CHAPTER 7

# Diagonalization and Quadratic Forms

CHAPTER CONTENTS

*In each part of Exercises 1–4, determine whether the matrix is orthogonal, and if so find it inverse.*

**1. a.** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

**Solution**

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I$$

$$\text{and } A^T A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I$$

therefore $A$ is an orthogonal matrix;

$$A^{-1} = A^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**3. b.** $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

## Solution

$$AA^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = I \quad \text{and } A^T A = I$$

therefore $A$ is an orthogonal matrix

$$A^{-1} = A^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

## Exercise Set 7.2

*In Exercises 1–6, find the characteristic equation of the given symmetric matrix, and then by inspection determine the dimensions of the eigenspaces.*

**2.** $\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$

**Solution**

$$\begin{vmatrix} \lambda-1 & 4 & -2 \\ 4 & \lambda-1 & 2 \\ -2 & 2 & \lambda+2 \end{vmatrix} = \lambda^3 - 27\lambda - 54 = (\lambda-6)(\lambda+3)^2$$

The characteristic equation is $\lambda^3 - 27\lambda - 54 = 0$ and the eigenvalues are $\lambda = 6$ and $\lambda = -3$.

The eigenspace for $\lambda = 6$ is one-dimensional; the eigenspace for $\lambda = -3$ is two-dimensional.

## Orthogonally Diagonalizing an $n \times n$ Symmetric Matrix

**Step 1.** Find a basis for each eigenspace of $A$.

**Step 2.** Apply the Gram–Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

**Step 3.** Form the matrix $P$ whose columns are the vectors constructed in Step 2. This matrix will orthogonally diagonalize $A$, and the eigenvalues on the diagonal of $D = P^T AP$ will be in the same order as their corresponding eigenvectors in $P$.

*In Exercises 7–14, find a matrix P that orthogonally diagonalizes A, and determine* $P^{-1}AP$.

**8.** $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

## Solution

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 4)$ therefore $A$ has eigenvalues 2 and 4.

so that the eigenspace $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_1 = 2$

so that the eigenspace $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_2 = 4$

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing

since the three vectors are already orthogonal.

the columns of a matrix $P$ that orthogonally diagonalizes $A$:

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^{-1}AP = P^{T}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

**12.** $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**Solution**

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ -1 & \lambda - 1 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda - 2)$ therefore $A$ has eigenvalues $0$ and $2$.

the eigenspace corresponding to $\lambda_1 = \lambda_2 = 0$ $\quad \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

the eigenspace corresponding to $\lambda_3 = 2$ $\quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing since the three vectors are already orthogonal.

the columns of a matrix $P$ that orthogonally diagonalizes $A$:

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}.$$

We have $P^{-1}AP = P^T AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$

**11.** $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

**Solution**

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda-3)^2$$

$A$ has eigenvalues $3$ and $0$.

the eigenspace corresponding to $\lambda_1 = \lambda_2 = 3$ $\quad \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

the eigenspace corresponding to $\lambda_3 = 0$ $\quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis

We apply the Gram-Schmidt process to find an orthogonal basis

$$\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \qquad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}.$$

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ orthogonally diagonalizes $A$

$$P^{-1}AP = P^{T}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**THEOREM 7.2.2** *If A is a symmetric matrix with real entries, then:*

(*a*) *The eigenvalues of A are all real numbers.*

(*b*) *Eigenvectors from different eigenspaces are orthogonal.*

### Orthogonally Diagonalizing an $n \times n$ Symmetric Matrix

*Step 1.* Find a basis for each eigenspace of $A$.

*Step 2.* Apply the Gram–Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

*Step 3.* Form the matrix $P$ whose columns are the vectors constructed in Step 2. This matrix will orthogonally diagonalize $A$, and the eigenvalues on the diagonal of $D = P^T AP$ will be in the same order as their corresponding eigenvectors in $P$.

## ▶ EXAMPLE 1 Orthogonally Diagonalizing a Symmetric Matrix

Find an orthogonal matrix $P$ that diagonalizes

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 4 & -2 & -2 \\ -2 & \lambda - 4 & -2 \\ -2 & -2 & \lambda - 4 \end{bmatrix} = (\lambda - 2)^2(\lambda - 8) = 0$$

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The eigenspace corresponding to $\lambda = 8$ has

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Applying the Gram–Schmidt

Applying the Gram–Schmidt

Normalizing the vectors u1 , u2 and u3

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Finally, using $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as column vectors, we obtain

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

# Practice Problem:

In each part of Exercises 1–4, determine whether the matrix is orthogonal, and if so find it inverse.

**1.** (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

**2.** (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

**3.** (a) $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

# Practice Problem:

In Exercises 7–14, find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$.

7. $A = \begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

9. $A = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$

10. $A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

# Quadratic Forms

Expressions of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 + (\text{all possible terms } a_kx_ix_j \text{ in which } i \neq j)$$

Thus, a general quadratic form on $R^2$ would typically be expressed as

$$a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 \tag{1}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T A\mathbf{x}$$

and a general quadratic form on $R^3$ as

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3 \tag{2}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^T A\mathbf{x}$$

In general, if $A$ is a symmetric $n \times n$ matrix and $\mathbf{x}$ is

an $n \times 1$ column vector of variables, then we call the function

$$Q_A(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

$$\mathbf{x}^T A \mathbf{x} = \mathbf{x} \cdot A\mathbf{x} = A\mathbf{x} \cdot \mathbf{x}$$

where $A$ is a diagonal matrix,

$$\mathbf{x}^T A \mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$$

**EXAMPLE 1** Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute $\mathbf{x}^T A\mathbf{x}$ for the following matrices:

a. $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ b. $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

**SOLUTION**

$$\mathbf{x}^T A\mathbf{x} = [\,x_1 \quad x_2\,]\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\,x_1 \quad x_2\,]\begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2.$$

$$\begin{aligned} \mathbf{x}^T A\mathbf{x} &= [\,x_1 \quad x_2\,]\begin{bmatrix} 3 & \mathbf{-2} \\ -2 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\,x_1 \quad x_2\,]\begin{bmatrix} 3x_1 - \mathbf{2}x_2 \\ -2x_1 + 7x_2 \end{bmatrix} \\ &= x_1(3x_1 - \mathbf{2}x_2) + x_2(-2x_1 + 7x_2) \\ &= 3x_1^2 - \mathbf{2}x_1x_2 - 2x_2x_1 + 7x_2^2 \\ &= 3x_1^2 - 4x_1x_2 + 7x_2^2 \end{aligned}$$

## ▶ EXAMPLE 1 Expressing Quadratic Forms in Matrix Notation

In each part, express the quadratic form in the matrix notation $\mathbf{x}^T A\mathbf{x}$, where $A$ is symmetric.

(a) $2x^2 + 6xy - 5y^2$ (b) $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_2$

Solution :

$$2x^2 + 6xy - 5y^2 = [x \quad y]\begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_3 = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# Practice Problem:

▶ In Exercises 1–2, express the quadratic form in the matrix notation $\mathbf{x}^T A\mathbf{x}$, where $A$ is a symmetric matrix. ◀

**1.** (a) $3x_1^2 + 7x_2^2$ (b) $4x_1^2 - 9x_2^2 - 6x_1x_2$

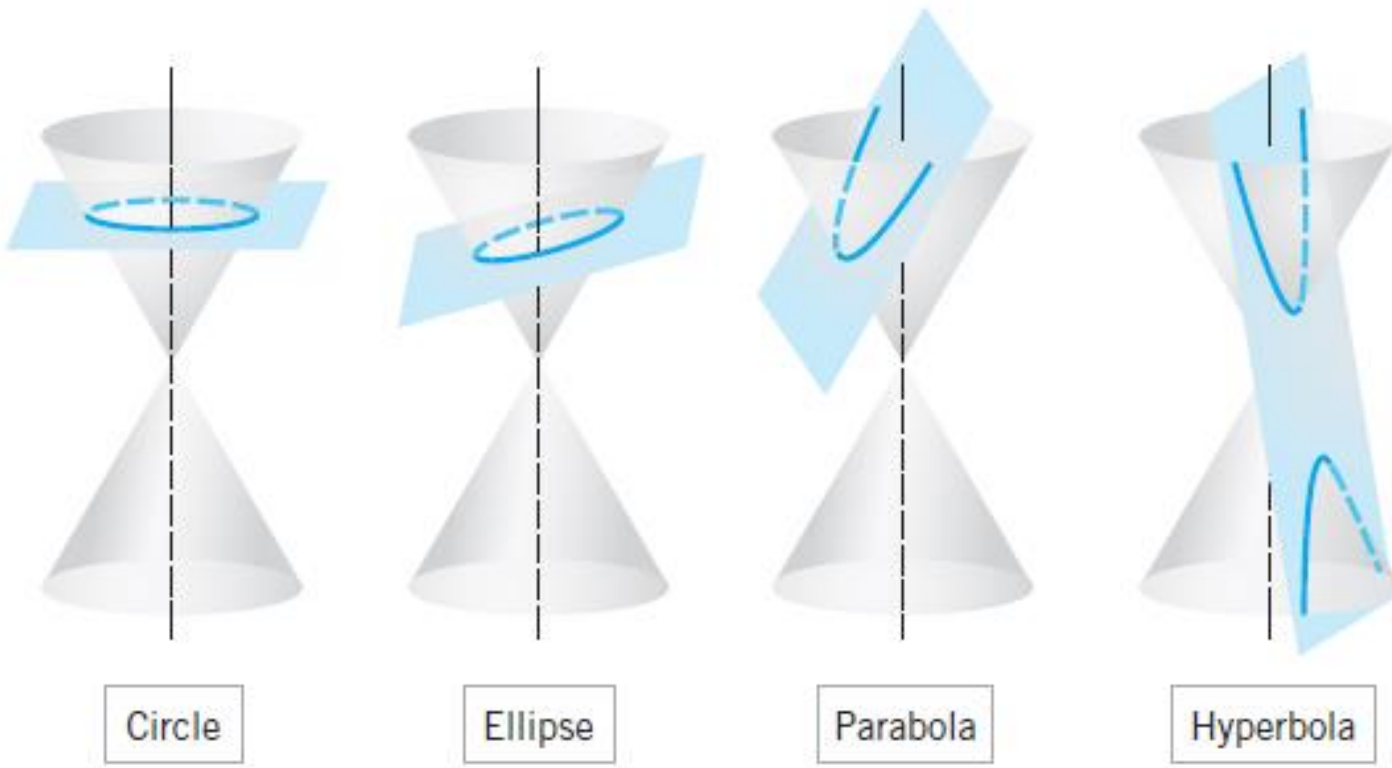
(c) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3$

**2.** (a) $5x_1^2 + 5x_1x_2$ (b) $-7x_1x_2$

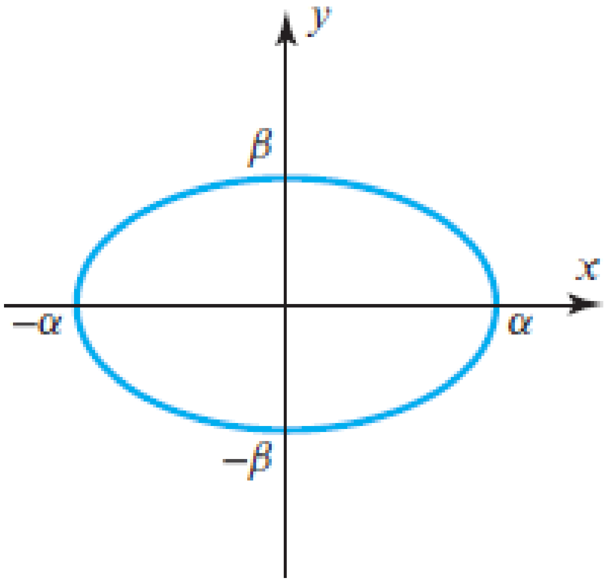
(c) $x_1^2 + x_2^2 - 3x_3^2 - 5x_1x_2 + 9x_1x_3$

# Quadratic Forms in Geometry

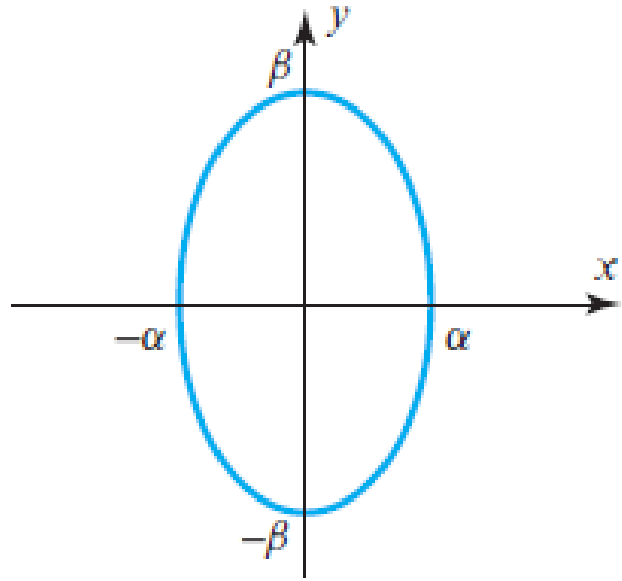
$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$



# Ellipse:

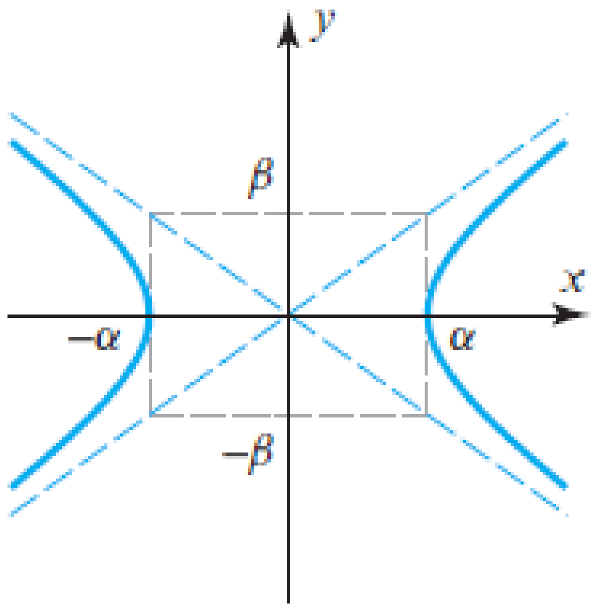



$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$
$$(\alpha \geq \beta > 0)$$

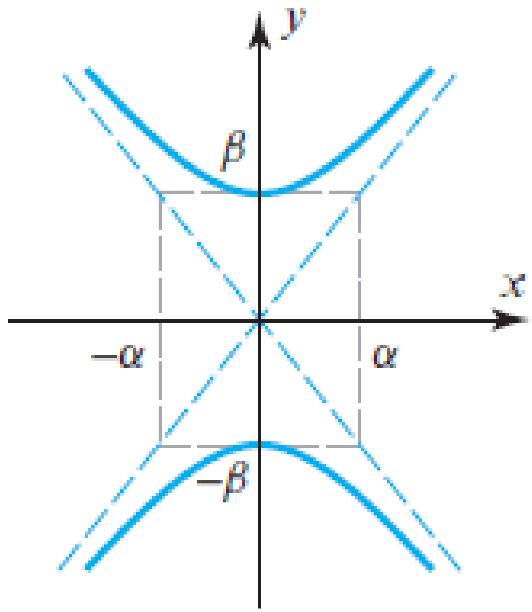



$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$
$$(\beta \geq \alpha > 0)$$

# Hyperbola:

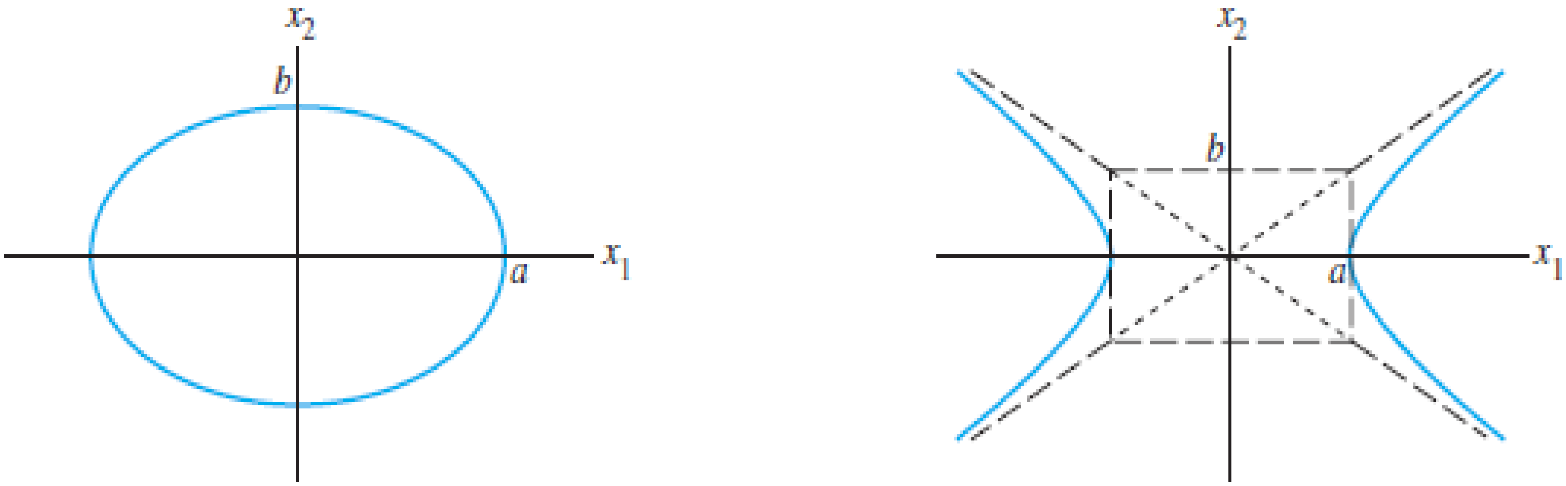



$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$
$$(\alpha > 0,\ \beta > 0)$$




$$\frac{y^2}{\beta^2} - \frac{x^2}{\alpha^2} = 1$$
$$(\alpha > 0,\ \beta > 0)$$

# Graphs:



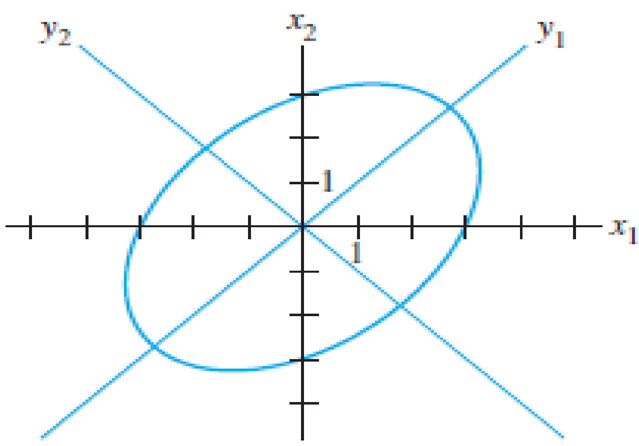

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \ a > b > 0$$

ellipse

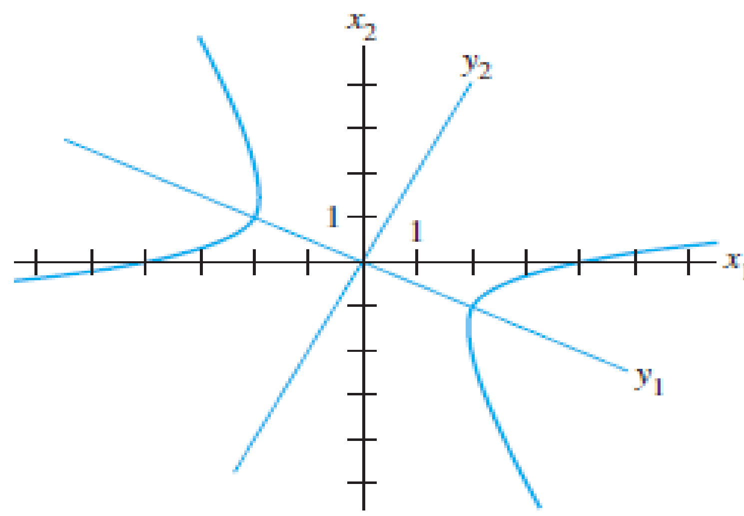
$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1, \ a > b > 0$$

hyperbola

**FIGURE 2** An ellipse and a hyperbola in standard position.




(a) $5x_1^2 - 4x_1x_2 + 5x_2^2 = 48$




(b) $x_1^2 - 8x_1x_2 - 5x_2^2 = 16$

**FIGURE 3** An ellipse and a hyperbola *not* in standard position.

### The Principal Axes Theorem

Let $A$ be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$ with no cross-product term.

# Change of Variable in a Quadratic Form

a quadratic form $\mathbf{x}^T A \mathbf{x}$

by making a substitution $\mathbf{x} = P\mathbf{y}$

If P is invertible then we call it change of variable

If P is orthogonal then we call it orthogonal change of variable

$$\mathbf{x}^T A \mathbf{x} = (P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T P^T A P \mathbf{y} = \mathbf{y}^T (P^T A P)\mathbf{y}$$

$$\mathbf{x}^T A \mathbf{x} = \mathbf{y}^T D \mathbf{y} = [y_1 \quad y_2 \quad \cdots \quad y_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

Thus, we have the following result, called the ***principal axes theorem***.

# General Quadratic form

Thus, a general quadratic form on $R^2$ would typically be expressed as

$$a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 \tag{1}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

and a general quadratic form on $R^3$ as

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3 \tag{2}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

# Problem:

Find an orthogonal change of variable that eliminates the cross product terms in the quadratic form $Q = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$, and express $Q$ in terms of the new variables.

The quadratic form can be expressed in matrix notation as

$$Q = \mathbf{x}^T A \mathbf{x} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation of the matrix $A$ is

$$\begin{vmatrix} \lambda - 1 & 2 & 0 \\ 2 & \lambda & -2 \\ 0 & -2 & \lambda + 1 \end{vmatrix} = \lambda^3 - 9\lambda = \lambda(\lambda + 3)(\lambda - 3) = 0$$

so the eigenvalues are $\lambda = 0, -3, 3$. We leave it for you to show that orthonormal bases for the three eigenspaces are

$$\lambda = 0: \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \lambda = -3: \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \lambda = 3: \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

Thus, a substitution $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

This produces the new quadratic form

$$Q = \mathbf{y}^T(P^TAP)\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -3y_2^2 + 3y_3^2$$

# Practice :

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In Exercises 5–8, find an orthogonal change of variables that eliminates the cross product terms in the quadratic form $Q$, and express $Q$ in terms of the new variables.

**5.** $Q = 2x_1^2 + 2x_2^2 - 2x_1x_2$

**6.** $Q = 5x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2$

**7.** $Q = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$

**8.** $Q = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$

**KEY CONCEPT** *For Your Notebook*

## Classifying Conics Using Their Equations

Any conic can be described by a **general second-degree equation** in $x$ and $y$: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. The expression $B^2 - 4AC$ is the **discriminant** of the equation and can be used to identify the type of conic.

| Discriminant | Type of Conic |
|---|---|
| $B^2 - 4AC < 0$, $B = 0$, and $A = C$ | Circle |
| $B^2 - 4AC < 0$ and either $B \neq 0$ or $A \neq C$ | Ellipse |
| $B^2 - 4AC = 0$ | Parabola |
| $B^2 - 4AC > 0$ | Hyperbola |

If $B = 0$, each axis of the conic is horizontal or vertical.

# Practice Problem:

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▶ In Exercises 11–12, identify the conic section represented by the equation. ◀

**11.** (a) $2x^2 + 5y^2 = 20$ (b) $x^2 - y^2 - 8 = 0$

(c) $7y^2 - 2x = 0$ (d) $x^2 + y^2 - 25 = 0$

**12.** (a) $4x^2 + 9y^2 = 1$ (b) $4x^2 - 5y^2 = 20$

(c) $-x^2 = 2y$ (d) $x^2 - 3 = -y^2$

▶ In Exercises 13–16, identify the conic section represented by the equation by rotating axes to place the conic in standard position. Find an equation of the conic in the rotated coordinates, and find the angle of rotation. ◀

**13.** $2x^2 - 4xy - y^2 + 8 = 0$ **14.** $5x^2 + 4xy + 5y^2 = 9$

**15.** $11x^2 + 24xy + 4y^2 - 15 = 0$ **16.** $x^2 + xy + y^2 = \frac{1}{2}$

## Classifying Conic Sections Using Eigenvalues

- $\mathbf{x}^T A\mathbf{x} = 1$ represents an ellipse if $\lambda_1 > 0$ and $\lambda_2 > 0$.
- $\mathbf{x}^T A\mathbf{x} = 1$ has no graph if $\lambda_1 < 0$ and $\lambda_2 < 0$.
- $\mathbf{x}^T A\mathbf{x} = 1$ represents a hyperbola if $\lambda_1$ and $\lambda_2$ have opposite signs.

In Example 3 we performed a rotation to show that the equation

$$5x^2 - 4xy + 8y^2 - 36 = 0$$

represents an ellipse with a major axis of length 6 and a minor axis of length 4. This conclusion can also be obtained by rewriting the equation in the form

$$\tfrac{5}{36}x^2 - \tfrac{1}{9}xy + \tfrac{2}{9}y^2 = 1$$

and showing that the associated matrix

$$A = \begin{bmatrix} \frac{5}{36} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{2}{9} \end{bmatrix}$$

has eigenvalues $\lambda_1 = \frac{1}{9}$ and $\lambda_2 = \frac{1}{4}$. These eigenvalues are positive, so the matrix $A$ is positive definite and the equation represents an ellipse. Moreover, it follows from (21) that the axes of the ellipse have lengths $2/\sqrt{\lambda_1} = 6$ and $2/\sqrt{\lambda_2} = 4$, which is consistent with Example 3.

# Problem:

Identify the conic whose equation is $5x^2 - 4xy + 8y^2 - 36 = 0$ by rotating the $xy$-axes to put the conic in standard position.

Find the angle $\theta$ through which you rotated the $xy$-axes

The given equation can be written in the matrix form

$$\mathbf{x}^T A \mathbf{x} = 36$$

where

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$$

The characteristic polynomial of $A$ is

$$\begin{vmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 8 \end{vmatrix} = (\lambda - 4)(\lambda - 9)$$

so the eigenvalues are $\lambda = 4$ and $\lambda = 9$. We leave it for you to show that orthonormal bases for the eigenspaces are

$$\lambda = 4: \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \quad \lambda = 9: \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

Thus, $A$ is orthogonally diagonalized by

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$[x' \quad y'] \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 36$$

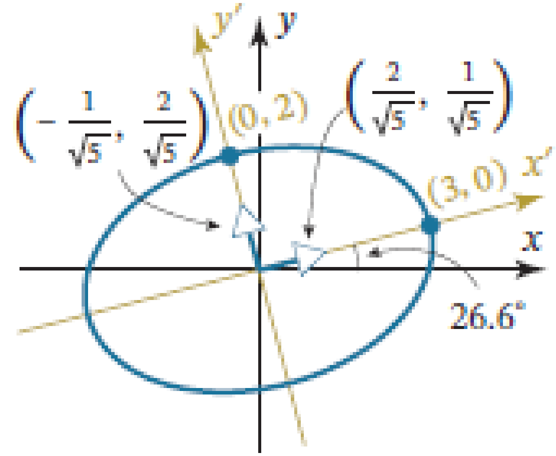



FIGURE 7.3.5

which we can write as

$$4x'^2 + 9y'^2 = 36 \quad \text{or} \quad \frac{x'^2}{9} + \frac{y'^2}{4} = 1$$

**For angle of rotation**

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\theta = \tan^{-1}\tfrac{1}{2} \approx 26.6°$$

**THEOREM 7.3.2** *If A is a symmetric matrix, then:*

(a) $\mathbf{x}^T A\mathbf{x}$ *is positive definite if and only if all eigenvalues of A are positive.*

(b) $\mathbf{x}^T A\mathbf{x}$ *is negative definite if and only if all eigenvalues of A are negative.*

(c) $\mathbf{x}^T A\mathbf{x}$ *is indefinite if and only if A has at least one positive eigenvalue and at least one negative eigenvalue.*

**THEOREM 7.3.3** *If A is a symmetric* $2 \times 2$ *matrix, then:*

(a) $\mathbf{x}^T A\mathbf{x} = 1$ *represents an ellipse if A is positive definite.*

(b) $\mathbf{x}^T A\mathbf{x} = 1$ *has no graph if A is negative definite.*

(c) $\mathbf{x}^T A\mathbf{x} = 1$ *represents a hyperbola if A is indefinite.*

# Practice Problem: David

7. Make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Give $P$ and the new quadratic form.

8. Let $A$ be the matrix of the quadratic form

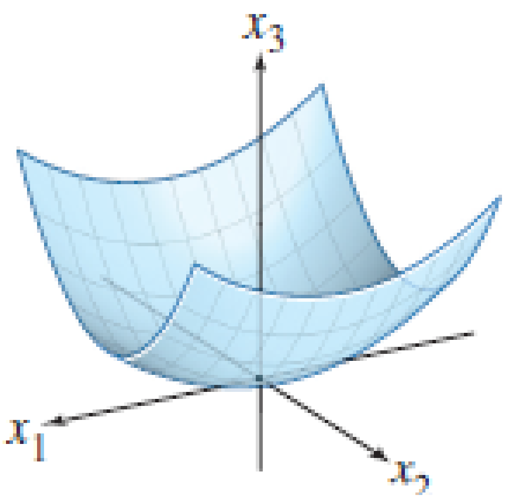
$$9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$$

It can be shown that the eigenvalues of $A$ are 3, 9, and 15. Find an orthogonal matrix $P$ such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^TA\mathbf{x}$ into a quadratic form with no cross-product term. Give $P$ and the new quadratic form.

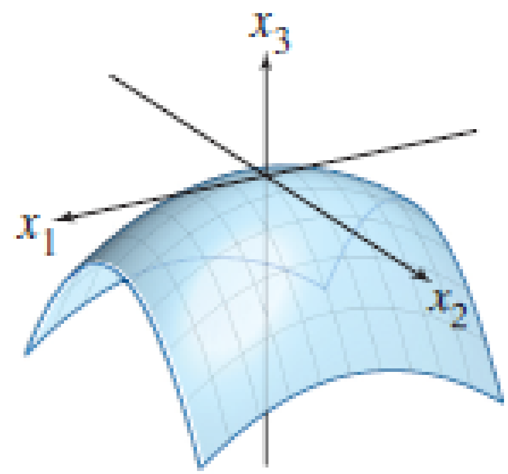
Classify the quadratic forms in Exercises 9–18. Then make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form into one with no cross-product term. Write the new quadratic form. Construct $P$ using the methods of Section 7.1.

9. $4x_1^2 - 4x_1x_2 + 4x_2^2$
10. $2x_1^2 + 6x_1x_2 - 6x_2^2$
11. $2x_1^2 - 4x_1x_2 - x_2^2$
12. $-x_1^2 - 2x_1x_2 - x_2^2$
13. $x_1^2 - 6x_1x_2 + 9x_2^2$
14. $3x_1^2 + 4x_1x_2$

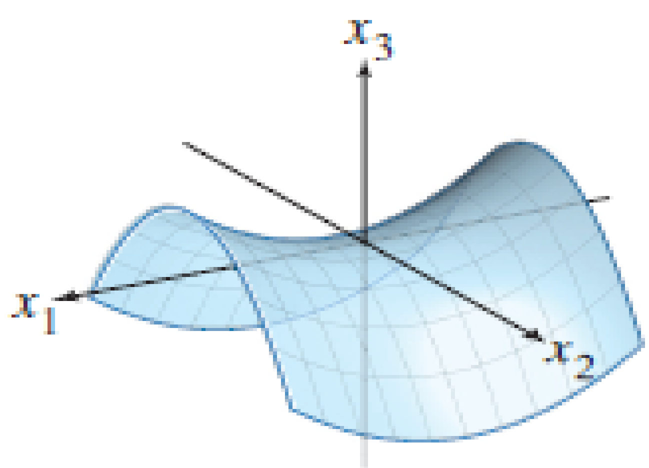
# Graph of quadratic forms:




Positive definite




Negative definite




Indefinite

## EXAMPLE 4 | Positive Definite Quadratic Forms

It is not usually possible to tell from the signs of the entries in a symmetric matrix $A$ whether that matrix is positive definite, negative definite, or indefinite. For example, the entries of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

are nonnegative, but the matrix is indefinite since its eigenvalues are $\lambda = 1, 4, -2$ (verify). To see this another way, write out the quadratic form as

$$\mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$$

We can now see, for example, that

$$\mathbf{x}^T A\mathbf{x} = 4 \quad \text{for} \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 1$$

and

$$\mathbf{x}^T A\mathbf{x} = -4 \quad \text{for} \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = -1$$

Is $Q(\mathbf{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ positive definite?

In Exercises 27–28, use Theorem 7.3.4 to classify the matrix as positive definite, negative definite, or indefinite.

**27.** (a) $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

**28.** (a) $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} -4 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$

7.5

# Hermitian, Unitary, and Normal Matrices

**Definition 1**

If $A$ is a complex matrix, then the ***conjugate transpose*** of $A$, denoted by $A^*$, is defined by

$$A^* = \overline{A}^T \tag{1}$$

## EXAMPLE 1 | Conjugate Transpose

Find the conjugate transpose $A^*$ of the matrix

$$A = \begin{bmatrix} 1+i & -i & 0 \\ 2 & 3-2i & i \end{bmatrix}$$

**Solution** We have

$$\overline{A} = \begin{bmatrix} 1-i & i & 0 \\ 2 & 3+2i & -i \end{bmatrix} \quad \text{and hence} \quad A^* = \overline{A}^T = \begin{bmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{bmatrix}$$

## Theorem 7.5.1

If $k$ is a complex scalar, and if $A$ and $B$ are complex matrices whose sizes are such that the stated operations can be performed, then:

(a) $(A^*)^* = A$

(b) $(A + B)^* = A^* + B^*$

(c) $(A - B)^* = A^* - B^*$

(d) $(kA)^* = \overline{k}A^*$

(e) $(AB)^* = B^*A^*$

### Definition 2

A square matrix $A$ is said to be ***unitary*** if

$$AA^* = A^*A = I \tag{2}$$

or, equivalently, if

$$A^* = A^{-1} \tag{3}$$

and it is said to be ***Hermitian**** if

$$A^* = A \tag{4}$$

## Theorem 7.5.2

If $A$ is a Hermitian matrix, then:

(a) The eigenvalues of $A$ are all real numbers.

(b) Eigenvectors from different eigenspaces are orthogonal.

## EXAMPLE 4 | Eigenvalues and Eigenvectors of a Hermitian Matrix

Confirm that the Hermitian matrix

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

has real eigenvalues and that eigenvectors from different eigenspaces are orthogonal.

**Solution** The characteristic polynomial of $A$ is

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 2 & -1-i \\ -1+i & \lambda - 3 \end{vmatrix} \\ &= (\lambda - 2)(\lambda - 3) - (-1-i)(-1+i) \\ &= (\lambda^2 - 5\lambda + 6) - 2 = (\lambda - 1)(\lambda - 4) \end{aligned}$$

so the eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 4$, which are real. Bases for the eigenspaces of $A$ can be obtained by solving the linear system

$$\begin{bmatrix} \lambda - 2 & -1-i \\ -1+i & \lambda - 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

with $\lambda = 1$ and with $\lambda = 4$. We leave it for you to do this and to show that the general solutions of these systems are

$$\lambda = 1: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -1-i \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda = 4: \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} \frac{1}{2}(1+i) \\ 1 \end{bmatrix}$$

Thus, bases for these eigenspaces are

$$\lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda = 4: \quad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{2}(1+i) \\ 1 \end{bmatrix}$$

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal since

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (-1-i)\left(\overline{\tfrac{1}{2}(1+i)}\right) + (1)(1) = \tfrac{1}{2}(-1-i)(1-i) + 1 = 0$$

and hence all scalar multiples of them are also orthogonal.

## EXAMPLE 5 | A Unitary Matrix

Use Theorem 7.5.3 to show that

$$A = \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(1+i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(-1+i) \end{bmatrix}$$

is unitary, and then find $A^{-1}$.

**Solution** We will show that the row vectors

$$\mathbf{r}_1 = \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{bmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(-1+i) \end{bmatrix}$$

are orthonormal. The relevant computations are

$$\|\mathbf{r}_1\| = \sqrt{\left|\tfrac{1}{2}(1+i)\right|^2 + \left|\tfrac{1}{2}(1+i)\right|^2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}} = 1$$

$$\|\mathbf{r}_2\| = \sqrt{\left|\tfrac{1}{2}(1-i)\right|^2 + \left|\tfrac{1}{2}(-1+i)\right|^2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}} = 1$$

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{r}_2 &= \left(\tfrac{1}{2}(1+i)\right)\left(\overline{\tfrac{1}{2}(1-i)}\right) + \left(\tfrac{1}{2}(1+i)\right)\left(\overline{\tfrac{1}{2}(-1+i)}\right) \\ &= \left(\tfrac{1}{2}(1+i)\right)\left(\tfrac{1}{2}(1+i)\right) + \left(\tfrac{1}{2}(1+i)\right)\left(\tfrac{1}{2}(-1-i)\right) = \tfrac{1}{2}i - \tfrac{1}{2}i = 0 \end{aligned}$$

Since we now know that $A$ is unitary, it follows that

$$A^{-1} = A^* = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1+i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(-1-i) \end{bmatrix}$$

You can confirm the validity of this result by showing that $AA^* = A^*A = I$.

## EXAMPLE 6 | Unitary Diagonalization of a Hermitian Matrix

Find a matrix $P$ that unitarily diagonalizes the Hermitian matrix

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

**Solution** We showed in Example 4 that the eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 4$ and that bases for the corresponding eigenspaces are

$$\lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda = 4: \quad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{2}(1+i) \\ 1 \end{bmatrix}$$

Since each eigenspace has only one basis vector, the Gram–Schmidt process is simply a matter of normalizing these basis vectors. We leave it for you to show that

$$\mathbf{p}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad \text{and} \quad \mathbf{p}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} \frac{1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Thus, $A$ is unitarily diagonalized by the matrix

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2] = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

Although it is a little tedious, you may want to check this result by showing that

$$P^*AP = \begin{bmatrix} \frac{-1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix} \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

## Exercise Set 7.5

*In Exercises 1–2, find $A^*$.*

**1.** $A = \begin{bmatrix} 2i & 1-i \\ 4 & 3+i \\ 5+i & 0 \end{bmatrix}$ **2.** $A = \begin{bmatrix} 2i & 1-i & -1+i \\ 4 & 5-7i & -i \end{bmatrix}$

*In Exercises 3–4, substitute numbers for the ×'s so that $A$ is Hermitian.*

**3.** $A = \begin{bmatrix} 1 & i & 2-3i \\ \times & -3 & 1 \\ \times & \times & 2 \end{bmatrix}$ **4.** $A = \begin{bmatrix} 2 & 0 & 3+5i \\ \times & -4 & -i \\ \times & \times & 6 \end{bmatrix}$

*In Exercises 5–6, show that $A$ is not Hermitian for any choice of the ×'s.*

**5.** **a.** $A = \begin{bmatrix} 1 & i & 2-3i \\ -i & -3 & \times \\ 2-3i & \times & \times \end{bmatrix}$

**b.** $A = \begin{bmatrix} \times & \times & 3+5i \\ 0 & i & -i \\ 3-5i & i & \times \end{bmatrix}$

*In Exercises 7–8, verify that the eigenvalues of the Hermitian matrix $A$ are real and that eigenvectors from different eigenspaces are orthogonal (see Theorem 7.5.2).*

**7.** $A = \begin{bmatrix} 3 & 2-3i \\ 2+3i & -1 \end{bmatrix}$ **8.** $A = \begin{bmatrix} 0 & 2i \\ -2i & 2 \end{bmatrix}$

*In Exercises 9–12, show that $A$ is unitary, and find $A^{-1}$.*

**9.** $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5}i \\ -\frac{4}{5} & \frac{3}{5}i \end{bmatrix}$ **10.** $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{bmatrix}$

*In Exercises 13–18, find a unitary matrix $P$ that diagonalizes the Hermitian matrix $A$, and determine $P^{-1}AP$.*

**13.** $A = \begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 3 & -i \\ i & 3 \end{bmatrix}$

**15.** $A = \begin{bmatrix} 6 & 2+2i \\ 2-2i & 4 \end{bmatrix}$ **16.** $A = \begin{bmatrix} 0 & 3+i \\ 3-i & -3 \end{bmatrix}$

**17.** $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}$

# ANY Questions?