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New types of aggregation functions for interval-valued fuzzy setting and preservation of pos-B and nec-B-transitivity in decision making problems

Urszula Bentkowska

Faculty of Mathematics and Natural Sciences, Interdisciplinary Centre for Computational Modelling, University of Rzeszów, Rzeszów, Poland

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ABSTRACT

In this contribution new types of aggregation functions for interval-valued fuzzy setting are introduced. They are called necessary and possible aggregation functions, respectively. In the monotonicity conditions for these aggregation functions the classical monotonicity for intervals is replaced with the new comparability relations. These relations follow naturally from the interpretations of interval-valued fuzzy sets and together with the classically used monotonicity for intervals, form a family of all possible approaches to define relations of comparability for intervals. Moreover, in this paper dependencies between considered families of aggregation functions are presented. Furthermore, transitivity properties of interval-valued fuzzy relations, based on these new comparability relations, are studied and preservation of them by possible and necessary aggregation functions are considered.

1. Introduction

Interval-valued fuzzy sets were introduced by Zadeh [33] as a generalization of the concept of a fuzzy set [32]. Interval valued fuzzy sets and relations have applications in diverse types of areas, for example in classification, image processing and multicriteria decision making.

In [20], a comparative study of the existing definitions of order relations between intervals, analyzing the level of acceptability and shortcomings from different points of view were presented. Comparability relations used in the interval-valued setting may be connected with ontic and epistemic interpretation of intervals [11,12]. Epistemic uncertainty represents the idea of partial or incomplete information. Simply, it may be described by means of a set of possible values of some quantity of interest, one of which is the right one. A fuzzy set represents in such approach incomplete information, so it may be

E-mail address: ududziak@ur.edu.pl

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called disjunctive [11]. On the other hand, fuzzy sets may be conjunctive and can be called ontic fuzzy sets [11]. In this situation the fuzzy set is used as representing some precise gradual entity consisting of a collection of items.

The aim of this work is to introduce new types of aggregation functions for interval-valued fuzzy setting. In the monotonicity condition of an aggregation function the natural (partial) order, from the family of intervals, is replaced with new comparability relations. These relations are really new comparability relations, since they have other properties than the natural order has. These relations follow naturally from the epistemic setting of interval-valued fuzzy sets and together with the natural order used in this setting form a family of all possible approaches to define comparability relations for intervals. The origin of these relations is other than the one presented in [3] for partial order used classically in interval-valued fuzzy setting.

Moreover, in this paper dependencies between the new introduced families of aggregation functions and known families of aggregation functions are studied for interval-valued fuzzy setting. Many examples of new types of aggregation functions are presented, including proper aggregation functions of the new type, i.e. not coinciding with the traditional families of aggregation functions. For this purpose, pre-aggregation functions turn out to be a useful tool. Characterization of one of the new families of aggregation functions is given in the case of decomposability. The important fact is that these new families of aggregation functions cannot be translated as a special case of aggregation functions on lattice. This follows from the fact that the family of intervals with each of the relation is not a partially ordered set, so it is not a lattice. Moreover, in this paper preservation of properties of interval-valued fuzzy relations by the new type of aggregation functions is discussed. These properties also follow from the interpretation of the new 'comparability' relations for the epistemic setting of fuzzy relations. We consider here only transitivity property which is one of the most interesting properties of relations. Other properties may be found in [26].

These considerations have possible applications in multicriteria (or similarly multiagent) decision making problems with intervals (not just numbers in [0, 1]). In virtue of using diverse approaches of defining the comparability relations for the intervals we may have applications depending on the presented problem from real-life situations. In such cases it may be interesting to use for aggregation of the given data (gathered as interval-valued fuzzy relations) adequate type of an aggregation function, which follows from the assumed interpretation.

The paper is structured as follows. Firstly, some concepts and results useful in further considerations are given (Section 2). Next, new types of aggregation functions for intervals are introduced and dependencies between them are studied (Section 3). Moreover, aggregation (by new types of aggregation functions) of interval-valued fuzzy relations having pos-B-transitivity and nec-B-transitivity are studied (Section 4). To finish, in Section 5 an algorithm presenting possible application is presented.

2. Interval-valued fuzzy relations

First, we recall definition of the lattice operations and the classical order for interval-valued fuzzy relations. We consider relations instead of sets, since we will concentrate ourselves on interval-valued fuzzy relations and their properties. Let X, Y, Z be non-empty sets.

Definition 1 (cf. [27,33]). An interval-valued fuzzy relation R between universes X, Y is a mapping R: $X \times Y \to L^I$ such that

$$R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)] \in L^I,$$

for all couples $(x, y) \in (X \times Y)$, where $L^I = \{[x_1, x_2] : x_1, x_2 \in [0,1], x_1 \leqslant x_2\}$.

The class of all interval-valued fuzzy relations between universes X, Y will be denoted by $\mathcal{IVFR}(X \times Y)$ or $\mathcal{IVFR}(X)$ for $X = Y$. The well-known classical monotonicity (partial order) for intervals is of the form

$$[x_1, y_1] \preceq [x_2, y_2] \Leftrightarrow x_1 \leqslant x_2, y_1 \leqslant y_2. \quad (1)$$

$\mathcal{IVFR}(X \times Y)$ with $\preceq$ is partially ordered and moreover it is a lattice. We will consider other comparability relations on L^I as a consequence on $\mathcal{IVFR}(X \times Y)$. To begin with, we recall the definition of an interval order [14–16] for crisp relations.

Definition 2 ([17], p. 42). A relation $R \subset X \times X$ is an interval order if it is complete and has the Ferrers property, i.e.:
$R(x, y)$ or $R(y, x)$, for $x, y \in X$,
$R(x, y)$ and $R(z, w) \Rightarrow R(x, w)$ or $R(z, y)$, for $x, y, z, w \in X$, respectively.

Now we consider the following comparability relations on L^I (cf. [25]):

$$[x_1, y_1] \preceq_\pi [x_2, y_2] \Leftrightarrow x_1 \leqslant y_2, \quad (2)$$

$$[x_1, y_1] \preceq_\nu [x_2, y_2] \Leftrightarrow y_1 \leqslant x_2. \quad (3)$$

These relations, including classical order, follow from the epistemic setting of interval-valued fuzzy relations and form the full possible set of interpretations of comparability relations on intervals. Detailed discussion on this subject is presented in [26] (cf. [25]). We will only briefly recall the interpretation of conditions (2) and (3). Relation (2) follows from the following interpretation of inclusion: there exists an instance in the left interval and there exists an instance in the right interval such that the inclusion of these intervals holds (possible inclusion). Relation (3) follows from the following interpretation of inclusion: for each instance in the left interval and for each instance in the right interval the inclusion of these intervals holds (necessary inclusion).

Proposition 1 (cf. [25]). *Relation $\preceq_\pi$ is an interval order and the relation $\preceq_\nu$ is antisymmetric and transitive on L^I. Moreover, $\preceq_\nu \Rightarrow \preceq \Rightarrow \preceq_\pi$.*

The converse implications do not hold, as can be seen from the following example.

Example 1. For intervals $A = [0.2, 0.3]$, $B = [0.1, 0.5]$ and $C = [0.4, 0.8]$. We observe that $A \preceq_\pi B$ but it is not true that $A \preceq B$ and similarly $B \preceq C$ but it is not true that $B \preceq_\nu C$.

Proposition 2 ([26]). *The boundary elements on L^I, with respect to each of the comparability relations (1)–(3), are $\mathbf{1} = [1, 1]$ and $\mathbf{0} = [0, 0]$.*

Proposition 3 ([26]). *The relation $\preceq_\pi$ on $\mathcal{IVFR}(X \times Y)$ is reflexive and the relation $\preceq_\nu$ on $\mathcal{IVFR}(X \times Y)$ is antisymmetric and transitive.*

Note that, relations (1)–(3) coincide on the family of fuzzy relations on a given set.

The classical monotonicity (partial order), which is traditionally used for intervals, is the consequence of the following interpretation. To each interval $[a, b] \in L^I$ we may assign uniquely a point $(a, b) \in \mathbb{R}^2$, so intervals (thanks to isomorphism) may be ordered by means of orders for points on $\mathbb{R}^2$. The usual partial order on $\mathbb{R}^2$, given by $(a, b) \leq (c, d) \Leftrightarrow a \leq c$ and $b \leq d$, induces the partial order (1) (cf. [3]). Classical order (1) is not complete (which is important for application reasons for example in decision making problems to compare alternatives and choose the best one). It is worth to recall that in the paper [3] the general method to build different linear orders, covering some of the known linear orders for intervals such as Xu and Yager's [31] or the lexicographical ones, is presented. The construction method that is proposed there is based on the use of two aggregation functions. In this sense, it provides an easy-to-use tool in order to generate new total orders between intervals (depending on the requirements of the real-life situations). Moreover, the presented method can be easily extended to use either more aggregation functions or functions other than aggregation functions as long as some basic properties are fulfilled.

In [26] there are considered new types of comparability relations to have the full view of the possible approaches of defining this concept for intervals. Note that classical order also follows, as one of the possibilities, from this interpretation of epistemic setting of intervals. There are also considered new types of properties of interval-valued fuzzy relations. The origin of them is the same as of the comparability relations and follows from the following concept.

Definition 3 ([26]). Relation $R \in \mathcal{IVFR}(X)$ has possible p-property (pos-p-property for short) if there exists at least one instance R^* of R which has property p. Relation $R \in \mathcal{IVFR}(X)$ has necessary p-property (nec-p-property for short) if for every instance R^* of R it has property p.

Transitivity properties connected with the new introduced comparability relations, and generalized to binary operation B based form (cf. [2]), are the following.

Definition 4 (cf. [5,25,26]). Let B: $[0, 1]^2 \to [0, 1]$ be a binary operation. A relation $R = [\underline{R}, \overline{R}] \in \mathcal{IVFR}(X)$ is:

- possibly B-transitive (pos-B-transitive), if

$$B(\underline{R}(x, y), \underline{R}(y, z)) \leqslant \overline{R}(x, z), \quad x, y, z \in X,$$

- necessarily B-transitive (nec-B-transitive), if

$$B(\overline{R}(x, y), \overline{R}(y, z)) \leqslant \underline{R}(x, z), \quad x, y, z \in X,$$

- B-transitive (cf. [25]), if

$$B(\underline{R}(x, y), \underline{R}(y, z)) \leqslant \underline{R}(x, z) \quad \text{and} \quad B(\overline{R}(x, y), \overline{R}(y, z)) \leqslant \overline{R}(x, z), \quad x, y, z \in X.$$

In the above definition arbitrary operation B: $[0, 1]^2 \to [0, 1]$ is applied in order to obtain the most general approach. However, it is natural to consider a fuzzy conjunction B, which generalizes the minimum used in the classical definition of transitivity. This approach is also a generalization of the crisp meaning of transitivity. Similarly to pos-B-transitivity and nec-B-transitivity we may also define other properties [26]. However, in this paper we recall only transitivities, which are one of the most important properties, to show their behavior in aggregation process (with the use of new types of aggregation functions which will be introduced in the sequel).

There are diverse approaches to define a fuzzy conjunction. We will apply the following.

Definition 5 (cf. [9]). An operation C: $[0, 1]^2 \to [0, 1]$ is called a fuzzy conjunction if it is increasing and

$$C(1, 1) = 1, \; C(0, 0) = C(0, 1) = C(1, 0) = 0.$$

A fuzzy conjunction is called a t-seminorm if it has a neutral element 1. A t-seminorm is called a t-norm if it is commutative and associative.

Example 2. Examples of fuzzy conjunctions are the following well-known t-norms (T for short) given here with their usually used abbreviation:

$T_M(x,y) = \min(x,y),\ T_P(x,y) = xy,\ T_L(x,y) = \max(x+y-1,0),$

$$T_D(x,y) = \begin{cases} x, & \text{if } y=1 \\ y, & \text{if } x=1 \\ 0, & \text{otherwise} \end{cases}.$$

These operations are comparable (linearly ordered in [0, 1]), i.e.

$$T_D \leqslant T_L \leqslant T_P \leqslant T_M. \tag{4}$$

Other examples and classes of fuzzy conjunctions are gathered for example in [2].

In order to determine connections between diverse types of B-dependent transitivity properties the following result is useful.

Proposition 4. *Let B_1, B_2: $[0,1]^2 \to [0,1]$ and $B_1 \le B_2$. If $R \in \mathcal{IVFR}(X)$ is pos-B_2-transitive (nec-B_2-transitive, B_2-transitive), then R is pos-B_1-transitive (nec-B_1-transitive, B_1-transitive).*

Proof. Let $x,\ y,\ z \in X$, R be pos-B_2-transitive. Then $B_2(\underline{R}(x,y), \underline{R}(y,z)) \leqslant \overline{R}(x,z)$ and by the fact that $B_1 \le B_2$ we have $B_1(\underline{R}(x,y), \underline{R}(y,z)) \leqslant B_2(\underline{R}(x,y), \underline{R}(y,z)) \leqslant \overline{R}(x,z)$ for $x,\ y,\ z \in X$. As a result R is pos-B_1-transitive. Justification for nec-B-transitivity and B-transitivity is similar. □

By (4) and Proposition 4, for example if $R \in \mathcal{IVFR}(X)$ is pos-T_M-transitive (nec-T_M-transitive, T_M-transitive), then it is also pos-B-transitive (nec-B-transitive, B-transitive) for $B \in \{T_P,\ T_L,\ T_D\}$.

3. New types of aggregation functions on L^I

Firstly, we recall the notion of an aggregation function on the unit interval [0, 1] and next on L^I.

Definition 6 ([4], p. 6)**.** Let $n \in \mathbb{N}$, $n \ge 2$. A function A: $[0,1]^n \to [0,1]$ which is increasing, i.e.

$$A(x_1,\ldots,x_n) \leqslant A(y_1,\ldots,y_n) \quad \text{for} \quad x_i, y_i \in [0,1], x_i \leqslant y_i, i = 1,\ldots,n$$

is called an aggregation function if

$$A(0,\ldots,0) = 0,\ A(1,\ldots,1) = 1.$$

Example 3. Examples of aggregation functions are conjunctions (with their diverse subfamilies) and quasi-linear means

$$A(x_1,\ldots,x_n) = \varphi^{-1}\Big(\sum_{k=1}^{n} w_k \varphi(x_k)\Big), \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1,$$

where $x_1,\ldots,x_n \in [0,1]$, $\varphi : [0,1] \to \mathbb{R}$ is a continuous, strictly increasing function.

Definition 7 (cf. [21])**.** An operation $\mathcal{A} : (L^I)^n \to L^I$ is called an aggregation function on L^I if it is increasing, i.e.

$$\underset{\mathbf{x}_i,\mathbf{y}_i \in L^I}{\forall} \mathbf{x}_i \preceq \mathbf{y}_i \Rightarrow \mathcal{A}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \preceq \mathcal{A}(\mathbf{y}_1,\ldots,\mathbf{y}_n) \tag{5}$$

and $\mathcal{A}(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{n\times}) = \mathbf{0}$, $\mathcal{A}(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n\times}) = \mathbf{1}$.

We introduce here new types of aggregation functions on L^I. We replace in the monotonicity condition (5) natural order $\preceq$ with the relations $\preceq_\pi$ and $\preceq_\nu$. Note that the obtained aggregation functions are not special cases of aggregation functions on lattices (well described in the literature), since relations $\preceq_\pi$ and $\preceq_\nu$ are not partial orders (cf. Proposition 1).

Definition 8. An operation $\mathcal{A} : (L^I)^n \to L^I$ is called a possible aggregation function (we will write for short pos-aggregation function) if

$$\underset{\mathbf{x}_i,\mathbf{y}_i \in L^I}{\forall} \mathbf{x}_i \preceq_\pi \mathbf{y}_i \Rightarrow \mathcal{A}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \preceq_\pi \mathcal{A}(\mathbf{y}_1,\ldots,\mathbf{y}_n) \tag{6}$$

and $\mathcal{A}(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{n\times}) = \mathbf{0}$, $\mathcal{A}(\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n\times}) = \mathbf{1}$.

Definition 9. An operation $\mathcal{A} : (L^I)^n \to L^I$ is called a necessary aggregation function (we will write for short nec-aggregation function) if

$$\underset{\mathbf{x}_i,\mathbf{y}_i \in L^I}{\forall} \mathbf{x}_i \preceq_\nu \mathbf{y}_i \Rightarrow \mathcal{A}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \preceq_\nu \mathcal{A}(\mathbf{y}_1,\ldots,\mathbf{y}_n) \tag{7}$$

and $\mathcal{A}(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{n\times})=\mathbf{0}$, $\mathcal{A}(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n\times})=\mathbf{1}$.

The family of all pos-aggregation functions will be denoted by $\mathcal{A}_\pi$ and the family of all nec-aggregation functions will be denoted by $\mathcal{A}_\nu$.

For the simplicity of notations we put all results for two-argument functions. We will present dependencies between the families of known aggregation functions on L^I and pos- and nec-aggregation functions. We will also give examples and representations of new types of aggregation functions on L^I. We will start with presenting the connections of the new families with decomposable operations (including representable aggregation functions).

Definition 10 (cf. [6]). Let $\mathcal{A}:(L^I)^2\to L^I$ be an aggregation function. $\mathcal{A}$ is said to be a representable aggregation function if there exist two aggregation functions A_1, A_2: $[0,1]^2\to[0,1]$, $A_1\le A_2$ such that, for every $[x_1,x_2], [y_1,y_2]\in L^I$ it holds that

$$\mathcal{A}([x_1,x_2],[y_1,y_2])=[A_1(x_1,y_1),A_2(x_2,y_2)].$$

Observe that lattice operations connected with natural order on L^I define representable aggregation functions on L^I, with $A_1=A_2=\min$ and $A_1=A_2=\max$. Moreover, many other examples of aggregation functions on L^I may be considered, such as:

- the representable product $\mathcal{A}_P([x_1,x_2],[y_1,y_2])=[x_1y_1,x_2y_2]$,
- the representable arithmetic mean $\mathcal{A}_{\text{mean}}([x_1,x_2],[y_1,y_2])=[\frac{x_1+y_1}{2},\frac{x_2+y_2}{2}]$,
- the representable geometric mean $\mathcal{A}_g([x_1,x_2],[y_1,y_2])=[\sqrt{x_1y_1},\sqrt{x_2y_2}]$,
- the representable product-mean $\mathcal{A}_{P,\text{mean}}([x_1,x_2],[y_1,y_2])=[x_1y_1,\frac{x_2+y_2}{2}]$.

Definition 11 ([10]). An operation $\mathcal{F}:(L^I)^2\to L^I$ is called decomposable if there exist operations F_1, F_2: $[0,1]^2\to[0,1]$ such that for all $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]\in L^I$

$$\mathcal{F}(\mathbf{x},\mathbf{y})=[F_1(x_1,y_1),F_2(x_2,y_2)].$$

Certainly, definition of decomposable functions makes sense only if $F_1\le F_2$. Note that a representable aggregation function is a particular instance of decomposable operation. Moreover, an aggregation function is decomposable if and only if it is representable, as the next result shows.

Theorem 1 (cf. [7]). *An operation $\mathcal{A}:(L^I)^2\to L^I$ is a decomposable aggregation function if and only if there exist aggregation functions A_1, A_2: $[0,1]^2\to[0,1]$ such that for all $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]\in L^I$ and $A_1\le A_2$*

$$\mathcal{A}(\mathbf{x},\mathbf{y})=[A_1(x_1,y_1),A_2(x_2,y_2)].$$

Since decomposability for aggregation functions is equivalent to representability, we will use in the sequel only the notion of decomposability. Decomposability is not the only possible way to build interval-valued aggregation functions.

Example 4 ([7]). Let $\mathbf{x},\mathbf{y}\in L^I$, $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]$, A_i: $[0,1]^2\to[0,1]$ be aggregation functions, $i\in\{1,\dots,4\}$, $A_1\le A_2$, $A_3\le A_4$. The following are aggregation functions on L^I:

$$\mathcal{A}(\mathbf{x},\mathbf{y})=[\min(A_1(x_1,y_2),A_1(x_2,y_1)),\max(A_2(x_1,y_2),A_2(x_2,y_1))],\tag{8}$$

$$\mathcal{A}(\mathbf{x},\mathbf{y})=[A_3(A_1(x_1,y_2),A_1(x_2,y_1)),A_4(A_2(x_1,y_2),A_2(x_2,y_1))].\tag{9}$$

Aggregation function (8) is a special case of (9).

Definition 12 (cf. [7]). Let $\mathbf{x}=[x_1,x_2],\mathbf{y}=[y_1,y_2]\in L^I$ and let A_1, A_2: $[0,1]^2\to[0,1]$, $A_1\le A_2$ be aggregation functions. The aggregation function $\mathcal{A}$ is called pseudomax A_1A_2-representable if

$$\mathcal{A}(\mathbf{x},\mathbf{y})=[A_1(x_1,y_1),\max(A_2(x_1,y_2),A_2(x_2,y_1))],\tag{10}$$

and pseudomin A_1A_2-representable if

$$\mathcal{A}(\mathbf{x},\mathbf{y})=[\min(A_1(x_1,y_2),A_1(x_2,y_1)),A_2(x_2,y_2)].\tag{11}$$

It is easy to check that the considered families of aggregation functions $\mathcal{A}_\pi$ and $\mathcal{A}_\nu$ have the same bounds, which are decomposable operations.

Proposition 5. *Decomposable aggregation functions $\mathcal{A}_w, \mathcal{A}_s:(L^I)^2\to L^I$ are respectively the lower and upper bounds of the families $\mathcal{A}_\pi$, $\mathcal{A}_\nu$ and the family of all aggregation functions on L^I, where*

$$\mathcal{A}_w(\mathbf{x},\mathbf{y})=\begin{cases}[1,1], & (x,y)=([1,1],[1,1])\\ [0,0], & \text{otherwise},\end{cases}$$

$$\mathcal{A}_s(\mathbf{x},\mathbf{y})=\begin{cases}[0,0], & (x,y)=([0,0],[0,0])\\ [1,1], & \text{otherwise},\end{cases}$$

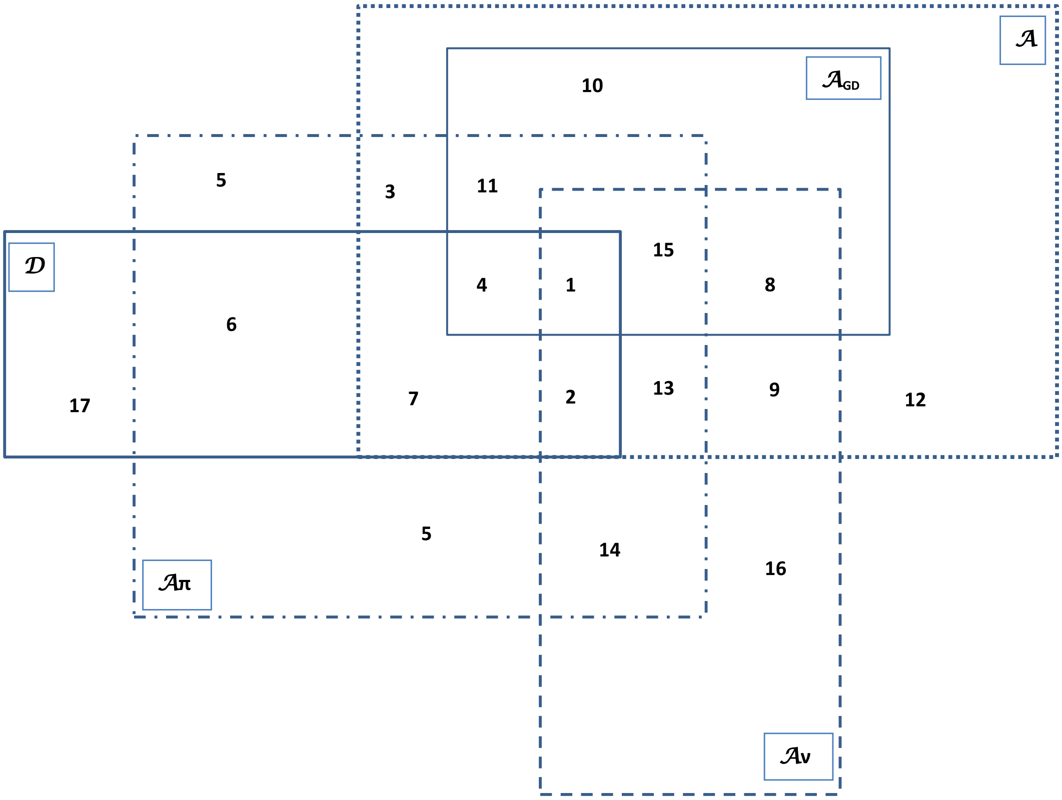

Fig. 1. Dependencies between families of aggregation functions.

Table 1
Guide to examples of aggregation functions.

Area	Statement	Area	Statement	Area	Statement
1	Example 7	7	Example 11	13	Remark 3
2	Example 8	8	Examples 16–18,20	14	Remark 2
3	Example 9 (e.g. $\mathcal{A}_5$)	9	Example 21	15	Remark 3
4	Example 9 (e.g. $\mathcal{A}_7$)	10	Examples 12, 14, 15	16	Remark 2
5	Example 10 (e.g. $\mathcal{A}_6$)	11	Examples 13, 19	17	Example 5
6	Example 10 (e.g. $\mathcal{A}_1$)	12	Example 6		

where $\mathcal{A}_w = [A_w, A_w]$ *and* $\mathcal{A}_s = [A_s, A_s]$, A_w *is the weakest aggregation function on* $[0, 1]$ *and* A_s *is the strongest aggregation function on* $[0, 1]$.

Implications given in Proposition 1 suggest that maybe similar implications can be obtained for families of aggregation functions, pos-aggregation functions and nec-aggregation functions. However, this is true only for decomposable operations (cf. Corollaries 1 and 2). In other cases these connections are more complicated. This is why we will give examples and representations of pos- and nec-aggregation functions. We will also provide connections between them and all presented here representations of aggregation functions on L^I, i.e. decomposable ones and those of the form (8)–(11) (these are all representations found by the author in the existing literature).

First, we will present these connections in Fig. 1 and next we will make the analysis of them providing suitable examples or proofs. We will also use the notation $\mathcal{A}_{GD}$ to gather aggregation functions on L^I of the type (8)–(11). Each family of considered operations is placed in a rectangle shape area. The guide to these examples is Table 1 where each area in Fig. 1 is numbered and equipped with the description where to find the example or justification. Fig. 1 is also drawn on the basis of Corollaries 1 and 2.

Example 5. We will start with giving an example of decomposable operation on L^I, $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [1 - x_1, 1]$, where $\mathbf{x} = [x_1, x_2]$, $\mathbf{y} = [y_1, y_2]$, which is not an aggregation function and similarly it is neither pos-aggregation function nor nec-aggregation function.

Example 6. The following non-decomposable operation is an aggregation function on L^I, where $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]$, $\mathcal{A}(\mathbf{x},\mathbf{y})=[y_1\frac{x_1+x_2}{2},\frac{x_1+x_2}{2}]$, but $\mathcal{A}\notin\mathcal{A}_{GD}$ and it also does not belong to other considered here families of aggregation functions, i.e. $\mathcal{A}\notin\mathcal{A}_\pi$, $\mathcal{A}\notin\mathcal{A}_\nu$. The same conditions are fulfilled by operation $\mathcal{A}(\mathbf{x},\mathbf{y})=[x_1y_1,x_1]$.

Example 7. Examples of decomposable functions which are at the same time aggregation functions on L^I and belong also to $\mathcal{A}_\pi$ and $\mathcal{A}_\nu$ are projections $\mathcal{P}_1$, $\mathcal{P}_2$. Let $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]$, then $\mathcal{P}_1(\mathbf{x},\mathbf{y})=[x_1,x_2]$, $\mathcal{P}_2(\mathbf{x},\mathbf{y})=[y_1,y_2]$. Note that they are also pseudomax A_1A_2-representable aggregation functions (with $A_1=A_2=P_1$ in (10) for $\mathcal{P}_1$ and $A_1=A_2=P_2$ in (10) for $\mathcal{P}_2$). Similarly, $\mathcal{P}_1$ and $\mathcal{P}_2$ are pseudomin A_1A_2-representable aggregation functions (they belong to the family $\mathcal{A}_{GD}$).

Further examples of such functions are $\mathcal{A}(\mathbf{x},\mathbf{y})=[x_1^2,x_2^2]$, where $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]$. More generally, the same conditions are fulfilled by functions $\mathcal{A}:(L^I)^2\to L^I$, $\mathcal{A}(\mathbf{x},\mathbf{y})=[A(x_1,x_1),A(x_2,x_2)]$ and analogously by functions $\mathcal{A}:(L^I)^2\to L^I$, $\mathcal{A}(\mathbf{x},\mathbf{y})=[A(y_1,y_1),A(y_2,y_2)]$, where A: $[0,1]^2\to[0,1]$ is an aggregation function.

Example 8. Let $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]$. An example of a function which is at the same time a decomposable aggregation function on L^I and belongs also to $\mathcal{A}_\pi$ and $\mathcal{A}_\nu$ but it does not belong to the family $\mathcal{A}_{GD}$ is $\mathcal{A}(\mathbf{x},\mathbf{y})=[x_1y_1,x_2y_2]$.

Theorem 2. *Let F_1, F_2: $[0,1]^2\to[0,1]$. If $\mathcal{F}:(L^I)^2\to L^I$ is a decomposable operation, $\mathcal{F}=[F_1,F_2]$, $F_1\le F_2$, F_1 is an aggregation function and $F_2(0,0)=0$, $F_2(1,1)=1$ (analogously $F_1\le F_2$, $F_1(0,0)=0$, $F_1(1,1)=1$, F_2 is an aggregation function), then $\mathcal{F}$ is a (decomposable) pos-aggregation function.*

Proof. According to definition of relation $\preceq_\pi$, for the notations $\mathbf{x}_1=[\underline{x_1},\overline{x_1}]$, $\mathbf{x}_2=[\underline{x_2},\overline{x_2}]$, $\mathbf{y}_1=[\underline{y_1},\overline{y_1}]$, $\mathbf{y}_2=[\underline{y_2},\overline{y_2}]$, and by Definition 11 assuming that $F_1\le F_2$ we have to prove

$$(\underline{x_1}\leqslant\overline{y_1},\ \underline{x_2}\leqslant\overline{y_2})\Rightarrow F_1(\underline{x_1},\underline{x_2})\leqslant F_2(\overline{y_1},\overline{y_2}).$$

By assumption F_1 is an aggregation function on $[0, 1]$, so it is increasing and if $\underline{x_1}\leqslant\overline{y_1}$, $\underline{x_2}\leqslant\overline{y_2}$, then $F_1(\underline{x_1},\underline{x_2})\leqslant F_1(\overline{y_1},\overline{y_2})\leqslant F_2(\overline{y_1},\overline{y_2})$. As a result for proving the monotonicity condition of $\mathcal{F}$ we need only increasingness of F_1. The proof for boundary conditions for $\mathcal{F}$ follows immediately from boundary conditions on F_1 and F_2. The second case may be proven analogously. □

Corollary 1. *If $\mathcal{F}:(L^I)^2\to L^I$ is a decomposable aggregation function, then $\mathcal{F}$ is a decomposable pos-aggregation function.*

The converse statement to Corollary 1 does not hold (certainly the same is true for Theorem 2). Namely, $\mathcal{A}_1$ in Example 10 shows that $\mathcal{F}$ need not be increasing and $\mathcal{A}_1$ in Example 9 shows that $\mathcal{F}$ need not be decomposable.

Example 9. Let $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]$ and A: $[0,1]^2\to[0,1]$ be an aggregation function, where for $\mathcal{A}_1$ and $\mathcal{A}_2$ let A have zero element 1, for $\mathcal{A}_3$ and $\mathcal{A}_4$ let A have zero element 0. The following functions are aggregation functions on L^I (non-decomposable) and they are also pos-aggregation functions (but they are not nec-aggregation functions and $\mathcal{A}_6,\mathcal{A}_5\notin\mathcal{A}_{GD}$):

$$\mathcal{A}_1(\mathbf{x},\mathbf{y})=\begin{cases}[1,1], & (\mathbf{x},\mathbf{y})=([1,1],[1,1])\\ [0,A(x_1,y_2)], & \text{otherwise}\end{cases}$$

$$\mathcal{A}_2(\mathbf{x},\mathbf{y})=\begin{cases}[1,1], & (\mathbf{x},\mathbf{y})=([1,1],[1,1])\\ [0,A(x_2,y_1)], & \text{otherwise}\end{cases}$$

$$\mathcal{A}_3(\mathbf{x},\mathbf{y})=\begin{cases}[0,0], & (\mathbf{x},\mathbf{y})=([0,0],[0,0])\\ [A(x_1,y_2),1], & \text{otherwise}\end{cases}$$

$$\mathcal{A}_4(\mathbf{x},\mathbf{y})=\begin{cases}[0,0], & (\mathbf{x},\mathbf{y})=([0,0],[0,0])\\ [A(x_2,y_1),1], & \text{otherwise}\end{cases}$$

$$\mathcal{A}_5(\mathbf{x},\mathbf{y})=\begin{cases}[1,1], & (\mathbf{x},\mathbf{y})=([1,1],[1,1])\\ [\frac{y_1\frac{x_1+x_2}{2}}{2},\frac{x_2+y_2}{2}], & \text{otherwise}\end{cases}$$

$$\mathcal{A}_6(\mathbf{x},\mathbf{y})=\begin{cases}[1,1], & (\mathbf{x},\mathbf{y})=([1,1],[1,1])\\ [\frac{x_1\frac{y_1+y_2}{2}}{2},\frac{x_2+y_2}{2}], & \text{otherwise}\end{cases}$$

The following functions are decomposable aggregation functions on L^I, they belong to $\mathcal{A}_{GD}$, and they are also pos-aggregation functions (but they are not nec-aggregation functions), where A: $[0,1]^2\to[0,1]$ is an aggregation function:

$$\mathcal{A}_7(\mathbf{x},\mathbf{y})=\begin{cases}[1,1], & (\mathbf{x},\mathbf{y})=([1,1],[1,1])\\ [0,A(x_2,y_2)], & \text{otherwise}\end{cases}$$

where $\mathcal{A}_7(\mathbf{x},\mathbf{y})=[A_w(x_1,y_1),A(x_2,y_2)]$ (A_w is the weakest aggregation function on $[0, 1]$).

$$\mathcal{A}_8(\mathbf{x},\mathbf{y})=\begin{cases}[0,0], & (\mathbf{x},\mathbf{y})=([0,0],[0,0])\\ [A(x_1,y_1),1], & \text{otherwise}\end{cases}$$

where $\mathcal{A}_8(\mathbf{x},\mathbf{y}) = [A(x_1,y_1), A_s(x_2,y_2)]$ (A_s is the strongest aggregation function on [0, 1]).

Example 10. Let $\mathbf{x} = [x_1, x_2]$, $\mathbf{y} = [y_1, y_2]$. The following functions on L^I are pos-aggregation functions but they are neither aggregation functions (cf. Definition 7) nor nec-aggregation functions (in $\mathcal{A}_1$ and $\mathcal{A}_2$ for the Lehmer mean we use convention $\frac{0}{0} = 0$):

$$\mathcal{A}_1(\mathbf{x},\mathbf{y}) = \begin{cases} [0,0], & (\mathbf{x},\mathbf{y}) = ([0,0],[0,0]) \\ [\frac{x_1^2+y_1^2}{x_1+y_1}, 1], & \text{otherwise} \end{cases}$$

$$\mathcal{A}_2(\mathbf{x},\mathbf{y}) = \begin{cases} [1,1], & (\mathbf{x},\mathbf{y}) = ([1,1],[1,1]) \\ [0, \frac{x_2^2+y_2^2}{x_2+y_2}], & \text{otherwise} \end{cases}$$

$$\mathcal{A}_3(\mathbf{x},\mathbf{y}) = [x_1 \cdot |2y_1 - 1|, x_2]$$

$$\mathcal{A}_4(\mathbf{x},\mathbf{y}) = [x_1 \cdot |2x_1 - 1|, x_2]$$

$$\mathcal{A}_5(\mathbf{x},\mathbf{y}) = [x_1 \cdot |2x_2 - 1|, x_2]$$

$$\mathcal{A}_6(\mathbf{x},\mathbf{y}) = [x_1 \cdot |2y_2 - 1|, x_2]$$

$$\mathcal{A}_7(\mathbf{x},\mathbf{y}) = [x_1 - (\max(0, x_1 - y_1))^2, x_2]$$

$$\mathcal{A}_8(\mathbf{x},\mathbf{y}) = [x_1 - (\max(0, x_1 - y_2))^2, x_2],$$

where $\mathcal{A}_1$–$\mathcal{A}_4$ and $\mathcal{A}_7$ are decomposable operations.

Other examples of proper functions from the family $\mathcal{A}_\pi$ may be created in a similar way as shown above, where instead of the Lehmer mean (or the weighted Lehmer mean) in Example 10 we may use a non-increasing, in a classical way, function. It was done in $\mathcal{A}_3$–$\mathcal{A}_8$ by applying $F(x,y) = x - (\max(0, x-y))^2$ or $F(x,y) = x \cdot |2y - 1|$ in the first coordinate. Such family of functions consists of recently introduced pre-aggregation functions.

Definition 13 ([22])**.** A function F: $[0, 1]^n \to [0, 1]$ is called a pre-aggregation function if it satisfies boundary conditions $F(0,\dots,0) = 0$, $F(1,\dots,1) = 1$ and there exists a real vector $\overrightarrow{r} = (r_1,\dots,r_n) \in [0,1]^n$, $\overrightarrow{r} \neq \overrightarrow{0}$ such that F is $\overrightarrow{r}$-increasing, i.e. for all $(x_1,\dots,x_n) \in [0,1]^n$ and for all $c > 0$ such that $(x_1 + cr_1,\dots,x_n + cr_n) \in [0,1]^n$ it holds

$$F(x_1 + cr_1,\dots,x_n + cr_n) \geqslant F(x_1,\dots,x_n).$$

Precisely speaking, we mean proper pre-aggregation functions (the ones which are not aggregation functions) where condition of monotonicity is weakened to the directional monotonicity condition with respect to some vector $\overrightarrow{r}$. In such case we obtain proper pos-aggregation functions. Note that directional monotonicity (monotonicity along some vector) is a generalization of weak monotonicity (increasingness) introduced in [30]. Examples of proper pre-aggregation functions are: the mode (defined as the function that gives back the value which appears most times in the considered n-tuple, or the smallest of the values that appears most times, in case there is more than one such a value) or functions $F(x,y) = x - (\max(0, x-y))^2$, $F(x,y) = x \cdot |2y-1|$. Other examples one can find in [22].

There exist decomposable operations $\mathcal{F} : (L^I)^2 \to L^I$ which are pos-aggregation functions but they are not aggregation functions (cf. Example 10). That is not the case for nec-aggregation functions which is shown by Theorem 3 (characterization of decomposable nec-aggregation functions).

Theorem 3. *Let $\mathcal{F} : (L^I)^2 \to L^I$ be a decomposable operation. $\mathcal{F}$ is a nec-aggregation function if and only if $F_1 = F_2$, F_1 is an aggregation function on* [0, 1].

Proof. We will prove the monotonicity condition.

Sufficiency. We have to show that

$$(\overline{x_1} \leqslant \underline{y_1},\ \overline{x_2} \leqslant \underline{y_2}) \Rightarrow F_2(\overline{x_1}, \overline{x_2}) \leqslant F_1(\underline{y_1}, \underline{y_2}), \tag{12}$$

which is fulfilled by assumption that F_1 is increasing and $F_1 = F_2$.

Necessity. We will show that $F_1 = F_2$. Since definition of decomposable operations makes sense only if $F_1 \leq F_2$, for indirect proof we suppose that $F_1 < F_2$. Let $\mathbf{x}_1 = [\underline{x_1}, \overline{x_1}]$, $\mathbf{x}_2 = [\underline{x_2}, \overline{x_2}]$, $\mathbf{y}_1 = [\underline{y_1}, \overline{y_1}]$, $\mathbf{y}_2 = [\underline{y_2}, \overline{y_2}]$ and $\overline{x_1} = \underline{y_1}$, $\overline{x_2} = \underline{y_2}$. We see that $\overline{x_1} \leqslant \underline{y_1}$, $\overline{x_2} \leqslant \underline{y_2}$ and by (12) we get $F_2(\overline{x_1}, \overline{x_2}) \leqslant F_1(\underline{y_1}, \underline{y_2})$, so $F_2(\overline{x_1}, \overline{x_2}) \leqslant F_1(\overline{x_1}, \overline{x_2})$. This a contradiction to $F_1 < F_2$. Now, knowing that $F_1 = F_2$, we will show that F_1 is increasing. By assumption that $\mathcal{F}$ is a decomposable nec-aggregation function it follows that equation (12) holds with $F_1 = F_2$ and it means that F_1 is increasing. The proof for boundary conditions is obvious. □

A decomposable operation $\mathcal{F}$ may be at the same time an aggregation function, a pos-aggregation function and a nec-aggregation function. Examples of such operations are $\mathcal{A}_s$ and $\mathcal{A}_w$, which have been already presented in Proposition 5. Moreover, by Corollary 1 and Theorem 3 we obtain the following result.

Corollary 2. *If $\mathcal{F}:(L^I)^2 \to L^I$ is a decomposable nec-aggregation function, then $\mathcal{F}$ is a decomposable aggregation function. Moreover, if $\mathcal{F}:(L^I)^2 \to L^I$ is a decomposable nec-aggregation function, then $\mathcal{F}$ is a decomposable pos-aggregation function.*

The converse statement to Corollary 2 is not true which is shown by the next example.

Example 11. Let $\mathbf{x}=[x_1,x_2], \mathbf{y}=[y_1,y_2]\in L^I$ and decomposable aggregation function be given by the formula $\mathcal{A}(\mathbf{x},\mathbf{y})=[\sqrt{x_1\cdot y_1}, \frac{x_2+y_2}{2}]$. $\mathcal{A}$ is a pos-aggregation function and it is not a nec-aggregation function. Namely, let us consider intervals $\mathbf{x}_1=[0.2,0.6]$, $\mathbf{x}_2=[0.3,0.5]$, $\mathbf{y}_1=[0.6,0.7]$, $\mathbf{y}_2=[0.5,0.6]$. Thus $\overline{x_1}\leqslant \underline{y_1}$ and $\overline{x_2}\leqslant \underline{y_2}$, but $0.55=\frac{\overline{x_1}+\overline{x_2}}{2} > \sqrt{\underline{y_1}\cdot\underline{y_2}}=\sqrt{0.3}$.

Now we will consider non-decomposable aggregation functions on L^I and their connections with the families $\mathcal{A}_\nu$ and $\mathcal{A}_\pi$.

Proposition 6. *If $A_1=A_2$, then pseudomax-A_1A_2 representable aggregation function $\mathcal{A}$ is a nec-aggregation function.*

Proof. Let $A_1=A_2=A$, $\overline{x_1}\leqslant \underline{y_1}$ and $\overline{x_2}\leqslant \underline{y_2}$. Thus since we have also $\underline{x_1}\leqslant\overline{x_1}$ and $\underline{x_2}\leqslant\overline{x_2}$, we get $A(\overline{x_1},\underline{x_2})\leqslant A(\underline{y_1},\underline{y_2})$ and $A(\underline{x_1},\overline{x_2})\leqslant A(\underline{y_1},\underline{y_2})$. As a result $\max(A(\underline{x_1},\overline{x_2}),A(\overline{x_1},\underline{x_2}))\leqslant A(\underline{y_1},\underline{y_2})$, which means that (7) is fulfilled and $\mathcal{A}$ is a nec-aggregation function. □

If $A_1<A_2$ we may not have at the same time pseudomax-A_1A_2 representable aggregation function and a nec-aggregation function.

Example 12. Let $\mathbf{x}=[x_1,x_2], \mathbf{y}=[y_1,y_2]\in L^I$ and $\mathcal{A}(\mathbf{x},\mathbf{y})=[x_1\cdot y_1, \max(\frac{x_1+y_2}{2},\frac{x_2+y_1}{2})]$. $\mathcal{A}$ is pseudomax-A_1A_2 representable aggregation function but $\mathcal{A}\notin\mathcal{A}_\nu$. Namely, to check condition (7) let us consider intervals $\mathbf{x}_1=[0.2,0.6]$, $\mathbf{x}_2=[0.3,0.5]$, $\mathbf{y}_1=[0.6,0.7]$, $\mathbf{y}_2=[0.5,0.6]$. Thus $0.45=\max(0.35,0.45)=\max(\frac{\underline{x_1}+\overline{x_2}}{2},\frac{\overline{x_1}+\underline{x_2}}{2}) > \underline{y_1}\cdot\underline{y_2}=0.3$. Moreover, $\mathcal{A}\notin\mathcal{A}_\pi$ since for $\mathbf{x}_1=\mathbf{x}_2=[0.6,0.8]$, $\mathbf{y}_1=\mathbf{y}_2=[0,0.6]$ in (6) we get $0.36=\underline{x_1}\cdot\underline{x_2} > \max(\frac{\underline{y_1}+\overline{y_2}}{2},\frac{\overline{y_1}+\underline{y_2}}{2})=0.3$.

Similarly, we may prove that:

Proposition 7. *If $A_1=A_2$, then pseudomin-A_1A_2 representable aggregation function $\mathcal{A}$ is a nec-aggregation function.*

If $A_1<A_2$ we may not have at the same time pseudomin-A_1A_2 representable aggregation function and a nec-aggregation function.

Example 13. Let $\mathbf{x}=[x_1,x_2], \mathbf{y}=[y_1,y_2]\in L^I$ and $\mathcal{A}(\mathbf{x},\mathbf{y})=[\min(x_1\cdot y_2, x_2\cdot y_1), \frac{x_2+y_2}{2}]$. $\mathcal{A}$ is pseudomin-A_1A_2 representable aggregation function but it is not a nec-aggregation function. Namely, to check condition (7) let us consider intervals $\mathbf{x}_1=[0.2,0.6]$, $\mathbf{x}_2=[0.3,0.5]$, $\mathbf{y}_1=[0.6,0.7]$, $\mathbf{y}_2=[0.5,0.6]$. Thus $0.55=\frac{\overline{x_1}+\overline{x_2}}{2} > \min(\underline{y_1}\cdot\overline{y_2},\overline{y_1}\cdot\underline{y_2})=\min(0.36,0.35)=0.35$. Moreover, it is easy to see that $\mathcal{A}\in\mathcal{A}_\pi$.

Proposition 8. *If $A_1=A_2$ and $A_3=A_4$ in (9), then $\mathcal{A}\in\mathcal{A}_\nu$.*

Proof. Let $\overline{x_1}\leqslant\underline{y_1}$ and $\overline{x_2}\leqslant\underline{y_2}$. By assumptions we have to show that $A_3(A_1(\underline{x_1},\overline{x_2}),A_1(\overline{x_1},\underline{x_2}))\leqslant A_3(A_1(\underline{y_1},\overline{y_2}),A_1(\overline{y_1},\underline{y_2}))$, which is fulfilled by the monotonicity of A_1 and A_3. □

Remark 1. Note that under assumptions of Proposition 8 we get the aggregation operator of the type $\mathcal{A}(\mathbf{x},\mathbf{y})=[\alpha(\mathbf{x},\mathbf{y}),\alpha(\mathbf{x},\mathbf{y})]$. So it is rather more suitable for fuzzy relations than for interval-valued fuzzy relations.

Example 14. If $A_1<A_2$ and $A_3<A_4$ in (9), then $\mathcal{A}$ may not be a nec-aggregation function. Let $A_1=A_3$ be the product and $A_2=A_4$ be the arithmetic mean. Thus it is not true that $A_4(A_2(\underline{x_1},\overline{x_2}),A_2(\overline{x_1},\underline{x_2}))\leqslant A_3(A_1(\underline{y_1},\overline{y_2}),A_1(\overline{y_1},\underline{y_2}))$. Namely, to check condition (7) let $\mathbf{x}_1=[0,0.5]$, $\mathbf{x}_2=[0.1,0.6]$, $\mathbf{y}_1=[0.5,0.8]$, $\mathbf{y}_2=[0.6,0.8]$, then

$$0.3=\frac{\frac{\underline{x_1}+\overline{x_2}}{2}+\frac{\overline{x_1}+\underline{x_2}}{2}}{2} > (\underline{y_1}\cdot\overline{y_2})\cdot(\overline{y_1}\cdot\underline{y_2})=0.192.$$

Moreover, $\mathcal{A}\notin\mathcal{A}_\pi$.

Example 15. Let $\mathcal{A}(\mathbf{x},\mathbf{y})=[x_1x_2,x_1]$, $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]$. This is a non-decomposable aggregation function on L^I, which belongs to $\mathcal{A}_{GD}$ (it fulfils (9) with $A_1=A_2=P_1$, $A_3=T_P$, $A_4=P_1$), but $\mathcal{A}$ is not a nec-aggregation function and $\mathcal{A}$ is not a pos-aggregation function.

Now, we will consider connections between the family of aggregation functions $\mathcal{A}_\pi$ and aggregation functions $\mathcal{A}_{GD}$ (cf. Example 4 and Definition 12).

Even if $A_1=A_2$, then pseudomin-A_1A_2 representable aggregation function (pseudomax-A_1A_2 representable aggregation function) may not be a pos-aggregation function. This is shown by the next two examples.

Example 16. Let $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]\in L^I$ and $\mathcal{A}(\mathbf{x},\mathbf{y})=[x_1\cdot y_1,\max(x_1\cdot y_2,x_2\cdot y_1)]$. Thus $\mathcal{A}$ is pseudomax A_1A_2-representable aggregation function but it is not a pos-aggregation function, i.e. let (using notations from Theorem 2)

$\mathbf{x}_1 = [0.7, 0.7]$, $\mathbf{x}_2 = [0.8, 0.8]$, $\mathbf{y}_1 = [0.4, 0.7]$, $\mathbf{y}_2 = [0.6, 0.8]$, then $\underline{x_1} \leqslant \overline{y_1}$ and $\underline{x_2} \leqslant \overline{y_2}$ but it is not valid that $0.56 = \underline{x_1} \cdot \underline{x_2} \leqslant \max(\underline{y_1} \cdot \overline{y_2}, \overline{y_1} \cdot \underline{y_2}) = 0.42$. Similar conclusion we get for $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [\frac{x_1+y_1}{2}, \max(\frac{x_1+y_2}{2}, \frac{x_2+y_1}{2})]$.

And vice versa, $\mathcal{A}_6$ from Example 9, which is a pos-aggregation function is not pseudomax A_1A_2-representable aggregation function.

Example 17. Let $\mathbf{x} = [x_1, x_2], \mathbf{y} = [y_1, y_2] \in L^I$ and $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [\min(x_1 \cdot y_2, x_2 \cdot y_1), x_2 \cdot y_2]$. Thus $\mathcal{A}$ is pseudomin A_1A_2-representable aggregation function but it is not a pos-aggregation function, i.e. let (using notations from Theorem 2) $\mathbf{x}_1 = [0.7, 0.8]$, $\mathbf{x}_2 = [0.8, 0.9]$, $\mathbf{y}_1 = [0.4, 0.7]$, $\mathbf{y}_2 = [0.6, 0.8]$, then $\underline{x_1} \leqslant \overline{y_1}$ and $\underline{x_2} \leqslant \overline{y_2}$ but it is not valid that $0.63 = \min(\underline{x_1} \cdot \overline{x_2}, \overline{x_1} \cdot \underline{x_2}) \leqslant \overline{y_1} \cdot \overline{y_2} = 0.56$.

And vice versa, $\mathcal{A}_5$ from Example 9, which is a pos-aggregation function is not pseudomin A_1A_2-representable aggregation function.

Example 18. For the same arguments as in Example 17, $A_1 = A_2 = T_P$, we see that an aggregation function of the form (8) (and certainly an aggregation function of the form (9)) may not be a pos-aggregation function, since it is not true that $0.63 = \min(\underline{x_1} \cdot \overline{x_2}, \overline{x_1} \cdot \underline{x_2}) \leqslant \max(\underline{y_1} \cdot \overline{y_2}, \overline{y_1} \cdot \underline{y_2}) = 0.42$.

And vice versa, $\mathcal{A}_5$ and $\mathcal{A}_6$ from Example 9, are pos-aggregation functions but they are not aggregation functions of the form (8) (and of the form (9)).

However, we may also consider the following operation:

Example 19. A non-decomposable aggregation function on L^I $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [x_1x_2, x_2]$, where $\mathbf{x} = [x_1, x_2]$, $\mathbf{y} = [y_1, y_2]$, fulfils the condition $\mathcal{A} \in \mathcal{A}_{GD}$ (it is of the form (9)), $\mathcal{A} \in \mathcal{A}_\pi$ but $\mathcal{A} \notin \mathcal{A}_\nu$.

Now we will give examples of aggregation functions belonging to $\mathcal{A}_\nu$ which may or may not belong to other considered in this paper families of functions. Especially, these functions are not decomposable ones and they are not of the form (8)–(11).

Example 20. Let $\mathbf{x} = [x_1, x_2]$, $\mathbf{y} = [y_1, y_2]$. The following aggregation function $\mathcal{A} : (L^I)^2 \to L^I$, where $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [x_1x_2, x_1x_2]$ is not decomposable and $\mathcal{A} \in \mathcal{A}_{GD}$ (it is enough to put $A_1 = A_2 = P_1$ and $A_3 = A_4 = T_P$ in (9)), $\mathcal{A} \in \mathcal{A}_\nu$ but $\mathcal{A} \notin \mathcal{A}_\pi$.

Further examples of such functions are $\mathcal{A} : (L^I)^2 \to L^I$, where $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [x_1^2, x_1x_2]$ is not decomposable, is of the form (9) ($A_1(a, b) = a^2$ for $a \in [0, 1]$, $A_2 = P_1$, $A_3 = \min$, $A_4 = T_P$, note that $A_4 \le A_3$), it belongs to $\mathcal{A}_\nu$, it does not belong to $\mathcal{A}_\pi$. Similarly, the same conditions are fulfilled by $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [x_1x_2, x_2^2]$.

Example 21. Let $\mathbf{x} = [x_1, x_2]$, $\mathbf{y} = [y_1, y_2]$. The aggregation function $\mathcal{A} : (L^I)^2 \to L^I$, $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [\frac{y_1 + \frac{x_1+x_2}{2}}{2}, \frac{x_2+y_2}{2}]$ is not decomposable, is not of the form (8)–(11), i.e. $\mathcal{A} \notin \mathcal{A}_{GD}$ it belongs to $\mathcal{A}_\nu$, it does not belong to $\mathcal{A}_\pi$.

Remark 2. There exist proper nec-aggregation functions. For example, $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [\frac{x_1+x_2}{2}, \frac{x_1^2+x_2^2}{x_1+x_2}]$, where $\mathbf{x} = [x_1, x_2]$, $\mathbf{y} = [y_1, y_2]$, is not an aggregation function on L^I and $\mathcal{A} \in \mathcal{A}_\nu$ and $\mathcal{A} \notin \mathcal{A}_\pi$, $\mathcal{A} \notin \mathcal{D}$. Similarly, analyzing the obtained results, we may assume existence of non-decomposable operations $\mathcal{A} : (L^I)^2 \to L^I$ not being aggregation functions on L^I and fulfilling $\mathcal{A} \in \mathcal{A}_\nu$ and $\mathcal{A} \in \mathcal{A}_\pi$.

Remark 3. If it comes to the problem of existence of non-decomposable aggregation functions $\mathcal{A}$ on L^I ($\mathcal{A} \notin \mathcal{D}$), which at the same time fulfil conditions $\mathcal{A} \in \mathcal{A}_\nu$ and $\mathcal{A} \in \mathcal{A}_\pi$, analyzing the obtained previously results, we may assume that such existence is possible (in both cases: $\mathcal{A} \in \mathcal{A}_{GD}$, $\mathcal{A} \notin \mathcal{A}_{GD}$).

To finish our analysis of connections between the families of operations considered in this paper we will give an example of operation which does not belong to any of them.

Example 22. Let $\mathbf{x} = [x_1, x_2]$, $\mathbf{y} = [y_1, y_2]$. The following function $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [x_2 \cdot |2x_2 - 1|, x_2]$ is not an aggregation function on L^I, $\mathcal{A} \notin \mathcal{A}_\nu$, $\mathcal{A} \notin \mathcal{A}_\pi$, $\mathcal{A} \notin \mathcal{D}$. $\mathcal{A}$ fulfils boundary conditions but it does not fulfil any of the monotonicity conditions (5), (6) and (7).

4. Preservation of *B*-transitivity properties by diverse types of aggregation functions on L^I

Now we will consider the aggregation of interval-valued fuzzy relations having pos-*B*-transitivity and nec-*B*-transitivity property and the problem of preservation of these properties, i.e. if we assume that $R_1, R_2 \in \mathcal{IVFR}(X)$ have some property, then $R_\mathcal{F} = \mathcal{F}(R_1, R_2)$ will also has the same property. Although $\mathcal{F} : (L^I)^2 \to L^I$ may be an aggregation function (of some type) we consider $\mathcal{F}$ with no pre-assumed properties. This approach enables to get more general results. We generally intend to consider the same type of property and aggregation function, namely based on the same type of comparability relation $\preceq_\pi$ or $\preceq_\nu$. However, to complete the information we also present the mixture of aggregation type and type of comparability relation. We will mainly consider subfamilies of aggregation functions which have the representations. To begin with, we recall the notion of dominance which will appear in conditions guaranteeing preservation of diverse types of transitivity.

Definition 14 (cf. [29]). Let $m, n \in \mathbb{N}$. A function F: $[0, 1]^m \to [0, 1]$ dominates function G: $[0, 1]^n \to [0, 1]$ ($F \gg G$) if for an arbitrary matrix $[a_{ik}] = A \in [0, 1]^{m\times n}$ the following inequality holds:

$$F(G(a_{11}, \ldots, a_{1n}), \ldots, G(a_{m1}, \ldots, a_{mn})) \geqslant G(F(a_{11}, \ldots, a_{m1}), \ldots, F(a_{1n}, \ldots, a_{mn})).$$

Here we present only a few examples of dominance between functions. Other examples on dominance and comprehensive study of literature on this subject are gathered for example in [2].

Example 23 (cf. [8,28]). A weighted geometric mean G_w dominates t-norm T_P, where

$$G_w(x_1, \ldots, x_n) = \prod_{k=1}^{n} x_k^{w_k}, \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1.$$

A weighted arithmetic mean M_w dominates t-norm T_L, where

$$M_w(x_1, \ldots, x_n) = \sum_{k=1}^{n} w_k x_k, \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1.$$

The aggregation function

$$A(t_1, \ldots, t_n) = \frac{p}{n} \sum_{k=1}^{n} t_k + (1-p) \min_{1\leqslant k\leqslant n} t_k$$

dominates T_L, where $p \in (0, 1)$. Let us consider projections P_k. Then $F \gg P_k$ and $P_k \gg F$ for any function F: $[0, 1]^n \to [0, 1]$.

Corollary 3 ([2]). *Minimum dominates any fuzzy conjunction. A weighted minimum*

$$A(t_1, \ldots, t_n) = \min_{1\leqslant k\leqslant n} \max(1 - w_k, t_k), \quad \max_{1\leqslant k\leqslant n} w_k = 1,$$

dominates any t-seminorm.

There exists a characterization theorem of increasing functions which dominate minimum.

Theorem 4. *(cf. [30, Proposition 5.1]) An increasing function F: $[0, 1]^n \to [0, 1]$ dominates minimum if and only if for each $t_1, \ldots, t_n \in [0, 1]$*

$$F(t_1, \cdots, t_n) = \min(f_1(t_1), \cdots, f_n(t_n)),$$

where f_k: $[0, 1] \to [0, 1]$ is increasing with $k = 1, \ldots, n$.

Example 24. Examples of functions which dominate minimum are: if $f_k(t) = t,\ k = 1, \ldots, n,$ then $F = \min,$ if for a certain $k \in \{1, \ldots, n\}$, function $f_k(t) = t$ and $f_i(t) = 1$ for $i \neq k$, then $F = P_k$-projections, if $f_k(t) = \max(1 - v_k, t),\ v_k \in [0, 1]$, $k = 1, \ldots, n,\ \max_{1\leqslant k\leqslant n} v_k = 1$, then F is a weighted minimum.

Dually, we get characterization theorem of increasing functions which are dominated by maximum.

Theorem 5 ([2]). *An increasing function F: $[0, 1]^n \to [0, 1]$ is dominated by maximum if and only if for each $t_1, \ldots, t_n \in [0, 1]$*

$$F(t_1, \ldots, t_n) = \max(f_1(t_1), \ldots, f_n(t_n)),$$

where f_k: $[0, 1] \to [0, 1]$ is increasing with $k = 1, \ldots, n$.

Example 25. Examples of functions which are dominated by maximum are: if $f_k(t) = t,\ k = 1, \ldots, n,$ then $F = \max,$ if for a certain $k \in \{1, \ldots, n\}$, function $f_k(t) = t$ and $f_i(t) = 1$ for $i \neq k$, then $F = P_k$ - projections, if $f_k(t) = \min(v_k, t),\ v_k \in [0, 1]$, $k = 1, \ldots, n,\ \max_{1\leqslant k\leqslant n} v_k = 1$, then F is a weighted maximum.

We present the next statement for general case of decomposable operations and then we put appropriate conclusion for considered here families of aggregation functions.

Theorem 6. *Let $\mathcal{F}$ be a decomposable operation and F_1 be increasing, $F_1 \gg B$, $F_1 \leq F_2$. If $R_1, R_2 \in \mathcal{IVFR}(X)$ are pos-B-transitive, then $R_{\mathcal{F}}(x, y) = \mathcal{F}(R_1(x, y), R_2(x, y))$ is pos-B-transitive.*

Proof. Let $x, y, z \in X$. According to assumptions for $k = 1, 2$ we have $B(\underline{R_k}(x, y), \underline{R_k}(y, z)) \leqslant \overline{R_k}(x, z)$. We will show that $R_{\mathcal{F}}$ is pos-B-transitive, i.e. $B(\underline{R_{\mathcal{F}}}(x, y), \underline{R_{\mathcal{F}}}(y, z)) \leqslant \overline{R_{\mathcal{F}}}(x, z)$. We obtain

$$B(\underline{R_{\mathcal{F}}}(x, y), \underline{R_{\mathcal{F}}}(y, z)) = B(F_1(\underline{R_1}(x, y), \underline{R_2}(x, y)), F_1(\underline{R_1}(y, z), \underline{R_2}(y, z))) \leqslant$$

$$F_1(B(\underline{R_1}(x, y), \underline{R_1}(y, z)), B(\underline{R_2}(x, y), \underline{R_2}(y, z))) \leqslant F_1(\overline{R_1}(x, z), \overline{R_2}(x, z)) \leqslant F_2(\overline{R_1}(x, z), \overline{R_2}(x, z)) = \overline{R_{\mathcal{F}}}(x, z).$$

□

Corollary 4. *Let $\mathcal{F}$ be a decomposable aggregation function, decomposable pos-aggregation function, or decomposable nec-aggregation function (then $F_1 = F_2$ according to Theorem 3) and $F_1 \gg B$. If $R_1, R_2 \in \mathcal{IVFR}(X)$ are pos-B-transitive, then $R_{\mathcal{F}}(x, y) = \mathcal{F}(R_1(x, y), R_2(x, y))$ is pos-B-transitive.*

Let us notice that dominance over operation B is not necessary for preservation of pos-B-transitivity.

Remark 4. Most of the pos-aggregation functions from Examples 9, 10 preserve pos-B-transitivity. For $\mathcal{A}_3$–$\mathcal{A}_8$ in Example 10 it is enough to assume that B is increasing. For $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_7$ in Example 9, under assumption that for input relations $R_1, R_2 \in \mathcal{IVFR}(X)$ we have $R_1(x, y) \neq [1, 1]$, $R_2(x, y) \neq [1, 1]$ for any $x, y \in X$, it is enough to assume that $B(0, 0) = 0$. For $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_8$ in Example 9, under assumption that for input relations $R_1, R_2 \in \mathcal{IVFR}(X)$ we have $R_1(x, y) \neq [0, 0]$, $R_2(x, y) \neq [0, 0]$ for any $x, y \in X$, we get straightforward (by definition of aggregation functions) preservation of pos-B-transitivity.

Theorem 7. *Let $\mathcal{F}$ be a decomposable operation and F_1 be increasing, $F_1 \gg B$, $F_1 = F_2$. If $R_1, R_2 \in \mathcal{IVFR}(X)$ are nec-B-transitive, then $R_{\mathcal{F}}(x, y) = \mathcal{F}(R_1(x, y), R_2(x, y))$ is nec-B-transitive.*

Proof. Let $x, y, z \in X$. According to assumptions for $k = 1, 2$ we have $B(\overline{R_k}(x, y), \overline{R_k}(y, z)) \leqslant \underline{R_k}(x, z)$. We will show that $R_{\mathcal{F}}$ is nec-B-transitive, i.e. $B(\overline{R_{\mathcal{F}}}(x, y), \overline{R_{\mathcal{F}}}(y, z)) \leqslant \underline{R_{\mathcal{F}}}(x, z)$. We obtain

$$B(\overline{R_{\mathcal{F}}}(x, y), \overline{R_{\mathcal{F}}}(y, z)) = B(F_1(\overline{R_1}(x, y), \overline{R_2}(x, y)), F_1(\overline{R_1}(y, z), \overline{R_2}(y, z))) \leqslant$$

$$F_1(B(\overline{R_1}(x, y), \overline{R_1}(y, z)), B(\overline{R_2}(x, y), \overline{R_2}(y, z))) \leqslant F_1(\underline{R_1}(x, z), \underline{R_2}(x, z)) \leqslant F_1(\underline{R_1}(x, z), \underline{R_2}(x, z)) = \underline{R_{\mathcal{F}}}(x, z).$$

□

Corollary 5. *Let $\mathcal{F}$ be a decomposable aggregation function such that $F_1 = F_2$, decomposable pos-aggregation function such that $F_1 = F_2$ or decomposable nec-aggregation function (then $F_1 = F_2$ according to Theorem 3), and $F_1 \gg B$. If $R_1, R_2 \in \mathcal{IVFR}(X)$ are nec-B-transitive, then $R_{\mathcal{F}}(x, y) = \mathcal{F}(R_1(x, y), R_2(x, y))$ is nec-B-transitive.*

Remark 5. Let us note that decomposability of $\mathcal{A}$ is not necessary for preservation of nec-B-transitivity property. The following aggregation function $\mathcal{A} : (L^I)^2 \to L^I$, $\mathcal{A}(\mathbf{x}, \mathbf{y}) = [\frac{y_1 + \frac{x_1 + x_2}{2}}{2}, \frac{x_2 + y_2}{2}]$ is not decomposable, belongs to $\mathcal{A}_\nu$ (cf. Example 21) and it preserves nec-B-transitivity for any B: $[0, 1]^2 \to [0, 1]$ such that B is dominated by the arithmetic mean M, i.e. $M \gg B$. Namely, if $R_1, R_2 \in \mathcal{IVFR}(X)$ are nec-B-transitive for such operation B, then $R_{\mathcal{A}}(R_1, R_2)$ is nec-B-transitive, since by the given assumptions it holds

$$B(\frac{\overline{R_1}(x, y) + \overline{R_2}(x, y)}{2}, \frac{\overline{R_1}(y, z) + \overline{R_2}(y, z)}{2}) \leqslant \frac{B(\overline{R_1}(x, y), \overline{R_1}(y, z)) + B(\overline{R_2}(x, y), \overline{R_2}(y, z))}{2} \leqslant$$

$$\frac{\underline{R_1}(x, z) + \underline{R_2}(x, z)}{2} \leqslant \frac{\underline{R_2}(x, z) + \frac{\underline{R_1}(x, z) + \overline{R_1}(x, z)}{2}}{2}.$$

Now we will turn to preservation of the new types of transitivity properties by other aggregation functions on L^I. We focus on pseudomin and pseudomax aggregation functions.

Theorem 8. *Let $\mathcal{A}$ be pseudomin A_1A_2-representable aggregation function and $A_1 = A_2$ (by Proposition 7 it means that $\mathcal{A} \in \mathcal{A}_\nu$). If $A_1 \gg B$, then $\mathcal{A}$ preserves nec-B-transitivity.*

Proof. Let $x, y, z \in X$. According to assumptions for $k = 1, 2$ we have $B(\overline{R_k}(x, y), \overline{R_k}(y, z)) \leqslant \underline{R_k}(x, z)$. We will show that $R_{\mathcal{A}}$ is nec-B-transitive, i.e. $B(\overline{R_{\mathcal{A}}}(x, y), \overline{R_{\mathcal{A}}}(y, z)) \leqslant \underline{R_{\mathcal{A}}}(x, z)$. Since, $A_1(\underline{R_1}(x, z), \underline{R_2}(x, z)) \leqslant A_1(\underline{R_1}(x, z), \overline{R_2}(x, z))$ and $A_1(\underline{R_1}(x, z), \underline{R_2}(x, z)) \leqslant A_1(\overline{R_1}(x, z), \underline{R_2}(x, z))$ we obtain

$$B(\overline{R_{\mathcal{A}}}(x, y), \overline{R_{\mathcal{A}}}(y, z)) = B(A_1(\overline{R_1}(x, y), \overline{R_2}(x, y)), A_1(\overline{R_1}(y, z), \overline{R_2}(y, z))) \leqslant$$

$$A_1(B(\overline{R_1}(x, y), \overline{R_1}(y, z)), B(\overline{R_2}(x, y), \overline{R_2}(y, z))) \leqslant A_1(\underline{R_1}(x, z), \underline{R_2}(x, z)) \leqslant$$

$$\min(A_1(\underline{R_1}(x, z), \overline{R_2}(x, z)), A_1(\overline{R_1}(x, z), \underline{R_2}(x, z))) = \underline{R_{\mathcal{A}}}(x, z).$$

Which means that $\mathcal{A}$ preserves nec-B-transitivity. □

Theorem 9. *Let $\mathcal{A}$ be pseudomax A_1A_2-representable aggregation function and $A_1 = A_2$ (by Proposition 6 it means that $\mathcal{A} \in \mathcal{A}_\nu$). If $A_1 \gg B$, B is increasing, then $\mathcal{A}$ preserves nec-B-transitivity.*

Proof. Let $x, y, z \in X$. According to assumptions for $k = 1, 2$ we have $B(\overline{R_k}(x, y), \overline{R_k}(y, z)) \leqslant \underline{R_k}(x, z)$. We will show that $R_{\mathcal{A}}$ is nec-B-transitive, i.e. $B(\overline{R_{\mathcal{A}}}(x, y), \overline{R_{\mathcal{A}}}(y, z)) \leqslant \underline{R_{\mathcal{A}}}(x, z)$. By monotonicity of operation B, and moreover by monotonicity of functions max and A_1, and by the idempotency of max we obtain

$$B(\overline{R_{\mathcal{A}}}(x, y), \overline{R_{\mathcal{A}}}(y, z)) =$$

$$B(\max(A_1(\underline{R_1}(x,y),\overline{R_2}(x,y)),A_1(\overline{R_1}(x,y),\underline{R_2}(x,y)),\max(A_1(\underline{R_1}(y,z),\overline{R_2}(y,z))),A_1(\overline{R_1}(y,z),\underline{R_2}(y,z)))) \leqslant$$

$$B(\max(A_1(\overline{R_1}(x,y),\overline{R_2}(x,y)),A_1(\overline{R_1}(x,y),\overline{R_2}(x,y)),\max(A_1(\overline{R_1}(y,z),\overline{R_2}(y,z))),A_1(\overline{R_1}(y,z),\overline{R_2}(y,z)))) =$$

$$B(A_1(\overline{R_1}(x,y),\overline{R_2}(x,y)),A_1(\overline{R_1}(y,z),\overline{R_2}(y,z))) \leqslant A_1(B(\overline{R_1}(x,y),\overline{R_1}(y,z)),B(\overline{R_2}(x,y),\overline{R_2}(y,z))) \leqslant$$

$$A_1(\underline{R_1}(x,z),\underline{R_2}(x,z)) = \underline{R_{\mathcal{A}}}(x,z).$$

It means that $\mathcal{A}$ preserves nec-B-transitivity. □

Now we will present preservation of pos-B-transitivity by pseudomin and pseudomax aggregation functions. Sufficient conditions for preservation of pos-B-transitivity in these cases are rather strong (but possible to be fulfilled), since they are connected with dominance of an aggregation function over minimum and maximum over an aggregation function (cf. Theorems 4 and 5).

Theorem 10. *Let $\mathcal{A}$ be pseudomax A_1A_2-representable aggregation function. If $A_1 \gg B$, $\max \gg A_2$, then $\mathcal{A}$ preserves pos-B-transitivity.*

Proof. Let $x, y, z \in X$. According to assumptions for $k = 1, 2$ we have $B(\underline{R_k}(x,y),\underline{R_k}(y,z)) \leqslant \overline{R_k}(x,z)$. We will show that $R_{\mathcal{A}}$ is pos-B-transitive, i.e. $B(\underline{R_{\mathcal{A}}}(x,y),\underline{R_{\mathcal{A}}}(y,z)) \leqslant \overline{R_{\mathcal{A}}}(x,z)$. By assumptions we obtain

$$B(\underline{R_{\mathcal{A}}}(x,y),\underline{R_{\mathcal{A}}}(y,z)) = B(A_1(\underline{R_1}(x,y),\underline{R_2}(x,y)),A_1(\underline{R_1}(y,z),\underline{R_2}(y,z))) \leqslant$$

$$A_1(B(\underline{R_1}(x,y),\underline{R_1}(y,z)),B(\underline{R_2}(x,y),\underline{R_2}(y,z))) \leqslant A_1(\overline{R_1}(x,z),\overline{R_2}(x,z)) \leqslant A_2(\overline{R_1}(x,z),\overline{R_2}(x,z)) =$$

$$A_2(\max(\underline{R_1}(x,z),\overline{R_1}(x,z)),\max(\overline{R_2}(x,z),\underline{R_2}(x,z))) \leqslant$$

$$\max(A_2(\underline{R_1}(x,z),\overline{R_2}(x,z)),A_2(\overline{R_1}(x,z),\underline{R_2}(x,z))) = \overline{R_{\mathcal{A}}}(x,z).$$

It means that $\mathcal{A}$ preserves pos-B-transitivity. □

Theorem 11. *Let $\mathcal{A}$ be pseudomin A_1A_2-representable aggregation function. If $A_1 \gg B$, $A_1 \gg \min$, B is increasing, then $\mathcal{A}$ preserves pos-B-transitivity.*

Proof. Let $x, y, z \in X$. According to assumptions for $k = 1, 2$ we have $B(\underline{R_k}(x,y),\underline{R_k}(y,z)) \leqslant \overline{R_k}(x,z)$. We will show that $R_{\mathcal{A}}$ is pos-B-transitive, i.e. $B(\underline{R_{\mathcal{A}}}(x,y),\underline{R_{\mathcal{A}}}(y,z)) \leqslant \overline{R_{\mathcal{A}}}(x,z)$. By assumptions we get

$$B(\underline{R_{\mathcal{A}}}(x,y),\underline{R_{\mathcal{A}}}(y,z)) =$$

$$B(\min(A_1(\underline{R_1}(x,y),\overline{R_2}(x,y)),A_1(\overline{R_1}(x,y),\underline{R_2}(x,y)),\min(A_1(\underline{R_1}(y,z),\overline{R_2}(y,z))),A_1(\overline{R_1}(y,z),\underline{R_2}(y,z)))) \leqslant$$

$$B(A_1(\min(\underline{R_1}(x,y),\overline{R_1}(x,y)),\min(\overline{R_2}(x,y),\underline{R_2}(x,y)),A_1(\min(\underline{R_1}(y,z),\overline{R_1}(y,z))),\min(\overline{R_2}(y,z),\underline{R_2}(y,z)))) =$$

$$B(A_1(\underline{R_1}(x,y),\underline{R_2}(x,y),A_1(\underline{R_1}(y,z),\underline{R_2}(y,z))) \leqslant A_1(B(\underline{R_1}(x,y),\underline{R_1}(y,z)),B(\underline{R_2}(x,y),\underline{R_2}(y,z))) \leqslant$$

$$A_1(\overline{R_1}(x,z),\overline{R_2}(x,z)) \leqslant A_2(\overline{R_1}(x,z),\overline{R_2}(x,z)) = \overline{R_{\mathcal{A}}}(x,z).$$

It means that $\mathcal{A}$ preserves pos-B-transitivity. □

There can be given many examples of aggregation functions (diverse types presented in this paper) preserving pos-B-transitivity and nec-B-transitivity. Some of them follow from Example 23 and Corollary 3. Below we give only one of the simplest example which covers all presented above theorems for the considered here transitivity properties.

Example 26. Since minimum dominates any fuzzy conjunction C (cf. Corollary 3), decomposable aggregation function $\mathcal{A}(\mathbf{x},\mathbf{y}) = [\min(x_1,y_1),\min(x_2,y_2)]$, where also $\mathcal{A} \in \mathcal{A}_\nu$ and $\mathcal{A} \in \mathcal{A}_\pi$, preserves pos-$C$-transitivity and nec-$C$-transitivity. Similarly, $\mathcal{A}(\mathbf{x},\mathbf{y}) = [\min(x_1,y_1),\max(\min(x_1,y_2),\min(x_2,y_1))]$ is pseudomax A_1A_2-representable aggregation function preserving nec-C-transitivity.

$\mathcal{A}(\mathbf{x},\mathbf{y}) = [\min(A(x_1,y_2),A(x_2,y_1)),A(x_2,y_2)]$ is a pseudomin A_1A_2-representable aggregation function, where $A_1 = A_2 = A$ is a weighted minimum, and it preserves nec-C-transitivity and pos-C-transitivity where C is an arbitrary t-seminrom (cf. Corollary 3, Theorem 4).

5. Application

The presented in this paper considerations, connected with preservation of transitivity properties by aggregation functions, have possible applications in multicriteria (or similarly multiagent) decision making problems with intervals. Interval-valued fuzzy relations in such setting represent the preferences. Let $card\ X = m,\ m \in \mathbb{N},\ X = \{x_1, \dots, x_m\}$ be a set of alternatives. In multicriteria decision making a decision maker has to choose among the alternatives with respect to a set of criteria. Let $K = \{k_1, \dots, k_n\}$ be the set of criteria on the base of which the alternatives are evaluated. $R_1, \dots, R_n$ be interval-valued fuzzy relations on a set X corresponding to each criterion represented by matrices, i.e. $R_k \in \mathcal{IVFR}(X),\ k = 1, \dots, n$, $n \in \mathbb{N}$. Similarly, if we consider multiagent decision making problems, relations $R_1, \dots, R_n$ represent the preferences of each agent (certainly, we can combine these two situations, i.e. many criteria and many agents). Relation $R_{\mathcal{F}} = \mathcal{F}(R_1, \dots, R_n)$ is supposed to help the decision maker to make up his/her mind. Some functions $\mathcal{F}$ may be more adequate for aggregation than the others since they may (or not) preserve the required properties of relations $R_1, \dots, R_n \in \mathcal{IVFR}(X)$. This is why preservation of these properties may be interested and required in aggregation process for multicriteria or multiagent decision making problems, like for example transitivity properties that may ensure consistency of choices. The comparability relations, and as a consequence adequate type of aggregation functions and transitivity properties considered in this paper, enable to use them if in decision making problem is required one of two interpretations: “possible” or “necessary”. In the presented multicriteria or multiagent decision making problems it is sometimes required that the given fuzzy relations representing the preferences are reciprocal. However, if R is not reciprocal, there are methods to transform it to the reciprocal one (cf. [1]) but we do not focus on this problem here. Similarly, we also do not consider ways of obtaining the best alternative in detail (cf. [1,18,19,23]).

We present here an algorithm to obtain the final solution from a given set of alternatives. We have the following inputs:

$X = \{x_1, \dots, x_m\}$, $\mathcal{F}$—an interval-valued aggregation function preserving pos-B-transitivity (or nec-B-transitivity), where $\mathcal{F} \in \mathcal{A}_\pi$ (or $\mathcal{F} \in \mathcal{A}_\nu$), $\mathcal{B} = \{B | B : [0, 1]^2 \to [0, 1]\}$—finite set of given comparable operations including constant operations $B \equiv 0$, $B \equiv 1$, $R_1, \dots, R_n \in \mathcal{IVFR}(X)$.

Algorithm—the steps:

1. Check the type of pos-B-transitivity (or nec-B-transitivity) of each R_k for $k = 1, \dots, n$
2. Fix the common type of pos-B_0-transitivity (or nec-B_0-transitivity) of each R_k for $k = 1, \dots, n$
3. Determine the relation $R_{\mathcal{F}}$ with the use of $\mathcal{F}$
4. Find the best alternative from the set X

Output: the aggregated interval-valued fuzzy relation $R_{\mathcal{F}}$ with pos-B_0-transitivity (or nec-B_0-transitivity).

Note that in point 2 of the algorithm we use the dependence presented in Proposition 4. Since we assume that $B \equiv 0$ belongs to the class $\mathcal{B}$, we always may determine the type of pos-B_0-transitivity (or nec-B_0-transitivity). For $B \equiv 0$ these transitivity conditions are fulfilled trivially, so the algorithm may be run to the end (however not giving an interesting result).

6. Conclusions

We introduced here new types of aggregation functions for interval-valued fuzzy setting based on comparability relations $\preceq_\pi$ and $\preceq_\nu$. We described dependencies between these types of aggregation functions and known aggregation functions on L^I. We also discussed preservation of pos-B and nec-B-transitivity properties (which derive from relations $\preceq_\pi$ and $\preceq_\nu$) by these new types of aggregation functions. For decision making problems we proposed to examine practically new approaches - depending on the interpretation “possible” or “necessary”. Namely, if we consider the “possible” approach, then relation $\preceq_\pi$ is involved and we proposed to apply in such case pos-B-properties of interval-valued fuzzy relations and a pos-aggregation function to aggregate the input relations. Analogously, for the “necessary” approach, with the relation $\preceq_\nu$ involved, we proposed to consider nec-B-properties of interval-valued fuzzy relations and a nec-aggregation function to aggregate the input relations.

From the theoretical point of view, in future work it will be interesting to find characterizations of the new families of aggregation functions (Theorem 3 gives characterization of decomposable aggregation functions from the family $\mathcal{A}_\nu$). Presented here results may be useful for it. New types of aggregation functions have possible applications in practical models like [24] (cf. [13,34]), where the process of aggregation of interval data is involved. It will be interesting to check whether applying pos- and nec-aggregation functions improves the obtained results.

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