LEARNING OBJECTIVES

After studying this chapter, you should be able to:

- Define Diffie–Hellman key exchange.
- Understand the man-in-the-middle attack.
- Present an overview of the Elgamal cryptographic system.
- Understand elliptic curve arithmetic.
- Present an overview of elliptic curve cryptography.
- Present two techniques for generating pseudorandom numbers using an asymmetric cipher.

This chapter begins with a description of one of the earliest and simplest PKCS: Diffie–Hellman key exchange. The chapter then looks at another important scheme, the Elgamal PKCS. Next, we look at the increasingly important PKCS known as elliptic curve cryptography. Finally, the use of public-key algorithms for pseudorandom number generation is examined.

10.1 DIFFIE–HELLMAN KEY EXCHANGE

The first published public-key algorithm appeared in the seminal paper by Diffie and Hellman that defined public-key cryptography [DIFF76b] and is generally referred to as Diffie–Hellman key exchange.[1] A number of commercial products employ this key exchange technique.

The purpose of the algorithm is to enable two users to securely exchange a key that can then be used for subsequent symmetric encryption of messages. The algorithm itself is limited to the exchange of secret values.

The Diffie–Hellman algorithm depends for its effectiveness on the difficulty of computing discrete logarithms. Briefly, we can define the discrete logarithm in the following way. Recall from Chapter 2 that a primitive root of a prime number p is one whose powers modulo p generate all the integers from 1 to $p - 1$. That is, if a is a primitive root of the prime number p, then the numbers

$$a \bmod p, a^2 \bmod p, \ \ldots \ , a^{p-1} \bmod p$$

are distinct and consist of the integers from 1 through $p - 1$ in some permutation.

For any integer b and a primitive root a of prime number p, we can find a unique exponent i such that

$$b \equiv a^i \pmod{p} \qquad \text{where } 0 \leq i \leq (p - 1)$$

[1]Williamson of Britain's CESG published the identical scheme a few months earlier in a classified document [WILL76] and claims to have discovered it several years prior to that; see [ELLI99] for a discussion.

The exponent i is referred to as the **discrete logarithm** of b for the base a, mod p. We express this value as $\text{dlog}_{a,p}(b)$. See Chapter 2 for an extended discussion of discrete logarithms.

The Algorithm

Figure 10.1 summarizes the Diffie–Hellman key exchange algorithm. For this scheme, there are two publicly known numbers: a prime number q and an integer α that is a primitive root of q. Suppose the users A and B wish to create a shared key.

User A selects a random integer $X_A < q$ and computes $Y_A = \alpha^{X_A} \bmod q$. Similarly, user B independently selects a random integer $X_B < q$ and computes $Y_B = \alpha^{X_B} \bmod q$. Each side keeps the X value private and makes the Y value available publicly to the other side. Thus, X_A is A's private key and Y_A is A's corresponding public key, and similarly for B. User A computes the key as $K = (Y_B)^{X_A} \bmod q$ and user B computes the key as $K = (Y_A)^{X_B} \bmod q$. These two calculations produce identical results:

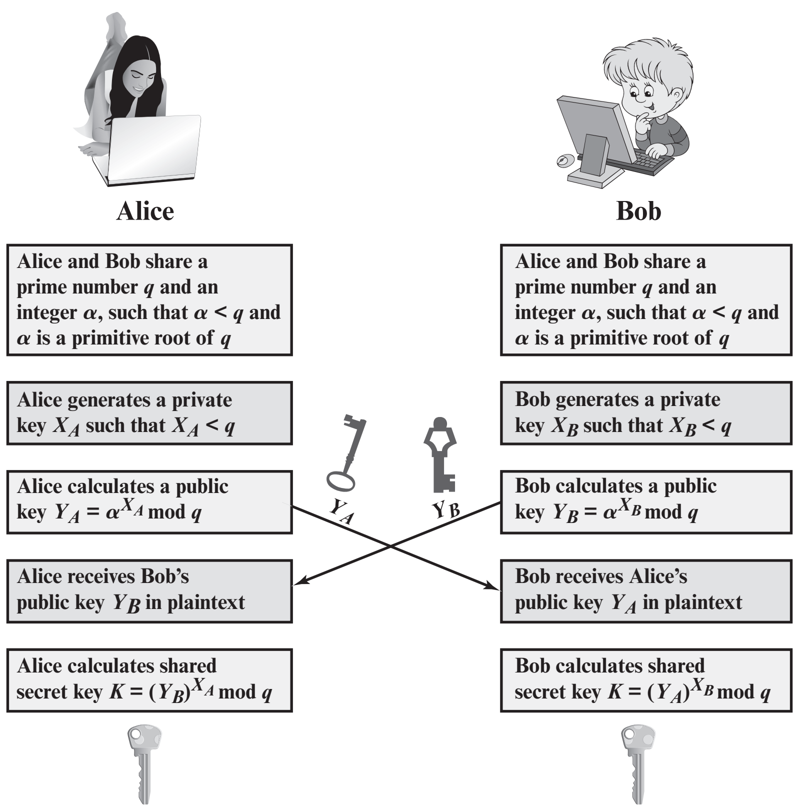

Figure 10.1 The Diffie–Hellman Key Exchange

$$\begin{aligned}
K &= (Y_B)^{X_A} \bmod q \\
&= (\alpha^{X_B} \bmod q)^{X_A} \bmod q \\
&= (\alpha^{X_B})^{X_A} \bmod q && \text{by the rules of modular arithmetic} \\
&= \alpha^{X_B X_A} \bmod q \\
&= (\alpha^{X_A})^{X_B} \bmod q \\
&= (\alpha^{X_A} \bmod q)^{X_B} \bmod q \\
&= (Y_A)^{X_B} \bmod q
\end{aligned}$$

The result is that the two sides have exchanged a secret value. Typically, this secret value is used as shared symmetric secret key. Now consider an adversary who can observe the key exchange and wishes to determine the secret key K. Because X_A and X_B are private, an adversary only has the following ingredients to work with: q, α, Y_A, and Y_B. Thus, the adversary is forced to take a discrete logarithm to determine the key. For example, to determine the private key of user B, an adversary must compute

$$X_B = \mathrm{dlog}_{\alpha,q}(Y_B)$$

The adversary can then calculate the key K in the same manner as user B calculates it. That is, the adversary can calculate K as

$$K = (Y_A)^{X_B} \bmod q$$

The security of the Diffie–Hellman key exchange lies in the fact that, while it is relatively easy to calculate exponentials modulo a prime, it is very difficult to calculate discrete logarithms. For large primes, the latter task is considered infeasible.

Here is an example. Key exchange is based on the use of the prime number $q = 353$ and a primitive root of 353, in this case $\alpha = 3$. A and B select private keys $X_A = 97$ and $X_B = 233$, respectively. Each computes its public key:

A computes $Y_A = 3^{97} \bmod 353 = 40$.
B computes $Y_B = 3^{233} \bmod 353 = 248$.

After they exchange public keys, each can compute the common secret key:

A computes $K = (Y_B)^{X_A} \bmod 353 = 248^{97} \bmod 353 = 160$.
B computes $K = (Y_A)^{X_B} \bmod 353 = 40^{233} \bmod 353 = 160$.

We assume an attacker would have available the following information:

$$q = 353;\ \alpha = 3;\ Y_A = 40;\ Y_B = 248$$

In this simple example, it would be possible by brute force to determine the secret key 160. In particular, an attacker E can determine the common key by discovering a solution to the equation $3^a \bmod 353 = 40$ or the equation $3^b \bmod 353 = 248$. The brute-force approach is to calculate powers of 3 modulo 353, stopping when the result equals either 40 or 248. The desired answer is reached with the exponent value of 97, which provides $3^{97} \bmod 353 = 40$.

With larger numbers, the problem becomes impractical.

Key Exchange Protocols

Figure 10.1 shows a simple protocol that makes use of the Diffie–Hellman calculation. Suppose that user A wishes to set up a connection with user B and use a secret key to encrypt messages on that connection. User A can generate a one-time private key X_A, calculate Y_A, and send that to user B. User B responds by generating a private value X_B, calculating Y_B, and sending Y_B to user A. Both users can now calculate the key. The necessary public values q and α would need to be known ahead of time. Alternatively, user A could pick values for q and α and include those in the first message.

As an example of another use of the Diffie–Hellman algorithm, suppose that a group of users (e.g., all users on a LAN) each generate a long-lasting private value X_i (for user i) and calculate a public value Y_i. These public values, together with global public values for q and α, are stored in some central directory. At any time, user j can access user i's public value, calculate a secret key, and use that to send an encrypted message to user A. If the central directory is trusted, then this form of communication provides both confidentiality and a degree of authentication. Because only i and j can determine the key, no other user can read the message (confidentiality). Recipient i knows that only user j could have created a message using this key (authentication). However, the technique does not protect against replay attacks.

Man-in-the-Middle Attack

The protocol depicted in Figure 10.1 is insecure against a man-in-the-middle attack. Suppose Alice and Bob wish to exchange keys, and Darth is the adversary. The attack proceeds as follows (Figure 10.2).

1. Darth prepares for the attack by generating two random private keys X_{D1} and X_{D2} and then computing the corresponding public keys Y_{D1} and Y_{D2}.
2. Alice transmits Y_A to Bob.
3. Darth intercepts Y_A and transmits Y_{D1} to Bob. Darth also calculates $K2 = (Y_A)^{X_{D2}} \bmod q$.
4. Bob receives Y_{D1} and calculates $K1 = (Y_{D1})^{X_B} \bmod q$.
5. Bob transmits Y_B to Alice.
6. Darth intercepts Y_B and transmits Y_{D2} to Alice. Darth calculates $K1 = (Y_B)^{X_{D1}} \bmod q$.
7. Alice receives Y_{D2} and calculates $K2 = (Y_{D2})^{X_A} \bmod q$.

At this point, Bob and Alice think that they share a secret key, but instead Bob and Darth share secret key $K1$ and Alice and Darth share secret key $K2$. All future communication between Bob and Alice is compromised in the following way.

1. Alice sends an encrypted message M: $E(K2, M)$.
2. Darth intercepts the encrypted message and decrypts it to recover M.
3. Darth sends Bob $E(K1, M)$ or $E(K1, M')$, where M' is any message. In the first case, Darth simply wants to eavesdrop on the communication without altering it. In the second case, Darth wants to modify the message going to Bob.

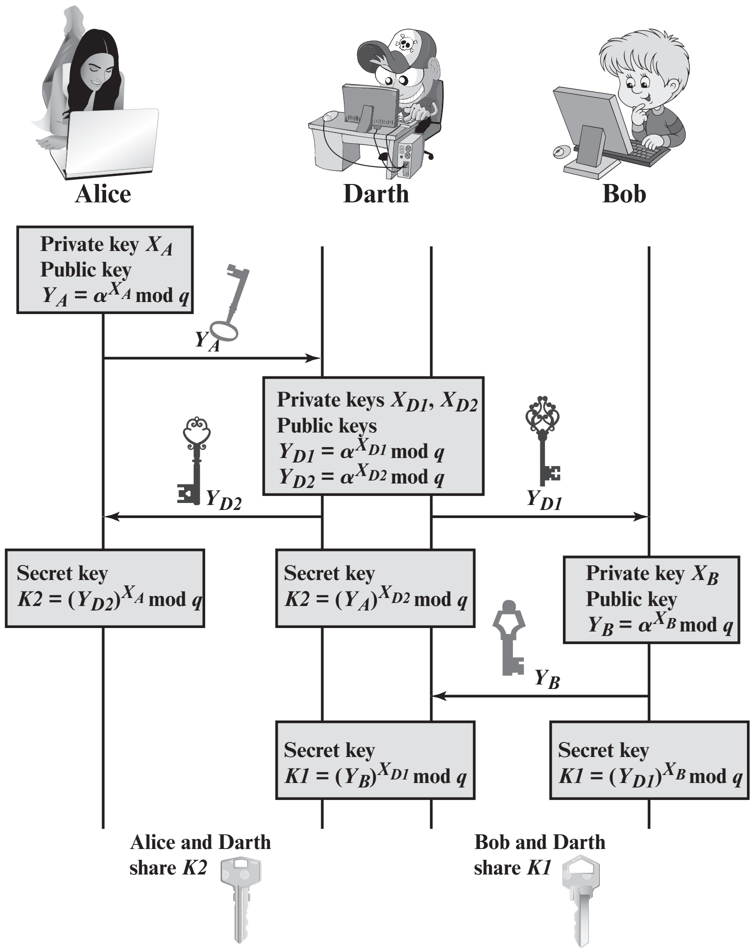

Figure 10.2 Man-in-the-Middle Attack

The key exchange protocol is vulnerable to such an attack because it does not authenticate the participants. This vulnerability can be overcome with the use of digital signatures and public-key certificates; these topics are explored in Chapters 13 and 14.

10.2 ELGAMAL CRYPTOGRAPHIC SYSTEM

In 1984, T. Elgamal announced a public-key scheme based on discrete logarithms, closely related to the Diffie–Hellman technique [ELGA84, ELGA85]. The Elgamal[2] cryptosystem is used in some form in a number of standards including the digital signature standard (DSS), which is covered in Chapter 13, and the S/MIME email standard (Chapter 19).

[2]For no apparent reason, most of the literature uses the term *ElGamal*, although Mr. Elgamal's last name does not have a capital letter G.