Information Sciences 179 (2009) 2102–2122

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Interval type-2 fuzzy membership function generation methods for pattern recognition

Byung-In Choi, Frank Chung-Hoon Rhee *

Computational Vision and Fuzzy Systems Laboratory, Department of Electronic Engineering, Hanyang University, 1271 Sa 1-Dong Sangnok-Gu, Ansan-Si, Gyeonggi-Do, Republic of Korea

ARTICLE INFO

Article history:
Received 18 January 2008
Accepted 29 April 2008

Keywords:
Fuzzy membership function generation
Interval type-2 fuzzy sets
Fuzzy *C*-means
Footprint of uncertainty

ABSTRACT

Type-2 fuzzy sets (T2 FSs) have been shown to manage uncertainty more effectively than T1 fuzzy sets (T1 FSs) in several areas of engineering [4,6–12,15–18,21–27,30]. However, computing with T2 FSs can require undesirably large amount of computations since it involves numerous embedded T2 FSs. To reduce the complexity, interval type-2 fuzzy sets (IT2 FSs) can be used, since the secondary memberships are all equal to one [21]. In this paper, three novel interval type-2 fuzzy membership function (IT2 FMF) generation methods are proposed. The methods are based on heuristics, histograms, and interval type-2 fuzzy *C*-means. The performance of the methods is evaluated by applying them to back-propagation neural networks (BPNNs). Experimental results for several data sets are given to show the effectiveness of the proposed membership assignments.

1. Introduction

Type-1 fuzzy sets (T1 FSs) have been successfully used to represent the uncertainty of pattern data in the area of pattern recognition [1–3,13,19,20,28,29,31]. The issue of automatic generation of T1 FMFs from pattern data is an important step in developing algorithms that can handle uncertainties. Several T1 FMF generation methods have been proposed [19,20,28,29,31]. Typical T1 FMFs were generated based on heuristics (e.g., concepts based on human perception and experts), histograms, probability, and entropy, to name a few. Additionally, algorithms based on the fuzzy nearest neighbor, back-propagation neural network, fuzzy *C*-means (FCM), robust agglomerative Gaussian mixture decomposition (RAGMD), and self-organizing feature map (SOFM) were used to generate T1 FMFs as well. However, if we cannot obtain satisfactory results using T1 FSs, employment of type-2 fuzzy sets (T2 FSs) for managing uncertainty may allow us to obtain more desirable results. It has been shown that T2 FSs can control the uncertainty of patterns more effectively than conventional T1 FSs due to the extra degree of freedom. However, the operations of T2 FSs such as type-reduction involve numerous embedded T2 FSs. Thus, an undesirable amoun of computations may be required to perform the operations of T2 FSs. To reduce the complexity, interval type-2 fuzzy sets (IT2 FSs) was proposed [21]. The reduction on the computational complexity is due to the property that all the secondary grades for IT2 FSs are uniformly weighted (i.e., all equal to one).

Generation methods for T2 FMF have been limitedly discussed although various algorithms based on the T2 FMF have been proposed [4,6,7,9,10,15,17,18,23–27]. In this paper, we focus on the generation of T2 FMFs for the use in classification algorithms. In particular, we consider interval type-2 fuzzy membership functions (IT2 FMFs) that are generated from sample data for the reasons mentioned above. We propose three methods based on heuristics, histograms, and interval type-2

* Corresponding author. Tel.: +82 11 9714 5296; fax: +82 31 501 7502.
E-mail address: frhee@fuzzy.hanyang.ac.kr (F. Chung-Hoon Rhee).

0020-0255/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.ins.2008.04.009

fuzzy *C*-means (IT2 FCM). For each method, the footprint of uncertainty (FOU) is only required to be obtained, since the FOU can completely describe an IT2 FMF (i.e., secondary grades are all equal to one).

To validate our proposed design methods, we apply our proposed IT2 FMFs generation methods to back-propagation neural networks (BPNNs). T1 FMF values for each pattern are computed from the interval centroids of the generated IT2 FMFs and are used as inputs to train the BPNN. The remainder of this paper is organized as follows. In Section 2, we briefly explain IT2 FSs. In Section 3, we present our three IT2 FMF generation methods. In Section 4, we explain how our proposed IT2 FMF generation methods can be applied in a back-propagation neural network (BPNN). Section 5 gives several examples showing the validity of our proposed methods. Finally, Section 6 gives the summary and conclusions.

2. Overview of interval type-2 fuzzy sets

The extension of T1 FSs to T2 FSs can be used to effectively describe uncertainties in situations where the available information is uncertain. T2 FSs include a secondary membership function to model the uncertainty of exact (crisp) T1 FSs. A T2 FS in the universal set $\mathbf{X}$, denoted as $\tilde{\mathbf{A}}$, can be characterized by a T2 FMF $\mu_{\tilde{\mathbf{A}}}(x,u)$ as

$$\tilde{\mathbf{A}} = \int_{x\in\mathbf{X}} \mu_{\tilde{\mathbf{A}}}(x)/x = \int_{x\in\mathbf{X}} \left[\int_{u\in J_x} f_x(u)/u\right] \Big/ x J_x \subseteq [0,1], \tag{1}$$

where $f_x(u)$ is the secondary membership function and J_x is the primary membership of x which is the domain of the secondary membership function [21]. Fig. 1 shows an example of a T2 FMF. The shaded region bounded by an upper and lower membership function is called the footprint of uncertainty (FOU) as shown in Fig. 1a. The FOU of $\tilde{\mathbf{A}}$ can be expressed by the union of all the primary memberships as

$$\mathrm{FOU}(\tilde{\mathbf{A}}) = \cup_{\forall x\in\mathbf{X}} J_x = \{(x,u) : u \in J_x \subseteq [0,1]\}. \tag{2}$$

The upper membership function (UMF) and lower membership function (LMF) of $\tilde{\mathbf{A}}$ are two T1 FMFs that bound the FOU. The UMF denoted by $\bar{\mu}_{\tilde{\mathbf{A}}}(x)$ is associated with the upper bound of FOU ($\tilde{\mathbf{A}}$) and the LMF denoted by $\underline{\mu}_{\tilde{\mathbf{A}}}(x)$ is associated with the lower bound of FOU ($\tilde{\mathbf{A}}$). By definition they can be represented as

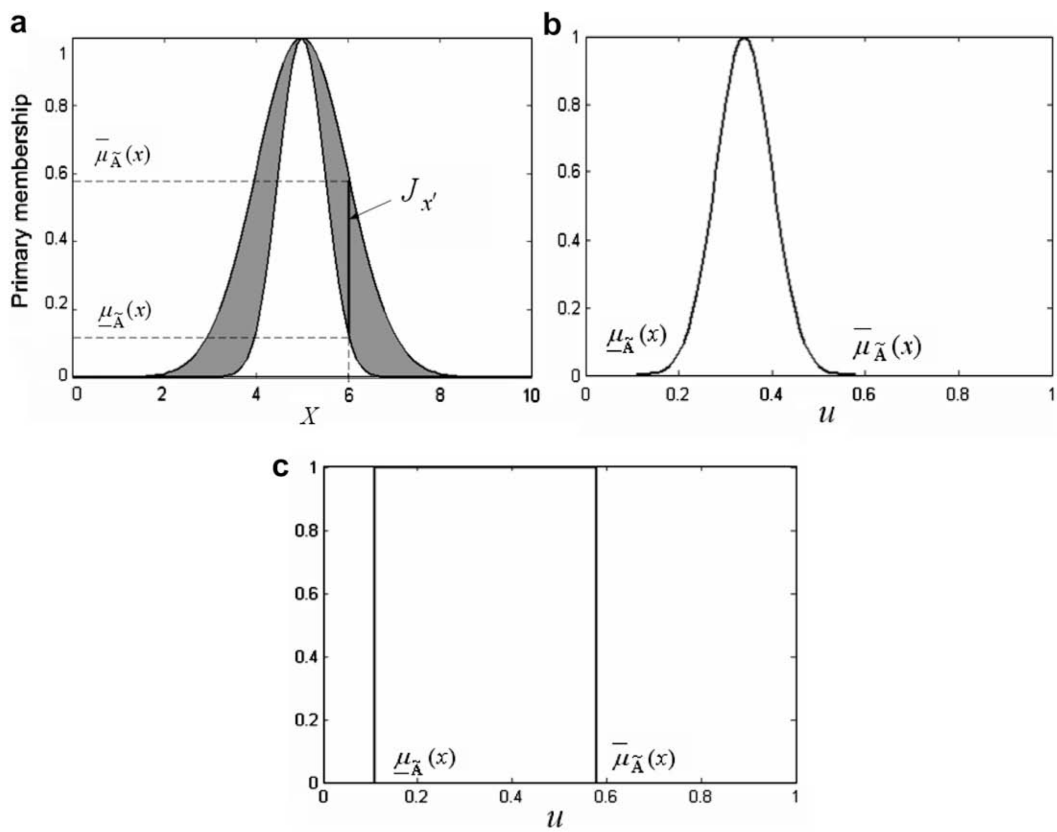

Fig. 1. Illustration of a T2 FMF: (a) T2 FMF, (b) Gaussian secondary membership function, and (c) interval secondary membership function.

$$\bar{\mu}_{\tilde{\mathbf{A}}}(x) \equiv \overline{\mathrm{FOU}(\tilde{\mathbf{A}})}\ \forall x \in \mathbf{X} \tag{3}$$

and

$$\underline{\mu}_{\tilde{\mathbf{A}}}(x) \equiv \underline{\mathrm{FOU}(\tilde{\mathbf{A}})}\ \forall x \in \mathbf{X}. \tag{4}$$

The secondary membership function is a vertical slice of $\mu_{\tilde{\mathbf{A}}}(x,u)$ as shown in Fig. 1b and c. Although T2 FSs may be useful in modeling uncertainty, where T1 FSs cannot, the operations of T2 FSs involve numerous embedded T2 FSs which consider all possible combinations of secondary membership values. Therefore, undesirably large amount of computations may be required. However, interval type-2 fuzzy sets (IT2 FSs) can be used to reduce the computational complexity. IT2 FSs are specific T2 FS whose secondary membership functions are interval sets expressed as

$$\tilde{\mathbf{A}} = \int_{x\in\mathbf{X}} \left[\int_{u\in J_x} 1/u\right] \Big/ x. \tag{5}$$

Fig. 1c shows the secondary membership function of an IT2 FSs. As shown in the figure, the secondary memberships are all uniformly weighted for each primary membership of x. Therefore, J_x can be expressed as

$$J_x = \left\{(x,u) : u \in \left[\underline{\mu}_{\tilde{\mathbf{A}}}(x), \bar{\mu}_{\tilde{\mathbf{A}}}(x)\right]\right\}. \tag{6}$$

Moreover, FOU($\tilde{\mathbf{A}}$) in (2) can also be expressed as

$$\mathrm{FOU}(\tilde{\mathbf{A}}) = \bigcup_{\forall x\in\mathbf{X}} \left[\underline{\mu}_{\tilde{\mathbf{A}}}(x), \bar{\mu}_{\tilde{\mathbf{A}}}(x)\right]. \tag{7}$$

As a result, IT2 FSs requires only simple interval arithmetic for computing.

3. Interval type-2 fuzzy membership function generation methods

In this section, we propose three methods for generating IT2 FMF automatically from pattern data. Our proposed methods are based on heuristics, histograms, and interval type-2 fuzzy *C*-means (IT2 FCM). The heuristic based design method simply generates the IT2 FMF using heuristic T1 FMFs and a scaling factor. The histogram based method uses suitable parameterized functions chosen to model the smoothed histogram for each class and feature extracted from sample data. The IT2 FCM based method uses the derived formulas of the IT2 FMFs in the IT2 FCM algorithm [7]. A detailed description of each method is discussed in the following sections.

3.1. Heuristic method

The heuristic method simply uses an appropriate predefined T1 FMF function, such as triangular, trapezoidal, Gaussian, *S*, or π function, to name a few, to initially represent the distribution of the pattern data. The following are some frequently used heuristic membership functions.

(1) Triangular function

$$\mu(x;a,b,c) = \begin{cases} 0 & \text{if} \quad x \leqslant a \\ \frac{x-a}{b-a} & \text{if} \quad a \leqslant x \leqslant b \\ \frac{c-x}{c-b} & \text{if} \quad b \leqslant x \leqslant c \\ 0 & \text{if} \quad x \leqslant c \end{cases}. \tag{8}$$

(2) Trapezoidal function

$$\mu(x;a,b,c,d) = \begin{cases} 0 & \text{if} \quad x \leqslant a \\ \frac{x-a}{b-a} & \text{if} \quad a \leqslant x \leqslant b \\ 1 & \text{if} \quad b \leqslant x \leqslant c \\ \frac{c-x}{c-b} & \text{if} \quad c \leqslant x \leqslant d \\ 0 & \text{if} \quad x \geqslant d \end{cases}. \tag{9}$$

(3) Gaussian function

$$\mu(x) = \mathrm{e}^{\frac{-(x-c)^2}{2\sigma^2}}. \tag{10}$$

(4) *S*-function

$$S(x;a,b,c)=\begin{cases}0 & \text{if} \quad x\leqslant a\\ 2\cdot\left(\frac{x-a}{c-a}\right)^2 & \text{if} \quad a<x\leqslant b\\ 1-2\cdot\left(\frac{x-a}{c-a}\right)^2 & \text{if} \quad b<x\leqslant c\\ 1 & \text{if} \quad x>c\end{cases}, \tag{11}$$

where $b=\dfrac{a+c}{2}$.

(5) π-function

$$\Pi(x;a,b,c)=\begin{cases}S(x;c-b,c-\frac{b}{2},c) & x\leqslant c\\ 1-S(x;c,c+\frac{b}{2},c+b) & x>c\end{cases}. \tag{12}$$

Once a T1 FMF that is suitable for a given pattern set is selected such as the ones in (8)–(12), the heuristic T1 FMF $\mu_i(x)$ for feature i is designed by determining the parameters of the function which are usually provided by experts. This becomes the UMF $\bar{u}_i(x)$ of the IT2 FMF. The LMF $\underline{u}_i(x)$ is obtained by scaling the UMF by a factor α between 0 and 1, which can be also be provided by an expert. The FOU of our heuristic method for feature i can be expressed as

$$\cup_{\forall x\in \mathbf{X}}[\underline{u}_i(x),\bar{u}_i(x)]=\cup_{\forall x\in \mathbf{X}}[\mu_i(x),\alpha\cdot\mu_i(x)],\quad 0<\alpha<1. \tag{13}$$

For input data of dimensions two or higher, we can obtain the overall FOU by taking intersections of all upper and lower memberships for all features. If we choose the min operation as intersection, the FOU can be expressed as

$$\cup_{\forall x\in \mathbf{X}}[\underline{u}(x),\bar{u}(x)]=\cup_{\forall x\in \mathbf{X}}[\min_i\{\underline{u}_i(x_i)\},\min_i\{\bar{u}_i(x_i)\}], \tag{14}$$

where $\underline{u}(x)$ and $\bar{u}(x)$ are the minimum UMF and LMF among all UMFs and LMFs for all features, respectively. Fig. 2 shows a simple example of triangular IT2 FMFs that are generated by our heuristic method for the sample data set in Fig. 2a. For the UMF, we select the height in (9) (parameter b) as the center location of the patterns of each class. Next, we compute the distance between the center and most distant pattern. The spread (parameter a and c) is selected by subtracting and adding this distance from the center. The LMF is obtained by adequately scaling the height of the UMF. Fig. 2b and c show the resulting UMF and LMF obtained by our proposed method for class "×" and class "O", respectively. The region between the UMF and corresponding LMF indicates the FOU for each class. Unfortunately, the shapes of heuristic FMFs are not flexible enough to model all types of data. The proposed heuristic method is summarized as follows.

Heuristic based IT2 FMF generation method

(1) Select a heuristic T1 FMF that is suitable for a given data set.
(2) Set the parameters for the membership function that is provided by an expert.
(3) Design the upper and lower membership functions using (13) and (14).

3.2. Histogram based method

Histograms of features can used to effectively describe the distribution of the feature values of the input data. Therefore, membership functions generated from histograms may be considered more suitable for arbitrary distributed data than from heuristics. We now propose a method, which designs IT2 FMFs by fitting parameterized functions to smoothed histograms of sample data. The proposed method is described as follows.

3.2.1. Histogram generation and smoothing

First, the histogram of a given sample data for each labeled class is obtained. Next, the histograms are smoothed using a symmetric window such as a hyper-cube or triangular window and then normalized. As an example, Fig. 3a shows the histogram of the data for Feature 2 of class "×" in Fig. 2a. Fig. 3b shows the smoothed and normalized histogram obtained by using a 3 point triangular window.

3.2.2. Polynomial function fitting

To extract the membership function, a suitable parameterized function is chosen to model the smoothed histograms. Approximate parameter values (such as the number, height, and location of peaks) used to determine the optimal parameter values of the function can be obtained by polynomial function (PF) fitting. Using a least squares approximation, the PF of the lowest possible degree (i.e., to avoid over fitting) is chosen such that the fit to each smoothed histogram has a reasonably small error. Fig. 4a shows the result of PF fitting for the smoothed histogram in Fig. 3b. The suitable degree of the PF is selected as the knee point of error. For this example, a 3rd degree polynomial function can be considered suitable as indicated in Fig. 4b.

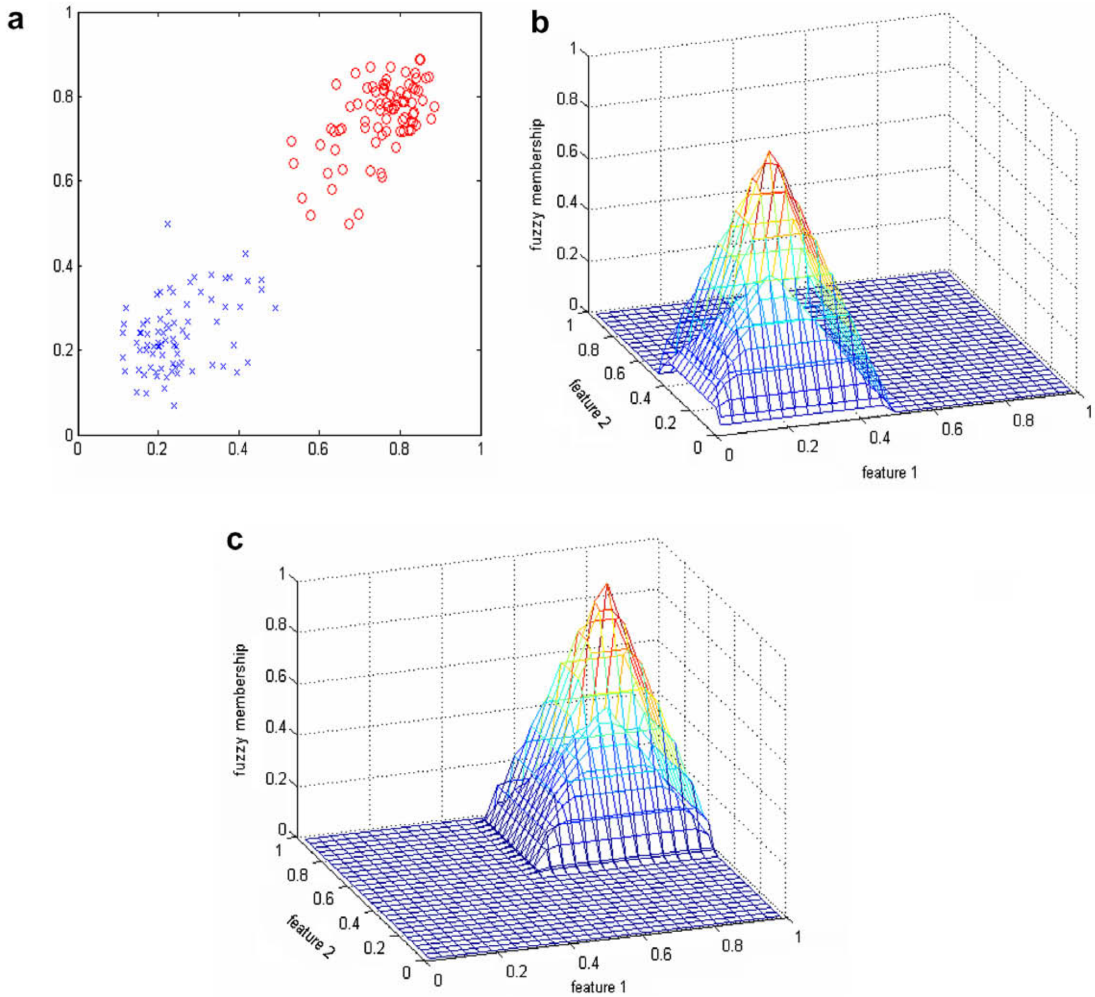

Fig. 2. Illustration of the heuristic based IT2 FMF generation method using a triangular IT2 FMF for a given sample data set: (a) scatter plot, (b) UMF and LMF representing the FOU of class "×", and (c) UMF and LMF representing the FOU of class "O".

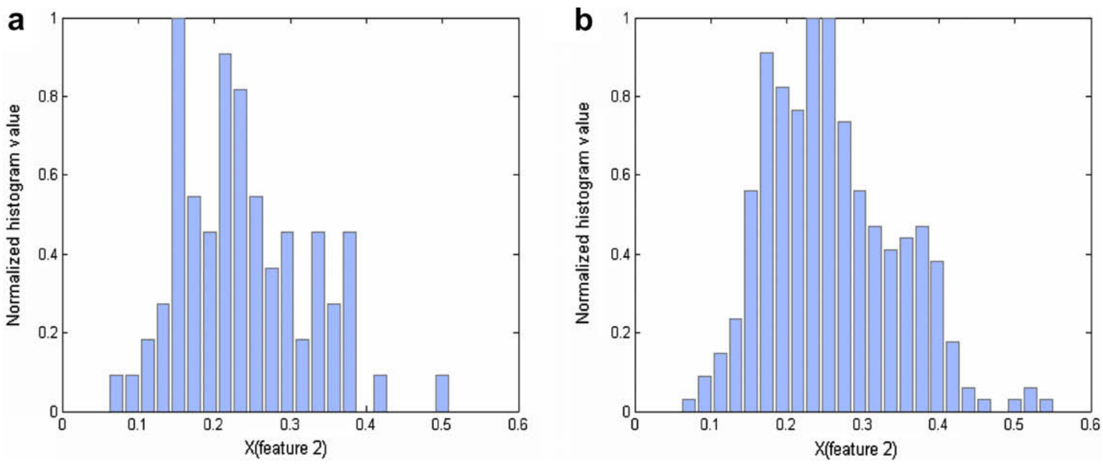

Fig. 3. Histograms for Feature 2 for patterns labeled class "×" in Fig. 2a: (a) histogram, and (b) smoothed histogram.

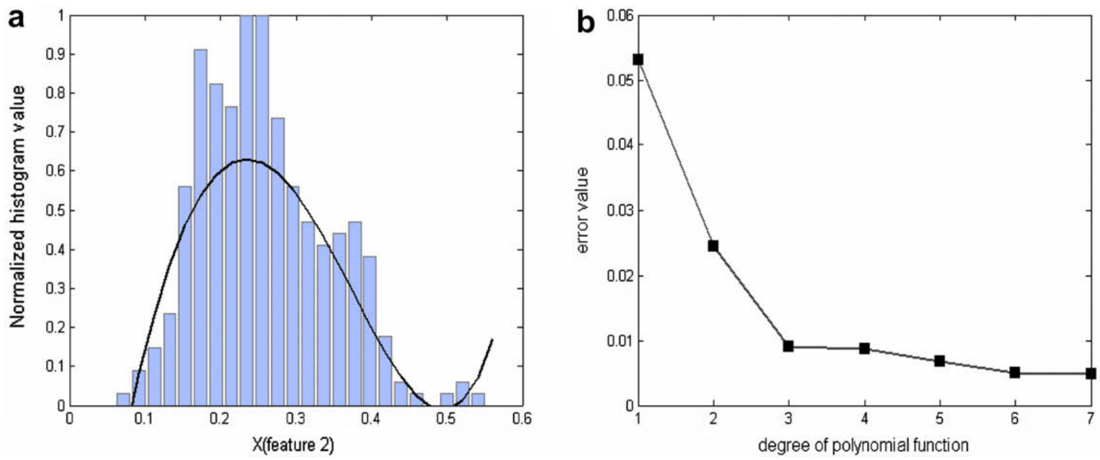

Fig. 4. Example of PF fitting of a function of: (a) 3rd degree, and (b) plot of the variation of average sum of squared errors.

To illustrate the above procedure, we shall consider Gaussian functions as suitable parameterized functions to model the IT2 FMFs, and describe a method for estimating the parameters for a reasonable approximation [28].

3.2.3. Gaussian function modeling

Consider a Gaussian function given by

$$G(\mathbf{x}) = a \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right), \tag{15}$$

where a is the height, $\boldsymbol{\mu}$ is mean vector, and $\boldsymbol{\Sigma}$ is the covariance matrix. If the histogram has N significant peaks, we can model it as the sum of Gaussian functions as

$$\tilde{G}(\mathbf{x}) = \sum_{i=1}^{N} G_i(\mathbf{x}). \tag{16}$$

To approximate the smoothed histogram as Gaussian functions, the objective function to be minimized can be considered as

$$J(\mathbf{p}) = \frac{1}{2}\left(\sum_{i=1}^{N} G_i(\mathbf{x}) - H(\mathbf{x})\right)^2, \tag{17}$$

where parameter vector $\mathbf{p}_i = (a_i, \boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ is for the ith Gaussian function $G_i(\mathbf{x})$ and $H(\mathbf{x})$ is the smoothed histogram of the input data. The gradient descent method can be used to estimate the parameter vector $\mathbf{p}_i$ which minimizes (17) by the following update rule:

$$\mathbf{p}_i^{\text{new}} = \mathbf{p}_i^{\text{old}} - \rho\frac{\partial J}{\partial \mathbf{p}_i}, \tag{18}$$

where ρ is a positive learning constant.

For Gaussian functions, the partial derivatives of J with respect to each component of $\mathbf{p}_i$ are

$$\frac{\partial J}{\partial a_i} = \left(\sum_{j=1}^{N} G_j(\mathbf{x}) - H(\mathbf{x})\right) \cdot \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i)\right), \tag{19}$$

$$\frac{\partial J}{\partial \boldsymbol{\mu}_i} = \frac{1}{2}\left(\sum_{j=1}^{N} G_j(\mathbf{x}) - H(\mathbf{x})\right) \cdot G_i(\mathbf{x}) \cdot (\mathbf{x}-\boldsymbol{\mu}_i)^{\mathrm{T}} \cdot (\boldsymbol{\Sigma}_i^{-\mathrm{T}} + \boldsymbol{\Sigma}_i^{-1}), \tag{20}$$

and

$$\frac{\partial J}{\partial \boldsymbol{\Sigma}_i} = \frac{1}{2}\left(\sum_{j=1}^{N} G_j(\mathbf{x}) - H(\mathbf{x})\right) \cdot G_i(\mathbf{x}) \cdot \left(\boldsymbol{\Sigma}_i^{-\mathrm{T}}(\mathbf{x}-\boldsymbol{\mu}_i) \cdot (\mathbf{x}-\boldsymbol{\mu}_i)^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-\mathrm{T}}\right). \tag{21}$$

For gradient descent methods, the choice of the initial values for the parameters is critical, due to the local minima problem. Therefore, for the case of Gaussian functions, we can use the following heuristic approach [28] that consists of four steps to obtain the initial parameters.

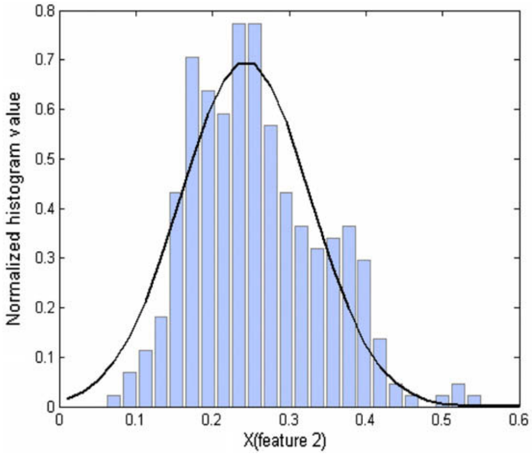

Fig. 5. Result of GF fitting.

Step 1: Generate the smoothed histograms for the given sample data set as explained in Section 3.2.1.
Step 2: Using least squares approximation, fit a polynomial function (PF) of the lowest possible degree (i.e., to avoid over fitting) such that the fit to each smoothed histogram has a reasonably small error.
Step 3: Calculate the extrema (maxima and minima) values for the PF in Step 2 and determine the number of Gaussians by the number of positive valued maxima, ignoring the ones that have small peaks.
Step 4: Initialize the heights of the Gaussians by the maxima values (peak values) and initialize the mean values of the Gaussians as the locations of these peaks. Initialize the standard deviation of each Gaussian as the shortest value among the distances between the mean of the Gaussian and the nearest minima or roots of the PF.

Fig. 5 shows the resulting membership function obtained by one-dimensional Gaussian function fitting.

3.2.4. Upper and lower membership function modeling

To obtain the generated IT2 FMF, the FOU needs to be determined. The UMF and LMF are obtained by again fitting GFs to the smoothed histograms values that are above and below the fitted GF obtained previously. Fig. 6 shows the resulting upper and lower Gaussian functions and the corresponding upper and lower histogram values [1](blue colored bars) used for fitting.

As the final step, to obtain the FOU, the UMFs and LMFs are designed by using the upper and lower GFs as discussed above. The UMF can be obtained by normalizing the height to 1. Moreover, the height of the LMF is scaled in proportion to the UMF. Fig. 7 shows the IT2 FMF obtained by our proposed method. The shaded region between the UMF and LMF indicates the FOU. As shown in the figure, our proposed method can effectively design IT2 FMFs based on the distribution of the input data. Fig. 8 shows the results obtained by the histogram based method for the sample data in Fig. 2a. Fig. 8a and b shows the UMF and LMF obtained for class "×" and class "O", respectively. Our proposed histogram based method is summarized as follows.

Histogram based IT2 FMF generation method

(1) Construct and smooth the histogram of the sample data for each labeled class.
(2) Perform PF fitting to obtain the approximate parameter values(e.g., the number, height, and location of peaks).
(3) Perform GF fitting using the values in step 2.
(4) Perform GF fitting for the upper and lower histogram values with respect to the GF in step 3.
(5) Determine UMF by normalizing the height of the upper GF and LMF by proportionally scaling the lower GF obtained in step 4.

For high-dimensional input data, the computational load of our proposed method can become undesirably high, due to the high-dimensional histogram smoothing process and fitting. As an alternative, first we find the one-dimensional UMF and LMF for each class label and feature by our proposed histogram based method. Next, we obtain the overall UMF and LMF by taking intersections of all UMF and LMF obtained for all features. If we use the min operation as our choice for intersection, the FOU can be expressed as

$$\left[f^L(\mathbf{x}), f^U(\mathbf{x})\right] = \left[\min_i\{f_i^L(\mathbf{x}_i)\}, \min_i\{f_i^U(\mathbf{x}_i)\}\right], \tag{22}$$

where f^U is the UMF, f^L is the LMF, and i is the feature number.

[1] For interpretation of the references to color in this figure, the reader is referred to the web version of this article.

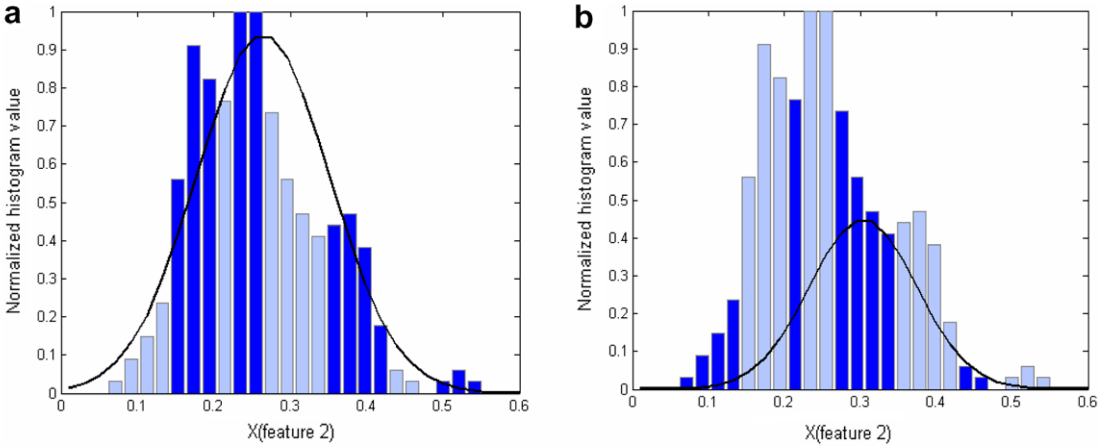

Fig. 6. Results of: (a) upper, and (b) lower GF fitting.

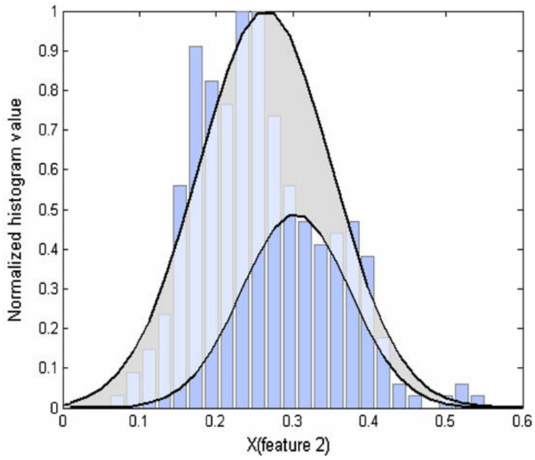

Fig. 7. Result of the generation of an IT2 FMF.

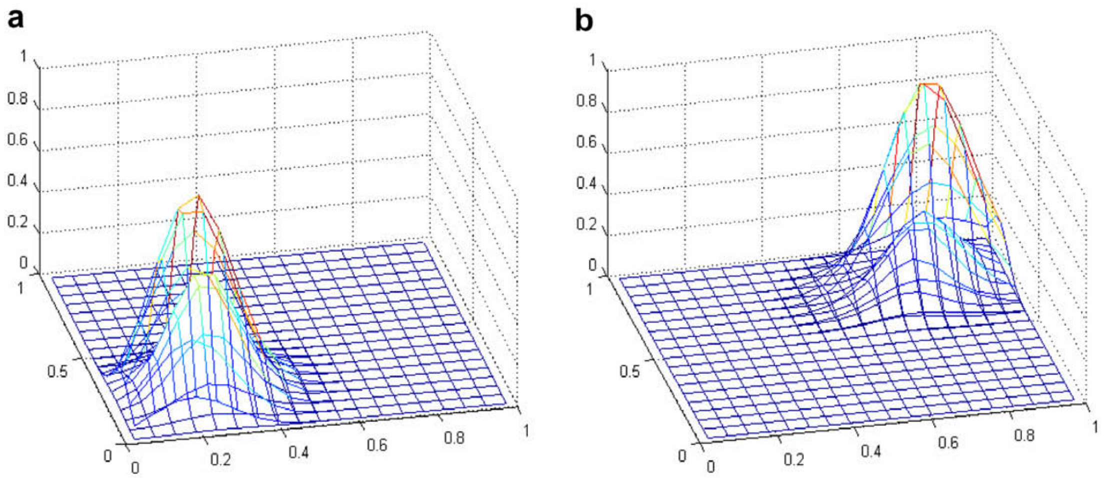

Fig. 8. Illustration of the histogram based IT2 FMF generation method for the sample data in Fig. 2a: (a) UMF and LMF representing the FOU of class "×", and (b) UMF and LMF representing the FOU of class "O".

3.3. Interval type-2 fuzzy C-means based method

The IT2 FCM algorithm was proposed to control the uncertainty of the fuzzifier m in FCM which affects the assignment of memberships for the patterns [7]. In this section, we propose an IT2 FMF generation method based on the IT2 FCM algorithm. First, the IT2 FCM algorithm is briefly explained, and then our IT2 FCM based method is described.

3.3.1. Interval type-2 fuzzy C-means algorithm

The FCM algorithm has been widely used for data partitioning [1]. FCM considers weight values (memberships) to control the contribution degree of a pattern in determining cluster prototypes. For example, a higher weighed pattern can play a more important role in determining the resulting cluster prototype. Thus, FCM can give more desirable cluster results than crisp C-means for pattern sets which contain overlapping clusters. The goal of FCM is to minimize the objective function

$$J(\mathbf{U};\mathbf{X},\mathbf{V}) = \sum_{j=1}^{C}\sum_{i=1}^{N} u_{ij}^{m} d_{ij}^{2} \quad \text{subject to} \quad \sum_{j=1}^{C} u_{ij} = 1, \tag{23}$$

where u_{ij} is the membership value of pattern $\mathbf{x}_i$ for cluster i, d_{ij}^2 is distance between $\mathbf{x}_i$ and the cluster prototype $\mathbf{v}_j$, and m represents the fuzzifier ($m > 1$). The partition matrix $\mathbf{U}$ represents the memberships for the patterns across each cluster having the elements u_{ij} and the matrix $\mathbf{V}$ is the collection of all cluster prototypes $\mathbf{v}_j$. The memberships and cluster prototypes that minimize the objective function in (23) can be obtained by

$$u_{ij} = \frac{(1/d_{ij}^2)^{1/(m-1)}}{\sum_{k=1}^{C}(1/d_{ik}^2)^{1/(m-1)}}, \tag{24}$$

and

$$v_j = \frac{\sum_{k=1}^{N}(u_{kj})^m x_k}{\sum_{k=1}^{N}(u_{kj})^m}. \tag{25}$$

However, in FCM, fuzzifier m affects the membership assignment for the patterns as illustrated in Fig. 9. As shown in the figure, suppose that there exist two cluster prototypes, namely centers $\mathbf{v}_1$ and $\mathbf{v}_2$, for a given pattern set (note that the patterns in which $\mathbf{v}_1$ and $\mathbf{v}_2$ are obtained are not shown). The membership plots corresponding to center $\mathbf{v}_1$ for patterns that lie in between the two cluster centers for various values of fuzzifier m can be shown as in Fig. 9b. When $m \to 1$, the memberships are maximally crisp (hard), that is, patterns that are located just left (right) of the maximum fuzzy boundary are assigned full (0) membership for center $\mathbf{v}_1(\mathbf{v}_2)$. Conversely, when $m \to \infty$, the memberships are maximally fuzzy, which means that patterns that are only located at the centers are assigned full (0) membership otherwise they are assigned memberships of 0.5. If the geometry of the clusters in a pattern set is of similar volume and density, change in fuzzifier m will not significantly affect the clustering result in FCM. However, if there is a significant difference in density among clusters in a pattern set, the choice of m will give inconsistent clustering results in the FCM. The reason is that an impertinent establishment of maximum fuzzy boundary in FCM can provide undesirable updating of the cluster centers. This was previously

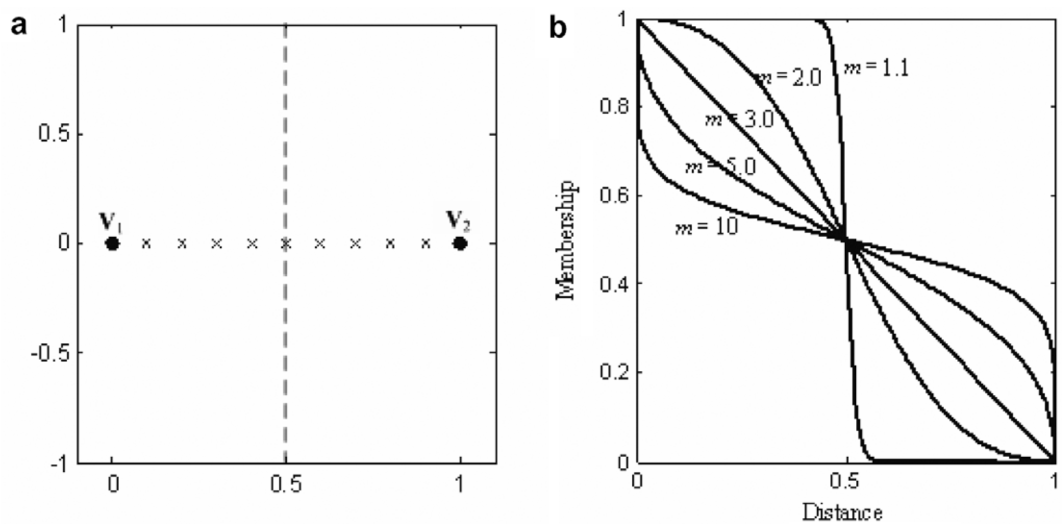

Fig. 9. Variation of fuzzy memberships for various m: (a) patterns between two centers, and (b) memberships corresponding to center $\mathbf{v}_1$ for patterns in (a) for various m.

emphasized in detail in [7]. A more desirable establishment of the maximum fuzzy region as in Fig. 10 can allow for desirable clustering results in the FCM. Again, due to the constraint on the memberships we cannot design this region with any particular single value of fuzzifier m to be used in the FCM. However, if we can somehow simultaneously incorporate various values of fuzzifier m, we may perhaps be able to design a desirable maximum fuzzy region such as the one in Fig. 10.

To overcome this problem caused by the uncertainty of fuzzifier m, the IT2 FCM algorithm was proposed [7]. In IT2 FCM, the maximum fuzzy boundary can be controlled by incorporating two values of fuzzifier m as shown in Fig. 10. By doing so, the management of uncertainty can further improve clustering results obtained by FCM algorithm. The IT2 FMF $J_{x_i} = [\underline{u}(x_i), \bar{u}(x_i)]$ in IT2 FCM can be expressed as

$$\bar{u}_j(\mathbf{x}_i) = \begin{cases} \frac{1}{\sum_{k=1}^{C}(d_{ij}/d_{ik})^{2/(m_1-1)}} & \text{if } \frac{1}{\sum_{k=1}^{C}(d_{ij}/d_{ik})^{2/(m_1-1)}} > \frac{1}{\sum_{k=1}^{C}(d_{ij}/d_{ik})^{2/(m_2-1)}} \\ \frac{1}{\sum_{k=1}^{C}\left(d_{ij}/d_{ik}\right)^{2/(m_2-1)}} & \text{otherwise} \end{cases} \tag{26}$$

and

$$\underline{u}_j(\mathbf{x}_i) = \begin{cases} \frac{1}{\sum_{k=1}^{C}(d_{ij}/d_{ik})^{2/(m_1-1)}} & \text{if } \frac{1}{\sum_{k=1}^{C}(d_{ij}/d_{ik})^{2/(m_1-1)}} \leqslant \frac{1}{\sum_{k=1}^{C}(d_{ij}/d_{ik})^{2/(m_2-1)}} \\ \frac{1}{\sum_{k=1}^{C}(d_{ij}/d_{ik})^{2/(m_2-1)}} & \text{otherwise} \end{cases}. \tag{27}$$

The procedure for updating cluster prototypes in IT2 FCM requires type-reduction and defuzzification using type-2 fuzzy operations. The generalized centroid (GC) type-reduction can be used as a type-reduction procedure. The centroid obtained by the type-reduction is shown as the following interval:

$$\mathbf{v}_{\tilde{\mathbf{X}}} = [\mathbf{v}_L, \mathbf{v}_R] = \sum_{u(x_1)\in J_{x_1}} \cdots \sum_{u(x_1)\in J_{x_1}} 1 \Big/ \frac{\sum_{i=1}^{N}\mathbf{x}_i u(\mathbf{x}_i)^m}{\sum_{i=1}^{N}u(\mathbf{x}_i)^m}. \tag{28}$$

The crisp center can be simply obtained by defuzzification as

$$\mathbf{v}_j = \frac{v_L + v_R}{2}. \tag{29}$$

3.3.2. Membership generation method using interval type-2 fuzzy C-means algorithm

For this method, we use (26) and (27) to generate IT2 FMF of patterns for each class. For pattern data that is labeled, it is not necessary to use unsupervised clustering as in FCM. Instead, we should perform "supervised" FCM clustering on each class separately. This is performed as follows.

Let the pth prototype obtained by FCM in class k be denoted by $\mathbf{v}_p^k$. Next, the minimum distance between the prototypes and input pattern $\mathbf{x}_j$ is

$$d_{kj}^* = \min_p\{d(\mathbf{x}_j, \mathbf{v}_p^k)\}, \quad p \in \{n_k\}, \tag{30}$$

where n_k is the number of prototypes used for class k. The UMF and LMF for class k and input pattern $\mathbf{x}_j$ can be expressed by modifying (26) and (27) as

$$\bar{u}_j(x_i) = \begin{cases} \frac{1}{\sum_{k=1}^{C}(d_{ij}^*/d_{ik}^*)^{2/(m_1-1)}} & \text{if } \frac{1}{\sum_{k=1}^{C}(d_{ij}^*/d_{ik}^*)^{2/(m_1-1)}} > \frac{1}{\sum_{k=1}^{C}(d_{ij}^*/d_{ik}^*)^{2/(m_2-1)}} \\ \frac{1}{\sum_{k=1}^{C}(d_{ij}^*/d_{ik}^*)^{2/(m_2-1)}} & \text{otherwise} \end{cases} \tag{31}$$

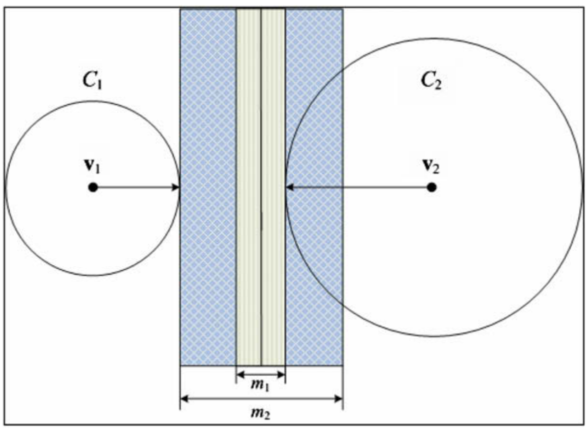

Fig. 10. Uncertain maximum fuzzy region represented by two fuzzifiers ($m_1 < m_2$).

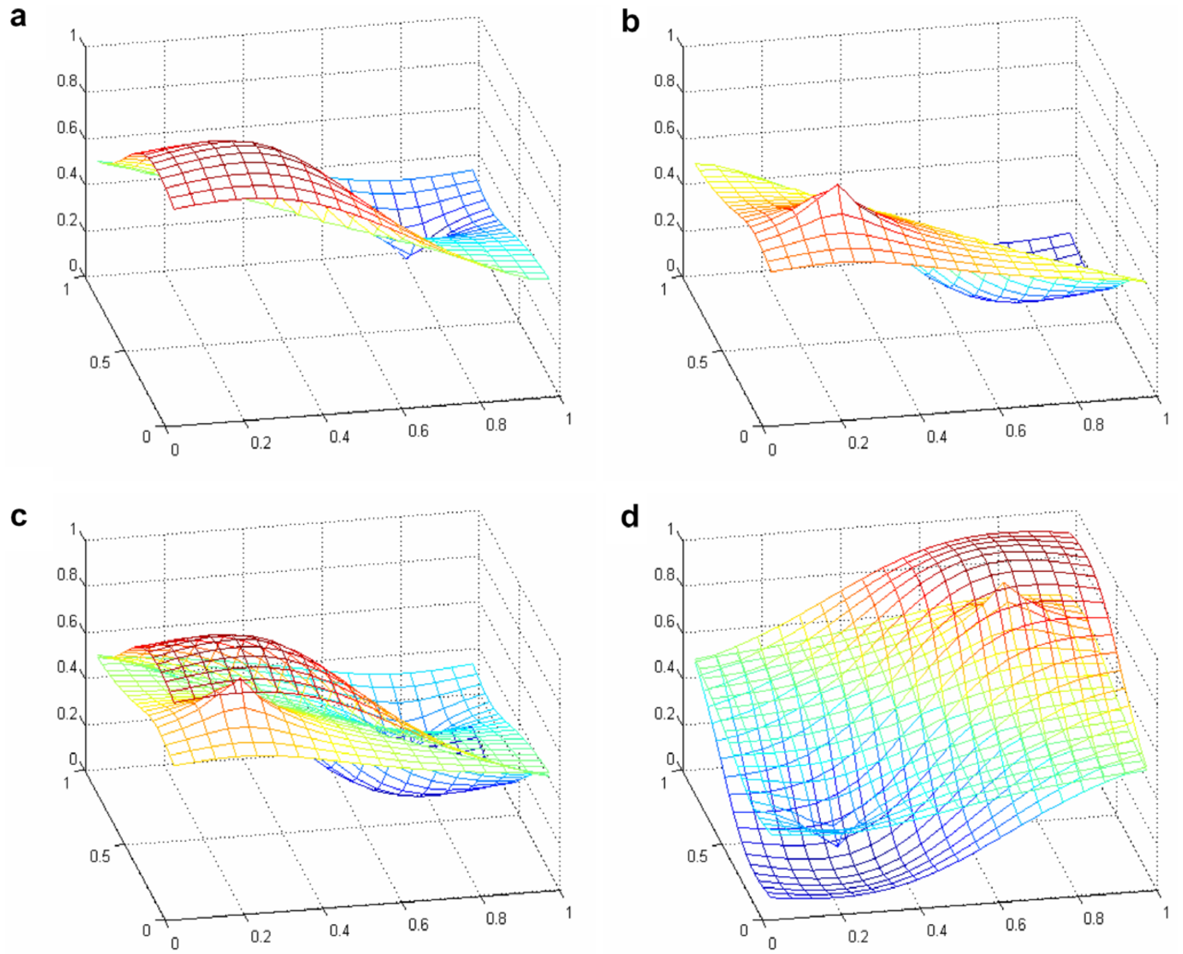

Fig. 11. Illustration of the IT2 FCM ($m_1 = 2$, $m_2 = 5$) based IT2 FMF generation method for the sample data in Fig. 2a: (a) UMF for class "×", (b) LMF for class "×", (c) FOU of class "×", and (d) FOU of class "O".

and

$$\underline{u}_j(x_i) = \begin{cases} \frac{1}{\sum_{k=1}^{C}(d_{ij}^*/d_{ik}^*)^{2/(m_1-1)}} & \text{if } \frac{1}{\sum_{k=1}^{C}(d_{ij}^*/d_{ik}^*)^{2/(m_1-1)}} \leqslant \frac{1}{\sum_{k=1}^{C}(d_{ij}^*/d_{ik}^*)^{2/(m_2-1)}} \\ \frac{1}{\sum_{k=1}^{C}(d_{ij}^*/d_{ik}^*)^{2/(m_2-1)}} & \text{otherwise} \end{cases}. \tag{32}$$

Fig. 11 shows the IT2 FMF obtained by our proposed IT2 FCM based method for the sample data in Fig. 2a. Fuzzifiers m_1 and m_2 are selected as 2 and 5 and the number of prototypes is chosen as one for each class. Fig. 11a and b show the generated UMF and LMF for class "×". As indicated, the membership values for the UMFs based on the lower fuzzifier value m_1 are selected when the memberships are above 0.5. On the contrary, the membership values based on the higher fuzzifier value m_2 are selected when the memberships are below 0.5. Fig. 11c and d show the FOU of the IT2 FMF for each labeled class. It is to be noted that the selection of fuzzifier values may affect the formation of the FOU.

Our proposed IT2 FCM based membership generation method can desirably control the uncertainty of fuzzifier m for pattern distributions of different structure and density as in IT2 FCM. Additionally, the proposed method is quite simple and can be used for all features simultaneously as in the case of high dimensional input data. Our proposed method is summarized as follows.

IT2 FCM based IT2 FMF generation method

(1) Select fuzzifiers m_1 and m_2.
(2) Perform IT2 FCM on each class separately.
(3) Find the minimum distance between prototypes of each class and input patterns.
(4) Design the UMF and LMF using (26) and (27).

4. Application to back-propagation neural networks

To evaluate the performance of the proposed IT2 FMF generation methods, we apply our proposed methods to back-propagation neural networks (BPNNs). In doing so, we extract T1 fuzzy membership values from the centroid of each IT2 FMF obtained by our proposed methods. The centroid is obtained by performing type-reduction. The type-reduction procedure gives the generalized centroid (GC) for the IT2 FMF. The generalized centroid is an IT1 FS for the centroid of embedded T2 FSs and is expressed as

$$GC = [c_l, c_r] = \int_{x_1 \in X} \cdots \int_{x_N \in X} \cdots \int_{\mu_i \in M_1} \cdots \int_{\mu_N \in M_N} 1 \Big/ \frac{\sum_{i=1}^{N} x_i \mu_i}{\sum_{i=1}^{N} \mu_i}, \tag{33}$$

where x_i is input feature pattern and μ_i is the primary membership for x_i. This GC can be obtained using the Karnik–Mendel iterative procedure [21]. Next, the fuzzy membership value of input pattern $\mathbf{x}$ for class j is obtained according to the distance between the centroid and input patterns as

$$f_j(\mathbf{x}) = \max_{\forall k}\{1 - d(\mathbf{x}, C_j^k)\}, \tag{34}$$

where C_j^k denotes the kth centroid of class j and $d(\cdot)$ is distance between the pattern and the centroid. The distance measure can be described as

$$d(\mathbf{x}, C_j^k) = \begin{cases} 0 & \text{if } c_{jl}^k \leqslant x_i \leqslant c_{jr}^k \text{ for all } i \\ \sqrt{\sum_{i=1}^{M} (\min\{|x_i - c_{jl}^k|, |x_i - c_{jr}^k|\})^2} & \text{otherwise} \end{cases}, \tag{35}$$

where M is the number of features, c_{jl}^k (c_{jr}^k) is the left (right) value of the interval centroid C_j^k. The equation above is used to compute the distance between input pattern and interval centroid using the minimum values among the distances between input pattern and left and right values of the interval centroid for all features. The fuzzy memberships extracted from the IT2 FMF can be considered suitable in describing the class memberships of patterns, since the uncertainty of pattern data are desirably controlled by the IT2 FMF. The extracted fuzzy membership values of the patterns are used as input training data to the BPNN. Fig. 12 shows the structure of our proposed IT2 FMF design method incorporated in the BPNN.

5. Experimental results

To illustrate the performance of our three proposed IT2 FMF generation methods, we apply them to back-propagation neural networks (BPNNs) and the classification results are given for several data sets. We first obtain the centroid from the generated IT2 FMF by performing type-reduction using the Karnik–Mendel iterative procedure as mentioned in the Section 4. Then, fuzzy membership values for patterns are extracted according to the distances between the centroids and patterns using (34) and (35). These fuzzy membership values are then used as inputs to the BPNN. The structure of the BPNN consists of three layers as shown in Fig. 13 [5,14]. We report the classification results by varying of the number of neurons in the hidden layer from 1 to 10. Results are given for the three proposed methods for several pattern sets. In addition, segmentation results for real images are also given. As a comparison, results are given for inputs to the BPNN that are from T1 FMF generation [20,24,28]. As in the generation of IT2 FMFs, fuzzy membership values for the patterns are extracted according to the distances between the centroids and patterns using (34) and (35), where in this case the centroid C_j^k (i.e., $c_{jl}^k = c_{jr}^k$) is simply considered as the peak (instead of interval) of the generated T1 FMF.

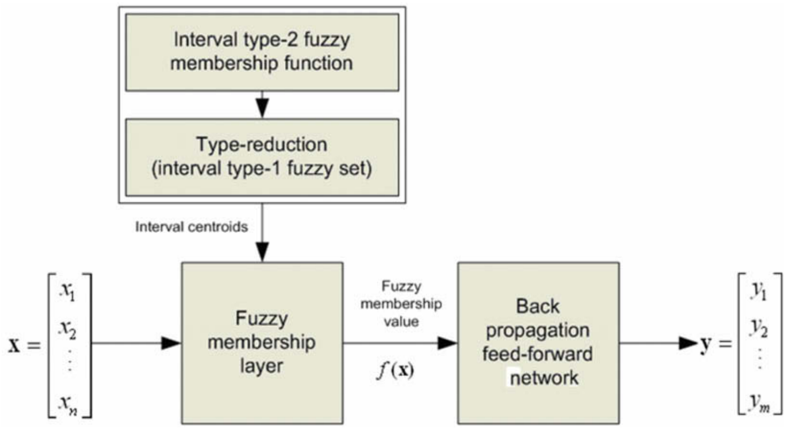

Fig. 12. Structure of the proposed IT2 FMF design method incorporated in a back-propagation neural network.

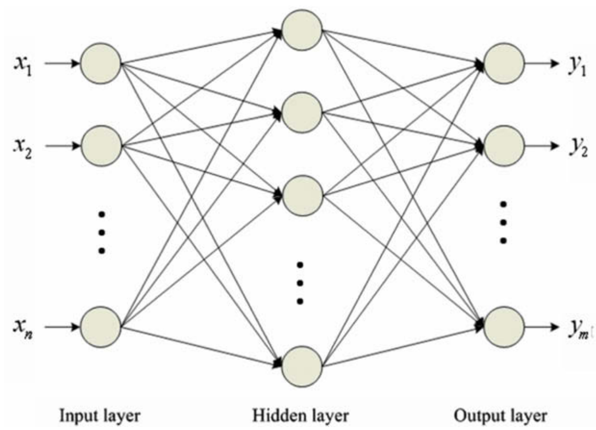

Fig. 13. Structure of a single hidden layer BPNN.

5.1. "Two-class" data set

The "two-class" data set that is collected from two Gaussian distributions consists of 242 patterns (121 patterns for each class) of 2 features and 2 classes. We use one pattern for testing and the remaining patterns for training. This procedure was repeated for all patterns. Fig. 14 shows the scatter plot of the "two-class" data set. We assume that the patterns located outside of the shaded region do not have uncertainty. Therefore, we exclude those patterns (104 patterns) when comparing the classification results. The BPNNs are trained using all of the sample patterns however classification results are reported for only the patterns in the shaded region (138 patterns). Fig. 15 shows the IT2 FMF obtained by our three proposed methods. As shown in Fig. 15a and b, the heuristic IT2 FMF was designed based on a triangular function. Fig. 15c and d shows the membership designed by the histogram based method. As shown in figure, 2D GF fitting was used. Fig. 15e and f displays the memberships designed by the IT2 FCM based method. Experiments for several possible combinations of fuzzifiers m_1 and m_2 were performed and the combination $m_1 = 2$ and $m_2 = 5$ gave the best results. Fig. 16 shows the classification results for the patterns in the shaded region by BPNNs using our proposed generated IT2 FMFs and generated T1 FMF. As indicated in the figure, in most cases the classification rates for our proposed IT2 FMF were slightly higher than those using the T1 FMF. On the average, the heuristic, histogram based, and IT2 FCM based method, gave improved classification rates of 0.4%, 1.2%, and 0.7%, respectively, when compared with the results by T1 FMF.

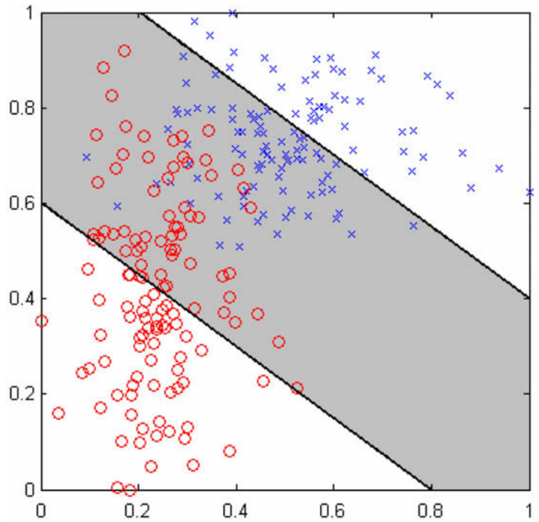

Fig. 14. Scatter plot for the "two class" data set.

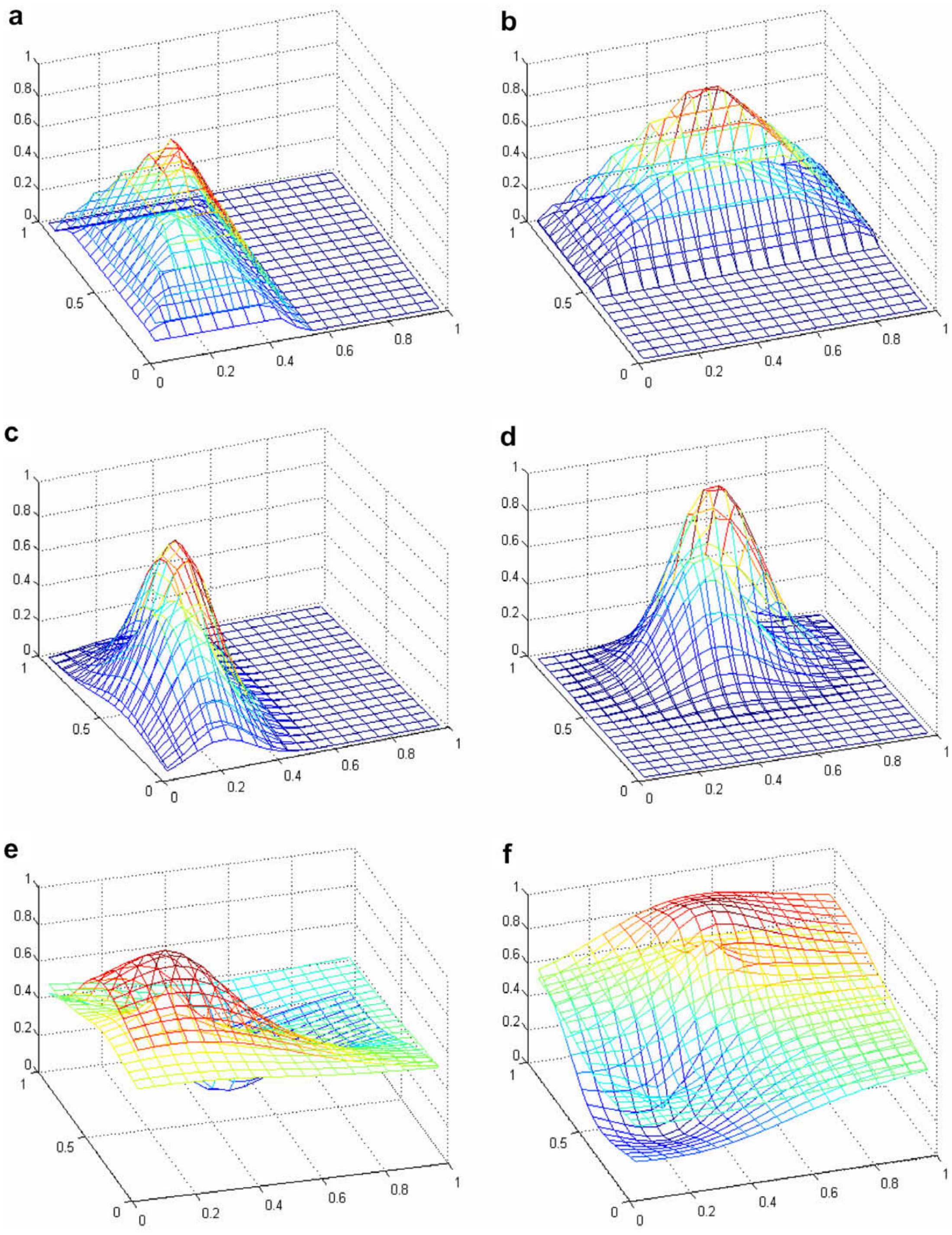

Fig. 15. IT2 FMFs generated by our proposed methods: (a) heuristic method for class "O", (b) class "+", (c) histogram based method for class "O", (d) class "+", (e) IT2 FCM based method for class "O", and (f) class "+" ($m_1 = 2$, $m_2 = 5$).

5.2. "T-shape" data set

The "T-shape" data set consists of 447 patterns (228 and 219 patterns for each class) of 2 features and 2 classes. Fig. 17 shows the scatter plot of the data set. As in the above example, we consider the classification results for the patterns (215 patterns) in the shaded region. We use the same training and testing procedures as mentioned above. Fig. 18 shows the classification results. As shown in Fig. 18a, the T2 FMF heuristic method slightly improved compared with the T1 FMF heuristic

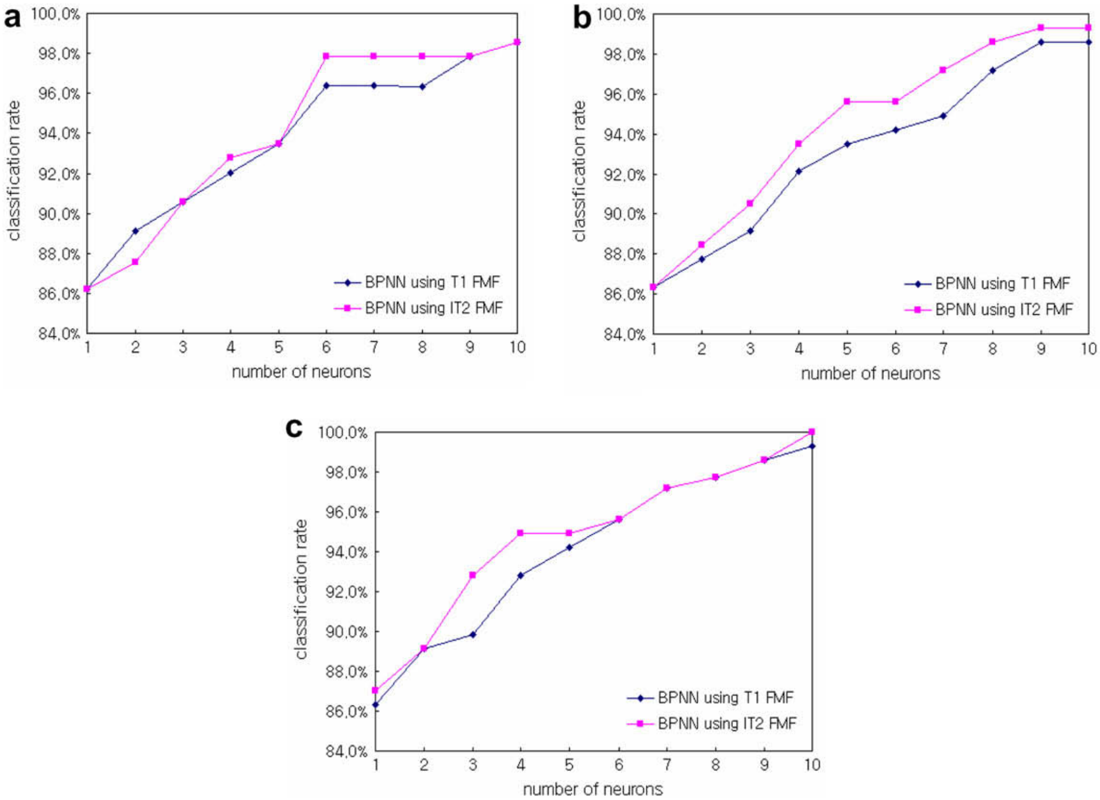

Fig. 16. Classification results for the ''two-class'' data set: (a) heuristic method, (b) histogram based method, and (c) IT2 FCM method ($m_1 = 2$, $m_2 = 5$).

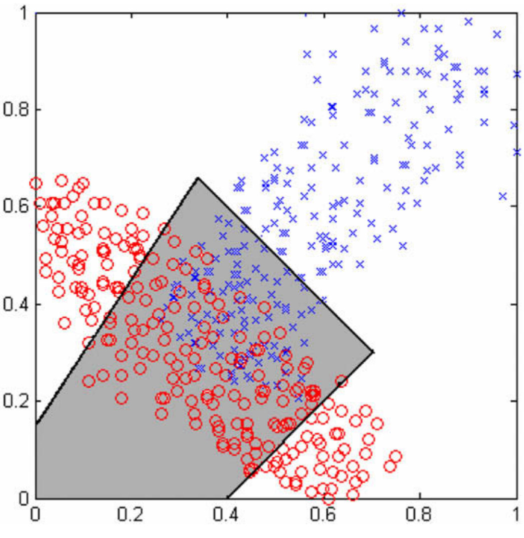

Fig. 17. Scatter plot for the ''T-shape'' data set.

method. However, as shown in Fig. 18b and c, the histogram based method and IT2 FCM based method reasonably improved in classification.

5.3. ''Pima Indian diabetes'' data set

We now give classification results for a high dimensional data set, namely ''Pima Indian diabetes'' data set. The data set consists of 768 patterns (500 and 268 patterns for each class) of 8 features and 2 classes. Fig. 19 shows the classification

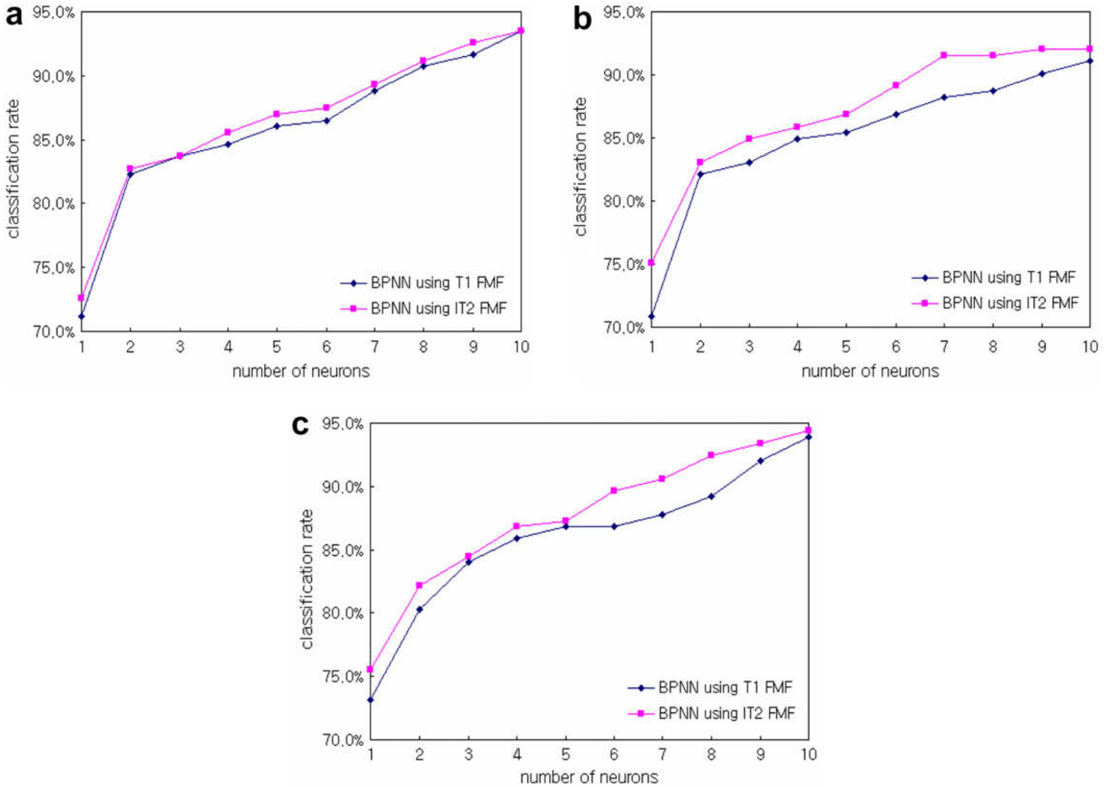

Fig. 18. Classification results for the "T-shape" data set: (a) heuristic method, (b) histogram based method, and (c) IT2 FS FCM method ($m_1 = 2$, $m_2 = 5$).

results. For the histogram based method, due to the difficulty of processing the eight-dimensional histogram and Gaussian function fitting, we generated the IT2 FMF by (22) using one-dimensional histogram and GF fitting for each feature. As in the above examples, our proposed methods gave significantly improved classification results compared with the generated T1 FMF. Average improvement in classification resulted in about 0.92% for the heuristic method, 2.16% for the histogram based method, and 1.5% for the IT2 FCM based method. As indicated in figure, the histogram based method showed the most improvement.

5.4. Real image segmentation

For the last example, we give segmentation results for 200×200 real scene images. The sample image consists of three regions, namely, road, forest, and sky. The gray levels of median and excess green filtered images were used as feature values. We randomly selected 100 pixels in each region of the image, and obtained the sample patterns as the feature values of the selected pixels. Fig. 20 shows the sample image and the scatter plot of the sample patterns. Next, BPNNs were trained the same as in the previous examples using the sample data. Then, all the pixels in the image were classified into three classes by the trained networks. For this example, we give segmentation results for the case of two hidden neurons. The results for networks with higher number of hidden neurons gave similar results.

5.5. Inference results

We present inference (segmentation) results involving one training image (see Fig. 20a) and one test image (see Fig. 21). Fig. 22a and b show the segmentation results for the training image using the heuristic T1 FMF and IT2 FMF method. Figs. 23 and 24 shows the segmentation results using the histogram based method and IT2 FS FCM based method, respectively. For all three IT2 FMF methods, one can visually see that the forest region was better classified than with the T1 FMF method. Figs. 25–27 shows the segmentation results for the test image. As with the training image, the results show significant improvement by applying all three methods.

6. Conclusions

In this paper, we proposed three IT2 FMF generation methods that were obtained from sample data. The heuristic method generated IT2 FMFs by simply incorporating heuristic type-1 FMFs provided by experts. The FOU was designed by assigning

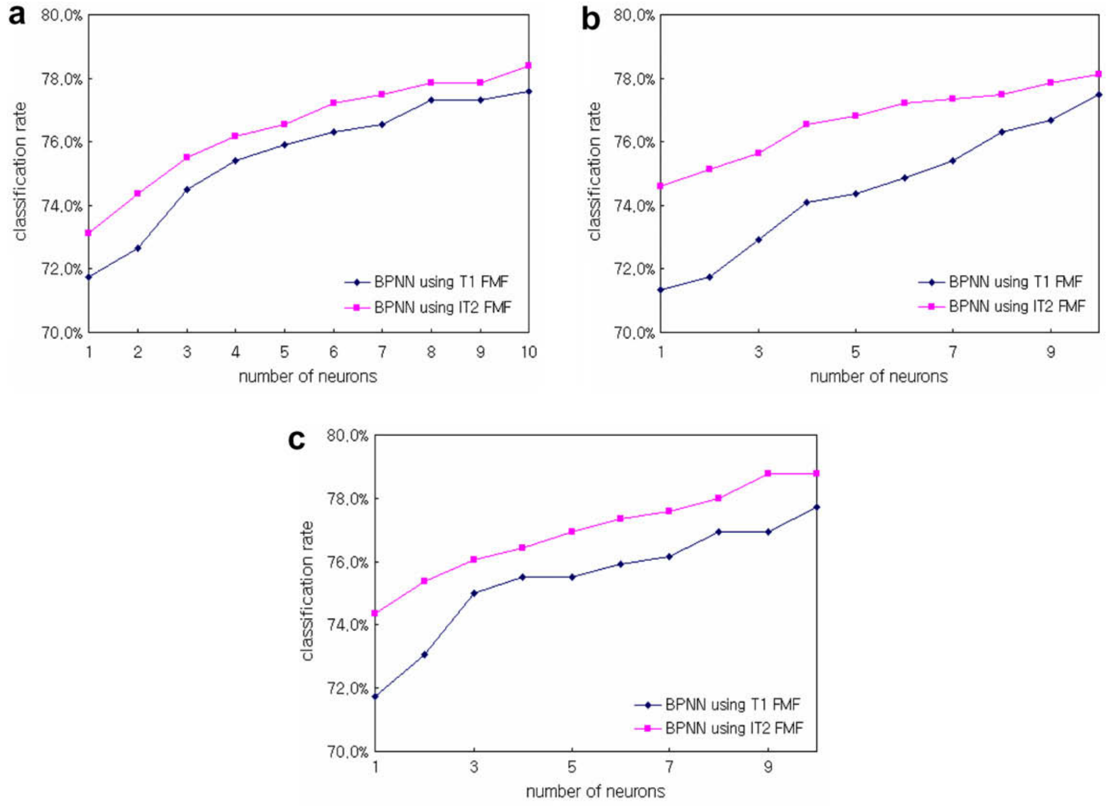

Fig. 19. Classification results for the "Pima Indians diabetes" data set: (a) heuristic method, (b) histogram based method, and (c) IT2 FCM method ($m_1 = 2$, $m_2 = 5$).

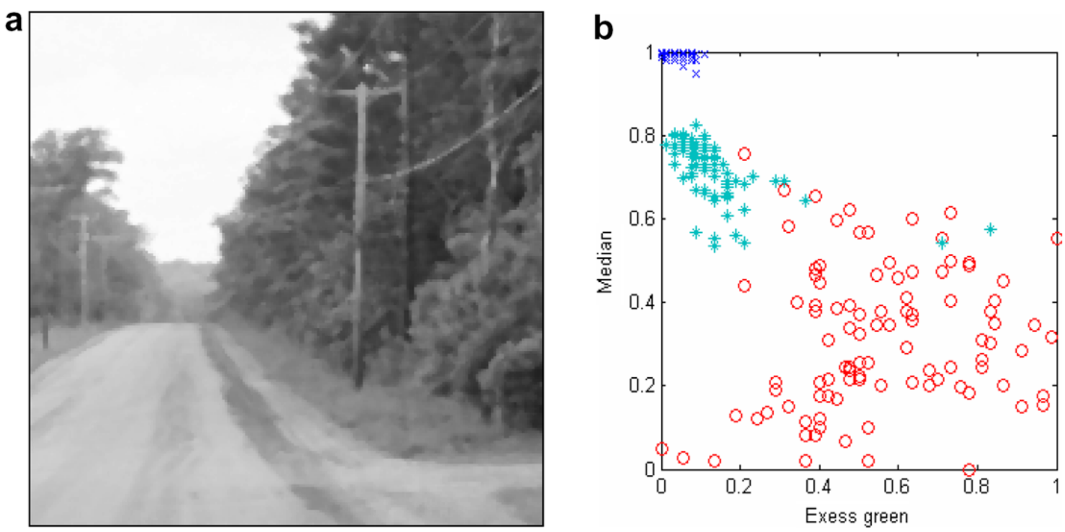

Fig. 20. Natural Scene 1 used for training sample patterns: (a) scene 1 image, and (b) scatter plot of sample patterns.

the UMF with the heuristic T1 FMF, and adequately scaling the UMF for the LMF. However, since the shapes of the heuristic FMF are not flexible, the obtained IT2 FMFs may not sufficiently represent the uncertainty for data that are complex. In the histogram based method, the FOU was designed by fitting Gaussian functions to the smoothed histograms of data. The UMF and LMF obtained by the histogram based method have shown to effectively represent the distribution of patterns better

Fig. 21. Natural scene 2 used for testing.

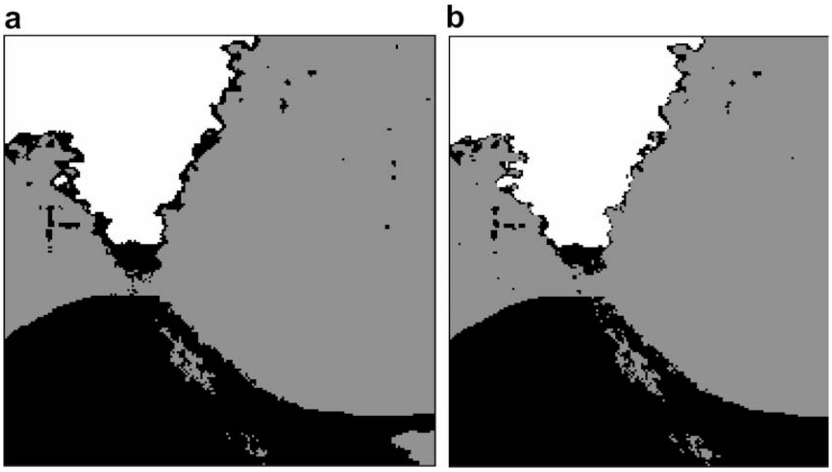

Fig. 22. Segmentation results by the heuristic method for scene 1: (a) T1 FMF, and (2) IT2 FMF.

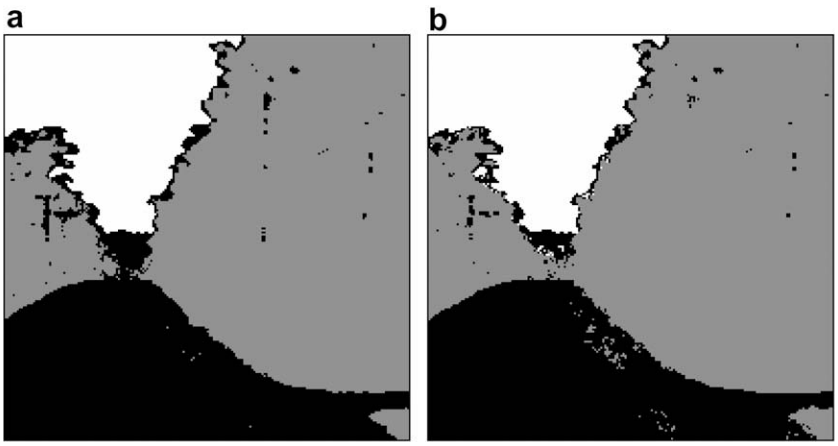

Fig. 23. Segmentation results by the histogram based method for scene 1: (a) T1 FMF, and (2) IT2 FMF.

than the heuristic method. In the IT2 FCM based method, the FOU was designed by using the UMF and LMF equations in the IT2 FCM. As with IT2 FCM, our proposed membership generation method can control the uncertainty of the fuzzifier m in FCM which undesirably affects the fuzzy memberships when the clusters are significantly different in size and density.

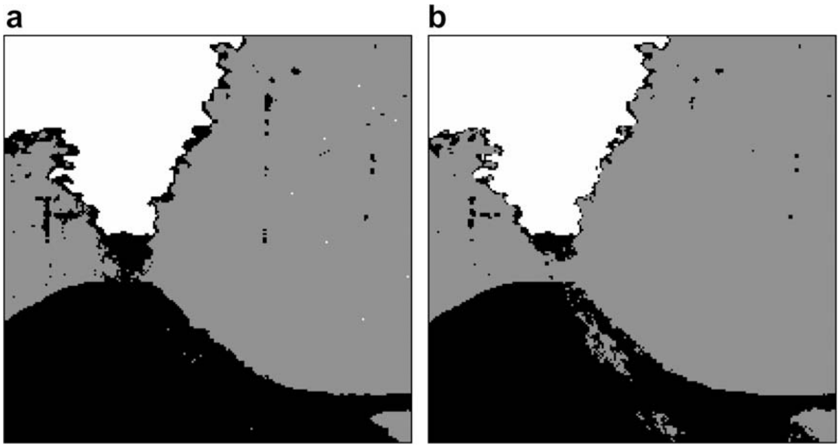

Fig. 24. Segmentation results by the IT2 FS FCM based method for scene 1: (a) T1 FMF, and (2) IT2 FMF.

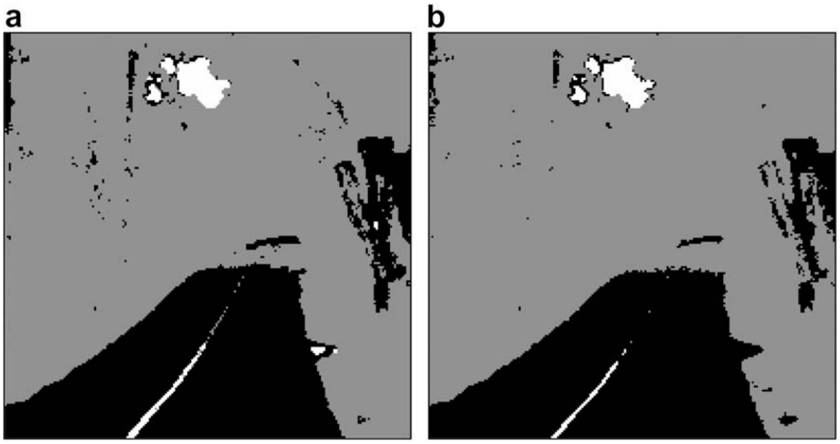

Fig. 25. Segmentation results by the heuristic based method for scene 2: (a) T1 FMF, and (2) IT2 FMF.

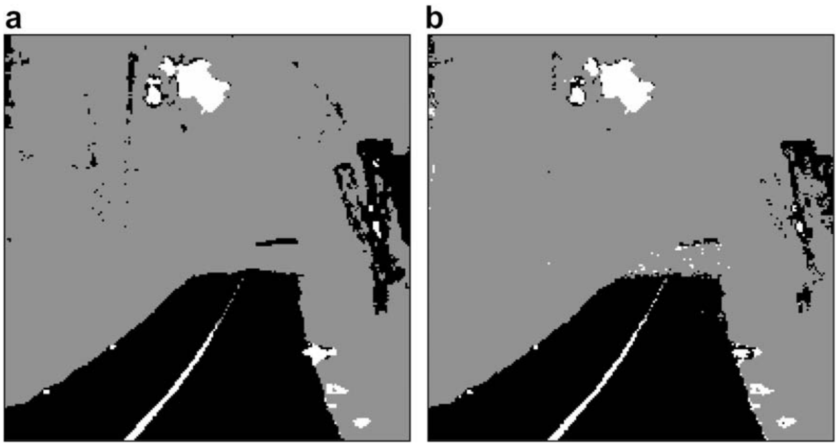

Fig. 26. Segmentation results by the histogram based method for scene 2: (a) T1 FMF, and (2) IT2 FMF.

Therefore, our proposed IT2 FCM based method can represent the uncertainty of fuzzifier m induced by arbitrary structures and densities of sample data.

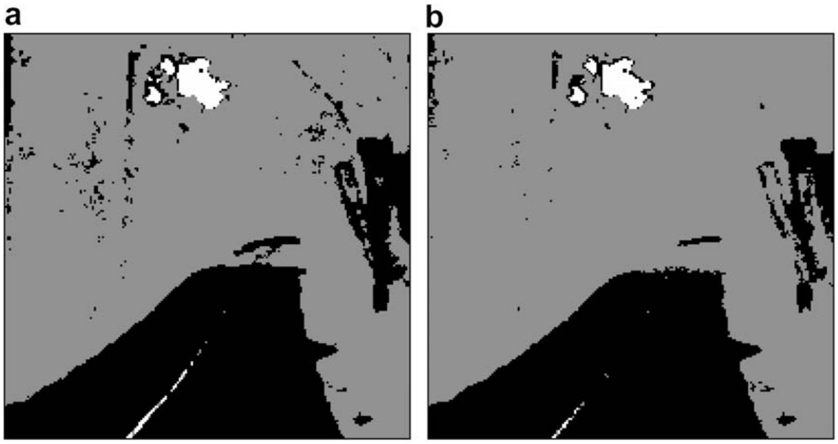

Fig. 27. Segmentation results by the IT2 FCM based method for scene 2: (a) T1 FMF, and (2) IT2 FMF.

Finally, we incorporated our proposed IT2 FMF generation methods into the BPNN, where T1 fuzzy membership values were computed from the centroids of the IT2 FMFs. Then, the T1 fuzzy membership values representing the sample data were used as inputs to the BPNN. The proposed membership assignment showed to improve the classification performance of the BPNN since the uncertainty of pattern data have been desirably controlled by the IT2 FMFs. Experimental results show that our IT2 FMF generation methods can effectively model the uncertainty of pattern data. Other methods of representing FMFs and the integration of IT2 FMFs into other types of neural networks are currently under investigation.

To comment on future studies, we proposed IT2 FMF generation methods to represent the uncertainty of sample data. We believe that the methods may be considered to be early attempts for effectively generating IT2 FMFs automatically from pattern sets. However, there exist areas of improvement as described in the following.

In the heuristic method, we need to select the proper heuristic T1 FMF for the sample data and optimally find the parameters of the selected heuristic T1 FMF. The proper shape and parameters of the heuristic T1 FMF can be critical for properly designing the IT2 FMF. Moreover, the scaling factor α which determines the height of the LMF needs to be carefully selected, since the FOU is dependent on the scaling factor. If an unsuitable value of α is chosen, the IT2 FMF may not suitably represent the uncertainty of the sample data.

In the histogram based method, we used symmetric Gaussian function fitting to the smoothed histograms of the sample data. However, non-symmetric Gaussian function fitting could be used to represent the distribution of sample data more effectively. Non-symmetric Gaussian functions can model the smoothed histograms with more flexibility since the symmetry constraint has been relaxed.

In the IT2 FCM based method, the selection of the fuzzifier m_1 and m_2 is very important in designing the FOU for the sample data distribution. In general, if we select unsuitable fuzzifier m_1 and m_2 for a pattern set, IT2 FCM can yield poor clustering results compared with FCM. For this reasoning, our proposed method may not suitably model the uncertainty of the sample data as well. Therefore, the selection of m_1 and m_2 is an important research area to which various methods can be applied (e.g., neural networks and genetic algorithm).

As a final note, the proposed methods presented can be used as major components of various applications in pattern recognition and computer vision. For example, they may be applied to the modeling of objects in images for object detection. We believe that our studies show that applying generated IT2 FMFs into various fuzzy classification systems can improve their performance.

Acknowledgements

This research was supported by the Defense Acquisition Program Administration and Agency for Defense Development, Korea, through the Image Information Research Center at Korea Advanced Institute of Science and Technology under the contract UD070007AD.

References

[1] J. Bezdek, Pattern Recognition with Fuzzy Objective Function Algorithms, Plenum, 1981.
[2] R. Dave, Fuzzy-shell clustering and applications to circle detection in digital images, International Journal of General Systems 16 (1990) 343–355.
[3] R. Duda, P. Hart, D. Stork, Pattern Classification, John Wiley & Sons, 2001.
[4] H. Hagras, A hierarchical type-2 fuzzy logic control architecture for autonomous mobile robots, IEEE Transactions on Fuzzy Systems 12 (4) (2004) 524–539.
[5] S. Haykin, Neural Networks: A Comprehensive Foundation, Prentice Hall, 1999.
[6] C. Hwang, F. Rhee, An interval type-2 fuzzy C spherical shells algorithm, in: Proceedings of the 2004 IEEE International Conference on Fuzzy Systems, 2004, pp. 1117–1122.

[7] C. Hwang, F. Rhee, Uncertain fuzzy clustering: interval type-2 fuzzy approach to *C*-means, IEEE Transactions on Fuzzy Systems 15 (1) (2007) 107–120.
[8] R. John, Embedded interval valued type-2 fuzzy sets, in: Proceedings of the 2002 IEEE International Conference on Fuzzy Systems, 2002, pp. 1316–1320.
[9] R. John, P. Innocent, M. Barnes, Neuro-fuzzy clustering of radiographic tiba image data using type-2 fuzzy sets, Information Sciences 125 (1–4) (2000) 65–82.
[10] N. Karnik, J. Mendel, Applications of type-2 fuzzy logic systems to forecasting of time series, Information Sciences 120 (1999) 89–111.
[11] N. Karnik, J. Mendel, Centroid of a type-2 fuzzy set, Information Sciences 132 (2001) 195–220.
[12] N. Karnik, J. Mendel, Operations on type-2 fuzzy sets, Fuzzy Sets and Systems 122 (2001) 327–348.
[13] G. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, 1995.
[14] A. Kulkarni, Computer Vision and Fuzzy-Neural Systems, Prentice Hall, 2001.
[15] Q. Liang, N. Karnik, J. Mendel, Connection admission control in ATM networks using survey-based type-2 fuzzy logic systems, IEEE Transactions on System, Man, and Cybernetics Part C: Applied Reviews 30 (2000) 329–339.
[16] Q. Liang, J. Mendel, Interval type-2 fuzzy logic systems: Theory and Design, IEEE Transactions on Fuzzy Systems 8 (5) (2000) 535–550.
[17] Q. Liang, J. Mendel, MPEG VBR video traffic modeling and classification using fuzzy techniques, IEEE Transactions on Fuzzy Systems 9 (1) (2001) 183–193.
[18] Q. Liang, J. Mendel, Overcoming time-varying co-channel interference using type-2 fuzzy adaptive filter, IEEE Transactions on Circuits and Systems-II: Analog and Digital Signal Processing 47 (2000) 1419–1428.
[19] M. Makrehchi, O. Basir, M. Karnel, Generation of fuzzy membership function using information theory measures and genetic algorithm, Lecture Notes in Computer Science 275 (2003) 603–610.
[20] S. Medasani, J. Kim, R. Krishnapuram, An overview of membership function generation techniques for pattern recognition, International Journal of Approximate Reasoning 19 (3–4) (1998) 391–417.
[21] J. Mendel, Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions, Prentice Hall, 2001.
[22] J. Mendel, R. John, Type-2 fuzzy set made simple, IEEE Transactions on Fuzzy Systems 10 (2) (2002) 117–127.
[23] F. Rhee, Uncertain fuzzy clustering: insights and recommendations, IEEE Computational Intelligence Magazine 2 (1) (2007) 44–56.
[24] F. Rhee, B. Choi, Interval type-2 fuzzy membership function design and its application to radial basis function neural networks, in: Proceedings of the 2007 IEEE International Conference on Fuzzy Systems, 2007, pp. 2047–2052.
[25] F. Rhee, C. Hwang, A type-2 fuzzy *C*-means clustering algorithm, in: Proceedings of the 2001 Joint Conference IFSA/NAFIPS, 2001, pp. 1919–1926.
[26] F. Rhee, C. Hwang, An interval type-2 fuzzy perceptron, in: Proceedings of the 2002 IEEE International Conference on Fuzzy Systems, 2002, pp. 1331–1335.
[27] F. Rhee, C. Hwang, An interval type-2 fuzzy K-nearest neighbor, in: Proceedings of the 2003 IEEE International Conference on Fuzzy Systems, 2003, pp. 802–807.
[28] F. Rhee, R. Krishnapuram, Fuzzy rule generation methods for high-level computer vision, Fuzzy Sets and Systems 60 (1993) 245–258.
[29] S. Wang, Generating fuzzy membership functions: a monotonic neural network, Fuzzy Sets and Systems 61 (1) (1994) 71–81.
[30] H. Wu, J. Mendel, Uncertainty bounds and their use in the design of interval type-2 fuzzy logic systems, IEEE Transactions on Fuzzy Systems 10 (5) (2002) 622–639.
[31] C. Yang, N. Bose, Generating fuzzy membership function with self-organizing feature map, Pattern Recognition Letters 27 (5) (2006) 356–365.