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Distributivity and conditional distributivity for T-uninorms

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ABSTRACT

Aggregation operators play an important role in many theoretical and practical aspects of applied mathematics. Recently, the focus is on operators with an annihilator, therefore the topic of this paper is distributivity, both conditional and regular, for certain classes of aggregation operators with this property. The characterization of all pairs (F, G) of aggregation operators that are satisfying distributivity law, on both whole and restricted domain, where F is a T-uninorm in $U_{\max}$, and G is a t-conorm or a uninorm from $U_{\min}$ or $U_{\max}$ is given.

1. Introduction

Aggregation operators have been intensively studied during the last decades, since they play essential role in many applications, from mathematics and natural sciences to economics and social sciences (see [13,15,20]). Lately, the attention of researchers is directed towards characterizations of pairs of aggregation operators that are satisfying the distributivity law. This topic has roots in [1] and the focus of current investigations is on t-norms and t-conorms [13], aggregation operators, quasi-arithmetic means [4], pseudo-arithmetical operations [2], uninorms and nullnorms [6,12,26,27,31,34,37], semi-t-operators and uninorms [7,8,29,35,36], 2-uninorms [9], Mayor's aggregation operators [4,18,30]. Recently researchers are investigating the problem of distributivity on the restricted domain since this particular approach produces a larger variety of solutions. This type of distributivity is known as the conditional distributivity or the restricted distributivity [3,16,17,20–24,32,33]. The significance of the considered topic follows not only from the theoretical point of view, but also from its applicability in the integration theory [21,33] and in the utility theory [10,11,15,19].

This paper follows two directions of investigation. The first aim is to extend research of [26,27] towards T-uninorms that are a generalization of disjunctive uninorms and nullnorms (t-operators). The second one is solving distributivity equations on the restricted domain for T-uninorms over uninorms. Since conditional distributivity of nullnorms over uninorms is considered in [17], this direction of research upgrades some previous results. Therefore, the main concern of this paper is how to solve functional equations

$$F(x, G(y, z)) = G(F(x, y), F(x, z)), \quad x, y, z \in [0, 1]$$

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and

$$F(x, G(y,z)) = G(F(x,y), F(x,z)), \quad x,y,z \in [0,1], \quad G(y,z) < 1$$

where F is a T-uninorm in $U_{\max}$ with annihilator $a \in (0, 1)$, and G is a t-conorm or a uninorm from the class $U_{\min} \cup U_{\max}$. Motivation for this line of investigation lies in possibility of obtaining new pairs of aggregation operators that further on can be applied in the utility theory for modeling some specific problems, e.g. problems that deals with existence of veto.

This paper is organized as follows. Some preliminary notions concerning aggregation operators, uninorms, T-uninorms, distributivity equations on the whole and restricted domain are given in the Section 2. Results on distributivity on the whole domain for T-uninorms are given in the third section. Distributivity on the restricted domain, i.e., conditional distributivity for T-uninorms is considered in the fourth section. Some concluding remarks are given in the fifth section.

2. Preliminaries

An overview of concepts relevant for the research that follows is given in this section (see [5,6,14,15,20,28]).

2.1. Aggregation operators

The starting point of this research is the notion of an aggregation operator in $[0, 1]^n$.

Definition 1 [15]**.** An aggregation operator in $[0, 1]^n$ is a function $A^{(n)}$: $[0, 1]^n \to [0, 1]$ that is nondecreasing in each variable and that fulfills the following boundary conditions

$$A^{(n)}(0,\dots,0) = 0 \quad \text{and} \quad A^{(n)}(1,\dots,1) = 1.$$

Of course, if required, the previous definition can be easily extended to an arbitrary real interval $[a, b]$. The integer n represents the number of input values of the aggregation in question. Since the focus of this paper is on binary aggregation operators, further the simple notation A will be used instead of $A^{(2)}$. Depending on the context of usage, some other properties can be required, e.g. associativity, commutativity, idempotency, decomposability, autodistributivity, bisymmetry, neutral and annihilator elements, etc., (see [15]).

In the sequel some classes of aggregation operators essential for the presented research are given. The first type of aggregation operators that is necessary is an aggregation operator with a neutral element, namely the uninorm.

Definition 2 [38]**.** A uninorm U: $[0, 1]^2 \to [0, 1]$ is binary aggregation operator that is commutative, associative, and for which there exists a neutral element $e \in [0, 1]$, i.e., $U(x, e) = x$ for all $x \in [0, 1]$.

Recall that when $e = 1$ uninorm U becomes a t-norm denoted by T, and when $e = 0$, U is a t-conorm denoted by S. Uninorms for which both functions $U(x, 0)$ and $U(x, 1)$ are continuous, except perhaps at the point e, are characterized in [14] by the following theorem.

Theorem 3 *[14]. Let U be a uninorm with a neutral element $e \in (0, 1)$ such that both functions $U(x, 1)$ and $U(x, 0)$ are continuous except at the point $x = e$.*

(i) *If $U(0,1) = 0$, then*

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{1}$$

where T is a t-norm, and S is a t-conorm.

(ii) *If $U(0,1) = 1$, then*

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise,} \end{cases} \tag{2}$$

where T is a t-norm, and S is a t-conorm.

A t-norm T from (1) (and (2)) is called the underlying t-norm of U and a t-conorm S is called the underlying t-conorm of U. The class of all uninorms of the form (1) consists of conjunctive uninorms and is denoted by $U_{\min}$, while the class of all uninorms of the form (2) consists of disjunctive uninorms and is denoted by $U_{\max}$. More on this subject can be found in [14,15,38].

Example 4. The first uninorms considered by Yager and Rybalov [38] are idempotent uninorms U_e^{min} and U_e^{max} from classes $U_{\min}$, and $U_{\max}$, respectively, of the following form

$$U_e^{min} = \begin{cases} \max & \text{on } [e,1]^2, \\ \min & \text{otherwise,} \end{cases} \tag{3}$$

and

$$U_e^{max} = \begin{cases} \min & \text{on } [0,e]^2, \\ \max & \text{otherwise.} \end{cases} \tag{4}$$

Uninorms (3) and (4) are only idempotent uninorms from classes $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$. The only idempotent t-norm and t-conorm are operators minimum and maximum, respectively.

Another type of aggregation operators necessary for the presented research is a T-uninorm, i.e., aggregation operator with an annihilator (absorbing element). An element $a \in [0, 1]$ is an annihilator for aggregation operator A if

$$A(a,x) = A(x,a) = a$$

for all $x \in [0, 1]$. Special aggregation operators with annihilator that are known under the name nullnorms (see [5]) or t-operators (see [25]) are given by the following definition.

Definition 5 [5]**.** A nullnorm V: $[0, 1]^2 \to [0, 1]$ is a binary aggregation operator that is commutative, associative and for which there exists an element $a \in [0, 1]$ such that

$$V(x,0) = x \ \text{ for } \ x \le a \quad \text{ and } \quad V(x,1) = x \ \text{ for } \ x \ge a.$$

It is clear that a from the previous definition is an annihilator for V.

Of the special interest for this paper are general commutative aggregation operators with an annihilator a, denoted with a-CAOA, that are studied in [28]. Representation for the associative class of a-CAOA operators is given by the following theorem from [28].

Theorem 6 *[28]***.** *A binary operator A: $[0, 1]^2 \to [0, 1]$ is an associative a-CAOA if and only if there are associative, commutative aggregation operators F and G with $F \le G, F(a, x) \le a \le G(a, x)$ for all $x \in [0, 1]$ and such that*

$$A(x,y) = med(a, F(x,y), G(x,y)), \qquad x, y \in [0,1],$$

where med is the standard median aggregation operator in statistics.

Further on, for any binary operator A: $[0, 1]^2 \to [0, 1]$, and any element $c \in [0, 1]$, the section A_c: $[0, 1] \to [0, 1]$ given by

$$A_c(x) = A(c,x)$$

will be denoted by A_c. As it will be seen from the following, the continuity (discontinuity) of sections A_0 and A_1 plays a crucial role for study the behavior of associative a-CAOA operators.

Definition 7 [28]**.** A binary operator A: $[0, 1]^2 \to [0, 1]$ will be called the T-uninorm if it is an associative a-CAOA satisfying the following properties:

- section A_1 is continuous and A_0 is not,
- there is $e \in (0, 1)$ such that e is idempotent element, the section A_e is continuous and $A_e(0) = 0$.

Theorem 8 *[28]***.** *Let A: $[0, 1]^2 \to [0, 1]$ be a binary operator. The following statements are equivalent.*

(i) *A is a T-uninorm.*

(ii) *There exists $a \in (0, 1]$, a t-norm T' and a disjunctive uninorm U' with neutral element $e' \in (0, 1)$ such that A is given by*

$$A(x,y) = \begin{cases} aU'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } \ (x,y) \in [0,a]^2, \\ a + (1-a)T'\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } \ (x,y) \in [a,1]^2, \\ a & \text{if } \ (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \end{cases} \tag{5}$$

(iii) *There exists $a \in (0, 1]$, a t-norm T and a disjunctive uninorm U with neutral element $e \in (0, 1)$ such that $U(x, a) \ge a$ for all $x \in [0, 1]$, $T \le U$ and*

$$A(x,y) = med(a, T(x,y), U(x,y)), \qquad x, y \in [0,1],$$

where med is the standard median aggregation operator in statistics.

Remark 9. Let A: $[0, 1]^2 \to [0, 1]$ be a T-uninorm.

- For $a = 1$ it becomes a disjunctive uninorm, i.e., $A = U'$.
- $a \neq 0$ in order to ensure the discontinuity of A_0, and $e < a$ since $A_e(0) = 0$.
- If $U' \in \mathcal{U}_{\max}$, then A is a T-uninorm in $\mathcal{U}_{\max}$.

Applying the previous theorem on some well-known t-norms and disjunctive uninorms, some interesting examples of T-uninorms can be constructed.

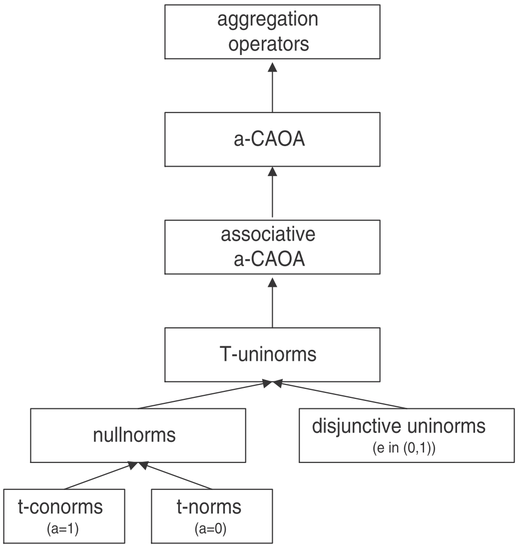

Fig. 1. Aggregation operators.

Example 10. Binary Operator A: $[0, 1]^2 \to [0, 1]$ given by

$$A(x, y) = \begin{cases} a & \text{if } (x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a], \\ \min(x, y) & \text{if } (x, y) \in [0, e]^2 \cup [a, 1]^2, \\ \max(x, y) & \text{otherwise}, \end{cases} \tag{6}$$

is an idempotent T-uninorm in $U_{\max}$ with annihilator a, obtained by (5) where t-norm $T' = \min$ and uninorm $U' = U_e^{\max}$.

More information on a-CAOA, more precisely, on families of associative a-CAOA such as nullnorms, S-uninorms, bi-uninorms can be found in [28].

2.2. Distributivity equations

Functional equations that are called left and right distributivity laws ([1], p. 318) are given by the following definition.

Definition 11. Let F, G: $[0, 1]^2 \to [0, 1]$ be two operators. F is distributive over G, if the following two laws hold:

(LD) F is left distributive over G ,i.e.,

$$F(x, G(y, z)) = G(F(x, y), F(x, z)), \quad \text{for all } x, y, z \in [0, 1]$$

and

(RD) F is right distributive over G ,i.e.,

$$F(G(y, z), x) = G(F(y, x), F(z, x)), \quad \text{for all } x, y, z \in [0, 1]$$

Of course, for a commutative F, laws (LD) and (RD) coincide.

The following two lemmas give answers to some questions regarding distributivity and will be used later on.

Lemma 12 *[6]. Let $X \neq \emptyset$, F: $X^2 \to X$ and let $e \in Y$, where $Y \subset X$, be a neutral element for the operator F on Y ($\forall_{x\in Y} F(e, x) = F(x, e) = x$). If the operator F is left or right distributive over some operator G: $X^2 \to X$ that fulfils $G(e, e) = e$, then G is idempotent on Y.*

Lemma 13 *[6]. All increasing functions F: $[0, 1]^2 \to [0, 1]$ are distributive (conditionally distributive) over* max *and* min.

2.3. Conditional distributivity

Since the problem of distributivity of a t-norm over a t-conorm gives us only the trivial solution (see [13]), i.e., t-conorm in question has to be $S_M = \max$, it was necessary to restrict the domain of distributivity in the following manner (see [20], p. 138).

Definition 14 [20]. A t-norm T is conditionally distributive (CD) over a t-conorm S if for all x, y, $z \in [0, 1]$ the following holds

(CD) $\quad T(x, S(y, z)) = S(T(x, y), T(x, z)),$ whenever $S(y, z) < 1.$

The previous definition can be extended to some more general aggregation operators.

Definition 15. Let F be a T-uninorm with annihilator $a \in (0, 1)$ and let G be a t-conorm or $G \in U_{min} \cup U_{max}$. F is conditionally distributive (CD) over G if for all $x, y, z \in [0, 1]$ the following holds

$$\text{(CD)} \qquad F(x, G(y, z)) = G(F(x, y), F(x, z)), \text{ whenever } G(y, z) < 1.$$

This type of distributivity is also known as the restricted distributivity [15] and, although the domain is only weakly restricted, the class of pairs of operators that fulfill (CD) is much wider. This can be nicely illustrated by the following well-known result form [20] (p. 138–140).

Theorem 16 *[20]. A continuous t-norm T and a continuous t-conorm S satisfy (CD), if and only if exactly one of the following cases is fulfilled:*

(i) $S = S_M$;
(ii) *there is a strict t-norm T^* and a nilpotent t-conorm S^* such that additive generator s of S^* satisfying $s(1) = 1$ is also multiplicative generator of T^*, and there is a $c \in [0, 1)$ such that for some continuous t-norm T^{**} we have*

$$S(x, y) = \begin{cases} c + (1-c)S^*\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x, y) \in [c, 1]^2, \\ \max(x, y) & \text{otherwise,} \end{cases} \tag{7}$$

and

$$T(x, y) = \begin{cases} cT^{**}\left(\frac{x}{c}, \frac{y}{c}\right) & \text{if } (x, y) \in [0, c]^2, \\ c + (1-c)T^*\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x, y) \in [c, 1]^2, \\ \min(x, y) & \text{otherwise.} \end{cases} \tag{8}$$

Remark 17.

(i) Due to isomorphisms between strict t-norms and product t-norm $T_P(x, y) = xy$ and nilpotent t-conorms and Lukasiewicz t-conorm $S_L(x, y) = \min\{x + y, 1\}$ (see [20]), the previous result is often reduced to the following pair given by Fig. 2.

$$S(x, y) = \begin{cases} c + (1-c)S_L\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x, y) \in [c, 1]^2, \\ \max(x, y) & \text{otherwise,} \end{cases} \tag{9}$$

and

$$T(x, y) = \begin{cases} cT^{**}\left(\frac{x}{c}, \frac{y}{c}\right) & \text{if } (x, y) \in [0, c]^2, \\ c + (1-c)T_P\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x, y) \in [c, 1]^2, \\ \min(x, y) & \text{otherwise.} \end{cases} \tag{10}$$

(ii) Some further generalizations of the Theorem 16 can be found in [16,22,32], where F is a uninorm, in [33] where F is a pseudo-multiplication, and in [17] where F is a nullnorm.

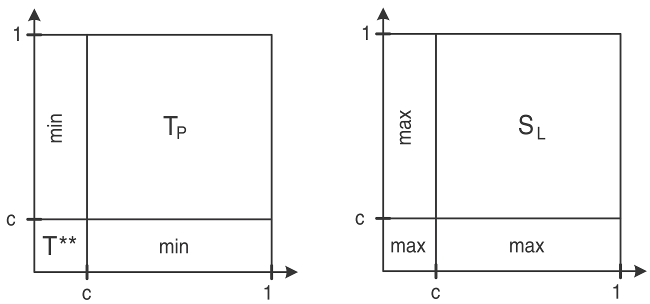

Fig. 2. Conditional distributivity: t-norm and t-conorm.

3. Distributivity for *T*-uninorms

Let F be a T-uninorm from U_{max} with an annihilator a, and let G be a t-norm, or a t-conorm or a uninorm from $U_{min} \cup U_{max}$. Since distributivity of F over G for $a = 1$, i.e., when F is a uninorm from U_{max}, is considered in [26,27], the further assumption is that $a \in (0, 1)$. Also, further on the neutral element of the underlying uninorm of F will denote by e. (Fig. 3)

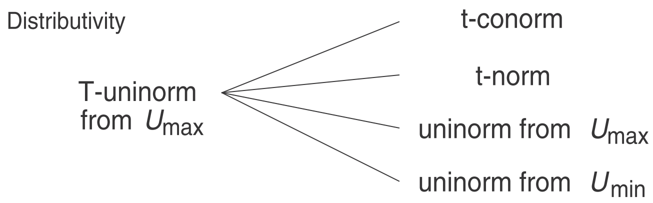

Fig. 3. Topic of Section 3.

Theorem 18. *Let F be a T-uninorm in U_{max} and S be a t-conorm. F is distributive over S if and only if $S = \max$.*

Proof. ($\Rightarrow$) There has to be show that S is indeed an idempotent operator, i.e., that $S(x, x) = x$ for all $x \in [0, 1]$:

- if $x \geq a$, then

$$x = F(x, 1) = F(x, S(1, 1)) = S(F(x, 1), F(x, 1)) = S(x, x),$$

- if $e < x \leq a$, then

$$x = F(x, 0) = F(x, S(0, 0)) = S(F(x, 0), F(x, 0)) = S(x, x).$$

Therefore, $S(x, x) = x$ for all $x \in (e, 1]$, and $S(e, e) \leq S(x, x) = x$. Also $S(e, e) \leq \inf_{x \in (e,1]} x = e$. Since S is a t-conorm, it holds $S(e, e) \geq e$, and therefore $S(e, e) = e$. On the other hand, Lemma 12 insures that S is an idempotent operator on $Y = [0, a]$. Consequently, S is idempotent, i.e., $S = \max$ (Fig. 3).

($\Leftarrow$) The (LD) follows from Lemma 13. □

Results on distributivity of a T-uninorm over a t-norm are given by the following theorem. As it can be seen, the sufficient and necessary condition requires some additional starting assumptions.

Theorem 19. *Let F be a T-uninorm in U_{max} and T be a t-norm.*

(i) *If F is distributive over T then $T(x, x) = x$ for all $x > e$.*
(ii) *Let the function $t(x) = T(x, x)$ be right-continuous at the point $x = e$. Then F is distributive over T if and only if $T = \min$.*

Proof. (i) Analogously to the proof of the previous theorem it can be shown that $T(x, x) = x$ for all $x > e$.

(ii) The right-continuity insures that $T(e, e) = e$. As in Theorem 18, it can be obtain that $T(x, x) = x$ for all $x \in [0, 1]$, i.e., $T = \min$. □

The previous results have considered distributivity of T-uninorms over t-conorms and t-norms. Now, this research can be extended to uninorms from the class U_{max}.

Lemma 20. *Let F be a T-uninorm in U_{max} and U be a uninorm from the class U_{max} with neutral element $e_1 \in (0, 1)$. If F is distributive over U then $e_1 < a$.*

Proof. Let suppose the opposite, i.e., $e_1 > a$. For $x = e_1,\ y = 0,\ z = 1$ assumed distributivity leads to the following contradiction

$$e_1 = F(e_1, 1) = F(e_1, U(0, 1)) = U(F(e_1, 0), F(e_1, 1)) = U(a, e_1) = a.$$

Therefore, $e_1 \leq a$.

If the assumption is now $e_1 = a$, then for $e < x < e_1 = a$ and $y = 0,\ z = 1$, from distributivity law follows

$$a = F(x, 1) = F(x, U(0, 1)) = U(F(x, 0), F(x, 1)) = U(x, a) = U(x, e_1) = x$$

which is again a contradiction. Hence, $e_1 < a$. □

The previous lemma has proved that $e_1 < a$. The following one will explain relation between neutral elements e and e_1, where e is the neutral element of the underlying uninorm of the observed T-uninorm and e_1 is the neutral element of the considered uninorm.

Lemma 21. *Let F be a T-uninorm in U_{max} and U be a uninorm from the class U_{max} with neutral element $e_1 \in (0, 1)$. If F is distributive over U then exactly one of the following cases holds:*

(i) $e_1 = e$;
(ii) $e < e_1$.

Proof. Let us suppose that $e_1 < e$. For $0 < x < e_1,\ y = e,\ z = 0$, the assumed distributivity leads to the following contradiction

$$x = F(x, e) = F(x, U(e, 0)) = U(F(x, e), F(x, 0)) = U(x, 0) = 0.$$

Therefore, either $e_1 = e$ or $e < e_1$ holds. □

Theorem 22. *Let F be a T-uninorm in $U_{\max}$ and U be a uninorm from the class $U_{\max}$ with a neutral element $e_1 \in (0, 1)$ and underlying t-norm T such that $T(x, x)$ is right-continuous at the point $x = e$. F is distributive over U if and only if $e_1 < a$ and exactly one of the following cases is fulfilled:*

(i) *$e_1 = e$, and U is an idempotent uninorm, i.e., $U = U_{e_1}^{\max}$,*
(ii) *$e < e_1, U = U_{e_1}^{\max}$, and F is given by*

$$F(x,y) = \begin{cases} eT_1\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (e_1 - e)S_1'\left(\frac{x-e}{e_1-e}, \frac{y-e}{e_1-e}\right) & \text{if } (x,y) \in [e,e_1]^2, \\ e_1 + (a - e_1)S_1''\left(\frac{x-e_1}{a-e_1}, \frac{y-e_1}{a-e_1}\right) & \text{if } (x,y) \in [e_1,a]^2, \\ a + (1-a)T'\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a], \\ \max & \text{otherwise,} \end{cases} \tag{11}$$

where T_1 and T' are t-norms, and S_1' and S_1'' are t-conorms (Fig. 4).

Proof. ($\Rightarrow$) Let F be a T-uninorm from the class $U_{\max}$ and U be uninorm from $U_{\max}$ that satisfy distributivity law. From Lemma 20 and Lemma 21 follows that $e_1 < a$, and either $e = e_1$ or $e < e_1$. The next step is to prove that U is an idempotent uninorm.

- For $x \geq a$, as in Theorem 18, holds $U(x, x) = x$.
- When $e = e_1$, since $U(e, e) = e$ according Lemma 12 holds $U(x, x) = x$ for $x \leq a$.
- When $e < e_1$, as in Theorem 19, holds $U(x, x) = x$ for $x \leq a$.

Consequently, in the both cases U is an idempotent uninorm, i.e., $U = U_{e_1}^{\max}$ and the claim (i) is proved.

Now, let us consider the structure of F for $e < e_1$. First, there has to be shown that $F(e_1, e_1) = e_1$. For $x = y = e_1$, $z = e$ from the distributivity law follows

$$e_1 = F(e_1, e) = F(e_1, U(e_1, e)) = U(F(e_1, e_1), F(e_1, e)) = U(F(e_1, e_1), e_1) = F(e_1, e_1).$$

For $e \leq x \leq e_1$ holds the following

$$e_1 = F(e, e_1) \leq F(x, e_1) \leq F(e_1, e_1) = e_1.$$

For $e_1 \leq x \leq a$, $y = e_1$, $z = e$ from the distributivity law follows

$$x = F(x, e) = F(x, U(e_1, e)) = U(F(x, e_1), F(x, e)) = U(F(x, e_1), x).$$

Since $F(x, e_1) \geq F(x, e) = x \geq e_1$, and $U = U_{e_1}^{\max}$ we have

$$x = U(F(x, e_1), x) = \max(F(x, e_1), x) = F(x, e_1).$$

Therefore,

$$F(x, e_1) = \begin{cases} e_1 & \text{for } e \leq x \leq e_1, \\ x & \text{for } e_1 \leq x \leq a. \end{cases} \tag{12}$$

Now, after considering (5) and (12), it is easy to show that F is given by (11).

($\Leftarrow$) It is enough to prove the claim (ii), since the proof for the claim (i) is analogous. Therefore, let F be a T-uninorm given by (11) and $U = U_{e_1}^{\max}$. To prove the distributivity law, we have to consider $4^3 = 64$ cases. However, directly from the Lemma 13 we have distributivity for $x \in [0, 1]$ and $(y, z) \in [0, e_1]^2 \cup [e_1, 1]^2$. Otherwise $U(y, z) = z$ for $y < e_1 < z$,

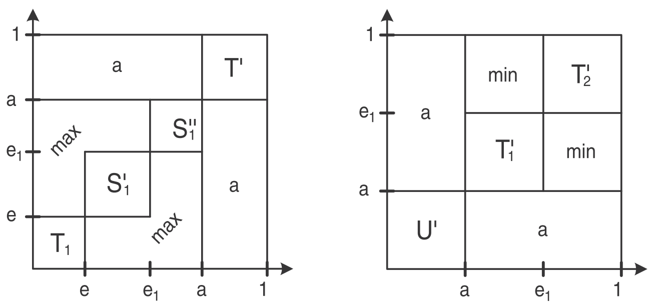

Fig. 4. Structure of T-uninorms given by (11) and (13).

and $L = F(x, U(y, z)) = F(x, z)$. We consider four cases for evaluation of the L and right side $R = U(F(x, y), F(x, z))$ of the distributivity law. Since $y < e_1 < z$, it follows $F(x, y) \le F(x, z)$.

- If $x \le e$, then $F(x, z) \in \{z, a\}$. Since $F(x, y) \le F(e, y) = y < e_1 < F(x, z)$ we obtain $R = \max(F(x, y), F(x, z)) = F(x, z)$.
- If $e \le x \le e_1$, then $F(x, z) \in \{z, a\}$ and as in the previous case we obtain $R = F(x, z)$.
- If $e_1 \le x \le a$, then $e_1 = F(e_1, 0) \le F(x, y) \le F(x, e_1) = x \le a$ and $e_1 \le x = F(x, e_1) \le F(x, z) \le F(a, z) = a$. Thus we obtain $R = \max(F(x, y), F(x, z)) = F(x, z)$.
- If $x \ge a$, then $F(x, y) \ge F(a, y) = a$ and $F(x, z) \ge F(a, z) = a$. Thus we obtain $R = \max(F(x, y), F(x, z)) = F(x, z)$.

As seen above, in all considered cases we obtain $L = R$ which proves that the distributivity law holds. □

Remark 23.

- The case (i) from the previous theorem also holds without assumption of right-continuity for the function $T(x, x)$ at the point $x = e$, while the case (ii), according to Theorem 19, has the following form:
- If F is distributive over U then $U(x, x) =$ for $x > e$ and F is given by (11).
- The restriction of the previous theorem for $a = 1$, i.e., for T-uninorm being just a uninorm from the class $U_{\max}$, has been shown in [26,27]. The case (i) generalizes the Proposition 6.2 from [26,27], and the case (ii) generalizes the Proposition 6.3 from [26].

Now, let us consider distributivity of a T-uninorm over a uninorm from the class $U_{\min}$.

Lemma 24. *Let F be a T-uninorm in $U_{\max}$ and U be a uninorm from the class $U_{\min}$ with neutral element $e_1 \in (0, 1)$. If F is distributive over U then $a < e_1$.*

Proof. Let us suppose opposite. Since $e < a$, the following three cases can be distinguished.

- If $e_1 \le e < a$ for $x = 0,\ y = e_1,\ z = a$, the assumed distributivity leads to a contradiction
$$a = F(0, a) = F(0, U(e_1, a)) = U(F(0, e_1), F(0, a)) = U(0, a) = \min(0, a) = 0.$$
- If $e < e_1 < a$ for $x = e_1,\ y = e,\ z = a$ the assumed distributivity leads to a contradiction
$$e_1 = F(e_1, e) = F(e_1, U(e, a)) = U(F(e_1, e), F(e_1, a)) = U(e_1, a) = a.$$
- If $a = e_1$ for $a < x < 1,\ y = 0,\ z = 1$ from distributivity law follows
$$a = F(x, 0) = F(x, U(0, 1)) = U(F(x, 0), F(x, 1)) = U(a, x) = U(e_1, x) = x$$
which is again a contradiction.

Therefore, $a < e_1$. □

Theorem 25. *Let F be a T-uninorm in $U_{\max}$ and U be a uninorm from the class $U_{\min}$ with neutral element $e_1 \in (0, 1)$ and underlying t-norm T such that $T(x, x)$ is right-continuous at the point $x = e$. F is distributive over U if and only if $a < e_1$, $U = U_{e_1}^{\min}$ and F is given by*

$$F(x, y) = \begin{cases} aU'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x, y) \in [0, a]^2, \\ a + (e_1 - a)T_1'\left(\frac{x-a}{e_1-a}, \frac{y-a}{e_1-a}\right) & \text{if } (x, y) \in [a, e_1]^2, \\ e_1 + (1 - e_1)T_2'\left(\frac{x-e_1}{1-e_1}, \frac{y-e_1}{1-e_1}\right) & \text{if } (x, y) \in [e_1, 1]^2, \\ a & \text{if } (x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a], \\ \min & \text{otherwise,} \end{cases} \tag{13}$$

where T_1', T_2' are t-norms, and U' is a uninorm from the class $U_{\max}$ (Fig. 4).

Proof. (⇒) Let F be a T-uninorm and U be a uninorm from the class $U_{\min}$ that satisfy the distributivity law. Lemma 24 insures that $a < e_1$. As in Theorem 22, there can be proved that $U(x, x) = x$ for all $x \in [0, 1]$, i.e., $U = U_{e_1}^{\min}$.

Now, the next step is to show that $F(e_1, e_1) = e_1$. For $x = y = e_1,\ z = 1$ from the distributivity low follows

$$e_1 = F(e_1, 1) = F(e_1, U(e_1, 1)) = U(F(e_1, e_1), F(e_1, 1)) = U(F(e_1, e_1), e_1) = F(e_1, e_1).$$

For $e_1 \le x \le 1$ we have

$$e_1 = F(e_1, e_1) \le F(x, e_1) \le F(1, e_1) = e_1$$

For $a \le x \le e_1,\ y = e_1,\ z = 1$ from the distributivity low follows

$$x = F(x, 1) = F(x, U(e_1, 1)) = U(F(x, e_1), x).$$

Since $F(x, e_1) \le F(x, 1) = x \le e_1$, and $U = U_{e_1}^{\min}$, there holds

$$x = U(F(x, e_1), x) = \min(F(x, e_1), x) = F(x, e_1).$$

Therefore,

$$F(x, e_1) = \begin{cases} e_1 & \text{for } \ x \geq e_1, \\ x & \text{for } \ a \leq x \leq e_1. \end{cases} \tag{14}$$

Now, it is easy to show that t-norm T' from the (5) is the ordinal sum of T'_1 and T'_2, i.e., F is given by (13).

($\Leftarrow$) Conversely, let F be a T-uninorm given by (13) and $U = U^{\min}_{e_1}$. As in Theorem 22, there can be proved that F is distributive over U. □

Remark 26. It has been proved in [26] (see Lemma 6.1) that if F is a uninorm from the class $U_{\max}$, i.e., a T-uninorm in $U_{\max}$ with annihilator $a = 1$, then there is no uninorm U from the class $U_{\min}$ such that F is distributive over U. Theorem 25 shows that, when T-uninorm in $U_{\max}$ has annihilator $a \in (0, 1)$, there is a uninorm $U = U^{\min}_{e_1} \in U_{\min}$ with neutral element $e_1 > a$ such that F is distributive over U.

4. Conditional distributivity for *T*-uninorms

Theorems shown in the previous section illustrate that the distributivity law is a very strong condition since it simplifies the structure of the inner operator considerably, i.e., the inner operator is being reduced to an idempotent operator. Thus, it seems to be reasonable direction of research to restrict the domain of the distributivity law, as given in Definition 15, in order to obtain some new solutions that are non-idempotent. Therefore, this section contains a counterparts of the Theorems 18, 22, 25 from the previous section for the restricted domain (Fig. 5). In order to give a characterization of all pairs (F, G) satisfying (CD) condition, according to Theorem 16, we have to suppose some kind of continuity for F and G.

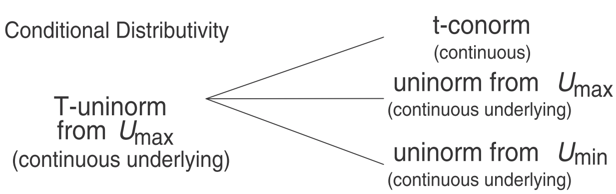

Fig. 5. Topic of Section 4.

Theorem 27. *Let F be a T-uninorm in $U_{\max}$ with continuous underlying t-norm T', and S be a continuous t-conorm. F is conditionally distributive over S if and only if exactly one of the following cases is fulfilled:*

(i) $S = S_M$;
(ii) *there is a $c \in [a, 1)$ such that S is of the form (9) and F is given by*

$$F(x, y) = \begin{cases} aU'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } \ (x, y) \in [0, a]^2, \\ a + (c - a)T_1\left(\frac{x-a}{c-a}, \frac{y-a}{c-a}\right) & \text{if } \ (x, y) \in [a, c]^2, \\ c + (1 - c)T_P\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } \ (x, y) \in [c, 1]^2, \\ a & \text{if } \ (x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a], \\ \min & \text{otherwise,} \end{cases} \tag{15}$$

where T_1 is a continuous t-norm, T_P is the product t-norm and U' is a uninorm from the class $U_{\max}$ (Fig. 6).

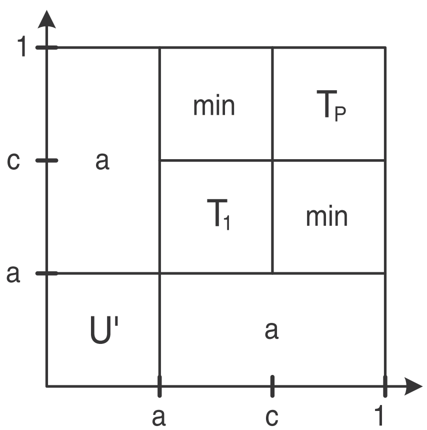

Fig. 6. Structure of T-uninorm from Theorem 27.

Proof. ($\Rightarrow$) Let F be conditionally distributive over S.

- For $x \le a$, as in Theorem 18, there can be shown that $S(x,x) = x$.
- Let $x \ge a$. If $c \in [a, 1)$ is an idempotent element of S, then for all $x \in [a, 1]$ holds

$$F(x,c) = F(x, S(c,c)) = S(F(x,c), F(x,c)),$$

  that is, $F(x, c)$ is also an idempotent element of S. Due to continuity of $F\mid_{[a,1]^2}$, range of function $F(., c)$ for input values from $[a, 1]$ is $[a, c]$. The previous imply that all elements from $[a, c]$ are idempotents of S.

Hence, either all elements from $[0, 1]$ are idempotent elements for t-conorm S and, therefore $S = S_M = \max$, or there is the largest nontrivial idempotent element $c \in [a, 1)$ of S, i.e.,

$$S(x,y) = \begin{cases} c + (1-c)S^*\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x,y) \in [c,1]^2, \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{16}$$

where S^* is a continuous Archimedean t-conorm. Now, as in Theorem 16, it can be proved that c is also an idempotent element of F, i.e., T-uninorm F on the square $[a, 1]^2$ is of the following form

$$F(x,y) = \begin{cases} a + (c-a)T_1\left(\frac{x-a}{c-a}, \frac{y-a}{c-a}\right) & \text{if } (x,y) \in [a,c]^2, \\ c + (1-c)T_2\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x,y) \in [c,1]^2, \\ \min(x,y) & \text{otherwise}, \end{cases} \tag{17}$$

where T_1 and T_2 are continuous t-norms. Also, in the same manner as in Theorem 16, it can be obtained that S^* is a nilpotent t-conorm, i.e., S is of the form (9), and that T_2 is a strict t-norm such that F is of the form (15).

($\Leftarrow$) Now, if the starting assumption is that S is a t-conorm of the form (9) and F a T-uninorm of the form (15), it can be easily shown that condition (CD) holds. For input values from $[c, 1]^2$ the problem is reduced to the pair (T_P, S_L) which satisfies (CD), and in all other cases it follows from Lemma 13. $\square$

Example 28. Operator F given by

$$F(x,y) = \begin{cases} \min & \text{if } (x,y) \in [0,\frac{1}{5}]^2 \cup [\frac{1}{4},\frac{1}{2}] \times [\frac{1}{4},1] \cup [\frac{1}{4},1] \times [\frac{1}{4},\frac{1}{2}], \\ 2xy - x - y + 1 & \text{if } (x,y) \in [\frac{1}{2},1]^2, \\ \frac{1}{4} & \text{if } (x,y) \in [0,\frac{1}{4}] \times [\frac{1}{4},1] \cup [\frac{1}{4},1] \times [0,\frac{1}{4}], \\ \max & \text{otherwise}, \end{cases} \tag{18}$$

is a T-uninorm in $U_{\max}$ with annihilator $a = \frac{1}{4}$, obtained by (15) where $U' = U^{\max}_{\frac{1}{5}}$, $T_1 = \min$ and $c = \frac{1}{2}$. The corresponding t-conorm is of the form (9).

Theorem 29. *Let F be a T-uninorm in $U_{\max}$ with a continuous underlying t-norm T', and U be a uninorm from the class $U_{\max}$ with a neutral element $e_1 \in (0, 1)$ and continuous underlying t-norm and t-conorm. F is conditionally distributive over U if and only if $e_1 < a$ and exactly one of the following cases is fulfilled:*

(i) *$e_1 = e$, and U is an idempotent uninorm, i.e., $U = U^{\max}_{e_1}$,*
(ii) *$e_1 = e$, and there is a $c \in [a, 1)$ such that F and U are given by*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e_1]^2, \\ c + (1-c)S_L\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x,y) \in [c,1]^2, \\ \max(x,y) & \text{otherwise} \end{cases} \tag{19}$$

and

$$F(x,y) = \begin{cases} aU'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ a + (c-a)T_2\left(\frac{x-a}{c-a}, \frac{y-a}{c-a}\right) & \text{if } (x,y) \in [a,c]^2, \\ c + (1-c)T_P\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x,y) \in [c,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a], \\ \min & \text{otherwise}, \end{cases} \tag{20}$$

(iii) *$e < e_1$, $U = U^{\max}_{e_1}$, and F is given by (11)*
(iv) *$e < e_1$, and there is a $c \in [a, 1)$ such that U is given by (19) and F is given by*

$$F(x,y) = \begin{cases} eT_1\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (e_1 - e)S'_1\left(\frac{x-e}{e_1-e}, \frac{y-e}{e_1-e}\right) & \text{if } (x,y) \in [e,e_1]^2, \\ e_1 + (a - e_1)S''_1\left(\frac{x-e_1}{a-e_1}, \frac{y-e_1}{a-e_1}\right) & \text{if } (x,y) \in [e_1,a]^2, \\ a + (c-a)T_3\left(\frac{x-a}{c-a}, \frac{y-a}{c-a}\right) & \text{if } (x,y) \in [a,c]^2, \\ c + (1-c)T_P\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x,y) \in [c,1]^2, \\ \min & \text{if } (x,y) \in [a,c] \times [c,1] \cup [c,1] \times [a,c], \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a], \\ \max & \text{otherwise}, \end{cases} \tag{21}$$

where U' is a uninorm from the class U_{max}, S_1', S_1'' are t-conorms, S_L is the Lukasiewicz t-conorm, T_1 is a t-norm, T_P is the product t-norm and T_2, T_3 are continuous t-norms (Fig. 7 and Fig. 8).

Proof. ($\Rightarrow$) Let F be conditionally distributive over U. The first step is to prove that $e_1 < a$. Let us suppose the opposite.

- Let $e_1 > a$. Now, for $x = e_1$, $z = 0$ and for an arbitrary $y \in (e_1, 1)$, the assumed (CD) insures the following

  $$F(e_1, U(y, 0)) = U(F(e_1, y), F(e_1, 0)).$$

  Since $y \in (e_1, 1)$, $e_1 > a$, $U(0, y) = y$ and $F(e_1, 0) = a$, the following is obtained

  $$F(e_1, y) = U(a, F(e_1, y)). \tag{22}$$

  Due to the assumption of continuity, (22) can be extend to $y = 1$ and we have

  $$e_1 = F(e_1, 1) = U(a, e_1) = a,$$

  which is in contradiction to the assumption $e_1 > a$.
- If the assumption is now $e_1 = a$, then for some x, y, z such that $e < x < e_1 = a < y < 1$ and $z = 0$, from (CD) follows

  $$a = F(x, y) = F(x, U(0, y)) = U(F(x, 0), F(x, y)) = U(x, a) = U(x, e_1) = x$$

  which is again a contradiction.

Therefore $e_1 < a$. As in Lemma 21, there can be proved that either $e = e_1$ or $e < e_1$ holds. In the sequel we suppose that $e < e_1$, since the case when $e = e_1$ is similar.

- For $x \le a$, as in Theorem 22, holds $U(x, x) = x$ and the structure of F on the square $[0, a]^2$ is given as in (21).
- For $x \ge a$, as in Theorem 27, there can be proved that either U is an idempotent uninorm and F is given by (11), or there is a $c \in [a, 1)$ such that U and F are given by (19) and (21), respectively.

($\Leftarrow$) On the other hand, if the observed T-uninorm F and uninorm U are of the form (21) and (19), respectively, the (CD) condition can be proved as in Theorem 22. □

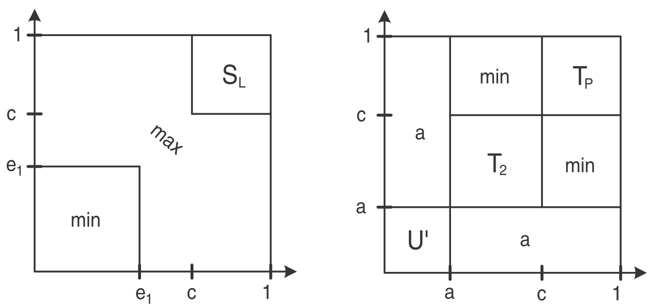

Fig. 7. Structure of conditionally distributive pair from Theorem 29(ii).

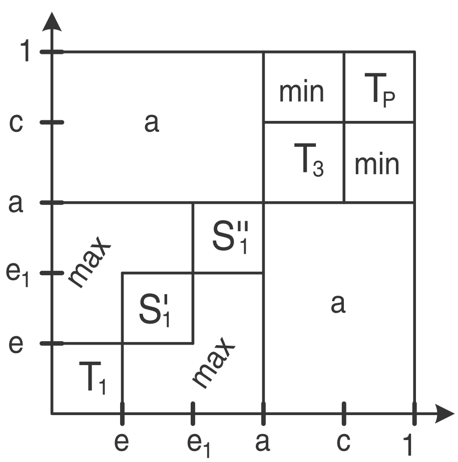

Fig. 8. Structure of the T-uninorm from Theorem 29 (iv).

Theorem 30. *Let F be a T-uninorm in $U_{\max}$ with a continuous underlying t-norm T', and U be a uninorm from the class $U_{\min}$ with a neutral element $e_1 \in (0, 1)$ and continuous underlying t-norm and t-conorm. F is conditionally distributive over U if and only if $a < e_1$ and exactly one of the following cases is fulfilled:*

(i) *$U = U_{e_1}^{\min}$, and F is given by (13);*
(ii) *there is a $c \in [e_1, 1)$ such that F and U are given by*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e_1) \times [0,1] \cup [0,1] \times [0,e_1), \\ c + (1-c)S_L\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x,y) \in [c,1]^2, \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{23}$$

and

$$F(x,y) = \begin{cases} aU'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ a + (e_1 - a)T_1'\left(\frac{x-a}{e_1-a}, \frac{y-a}{e_1-a}\right) & \text{if } (x,y) \in [a,e_1]^2, \\ e_1 + (c-e_1)T_1''\left(\frac{x-e_1}{c-e_1}, \frac{y-e_1}{c-e_1}\right) & \text{if } (x,y) \in [e_1,c]^2, \\ c + (1-c)T_P\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \text{if } (x,y) \in [c,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a], \\ \min & \text{otherwise}, \end{cases} \tag{24}$$

where U' is a uninorm from the class $U_{\max}$, S_L is the Lukasiewicz t-conorm, T_P is the product t-norm and T_1', T_1'' are continuous t-norms (Fig. 9).

Proof. ($\Rightarrow$) Let F be conditionally distributive over U.

- As in Theorem 25 holds $a < e_1$.
- For $x \le a$, as in Theorem 25, there can be proved that $U(x,x) = x$.
- Since $U(e_1, e_1) = e_1 < 1$, (CD) insures the following

  $$F(x, e_1) = F(x, U(e_1, e_1)) = U(F(x, e_1), F(x, e_1)),$$

  i.e., $F(x, e_1)$ is an idempotent element for U. Due to the continuity of function $F(., e_1)$: $[a, 1] \to [a, e_1]$, all elements from $[a, e_1]$ are idempotent for U. Since U is a uninorm, and that previously was shown that $U(x,x) = x$ for $x \le a$, follows that $U = \min$ on the square $[0, e_1]^2$.
- Let us prove that e_1 is an idempotent element of F. For $x = e_1$, $z = e_1$, and an arbitrary $y \in (e_1, 1)$ from equation (CD) follows

  $$F(e_1, y) = F(e_1, U(y, e_1)) = U(F(e_1, y), F(e_1, e_1)).$$

  Due to the assumption of continuity, the previous equality can be extended to $y = 1$ and $e_1 = F(e_1, 1) = U(e_1, F(e_1, e_1)) = F(e_1, e_1)$. Now, since $T' = F\,|_{[a,1]^2}$ is a continuous t-norm immediately follows that T' is ordinal sum T_1' and T_2', i.e., F is given by (13).

  Therefore, it is proved that $U = \min$ on the square $[0, e_1]^2$, and F is given by (13).

- For $x \ge e_1$, on the same way as in Theorem 27 when $x \ge a$, there can be proved that either U is an idempotent uninorm, i.e., $U = U_{e_1}^{\min}$ and F is given (13), or there is a $c \in [e_1, 1)$ such that U, F are given by (23) and (24) respectively.

($\Leftarrow$) On the other hand, if the observed T-uninorm F and uninorm U are of the form (24) and (23), respectively, the (CD) condition is proved as in previous theorems. □

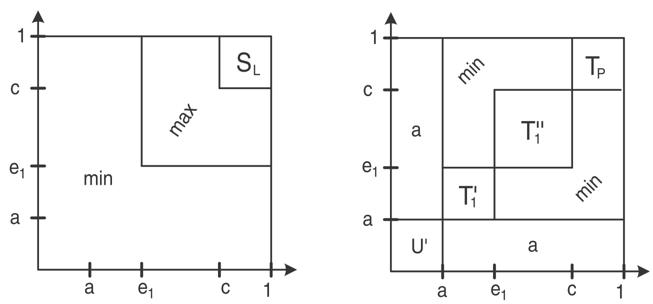

Fig. 9. Structure of conditionally distributive pair from Theorem 30(ii).

5. Conclusions

Distributivity and conditional distributivity of a T-uninorm in $U_{\max}$ with an annihilator $a \in (0, 1)$, over t-conorm and uninorm from the class $U_{\min} \cup U_{\max}$ is considered through this paper. Results from the third section, when distributivity law hold on the whole domain, extend and upgrade the corresponding ones from [26,27]. It turned out that the distributivity law is a rather strong condition, since it simplifies the structure of the inner operator considerably, i.e., it is being reduced to an idempotent operator. Consequently, in the Section 4 we have given a characterization of all pairs (F, G) satisfying distributivity law on the restricted domain. Now, the conditional distributivity produces a larger variety of solutions. Research in the fourth section is the continuation investigation of conditional distributivity for aggregation operators with annihilator [17,23]. In the forthcoming work the distributivity and conditional distributivity for the other classes of T-uninorms and uninorms will be considered. Also, the further research will be focused on the obtained structures and possible application to utility theory.

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