Uncertain Fuzzy Clustering: Interval Type-2 Fuzzy Approach to C-Means

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***Abstract*—In many pattern recognition applications, it may be impossible in most cases to obtain perfect knowledge or information for a given pattern set. Uncertain information can create imperfect expressions for pattern sets in various pattern recognition algorithms. Therefore, various types of uncertainty may be taken into account when performing several pattern recognition methods. When one performs clustering with fuzzy sets, fuzzy membership values express assignment availability of patterns for clusters. However, when one assigns fuzzy memberships to a pattern set, imperfect information for a pattern set involves uncertainty which exist in the various parameters that are used in fuzzy membership assignment. When one encounters fuzzy clustering, fuzzy membership design includes various uncertainties (e.g., distance measure, fuzzifier, prototypes, etc.). In this paper, we focus on the uncertainty associated with the fuzzifer parameter m that controls the amount of fuzziness of the final C-partition in the fuzzy C-means (FCM) algorithm. To design and manage uncertainty for fuzzifier m, we extend a pattern set to interval type-2 fuzzy sets using two fuzzifiers m_1 and m_2 which creates a footprint of uncertainty (FOU) for the fuzzifier m. Then, we incorporate this interval type-2 fuzzy set into FCM to observe the effect of managing uncertainty from the two fuzzifiers. We also provide some solutions to type-reduction and defuzzification (i.e., cluster center updating and hard-partitioning) in FCM. Several experimental results are given to show the validity of our method.**

***Index Terms*—Fuzzy C-means (FCM), fuzzy clustering, interval type-2 fuzzy sets, type-2 fuzzy sets.**

I. INTRODUCTION

TYPE-2 fuzzy sets allow us to model various uncertainties, which cannot be appropriately managed by type-1 fuzzy sets [1]–[4]. Of course, employment of type-2 fuzzy sets usually increases the computational complexity in comparison with type-1 fuzzy sets due to the additional dimension of having to compute secondary grades for each primary membership. However, if we cannot obtain satisfactory results using type-1 fuzzy sets, employment of type-2 fuzzy sets for managing uncertainty may allow us to obtain desirable results. Therefore, if the employment of type-2 fuzzy sets can provide significant improvement on performance (depending on the application), the increase of computational complexity due to type-2 fuzzy sets may be a small price to pay [3]. Also, if we simplify type-2 fuzzy sets to interval type-2 fuzzy sets, the computational complexity can be significantly reduced in appropriate applications that are applicable. The reduction on the computational complexity is due to the property that all the secondary grades for an interval type-2 fuzzy set are uniformly weighted (i.e., all equal to one).

In general, the management of uncertainty using type-2 fuzzy sets may be applied to various fields where we cannot obtain satisfactory performance with ordinary type-1 fuzzy sets. For example, applications include the classification of coded video streams [5], co-channel interference elimination from nonlinear time-varying communication channels [6], connection admission control [7], control of mobile robots [8], [9], decision making [10], [11], equalization of nonlinear fading channels [12], interval-representation of sets [13], forecasting of time-series [14], diagnosis of diseases [15], inference engine design [16]–[18], learning linguistic membership grades [19], management surveys [20], pre-processing radiographic images [21], medical applications [22], transport scheduling [23], and the list goes on. Most of the published studies focus mainly on type-2 fuzzy logic systems (FLS) and several interesting results have been reported. However, very little work (compared to other areas) has been done in applying type-2 fuzzy sets to the area of pattern recognition. We believe that this is a promising area of research and therefore, we will focus on a particular area, namely, clustering.

Most of the type-2 fuzzy sets research in pattern recognition relate to parameter uncertainty. For example, Ozkan and Turksen have considered uncertainties of various parameters from imperfect information of patterns when applying fuzzy C-means (FCM). In particular, they focused on the uncertainty regarding the fuzzifier parameter m that determines the shape of the membership function [24]. Their research talks about a method that evaluates fuzzifier m according to entropies after removing uncertainties from all other parameters (e.g., number of clusters, cluster centers, etc.). Rhee and Hwang have studied on how to define and manage uncertainty for distance measures when fuzzy membership functions are designed in pattern classification and clustering [25]–[28]. Most of the research was related to uncertain distance measures. However, when one achieves fuzzy clustering in the pattern recognition area, fuzzy membership design includes various types of uncertainty (e.g., distance measure, fuzzy degree, type of prototype, etc).

In this paper, we represent and manage uncertainties in the area of pattern recognition, in particular, clustering. If we can properly identify the various types of uncertainties embedded in various clustering methods, expectation for developing algorithms to improve the performance may be highly considered. Our focus will be on prototype-based clustering, namely, FCM.

Manuscript received February 3, 2005; revised December 18, 2005, March 20, 2006, and June 16, 2006. This work was supported by the research fund of Hanyang University under Grant HY-2002-T.

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Digital Object Identifier 10.1109/TFUZZ.2006.889763

For this, we consider the possibility of the existence of uncertainty on the representation of a fuzzy set in FCM. In particular, we focus on the representation and management of uncertainty which is present in the fuzzy memberships of the patterns which are associated with the varying of fuzzifier m. This fuzzifer m controls the amount of fuzziness of the final C-partition in FCM. Therefore, to design and manage the uncertainty for fuzzifier m, we extend a pattern set to interval type-2 fuzzy sets using two fuzzifiers m_1 and m_2 which creates a footprint of uncertainty (FOU) for the fuzzifier parameter m. Then, we incorporate this interval type-2 fuzzy set into FCM to observe the effect of managing uncertainty from the two fuzzifiers. We also provide some solutions to type-reduction and defuzzification for each step (i.e., cluster center updating and hard-partitioning) in FCM. As a result, this gives us our proposed interval type-2 fuzzy FCM method.

The remainder of this paper is organized as follows. In Section II, we define the *certainty* for fuzzifier m in FCM through some illustrations. In Section III, we define the *uncertainty* for fuzzifier m in FCM, explaining how it can be established. In Section IV, we discuss how to extend a pattern set to an interval type-2 fuzzy set in order to manage the defined uncertainty. In Section V, we present how to apply an interval type-2 fuzzy set to FCM and we propose a hard-partitioning method to obtain final clustering results. In Section VI, we provide several examples showing the validity of our proposed method. Finally, Section VII gives the summary and conclusions.

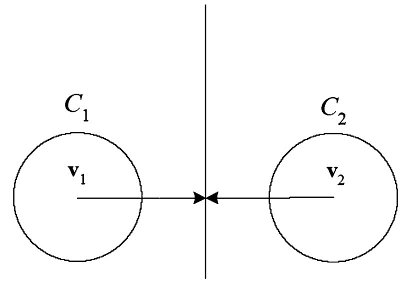


Fig. 1. Crisp membership assignment in FCM.

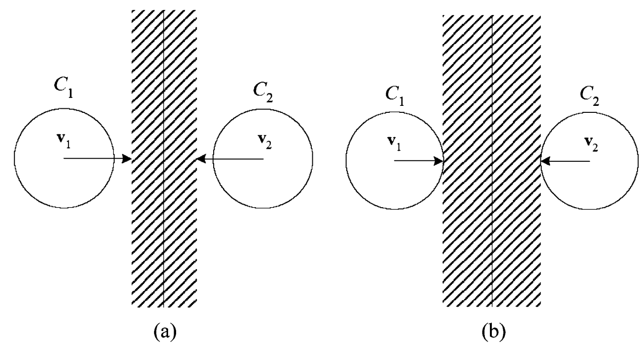


Fig. 2. Maximum fuzzy region for two clusters of similar volume by fuzzifier m of (a) low degree and (b) high degree.

II. CERTAINTY IN THE FUZZIFIER m IN THE FCM ALGORITHM

In general, objective function-based clustering methods are used to minimize the distance between a pattern and cluster prototype such as a point, line, hyperellipsoid, or quadric shell, and the objective is to determine the prototype parameters such as center and radius [29]. In these approaches, iterative algorithms such the C-means is used to seek C partitions that are representative of the pattern set. If a pattern set forms "compact" clusters and each cluster is reasonably separable from each other, desirable clustering results can be obtained using C-means [30]. However, in real world applications, pattern sets of such may be seldom encountered. For example, in several practical cases, a few nonprototypical patterns in a pattern set (e.g., remote patterns distant from the cluster prototype) can cause the cluster prototypes to end up as "poor" representatives of the cluster partitions, since they contribute "fully" in the prototype update. Hence, these patterns may cause undesirable clustering results. To overcome this drawback of C-means, FCM-based clustering algorithms which consider weight values (memberships) to control the contribution degree of a pattern in determining the parameters of cluster prototypes were developed. A high weighted pattern compared to others can play an important role in determining the resulting cluster prototype parameters. On the other hand, the role of a pattern is reduced for patterns that are far from the cluster centers in comparison with highly weighted patterns [31].

These approaches can give desirable clustering results for pattern sets that are not well separable, that is, pattern distributions that seem to contain overlapping clusters. The clustering results are due to the fuzzy membership assignment that considers relative distance of the patterns across all clusters. For example, a pattern that is distant from a cluster contributes less in the updating procedure of the cluster prototypes, whereas a pattern located near contributes more. If we consider point prototypes in describing the clusters for a pattern set, the Euclidean distance measure is used to measure the distance between the cluster prototype (center) and patterns. As an example, Fig. 1 shows two cluster boundaries C_1 and C_2, where the clusters are considered to be identical in structure and density.

The maximally fuzzy membership locations (vertical line) are where patterns are equally distant from the two cluster centers, that is, the relative distance between a pattern and each cluster center equaling 0.5. In most FCM based methods, those patterns will contribute (i.e., help in locating the cluster centers) relatively less than a pattern that is located away from the maximally fuzzy membership location during the cluster center update. As a result, the vertical line in the figure can be considered as a "decision" boundary where patterns located to the left (right) of the boundary belongs to cluster C_1 (C_2). This boundary can be expanded by a fuzzifier parameter m as shown in Fig. 2. If the value of fuzzifier m is increased then the maximum fuzzy boundary becomes wider. The width of the boundary indicates the range one desires to assign maximally fuzzy memberships (based on relative distance) to the patterns.

In general, several studies show that reasonable clustering results can be obtained using $m = 2.0$, however there is hardly no change in the clustering results regardless of fuzzzifier m for types of problems as shown in Fig. 2.

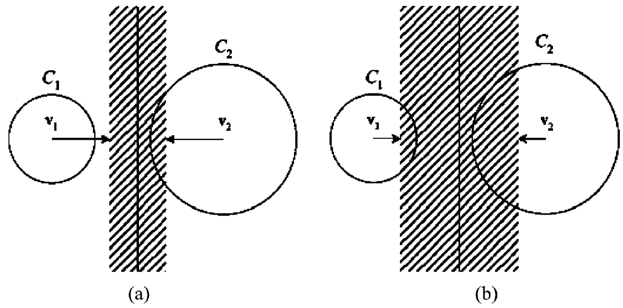


Fig. 3. Maximum fuzzy region for two clusters of different volume by fuzzifier m of (a) low degree and (b) high degree.

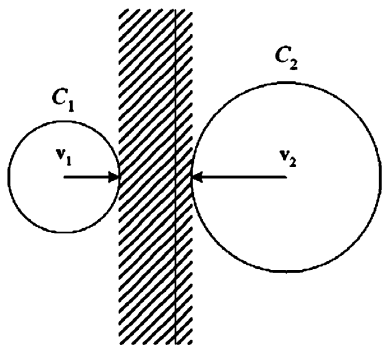


Fig. 4. Desirable maximum fuzzy region.

FCM based methods are known to perform well on pattern sets that contain clusters that are of similar volume of hyperspherical shape and density. However, if the clusters in a pattern set are of different density (i.e., different cluster size/different number of patterns ratios or same clusters size/different number of patterns ratios or different cluster size/same number of patterns ratios), performance of FCM may significantly differ depending on the choice of fuzzifier m. Hence, a good choice of m may be considered to be related to the distribution of the patterns for the pattern set.

For the case as shown in Fig. 3, the maximum fuzzy boundary can cause undesirable clustering results if fuzzifier m is not assigned properly due to the difference in volume between two clusters.

Patterns that are on the left side of the maximum fuzzy boundary that is part of cluster C_2 intuitively contributes more for cluster C_1 rather than cluster C_2. This suggests that appropriate control of fuzzifier m is needed to improve the clustering performance in FCM for pattern distributions that contain clusters that differ significantly in size. If we set fuzzifier m to a large value, a wide maximum fuzzy boundary such as in Fig. 3 can be obtained. This may seem to be desirable for the viewpoint of cluster C_1, however, the estimated cluster center $\mathbf{v}_1$ will tend to move toward C_2. Hence, $\mathbf{v}_1$ will deviate away from the ideal center of cluster C_1. This is also not desirable for the viewpoint of cluster C_2 (an undesirable cluster center may result for C_2). Therefore, the ideal situation is to have the maximum fuzzy region with a wide left region and narrow right region with respect to the maximum fuzzy membership locations as the one shown in Fig. 4.

For each region with respect to the vertical line, the relative distance is 0.5. Unfortunately, a maximum fuzzy region of such cannot be represented by FCM since the change of fuzzifier m affects all clusters equally.

In this paper, we define and manage uncertainty, which exists on establishment of the maximum fuzzy boundary in FCM. We use two fuzzifiers to extend memberships of a pattern set to an interval type-2 fuzzy set and apply it to FCM. This will allow us to define and manage the uncertainty of fuzzifier m in FCM. In order to manage the uncertainty, we apply interval type-2 fuzzy sets instead of type-2 fuzzy sets to help reduce the computational complexity and simultaneously improve clustering performance in FCM. In the next section, we define the *uncertainty* for fuzzifier m in the FCM algorithm.

III. Uncertainty in the Fuzzifier m in the FCM Algorithm

Conventional FCM is an iterative algorithm which assigns fuzzy memberships to pattern data and updates the prototype parameters of the cluster according to the assigned memberships. The assigned memberships play a role as weight values, that is, they represent the contribution degree of a pattern on estimating the cluster prototypes. The amount of contribution depends on the selection of fuzzifier m. Details are briefly described as follows.

Let the updates for center $\mathbf{v}_j$ and membership $u_j\ (\mathbf{x}_i)$ for pattern $\mathbf{x}_i$ be described as

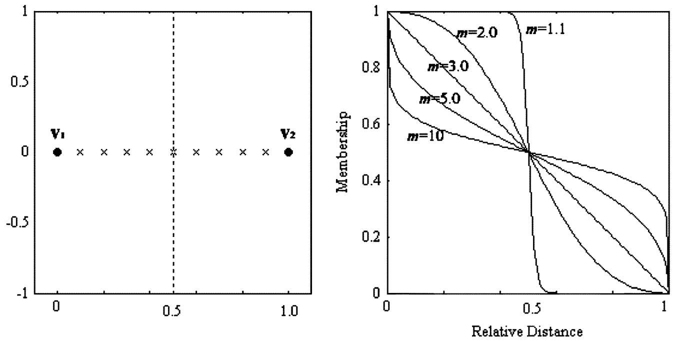
$$\mathbf{v}_j = \frac{\sum_{i=1}^{N} u_j(\mathbf{x}_i)^m \mathbf{x}_i}{\sum_{i=1}^{N} u_j(\mathbf{x}_i)^m}, \qquad j = 1, \ldots, C \tag{1}$$

where

$$u_j(\mathbf{x}_i) = \frac{1}{\sum_{k=1}^{C} \left(\frac{d_{ji}}{d_{ki}}\right)^{2/(m-1)}}. \tag{2}$$

Distance $d_{ji}\ (d_{ki})$ denotes the distance between cluster $j\ (k)$ and pattern $\mathbf{x}_i$. Equation (2) presents the fuzzy membership of pattern $\mathbf{x}_i$ that is assigned by the relative distance between pattern $\mathbf{x}_i$ and cluster center $\mathbf{v}_j$. This value depends on the relative distance between pattern $\mathbf{x}_i$ and all of the cluster centers, and fuzzifier m. Fig. 5 illustrates the variation in the memberships according to the relative distance for various values of m.

As shown in Fig. 5(a), suppose that there exists two cluster centers $\mathbf{v}_1$ and $\mathbf{v}_2$ for a given pattern set (note that the patterns in which $\mathbf{v}_1$ and $\mathbf{v}_2$ are obtained are not shown). The membership plots corresponding to center $\mathbf{v}_1$ for patterns that lie in between the two cluster centers for various values of fuzzifier m can be shown as in Fig. 5(b). When $m \to 1$, the memberships are maximally crisp (hard), that is, patterns that are located just left (right) of the maximum fuzzy boundary are assigned full (0) membership for center $\mathbf{v}_1\ (\mathbf{v}_2)$. Conversely, when $m \to \infty$, the memberships are maximally fuzzy, which means that for patterns that are only located at the centers are assigned full (0) membership otherwise they are assigned memberships of 0.5. As mentioned earlier, if the geometry of the clusters in a pattern set is of similar volume and density, change in fuzzifier m



(a) (b)

Fig. 5. Example illustrating the variation of fuzzy memberships. (a) Pattern set consisting of two centers. (b) Membership plots corresponding to center $\mathbf{v}_1$ for patterns located between $\mathbf{v}_1$ and $\mathbf{v}_2$ in (a) for various values of m.

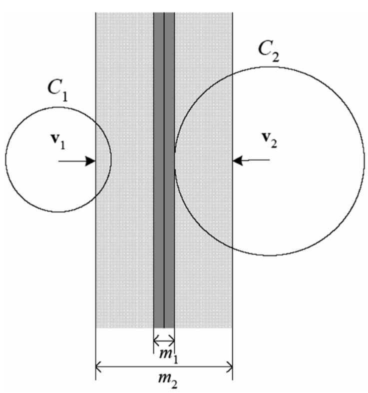
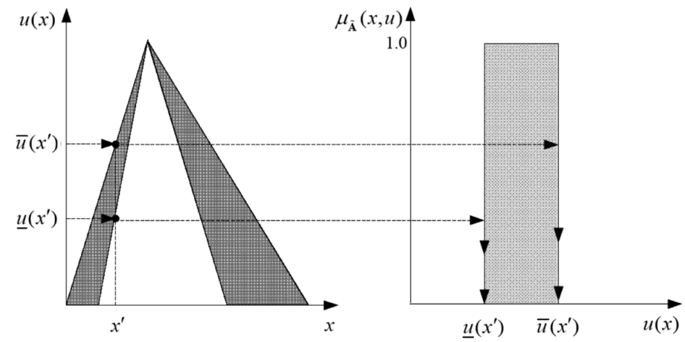


Fig. 6. Uncertain maximum fuzzy region by two fuzzifiers $(m_1 < m_2)$.

will not significantly affect the clustering result in FCM. However, if there is a significant difference in density among clusters in a pattern set, the choice of m will give inconsistent clustering results. The reason is that an impertinent establishment of maximum fuzzy boundary in FCM can provide undesirable updating of the cluster centers. This was previously emphasized in the explanation of Fig. 3. A more desirable establishment of the maximum fuzzy region as in Fig. 4 can allow for desirable clustering results in the FCM. Again, due to the constraint on the memberships we cannot design this region with any particular single value of fuzzifier m to be used in the FCM. However, if we can somehow simultaneously incorporate various values of fuzzifier m, we may perhaps be able to design a desirable maximum fuzzy region such as the one in Fig. 4. This is shown in Fig. 6.

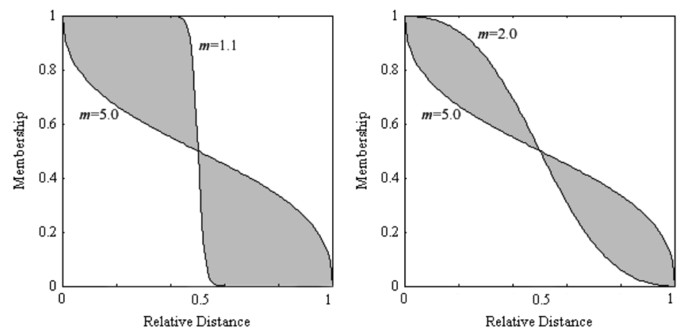
Unlike Figs. 3 and 4, the established maximum fuzzy region in Fig. 6 should be considered uncertain although we establish it with two fuzzifiers m_1 and m_2 which represent different fuzzy degrees. Therefore, we should consider memberships for a pattern as uncertain (fuzzy) instead of as certain (crisp) in the FCM. In other words, memberships for a pattern should not have to rely on a single specific fuzzifier m as in FCM. In this paper, we define uncertainty of fuzzy memberships for a pattern associated to a cluster prototype as an *uncertain* maximum fuzzy boundary such as the case in Fig. 6.

Now, we should properly manage the defined uncertainty so that the uncertain maximum fuzzy boundary approximates to a



(a) (b)

Fig. 7. Example of (a) an interval type-2 fuzzy set and (b) the vertical slice of sample x'.



(a) (b)

Fig. 8. Footprint of uncertainty (FOU) of an interval type-2 fuzzy set for (a) $m = 1.1$ and 5.0 and (b) $m = 2.0$ and 5.0.

desirable one such as in Fig. 4. The appropriate management of defined uncertainty may hopefully improve the clustering performance in FCM. This is explained in the following section by incorporating interval type-2 fuzzy sets into the FCM.

IV. Extension to Interval Type-2 Fuzzy Sets

When we represent uncertainty for a fuzzy set with an interval type-2 fuzzy set, the primary membership $J_{\mathbf{x}_i}$ of a pattern $\mathbf{x}_i$ can be represented as a membership interval with all secondary grades of the primary memberships equaling to one [1]. An interval type-2 fuzzy set $\tilde{\mathbf{A}}$ can be represented as

$$\tilde{\mathbf{A}} = \{((\mathbf{x}, u), \mu_{\tilde{\mathbf{A}}}(\mathbf{x}, u)) | \forall \mathbf{x} \in \mathbf{A}, \forall u \in J_{\mathbf{x}} \subseteq [0,1] \quad \mu_{\tilde{\mathbf{A}}}(\mathbf{x}, u) = 1\}. \quad (3)$$

Fig. 7 illustrates an example of an interval type-2 fuzzy set where the gray shaded region indicates the footprint of uncertainty (FOU).

In the figure, the membership value for a 1-D sample x' is represented by the interval between upper $(\bar{u}(x'))$ and lower $(\underline{u}(x'))$ membership. Therefore, each x' has a primary membership interval as

$$J_{x'} = [\underline{u}(x'), \bar{u}(x')] \quad (4)$$

and the vertical slice of x' is shown in Fig. 7(b), where the secondary grade for the primary membership of each x' equals one, according to the property of interval type-2 fuzzy sets.

In general, computation of fuzzy memberships in FCM is achieved by computing the relative distance among the patterns and cluster centers according to (2), mentioned above. Now, to define the interval of primary membership for a pattern, we define the lower and upper interval memberships using two different values of m. The primary memberships that extend pattern $\mathbf{x}_i$ by interval type-2 fuzzy sets becomes (5), as shown at the bottom of the next page.

In (5), m_1 and m_2 are fuzzifiers which represent different fuzzy degrees. We define the interval of a primary membership for a pattern, as the highest and lowest primary membership for a pattern. These values are denoted by upper and lower membership for a pattern, respectively. The uses of fuzzifiers which represent different fuzzy degrees give different objective functions to be minimized in FCM as

$$J_{m_1}(\mathbf{U},\mathbf{v}) = \sum_{i=1}^{N}\sum_{j=1}^{C} u_j(\mathbf{x}_i)^{m_1} d_{ji}^2$$

and

$$J_{m_2}(\mathbf{U},\mathbf{v}) = \sum_{i=1}^{N}\sum_{j=1}^{C} (u_j(\mathbf{x}_i)^{m_2} d_{ji}^2. \tag{6}$$

Fig. 8 illustrates an example of an interval type-2 fuzzy set according to (5) for a two cluster case as in Fig. 5(a) for fuzzifier combinations of $m_1 = 1.1$ and $m_2 = 5.0$, and $m_1 = 2.0$ and $m_2 = 5.0$.

The difference in fuzzy memberships denotes the FOU as indicted by the shaded gray areas. The membership for patterns located at each current cluster center (the center location at the current iteration) is set to 1.0 and set to 0 for the other corresponding center regardless of the fuzzy degree in a pattern set. Hence, there are no FOU for these patterns. For patterns, which are relatively equal distance from the two cluster centers, memberships are set to 0.5. There are also no FOU for these patterns. Hence, it is reasonable to consider that there exists no uncertainty for the maximum and minimum fuzzy points. The application of interval type-2 fuzzy sets in FCM occurs in the modification of the center-updating and hard-partition procedure. This is discussed in the following section.

V. TYPE-REDUCTION AND DEFUZZIFICATION FOR INTERVAL TYPE-2 FCM

In this paper, we compute an interval of primary memberships for a pattern with two fuzzifiers m_1 and m_2 which represent different degrees of membership. The establishment of the membership interval for a pattern set is to represent and manage the uncertainty which occurs by the use of fuzzifiers m_1 and m_2 in FCM. Hence, we extend a pattern set to interval type-2 fuzzy sets. The uncertainty defined in an interval type-2 fuzzy set should be managed appropriately through all steps of FCM. For this, we manage the uncertainty of fuzzifier m in the FCM algorithm by the following two procedures:

i) procedure for updating cluster centers;
ii) procedure for hard-partitioning (defuzzification) in the final clustering decision.

We begin by modifying the procedure for updating cluster centers by the extension of interval type-2 fuzzy sets while executing the FCM algorithm. For this, we employ type reduction and defuzzification methods using type-2 fuzzy operations [1]. Among the various choices of type-reduction methods, we use the generalized centroid (GC) type-reducer since it is similar to the center-update method in prototype based clustering algorithms. It is to be noted that there exists other type-reduction methods. For example, if we choose height type-reduction, computation complexity can be reduced. However, due to its simple estimation, this can give us inaccurate cluster centers leading to undesirable clustering results. Therefore, we consider only the centroid type-reducer in this paper. Details on other methods can be found in [1].

The centroid of a type-1 fuzzy set can be computed as

$$\mathbf{v_x} = \frac{\sum_{i=1}^{N} x_i u(x_i)}{\sum_{i=1}^{N} u(x_i)}. \tag{7}$$

If a type-2 fuzzy set $\tilde{\mathbf{X}}$ whose domain is discretized into N points, $\mathbf{x}_1, \ldots, \mathbf{x}_N$, we can similarly obtain the centroid of a type-2 fuzzy set $\tilde{\mathbf{X}}$ by the extension principle as

$$\mathbf{v}_{\tilde{\mathbf{X}}} = \sum\nolimits_{u(x_1)\in J_{x_1}} \cdots \sum\nolimits_{u(x_N)\in J_{x_N}} ([f(u(x_1)) * \cdots * f(u(x_N))]) \Big/ \frac{\sum_{i=1}^{N} \mathbf{x}_i u(\mathbf{x}_i)}{\sum_{i=1}^{N} u(\mathbf{x}_i)} \tag{8}$$

Since we consider interval type-2 fuzzy sets, we can replace $f(u(\mathbf{x}_i))$ by 1.0 for pattern set $\mathbf{X}$ and to apply this to fuzzy clus-

$$\bar{u}_j(\mathbf{x}_i) = \begin{cases} \dfrac{1}{\sum_{k=1}^{C} (d_{ji}/d_{ki})^{2/(m_1-1)}}, & \text{if } \dfrac{1}{\sum_{k=1}^{C}(d_{ji}/d_{ki})} < \dfrac{1}{C} \\ \dfrac{1}{\sum_{k=1}^{C} (d_{ji}/d_{ki})^{2/(m_2-1)}}, & \text{otherwise} \end{cases}$$

and

$$\underline{u}_j(\mathbf{x}_i) = \begin{cases} \dfrac{1}{\sum_{k=1}^{C} (d_{ji}/d_{ki})^{2/(m_1-1)}}, & \text{if } \dfrac{1}{\sum_{k=1}^{C}(d_{ji}/d_{ki})} \geq \dfrac{1}{C} \\ \dfrac{1}{\sum_{k=1}^{C} (d_{ji}/d_{ki})^{2/(m_2-1)}}, & \text{otherwise} \end{cases} \tag{5}$$

tering, we employ fuzzy degree m in (8). This can be rewritten as

$$\mathbf{v}_{\tilde{\mathbf{X}}} = [\mathbf{v}_L, \mathbf{v}_R] = \sum_{u(x_1)\in J_{x_1}} \cdots \sum_{u(x_N)\in J_{x_N}} 1 \bigg/ \frac{\sum_{i=1}^{N} \mathbf{x}_i u(\mathbf{x}_i)^m}{\sum_{i=1}^{N} u(\mathbf{x}_i)^m}. \quad (9)$$

Details on how to obtain the previous equations can be found in [1].

As shown in (9), estimated centers by the centroid type-reduction are represented by interval $[\mathbf{v}_L, \mathbf{v}_R]$. Basically, all centers among the interval are needed to be computed in order to represent interval $[\mathbf{v}_L, \mathbf{v}_R]$. For this, we consider embedded type-2 fuzzy sets for an interval type-2 fuzzy set that are represented by all upper and lower values of primary membership $J_{\mathbf{x}_i}$. We can organize $J_{\mathbf{x}_i}$ for a pattern $\mathbf{x}_i$ with $\bar{u}(\mathbf{x}_i)$ and $\underline{u}(\mathbf{x}_i)$. A simple example illustrating embedded type-2 fuzzy sets is shown in Fig. 9.

Consider an example that consists of two patterns of 1-D, that is, x_1 and x_2 shown in Fig. 9(a). If we consider a type-1 fuzzy set that represents the two patterns as $\mathbf{A} = u(x_1)/x_1 + u(x_2)/x_2$, then its corresponding interval type-2 fuzzy set can be expressed as

$$\tilde{\mathbf{A}} = \sum_{x\in X} \mu_{\tilde{\mathbf{A}}}(x)/x = \sum_{x\in X} \left[\int_{u\in J_x} f_x(u)/u\right] \bigg/ x, \quad J_x \subseteq [0,1]. \quad (10)$$

Since $\tilde{\mathbf{A}}$ is an interval type-2 fuzzy set, all of the secondary grades amount to 1.0. Hence, we can arrange (10) as

$$\begin{aligned}\tilde{\mathbf{A}} &= \sum_{x\in \mathbf{X}} \left[\int_{J_x} 1.0/u\right] \bigg/ x \\ &= \left(\int_{J_{x_1}} 1.0/u(x_1)\right) \bigg/ x_1 + \left(\int_{J_{x_2}} 1.0/u(x_2)\right) \bigg/ x_2. \end{aligned} \quad (11)$$

Furthermore, although there are an infinite number of embedded type-2 fuzzy sets, for computation purposes, if we discretize the primary memberships in (11) we obtain

$$\begin{aligned}\tilde{\mathbf{A}} = 1.0/\bar{u}(x_1)/x_1 &+ 1.0/\underline{u}(x_1)/x_1 \\ &+ 1.0/\bar{u}(x_2)/x_2 + 1.0/\underline{u}(x_2)/x_2. \end{aligned} \quad (12)$$

We now can estimate all of the embedded type-2 fuzzy sets present in $\tilde{\mathbf{A}}$, since there exists a finite number. There are two primary memberships since each pattern has an upper and lower membership. Hence, this yields four embedded type-2 fuzzy sets for this example. This is illustrated in Fig. 9(b)–(e) and expressed as follows:

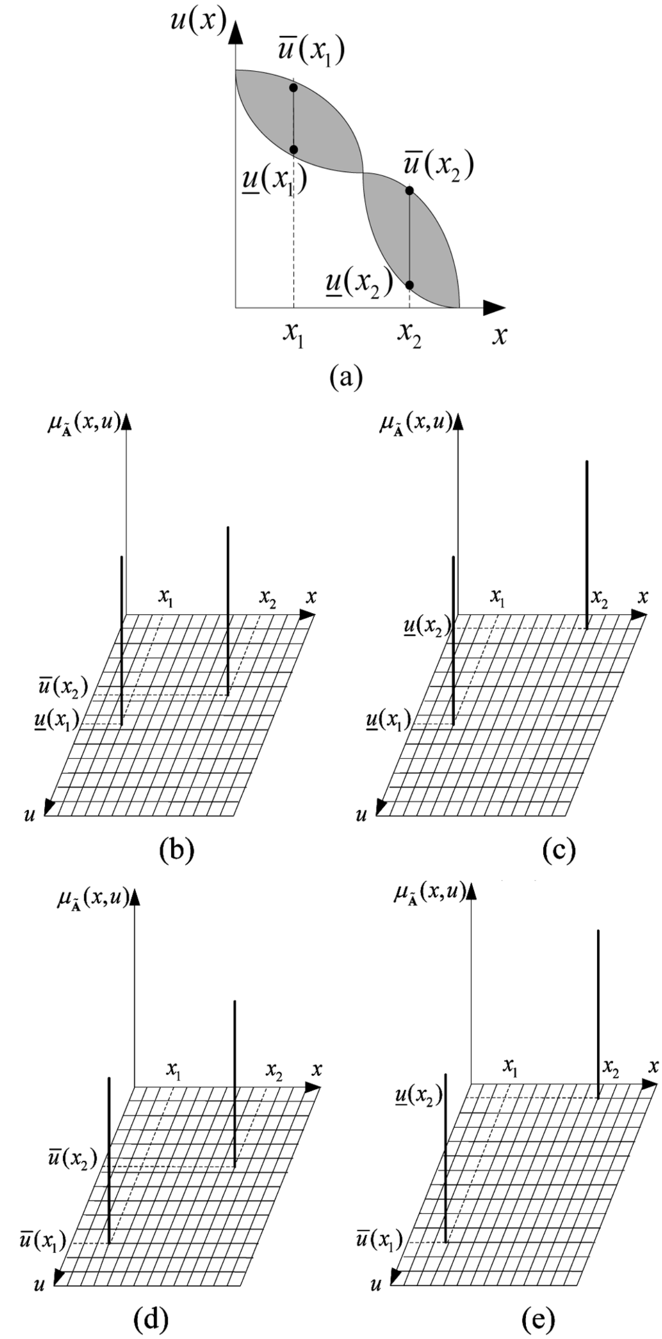


Fig. 9. Example of embedded fuzzy sets for an interval type-2 fuzzy set. (a) Interval type-2 fuzzy set. (b)–(e) Embedded fuzzy sets of (a).

$$\begin{aligned}\tilde{\mathbf{A}}_{e1} &= 1.0/\underline{u}(x_1)/x_1 + 1.0/\bar{u}(x_2)/x_2 \\ \tilde{\mathbf{A}}_{e2} &= 1.0/\underline{u}(x_1)/x_1 + 1.0/\underline{u}(x_2)/x_2 \\ \tilde{\mathbf{A}}_{e3} &= 1.0/\bar{u}(x_1)/x_1 + 1.0/\bar{u}(x_2)/x_2 \\ \tilde{\mathbf{A}}_{e4} &= 1.0/\bar{u}(x_1)/x_1 + 1.0/\underline{u}(x_2)/x_2. \end{aligned} \quad (13)$$

In addition, since the secondary grade of all the interval type-2 fuzzy sets are 1.0 each embedded type-2 fuzzy set naturally becomes an embedded type-1 fuzzy set. Hence,

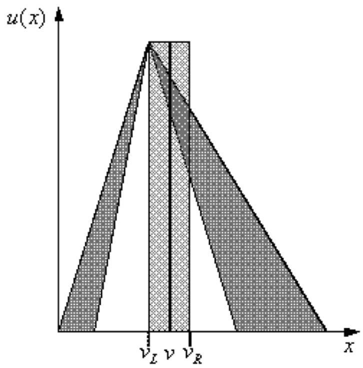


Fig. 10. Example of a crisp center and interval type-1 fuzzy center.

type-reduction for all of the embedded type-2 fuzzy sets can be easily obtained and described as

$$\begin{aligned}\mathbf{A}_{e1} &= \underline{u}(x_1)/x_1 + \bar{u}(x_2)/x_2\\ \mathbf{A}_{e2} &= \underline{u}(x_1)/x_1 + \underline{u}(x_2)/x_2\\ \mathbf{A}_{e3} &= \bar{u}(x_1)/x_1 + \bar{u}(x_2)/x_2\\ \mathbf{A}_{e4} &= \bar{u}(x_1)/x_1 + \underline{u}(x_2)/x_2. \end{aligned} \tag{14}$$

In general, there exists 2^N embedded fuzzy sets in an interval type-2 fuzzy set that consist of N patterns. If the number of patterns in a pattern set is N, then the number of embedded fuzzy sets for an interval type-2 fuzzy set can be obtained as

$$\prod_{i=1}^{N} |J_{x_i}| = 2^N \tag{15}$$

where $|\cdot|$ denotes the cardinality. There are two primary memberships for a pattern, each having an upper and lower membership. Therefore, the number of embedded fuzzy sets is 2^N for N patterns. Also, (15) denotes the number of center intervals that can be obtained from the embedded fuzzy sets as in (14). The union of estimated center intervals from the embedded fuzzy sets constitutes an interval type-1 fuzzy set as

$$\begin{aligned}\mathbf{Y} &= \{1/\mathbf{v}_1, \ldots, 1/\mathbf{v}_{2^N}\}\\ &= 1.0/[\mathbf{v}_L, \mathbf{v}_R] \quad \mathbf{v}_L \leqslant \forall \mathbf{v}_i \leqslant \mathbf{v}_R. \end{aligned} \tag{16}$$

From this approach, we can obtain a type-reduced fuzzy set for an interval type-2 fuzzy set. However, this approach should be avoided in practical cases since the amount of computation can drastically increase if the number of patterns increases. Therefore, we consider an alternative approach which can reduce the computational load. Karnik *et al.* have suggested an iterative algorithm which estimates both ends of an interval, that is, estimating $\mathbf{v}_R$ and $\mathbf{v}_L$ without using all of the embedded fuzzy sets for an interval type-2 fuzzy set [1]. These can be estimated by arranging the pattern set in ascending order and finding the location between the upper and lower membership for a pattern set. It is required for an ascending ordered pattern set to represent a left value $\mathbf{v}_L$ and a right value $\mathbf{v}_R$ of an interval cluster center. For our proposed method, we apply the iterative algorithm to estimate the cluster centers. The iterative algorithm is used to estimate the maximum value $\mathbf{v}_R$ and minimum value $\mathbf{v}_L$. This is described as follows.

Iterative Algorithm for Finding Maximum of Center

$\mathbf{v}_j : \mathbf{v}_R$
FOR all patterns $\mathrm{x}_i = (\mathrm{x}_{i1}, \ldots, x_{iM})$ **DO**
  Compute primary membership $u_j(\mathrm{x}_i)$ using (5);
END FOR
Set fuzzifier $m =$ arbitrary;
Compute centroid $\mathrm{v}'_j = (v'_{j1}, \ldots v'_{jM})$ by (7) and

$$u_j(\mathbf{x}_i) = \frac{\bar{u}(\mathbf{x}_i) + \bar{u}(\mathbf{x}_i)}{2};$$

Sort pattern indexes for all N patterns $(i = 1, \ldots, N)$ in each of M features $(l = 1, \ldots, M)$ in ascending order;
(i.e., Sorted feature $1 : x_{11} \leq \cdots \leq_{N1}$

$\vdots$

  Sorted feature $M : x_{1M} \leq \cdots \leq x_{NM})$
Set comparison = FALSE;
FOR all features **DO**
  WHILE (comparison = FALSE)
    Find interval index $k (1 \leq k \leq N-1)$ such that $x_{kl} = v_{jl'} = x_{(k+1)l}$;
    FOR all patterns **DO**
      IF $(i \leq k)$, **THEN**
        Set primary membership $u_j(\mathbf{x}_i) = \underline{u}(\mathbf{x}_i)$;
      ELSE
        Set primary membership $u_j(\mathbf{x}_i) = \bar{u}(\mathbf{x}_i)$;
      END IF
    END FOR
    Compute maximum of center candidate v_{jl}'' using (10);
    IF $(v_{jl}'' = v_{jl}'')$, **THEN**
      Set comparison = TRUE;
    ELSE
      Set $\mathrm{v}'_{jl} = \mathrm{v}_{jl}''$;
    END IF
  END WHILE;
END FOR
Set $v_R = v_j'' = (v_{j1}'', \ldots, v_{jM}'')$;

The minimum of center $\mathbf{v}_j$ (i.e., $\mathbf{v}_L$), can be obtained using the previous procedure and replacing the first "IF statement" by

IF $(i \leqslant k)$, **THEN**
  Set primary membership $u_j(\mathbf{x}_i) = \bar{u}(\mathbf{x}_i)$;
ELSE
  Set primary membership $u_j(\mathbf{x}_i) = \underline{u}(\mathbf{x}_i)$;
END IF

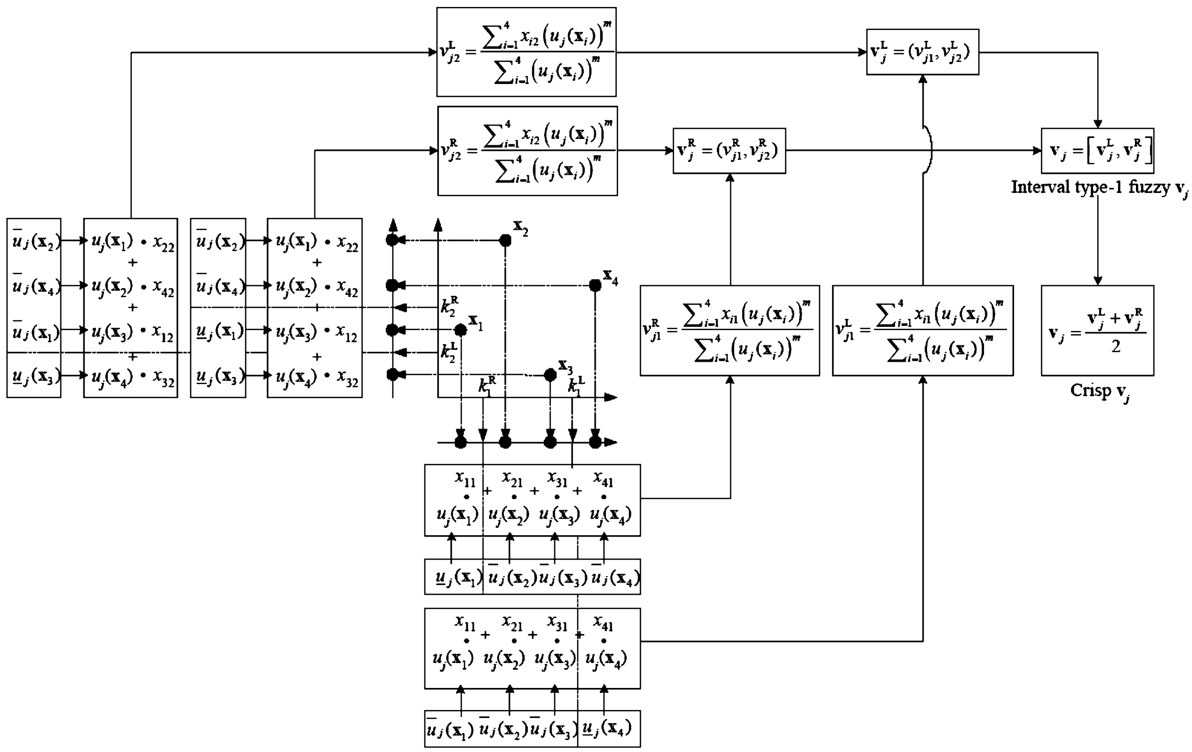


Fig. 11. Example of type-reduction and hard partitioning for center $\mathbf{v}_j$.

The estimated interval type-1 fuzzy set for the cluster centers can be represented as

$$\mathbf{v}_j = 1.0/[\mathbf{v}_L, \mathbf{v}_R]. \tag{17}$$

A crisp center for estimated center $\mathbf{v}_j$ is simply obtained by a defuzzification method for a fuzzy set since estimated center $\mathbf{v}_j$ is an interval type-1 fuzzy set. This can be computed as

$$\mathbf{v}_j = \frac{\sum_{\mathbf{v}\in J_{\mathbf{Y}_j}} (u(\mathbf{v}))\mathbf{v}}{\sum_{\mathbf{v}\in J_{\mathbf{Y}_j}} (u(\mathbf{v}))} = \frac{\mathbf{v}_L + \mathbf{v}_R}{2}. \tag{18}$$

When applying the previous procedures (i.e., type-reduction and defuzzification) as for the example in Fig. 7, the center of an interval type-2 fuzzy set can be obtained as shown in Fig. 10.

Fig. 11 gives an example illustrating the procedure of type-reduction and defuzzification for center $\mathbf{v}_j$.

The estimated center from type-reduction and defuzzification signifies the management of uncertainty which is represented by a membership interval that applies two different values of m in FCM. In this paper, we manage the membership interval according to two different values of m. This is achieved by the modified procedure of center-updating which contains type-reduction and defuzzification. The management of uncertainty with an interval type-2 fuzzy set can yield more desirable cluster centers by our proposed method compared with FCM that uses a single value of m in the case when pattern distributions contain partitions of different size volumes. This is greatly due to the extra degree of freedom where two values of m are used. These values are chosen to appropriately describe the fuzzy degree for the clusters.

As a final clustering result, estimation of the cluster centers is obtained by hard-partitioning of the fuzzy memberships for a pattern. In other words, a pattern is assigned to a cluster according to a fuzzy membership. Hence, we can consider hard-partitioning as defuzzification for a fuzzy set. Hard-partitioning in FCM can be summarized as

$$\begin{aligned}&\textbf{IF } (u_j(\mathbf{x}_i) > u_k(\mathbf{x}_i)), \quad \text{for } k = 1,\ldots,C \quad \text{and } j \neq k\\ &\qquad\textbf{THEN } \mathbf{x}_i \text{ is assigned to cluster } j.\end{aligned} \tag{19}$$

Unfortunately, we cannot apply (19) directly to our proposed method since the pattern set is extended to an interval type-2 fuzzy set. We need to reduce the type of an interval type-2 fuzzy set before hard-partitioning. Type-reduction before hard-partition should be handled carefully since upper and lower membership $\bar{u}(\mathbf{x}_i)$ and $\underline{u}(\mathbf{x}_i)$ cannot be used directly in achieving type-reduction before hard-partitioning.

In our proposed method, we perform type-reduction in order to estimate cluster centers. For this, left memberships $u_j^L(\mathbf{x}_i)$ and right memberships $u_j^R(\mathbf{x}_i)$ for all patterns have already been estimated to organize $\mathbf{v}_L$ and $\mathbf{v}_R$, respectively. Therefore, type-reduction can be achieved using $u_j^L(\mathbf{x}_i)$ and $u_j^R(\mathbf{x}_i)$ to partition a pattern set into clusters. In this approach, type-reduction and hard-partitioning can be obtained as follows:

$$\text{Type} - \text{reduction}: \quad u_j(\mathbf{x}_i) = \frac{u_j^R(\mathbf{x}_i) + u_j^L(\mathbf{x}_i)}{2}, \quad j = 1,\ldots,C \tag{20}$$

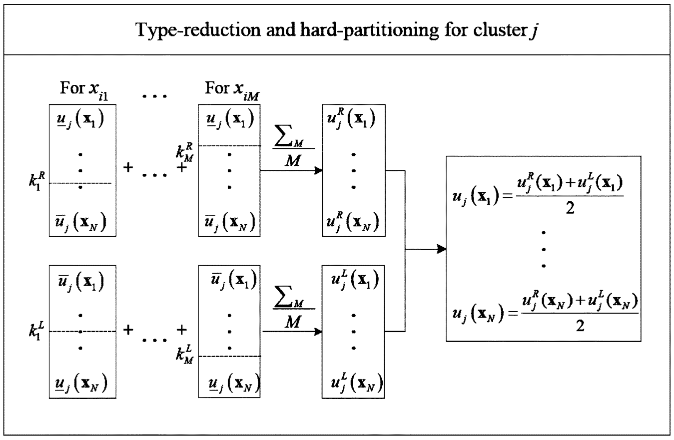


Fig. 12. Type-reduction for pattern set $\mathbf{X}$.

and

Hard partitioning :

$$\textbf{If}\,(u_j(\mathbf{x}_i) > u_k(\mathbf{x}_i)), \quad \text{for } k = 1, \ldots, C \text{ and } j \neq k$$
$$\textbf{Then } \mathbf{x}_i \text{ is assigned to cluster } j. \quad (21)$$

In (20), memberships $u_j^L(\mathbf{x}_i)$ and $u_j^R(\mathbf{x}_i)$ are also different according to each feature for a pattern. Therefore, a representative value for left and right membership needs to be computed for each feature. This can be obtained as

$$u_j^R(\mathbf{x}_i) = \frac{\sum_{l=1}^{M} u_{jl}(\mathbf{x}_i)}{M}$$
$$\text{where } u_{jl}(\mathbf{x}_i) = \begin{cases} \bar{u}_j(\mathbf{x}_i), & \text{if } x_{il} \text{ uses } \bar{u}_j(\mathbf{x}_i) \text{ for } \mathbf{v}_j^R \\ \underline{u}_j(\mathbf{x}_i), & \text{otherwise} \end{cases} \quad (22)$$

and

$$u_j^L(\mathbf{x}_i) = \frac{\sum_{l=1}^{M} u_{jl}(\mathbf{x}_i)}{M}$$
$$\text{where } u_{jl}(\mathbf{x}_i) = \begin{cases} \bar{u}_j(\mathbf{x}_i), & \text{if } x_{il} \text{ uses } \bar{u}_j(\mathbf{x}_i) \text{ for } \mathbf{v}_j^L \\ \underline{u}_j(\mathbf{x}_i), & \text{otherwise.} \end{cases}$$

In (22), M denotes the number of features for pattern $\mathbf{x}_i$. Fig. 12 illustrates the procedure of type-reduction before hard-partitioning. Our overall proposed interval type-2 method can be summarized in Fig. 13.

As a final comment, our proposed method expresses the uncertainty for a pattern set with an interval of fuzzy membership which is obtained by using two fuzzifiers m_1 and m_2. However, if the same value for the two fuzzifiers is used $(m_1 = m_2)$, we can naturally obtain a type-1 fuzzy set which has no uncertainty for its memberships (see Fig. 14).

In this case, the result of our proposed method becomes exactly the same as FCM since the results of center updating and hard-partitioning in our proposed method are identical to the results in FCM. Therefore, we conclude that our proposed method can be considered as a generalized form of FCM.

Now, to comment on computational complexity, our proposed method definitely has higher complexity than FCM. The major factor is due to the iterative algorithm that is required for type-reduction. This can be explained as follows. Assuming N number of patterns and M number of features for each pattern where $N \gg M$, the required computations for estimating a cluster center by each feature is N^2 since both the number of patterns and the number of memberships are N. To be exact, the amount is MN^2 since this is performed for all M features. However, the number of features for a pattern is usually relatively much smaller than the number of patterns for a pattern set in practical cases. Therefore, we can conclude that the order of computation for $N \gg M$ approximates to $O(N^2)$. Next, if we apply interval type-2 fuzzy sets in FCM, we should consider a couple of factors of computation. That is, the estimation of the left $(\mathbf{v}_L)$ and right $(\mathbf{v}_R)$ value for an interval cluster center when performing type-reduction during center-updating by the Karnik–Mendel iterative algorithm. The order of computation for the iterative algorithm can be computed by the following three steps:

Step 1) ascending ordering of a pattern set

Step 2) estimation of a cluster center in accordance with lower or upper membership of a pattern set;

Step 3) detection of center location in a pattern set.

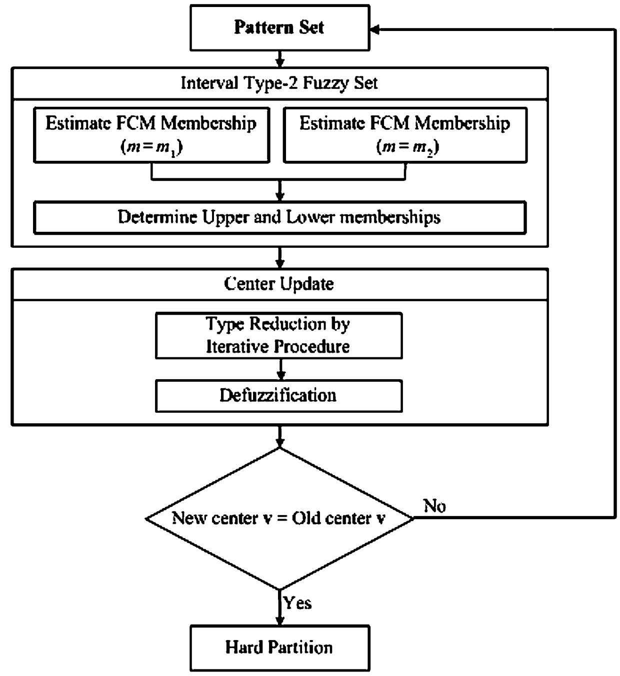


Fig. 13. Interval type-2 FCM algorithm.

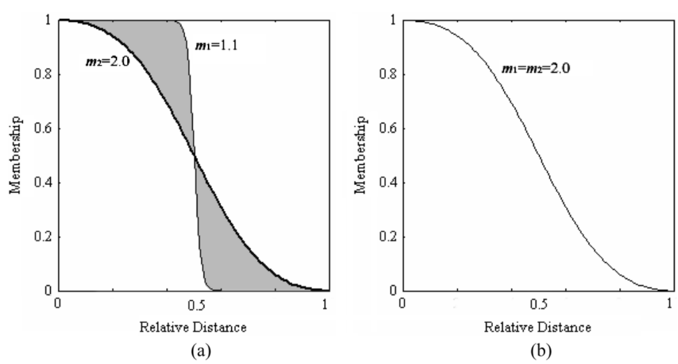


Fig. 14. Description of uncertainty in accordance with m. (a) $m_1 \neq m_2$. (b) $m_1 = m_2$.

In step 1), the order of computation for a typical sorting algorithm is $N\ (N-1)$ which is based on full search sorting. The ascending ordering is also applied for each feature. Hence, the complexity becomes $MN(N-1)$. If $N \gg M$, the complexity becomes $O\ (N^2)$. In step 2), the estimation of a cluster center is the same as in FCM. Therefore, the order of computation can also be approximated as $O\ (N^2)$. In step 3), a comparison of the difference between a cluster center and each pattern is required to find the location of a cluster center in a pattern set. From this point of view, the comparison occurs N times in the limit and as before if $N \gg M$ the complexity becomes $O\ (N^2)$. Now, the total computations (adding the three steps) required for the iteration algorithm for a single iteration is $O\ (N^2)$. If the number of required iterations is R, the order of computation becomes $R \times O\ (N^2)$. In [1], it has already been proved that the Karnik–Mendel method converges at most N times. Therefore, the order of computation for our proposed method becomes $N \times O\ (N^2) = O(N^3)$. However, when applying the Karnik–Mendel method for most typical pattern sets the authors claim that the method converges within three or four iterations. Consequently, the computation of our proposed method is much lower in the practical cases since $R \ll N$. For the worst case, the order of computation in our proposed method is considered much larger than in FCM. However, this may be considered as a small price to pay for obtaining improved clustering performance. Furthermore, for certain practical cases the order of computation can be approximated to $O\ (N^2)$.

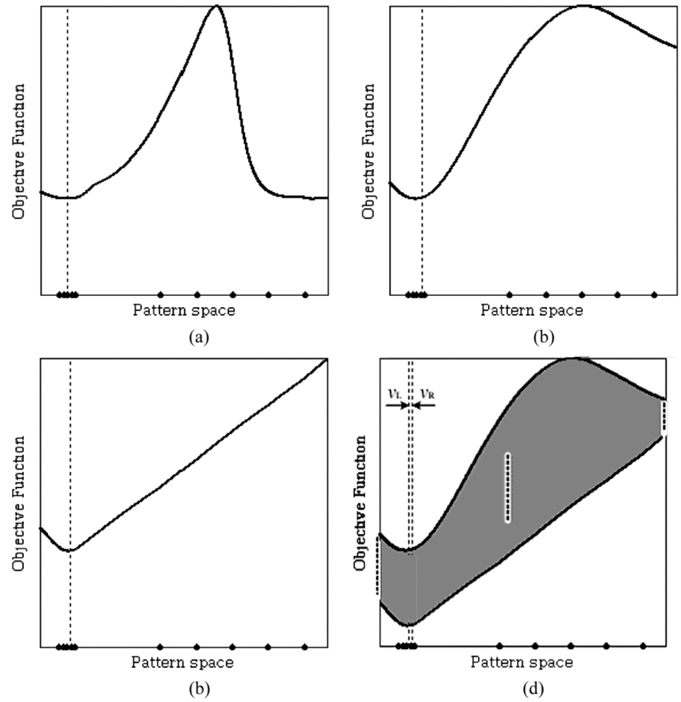


Fig. 15. Subobjective function for left cluster $(J_1 = \sum_{j=1}^{N}(u_{1j})^m d_{1j}^2)$. (a) J_1 with $m = 1.1$. (b) J_1 with $m = 2.0$. (c) J_1 with $m = 10.0$. (d) J_1 with $m_1 = 2.0$ and $m_2 = 10.0$ (vertically dotted lines are final cluster centers).

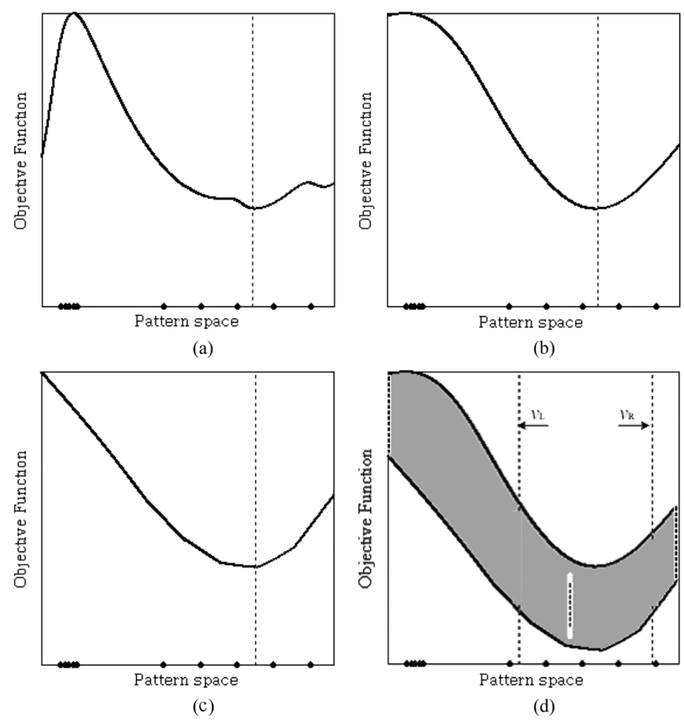


Fig. 16. Sub-objective function for right cluster $(J_2 = \sum_{j=1}^{N}(u_{2j})^m d_{2jf}^2)$. (a) J_2 with $m = 1.1$. (b) J_2 with $m = 2.0$. (c) J_2 with $m = 10.0$. (d) J_2 with $m_1 = 2.0$ and $m_2 = 10.0$ (vertically dotted lines are final cluster centers).

VI. Experimental Results

In this section, we compare the results between the type-1 FCM and our proposed interval type-2 FCM method for several pattern sets. The fuzzifier m for the type-1 FCM is selected by increasing the value by 1 in the interval $[1.1, 10.0]$, resulting in ten experiments for a pattern set. For our proposed method, we select values for m_1 and m_2 also by increasing each value by 1 in the interval $[1.1, 10.0]$. Experiments for all possible combinations of m_1 and m_2 are performed, which results in 45 experiments for a pattern set. It is to be noted that since the type-1 FCM and our proposed method is the same when $m_1 = m_2$ and due to symmetry in FOU, 45 experiments will cover all combinations. Before presenting results for several pattern sets, Figs. 15 and 16 illustrates the final value of a sub-objective function for two 1-D clusters and the interval of each cluster center which have different volume. The gray shaded region (i.e., FOU) in Fig. 15(d) and 16(d) represent the convergence interval for the sub-objective function for an interval type-2 fuzzy set. Convergence curves of sub-objective function for left center $\mathbf{v}_L$ and right center $\mathbf{v}_R$ are drawn within the convergence interval. This indicates that our interval type-2 FCM method can better locate the desirable cluster centers between the two sub-objective functions.

A. Identical "Squares" Data

Our first experiment shows the clustering results for eight patterns that are distributed as two identical squares in structure and density as shown in Fig. 17.

In general, FCM performs well for pattern sets that contain partitions of similar volume and similar number of patterns. As expected, both FCM and the interval type-2 FCM method gave desirable results. For our proposed method, all of the clustering

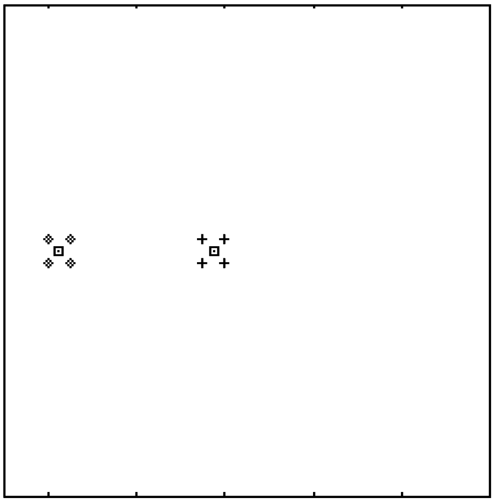


Fig. 17. Clustering results of FCM and interval type-2 FCM (all combinations of m_1 and m_2) on a pattern set consisting of two identical square distributed patterns.

results for the 45 combinations of m_1 and m_2 gave identical results since the two clusters are assigned equal or similar degrees of fuzziness due to the structure and density of the patterns. Hence, the management of uncertainty is not required for this problem.

B. Different Volume/Equal Low Number "Squares" Data

By changing the volume of clusters in a pattern set we observe the effectiveness of our proposed method in comparison with FCM. Fig. 18 shows a simple example that illustrates the effect of fuzzifier m for a pattern set that contains clusters of different volume and equal number of patterns.

The pattern set contains two clusters each containing 9 patterns with different cluster density. Fig. 18(a) shows the final cluster partition for the best possible clustering result using FCM ($m = 4.0$). Fig. 18(b) shows the FCM result using $m = 2.0$ which is generally used when performing FCM. As shown in the figure, the three left vertical patterns of the big cluster are located closer towards the small cluster from the midpoint of the two ideal cluster centers (locations where pattern memberships are maximally fuzzy, i.e., relative distance is 0.5). Therefore, with respect to the small cluster, memberships assigned to those patterns are larger when $m = 2.0$, than for when $m = 4.0$. This is due to the fact that, the maximally fuzzy boundary is less wide when $m = 2.0$, than for when $m = 4.0$. Hence, the center of the small cluster will eventually move more to the right than for the case when $m = 4.0$. As a result, final partitioning of FCM using $m = 4.0$ gives better (but not desirable) results than for when $m = 2.0$. However, in Fig. 18(c), the result of interval type-2 FCM by establishing a FOU using $m_1 = 2.0$ and $m_2 = 10.0$ gives desirable results with cluster centers located at their prototypical locations. The reason for this is that our proposed method utilizes the appropriate value of m during center updating as explained earlier. As a result, undesirable partitioning of the data as in the FCM can be avoided. Table I gives the clustering result of our proposed method using various combinations of m_1 and m_2 to illustrate the sensitivity of the method.

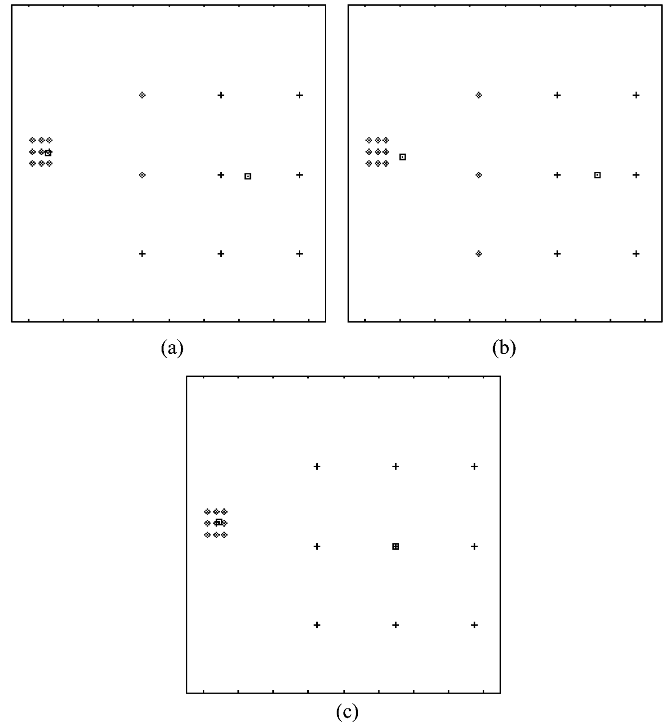


Fig. 18. Clustering results on a pattern set consisting of two square distributed patterns of different volume. (a) FCM with $m = 4.0$. (b) FCM $m = 2.0$. (c) Interval type-2 FCM with $m_1 = 2.0$ and $m_2 = 10.0$.

TABLE I
NUMBER OF PATTERNS CORRECTLY CLUSTERED FOR INTERVAL TYPE-2 FCM

m_2 \ m_1	1.1	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
1.1	15	15	15	15	18	18	18	18	18	18
2.0	15	15	15	15	15	15	15	15	18	18
3.0	15	15	15	16	15	15	15	15	18	18
4.0	15	15	16	16	16	16	15	17	15	18
5.0	18	15	15	16	16	16	16	15	17	15
6.0	18	15	15	16	16	16	16	16	16	17
7.0	18	15	15	15	16	16	16	16	16	16
8.0	18	15	15	17	15	16	16	16	16	16
9.0	18	18	18	15	17	16	16	16	16	16
10.0	18	18	18	18	15	17	16	16	16	16

As mentioned previously, since interval type-2 FCM becomes identical with type-1 FCM when $m_1 = m_2$, the diagonal elements represent clustering results for FCM. Fig. 19 shows the recognition plots that compare both methods.

The plot for the interval type-2 method shows the best result for each value of m (i.e., best result from fixing m_1 and varying m_2 or varying m_1 and fixing m_2). From the figure, it shows that our method outperforms FCM in all cases.

C. Gaussian Random Data With Noise

The remaining experiments are shown for more practical pattern sets. In the next example, we perform FCM and our proposed method for a pattern set which is constituted by two clusters (25 patterns for each cluster) of similar volume. As expected in Fig. 20, FCM and our proposed method gave identical results.

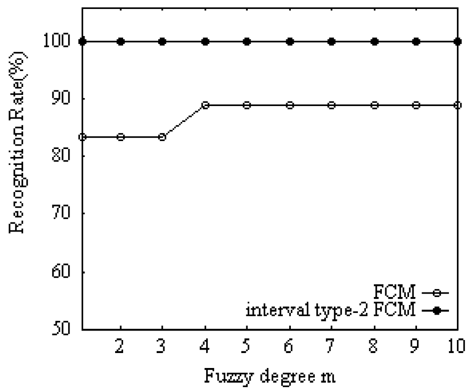


Fig. 19. Recognition plot for FCM and interval type-2 FCM clustering results (m and $m_1 = 1.1$–10.0, $m_2 =$ value for best result).

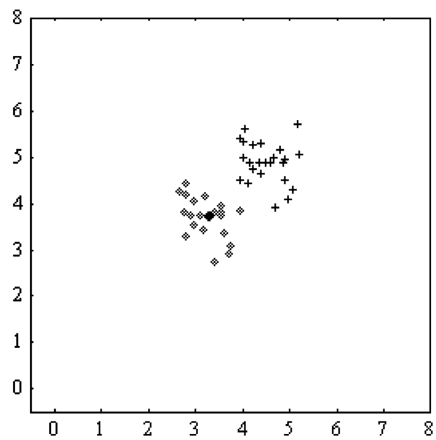


Fig. 20. Clustering result for Gaussian random number data for FCM and (a) interval type-2 FCM.

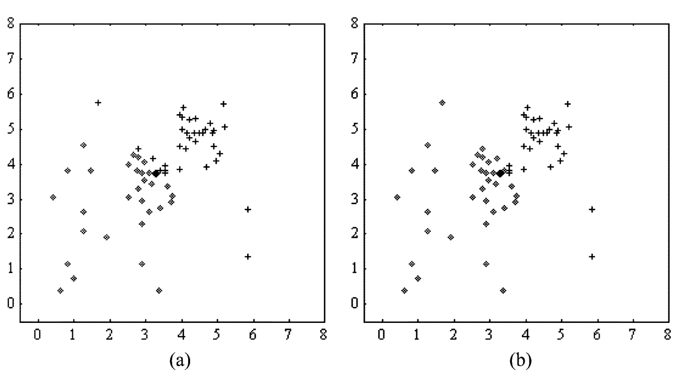


Fig. 21. Gaussian random number data with noise: results for (a) FCM (b) interval type-2 FCM.

That is, since the two clusters are of similar volume and density, they receive similar fuzzy degrees for similar volumes as in the example mentioned previously. However, when we add 20 noise patterns (40% noise) to the lower cluster to obtain a difference in volume compared to the upper cluster, the result of the proposed method gives more suitable results than FCM (see Fig. 21). Due to the increase in number of patterns in the lower cluster, performance of the proposed method is slightly reduced (three misclassified patterns).

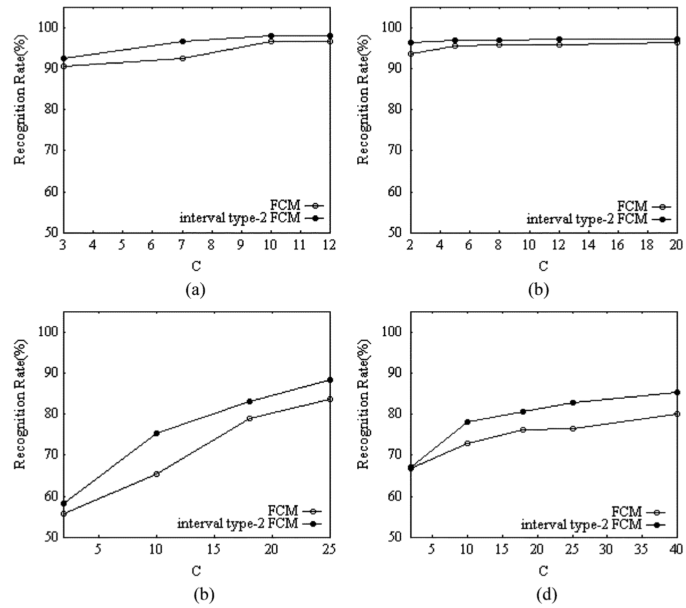


Fig. 22. Recognition rate for data sets (a) "Iris Plant" (b) "Wisconsin Breast Cancer" (c) Sonar: Mines versus Rocks," and (d) "Pima Indians Diabetes."

TABLE II
DESCRIPTION OF DATA USED FOR VARIOUS HIGH-DIMENSIONAL EXAMPLES

Data Set	Features	Patterns	Classes	Fuzzifier m, m_1, m_2
Iris plant	4	150	3	2.0, 2.0, 9.0
Wisconsin breast cancer	8	699	2	4.0, 8.0, 10.0
Pima Indians diabetes	8	768	2	1.1, 1.1, 8.0
Sonar: mines vs. rocks	60	208	2	2.0, 2.0, 4.0

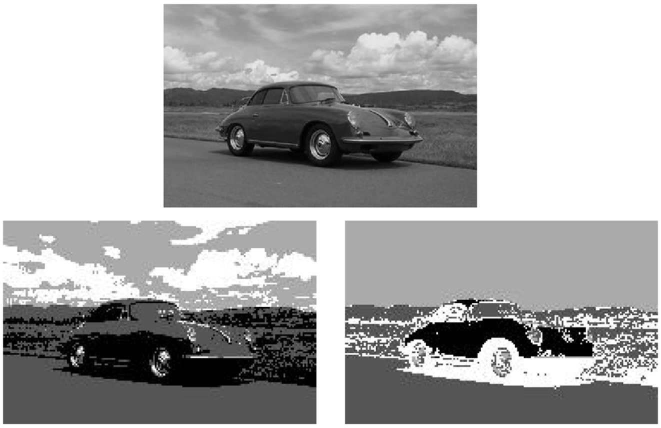
D. High Dimensional Data Sets

For our next experiment, we present results for four examples of high dimensional data, namely, "Iris plant," "Wisconsin breast cancer," "Sonar: mines versus rocks," and "Pima Indians diabetes" [32]. We used a "jack-knife" procedure, where 75% of the data from each class was used for clustering and the remaining 25% for testing. This was repeated four times for various numbers of initial clusters. Table II gives the experimental conditions and the description of the data sets used. In general, since the increase in number of initial clusters leads to improvement in the recognition rate, we increased the number of initial clusters for each experiment to see what kind of result can be obtained. Fig. 22 shows the comparison of the two methods for each data by listing the overall recognition rate.

E. Image Segmentation

For our final examples, we apply our clustering method to a problem that involves 364×236 images, namely, "car," and "wolf." We extracted the color information (R, G, B) from each image and constructed three-dimentional pattern vectors. Fig. 23 shows an image that is considered to contain four regions, namely, sky, road, grass, and car.

Fig. 23(a) shows the best result of FCM, where each pixel is labeled a gray scale that corresponds to the gray scale of the closest resulting cluster center. Results show the car region being highly shared with the road and grass. However, as shown



(a) (b)

Fig. 23. Segmented results for "Car" image for (a) FCM and (b) interval type-2 FCM.

(a) (b)

Fig. 24. Segmented results for "Wolf" image for (a) FCM and (b) interval type-2 FCM.

in Fig. 23(b), the proposed method gives better results where the regions are more distinct.

Fig. 24 shows results for "Wolf" image. This image is considered to have three distinctive regions, namely, wolf, grass, and background. For the FCM result shown in Fig. 24(a), two of the clusters centers almost overlap hence the wolf and grass becomes almost indistinguishable. Whereas shown in Fig. 24(b), the proposed method gives three distinct regions.

VII. Conclusion

Normally, good performance of FCM is not guaranteed for a pattern set that is constituted by clusters with different densities (different volumes with different number of patterns). For this, it is rarely the case that a single value of fuzzifier m can properly represent the pattern memberships for all of the clusters. If the volume of clusters in a pattern set is increased (i.e., initial cluster C lowered) or the number of patterns in each cluster is decreased, the fuzzy degree of patterns in a cluster should be properly adjusted. In FCM, monotonically increasing or decreasing of fuzzy degree of patterns with fuzzifier m can cause loss of performance.

In this paper, we extended a pattern set into an interval type-2 fuzzy set by incorporating two different values of fuzzifier m (m_1 and m_2) and modified the procedure of FCM. The purpose was to represent and manage uncertainty which occurs when determining the fuzzy degree for clusters with different volume for a given pattern set. Consequently, the management of uncertainty aids the cluster centers to converge to a more reasonable position in our proposed method. Center-updating and hard partition was modified in the FCM by incorporating interval type-2 fuzzy sets. From this approach, we can obtain improved clustering results. Also, we verified that our proposed interval type-2 FCM method resulted to be a generalized form of FCM when we set the two fuzzifiers equal $(m_1 = m_2)$. Several examples for clusters with different volume illustrated the effectiveness of our proposed method.

As a further study, we will apply this approach to various types of clustering methods which are objective function based. As shown in several studies, another factor of loss of performance in FCM is the number of patterns in a cluster. The reason for this is that the estimation of geometric means in FCM is affected by the number of patterns in each cluster. This can cause final cluster centers to end up in undesirable locations. Hence, this can cause the result of unrepresentative partitioning of a pattern set. Therefore, applying our method on a possibilistic approach [33] in order to further improve the performance of FCM for a pattern set which is constituted by clusters with different number of patterns is under study and will be reported shortly.

Acknowledgment

The authors would like to thank the anonymous referees for their helpful suggestions in the improvement of this manuscript.

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