The collapsing method of defuzzification for discretised interval type-2 fuzzy sets

Sarah Greenfield, Francisco Chiclana, Simon Coupland *, Robert John

Centre for Computational Intelligence, School of Computing, De Montfort University, Leicester, LE1 9BH, United Kingdom

ARTICLE INFO

ABSTRACT

This paper proposes a new approach for defuzzification of interval type-2 fuzzy sets. The collapsing method converts an interval type-2 fuzzy set into a type-1 *representative embedded set (RES)*, whose defuzzified values closely approximates that of the type-2 set. As a type-1 set, the RES can then be defuzzified straightforwardly. The novel *representative embedded set approximation (RESA)*, to which the method is inextricably linked, is expounded, stated and proved within this paper. It is presented in two forms: *Simple RESA:* this approximation deals with the most simple interval FOU, in which a vertical slice is discretised into 2 points. *Interval RESA:* this approximation concerns the case in which a vertical slice is discretised into 2 or more points. The collapsing method (simple RESA version) was tested for accuracy and speed, with excellent results on both criteria. The collapsing method proved more accurate than the Karnik–Mendel iterative procedure (KMIP) for an asymmetric test set. For both a symmetric and an asymmetric test set, the collapsing method outperformed the KMIP in relation to speed.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

1.1. Type-2 fuzzy inferencing systems

This paper concerns itself with the process of defuzzification in type-2 fuzzy inferencing systems (FISs). Type-2 FISs employ type-2 fuzzy sets, which were originally proposed by Zadeh [25]; their advantage over type-1 fuzzy sets is their ability to model second-order uncertainties [7]. There is now a significant body of theoretical research on the properties of type-2 fuzzy sets (e.g. [18,19,5,10,14,16,23]), and many applications have been developed (e.g. [17,26,8,6]).

An FIS of any type is a computerised system that uses fuzzy sets and rules to support decision making. FISs are very versatile and have been developed for a wide variety of applications [4,6,12]. There are five main stages to any FIS: fuzzification, antecedent computation, implication, aggregation and defuzzification. In the case of a type-2 FIS (where at least one fuzzy set is type-2), defuzzification consists of two parts – type-reduction and defuzzification proper, as shown in Fig. 1. Type-reduction is the procedure by which a type-2 fuzzy set is converted to a type-1 fuzzy set, known as the type-reduced set (TRS). The TRS is then easily defuzzified to give a crisp number. When implementing a type-2 FIS researchers discretise the membership functions. The work presented here is concerned only with defuzzification of *discretised* type-2 fuzzy sets.

1.2. Defuzzification

The defuzzification techniques available for discretised type-2 sets are:

---

* Corresponding author. Tel.: +44 116 207 8419; fax: +44 116 207 8159.
E-mail addresses: sarahg@dmu.ac.uk (S. Greenfield), chiclana@dmu.ac.uk (F. Chiclana), simonc@dmu.ac.uk (S. Coupland), rij@dmu.ac.uk (R. John).
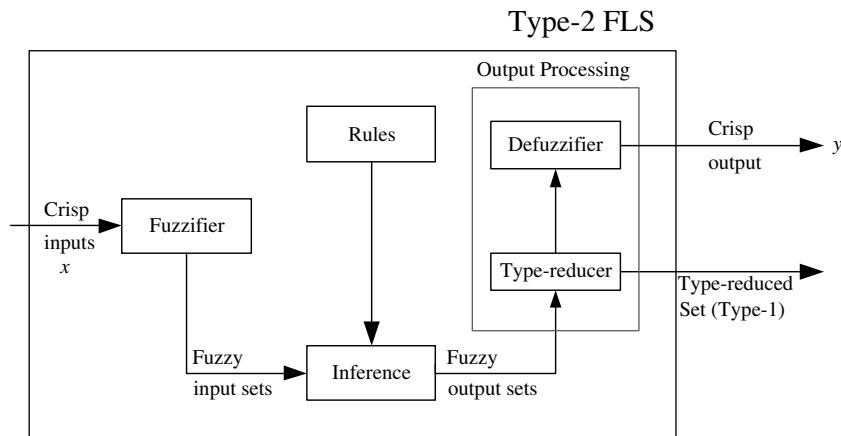
Type-2 FLS

Fig. 1. Type-2 FIS (from [11]).

Exhaustive defuzzification: As we have seen, for a type-2 FIS, the defuzzification stage consists of two parts, type-reduction and defuzzification proper. This is the type-reduction algorithm originally described by Mendel [11, pp. 248–252].

1. All possible type-2 embedded sets [11, Definitions 3–10, p. 98] are enumerated.
2. For each embedded set the minimum secondary membership grade is found.
3. For each embedded set the domain value of the type-1 centroid of the type-2 embedded set is calculated.
4. For each embedded set the secondary grade is paired with the domain value to produce a set of ordered pairs (x,z). It is possible that for some values of x there will be more than one corresponding value of z.
5. For each domain value, the maximum secondary grade is selected. This creates a set of ordered pairs $(x,zMax)$ such that there is a one-to-one correspondence between x and $zMax$. This completes the type-reduction of the type-2 set to the type-1 TRS.

(In the interval case, since the minimum secondary grade is invariably 1, the algorithm reduces to a simple calculation of the mean of the domain values.) The resultant TRS, as with any type-1 fuzzy set, is readily defuzzified by finding its centroid.Thus type-reduction involves the processing of *all* the embedded sets within the type-2 set. This is why we term the procedure 'exhaustive defuzzification'. Embedded sets are very numerous. For instance, when a prototype type-2 FIS performed an inference using sets which had been discretised into 51 slices across both the x and y-axes, the number of embedded sets in the aggregated set was calculated to be in the order of 2.9×10^{63}. Though individually easily processed, embedded sets in their totality give rise to a processing bottleneck simply by virtue of their high cardinality. As a consequence, exhaustive defuzzification is a technique that is impractical to implement.

Karnik–Mendel iterative procedure: the most widely adopted method for type-reducing an interval type-2 fuzzy set is the Karnik–Mendel iterative procedure (KMIP) [7]. The result of type-reduction of an interval type-2 fuzzy set is an interval type-1 set where the centroid lies between the two endpoints. The iterative procedure is an efficient method for finding these endpoints. The centroid of the type-1 set (i.e. the defuzzified value of the type-2 set) is taken to be the centre of this interval. Greenfield and John [3] have proposed an extension of this procedure to generalised type-2 fuzzy sets.

Sampling method: the sampling method of defuzzification [2] is an efficient, cut-down alternative to exhaustive defuzzification. In this technique, only a relatively small random sample of the totality of embedded sets is processed. The resultant defuzzified value, though surprisingly accurate, is nonetheless an approximation.

The use of uncertainty bounds for defuzzification of a type-2 interval set is an interesting approach. In [24], Wu and Mendel provided a closed form approximation of the centroid of a type-2 interval fuzzy set. Other researchers [21,22] have developed a novel approach using uncertainty measures for implementing interval type-2 fuzzy logic in control applications and applied this approach in a number of applications.

Both the sampling method and the collapsing method introduced in this paper are inspired by the goal of making exhaustive defuzzification computationally simpler.

For continuous generalised type-2 fuzzy sets, Coupland and John [1] proposed a geometric approach to defuzzification. However the research here concentrates solely on the discretised case.

1.3. Reversal of blurring

Mendel and John [10, p. 118] describe the transformation from a type-1 to a type-2 membership function by a process of blurring:

"Imagine blurring the type-1 membership function [...] by shifting the points [...] either to the left or the right, and not necessarily by the same amounts, [...]. Then, at a specific value of x, say x', there no longer is a single value for the mem-

bership function (u'); instead the membership function takes on values wherever the vertical line [$x = x'$] intersects the blur. These values need not all be weighted the same; hence, we can assign an amplitude distribution to all of these points. Doing this for all $x \in X$, we create a three-dimensional membership function – a type-2 membership function – that characterizes a type-2 fuzzy set."

The collapsing method of defuzzification is a response to the challenge of reversing blurring.

1.4. Embedded sets

An interesting feature of an FIS is that embedded sets only appear during the final defuzzification stage (though theoretically they could be employed in the earlier stages, but to do so would be impractical). A defuzzification technique that reversed blurring would obviate the need for processing embedded sets, which would be of enormous practical advantage, as well as being satisfying from a theoretical perspective.

Though the collapsing algorithm does not require processing of embedded sets, it involves the creation, and subsequent defuzzification, of a single type-1 set which is embedded. Moreover, the embedded set concept is used in the proof of the results (Sections 3.2 and 4.3) upon which the method is based.

1.5. Overview and preliminaries

We propose a straightforward, iterative defuzzification technique for a discretised interval type-2 fuzzy set. We approach this result in two stages:

Simple RES: we consider the simple interval case, in which each vertical slice consists of two points, corresponding to the lower membership function and the upper membership function (Sections 2 and 3).
Interval RES: we then consider the more complex interval case, whereby each vertical slice consists of a finite number of points, whose primary membership grades are not necessarily evenly spaced (Section 4).

The discussion in this article relies on certain assumptions and definitions.

Defuzzification method: the analysis presented in this paper presupposes that the centroid method of defuzzification [9, p. 336] is adopted.
Scalar cardinality: the concept of *scalar cardinality* is frequently encountered in the following analysis. For type-1 fuzzy sets, Klir and Folger [8, p. 17] define scalar cardinality as follows:

Definition 1 (*Scalar cardinality*). The *scalar cardinality* of a fuzzy set *A* defined on a finite universal set *X* is the summation of the membership grades of all the elements of *X* in *A*. Thus,

$$|A| = \sum_{x \in X} \mu_A(x).$$

To distinguish scalar cardinality from cardinality in the classical sense of the number of members of a set, we adopt the '$\| \ \|$' symbol for scalar cardinality, i.e. $\|A\|$ represents the scalar cardinality of *A*.

Discretisation: for type-2 sets there is more than one method of discretisation. Two useful techniques are the standard method and the grid method.

Standard method: In this discretisation technique, the domain of the type-2 fuzzy set is sliced vertically at even intervals. The discretisation tends to be coarse, e.g. a slice separation of 0.1, giving rise to 11 slices. Each of the slices generated intersects the FOU[1]; each line of intersection (within the FOU) is itself sliced the same number of times, at even intervals parallel to the $x - z$ plane. This results in different co-domain separations of slices, according to the vertical slice.
Grid method: The grid method of discretisation is a simple procedure in which the $x - y$ plane is evenly divided into a rectangular grid. The fuzzy set surface, consisting of the secondary membership grades corresponding to each grid point (x,y) in the FOU, may be represented by a matrix of the secondary grades, in which the x and y co-ordinates are implied by the secondary grade's position within the matrix.
The collapsing method of defuzzification is compatible with either discretisation style.

2. Collapsing as reverse blurring: the representative embedded set

In this section we introduce the idea of the representative embedded set (RES), a concept which is inextricably linked to the collapsing method of defuzzification that is the subject of this paper.

---

[1] FOU stands for 'footprint of uncertainty', the projection of the type-2 fuzzy set surface onto the $x - y$ plane [10].

Consider an *interval* type-2 fuzzy set discretised such that every secondary membership function has 2 points. It is helpful to think of the interval type-2 fuzzy set as a *blurred* type-1 membership function [10, p. 118]. The *collapsing technique* is the reversal of this blurring process in order to derive a type-1 fuzzy set from an interval type-2 fuzzy set. The type-1 set's membership function is calculated so that its defuzzified value is equal to that of the interval type-2 fuzzy set. It is a simple matter to defuzzify a type-1 set, and to do so would be to find the defuzzified value of the original interval type-2 fuzzy set. Hence the collapsing process reduces the computational complexity of interval type-2 defuzzification. We term this special type-1 set the *'representative embedded set'*. It is a *representative set* because it has the same defuzzified value as the original interval type-2 fuzzy set. It is an embedded set because it lies within the footprint of uncertainty (FOU) of the interval type-2 fuzzy set.

We formally define the concepts of a representative set and a representative embedded set.

Definition 2 (*Representative set*). Let $\widetilde{F}$ be a type-2 fuzzy set with defuzzified value $X_{\widetilde{F}}$. Then type-1 fuzzy set R is a *representative set (RS)* of $\widetilde{F}$ if its defuzzified value (X_R) is equal to that of $\widetilde{F}$, i.e. $X_R = X_{\widetilde{F}}$.

Definition 3 (*Representative embedded set*). Let $\widetilde{F}$ be a type-2 fuzzy set with defuzzified value $X_{\widetilde{F}}$. Then type-1 fuzzy set R is a *representative embedded set (RES)* of $\widetilde{F}$ if it is a RS of $\widetilde{F}$ *and* its membership function lies within the FOU of $\widetilde{F}$.

We now consider the simple case of an interval type-2 set whereby the vertical slices are discretised into more than two points.

3. RES of an interval type-2 fuzzy set with secondary membership function discretised into two points (simple RES)

An interval type-2 fuzzy set is a type-2 fuzzy set in which every secondary membership grade takes the value 1. Because of this, such a set is completely specified by its footprint of uncertainty [10, Definition 5, p. 119]. In the analysis which follows, to speak in terms of the FOU of an interval type-2 fuzzy set is equivalent to referring to the interval type-2 fuzzy set itself. It is assumed that the domain is discretised into an arbitrary number m of vertical slices, and the co-domain into 2 slices, which are the end-points of the secondary domain.

The objective of this analysis is to derive an expression for the membership function of an RES in terms of the upper and lower membership functions of the interval set to be defuzzified. We term this RES the 'Simple RES'. Our strategy is two-stage:

1. We derive a formula for the special case of the interval FOU which has only one blurred vertical slice. we call this the 'simple solitary collapsed slice lemma.'
2. We generalise this formula to the typical interval FOU with secondary membership function having two points, one corresponding to the upper membership function (U) and the other one to the lower membership function (L). We call this the 'simple representative embedded set approximation'. It is an approximation because not every embedded set is taken into account in deriving the RES.

3.1. Simple solitary collapsed slice lemma

In this subsection we concentrate on the derivation of the RES for a special case of an interval FOU formed by (upwardly) blurring the membership function of a type-1 fuzzy set (A) at a single domain value x_I (Fig. 2), to create a vertical slice which is an interval as opposed to a point. This interval is the secondary domain at x_I. The FOU formed by this blurring is depicted in Fig. 3, and consists of the shaded triangular region plus the line A. We derive a formula for the membership function of the RES of this somewhat unusual interval FOU, in terms of the original type-1 membership function and the amount of blurring. Let A be a non-empty type-1 fuzzy set that has been discretised into m vertical slices (at $x_1, x_2, \ldots, x_m$). We calculate X_A, the defuzzified value of A, by finding the centroid of A:

$$X_A = \frac{\sum_{i=1}^{i=m} \mu_A(x_i) x_i}{\sum_{i=1}^{i=m} \mu_A(x_i)} = \frac{\sum_{i=1}^{i=m} \mu_A(x_i) x_i}{\|A\|}.$$

Now suppose the membership function of A is blurred upwards at domain value x_I, so that x_I, instead of corresponding to the point $\mu_A(x_I)$, corresponds to the co-domain range $[\mu_A(x_I), \mu_A(x_I) + b_I]$. Let B (Fig. 2) be a type-1 fuzzy set whose membership function is the same as that of A apart from at the point x_I, for which $\mu_B(x_I) = \mu_A(x_I) + b_I$. X_B, the defuzzified value of B, may be calculated:

$$X_B = \frac{\sum \mu_B(x_i) x_i}{\sum \mu_B(x_i)} = \frac{\sum \mu_A(x_i) x_i + b_I x_I}{\sum \mu_A(x_i) + b_I} = \frac{\|A\| X_A + b_I x_I}{\|A\| + b_I} = X_A + \frac{b_I(x_I - X_A)}{\|A\| + b_I}.$$

Let $\widetilde{F}$ (Fig. 3) be an interval type-2 fuzzy set whose lower membership function is A and upper membership function is B. Exhaustive defuzzification (Section 1.2) requires that *all* the embedded sets of a type-2 fuzzy set be processed to form the type-reduced set. $\widetilde{F}$ contains only two embedded sets, namely A and B. Therefore, we find the defuzzified value of $\widetilde{F}$ by
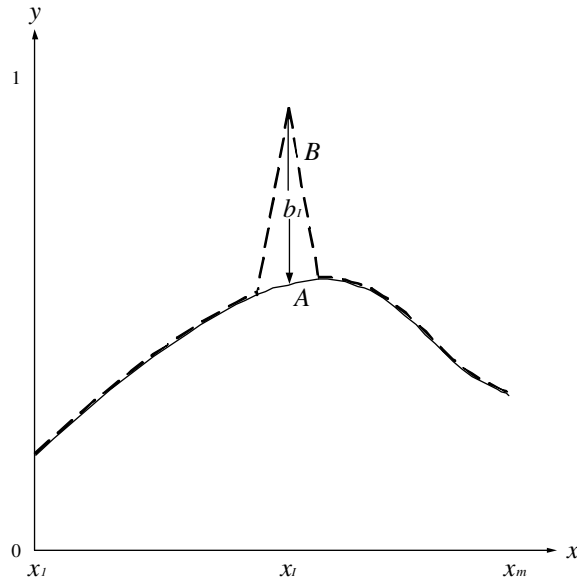
Fig. 2. At $x = x_I$ the membership function of type-1 fuzzy set A has been blurred, increasing the membership grade by the amount b_I, creating a new type-1 fuzzy set B.
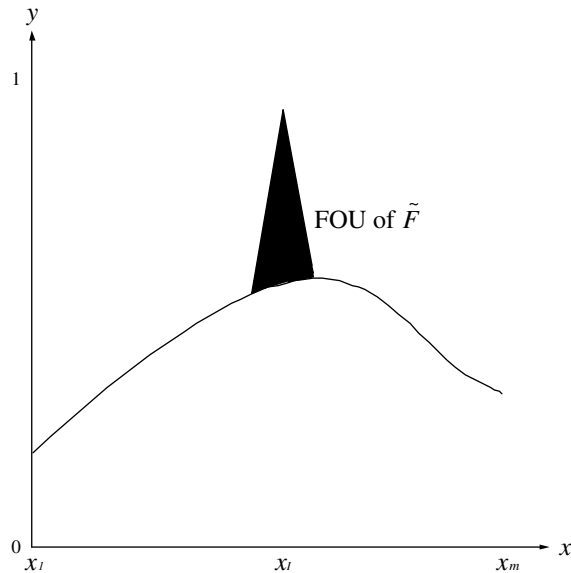

Fig. 3. FOU of interval type-2 fuzzy set $\widetilde{F}$, which consists of the original line of type-1 set A plus the triangular region.

calculating the mean of X_A and X_B, i.e. $\frac{1}{2}(X_A + X_B)$. Let $X_{\widetilde{F}}$ be the defuzzified value of $\widetilde{F}$. $X_{\widetilde{F}}$ will be expressed in terms of $\|A\|$, X_A, x_I and b_I, all of which are known values:

$$X_{\widetilde{F}} = \frac{1}{2}(X_A + X_B) = \frac{1}{2}\left(X_A + X_A + \frac{b_I(x_I - X_A)}{\|A\| + b_I}\right) = X_A + \frac{b_I(x_I - X_A)}{2(\|A\| + b_I)}.$$

Let R be the RES of $\widetilde{F}$ such that the membership function of R is the same as that of A for all domain values x_i apart from x_I. At this point the membership function deviates from that of A so that $\mu_R(x_I)$ takes the value $\mu_A(x_I) + r_I$. Fig. 4 depicts the membership function of R. Following the same chain of reasoning as in the derivation of X_B, we work out an expression for X_R in terms of $\|A\|$, X_A, x_I and r_I, where $\|A\|$, X_A, x_I are known values:
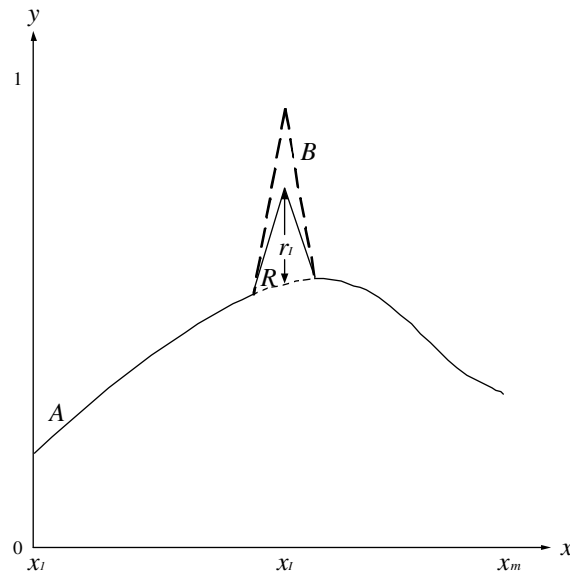
$$X_R = X_A + \frac{r_I(x_I - X_A)}{\|A\| + r_I}.$$

Fig. 4. *R*, the representative embedded set of $\widetilde{F}$, is indicated by the undashed line.

The defuzzified values X_R and $X_{\widetilde{F}}$ are by definition equal, and by equating these values we are able to obtain a formula for r_I in terms of $\|A\|$ and b_I:

$$X_R = X_{\widetilde{F}} \Rightarrow r_I = \frac{b_I \|A\|}{2\|A\| + b_I}.$$

We have arrived at the membership function of *R*, and in so doing proved the Simple Solitary Collapsed Slice Lemma (Simple SCSL):

Lemma 4 (Simple solitary collapsed slice lemma). *Let A be a non-empty discretised type-1 fuzzy set which has been blurred upwards by amount b_I at a single point x_I to form the FOU of interval type-2 fuzzy set $\widetilde{F}$. Then R, the RES of $\widetilde{F}$, has a membership function such that*

$$\mu_R(x_i) = \begin{cases} \mu_A(x_i) + \frac{\|A\|b_I}{2\|A\| + b_I} & \text{if } i = I, \\ \mu_A(x_i) & \text{otherwise}. \end{cases}$$

3.2. Simple representative embedded set approximation

We extend the simple solitary collapsed slice lemma to the typical situation in which every point of the type-1 membership function has been blurred. First we present the concept behind the approximation: How an interval type-2 fuzzy set may be collapsed to create an approximation to an RES.

In Section 3.1, we have considered an extremely atypical interval FOU whose membership function follows the course of a type-1 fuzzy set apart from at one point x_I, at which its membership grade opens up to form a secondary domain $[\mu(x_I), \mu(x_I) + b_I]$. We have done this to provide a simple yet illustrative example of the collapsing process, as a basis for generalisation to the typical interval FOU. The simple SCSL (Section 3.1) tells us how to calculate the RES for this special case of an interval type-2 set.

Now we proceed to look at the typical interval FOU, in which the upper membership grade is greater than the lower membership grade at a minimum of 2 points. The difference between the lower and upper membership grades at any given point is the amount of blur (b_i) at that point, i.e. $\mu_U(x_i) - \mu_L(x_i) = b_i$. The Simple SCSL does not apply in this situation. However, this lemma may be applied repeatedly to FOUs assembled in stages using slices taken from the interval set.

3.2.1. Collapsing the 1st FOU to form RES R_1

The first interval FOU to be collapsed (Fig. 5) comprises the slice at x_1, plus the rest of the lower membership function *L*, (represented by the shaded triangular region plus the line *L*). The lower membership function of the FOU is the line *L*, and the upper membership function starts (at x_1) at the line *U*, but immediately descends to *L* (slice x_2), after which it follows the course of *L* (slices $x_2, \ldots, x_m$). The Simple SCSL tells us that this interval set may be collapsed into its RES R_1, depicted in Fig. 6. The collapse increases the membership grade $\mu_L(x_1)$ by r_1 to $\mu_{R_1}(x_1)$.
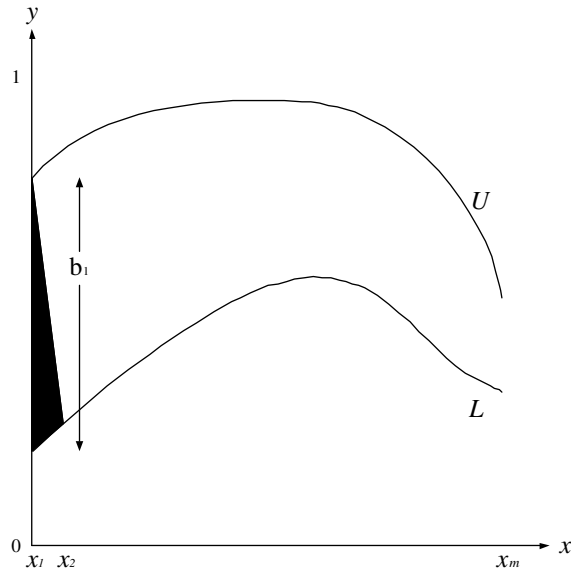
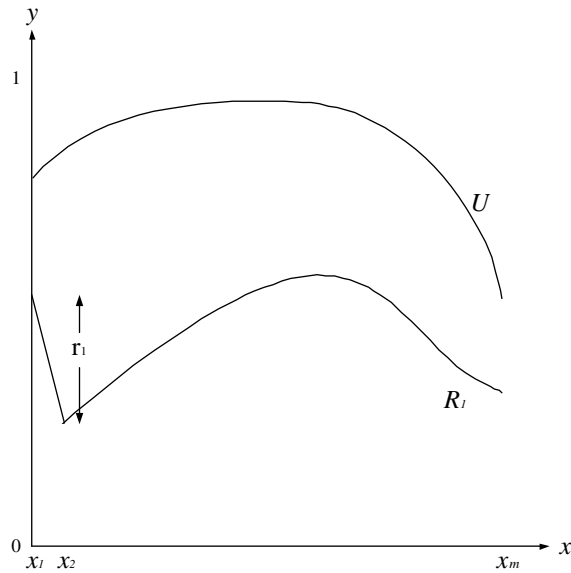**Fig. 5.** The first slice in interval type-2 fuzzy set $\widetilde{F}$.

Fig. 6. The first slice collapsed, creating an RES R_1 for the interval type-2 fuzzy set $\widetilde{F}$.

3.2.2. Collapsing the 2nd FOU to form RES R_2

We now move on to the second FOU. Fig. 7 shows this FOU before it is collapsed. The Simple SCSL is re-applied, but instead of the lower membership function being L, it is now R_1. The RES of the second FOU is R_2, which is depicted in Fig. 8.

3.2.3. Collapsing the $(k + 1)$th FOU to form RES $R_{(k+1)}$

Suppose FOUs $1,\ldots,k$ have been collapsed in turn, with R_k being the most recently formed RES. Then it is the turn of the $(k + 1)$th FOU to be collapsed. The lower membership function is R_k. This situation prior to the $(k + 1)$th FOU's collapse is represented in Fig. 9; the situation after the collapse in Fig. 10.

3.2.4. Collapsing the mth FOU to form an approximation for the RES of the entire interval type-2 fuzzy set

Suppose FOUs $1,\ldots,m - 1$ have been collapsed in turn, with R_{m-1} being the most recently formed RES. Then it is the turn of the mth FOU to be collapsed. The lower membership function is R_{m-1}, and the slice to be collapsed is slice m at x_m. After the
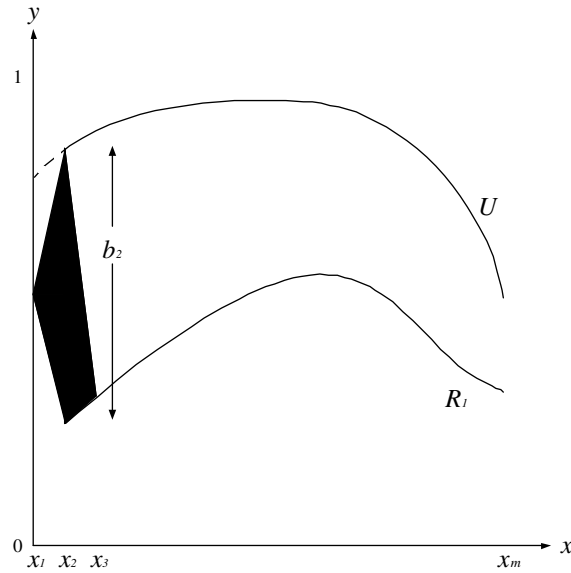
Fig. 7. For the interval type-2 fuzzy set $\widetilde{F}$, the first slice is collapsed, and the second slice is shown.
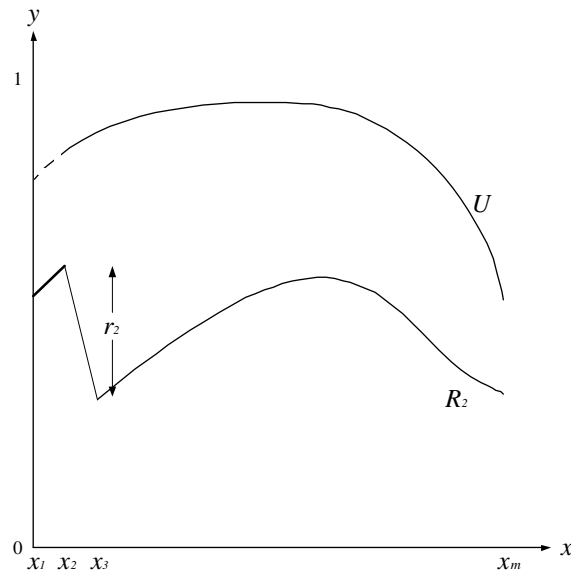

Fig. 8. Slices 1 and 2 collapsed, creating RES R_2 for the interval type-2 fuzzy set $\widetilde{F}$.

collapse the new lower membership function is R_m. As the mth slice is the final slice, then R_m is the RES (R) of the FOU of R_m. The RES of R_m is an approximation to the RES of $\widetilde{F}$. The RES of R_m is an approximation to the RES of the original type-2 fuzzy set because not all the embedded sets have been taken into account simultaneously. (Table 1 records test results showing the collapsing method to be a close approximation.)

We now state and prove the simple representative embedded set approximation (simple RESA).

Theorem 5 (Simple representative embedded set approximation). *The membership function of the embedded set R derived by dynamically collapsing slices of a discretised type-2 interval fuzzy set $\widetilde{F}$, having lower membership function L, and upper membership function U, is*
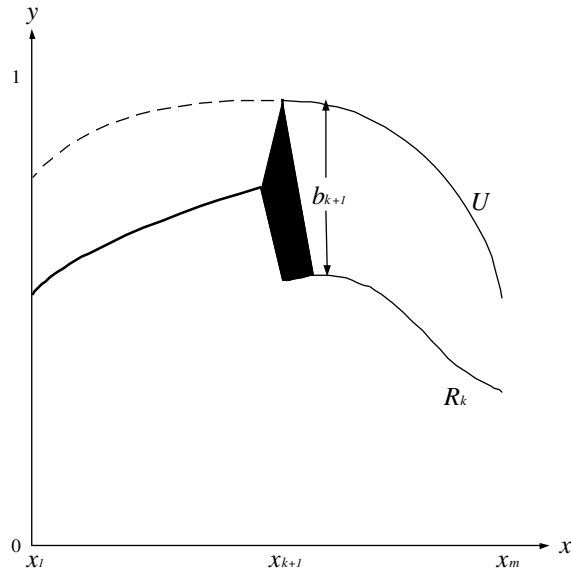
$$\mu_R(x_i) = \mu_L(x_i) + r_i$$

Fig. 9. Slices 1 to k collapsed, slice $k + 1$ about to be collapsed, for interval type-2 fuzzy set $\widetilde{F}$.
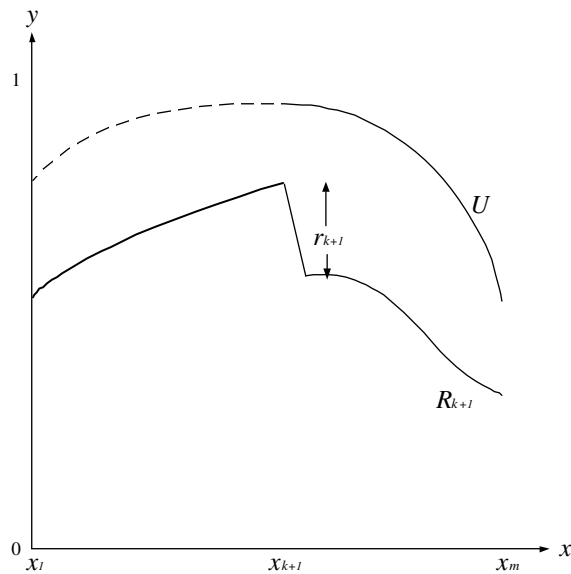
Fig. 10. Slices 1 to $k + 1$ collapsed, creating RES R_{k+1} for interval type-2 fuzzy set $\widetilde{F}$.

with

$$r_i = \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}$$

and $b_i = \mu_U(x_i) - \mu_L(x_i), r_0 = 0$.

Proof. Proof by induction on the number of collapsing vertical slices will be used. Let R_1 be the type-1 fuzzy set formed by collapsing slice 1, R_2 by collapsing slices 1 and 2, and R_i by collapsing slices 1–i. R_m is the approximate RES, R, of $X_{\widetilde{F}}$.

Basis (collapsing the 1st slice to form R_1): Figs. 5 and 6 depict the collapse of the first slice. The resultant RES is R_1. For R_1, $i = 1$, and $\sum_1^{i-1} r_j = 0$. We need to prove that

Table 1
Defuzzified values obtained for the symmetrical triangular test set

Number of slices	Defuzzified value	Karnik–Mendel iterative procedure	Collapsing method
11	0.4	0.39999999999999900000	0.40013590914337500000
21	0.4	0.40000000000000000000	0.40005971885412300000
51	0.4	0.40000000000000000000	0.40002308062313800000
101	0.4	0.39999999999999900000	0.40001153259271900000
201	0.4	0.39999999999999900000	0.40000577734692600000
501	0.4	0.40000000000000000000	0.40000231530285900000
1001	0.4	0.39999995864537600000	0.40000115852345500000
10,001	0.4	0.39999999999999900000	0.40000011593706100000
100,001	0.4	0.39999999999482100000	0.40000001159458700000

$$\mu_{R_1}(x_1) = \mu_L(x_1) + \frac{\|L\|b_1}{2\|L\| + b_1}$$

but this is actually what we have when we apply the simple SCSL for $i = 1$.

Induction hypothesis: assume the theorem is true for R_k, i.e. that slices $1, \ldots, k$ have been collapsed to form type-1 fuzzy set R_k, and that

$$\mu_{R_k}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}$$

(In this formula, for $i > k$, $b_i = 0$.)

Induction step: now we collapse slice $(k + 1)$, which is a single slice. Applying the solitary collapsed slice lemma to R_k we have

$$r_{k+1} = \frac{\|R_k\|b_{k+1}}{2\|R_k\| + b_{k+1}}.$$

We need to prove that $\forall i$

$$\mu_{R_{k+1}}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}.$$

The proof will be split into three cases.

Case 1: $1 \leqslant i \leqslant k$

$$\mu_{R_{k+1}}(x_i) = \mu_{R_k}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}.$$

Case 2: $i = k + 1$

$$\mu_{R_{k+1}}(x_i) = \mu_L(x_i) + r_{k+1} = \mu_L(x_i) + \frac{\|R_k\|b_i}{2\|R_k\| + b_i}.$$

We know that

$$\|R_k\| = \sum_{j=1}^{m} \mu_{R_k}(x_j) = \sum_{j=1}^{k} (\mu_L(x_j) + r_j) + \sum_{j=k+1}^{m} \mu_L(x_j) = \|L\| + \sum_{j=1}^{k} r_j$$

and therefore we conclude that

$$\mu_{R_{k+1}}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{k} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{k} r_j\right) + b_i}.$$

Case 3: $i > k + 1$

$$\mu_{R_{k+1}}(x_i) = \mu_{R_k}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}.$$

Conclusion: Drawing the three cases together, we conclude that $\forall i$,

$$\mu_{R_{k+1}}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}. \qquad \square$$

4. Interval representative embedded set

In this section, we shall derive the RES for an interval FOU, $\widetilde{F}$, formed by (upwardly) blurring the membership function of a type-1 fuzzy set (A) at a single domain value x_I, to create a vertical slice which is an interval as opposed to a single point ($\mu_A(x_I)$), discretised with $n(n \geqslant 2)$ points B_0 ($=\mu_A(x_I)$), $B_1, B_2, \ldots, B_{n-1}$ at distance b_0 (=0), $b_1, b_2, \ldots, b_{n-1}$ from $\mu_A(x_I)$ (Fig. 11). We seek an approximate formula for the membership function of the RES of this interval FOU, in terms of the original type-1 membership function and $b_1, b_2, \ldots, b_{n-1}$.

Exhaustive defuzzification requires that *all* the embedded sets of a type-2 fuzzy set be processed to form the type-reduced set. $\widetilde{F}$ contains n embedded sets, namely A ($=B_0$), $B_1, B_2, \ldots, B_{n-1}$. We therefore find the defuzzified value of $\widetilde{F}$ by calculating the mean of X_A and $X_{B_1}, X_{B_2}, \ldots, X_{B_{n-1}}$, where X_{B_i} is the defuzzified value of B_i. I.e. $X_{\widetilde{F}} = \frac{1}{n}(X_A + X_{B_1} + X_{B_2} + \cdots + X_{B_{n-1}})$.

4.1. Interval solitary collapsed slice lemma

Let R be the RES of $\widetilde{F}$ such that the membership function of R is the same as that of A for all domain values x_i apart from x_I. At this point the membership function deviates from that of A so that $\mu_R(x_I)$ takes the value $\mu_A(x_I) + r_I$. From Section 3.1 we have

$$X_R = X_A + \frac{r_I(x_I - X_A)}{\|A\| + r_I}$$

and

$$X_{B_i} = X_A + \frac{b_i(x_I - X_A)}{\|A\| + b_i} \quad \forall i = 1, \ldots, n-1.$$

We know that

$$X_R = \frac{1}{n}(X_A + X_{B_1} + X_{B_2} + \cdots + X_{B_{n-1}})$$

from which we obtain

$$X_A + \frac{r_I(x_I - X_A)}{\|A\| + r_I} = \frac{1}{n} \sum_{i=0}^{n-1} \left(X_A + \frac{b_i(x_I - X_A)}{\|A\| + b_i}\right)$$
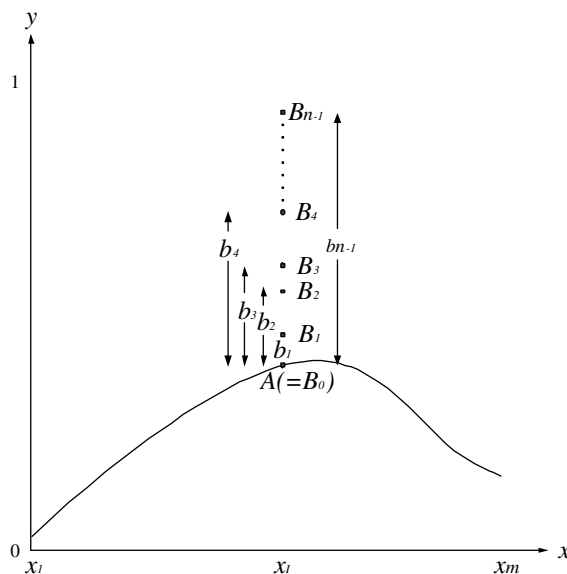

Fig. 11. A vertical slice, discretised into more than two co-domain points.

and simplifying

$$\frac{r_I}{\|A\| + r_I} = \frac{1}{n} \sum_{i=0}^{n-1} \frac{b_i}{\|A\| + b_i}.$$

Denoting $C = \sum_{i=0}^{n-1} \frac{b_i}{\|A\| + b_i}$ we get

$$r_I = \frac{C \cdot \|A\|}{n - C}.$$

Using the notation: $w_i = \frac{1}{\|A\| + b_i}$ and $\bar{w}_i = \frac{w_i}{\sum_{i=0}^{n-1} w_i}$ we finally arrive at

$$r_I = \sum_{i=0}^{n-1} \bar{w}_i b_i.$$

This result is the simple SCSL generalised for the case where the secondary membership function is discretised into more than two points. We call it the interval solitary collapsed slice lemma (interval SCSL):

Lemma 6 (Interval solitary collapsed slice lemma). *Let $\widetilde{F}$ be the interval FOU formed by (upwardly) blurring the membership function of a type-1 fuzzy set (A) at a single domain value x_I, to create a vertical slice which is an interval as opposed to a single point $(\mu_A(x_I))$, discretised with $n(n \geqslant 2)$ points $B_0^I, B_1^I, B_2^I, \ldots, B_{n-1}^I$ with primary membership grades at distances $b_0^I(=0), b_1^I, b_2^I, \ldots, b_{n-1}^I$ from $\mu_L(x_I)$. Then R, the RES of $\widetilde{F}$, has a membership function such that*

$$\mu_R(x_j) = \begin{cases} \mu_A(x_I) + r_I & \text{if } j = I, \\ \mu_A(x_j) & \text{otherwise}, \end{cases}$$

where $r_I = \sum_{i=0}^{n-1} \bar{w}_i^I \cdot b_i^I$, $\bar{w}_i^I = \frac{w_i^I}{\sum_{i=0}^{n-1} w_i^I}$, and $w_i^I = \frac{1}{\|A\| + b_i^I}$.

4.2. The interval SCSL as a generalisation of the simple SCSL

In the following we show that Lemma 6, the interval solitary collapsed slice lemma, generalises Lemma 4, the simple solitary collapsed slice lemma.

In the simple case we have at a single domain value x_I, $n = 2$ points B_0^I, B_1^I, with primary membership grades at distances $b_0^I = 0$, $b_1^I = b_I = \mu_U(x_I) - \mu_L(x_I)$ from $\mu_L(x_I)$. In this case, we have

$$w_0^I = \frac{1}{\|A\| + b_0^I} = \frac{1}{\|A\|},$$

$$w_1^I = \frac{1}{\|A\| + b_I} \quad \text{and}$$

$$w_0^I + w_1^I = \frac{2\|A\| + b_I}{\|A\|(\|A\| + b_I)}.$$

$$r_I = \frac{w_0^I b_0^I + w_1^I b_1^I}{w_0^I + w_1^I} = \frac{\frac{1}{\|A\| + b_I} b_I}{\frac{2\|A\| + b_I}{\|A\|(\|A\| + b_I)}} = \frac{\|A\| b_I}{2\|A\| + b_I}.$$

4.3. Interval representative embedded set approximation

Corresponding to the interval SCSL, the interval representative embedded set approximation is obtained following a similar line of reasoning to that employed in sub Section 3.2:

Theorem 7 (Interval representative embedded set approximation). *Let $\widetilde{F}$ be an interval type-2 fuzzy set whose domain X is discretised into N vertical slices, and whose lower and upper membership functions, L and U, are also discretised using n $(n \geqslant 2)$ points $B_0^I, B_1^I, B_2^I, \ldots, B_{n-1}^I$, with primary membership grades at distances $b_0^I(=0), b_1^I, b_2^I, \ldots, b_{n-1}^I(= \mu_U(x_I) - \mu_L(x_I))$ from $\mu_L(x_I)$. The membership function of the representative embedded set R approximates to*

$$\mu_R(x_I) \approx \mu_L(x_I) + r_I \quad \forall I = 1, \ldots, N,$$

where $\forall I$: $r_I = \sum_{i=0}^{n-1} \bar{w}_i^I \cdot b_i^I$; $\bar{w}_i^I = \frac{w_i^I}{\sum_{i=0}^{n-1} w_i^I}$; $w_i^I = \frac{1}{\|L\| + \Re_{I-1} + b_i^I}$; and $\Re_{I-1} = \sum_{k=0}^{I-1} r_k$ with $\Re_0 = 0$.

Result 8 provides the formula for calculating the approximate defuzzified value of an interval type-2 fuzzy set using the collapsing method.

Theorem 8 (Defuzzified value of a discretised interval type-2 FS). *Let $\widetilde{F}$ be an interval type-2 fuzzy set discretised with N points in its domain X, and whose lower and upper membership functions, L and U, are also discretised using n $(n \geqslant 2)$ points $B_0^I, B_1^I, B_2^I, \ldots, B_{n-1}^I$, with primary membership grades at distances $b_0^I(= 0), b_1^I, b_2^I, \ldots, b_{n-1}^I(= \mu_U(x_I) - \mu_L(x_I))$ from $\mu_L(x_I)$. The defuzzified value of $\widetilde{F}$ approximates to*

$$X_{\widetilde{F}} \approx X_L + \frac{\sum_{I=1}^{N} r_I(x_I - X_L)}{\|L\| + \sum_{I=1}^{N} r_I},$$

where $\forall I$: $r_I = \sum_{i=0}^{n-1} \bar{w}_i^I \cdot b_i^I$; $\bar{w}_i^I = \frac{w_i^I}{\sum_{i=0}^{n-1} w_i^I}$; $w_i^I = \frac{1}{\|L\| + \Re_{I-1} + b_i^I}$; and $\Re_{I-1} = \sum_{k=0}^{I-1} r_k$ with $\Re_0 = 0$.

The next section describes how the collapsing method was tested.

5. Testing the collapsing method

5.1. Experimental methodology

The collapsing method was tested both for accuracy and speed. Two test sets were formed, one having reflectional symmetry, the other asymmetric, both with their primary domains scaled from 0 to 1. Each set was created in nine versions, reflecting different degrees of discretisation of the domain, the coarsest employing 11 slices, the finest 100,001 slices.

Symmetric triangular test set: this test set consists of triangular lower and upper membership functions, and has, owing to its symmetry, a defuzzified value of 0.4.
Asymmetric Gaussian test set: this test set consists of Gaussian lower and upper membership functions placed in such a way as to give an FOU lacking in symmetry. There is no way of knowing its defuzzified value other than by exhaustive defuzzification (Section 1.2).

Each version of each test set was defuzzified using versions coded in C of both the Karnik–Mendel iterative procedure (Section 1.2) and the collapsing method. For both defuzzification approaches 30 defuzzifications of each version of each test set were performed and their timings averaged. These experiments were carried out on a Intel 2 core CPU 4300 with a clock speed of 1.80 GHz, 1 GB RAM, running the Vista operating system. In addition the asymmetric set was exhaustively defuzzified at the coarser discretisations, in order to obtain accurate results for comparison. Defuzzified values from finer discretisations of the test set were not obtained since they required a prohibitively long computation.

5.2. Results

Tables 1 and 2 show the actual defuzzified values (where known) against those obtained by the KMIP and the collapsing method. Tables 3 and 4 compare the defuzzification times taken by the KMIP and collapsing method.

Table 2
Defuzzified values obtained for the asymmetrical Gaussian test set

Number of slices	Defuzzified value	Karnik–Mendel iterative procedure	Collapsing method
11	0.3074987724	0.31039553169254100000	0.30824331833942600000
21	0.3142610070	0.31738356336515200000	0.31462785068788000000
51		0.32148294233936500000	0.31877362815103700000
101		0.32293125067512200000	0.32021342566277500000
201		0.32367551173575500000	0.32094477267063100000
501		0.32411955398505900000	0.32138732724623000000
1001		0.32426849198305300000	0.32153547696432100000
10,001		0.32440227394119200000	0.32166908337487100000
100,001		0.32441565472096200000	0.32168245820705000000

Table 3
Defuzzification times for the symmetrical triangular test set

Number of slices	Mean time for Karnik–Mendel iterative procedure	Mean time for collapsing method	KMIP time/collapsing time
11	0.00000506814879172260	0.00000192296320714877	2.636
21	0.00000718666757925937	0.00000252592624667846	2.845
51	0.0000098103716161318 4	0.00000373185232573786	2.629
101	0.0000180074096940625 0	0.00000565481553288663	3.184
201	0.00002283111401030010	0.00000904444559294547	2.524
501	0.00005165926581916600	0.00002009333588486800	2.571
1001	0.00011294964397244100	0.00004694963559148810	2.406
10,001	0.00125553201128448000	0.00044156450051612700	2.843
100,001	0.01300987276315840000	0.00550875181063515000	2.362

Table 4
Defuzzification times for the asymmetrical Gaussian test set

Number of slices	Mean time for Karnik–Mendel iterative procedure	Mean time for collapsing method	KMIP time/collapsing time
11	0.00000554074144432695	0.00000206962989243977	2.677
21	0.00000684444531358036	0.00000270518552870081	2.530
51	0.00000919111227823647	0.00000386222271266320	2.380
101	0.00001430814996505600	0.00000598074150019998	2.392
201	0.00002822518876933610	0.00000995703830142284	2.835
501	0.00006086667439576810	0.00002400444749262820	2.536
1001	0.00013221186864066000	0.00004747111713918940	2.785
10,001	0.00126398979013626000	0.00048756895080240600	2.592
100,001	0.01279743421766360000	0.00536135179191768000	2.387

5.3. Discussion of results

5.3.1. Accuracy

For the symmetric test set with a known defuzzified value of 0.4, the results generated by the KMIP were exceedingly accurate, in some instances absolutely accurate. The accuracy of the collapsing method was good, and improved with finer discretisation, with the values decreasing monotonically towards 0.4.

In contrast, for the asymmetric test set, the collapsing method generated results closer than those of the KMIP to those (where available) produced by exhaustive defuzzification. Both the KMIP and collapsing method's results increase monotonically with finer domain discretisation, and appear to converge.

5.3.2. Speed

For both test sets, at all domain discretisations, the collapsing method performed more than twice as quickly as the KMIP.

6. Conclusions and further work

The collapsing method of defuzzification generates a type-1 embedded set from an interval type-2 fuzzy set that is deemed representative in that it has approximately the same defuzzified value as the original type-2 set. The collapsing form of type-reduction is computationally simple compared with the conventional, exhaustive type-reduction. Collapsing compares well with the established Karnik–Mendel iterative procedure in terms of both speed and accuracy.

Future work will consider the following:

Generalised REST: we would like to extend the interval REST to *generalised* type-2 fuzzy sets.
Other methods of defuzzification: there is no obvious reason why this technique may not be extended to other methods of defuzzification besides the centroid method.
Approximating an RES: it is easy to calculate an approximation for the Interval RES, simply by averaging the primary membership grades. It would be useful to investigate (both mathematically and experimentally), the extent to which accuracy is lost this approximation.
Fuzzy sets of higher type than 2: it would be interesting to explore the possibility of collapsing a type-n fuzzy set to create a type $-(n-1)$ fuzzy set. In this way, by a process of successive collapse, a fuzzy set of any type might be type-reduced to a type-1 set.
Varying the order of slice collapse: we have described the slices collapsing in the order x_1–x_m. However the slices may be collapsed in any order; a different RES will result depending on the collapse order. We intend to investigate whether the collapse order makes a difference to the accuracy of the resultant defuzzified value.
The continuous case of the RESA: Table 1 shows how, as discretisation becomes finer, the defuzzified value obtained by the collapsing method decreases monotonically, converging towards the known defuzzified value of 0.4. This leads us to anticipate that in the continuous case collapsing will obtain an exact RES for the original type-2 set. We hope to look into this further.

References

[1] Simon Coupland, Robert I. John, Geometric Type-1 and Type-2 Fuzzy Logic Systems, IEEE Transactions on Fuzzy Systems 15 (1) (2007) 3–15. February.
[2] Sarah Greenfield and Robert I. John, Stratification in the Type-Reduced Set and the Generalised Karnik-Mendel Iterative Procedure, In Proc. IPMU 2008, Malaga, June 2008.
[3] Sarah Greenfield, Robert I. John, and Simon Coupland, A Novel Sampling Method for Type-2 Defuzzification. In Proc. UKCI 2005, London, September 2005 pages 120–127.
[4] Hani Hagras, A Hierarchical Type-2 Fuzzy Logic Control Architecture for Autonomous Mobile Robots, IEEE Transactions on Fuzzy Systems 12 (2004) 524–539.
[5] Dug Hun Hong, Sungho Lee, Some Algebraic Properties and a Distance Measure for Interval-Valued Fuzzy Numbers, Information Sciences 148 (1-4) (2002) 1–10. December.

[6] Peter R. Innocent, Robert I. John, Computer Aided Fuzzy Medical Diagnosis, Information Sciences 162 (2) (2004) 81–104.
[7] Robert I. John, Simon Coupland, Type-2 Fuzzy Logic: A Historical View, IEEE Computational Intelligence Magazine 2 (1) (2007) 57–62. February.
[8] Robert I. John, Peter R. Innocent, Modelling Uncertainty in Clinical Diagnosis using Fuzzy Logic, IEEE Transactions on System, Man and Cybernetics - Part B: Cybernetics 35 (5) (2005) 1340–1350. ISSN 1083-4419.
[9] Robert I. John and Sarah Lake, Modelling Nursing Perceptions Using Type-2 Fuzzy Sets, In Proceedings of EUROFUSE 2001 Workshop on Preference Modelling and Applications, 2001, pp. 241–246.
[10] Nilesh N. Karnik, Jerry M. Mendel, Centroid of a Type-2 Fuzzy Set, Information Sciences 132 (2001) 195–220.
[11] George J. Klir, Tina A. Folger, Fuzzy Sets, Uncertainty, and Information, Prentice-Hall International, 1992.
[12] George J. Klir, Bo Yuan, Fuzzy Sets and Fuzzy Logic, Prentice-Hall P T R, 1995.
[13] Jerry M. Mendel, Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions, Prentice-Hall PTR, 2001.
[14] Jerry M. Mendel, Advances in Type-2 Fuzzy Sets and Systems, Information Sciences 177 (1) (2007) 84–110. January.
[15] Jerry M. Mendel, Robert I. John, Type-2 Fuzzy Sets Made Simple, IEEE Transactions on Fuzzy Systems 10 (2) (2002) 117–127.
[16] Jerry M. Mendel, Robert I. John, Feilong Liu, Interval Type-2 Fuzzy Logic Systems Made Simple, IEEE Transactions on Fuzzy Systems 14 (6) (2006) 808–821. December.
[17] Harvey B. Mitchell, Pattern recognition Using Type-II Fuzzy Sets, Information Sciences 170 (2005) 409–418.
[18] Masaharu Mizumoto, Kokichi Tanaka, Some Properties of Fuzzy Sets of Type 2, Information and Control 31 (1976) 312–340.
[19] Masaharu Mizumoto, Kokichi Tanaka, Fuzzy Sets of Type 2 Under Algebraic Product and Algebraic Sum, Fuzzy Sets and Systems 5 (3) (1981) 277–290. May.
[20] Turhan Ozen and Jonathan M. Garibaldi, Investigating Adaptation in Type-2 Fuzzy logic Systems Applied to Umbilical Acid-Base Assessment, In Proceedings of 2003 European Symposium on Intelligent Technologies (EUNITE 2003), Oulu, Finland, July 2003, pages 289–294.
[21] Roberto Sepúlveda, Oscar Castillo, Patricia Melin, Oscar Montiel, Analysis and Design of Intelligent Systems Using Soft Computing Techniques, volume 41 of Advances in Soft Computing, chapter, An Efficient Computational Method to Implement Type-2 Fuzzy Logic in Control Applications, Springer, Berlin, 2007. pp. 45–52.
[22] Roberto Sepúlveda, Oscar Castillo, Patricia Melin, Antonio Rodríguez-Díaz, Oscar Montiel, Experimental Study of Intelligent Controllers under Uncertainty using Type-1 and Type-2 Fuzzy Logic, Information Sciences 177 (10) (2007) 2023–2048. May.
[23] Dongrui Wu, Jerry M. Mendel, Uncertainty Measures for Interval Type-2 Fuzzy Sets, Information Sciences 177 (23) (2007) 5378–5393.
[24] Hongwei Wu, Jerry M. Mendel, Uncertainty Bounds and Their Use in the Design of Interval Type-2 Fuzzy Logic Systems, IEEE Transactions on Fuzzy Systems 10 (5) (2002) 622–639. October.
[25] Lotfi A. Zadeh, The Concept of a Linguistic Variable and its Application to Approximate Reasoning – II, Information Sciences 8 (1975) 301–357.
[26] Shang-Ming Zhou, Francisco Chiclana, Robert. I. John, and Jonathan M. Garibaldi, Type-2 OWA Operators - Aggregating Type-2 Fuzzy Sets in Soft Decision Making, In Proc. of IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Hongkong, China, 2008, pages 625–630.