arithmetik, Marburg, 1898, p. 18, with no indication of its mathematical significance. There are numerous other works on Aristotle's general mathematical thought, and more particularly on his concept of number and his notion of infinity; but these appear to contain no reference to his statement on division by zero. See, for example, Max Dehn, "Raum, Zeit, Zahl bei Aristoteles, vom mathematischen Standpunkt aus," Scientia, LX, 1936, 12–21, 69–74, supp. pp. 1–9, 32–36; Abel Burja, "Sur les connoissances mathématiques d'Aristote," Preussische Akademie der Wissenschaften, Berlin, Mémoires de l'académie royale des sciences et belles lettres, 1790–1791, pp. 257–276; J. L. Heiberg, "Mathematisches zu Aristoteles," Abhandlungen zur Geschichte der mathematischen Wissenschaften, 18, 1904, 1–49; Gaston Milhaud, "Aristote et les mathématiques," Archiv für Geschichte der Philosophie, 16, 1903, 367–392; Paul Mansion, Aristote et les mathématiques," Revue de Philosophie, 3, 1903, 832–834; Abraham Edel, Aristotle's Theory of the Infinite, New York, 1934. I have not had an opportunity to examine the following: Josephus Blancanus, Aristotelis Loca Mathematica ex Universis Ipsius Operibus Collecta, et Explicata, Bononiae, 1615; P. J. Monzó, De Locis apud Aristotelem Mathematicis, Valencia, 1556; Leo Reiche, Das Problem des Unendlichen bei Aristoteles, Breslau, 1911; Julius Stenzel, Zahl und Gestalt bei Plato und Aristoteles, Leipzig, 1924; R. Stölze, Die Lehre vom Unendlichen bei Aristoteles, Würzburg, 1882. The excellent source book by Ivor Thomas, Selections Illustrating the History of Greek Mathematics, 2 vols., Cambridge, Mass., 1939–1941, devotes more than a dozen pages to Aristotle, but does not include this material. So far as I am aware, there is no reference to this point in any history or textbook of mathematics.

§ That this notorious inference is incorrect need not concern us here; but it should perhaps be noted that only in the time of Newton was a more accurate study made of the difficult problem proportional to the density of the medium—such as air or water—through which the object moved. In modern symbolism this principle may be represented in the form of an equation, $v=k/\delta$, where v, the speed, is a function of δ, the density of the medium, and k is a factor of proportionality. Aristotle then raised the question as to what would be the speed of a body in a vacuum—that is, when $\delta=0$. In answer he wrote:

> Now there is no ratio in which the void is exceeded by body, as there is no ratio of 0 to a number. For if 4 exceeds 3 by 1, and 2 by more than 1, and 1 by still more than it exceeds 2, still there is no ratio by which it exceeds 0; for that which exceeds must be divisible into the excess + that which is exceeded, so that 4 will be what it exceeds 0 by +0. For this reason, too, a line does not exceed a point—unless it is composed of points!*

3. Impossibility of division by zero. In this quotation it is quite evident that Aristotle had the arithmetical zero in mind. It is clearly distinguished from the philosophical void, and it is regarded as akin to, although not actually one of, the numbers. Zero is looked upon as bearing to number the same relationship as does a point to a line. Moreover, the impossibility of division by zero is here explicitly stated almost fifteen hundred years before the time of Bhaskara. The argument by which Aristotle excluded division by zero is based largely on the traditional meanings of words, and hence it differs from the modern point of view. But there can be no doubt as to his correct understanding of the situation. If division by zero were possible, then the result would exceed every integer. That this is Aristotle's position is clear from his inference with respect to motion in a vacuum:

> But if a thing moves through the thickest medium such and such a distance in such and such a time, it moves through the void with a speed beyond any ratio.†

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of the influence of a resisting medium. Indeed, the assumption of Aristotle was applied to the velocity of light both by Descartes in illustrating the law of refraction and by Fermat in demonstrating that the principle of "least time" satisfies this law.

* Physica IV. 8. 215b. The translation is taken from The Works of Aristotle, ed. by W. D. Ross and J. A. Smith, 11 vols., Oxford, 1908–1931, vol. II. The same translation is found in The Basic Works of Aristotle, ed. by Richard McKeon, 1941. This rendering of the Greek is misleading, however, in the use of symbols for numbers and operations. Aristotle did not here use the Greek symbol *o* for zero, but rather the word *οὐδεν*, corresponding to the Hindu *sunya*, "void," and the Arabic *sifra* (whence our *cipher*). In this respect the translation given by P. H. Wicksteed and F. M. Cornford, Aristotle, The Physics, 2 vols., London, 1929–1934 is somewhat more cautious, if also more diffuse: "But the nonexistent substantiality of vacuity cannot bear any ratio whatever to the substantiality of any material substance, any more than zero can bear a ratio to a number. For if we divide a constant quantity c (that which exceeds) into two variable parts, a (the excess) and b (the exceeded), then, as a increases, b will decrease and the ratio $a:b$ will increase; but when the whole of c is in section a there will be none of c for section b; and it is absurd to speak of 'none of c' as 'a part of c.' So the ratio $a:b$ will cease to exist, because b has ceased to exist and only a is left, and there is no proportion between something and nothing. (And in the same way there is no such thing as the proportion between a line and a point, because, since a point is no part of a line, taking a point is not taking any of the line.)"

† *Ibid.* The translation is again that of Ross and Smith. Wicksteed and Cornford render it:

He easily establishes this fact through a reductio ad absurdum: If there were a a ratio, it would follow that a body in a given time would traverse a certain distance whatever the density of the medium, a result which contradicts the principle of the premise,* $v = k/\delta$.

4. Aristotle vs. Bhaskara. It is illuminating to compare these passages from Aristotle with the much vaunted statement on the subject of division by zero given by Bhaskara:

> Statement: Dividend 3. Divisor 0. Quotient the fraction 3/0.
>
> This fraction, of which the denominator is cipher, is termed an infinite quantity.
>
> In this quantity consisting of that which has cipher for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth.†

A comparison of the above quotations from Aristotle and Bhaskara will show that the view of the former comes closer to the modern attitude than does the much more recent dictum of the latter. In contemporary mathematics it is indeed customary to say that the quotient of any number (other than zero) divided by zero is infinity. This language, however, is conventionally interpreted as signifying two things: first, that in this case the quotient is not defined and hence the indicated division is impossible; second, that as a variable divisor tends indefinitely closely toward zero, the (non-zero) dividend remaining constant, the quotient increases without limit. Both of these aspects are present in Aristotle's statement, the former categorically stated and the latter clearly implied.‡ Bhaskara, on the other hand, did not assert the impossibility of division by zero; nor is there in his form of expression any indication of the indefinite increase of $3 \div x$ as x tends towards 0. He looked upon $3 \div 0$ as a number having unusual properties, but very definitely a fixed quantity, much as Wallis in 1665 boldly but uncritically wrote $1/0 = \infty$. Bhaskara's further remark that 3/0 remains unchanged, whatever be added to or taken from it, is sometimes interpreted as a significant anticipation of the properties of the modern infinite; but it must be remarked that the contemporary viewpoint of infinity is con-

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"But if movement through the finest medium covers a given distance in a given time, movement through the void is out of all proportion."

* It was this obvious contradiction which led Aristotle to deny the existence of the void. This fact indicates that the Peripatetic doctrine, "Nature abhors a vacuum," was not based on animistic or even teleological notions, but was instead a logical consequence of a physical principle which science ultimately found cause to reject.

† Colebrooke, *op. cit.*, pp. 137–138.

‡ The implication in this connection is more clearly expressed in the Wicksteed-Cornford translation than in that of Ross and Smith. See note above.

cerned with infinite sets and one-to-one correspondence, or with improper geometrical elements, rather than with magnitudes and equality in the ordinary sense. Hence Bhaskara's naïve assumption (typical of Hindu arithmetic) that division by zero results in a quasi-theological ineffable fixed infinity is less to be admired than Aristotle's critical recognition (characteristic of Greek thought) of the impossibility of division by zero.*

5. Judgment suspended. That Aristotle was the first person to discuss this question is not unlikely, but this cannot be held as established. One may perhaps be permitted to conjecture from the very casualness of Aristotle's language in this connection that the arithmetic of zero was more or less well known in his day. However, there seems to be no other extant reference to division by zero before the time of Brahmagupta. As a matter of fact, the Greeks, Aristotle included, did not look upon zero as a number in the strict sense of the word, for this status was reserved exclusively for the natural numbers.† Statements on the arithmetic of zero are rare in ancient times,‡ and indeed zero was not fully accepted in algebra until the sixteenth and seventeenth centuries.§ Aristotle himself seems never to have had occasion to consider the quotient of zero divided by zero. The earliest, but quite inadequate, consideration of this problem traditionally has been ascribed to Brahmagupta:

> Positive, divided by positive, or negative by negative, is affirmative. Cipher, divided by cipher, is nought. Positive, divided by negative, is negative. Negative, divided by affirmative, is negative. Positive, or negative, divided by cipher, is a fraction with that for denominator: or cipher divided by negative or affirmative.||

Tradition in this particular respect may prove to be trustworthy, but it necessarily must be rejected with respect to the more general problem. Historical evidence points to Aristotle, rather than to Brahmagupta, as the one who first considered division by zero.

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* Incidentally, one finds in Aristotle's Physica a discussion of the infinite which, while differing markedly from the modern treatment, shows an admirable grasp of the problems.

† The integers one and two frequently were excluded, together with zero, from the realm of number. This situation, however, does not alter the argument of this paper. That the Greeks had a thorough knowledge of the arithmetic of ratios and of the integers one and two—even though these were not strictly defined as numbers—indicates that a corresponding arithmetic of zero was quite possible, notwithstanding the nice distinctions which placed zero beyond the scope of the word number.

‡ Nicomachus (c. 100 A.D.) in his Arithmetica referred only once, and then somewhat equivocally, to zero in saying that the sum of nothing added to nothing is nothing. See Nicomachus of Gerasa, Introduction to Arithmetic, transl. by M. L. D'Ooge, with Studies in Greek Arithmetic by F. E. Robbins and L. C. Karpinski, New York, 1926, pp. 48, 120, 237–238. Cf. Johannes Tropfke, Geschichte der Elementar-Mathematik, Vol. II, Leipzig, 1921, pp. 56–59.

§ See Tropfke, *loc. cit. Cf.* Encyclopédie des sciences mathématiques, I, 1^1, p. 133, note 147; also G. Eneström, "Über die Anfänge der Benutzung von Null als eine wirkliche Grösse," Bibliotheca Mathematica (3), VII, 1906–1907, 309.

|| Colebrooke, *op. cit.*, pp. 339–340.