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# Aggregation of diverse types of fuzzy orders for decision making problems

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ABSTRACT

In this paper, conditions for $n$-argument functions to preserve diverse types of fuzzy orders during aggregation process are presented. Namely, the following fuzzy orders are studied: total preorder, total order, strict total order, partial preorder, partial order, strict partial order, interval order or semiorder and their preservation in aggregation process along with other connections between input fuzzy relations and the aggregated one are presented. Moreover, dependencies between diverse classes and also among the given class of fuzzy orders are presented. The considered fuzzy orders are compound properties depending in their definitions on arbitrary binary operations on [0, 1], including fuzzy conjunctions and disjunctions (with special cases of triangular norms and triangular conorms). Finally, an application of the obtained result is presented by the given algorithm.



## 1. Introduction

In this contribution we pay attention to fuzzy orders which are transitive, like for example partial order, or without transitivity property like interval orders or semiorders, whose definitions are based on the notion of Ferrers property. Ferrers property is less demanding than transitivity property used in the most of orders.

In 1956 Luce [35] introduced the notion of a semiorder to model a situation of intransitive indifference with a “threshold of discrimination or perception”. This was done for preference structure consisting of strict preference relation, indifference relation and incomparability relation. Later on, Fishburn [23,24] extended the notion of semiorder to that of an interval order, whose underlying idea is to assign an ”interval of evaluations” to each alternative. What makes interval orders and semiorders so interesting, is the fact that the strict preference relation is transitive, whereas the indifference relation may

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be intransitive. The famous sugar-coffee example [35] indicated that intransitivity of indifference can indeed be a very reasonable assumption. Interval orders and semi-orders play a central role in the modelling of intransitive indifference. Namely, according to Luce, in some situations small differences may be considered as not significative for decision makers. For example, "let us imagine a decision maker looking for a new car and comparing two cars with prices equal to 20,000 euro for the first one and 20,500 euro for the second one. It may be not surprising if the decision makers says that he/she is indifferent with these two prices. Such an idea of an indifference threshold cannot be handled by total orders or preorders. Interval orders are used in such situations. The special case where the length of intervals is constant (hence the indifference threshold is constant) corresponds to a semiorder" [38]. Interval orders and semiorders, with Ferrers property, which is a form of pseudo-transitivity, are able to model various types of situations in a wide range of applied fields, e.g., decision-aid, mathematical psychology, choice theory under risk, genetics, information storage, scheduling, mathematical programming, archeology, etc.

In [27] fuzzy orders are discussed and theorems which can be viewed as fuzzy analogs of Szpilrajn's theorem are considered. Fuzzy order structures, such as linear orders, semi-orders and interval orders are often used to model preferences in decision making problems, where the information about the alternatives provided by the experts may be represented in three different ways: preference orderings of the alternatives (alternatives are ordered from the best to the worst without any supplementary information – preferences are modelled qualitatively by means of ordering relations), fuzzy preference relations (the usual case – each expert provides a fuzzy binary relation reflecting the degree to which an alternative is preferred to another), utility functions (expert supplies a real for each alternative, i.e. a function which associates each alternative with a real number indicating the performance of that alternative according to this point of view – preferences are modelled quantitatively by means of utility functions). We will consider the second approach when the information may be represented by means of fuzzy preference relations which represent some type of order relations (i.e. in a sense this is a combination of the first and second approach). A decision maker indicates a number from the unit interval [0, 1] describing his/her degree of preference $x \succ y$. Thus a fuzzy order (preference) relation may be obtained in which for every pair of alternatives $x$, $y$ a number from [0, 1] is specified indicating a degree of preference $x \succ y$. We will examine the assumptions that guarantee preservation of these orders in aggregation process (cf. [11,16,25,29,39,44]). Such considerations follow from the practical situations in fuzzy preference modelling, multiagent or multicriteria decision making problems and solving other issues related to imprecise and uncertain information. In decision making problems a set $X = \{x_1, \dots, x_m\}$ represents a set of objects, where $m \in \mathbb{N}$. In multicriteria decision making there is also considered a set $K = \{k_1, \dots, k_n\}$ of criteria under which the objects are supposed to be evaluated. Fuzzy relations $R_1, \dots, R_n$ reflect judgements of decision makers. An application of such considerations is presented in [39]. From practical point of view it is important that the numerical data do not have to come from the interval [0, 1] (numerical data may be transformed/rescaled to [0, 1], cf. [39]). In multiagent decision making problems fuzzy relations $R_1, \dots, R_n$ represent opinion of each agent over given alternatives. There are many voting methods which require ordered preferences of voters [31,42,43]. Certainly, we may have combination of two situations, i.e. many criteria and many agents.

It is worth to mention that there are two meanings of fuzzy preference relations [20]. If the unit interval is viewed as a bipolar scale then $R(x, y) = 1$ means full strict preference of $x$ over $y$, and is equivalent to $R(y, x) = 0$, which means full negative preference. In such situation indifference is modelled by condition $R(x, y) = R(y, x) = 0.5$ and more generally by the reciprocity property $R(x, y) + R(y, x) = 1$ which generalizes completeness (connectedness). In the other convention the unit interval is viewed as a negative unipolar scale, with neutral upper end, and the preference status between $x$ and $y$ is judged by simultaneous checking the values of $R(x, y)$ and $R(y, x)$. In this case $R(x, y)$ evaluates weak preference and completeness means in the classical form $\max(R(x, y), R(y, x)) = 1$. Indifference is when $R(x, y) = R(y, x) = 1$ and incomparability when $R(x, y) = R(y, x) = 0$. The above two interpretations entail the appropriate form of transitivity properties that should be considered [12,13] and similarly the procedures for choosing the best alternative [17,33]. In this paper we will follow the latter approach. We will weaken the condition of completeness to the form $B(R(x, y), R(y, x)) = 1$, where $B$: $[0, 1]^2 \to [0, 1]$ is an arbitrary operation. However, when choosing concrete $B$ in examples, to keep the meaning of the given property, we will consider fuzzy disjunctions (which generalize the operation max and follow from the meaning of crisp completeness).

In this contribution the considered aggregation process involves also an $n$-argument function $F$. With the use of given fuzzy relations $R_1, \dots, R_n$ and the function $F$, we consider a new fuzzy relation $R_F = F(R_1, \dots, R_n)$ representing a final decision on evaluated objects (after considering the involved criteria). Although we focus on aggregation functions, the aim of this paper is to give the results under the weakest assumptions on $F$ used for the aggregation process. Therefore, we present our considerations with an arbitrary $n$-ary functions and later give examples of aggregation functions. It is worth to mention that aggregation of preference relations based on fuzzy partial orders was considered in [2] and aggregation of fuzzy orders for multi-objective linear programming was applied in [30].

The notions of fuzzy relation properties, in their simplest forms, may involve functions min and max (as it was already mentioned for completeness). These ones were generalized by the use of a t-norm and t-conorm, respectively [25, Chapter 2.5]. In particular, the following properties were examined: $T$-asymmetry, $T$-antisymmetry, $S$-connectedness, $T$-transitivity, negative $S$-transitivity, $T-S$-semitransitivity, and $T-S$-Ferrers property of fuzzy relations, where $T$ is a t-norm and $S$ a t-conorm, also with regard to their preservation in aggregation process [21]. However, the assumptions put on widely used t-norms are not always necessary or desired. This is why a lot of definitions of binary operations which can play a role of weaker fuzzy connectives were introduced and studied, for example fuzzy conjunctions: weak t-norms, overlap functions, t-seminorms (or semicopulas, or conjunctors), and pseudo-t-norms, sometimes along with their dual disjunctions also in the context of their preservation in aggregation process [4].

To sum up, in this article we consider fuzzy order relations whose definitions are based on arbitrary binary operations on [0, 1] including fuzzy conjunctions and disjunctions. We study dependencies between these types of orders among the given class and also between diverse classes. The main aim of the paper is to study preservation of diverse types of fuzzy order relations in aggregation process and also to prove other connections between the properties of individual fuzzy relations and the aggregated one. These properties may depend on operations $B$ involved, on aggregation functions $F$, and on connections between classes of fuzzy relations, which is analyzed in this paper.

In Section 2, we provide basic definitions and results concerning $n$-ary functions on [0, 1] including aggregation functions, fuzzy connectives and dominance between functions. Next, in Section 3, we present basic information about fuzzy relations, its classes, dependencies between properties of fuzzy relations and some results related to preservation of basic fuzzy relation properties in aggregation process. Furthermore, in Sections 4 and 5 we put the main results of this contribution connected with preservation of fuzzy interval orders, fuzzy semiorders and diverse types of transitive fuzzy orders in aggregation process. Finally, in Section 6 there is given an algorithm where the provided theoretical results may be useful.

## 2. Preliminaries

In this section we present the notions and properties of $n$-ary functions on [0, 1], fuzzy connectives and dominance between operations.

**Definition 1** ([10]). Let $n \in \mathbb{N}$. A function $A$: $[0, 1]^n \to [0, 1]$ which is increasing, i.e.

$$A(x_1, \ldots, x_n) \leqslant A(y_1, \ldots, y_n) \quad \text{for} \quad x_i, y_i \in [0, 1], x_i \leqslant y_i, i = 1, \ldots, n$$

is called an aggregation function if $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

**Example 1.** Aggregation functions are:

- median

$$\operatorname{med}(x_1, \ldots, x_n) = \begin{cases} \frac{s_k + s_{k+1}}{2}, & \text{for } n = 2k \\ s_{k+1}, & \text{for } n = 2k+1 \end{cases}, \tag{1}$$

where $(s_1, \ldots, s_n)$ is the increasingly ordered sequence of the values $x_1, \ldots, x_n$, so $s_1 \leqslant \cdots \leqslant s_n$,

- arithmetic mean

$$A(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n}, \tag{2}$$

- a weighted arithmetic mean

$$A_w(x_1, \ldots, x_n) = \sum_{k=1}^{n} w_k x_k, \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1, \tag{3}$$

- a weighted geometric mean

$$G_w(x_1, \ldots, x_n) = \prod_{k=1}^{n} x_k^{w_k}, \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1, \tag{4}$$

- a weighted harmonic mean

$$H_w(x_1, \ldots, x_n) = \begin{cases} 0, & \underset{1 \leqslant k \leqslant n}{\exists} x_k = 0 \\ \left( \sum\limits_{k=1}^{n} \frac{w_k}{x_k} \right)^{-1}, & \text{otherwise} \end{cases}, \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1, \tag{5}$$

- a weighted quadratic mean

$$Q_w(x_1, \ldots, x_n) = \sqrt{\sum_{k=1}^{n} w_k x_k^2}, \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1, \tag{6}$$

- a weighted power mean

$$P_w^{(r)}(x_1, \ldots, x_n) = \begin{cases} 0, & r < 0, \ \underset{1 \leqslant k \leqslant n}{\exists} x_k = 0 \\ G_w(x_1, \ldots, x_n), & r = 0 \\ \left( \sum\limits_{k=1}^{n} w_k x_k^r \right)^{\frac{1}{r}}, & \text{otherwise} \end{cases}, \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1, \tag{7}$$

- a quasi-arithmetic mean

$$M_\varphi(x_1,\dots,x_n)=\varphi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}\varphi(x_k)\right), \tag{8}$$

- a quasi-linear mean

$$F(x_1,\dots,x_n)=\varphi^{-1}\left(\sum_{k=1}^{n}w_k\varphi(x_k)\right),\quad \text{for}\quad w_k>0,\sum_{k=1}^{n}w_k=1, \tag{9}$$

where $x_1,\dots,x_n\in[0,1]$, $\varphi:[0,1]\to\mathbb{R}$ is a continuous, strictly increasing function.

Note that means (2)–(6) are examples of weighted power means. Quasi-linear means include all means (2)–(7) and are obtained for an adequate function $\varphi$. A quasi-arithmetic mean is a special case of a quasi-linear mean for equal weights. Moreover, the listed above special cases of quasi-linear means are linearly ordered (for a fixed set of weights $w$) and have limit cases min and max respectively, i.e.

$$\min\leqslant H_w\leqslant G_w\leqslant A_w\leqslant Q_w\leqslant\max. \tag{10}$$

**Definition 2.** Let $n\in\mathbb{N}$. We say that a function $F$: $[0,1]^n\to[0,1]$:

- has a zero element $z\in[0,1]$ if for each $k\in\{1,\dots,n\}$ and each $x_1,\dots,x_{k-1},x_{k+1},\dots,x_n\in[0,1]$ one has

$$F(x_1,\dots,x_{k-1},z,x_{k+1},\dots,x_n)=z,$$

- is without zero divisors if it has a zero element $z$ and

$$\underset{x_1,\dots,x_n\in[0,1]}{\forall}\ (F(x_1,\dots,x_n)=z\ \Rightarrow\ (\underset{1\leqslant k\leqslant n}{\exists}x_k=z)).$$

**Definition 3** ([22])**.** An operation $C$: $[0,1]^2\to[0,1]$ is called a fuzzy conjunction if it is increasing with respect to each variable and $C(1,1)=1$, $C(0,0)=C(0,1)=C(1,0)=0$. An operation $D$: $[0,1]^2\to[0,1]$ is called a fuzzy disjunction if it is increasing with respect to each variable and $D(0,0)=0$, $D(1,1)=D(0,1)=D(1,0)=1$.

Note, that there are also other ways of defining fuzzy conjunctions and disjunctions (cf. [41]).

**Corollary 1.** *A fuzzy conjunction $C$ has a zero element 0. A fuzzy disjunction $D$ has a zero element 1. If a fuzzy conjunction $C$ has a neutral element 1, then $C\le\min$. If a fuzzy disjunction $D$ has a neutral element 0, then $D\ge\max$.*

Fuzzy conjunctions and disjunctions are examples of aggregation functions.

**Example 2** ([4])**.** Consider the following family of fuzzy conjunctions $C^\alpha$ and disjunctions $D^\alpha$ for $\alpha\in[0,1]$:

$$C^\alpha(x,y)=\begin{cases}1, & \text{if } x=y=1\\ 0, & \text{if } x=0 \text{ or } y=0,\\ \alpha & \text{otherwise}\end{cases}\qquad D^\alpha(x,y)=\begin{cases}0, & \text{if } x=y=0\\ 1, & \text{if } x=1 \text{ or } y=1.\\ \alpha & \text{otherwise}\end{cases}$$

Operations $C^0$ and $C^1$ are the least and the greatest fuzzy conjunction, respectively, where

$$C^0(x,y)=\begin{cases}1, & \text{if } x=y=1\\ 0, & \text{else}\end{cases},\ C^1(x,y)=\begin{cases}0, & \text{if } x=0 \text{ or }\ y=0\\ 1, & \text{else}\end{cases}.$$

Operations $D^0$ and $D^1$ are the least and the greatest fuzzy disjunction, respectively, where

$$D^0(x,y)=\begin{cases}1, & \text{if } x=1 \text{ or }\ y=1\\ 0 & \text{otherwise}\end{cases},\ D^1(x,y)=\begin{cases}0, & \text{if } x=y=0\\ 1 & \text{otherwise}\end{cases}.$$

There are distinguished some families of conjunctions and disjunctions. We will recall some of them also including well-known conjunctions which are triangular norms and disjunctions which are triangular conorms.

**Definition 4.** An operation $C$: $[0,1]^2\to[0,1]$ is called:

- an overlap function [8] if it is a commutative, continuous fuzzy conjunction without zero divisors, fulfilling condition $C(x,y)=1$ if and only if $xy=1$,
- a t-seminorm [28] (a semicopula [1], a conjunctor [14]) if it is a fuzzy conjunction and has a neutral element 1,
- a t-norm [46] if it is a commutative, associative, increasing operation with a neutral element 1.

**Definition 5.** An operation $D$: $[0,1]^2\to[0,1]$ is called:

- a grouping function [9] if it is a commutative, continuous fuzzy disjunction without zero divisors, fulfilling condition $D(x,y)=0$ if and only if $x=y=0$,
- a t-semiconorm if it is a fuzzy disjunction and has a neutral element 0,

- a t-conorm [34] if it is a commutative, associative, increasing operation with a neutral element 0,
- a strict t-conorm $S$: $[0, 1]^2 \to [0, 1]$ if it is a t-conorm which is continuous and strictly increasing in $[0, 1)^2$.

**Example 3** ([34]). The four well-known examples of t-norms $T$ and corresponding t-conorms $S$ are:

$$T_M(s,t) = \min(s,t), \qquad T_P(s,t) = st,$$
$$S_M(s,t) = \max(s,t), \qquad S_P(s,t) = s + t - st,$$

$$T_L(s,t) = \max(s+t-1, 0), \qquad S_L(s,t) = \min(s+t, 1),$$

$$T_D(s,t) = \begin{cases} s, & t = 1 \\ t, & s = 1 \\ 0, & otherwise \end{cases}, \quad S_D(s,t) = \begin{cases} s, & t = 0 \\ t, & s = 0 \\ 1, & otherwise \end{cases}$$

for $s, t \in [0, 1]$. These operations are comparable (linearly ordered). It holds:

$$T_D \leqslant T_L \leqslant T_P \leqslant T_M, \qquad S_M \leqslant S_P \leqslant S_L \leqslant S_D. \tag{11}$$

Triangular norms and conorms are important operations and their properties are well examined. We may distinguish some important families of these operations which will be applied in the sequel.

**Definition 6** ([34]). A t-norm $T$ is called nilpotent if it is continuous and each $x \in (0, 1)$ is a nilpotent element of $T$, i.e. for each $x \in (0, 1)$ there exists $n \in \mathbb{N}$ such that $x_T^{(n)} = 0$.

**Theorem 1** ([34]). *Any nilpotent t-norm is isomorphic to the Łukasiewicz t-norm $T_L$, i.e.*

$$T(x,y) = \varphi^{-1}(T_L(\varphi(x), \varphi(y))), \quad x, y \in [0, 1],$$

*where $\varphi$: $[0, 1] \to [0, 1]$ is an increasing bijection.*

**Definition 7** (cf. [32]). A rotation invariant $t$-norm is a $t$-norm $T$ that verifies for all $x, y, z \in [0, 1]$

$$T(x,y) \leqslant z \Leftrightarrow T(x, 1-z) \leqslant 1-y.$$

**Proposition 1** ([15]). *All rotation invariant t-norms $T$ fulfil the property*

$$T_L \leqslant T \leqslant T_{nM},$$

*where $T_{nM}$ is the nilpotent minimum [26] and*

$$T_{nM}(x,y) = \begin{cases} 0, & \text{if } x + y \leqslant 1 \\ \min(x,y), & \text{otherwise} \end{cases}.$$

Rotation invariant $t$-norms satisfy the property $T(x,y) > 0 \Leftrightarrow x + y > 1$, i.e. the lower-left triangle of the unit square constitutes the zero divisors of $T$. We will not go into further details with this family of t-norms, however this subject was developed for example in [36]. Some of the presented classes of operations are dual to the others.

**Definition 8** ([10]). Let $F$: $[0, 1]^n \to [0, 1]$. A function $F^d$ is called a dual function to $F$, if for all $x_1, \ldots, x_n \in [0, 1]$

$$F^d(x_1, \ldots, x_n) = 1 - F(1 - x_1, \ldots, 1 - x_n).$$

$F$ is called a self-dual function, if it holds $F = F^d$.

Fuzzy disjunctions are dual to fuzzy conjunctions, grouping functions are dual to overlap functions, t-conorms are dual functions to t-norms, in particular $S_L$ is dual to $T_L$, $\max$ is dual to $\min$.

Now, we recall the notion of dominance. Dominance may be treated as a relation between operations defined on the same domain.

**Definition 9** ([47]). Let $m, n \in \mathbb{N}$. A function $F$: $[0, 1]^m \to [0, 1]$ dominates $G$: $[0, 1]^n \to [0, 1]$ ($F \gg G$) if for an arbitrary matrix $[a_{ik}] = A \in [0, 1]^{m \times n}$ the following inequality holds

$$F(G(a_{11}, \ldots, a_{1n}), \ldots, G(a_{m1}, \ldots, a_{mn})) \geqslant G(F(a_{11}, \ldots, a_{m1}), \ldots, F(a_{1n}, \ldots, a_{mn})).$$

There are many examples of dominance. We recall only some samples of them.

**Example 4** ([4]). A weighted arithmetic mean dominates t-norm $T_L$ and a weighted arithmetic mean is dominated by $S_L$. Minimum dominates any fuzzy conjunction and any fuzzy disjunction dominates maximum. A weighted geometric mean dominates t-norm $T_P$. A weighted minimum ([18])

$$F(x_1, \ldots, x_n) = \min_{1 \leqslant k \leqslant n} \max(1 - w_k, x_k), \quad \max_{1 \leqslant k \leqslant n} w_k = 1, \tag{12}$$

is an aggregation function which dominates any $t$-seminorm. Any $t$-semiconorm dominates a weighted maximum ([18])

$$F(x_1,\dots,x_n)=\max_{1\leqslant k\leqslant n}\min(w_k,x_k),\quad \max_{1\leqslant k\leqslant n} w_k=1, \tag{13}$$

which is another example of aggregation function.

**Example 5.** If $r\leq 1$, then $P_w^{(r)}\gg +$. If $r\geq 1$, then $+\gg P_w^{(r)}$. As a result, a weighted arithmetic mean dominates operation + and operation + dominates a weighted arithmetic mean [3]. Moreover, $\min\gg +$ and $+\gg\max$. Let us also note that condition $F\gg +$ is equivalent to superadditivity of $F$, and condition $+\gg F$ is equivalent to subadditivity of $F$ (for notions of superadditive and subadditive functions please see [10]). Moreover, it is easy to check that if $F_1\gg +$ and $F_2\gg +$, then a convex combination of these functions also dominates + , i.e. $F_3\gg +$, where $F_3=pF_1+(1-p)F_2$, $p\in(0,1)$. As a result, an aggregation function

$$F(x_1,\dots,x_n)=\frac{p}{n}\sum_{k=1}^{n}x_k+(1-p)\min_{1\leqslant k\leqslant n}x_k \tag{14}$$

dominates + , where $p\in(0,1)$. A function (14) dominates also $t$-norm $T_L$ (cf [4].).

Similarly, if $+\gg F_1$ and $+\gg F_2$, then $+\gg F_3$, where $F_3=pF_1+(1-p)F_2$, $p\in(0,1)$. As a result, an aggregation function

$$F(x_1,\dots,x_n)=\frac{p}{n}\sum_{k=1}^{n}x_k+(1-p)\max_{1\leqslant k\leqslant n}x_k \tag{15}$$

is dominated by + , where $p\in(0,1)$.

**Proposition 2** (cf. [44])**.** *Let us consider increasing operations $F, G$: $[0,1]^n\to[0,1]$, a bijection $\varphi$: $[0,1]\to[0,1]$ and*

$$F_\varphi(x,y)=\varphi^{-1}(F(\varphi(x_1),\dots,\varphi(x_n))),\ x_1,\dots,x_n\in[0,1]. \tag{16}$$

*If $\varphi$ is increasing, then $F\gg G\Leftrightarrow F_\varphi\gg G_\varphi$.*
*If $\varphi$ is decreasing, then $F\gg G\Leftrightarrow F_\varphi\ll G_\varphi$.*

**Corollary 2.** *A quasi-linear mean $(A_w)_\varphi$ dominates a nilpotent t-norm $T=(T_L)_\varphi$, where $\varphi$: $[0,1]\to[0,1]$ is a given continuous, strictly increasing function.*

**Proof.** Since any weighted arithmetic mean $A_w$ dominates t-norm $T_L$ (cf. Example 4) and quasi-linear means may be treated as isomorphic operations to weighted arithmetic means (cf. Example 1), by Theorem 1 and Proposition 2 a quasi-linear mean $(A_w)_\varphi$ dominates adequate nilpotent t-norm $T=(T_L)_\varphi$, where $(A_w)_\varphi$ and $T=(T_L)_\varphi$ are generated by the same continuous, strictly increasing function $\varphi$: $[0,1]\to[0,1]$. □

## 3. Fuzzy relations

In this section we will present some basic properties of fuzzy relations along with classes of fuzzy relations and dependencies between them. We will also recall some results concerning preservation of fundamental properties of fuzzy relations in aggregation process.

### 3.1. Basic properties of fuzzy relations

Here we give the notion of a fuzzy relation, some properties of fuzzy relations and dependencies between them.

**Definition 10** ([50])**.** A fuzzy relation on a set $X\neq\emptyset$ is an arbitrary function $R$: $X\times X\to[0,1]$. The family of all fuzzy relations on $X$ is denoted by $\mathcal{FR}(X)$.

**Definition 11** (cf. [50])**.** Let $B$: $[0,1]^2\to[0,1]$. A sup–$B$–composition of relations $R,W\in\mathcal{FR}(X)$ is the relation $(R\circ_B W)\in\mathcal{FR}(X)$ such that for any $(x,z)\in X\times X$ it holds

$$(R\circ_B W)(x,z)=\sup_{y\in X}B(R(x,y),W(y,z)). \tag{17}$$

An inf–$B$–composition of relations $R,W\in\mathcal{FR}(X)$ is the relation $(R\circ'_B W)\in\mathcal{FR}(X)$ such that for any $(x,z)\in X\times X$ it holds

$$(R\circ'_B W)(x,z)=\inf_{y\in X}B(R(x,y),W(y,z)). \tag{18}$$

In this section $B$, $B_1$, $B_2$ denote binary operations on the unit interval, i.e. $B$, $B_1$, $B_2$: $[0,1]^2\to[0,1]$. We will consider properties of fuzzy relations introduced in [4], which are generalizations of the ones considered in [25,40], where t-norms and t-conorms are replaced with arbitrary binary operations. If it comes to transitivity property, probably the most known property whose definition is based on other operation than min or max, Zadeh introduced $T_P$-transitivity in [51] and later Bezdek and Harris introduced other $B$-transitivity types, where $B\in\{T_L,\max,S_P,A\}$ (cf. [6]). There are also other ways of defining properties for fuzzy relations. For example, if it comes to connectedness (completeness or linearity) three possible ways are presented and consequences of each approach and dependencies between them are studied in [7]. Diverse approaches of defining the Ferrers property for the fuzzy environment were presented in [15].

Now, we give the list of properties, which will be considered in this paper, whose definitions are based on binary operations.

**Definition 12.** Let $B$, $B_1$, $B_2$: $[0, 1]^2 \to [0, 1]$ be binary operations. Relation $R \in \mathcal{FR}(X)$ is:

- reflexive, if $\underset{x \in X}{\forall} R(x, x) = 1$,
- irreflexive, if $\underset{x \in X}{\forall} R(x, x) = 0$,
- totally $B$-connected, if $\underset{x, y \in X}{\forall} B(R(x, y), R(y, x)) = 1$,
- $B$-connected, if $\underset{x, y \in X, x \neq y}{\forall} B(R(x, y), R(y, x)) = 1$,
- $B$-asymmetric, if $\underset{x, y \in X}{\forall} B(R(x, y), R(y, x)) = 0$,
- $B$-antisymmetric, if $\underset{x, y \in X, x \neq y}{\forall} B(R(x, y), R(y, x)) = 0$,
- $B$-transitive, if $\underset{x, y, z \in X}{\forall} B(R(x, y), R(y, z)) \leqslant R(x, z)$,
- negatively $B$-transitive, if $\underset{x, y, z \in X}{\forall} B(R(x, y), R(y, z)) \geqslant R(x, z)$,
- $B_1 - B_2$-Ferrers, if $\underset{x, y, z, w \in X}{\forall} B_1(R(x, y), R(z, w)) \leqslant B_2(R(x, w), R(z, y))$,
- $B_1 - B_2$-semitransitive, if $\underset{x, y, z, w \in X}{\forall} B_1(R(x, w), R(w, y)) \leqslant B_2(R(x, z), R(z, y))$.

Using the above definition we present notions of fuzzy order relations.

**Definition 13.** Let $B$, $B_i$: $[0, 1]^2 \to [0, 1]$ for $i \in \{1, 2, 3, 4\}$. Relation $R \in \mathcal{FR}(X)$ is called:

- a fuzzy total $B_1 - B_2$-preorder, if it is reflexive, totally $B_1$-connected and $B_2$-transitive,
- a fuzzy total $B_1 - B_2 - B_3$-order, if it is reflexive, $B_1$-antisymmetric, totally $B_2$-connected and $B_3$-transitive,
- a fuzzy strict total $B_1 - B_2$-order, if it is irreflexive, $B_1$-connected and $B_2$-transitive,
- a fuzzy partial $B$-preorder, if it is reflexive and $B$-transitive,
- a fuzzy partial $B_1 - B_2$-order, if it is reflexive, $B_1$-antisymmetric and $B_2$-transitive,
- a fuzzy strict partial $B$-order, if it is irreflexive and $B$-transitive,
- a fuzzy interval $B - B_1 - B_2$-order, if it is totally $B$-connected and $B_1 - B_2$-Ferrers,
- a fuzzy $B - B_1 - B_2 - B_3 - B_4$-semiorder, if it is totally $B$-connected, $B_1 - B_2$-semitransitive and $B_3 - B_4$-Ferrers.

If operations $B$ coincide in the listed above properties we will use the following notations (we show only examples of such coincidences).

**Definition 14.** Let $B$, $B_i$: $[0, 1]^2 \to [0, 1]$ for $i \in \{1, 2\}$. Relation $R \in \mathcal{FR}(X)$ is called:

- a fuzzy total $B_1 - B_2$-order, if it is reflexive, $B_1$-antisymmetric, totally $B_2$-connected and $B_1$-transitive,
- a fuzzy partial $B$-order, if it is reflexive, $B$-antisymmetric and $B$-transitive,
- a fuzzy interval $B_1 - B_2$-order, if it is totally $B_2$-connected and $B_1 - B_2$-Ferrers,
- a fuzzy $B - B_1 - B_2$-semiorder, if it is totally $B$-connected, $B_1 - B_2$-semitransitive and $B_1 - B_2$-Ferrers,
- a fuzzy $B_1 - B_2$-semiorder, if it is totally $B_2$-connected, $B_1 - B_2$-semitransitive and $B_1 - B_2$-Ferrers.

**Remark 1.** What operations $B$: $[0, 1]^2 \to [0, 1]$ should/can we use in the presented classes? For example, for fuzzy total order $R \in \mathcal{FR}(X)$, let us take $B$-antisymmetry, total $B$-connectedness and $B$-transitivity for a given $B$: $[0, 1]^2 \to [0, 1]$. Then from $B$-antisymmetry for $x$, $y$, $x \neq y \in X$, $B(R(x, y), R(y, x)) = 0$ and from total $B$-connectedness for $x$, $y \in X$, $B(R(x, y), R(y, x)) = 1$. Which leads us to a contradiction. This justifies our approach that although we consider given properties in the most general version, i.e. with operations $B$: $[0, 1]^2 \to [0, 1]$, it is natural to put examples with a fuzzy conjunction $B$ in $B$-transitivity property, fuzzy disjunction $B$ in total $B$-connectedness property, a fuzzy conjunction $B_1$ and a fuzzy disjunction $B_2$ in the Ferrers and semitransitivity property, etc. (using conjunctions we enclose t-norms and using disjunctions we enclose t-conorms and this approach is a generalization of the given properties and their crisp meaning). However, in order to show the weakest assumptions on operations $B$ we will state the results in the most general assumption, i.e. starting just with operations $B$: $[0, 1]^2 \to [0, 1]$.

Considering diverse order relations we gather a few properties together. This may affect the form of the relation. In order to explain it we will now concern total max -connectedness (this notion is also named as strong completeness, cf [25].) and some dependencies between classes of fuzzy relations.

**Proposition 3.** *Let B have a zero element 1 and no zero divisors. Then total B-connectedness (B-connectedness) is equivalent to total* max *-connectedness (*max *-connectedness).*

**Proof.** Let $R \in \mathcal{FR}(X)$, $B$ have a zero element 1 and no zero divisors. Total $B$-connectedness is equivalent to

$$B(R(x,y),R(y,x)) = 1 \Leftrightarrow R(x,y) = 1 \vee R(y,x) = 1 \Leftrightarrow \max(R(x,y),R(y,x)) = 1,$$

which is equivalent to the fact that $R$ is totally max -connected. The proof for connectedness property is similar. □

In [7] and [25] (p. 52) a weaker version of this result was presented for $S$ being a $t$-conorm without zero divisors. In such case total $S$-connectedness is equivalent to total max -connectedness. And we have similar result for asymmetry (antisymmetry).

**Proposition 4.** *Let B have a zero element 0 and no zero divisors. Then B-asymmetry (B-antisymmetry) is equivalent to* min *-asymmetry (*min *-antisymmetry).*

**Proof.** Let $R \in \mathcal{FR}(X)$, $B$ have a zero element 0 and no zero divisors. $B$-asymmetry is equivalent to

$$B(R(x,y),R(y,x)) = 0 \Leftrightarrow R(x,y) = 0 \vee R(y,x) = 0 \Leftrightarrow \min(R(x,y),R(y,x)) = 0,$$

which is equivalent to the fact that $R$ is min -asymmetric. The proof for antisymmetry is analogous. □

In [25] (p. 50), we have a weaker result, with $T$ being a $t$-norm without zero divisors. In such case $T$-asymmetry ($T$-antisymmetry) is equivalent to min -asymmetry (min -antisymmetry). Now we give some other dependencies – this time between classes of fuzzy relation properties. By Proposition 4 we have

**Proposition 5.** *Let B have a zero element 0 and no zero divisors (alternative assumption is idempotency of B). Then B-asymmetry implies irreflexivity.*

**Proof.** It is enough to analyze the proof of Proposition 4 for $x = y$. □

Analogously, by Proposition 3 we get

**Proposition 6.** *Let B have a zero element 1 and no zero divisors (alternative assumption is idempotency of B). Then total B-connectedness implies reflexivity.*

**Proposition 7.** *If $R \in \mathcal{FR}(X)$ is irreflexive and B-transitive, then it is B-asymmetric.*

**Proof.** Let $x, y, z \in X$. Applying definition of $B$-transitivity for $x = z$ and the fact that $R$ is irreflexive we get $B(R(x,y),R(y,x)) \leqslant R(x,x) = 0$. As a result $B(R(x,y),R(y,x)) = 0$, which means that $R$ is $B$-asymmetric. □

In [25] (p. 64) we have the weaker version of this result for t-norms $T$, i.e. if $R \in \mathcal{FR}(X)$ is irreflexive and $T$-transitive, then it is $T$-asymmetric. By Propositions 3 and 4, generalizing the result from [25] (cf. Theorem 4.15) we have

**Proposition 8.** *Let $B_1 \neq B_2$, $B_1$ have a zero element 0 and be without zero divisors, $B_2$ have a zero element 1 and be without zero divisors (alternative assumption is that $B_1$, $B_2$ are idempotent operations). If $R \in \mathcal{FR}(X)$ fulfils one of the following pairs of properties:*

- *$B_1$-antisymmetry, total $B_2$-connectedness,*
- *$B_1$-asymmetry, $B_2$-connectedness,*

*then $R(x, y) \in \{0, 1\}$ for $x, y \in X$, which means that R is a crisp relation.*

**Proof.** Let $x, y \in X$. If $R$ is $B_1$-antisymmetric and totally $B_2$-connected, then by Proposition 6 for $x = y$ we get $R(x,x) = 1$. If $x \neq y$, then by $B_1$-antisymmetry we have $B_1(R(x,y),R(y,x)) = 0$ and from total $B_2$-connectedness we have $B_2(R(x,y),R(y,x)) = 1$, which by Propositions 3 and 4 is equivalent to $\min(R(x,y),R(y,x)) = 0$ and $\max(R(x,y),R(y,x)) = 1$, respectively. This means that $R(x, y) \in \{0, 1\}$. The second property may be proven analogously. □

As a result by Proposition 8 and definition of fuzzy orders we get

**Corollary 3.** *Let $B_1 \neq B_2$, $B_1$ have a zero element 0 and be without zero divisors, $B_2$ have a zero element 1 and be without zero divisors (or alternatively $B_1$, $B_2$ are idempotent operations). If $R \in \mathcal{FR}(X)$ is a fuzzy total $B_1 - B_2 - B_3$-order, then $R(x, y) \in \{0, 1\}$ for $x, y \in X$.*

**Corollary 4.** *Let $B_1 \neq B_2$, $B_1$ have a zero element 1 and be without zero divisors, $B_2$ have a zero element 0 and be without zero divisors (or alternatively $B_1$, $B_2$ are idempotent operations). If $R \in \mathcal{FR}(X)$ is a fuzzy strict total $B_1 - B_2$-order, then $R(x, y) \in \{0, 1\}$ for $x, y \in X$.*

**Proof.** If $R \in \mathcal{FR}(X)$ is a fuzzy strict total $B_1 - B_2$-order, then it is irreflexive, $B_1$-connected and $B_2$-transitive, and then by Proposition 7, it is $B_2$-asymmetric. As a result, by Proposition 8, $R(x, y) \in \{0, 1\}$ for $x, y \in X$. □

We see that $R \in \mathcal{FR}(X)$ fulfilling assumptions of Proposition 8 is a characteristic function of some crisp relation $S$ described on a set $X$, so $R = \chi_S$, where $S \subset X \times X$. Assumption $B_1 \neq B_2$ is due to our approach that the most useful is to consider fuzzy disjunctions and conjunctions in appropriate definitions (cf. Remark 1). Some properties may be characterized with the use of composition.

**Corollary 5.** *Let $B_1, B_2$: $[0, 1]^2 \to [0, 1]$. A relation $R \in \mathcal{FR}(X)$ is $B_1$–$B_2$–semitransitive if and only if $R \circ_{B_1} R \leqslant R \circ'_{B_2} R$.*

**Proof.** Let $R \in \mathcal{FR}(X)$. Directly by (17) and (18) and definition of $B_1$–$B_2$–semitransitivity we get

$$R \circ_{B_1} R \leqslant R \circ'_{B_2} R \Leftrightarrow \underset{x,y\in X}{\forall} \sup_{w\in X} B_1(R(x,w), R(w,y)) \leqslant \inf_{z\in X} B_2(R(x,z), R(z,y))$$

$$\Leftrightarrow \underset{x,y,z,w\in X}{\forall} B_1(R(x,w), R(w,y)) \leqslant B_2(R(x,z), R(z,y)),$$

which means that $R$ is $B_1$–$B_2$–semitransitive. □

Similarly we get

**Corollary 6.** *Let $B$: $[0, 1]^2 \to [0, 1]$. A relation $R \in \mathcal{FR}(X)$ is $B$–transitive if and only if $R \circ_B R \leqslant R$. A relation $R \in \mathcal{FR}(X)$ is negatively $B$–transitive if and only if $R \circ'_B R \geqslant R$.*

As a result, by Corollary 5 and Corollary 6, we see that semitransitivity is a weaker property than transitivity.

**Proposition 9.** *Let $B_1, B_2$: $[0, 1]^2 \to [0, 1]$. If $R \in \mathcal{FR}(X)$ is $B_1$-transitive (respectively negatively $B_2$-transitive), then $R$ is $B_1$–$B_2$–semitransitive for $B_2 = (B_1)^d$ (respectively $B_1 = (B_2)^d$).*

**Proof.** If $R \in \mathcal{FR}(X)$ is $B_1$-transitive, then $R \circ_{B_1} R \leqslant R$, and also by duality principle, $R$ is negatively $B_2$-transitive for $B_2 = (B_1)^d$, so $R \leqslant R \circ'_{B_2} R$. As a result we get $B_1$–$B_2$–semitransitivity of $R$. □

There are many connections between properties of fuzzy relations. These are dependencies between diverse classes of properties (as for example transitivity and semitransitivity), which were above presented, and dependencies inside one class, for example semitransitivity with respect to diverse operations $B$, which will be given below. In this section there are presented only some of the dependencies. Other results are provided in next sections.

**Proposition 10.** *Let $B, B^*$: $[0, 1]^2 \to [0, 1]$, $B \le B^*$. Thus if $R \in \mathcal{FR}(X)$ is totally $B$-connected ($B$-connected), then it is also totally $B^*$-connected ($B^*$-connected).*

**Proof.** Let $x, y \in X$. By assumption $B(R(x,y), R(y,x)) = 1$. Since $B, B^*$: $[0, 1]^2 \to [0, 1]$, $B \le B^*$, we get $B^*(R(x,y), R(y,x)) = 1$. □

Analogously we may prove that

**Proposition 11.** *Let $B, B^*$: $[0, 1]^2 \to [0, 1]$, $B^* \le B$. Thus if $R \in \mathcal{FR}(X)$ is $B$-asymmetric ($B$-antisymmetric, $B$-transitive), then it is also $B^*$-asymmetric ($B^*$-antisymmetric, $B^*$-transitive).*

**Proposition 12.** *Let $B_i, B_i^* : [0,1]^2 \to [0,1]$, $i = 1, 2$, $B_1^* \leqslant B_1$, $B_2 \leqslant B_2^*$. Thus if $R \in \mathcal{FR}(X)$ is $B_1 - B_2$-Ferrers ($B_1 - B_2$-semitransitive), then it is also $B_1^* - B_2^*$-Ferrers ($B_1^* - B_2^*$-semitransitive).*

**Proof.** Let $x, y, z, w \in X$. We will prove the dependence for the Ferrers property. Thus by assumptions

$$B_1^*(R(x,y), R(z,w)) \leqslant B_1(R(x,y), R(z,w)) \leqslant B_2(R(x,w), R(z,y)) \leqslant B_2^*(R(x,w), R(z,y)),$$

which means that $R$ has $B_1^* - B_2^*$-Ferrers property. The proof for $B_1^* - B_2^*$-semitransitivity is analogous. □

### *3.2. Preservation of fuzzy relation properties in aggregation process*

We will present the results with the minimal assumptions on $F$ acting as an “aggregation function”. However, not all aggregation operators make sense in any context. As it is stated in [19]: “The problem of preference aggregation consists in deriving a global preference profile achieving a consensus between the preference profiles supplied by the various sources. This new preference profile may be different from all the input ones provided that it remains close to all of them. Generally, averaging (compensative) operators are natural candidates for preference merging. Several types of preference merging problems exist. Among them there is multiagent fusion and multicriteria decision making. Multiagent fusion consists in finding a consensus among individuals expressing their preferences on a set of candidate choices. The aim is often to determine the average opinion likely to avoid extreme positions. This problem has been widely studied in social choice theory [45]. Multicriteria decision making consists in rating and ranking individual decisions from several points of view. Multiple goals are involved and the problem is to find trade-offs between them. These problems are mathematically very similar: agents, criteria, and states play the same role of sources supplying preference profiles across potential decisions (candidates, objects, uncertain acts). In the setting of preference merging, it is very natural to require idempotent aggregation operations. If all individuals have the same preference profiles, this preference profile should be the global one. Given the natural assumption of increasingness in the wide sense for aggregation operations, it is clear that they can only be averaging ones." Averaging functions $F$: $[0, 1]^n \to [0, 1]$ (compensative functions or in other words functions fulfilling the Pareto principle) are the ones which fulfil the property $\min(x_1, \dots, x_n) \leqslant F(x_1, \dots, x_n) \leqslant \max(x_1, \dots, x_n)$ for $x_1, \dots, x_n \in [0, 1]$. Examples of such functions are: quasi-linear means, minimum, maximum, median, weighted minimum and weighted maximum (which may be calculated as a median [18]), aggregation functions (14) and (15). Note that quasi-linear means are one of the most often appearing aggregation functions in decision making real-life situations [52]. In this paper, after giving the result with the weakest assumptions on $F$, we will put examples of averaging functions preserving the considered property.

**Definition 15.** Let $F$: $[0, 1]^n \to [0, 1]$, $R_1, \dots, R_n \in \mathcal{FR}(X)$. An aggregated fuzzy relation $R_F \in \mathcal{FR}(X)$ is described by the formula $R_F(x, y) = F(R_1(x, y), \dots, R_n(x, y))$, $x, y \in X$. A function $F$ preserves a property of fuzzy relations if for every $R_1, \dots, R_n \in \mathcal{FR}(X)$ having this property, $R_F$ also has this property.

Preservation of the properties listed in Definition 12 was considered in [4]. We recall here the results from this paper (Propositions 13 – 18) which will be useful in the sequel.

**Proposition 13.** *Let $R_1, \dots, R_n \in \mathcal{FR}(X)$ be reflexive. The relation $R_F$ is reflexive (irreflexive), if and only if the function $F$ satisfies the condition $F(1, \dots, 1) = 1$ ($F(0, \dots, 0) = 0$).*

**Proposition 14.** *Let card $X \geq 2$, $B$ have a zero element 0 and be without zero divisors. A function $F$ preserves $B$-asymmetry ($B$-antisymmetry) if and only if it satisfies the following condition for all $s, t \in [0, 1]^n$*

$$\forall_{1 \leqslant k \leqslant n} \min(s_k, t_k) = 0 \Rightarrow \min(F(s), F(t)) = 0. \tag{19}$$

**Example 6.** Let $B$ be a fuzzy conjunction without zero divisors (e.g. a strict t-norm or an overlap function). The function $F = \min$ or a weighted minimum preserve $B$-asymmetry ($B$-antisymmetry). Functions $F$ which have a zero element $z = 0$ with respect to a certain coordinate, i.e.

$$\exists_{1 \leqslant k \leqslant n} \forall_{i \neq k} \forall_{t_i \in [0,1]} F(t_1, \dots, t_{k-1}, 0, t_{k+1}, \dots, t_n) = 0$$

fulfil (19), so they preserve $B$-asymmetry ($B$-antisymmetry). In particular, a weighted geometric mean fulfils (19). As another example we may consider median. If a function $F$ fulfils the condition

$$\forall_{t \in [0,1]^n} \text{card}\{k : t_k = 0\} > \frac{n}{2} \Rightarrow F(t) = 0, \tag{20}$$

then we also get (19) (e.g. median fulfils condition (20)). However, the above condition is not necessary for (19), because it does not cover the projections which preserve $B$-asymmetry ($B$-antisymmetry).

**Proposition 15.** *Let card $X \geq 2$, $B$ have a zero element 1 and be without zero divisors. A function $F$ preserves total $B$-connectedness ($B$-connectedness) if and only if it satisfies the following condition for all $s, t \in [0, 1]^n$*

$$\forall_{1 \leqslant k \leqslant n} \max(s_k, t_k) = 1 \Rightarrow \max(F(s), F(t)) = 1. \tag{21}$$

**Example 7.** Let $B$ be a fuzzy disjunction without zero divisors (e.g. a strict $t$-conorm or a grouping function). Examples of functions fulfilling (21) for all $s, t \in [0, 1]^n$ are $F = \max$, $F = $med, a weighted maximum or functions $F$ with a zero element $z = 1$ with respect to a certain coordinate, i.e.

$$\exists_{1 \leqslant k \leqslant n} \forall_{i \neq k} \forall_{t_i \in [0,1]} F(t_1, \dots, t_{k-1}, 1, t_{k+1}, \dots, t_n) = 1.$$

The dual property for (20) have the form

$$\forall_{t \in [0,1]^n} \text{card}\{k : t_k = 1\} > \frac{n}{2} \Rightarrow F(t) = 1. \tag{22}$$

Note that according to Proposition 3 preservation of total max-connectedness (max-connectedness), which are typical Zadeh's properties, is equivalent to preservation of total $B$-connectedness ($B$-connectedness) for operations $B$ with a zero element 1 and without zero divisors. Analogous situation is with respect to min-asymmetry (min-antisymmetry) and $B$-asymmetry ($B$-antisymmetry) for operations $B$ with a zero element 0 and without zero divisors (cf. Proposition 4).

**Proposition 16.** *Let card $X \geq 3$, $B$ have a zero element $z = 0$, $F$: $[0, 1]^n \to [0, 1]$ be increasing. Function $F$ preserves $B$-transitivity if and only if $F \gg B$, which means that for all $(s_1, \dots, s_n), (t_1, \dots, t_n) \in [0, 1]^n$*

$$F(B(s_1, t_1), \dots, B(s_n, t_n)) \geqslant B(F(s_1, \dots, s_n), F(t_1, \dots, t_n)).$$

Below we also recall a sufficient condition for preservation of $B$-transitivity, which shows that assumption on $B$ to have a zero element is not needed in this case.

**Proposition 17.** *If an increasing function $F$: $[0, 1]^n \to [0, 1]$ dominates $B$, then it preserves $B$-transitivity.*

By results on dominance we obtain the following examples of functions which preserve $B$-transitivity.

**Example 8.** Each quasi–linear mean preserves $T_D$-transitivity (cf. [44]). Moreover, for $n = 2$ arbitrary t-norm $F = T$ preserves $T_D$-transitivity (cf[44].). A weighted geometric mean preserves $T_P$-transitivity, a weighted arithmetic mean preserves $T_L$-transitivity (cf. Example 4). A function $F$ described by the formula (14) preserves $T_L$-transitivity (cf. Example 5). Minimum preserves $C$-transitivity for any fuzzy conjunction $C$ (cf. Example 4). A weighted minimum preserves $C$-transitivity for any t-seminorm $C$ (cf. Example 4).

By Example 8 and Corollary 2 we get

**Corollary 7.** *Any quasi-linear mean $(A_w)_\varphi$ preserves $T$-transitivity for a nilpotent t-norm $T = (T_L)_\varphi$, where $\varphi : [0, 1] \to \mathbb{R}$ is a given continuous, strictly increasing function (cf. [25], Theorem 5.11).*

Now we recall the result for preservation of $B_1 - B_2$-Ferrers property and $B_1 - B_2$-semitransitivity. In this case we have only sufficient conditions.

**Proposition 18.** *If a function $F$: $[0, 1]^n \to [0, 1]$, which is increasing in each of its arguments fulfils $F \gg B_1$ and $B_2 \gg F$, then it preserves $B_1 - B_2$-Ferrers property ($B_1 - B_2$-semitransitivity).*

The following lemma will help with obtaining wide family of operations preserving Ferrers and semitransitivity properties.

**Lemma 1.** *Let $B$: $[0, 1]^2 \to [0, 1]$ and $B^d$ be a corresponding dual operation. If $F$: $[0, 1]^n \to [0, 1]$ is a self-dual function, then $F \gg B$ implies $B^d \gg F$.*

The condition given in Lemma 1 is only the sufficient one. Let us consider projections $F = P_k$, $B = T$ being a t-norm, $S = T^d$. Then $S \gg P_k$ and $P_k \gg T$, but $F \neq F^d$.

**Example 9.** Since weighted arithmetic means dominate $T_L$ (cf. Example 4), and by the fact that weighted arithmetic means are self-dual functions (cf. Definition 8), then by Proposition 18 and Lemma 1 it follows that any weighted arithmetic mean preserves $B_1 - B_2$-Ferrers property ($B_1 - B_2$-semitransitivity) for $t$-norm $T_L = B_1$ and $t$-conorm $S_L = B_2$.

By Example 9 and Corollary 2 we have

**Corollary 8.** *Any quasi-linear mean $(A_w)_\varphi$ preserves $T - S$-Ferrers property ($T - S$-semitransitivity) for a nilpotent t-norm $T = (T_L)_\varphi$, where $\varphi : [0, 1] \to \mathbb{R}$ is a given continuous, strictly increasing function and $S = T^d$.*

Conditions given in Proposition 18 are only the sufficient ones.

**Example 10.** Let us consider function $F(s, t) = st$ (so $F = T_P$) and fuzzy relations presented by the matrices

$$R_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Relations $R_1, R_2$ are min − max-Ferrers (cf. [25]) and min − max-semitransitive. Moreover $R = F(R_1, R_2)$ is both min − max-Ferrers and min − max-semitransitive, where $R \equiv 0$. However, it is not true that $F \gg \min$ (the only $t$-norm that dominates minimum is minimum itself).

The other result for preservation of $B_1 - B_2$-Ferrers property for $B_1$ being a t-norm and $B_2$ being a t-conorm (and also $T$-transitivity) for quasi-arithmetic means one can find in [25] (p. 140–141).

Since the results for preservation of total $B$-connectedness ($B$-connectedness) and $B$-asymmetry ($B$-antisymmetry), i.e. characterization theorems (including the sufficiency conditions), are given only for operations $B$ without appropriate zero divisors, we present here result for preservation of these properties for other type of operations $B$. Namely, these are widely used t-conorm $S_L$ and t-norm $T_L$, respectively.

**Proposition 19.** *If $F$: $[0, 1]^n \to [0, 1]$ is increasing, have an idempotent element 1 and $F \gg +$, then $F$ preserves $T_L$-asymmetry ($T_L$-antisymmetry).*

**Proof.** Let $R_1, \ldots, R_n \in \mathcal{FR}(X)$ be $T_L$-asymmetric, $x$, $y \in X$. It means that $T_L(R_i(x, y), R_i(y, x)) = 0$ for $i = 1, \ldots, n$, so $\max(R_i(x, y) + R_i(y, x) - 1, 0) = 0$ for $i = 1, \ldots, n$. As a result $R_i(x, y) + R_i(y, x) \leqslant 1$ for $i = 1, \ldots, n$. We will show that $R_F$ is $T_L$-asymmetric, so $T_L(R_F(x, y), R_F(y, x)) = 0$. This is equivalent to the inequality $R_F(x, y) + R_F(y, x) \leqslant 1$. We get

$$R_F(x, y) + R_F(y, x) = F(R_1(x, y), \ldots, R_n(x, y)) + F(R_1(y, x), \ldots, R_n(y, x))$$

$$\leqslant F(R_1(x, y) + R_1(y, x), \ldots, R_n(x, y) + R_n(y, x)) \leqslant F(1, \ldots, 1) = 1.$$

The proof for $T_L$-antisymmetry is analogous. □

Examples of functions which preserve $T_L$-asymmetry and $T_L$-antisymmetry are: a weighted arithmetic mean, a weighted geometric mean, a weighted harmonic mean, minimum, aggregation function (14) (cf. Example 5) and triangular norms $T_P$ and $T_L$. Note that $T_L$-asymmetry with its equivalent condition $R(x,y)+R(y,x)\leqslant 1$ for $x, y\in X$ was considered in [39] under the name 'moderate asymmetry'. Analogously to Proposition 19 we get

**Proposition 20.** *If $F$: $[0,1]^n\to[0,1]$ is increasing, have an idempotent element 1 and $+\gg F$, then $F$ preserves total $S_L$-connectedness ($S_L$-connectedness).*

Examples of functions which preserve total $S_L$-connectedness and $S_L$-connectedness are: a weighted arithmetic mean, a weighted quadratic mean, maximum, an aggregation function (15) (cf. Example 5) and triangular conorms $S_P$ and $S_L$. Total $S_L$-connectedness with its equivalent condition $R(x,y)+R(y,x)\geqslant 1$ for $x, y\in X$ was considered in [39] under the name 'moderate comparability' and along with moderate asymmetry it was examined the preservation of these properties in the context of aggregation of diverse types of fuzzy orders. However, these orders in almost all cases do not coincide with the ones considered in this contribution. Only one property, i.e. 'preorder' considered in [39], coincides in our terminology with fuzzy $T_L$-preorder.

Let us note that if $R\in FR(X)$ fulfils property $R(x,y)+R(y,x)=1$ for $x\neq y$, with assumption of reflexivity, we do not have reciprocity on the diagonal. Moreover, with this assumption $R$ is $S_L$-connected and $T_L$-antisymmetric (if $R$ is reflexive, then $R$ could not be $T_L$-asymmetric).

### 3.3. Connections between input relations and the aggregated one

Here we give results of some other kind. Namely, we will not focus on preservation of properties in aggregation process but we will ask what properties will have the aggregated fuzzy relation $R_F$ if $F$ does not preserve the given property. We will approach this problem in two ways. Firstly, with respect to the dependencies between operations $B$ (comparability of them), secondly with respect to the dependencies between functions $F$ (comparability of them).

**Proposition 21.** *Let $B_1$, $B_2$, $B_*$: $[0,1]^2\to[0,1]$, $B_1\le B_2$. If $R_1,\dots,R_n\in\mathcal{FR}(X)$ are $B_2$-transitive ($B_2$-asymmetric, $B_2$-antisymmetric) and $F$: $[0,1]^n\to[0,1]$ preserves $B_1$-transitivity ($B_1$-asymmetry, $B_1$-antisymmetry), then $R_F$ is $B_1$-transitive ($B_1$-asymmetric, $B_1$-antisymmetric). Moreover, $R_F$ is $B_*$-transitive ($B_*$-asymmetric, $B_*$-antisymmetric) for any $B_*\le B_1$.*

**Proof.** Let $B_1$, $B_2$: $[0,1]^2\to[0,1]$, $B_1\le B_2$. If $R_1,\dots,R_n\in\mathcal{FR}(X)$ are $B_2$-transitive, then by Proposition 11 they are also $B_1$-transitive. Since $F$ preserves $B_1$-transitivity, $R_F=F(R_1,\dots,R_n)$ is $B_1$-transitive. Using again Proposition 11 we see that $R_F$ is $B_*$-transitive for $B_*\le B_1$. Justification for other properties is analogous. □

In particular, if $R_1,\dots,R_n\in\mathcal{FR}(X)$ are $B$-transitive for $T_P\le B$, then they are also $T_P$-transitive. If $F$ is a weighted geometric mean, which preserves $T_P$-transitivity, then $R_F$ is $B_*$-transitive for arbitrary $B_*\le T_P$. Using similar considerations we see that, if $R_1,\dots,R_n\in\mathcal{FR}(X)$ are min-transitive (in literature it is used a short form "transitive"), then $R_F$, for a weighted geometric mean $F$, is $T_L$-transitive (cf. Proposition A.1, [39]). Other detailed results concerning preservation of $B$-transitivity, where $B$ is a $t$-norm and an aggregation function $F$ is a quasi-linear mean, are given in [48].

Analogously, by Proposition 10 we have

**Proposition 22.** *Let $B_1$, $B_2$, $B_*$: $[0,1]^2\to[0,1]$, $B_2\le B_1$. If $R_1,\dots,R_n\in\mathcal{FR}(X)$ are totally $B_2$-connected ($B_2$-connected) and $F$: $[0,1]^n\to[0,1]$ preserves total $B_1$-connectedness ($B_1$-connectedness), then $R_F$ is totally $B_1$-connected ($B_1$-connected). Moreover, $R_F$ is totally $B_*$-connected ($B_*$-connected) for any $B_1\le B_*$.*

Let $R_1,\dots,R_n\in\mathcal{FR}(X)$ be totally max-connected. By Example 7, $F=$med preserves total max-connectedness. As a result, by (11), $R_F$ is totally $B$-connected for $B\in\{\max, S_P, S_L, S_D\}$.

Similarly to the previous results, by Proposition 12 we get

**Proposition 23.** *Let $B_i, B_i^*:[0,1]^2\to[0,1],\ i=1,2,\ B_1^*\leqslant B_1,\ B_2\leqslant B_2^*$. If $R_1,\dots,R_n\in\mathcal{FR}(X)$ are $B_1-B_2$-Ferrers ($B_1-B_2$-semitransitive) and $F$: $[0,1]^n\to[0,1]$ preserves $B_1^*-B_2^*$-Ferrers property ($B_1^*-B_2^*$-semitransitivity), then $R_F$ is $B_1^*-B_2^*$-Ferrers ($B_1^*-B_2^*$-semitransitive).*

If $R_1,\dots,R_n\in\mathcal{FR}(X)$ are $T_L-S_L$-Ferrers or $T_L-S_L$-semitransitive then by Example 9 a weighted arithmetic mean $F$ preserves these properties and by (11) $R_F$ is for example $T_D-S_D$-Ferrers and $T_D-S_D$-semitransitive.

Now, we will consider dependencies between aggregating functions $F$: $[0,1]^n\to[0,1]$ and their influence on aggregated fuzzy relation $R_F$. Examples of the mentioned dependencies are presented by (10). We will give the results for properties listed in Definition 12 along with some applications to the concrete operations $B$ and functions $F$.

**Proposition 24.** *If $R_1,\dots,R_n\in\mathcal{FR}(X)$ are reflexive and $F_2$ preserves reflexivity, then $R_{F_1}\in\mathcal{FR}(X)$, for any function $F_1$ such that $F_2\le F_1$, also is reflexive.*

**Proof.** Let $R_1,\dots,R_n\in\mathcal{FR}(X)$ be reflexive and if $F_2$ preserves reflexivity, then $R_{F_2}\in\mathcal{FR}(X)$ is reflexive which means that for any $x\in X$ we have $R_{F_2}(x,x)=1$. This is equivalent to $F_2(R_1(x,x),\dots,R_n(x,x))=1$ and by assumption $F_2\le F_1$ yields $F_1(R_1(x,x),\dots,R_n(x,x))\geqslant F_2(R_1(x,x),\dots,R_n(x,x))=1$ for any $x\in X$. As a result $R_{F_1}(x,x)=1$ for any $x\in X$, which proves reflexivity of $R_{F_1}$. □

Analogously we get

**Proposition 25.** *If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are irreflexive and $F_2$ preserves irreflexivity, then $R_{F_1}$, for any function $F_1$ such that $F_1 \le F_2$, also is irreflexive.*

Examples of functions which preserve reflexivity and irreflexivity are quasi-linear means (cf. Proposition 13). By the above results if $R_1, \dots, R_n \in \mathcal{FR}(X)$ are reflexive, then $R_F$ for $F$ being a fuzzy disjunction with a neutral element zero is reflexive (cf. Corollary 1). Similarly, if $R_1, \dots, R_n \in \mathcal{FR}(X)$ are irreflexive, then $R_F$ for $F$ being a fuzzy conjunction with a neutral element one is irreflexive (cf. Corollary 1).

**Proposition 26.** *Let $B$: $[0, 1]^2 \to [0, 1]$ be increasing. If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are totally $B$-connected ($B$-connected) and $F_2$ preserves total $B$-connectedness ($B$-connectedness), then $R_{F_1} \in \mathcal{FR}(X)$, for any function $F_1$ such that $F_2 \le F_1$, also is totally $B$-connected ($B$-connected).*

**Proof.** Let $R_1, \dots, R_n \in \mathcal{FR}(X)$ be totally $B$-connected and if $F_2$ preserves total $B$-connectedness, then $R_{F_2} \in \mathcal{FR}(X)$ is totally $B$-connected which means that for any $x$, $y \in X$ we have $B(R_{F_2}(x, y), R_{F_2}(y, x)) = 1$. This is equivalent to $B(F_2(R_1(x, y), \dots, R_n(x, y)), F_2(R_1(y, x), \dots, R_n(y, x))) = 1$ and since $B$ is increasing and $F_2 \le F_1$, $B(F_1(R_1(x, y), \dots, R_n(x, y)), F_1(R_1(y, x), \dots, R_n(y, x))) = 1$ for any $x$, $y \in X$. As a result $B(R_{F_1}(x, y), R_{F_1}(y, x)) = 1$ for any $x$, $y \in X$, which proves total $B$-connectedness of $R_{F_1}$. The proof for $B$-connectedness is analogous. $\square$

Similarly we may justify

**Proposition 27.** *Let $B$: $[0, 1]^2 \to [0, 1]$ be increasing. If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are $B$-asymmetric ($B$-antisymmetric) and $F_2$ preserves $B$-asymmetry ($B$-antisymmetry), then $R_{F_1}$, for any function $F_1$ such that $F_1 \le F_2$, also is $B$-asymmetric ($B$-antisymmetric).*

As a result if $R_1, \dots, R_n \in \mathcal{FR}(X)$ are totally $S_L$-connected ($S_L$-connected) and since a weighted arithmetic mean preserves this property, then $R_F$ for $F$ being a weighted quadratic mean, maximum (cf. Proposition A.13, [39]) or a fuzzy disjunction with a neutral element zero is totally $S_L$-connected ($S_L$-connected).

If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are $B$-asymmetric ($B$-antisymmetric), where $B$ is a fuzzy conjunction without zero divisors (e.g. a strict $t$-norm or an overlap function), then since a weighted geometric mean preserves this property (cf. Example 6), then $R_F$ for $F$ being a weighted harmonic mean, minimum or a fuzzy conjunction with a neutral element one is $B$-asymmetric ($B$-antisymmetric).

If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are $T_L$-asymmetric ($T_L$-antisymmetric) and since a weighted arithmetic mean preserves this property, then $R_F$ for $F$ being a weighted geometric mean, a weighted harmonic mean, minimum (cf. Proposition A.11, [39]) or a fuzzy conjunction with a neutral element one is $T_L$-asymmetric ($T_L$-antisymmetric).

If it comes to the properties of transitivity, negative transitivity, Ferrers property and semitransitivity similar results, for the comparable functions $F$, with the methods presented above cannot be obtained. This is due to the formulas describing these properties. However, we may obtain some results using other dependencies.

**Example 11.** If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are min-transitive, then $R_F$ is $T_L$-transitive for $F \in \{\min, H_w, G_w, A_w\}$ (cf. Proposition A.8, [39]). Note that, a weighted arithmetic mean $A_w$ preserves $T_L$-transitivity (cf. Example 8), and each given in this example function $F$ fulfils the inequality $F \le A_w$.

## 4. Fuzzy interval orders and fuzzy semiorders

### 4.1. Preservation of fuzzy interval orders and fuzzy semiorders

Firstly, in this section we will present some dependencies between the considered orders. By Definitions 13 and 14 we see that the relation $R \in \mathcal{FR}(X)$ is a fuzzy $B - B_1 - B_2$-semiorder (fuzzy $B_1 - B_2$-semiorder) if it is a $B_1 - B_2$-semitransitive fuzzy interval $B - B_1 - B_2$-order (fuzzy interval $B_1 - B_2$-order).

By Propositions 10 and 12, applying notions from Definitions 13 and 14, we may simplify the notions of fuzzy interval and semiorders.

**Corollary 9.** *Let $B_1$, $B_2$, $B_3$, $B_4$, $B^*$, $B^\diamond$, $B$: $[0, 1]^2 \to [0, 1]$. If there exist $B^*$, $B^\diamond$ such that $B^* \le B_1$, $B^* \le B_3$, $B^\diamond \ge B_2$, $B^\diamond \ge B_4$, $B^\diamond \ge B$, and $R \in \mathcal{FR}(X)$ is a fuzzy $B - B_1 - B_2 - B_3 - B_4$-semiorder, then it is a fuzzy $B^* - B^\diamond$-semiorder.*

**Corollary 10.** *Let $B$, $B_1$, $B_2$, $B^\diamond$: $[0, 1]^2 \to [0, 1]$. If there exists $B^\diamond$ such that $B^\diamond \ge B_2$, $B^\diamond \ge B$, and $R \in \mathcal{FR}(X)$ is a fuzzy interval $B - B_1 - B_2$-order, then it is a fuzzy interval $B_1 - B^\diamond$-order.*

**Example 12.** Let $R \in \mathcal{FR}(X)$, where

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0.2 & 1 & 0.2 & 1 \end{bmatrix}.$$

Relation $R$ is $\min - \max$-Ferrers and totally $\max$-connected, so it is a fuzzy interval $\min - \max$-order. By Propositions 10 and 12, relation $R$ is a fuzzy interval $B^* - B_1^* - B_2^*$-order for any $B^*, B_i^* : [0,1]^2 \to [0,1]$, $i = 1, 2$ such that $\max \leq B^*$, $B_1^* \leqslant \min$, $\max \leqslant B_2^*$. As a special case, when $B^* = B_2^*$, relation $R$ is a fuzzy interval $B_1^* - B_2^*$-order. By Corollary 1, we see that there are many examples of such operations $B^*, B_2^*$, namely t-semiconorms (including all t-conorms), and similarly operations $B_1^*$, namely t-seminorms (including all t-norms). However, $R$ is neither $T_D$-transitive nor $T_D - S_D$-semitransitive (so by Corollary 1 and dependencies between t-norms/t-conorms, $R$ does not have these properties for any operation $B_1 \geq T_D$, $B_2 \leq S_D$, including t-norms/t-conorms, respectively). But $R$ is $C^0 - D^1$-semitransitive, where $C^0$ is the weakest fuzzy conjunction and $D^1$ is the strongest fuzzy disjunction (note that $R$ is not $C^0$-transitive, so it is none of other fuzzy orders than interval and semiorder ones). Since $R \circ_B R \equiv 1$ for $B = C^0$ (cf. Corollary 5), then only the strongest fuzzy disjunction could work as a corresponding operation $B_2$ in the semitransitivity property. As a result, according to Corollary 9, $R$ is a fuzzy $C^0 - D^1$-semiorder.

Now, we will give results for preservation of fuzzy interval orders and fuzzy semiorders.

**Proposition 28.** *Let $B_1$ have an idempotent element 1. A reflexive $B_1 - B_2$-Ferrers relation is totally $B_2$-connected.*

**Proof.** If $R$ is a reflexive, $B_1 - B_2$-Ferrers fuzzy relation, then we get

$$1 = B_1(1, 1) = B_1(R(x, x), R(y, y)) \leqslant B_2(R(x, y), R(y, x)),$$

which means that $B_2(R(x, y), R(y, x)) = 1$ and $R$ is totally $B_2$-connected. □

**Corollary 11.** *Let $T$ be a t-norm and $S$ a t-conorm. A reflexive $T - S$-Ferrers relation is totally $S$-connected.*

**Proposition 29** ([5])**.** *Let $B_1$ have a zero element 0, an idempotent element 1 and for each $x, y \in [0, 1]$ such that $x + y > 1$ fulfil $B_1(x, y) = B_1(y, x)$ and let $B_2$ be dual to $B_1$ such that $B_1 \leq B_2$. The following assertions are equivalent:*
*(1) A reflexive $B_1 - B_2$-Ferrers relation is totally $S_L$-connected.*
*(2) Operation $B_1$: $[0, 1]^2 \to [0, 1]$ fulfils $B_1(x, y) > 0$ for any pair $(x, y) \in [0, 1]^2$ such that $x + y > 1$.*

**Corollary 12** ([15])**.** *Let us consider a t-norm $T$ and its dual t-conorm $S$. The following assertions are equivalent:*
*(1) A reflexive $T - S$-Ferrers relation is totally $S_L$-connected.*
*(2) The t-norm $T$ fulfils $T(x, y) > 0$ for any pair $(x, y) \in [0, 1]^2$ such that $x + y > 1$.*

The above corollary applies to all rotation invariant *t*-norms (cf. Propositions 1 and 12), in particular to $T_L$, and hence to any $T \geq T_L$. As a result it applies also to t-norms without zero divisors, although in that case the much stronger result holds (cf[15].). It will be presented in Proposition 30.
By Proposition 3 we get

**Proposition 30** ([5])**.** *Let a commutative operation $B_1$ have a zero element 0, an idempotent element 1 and let $B_2$ be dual to $B_1$ such that $B_1 \leq B_2$. The following assertions are equivalent:*
*(1) A reflexive $B_1 - B_2$-Ferrers relation is totally* $\max$*-connected.*
*(2) $B_1$ has no zero divisors.*

**Corollary 13** ([15])**.** *Let $T$ be a t-norm and $S$ its dual t-conorm. Then the following conditions are equivalent:*
*(1) A reflexive $T - S$- Ferrers relation is totally* $\max$*-connected.*
*(2) The t-norm $T$ has no zero divisors.*

The above results simplify the considerations on aggregation of fuzzy interval orders (condition on $F$ for preservation of reflexivity is much easier than the one for total connectedness). Applying these results and results from Section 3 we get adequate statements for preservation of fuzzy interval orders and fuzzy semiorders which will be presented here only for the simplest cases with regard to the notations (cf. Definition 14, Corollaries 9 and 10).

**Theorem 2.** *Let $T$ be a rotation invariant t-norm, $R_1, \dots, R_n \in \mathcal{FR}(X)$ be reflexive and $T - S_L$-Ferrers (and $T - S_L$-semitransitive). If a function $F$: $[0, 1]^n \to [0, 1]$, which is increasing in each of its arguments, fulfils $F(1, \dots, 1) = 1$, $F \gg T$ and $S_L \gg F$, then $R_F = F(R_1, \dots, R_n)$ is a fuzzy interval $T - S_L$-order (fuzzy $T - S_L$-semiorder).*

**Proof.** Since sufficient conditions for preservation of $B_1 - B_2$-Ferrers property and $B_1 - B_2$-semitransitivity are the same (cf. Proposition 18), by Corollary 12 assuming that $T$ is a rotation invariant t-norm, for obtaining $R_F$ which is a fuzzy interval $T - S_L$-order and a fuzzy $T - S_L$-semiorder it is enough to assume that $F$ preserves reflexivity (cf. Proposition 13) and $T - S_L$-Ferrers property (cf. Proposition 18). □

By Theorem 2 and Proposition 23 we get

**Corollary 14.** *Let $B_1^*, B_2^* : [0, 1]^2 \to [0, 1]$, $B_1^* \leqslant T$, $S_L \leqslant B_2^*$, $T$ be a rotation invariant t-norm, $R_1, \dots, R_n \in \mathcal{FR}(X)$ be reflexive and $T - S_L$-Ferrers (and $T - S_L$-semitransitive). If a function $F$: $[0, 1]^n \to [0, 1]$, which is increasing in each of its arguments, fulfils $F(1, \dots, 1) = 1$, $F \gg T$ and $S_L \gg F$, then $R_F = F(R_1, \dots, R_n)$ is a fuzzy interval $B_1^* - B_2^*$-order (fuzzy $B_1^* - B_2^*$-semiorder).*

Since $T_L$ is an example of a rotation invariant *t*-norm, by Theorem 2 and Proposition 23 we get the following results.

**Example 13.** Let $B_1^*, B_2^* : [0,1]^2 \to [0,1]$, $B_1^* \leqslant T_L$, $S_L \leqslant B_2^*$, $R_1, \dots, R_n \in \mathcal{FR}(X)$ be reflexive and $T_L - S_L$-Ferrers (and $T_L - S_L$-semitransitive). If a function $F$: $[0, 1]^n \to [0, 1]$, which is increasing in each of its arguments fulfils $F(1, \dots, 1) = 1$, $F \gg T_L$ and $S_L \gg F$, then $R_F = F(R_1, \dots, R_n)$ is a fuzzy interval $T_L - S_L$-order (fuzzy $T_L - S_L$-semiorder). Moreover, $R_F$ is a fuzzy interval $B_1^* - B_2^*$-order (fuzzy $B_1^* - B_2^*$-semiorder).

By Corollary 8 and Example 13 we obtain the following two statements

**Corollary 15.** *Let $R_1, \dots, R_n \in \mathcal{FR}(X)$ be reflexive and $T_L - S_L$-Ferrers (and $T_L - S_L$-semitransitive). Then fuzzy relation $R_F = F(R_1, \dots, R_n)$ is a fuzzy interval $T_L - S_L$-order (fuzzy $T_L - S_L$-semiorder), where F is a weighted arithmetic mean. Moreover, fuzzy relation $R_F$ is a fuzzy interval $T - S$-order (fuzzy $T - S$-semiorder), where F is a quasi-linear mean and T is a nilpotent t-norm, $S = T^d$.*

**Corollary 16.** *Let $B_1^*, B_2^* : [0,1]^2 \to [0,1]$, $B_1^* \leqslant T_L$, $S_L \leqslant B_2^*$, $R_1, \dots, R_n \in \mathcal{FR}(X)$ be reflexive and $T_L - S_L$-Ferrers (and $T_L - S_L$-semitransitive). Then fuzzy relation $R_F = F(R_1, \dots, R_n)$ is a fuzzy interval $B_1^* - B_2^*$-order (fuzzy $B_1^* - B_2^*$-semiorder), where F is a weighted arithmetic mean (or more generally a quasi-linear mean).*

Applying Corollary 13 for $T = \min$ and $S = \max$, by Propositions 13 and 18 we obtain

**Example 14.** Let $B_1^*, B_2^* : [0,1]^2 \to [0,1]$, $B_1^* \leqslant \min$, $\max \leqslant B_2^*$, $R_1, \dots, R_n \in \mathcal{FR}(X)$ be reflexive and $\min - \max$-Ferrers (and $\min - \max$-semitransitive). If a function $F$: $[0, 1]^n \to [0, 1]$, which is increasing in each of its arguments fulfils $F(1, \dots, 1) = 1$, $F \gg \min$ and $\max \gg F$, then $R_F = F(R_1, \dots, R_n)$ is a fuzzy interval $\min - \max$-order (fuzzy $\min - \max$-semiorder). Moreover, (by Proposition 23) $R_F$ is a fuzzy interval $B_1^* - B_2^*$-order (fuzzy $B_1^* - B_2^*$-semiorder).

Projections dominate minimum and are dominated by maximum ([4]), so they fulfil assumptions on $F$ in the above example. Fuzzy conjunctions with neutral element one are examples of $B_1^* \leqslant \min$ and fuzzy disjunctions with neutral element zero are examples of $B_2^* \geqslant \max$ (cf. Corollary 1).

## 5. Orders with transitivity property

### 5.1. Preservation of transitive orders in aggregation process

We will present here the results for preservation of orders based in their notions on transitivity property. We start with examples of such orders.

**Example 15.** The following $R \in \mathcal{FR}(X)$ is reflexive, $S_L$-connected and $T_L$-transitive ($R \circ_{T_L} R = R$, cf. Corollary 6), where

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 1 & 1 & 0.6 \\ 0.7 & 0.5 & 1 \end{bmatrix}.$$

Relation $R$ is a fuzzy total $S_L - T_L$-preorder but not a fuzzy total $S_L - T_L$-order, since $R$ is neither $T_L$-antisymmetric nor $T_D$-antisymmetric ($R$ is $C_0$-antisymmetric for the weakest fuzzy conjunction $C_0$).

Note, that in [49] it is shown that $T_L$-transitivity is the most suitable notion for fuzzy orderings (with transitivity requirement with assumption that transitivity is t-norm dependent).

In virtue of connections between classes of fuzzy relations we may simplify the process of checking the type of a property. Namely, by Proposition 6 we have

**Corollary 17.** *Let $B_i$: $[0, 1]^2 \to [0, 1]$ for $i \in \{1, 2, 3\}$, $R \in \mathcal{FR}(X)$:*

- *If $B_1$ has a zero element 1 and no zero divisors (or alternatively $B_1$ is idempotent), R is totally $B_1$-connected and $B_2$-transitive, then R is a fuzzy total $B_1 - B_2$-preorder.*
- *If $B_2$ has a zero element 1 and no zero divisors (or alternatively $B_2$ is idempotent), R is $B_1$-antisymmetric, totally $B_2$-connected and $B_3$-transitive, then R is a fuzzy total $B_1 - B_2 - B_3$-order.*

Let $B_1 \neq B_2$, $B_1$ have a zero element 0 and be without zero divisors, $B_2$ have a zero element 1 and be without zero divisors. We should remember that if a function $F$ preserves a fuzzy total $B_1 - B_2 - B_3$-order for such operations, then the aggregated fuzzy relation $R_F$ is a characteristic function of a crisp relation (cf. Corollary 3). However, that is not the case for other operations $B$.

It is worth to mention that by Propositions 10 and 11, applying notions from Definitions 13 and 14, we may simplify the notions of fuzzy total and partial orders.

**Corollary 18.** *Let $B_1$, $B_2$, $B_3$, $B^*$: $[0, 1]^2 \to [0, 1]$. If there exists $B^*$ such that $B^* \le B_1$, $B^* \le B_3$ and $R \in \mathcal{FR}(X)$ is a fuzzy total $B_1 - B_2 - B_3$-order, then it is a fuzzy total $B^* - B_2$-order.*

**Corollary 19.** *Let $B_1$, $B_2$, $B^*$: $[0, 1]^2 \to [0, 1]$. If there exists $B^*$ such that $B^* \le B_1$, $B^* \le B_2$ and $R \in \mathcal{FR}(X)$ is a fuzzy partial $B_1 - B_2$-order, then it is a fuzzy partial $B^*$-order.*

Now, we turn to the problem of preservation of fuzzy orders in aggregation process. We will apply the notions from Definition 13. It is worth to mention that all given theorems are characterizations. We start with fuzzy preorders.

**Theorem 3.** *Let* $\operatorname{card} X \geqslant 3$, $B$: $[0, 1]^2 \to [0, 1]$ *have a zero element 0. An increasing function* $F$: $[0, 1]^n \to [0, 1]$ *preserves a fuzzy partial B-preorder if and only if* $F(1, \ldots, 1) = 1$ *and* $F \gg B$.

**Proof.** By Proposition 13 we have condition $F(1, \ldots, 1) = 1$ for preservation of reflexivity and by Proposition 16 we get the condition $F \gg B$ for preservation of $B$-transitivity. As a result justification for a fuzzy partial $B$-preorder is completed. □

By Proposition 17 and Example 4 we get

**Corollary 20.** *A quasi-linear mean preserves a fuzzy partial T-preorder for an appropriate nilpotent t-norm T (cf. Corollary 2), in particular a weighted arithmetic mean preserves a fuzzy partial* $T_L$*-preorder. Moreover, a weighted geometric mean preserves a fuzzy partial* $T_P$*-preorder, a function (14) preserves a fuzzy partial* $T_L$*-preorder, minimum preserves a fuzzy partial C-preorder for any fuzzy conjunction C. A weighted minimum preserves a fuzzy partial C-preorder for any fuzzy t-seminorm C.*

**Theorem 4.** *Let* $\operatorname{card} X \geqslant 3$, $B_2$: $[0, 1]^2 \to [0, 1]$ *have a zero element 0,* $B_1$: $[0, 1]^2 \to [0, 1]$ *have a zero element 1 and be without zero divisors. An increasing function* $F$: $[0, 1]^n \to [0, 1]$ *preserves a fuzzy total* $B_1 - B_2$*-preorder if and only if* $F(1, \ldots, 1) = 1$, $F \gg B_2$ *and F fulfils (21).*

**Proof.** By Proposition 13 we have condition for preservation of reflexivity, by Proposition 15 we have condition for preservation of total $B_1$-connectedness and by Proposition 16 we obtain condition for preservation of $B_2$-transitivity. As a result proof for a fuzzy total $B_1 - B_2$-preorder is completed. □

By Propositions 17 and 20 and Example 4 we get the following result for preservation of a fuzzy total $B_1 - B_2$-preorder for operation $B_1$ with zero divisors (cf. Theorem 4).

**Corollary 21.** *A weighted arithmetic mean preserves a fuzzy total* $S_L - T_L$*-preorder.*

Analogously to the results for fuzzy preorders we obtain the statements for fuzzy strict orders. The difference is only in assumption of idempotent element 0 which stems from the requirement of irreflexivity property for fuzzy strict orders.

**Theorem 5.** *Let* $\operatorname{card} X \geqslant 3$, $B$: $[0, 1]^2 \to [0, 1]$ *have a zero element 0. An increasing function* $F$: $[0, 1]^n \to [0, 1]$ *preserves a fuzzy strict partial B-order if and only if* $F(0, \ldots, 0) = 0$ *and* $F \gg B$.

**Proof.** By Proposition 13 we have condition $F(0, \ldots, 0) = 0$ for preservation of irreflexivity. By Proposition 16 we get the condition $F \gg B$ for preservation of $B$-transitivity. As a result, by Definition 13 we get the required properties. □

By Proposition 17 and Example 4 we get

**Corollary 22.** *A quasi-linear mean preserves a fuzzy strict partial T-order for an appropriate nilpotent t-norm T (cf. Corollary 2), in particular a weighted arithmetic mean preserves a fuzzy strict partial* $T_L$*-order. Moreover, a weighted geometric mean preserves a fuzzy strict partial* $T_P$*-order, a function (14) preserves a fuzzy strict partial* $T_L$*-order, minimum preserves a fuzzy strict partial C-order for any fuzzy conjunction C and a weighted minimum preserves a fuzzy strict partial C-order for any t-seminorm C.*

By Propositions 13, 15 and 16 we get

**Theorem 6.** *Let* $\operatorname{card} X \geqslant 3$, $B_2$: $[0, 1]^2 \to [0, 1]$ *have a zero element 0,* $B_1$: $[0, 1]^2 \to [0, 1]$ *have a zero element 1 and be without zero divisors. An increasing function* $F$: $[0, 1]^n \to [0, 1]$ *preserves a fuzzy strict total* $B_1 - B_2$*-order if and only if* $F(0, \ldots, 0) = 0$, $F \gg B_2$ *and F fulfils (21).*

Below there is given an example of an aggregation function which preserves a fuzzy strict total $B_1 - B_2$-order for operation $B_1$: $[0, 1]^2 \to [0, 1]$ with zero divisors (cf. Theorem 6).

**Example 16.** Since a weighted arithmetic mean preserves irreflexivity (cf. Proposition 13), $T_L$-transitivity (cf. Example 4, Proposition 17), $S_L$-connectedness (cf. Example 5, Proposition 20), by Definition 13 a weighted arithmetic mean preserves a fuzzy strict total $S_L - T_L$-order.

Let $B_1 \neq B_2$, $B_1$ have a zero element 1 and be without zero divisors, $B_2$ have a zero element 0 and be without zero divisors. We should remember that if a function $F$ preserves a fuzzy strict total $B_1 - B_2$-order for such operations, then the aggregated fuzzy relation $R_F$ is a characteristic function of a crisp relation (cf. Corollary 4).

Now, we will consider preservation of fuzzy partial and total orders.

**Theorem 7.** *Let* $\operatorname{card} X \geqslant 3$, $B_1, B_2$: $[0, 1]^2 \to [0, 1]$ *have a zero element 0 and* $B_1$ *be without zero divisors. An increasing function* $F$: $[0, 1]^n \to [0, 1]$ *preserves a fuzzy partial* $B_1 - B_2$*-order if and only if* $F(1, \ldots, 1) = 1$, *F fulfils (19) and* $F \gg B_2$.

**Proof.** By Proposition 13 we have preservation of reflexivity, by Proposition 14 we have preservation of $B_1$-antisymmetry and Proposition 16 we get the preservation of $B_2$-transitivity. Finally, we have completed proof for preservation of a fuzzy partial $B_1 - B_2$-order. □

By Example 4, Example 6 and Proposition 16 we obtain

**Corollary 23.** *Let $B_1$ be an overlap function or a strict t-norm. A weighted geometric mean preserves a fuzzy partial $B_1 - T_P$-order (fuzzy partial $T_P$-order), minimum preserves a fuzzy partial $B_1 - C$-order for any fuzzy conjunction $C$ and a weighted minimum preserves a fuzzy partial $B_1 - C$-order for any t-seminorm $C$.*

Below there are given examples of aggregation functions which preserve a fuzzy partial $B_1 - B_2$-order for operation $B_1$: $[0, 1]^2 \to [0, 1]$ with zero divisors (cf. Theorem 7).

**Example 17.** Since a weighted arithmetic mean and a function (14) preserve reflexivity (cf. Proposition 13), $T_L$-antisymmetry (cf. Example 5, Proposition 19) and $T_L$-transitivity (cf. Example 4, Proposition 17), they preserve a fuzzy partial $T_L$-order (cf. Definition 14). Similarly, we may see that a weighted geometric mean preserves a fuzzy partial $T_L - T_P$-order.

**Theorem 8.** *Let* $\operatorname{card} X \geqslant 3$, $B_1$, $B_3$: $[0, 1]^2 \to [0, 1]$ *have a zero element 0 and $B_1$ be without zero divisors,* $B_2$: $[0, 1]^2 \to [0, 1]$ *have a zero element 1 and be without zero divisors. An increasing function* $F$: $[0, 1]^n \to [0, 1]$ *preserves a fuzzy total $B_1 - B_2 - B_3$-order if and only if* $F(1, \dots, 1) = 1$, $F \gg B_3$ *and F fulfils conditions (19) and (21).*

**Proof.** By Proposition 13 we have preservation of reflexivity, by Proposition 14 we have preservation of $B_1$-antisymmetry, by Proposition 15 we have preservation of $B_2$-connectedness and by Proposition 16 we get the preservation of $B_3$-transitivity. Finally, we have completed proof for preservation of a fuzzy total $B_1 - B_2 - B_3$-order. □

**Example 18.** A weighted arithmetic mean preserves a fuzzy total $T_L - S_L$-order (cf. Propositions 13, 17, 19, and 20). Note that operations $S_L$ and $T_L$ have zero divisors (cf. Theorem 8, Definitions 13 and 14).

### *5.2. Other connections between input relations and the aggregated one*

We may observe diverse dependencies between transitive fuzzy orders. Firstly, each fuzzy total order is an adequate fuzzy partial order. Moreover, a fuzzy total $B_1 - B_2 - B_3$-order is a fuzzy total $B_2 - B_3$-preorder and a fuzzy partial $B_1 - B_2$-order is a fuzzy partial $B_2$-preorder.

We will give here some samples of possible results concerning the connections of aggregated fuzzy relation $R_F$ and input relations $R_1, \dots, R_n \in \mathcal{FR}(X)$, i.e. not necessarily preservation of properties of $R_1, \dots, R_n \in \mathcal{FR}(X)$.

**Corollary 24.** *Let* $\operatorname{card} X \geqslant 3$, $B_2$: $[0, 1]^2 \to [0, 1]$ *have a zero element 0,* $B_1$: $[0, 1]^2 \to [0, 1]$ *have a zero element 1 and be without zero divisors. If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are fuzzy total $B_1 - B_2$-preorders, then for an increasing function* $F$: $[0, 1]^n \to [0, 1]$ *such that* $F(1, \dots, 1) = 1$ *and* $F \gg B_2$, *$R_F$ is a fuzzy partial $B_2$-preorder.*

**Proof.** It follows from Theorem 4 and from the fact that a fuzzy total preorder is a fuzzy partial preorder. □

For example, let $R_1, \dots, R_n \in \mathcal{FR}(X)$ be fuzzy total $S_P - T_L$-preorders, and $F$ be a weighted arithmetic mean. Thus $R_F$ is a fuzzy partial $T_L$-preorder.

**Corollary 25.** *Let* $\operatorname{card} X \geqslant 3$, $B_1$, $B_3$: $[0, 1]^2 \to [0, 1]$ *have a zero element 0 and $B_1$ be without zero divisors,* $B_2$: $[0, 1]^2 \to [0, 1]$ *have a zero element 1 and be without zero divisors. If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are fuzzy total $B_1 - B_2 - B_3$-orders, then for an increasing function* $F$: $[0, 1]^n \to [0, 1]$ *such that* $F(1, \dots, 1) = 1$, $F \gg B_3$ *and F fulfils condition (21), $R_F$ is a fuzzy total $B_2 - B_3$-preorder.*

**Proof.** It follows from Theorem 8 and by the fact that a fuzzy total order is a fuzzy total preorder. □

Applying considerations presented in Section 3.3 we will give now suitable results for fuzzy total and partial orders.

**Corollary 26.** *Let* $B_1$, $B_2$, $B_*$: $[0, 1]^2 \to [0, 1]$, $B_* \le B_1 \le B_2$. *If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are fuzzy partial $B_2$-preorders and* $F$: $[0, 1]^n \to [0, 1]$ *preserves a fuzzy partial $B_1$-preorder, then $R_F$ is a fuzzy partial $B_*$-preorder.*

**Proof.** It is a consequence of Proposition 21 twice applied. □

By Corollary 20 and condition (11) we see that for example if $R_1, \dots, R_n \in \mathcal{FR}(X)$ are fuzzy partial min-preorders and $F$ is a weighted geometric mean, then $R_F$ is a fuzzy partial $T_L$-preorder.

Moreover, similarly to Corollary 26, for total preorders we get

**Corollary 27.** *Let* $B_1, B_2, B_*, B_1', B_2', B_*' : [0,1]^2 \to [0,1]$, $B_* \le B_1 \le B_2$ *and* $B_2' \leqslant B_1' \leqslant B_*'$. *If $R_1, \dots, R_n \in \mathcal{FR}(X)$ are fuzzy total $B_2' - B_2$-preorders and* $F$: $[0, 1]^n \to [0, 1]$ *preserves a fuzzy total $B_1' - B_1$-preorder, then $R_F$ is a fuzzy total $B_*' - B_*$-preorder.*

For example, by Corollary 21, we see that if $R_1, \dots, R_n \in \mathcal{FR}(X)$ are fuzzy total $S_P$ − min-preorders and $F$ is a weighted arithmetic mean, then $R_F$ is a fuzzy total $S_L - T_L$-preorder.

## 6. Application

To find the solution alternative in decision making problems we may apply for example nondominance method [37] which may be applied to a fuzzy relation $R$ on a finite set $X$ which is a fuzzy partial min-preorder. We may also apply methods considered in [39], where also a numerical example dealing with a real-life multicriteria decision making problem (concerning water supplies in China) is given. If $R \in \mathcal{FR}(X)$ is a fuzzy partial $T_L$-preorder, then it is possible to antisymmetrize it to $R' \in \mathcal{FR}(X)$ according to the rule: if $R(x, y) \geq R(y, x)$, then $R'(x, y) = R(x, y)$ and $R'(y, x) = 0$. As a result $R'$ is a fuzzy partial $\min - T_L$-order (cf. [17,39], "fuzzy partial order"). It is possible to represent $R'$ as a triangular matrix. Due to the min-antisymmetry and $T_L$-transitivity the graph corresponding to $R'$ has no cycle. $R'$ may also have (but not necessarily [39]) a linearity property of the type: for any $x, y \in X$, $x \neq y$, either $R(x, y) > 0$ or $R(y, x) > 0$ ([17]). If $R'$, being a fuzzy partial $\min - T_L$-order, is linear in the above sense, then any $\alpha$-cut of such fuzzy linear order $R'$ is a non-fuzzy linear order. As a result there is possible a ranking of alternatives, i.e. obtaining an order of alternatives (in a crisp sense).

A fuzzy partial $T_L$-preorder is a special case of fuzzy orders considered in this paper. Moreover, many types of fuzzy orders, also having more general properties, may be treated as fuzzy partial $T_L$-preorders or after aggregation process are fuzzy partial $T_L$-preorders (cf. Corollaries 24 – 27). As a result, due to the recalled properties, there is possible the choice or ranking of the alternatives. Strict orders are not applicable to our approach of decision making problems, since they are not reflexive. Fuzzy interval orders and semiorders are not fuzzy partial $T_L$-preorders, so there is an open question how in these cases we may choose the alternatives.

We propose an algorithm where fuzzy order relations $R_1, \dots, R_n \in \mathcal{FR}(X)$ represent the opinions over alternatives. In (Step 2) of the algorithm we use dependencies presented in Propositions 10–12 and results from Section 5, in particular Corollaries 18 and 19. As a result we may fix the common type of a fuzzy order for all relations $R_1, \dots, R_n \in \mathcal{FR}(X)$ and then aggregate these relations in order to obtain the final result. Since we assume that $B \equiv 0$ belongs to the class $\mathcal{B}$, we always may fix the type of fuzzy order depending on operation $B$. For $B \equiv 0$, $B$-transitivity condition is fulfilled trivially, so with the assumption that $R$ is reflexive, the algorithm may be run to the end (however not giving an interesting result - not always guaranteeing a possibility to choose an alternative). Relations $R_1, \dots, R_n$ in this case are fuzzy partial $B$-preorders for $B \equiv 0$. In (Step 3) of the algorithm we may apply adequate function $F$, to aggregate input relations $R_1, \dots, R_n \in \mathcal{FR}(X)$ (to choose $F$ we may use the results of Sections 3.2 and 3.3, and Section 5).

**Inputs:** $X = \{x_1, \dots, x_m\}$ - a set of alternatives, $R_1, \dots, R_n \in \mathcal{FR}(X)$ fuzzy order relations, $\mathcal{F} = \{F | F : [0, 1]^n \to [0, 1]\}$ - a finite set of aggregation functions preserving fuzzy orders (including functions preserving fuzzy partial $T_L$-preorders), $\mathcal{B} = \{B | B : [0, 1]^2 \to [0, 1]\}$ - a finite set of given comparable operations including constant operations $B \equiv 0$, $B \equiv 1$.

(**Step 1**) Checking the type of a fuzzy order of each $R_k$ for $k = 1, \dots, n$
(**Step 2**) Fixing the common fuzzy order type of $R_1, \dots, R_n \in \mathcal{FR}(X)$
(**Step 3**) Aggregation of given relations $R_1, \dots, R_n \in \mathcal{FR}(X)$
(**Step 4**) Ordering the alternatives in a non-increasing way

**Output:** the aggregated fuzzy relation $R_F$ with the property stated in (Step 2) or other required property according to the choice of aggregation function $F$.

Now, we will give a numerical example illustrating the steps of the algorithm. Before we give an example it is worth to mention that in [20] it is stressed that "if the fuzzy relations are considered along with the meaning of membership grades to fuzzy relations crucial assumption is card $X \geqslant 4$. Practically a fuzzy relation makes sense only if it is meaningful to compare $R(x, y)$ to $R(z, w)$ for 4-tuples of acts $(x, y, z, w)$, that is, in the scope of preference modelling, to decide whether $x$ is preferred (or not) to $y$ in the same way or not as $z$ is preferred to $w$".

Let $X = \{x_1, x_2, x_3, x_4\}$ and be given fuzzy relations $R_1, R_2 \in \mathcal{FR}(X)$ which present preferences over alternatives $x_1$, $x_2$, $x_3$, $x_4$. These fuzzy relations are aggregated with the use of a weighted arithmetic mean $F(s, t) = 0.6s + 0.4t$, so $R_F = F(R_1, R_2)$, where

$$R_1 = \begin{bmatrix} 1 & 0.2 & 0.3 & 0.5 \\ 1 & 1 & 0.3 & 0.5 \\ 0.7 & 0.6 & 1 & 0.6 \\ 0.6 & 0.6 & 0.5 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, R_F = \begin{bmatrix} 1 & 0.12 & 0.18 & 0.3 \\ 1 & 1 & 0.18 & 0.3 \\ 0.82 & 0.76 & 1 & 0.36 \\ 0.76 & 0.76 & 0.7 & 1 \end{bmatrix}.$$

We see that $R_1$ is reflexive, $T_L$-transitive ($R \circ_{T_L} R = R$, cf. Corollary 6), totally $S_D$-connected, $C^0$-antisymmetric. While $R_2$ is reflexive, $G$-transitive (where $G$ is a geometric mean, so it as an overlap function), totally max-connected, min-antisymmetric. As a result both $R_1$ and $R_2$ are fuzzy partial $T_L$-preorders (cf. Proposition 11 and $T_L \leq G$). Since a weighted arithmetic mean preserves this type of order (cf. Theorem 3), $R_F$ is also a fuzzy partial $T_L$-preorder. This is why we may apply methods recalled at the beginning of this section, so we antisymmetrize $R_F$ to $R'_F \in \mathcal{FR}(X)$, where

$$R'_F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0.82 & 0.76 & 1 & 0 \\ 0.76 & 0.76 & 0.7 & 1 \end{bmatrix}.$$

We see that $R_F'$ has a linearity property of the type: for any $x, y \in X$, $x \neq y$, either $R_F'(x, y) > 0$ or $R_F'(y, x) > 0$. As a consequence, since $R_F'(x_4, x_3) \leqslant R_F'(x_3, x_2) \leqslant R_F'(x_2, x_1)$, we obtain a linear order of alternatives $x_4 \prec x_3 \prec x_2 \prec x_1$. In order to check if the choice is independent of the aggregation function we may use other aggregation functions (the ones preserving the given fuzzy order). It may happen that orders of alternatives will be different for other aggregation functions. In such case we may take as many aggregation functions as alternatives we have plus one. In this way we are certain that at least one of the alternatives appears twice in the first position.

## 7. Conclusions

In this paper diverse types of fuzzy orders were considered in the context of preservation of these orders in aggregation process. There were considered transitive fuzzy orders as well as non-transitive ones and their preservation in aggregation process. Moreover, other connections (than the mentioned preservation of properties) between input fuzzy relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ and the output fuzzy relation $R_F$ were given. There was also presented an algorithm which shows the application of the presented results.

For the future work it will be worth to examine how the type of a fuzzy order of an aggregated fuzzy relation $R_F$ may influence and improve the possibility to rank and choose the alternatives in decision making situations.

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## References

[1] B. Bassan, F. Spizzichino, Dependence and multivariate aging: the role of level sets of the survival function, in: System and Bayesian Reliability, Series of Quality Reliability Engineering Statistics, vol. 5, World Scientific Publishers, River Edge, NJ, 2001, pp. 229–242.
[2] G. Beliakov, S. James, T. Wilkin, Aggregation and consensus for preference relations based on fuzzy partial orders, Fuzzy Optim. Decis. Making (2016) 1–20.
[3] U. Bentkowska, J. Drewniak, R. Mesiar, B. De Baets M. Baczyński (eds.) Aggregations of *MN*-convex functions on complete lattices, Proc. 8th Internat. Summer School on Aggregation Operators (AGOP 2015), Katowice (Poland), 2015. 55–60.
[4] U. Bentkowska, A. Król, Preservation of fuzzy relation properties based on fuzzy conjunctions and disjunctions during aggregation process, Fuzzy Sets Syst. 291 (2016) 98–113.
[5] U. Bentkowska, A. Król, B. De Baets, Conjunction and disjunction based fuzzy interval orders in aggregation process, Tatra Mountains Math. Publ. 66 (2016) 13–24.
[6] J.C. Bezdek, J.D. Harris, Fuzzy partitions and relations: an axiomatic basis for clustering, Fuzzy Sets Syst. 1 (1978) 111–127.
[7] U. Bodenhofer, F. Klawonn, A formal study of linearity axioms for fuzzy orderings, Fuzzy Sets Syst. 145 (2004) 323–354.
[8] H. Bustince, J. Fernandez, R. Mesiar, J. Montero, R. Orduna, Overlap functions, Nonlinear Anal.-Theor. 72 (2010) 1488–1499.
[9] H. Bustince, M. Pagola, R. Mesiar, E. Hullermeier, F. Herrera, Grouping, overlap functions and generalized bi-entropic functions for fuzzy modeling of pairwise comparison, IEEE Trans. Fuzzy Syst. 20 (3) (2012) 405–415.
[10] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation operators: properties, classes and construction methods, in: T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators vol. 97: Studies in Fuzziness and Soft Computing, Physica-Verlag, Heildelberg, 2002, pp. 3–104.
[11] B. De Baets, R. Mesiar, T–partitions, Fuzzy Sets Syst. 97 (1998) 211–223.
[12] B. De Baets, H. De Meyer, Transitivity frameworks for reciprocal relations: cycle transitivity versus FG-transitivity, Fuzzy Sets Syst. 152 (2005) 249–270.
[13] B. De Baets, H. De Meyer, B. De Schuymer, S. Jenei, Cyclic evaluation of transitivity of reciprocal relations, Soc. Choice Welfare 26 (2006) 217–238.
[14] B. De Baets, S. Janssens, H. De Meyer, Meta-theorems on inequalities for scalar fuzzy set cardinalities, Fuzzy Sets Syst. 157 (2006) 1463–1476.
[15] S. Díaz, B. De Baets, S. Montes, On the ferrers property of valued interval orders, Top 19 (2011) 421–447.
[16] J. Drewniak, U. Dudziak, Preservation of properties of fuzzy relations during aggregation processes, Kybernetika 43 (2) (2007) 115–132.
[17] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
[18] D. Dubois, H. Prade, Weighted minimum and maximum operations in fuzzy set theory, Inf. Sci 39 (1986) 205–210.
[19] D. Dubois, H. Prade, On the use of aggregation operations in information fusion processes, Fuzzy Sets Syst. 142 (2004) 143–161.
[20] D. Dubois, The role of fuzzy sets in decision sciences: old techniques and new directions, Fuzzy Sets Syst. 184 (2011) 3–28.
[21] U. Dudziak, Preservation of t-norm and t-conorm based properties of fuzzy relations during aggregation process, September 11–13, Atlantis Press, d-side pub., University of Milano-Bicocca, Milan (Italy),. Proceedings of the 8th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT-2013), 376–383.
[22] J. Drewniak, A. Król, A survey of weak connectives and the preservation of their properties by aggregations, Fuzzy Sets Syst. 161 (2010) 202–215.
[23] P.C. Fishburn, Intransitive indifference with unequal indifference intervals, J. Math. Psych. 7 (1970) 144–149.
[24] P.C. Fishburn, Interval representations for interval orders and semiorders, J.Math. Psych 10 (1973) 91–105.
[25] J. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Acad. Publ., Dordrecht, 1994.
[26] J. Fodor, Contrapositive symmetry of fuzzy implications, Fuzzy Sets Syst. 69 (1995) 141–156.
[27] J. Fodor, S. Ovchinnikov, On aggregation of t-transitive fuzzy binary relations, Fuzzy Sets Syst. 72 (1995) 135–145.
[28] F.S. Garcia, P.G. Alvarez, Two families of fuzzy integrals, Fuzzy Sets Syst. 18 (1986) 67–81.
[29] O. Grigorenko, J. Lebedinska, On another view of aggregation of fuzzy relations, in: S. Galichet, J. Montero, G. Mauris (Eds.), Procedings of 7th Conference EUSFLAT-2011 and LFA-2011, July 18–22, d-side pub., Aix-les-Bains (France), pp. 21–27.
[30] O. Grigorenko, Involving fuzzy orders for multi-objective linear programming, Math. Modell. Anal. 17 (3) (2012) 366–382.
[31] J.K. Hodge, R.E. Klima, The mathematics of voting and elections, Am. Math. Soc.(2005).
[32] S. Jenei, Geometry of left-continuous t-norms with strong induced negations, Belg. J. Oper. Res. Statist. Comput. Sci. 38 (1998) 5–16.
[33] L. Kitainik, Fuzzy Decision Procedures with Binary Relations, Kluwer Academic, Dordrecht, 1993.
[34] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Acad. Publ., Dordrecht, 2000.
[35] R.D. Luce, Semiorders and a theory of utility discrimination, Econometrica 24 (1956) 178–191.
[36] K.C. Maes, B. De Baets, Rotation-invariantt-norms: where triple rotation and rotation-annihilation meet, Fuzzy Sets Syst. 160 (2009) 1998–2016.

[37] S.A. Orlovsky, Decision-making with a fuzzy preference relation, Fuzzy Sets Syst. 1 (3) (1978) 155–167.
[38] M. Öztürk, A. Tsoukiàs, A valued ferrers relation for interval comparison, Fuzzy Sets Syst. 266 (2015) 47–66.
[39] V. Peneva, I. Popchev, Properties of the aggregation operators related with fuzzy relations, Fuzzy Sets Syst. 139 (2003) 615–633.
[40] P. Perny, M. Roubnes, Fuzzy preference modeling, in: R. Słowiński (Ed.), Fuzzy sets in decision analysis, operations, research and statistics, Kluwer Acad. Publ., Dordrecht, 1998, pp. 3–30.
[41] A. Pradera, G. Beliakov, H. Bustince, B. De Baets, Areview of the relation ships between implication, negation and aggregation functions from the point of view of material implication, Inf. Sci. 329 (2016) 357–380.
[42] E.A. Robinson, D.H. Ullman, A Mathematical Look at Politics, CRC Press, 2011.
[43] D.G. Saari, Chaotic elections, A mathematician looks at voting American Mathematical Society, 2001.
[44] S. Saminger, R. Mesiar, U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, Internat. J. Uncertain., Fuzziness, Knowl.-Based Syst. 10 (Suppl.) (2002) 11–35.
[45] A.K. Sen, Social choice theory, in: K. Arrow, M. Intrilligator (Eds.), Handbook of Mathematical Economics, Elsevier, Amsterdam, 1986, pp. 1173–1181. Chapter 22
[46] B. Schweizer, A. Sklar, Associative functions and statistical triangle inequality, Publ. Math. Debrecen 8 (1961) 169–186.
[47] B. Schweizer, A. Sklar, Probabilistic metric spaces, north holland, new york, 1983.
[48] J. Sobera, Aggregation of $*$-transitive fuzzy relations by quasi -linear means, in: Recent Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Genaralized Nets and Related Topics, Vol. I: Foundations, Polish Academy of Sciences, Warsaw, 2011, pp. 209–222.
[49] P. Venugopalan, Fuzzy ordered sets, Fuzzy Sets Syst. 46 (1992) 221–226.
[50] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.
[51] L.A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.
[52] H.J. Zimmermann, P. Zysno, Decisions and evaluations by hierarchical aggregation of information, Fuzzy Sets Syst. 10 (1983) 243–260.