Complexity Analysis

Problem 1

```
i = 1
while i < n:
    i = i * 2
```

$\to$ k times

While loop exit

$2^k > n \longrightarrow 2^k = n \longrightarrow$

$n = 16$

$1 < 16$

$i = 1$ ✓ 2^0

$i = 2$ ✓ 2^1

$i = 4$ ✓ 2^2

$i = 8$ ✓ 2^3

$\longrightarrow i = 16$ 2^4

$\log_2 2^3 = 3\cancel{\log_2 2}$

$= 3$

Logarithmic Properties

1. $\log_a a = 1$
2. $\log_a m^n = n \log_a m$
3. $\log_a mn = \log_a m + \log_a n$
4. $\log_a \frac{m}{n} = \log_a m - \log_a n$
5. Higher the base $\rightarrow$ Lower the value

2^0 —— $i=1$

2^1 —— $i=2$

2^2 —— $i=4$

2^3 —— $i=8$

|

2^k —— n

$2^k = n$

$\log_2(2^k) = \log_2 n$

$k\cancel{\log_2 2} = \log_2 n$

$\boxed{k = \log_2 n}$ $\rightarrow$

$\log_2 16$

$\Rightarrow \log_2 2^4$

$\Rightarrow 4\cancel{\log_2 2}$

$\Rightarrow 4$

Time complexity =

$O(\log n)$

Problem 2

$i = n$

while $i > 0$:

$\boxed{i = i - 1}$

$n = 5$

$i = 5$

$i = 4$
$i = 3$
$i = 2$
$i = 1$
$i = 0$

$\approx n/3$

$n = 9$

$O(n)$

$9/3 = 3$

$i = 6$
$i = 3$
$i = 0$

$i = n$

while $i > 0$:

$\boxed{i = i - 3}$

$i = 10$
$i = 7$
$i = 4$
$i = 1$
$i = -2$

4

$i = 1$

while $i < n$:

$i = i * 10$

$10^k = n$

$\log_{10} 10^k = \log_{10} n$

$k \cancel{\log_{10} 10} = \log_{10} n$

$k = \log_{10} n$

$O(\log n)$

Problem 3

```
for i in range(n):        # n
    k = 2                 # loglog n
    while k < n:
        k = k * k         # x times
```

$n = 64$

very small

$O(n \cdot \log\log n)$

$\approx O(n)$

$2 < 64$

$k = 4 \leftarrow 2^2$

$4 < 64$

$k = 16 \leftarrow 2^{2^2}$

$16 < 64$

$k = 256 \leftarrow 2^{2^3}$

2^{2^x}

$$2^{2^x} \geq n \longrightarrow 2^{2^x} = n$$

$$\log_2(2^{2^x}) = \log_2 n$$

$$2^x \cancel{\log_2 2} = \log_2 n$$

$$\boxed{2^x = \log_2 n}$$

$$\log_2(2^x) = \log_2(\log_2 n)$$

$$x \cancel{\log_2 2} = \log_2(\log_2 n)$$

$$x = \log_2(\log_2 n)$$

Asymptotic Notation

Big O

↳ worst case scenario

$f(n) = O(g(n))$ if

$f(n) <= c \cdot g(n)$

$\forall n, \ n \geq n_0$

$c > 0$

$n_0 \geq 1$

① $f(n) = 5n, \ g(n) = n$

$5n = O(n)$ True

$$\boxed{5n <= c \cdot n}$$

$c = 5$

$$\left.\begin{matrix} n/3 \\ 3n \\ 5n \end{matrix}\right\} \rightarrow O(n)$$

② $f(n) = n$

$g(n) = 5n$

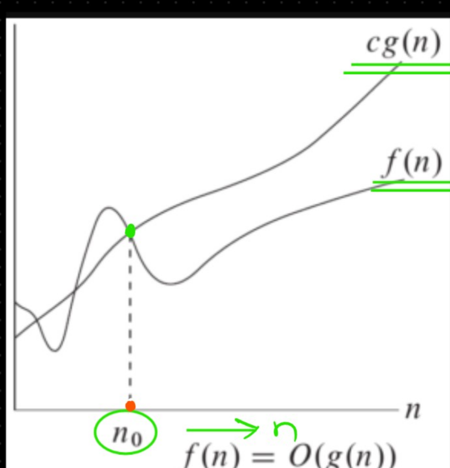
$n = O(5n) \rightarrow$ True

$$\boxed{n <= c \cdot 5n}$$

$c = 1/5$

Experimentally

Time ↑



$f(n) = O(g(n))$ Mathematical intuition

$f(n) <= c \cdot g(n)$

$n \geq n_0$

③ $f(n) = n^2, \quad g(n) = n$

No

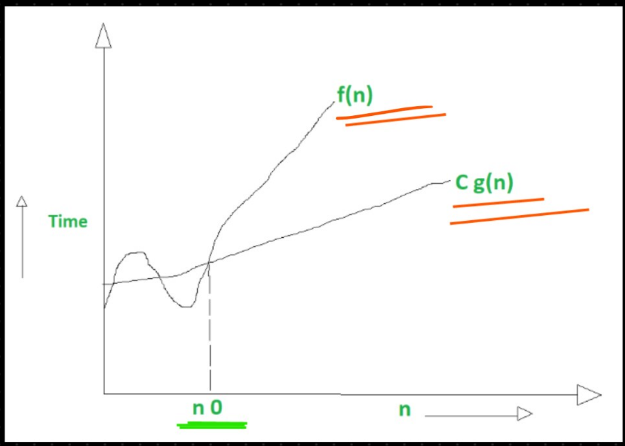
$n^2 = O(n)$

$n^2 \leq c \cdot n$

$c = n$ Not constant

Omega Notation (Ω)

Best case scenario



$f(n) = \Omega(g(n))$

$f(n) \geq c \cdot g(n)$

$\forall n \geq n_0$

$c > 0, \; n_0 \geq 1$

Example 1

$f(n) = n, \quad g(n) = 5n$ true

$n = \Omega(5n)$

$n >= c \cdot 5n$ $\quad c = 1/5$

Example 2

$f(n) = 5n, \quad g(n) = n$

$5n = \Omega(n)$

$5n >= c \cdot n$

$c = 5$

$f(n) = n, \quad g(n) = n^2$

Example 3

$n = \Omega(n^2)$ Not true

$n \geq c \cdot n^2 \longrightarrow c = 1/n$

$f(n)$	$O(g(n))$
$n/3 \longrightarrow$	$O(n)$

Why?? $\begin{cases} f(n) = n/3 \\ g(n) = n \end{cases}$

$f(n) <= c \cdot g(n)$

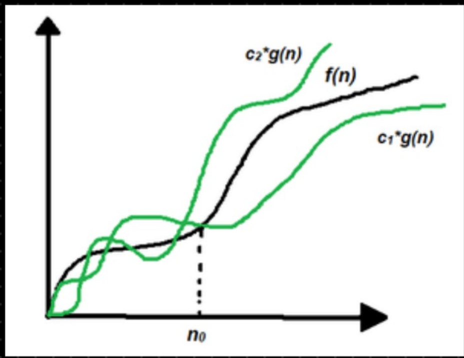
$\boxed{n/3 <= c \cdot n}$

$c = 1/3$

true

Theta Notation

→ average case Scenario → (Big O, Omega)



① $f(n) = n$

$g(n) = 5n$

$f(n) <= c_1 \cdot g(n)$

$f(n) >= c_2 \cdot g(n)$

$n = \Theta(5n)$ — true

① Big O

$\boxed{n <= c_1 \cdot 5n}$

$c_1 = 1/5$

AND

② Omega

$n >= c_2 \cdot 5n$

$c_2 = 1/5$

f(n) g(n)

$2^{n+5} = O(2^n) \longrightarrow$ Yes/No ✓

$\boxed{2^n \cdot 2^5 <= c \cdot (2^n)}$ — Yes

$c = 2^5$

$2^{2n} = O(2^n)$ — No

$c = 2^n$

$\boxed{2^{2n} <= c \cdot 2^n}$

Complexity classes

1. $O(1) \longrightarrow$ Constant time complexity

Binary search ———— $O(\log \log n)$

2. $O(\log n) \longrightarrow$ Logarithmic time complexity

Linear search

3. $O(n) \longrightarrow$ Linear time complexity

$O(n \log n)$

4. $O(n^2) \longrightarrow$ Quadratic time complexity

5. $O(n^3) \longrightarrow$ cubic time complexity

6. $O(n^c) \longrightarrow$ Polynomial time complexity

$c > 3$

7. $O(c^n) \longrightarrow$ Exponential time complexity

$c > 1$

$O(2^n)$

$$\log_2 n > \log_3 n$$

$$\log_3 n = O(\log_2 n) \text{ — Yes}$$

$$\log_3 n <= c \cdot \log_2 n$$

$$c = 1$$

$$2^n = O(3^n)$$

$$2^n <= c \cdot 3^n$$

$$2^n <= c \cdot (2 * 1.5)^n$$

$$2^n <= c \cdot 2^n * \underline{\underline{1.5^n}} \rightarrow \text{True}$$

$$c = 1$$