Conductive and Convective Heat Transfer & Thermal Flux Analysis of Composite Cylindrical Pressure Vessel By Finite Difference Method

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Abstract: -

This study mainly consists of the thermal flux analysis of the composite cylindrical pressure vessel made of two materials. The study uses computational method for the analysis of the heat flux in the cylinder. One of the main advantages of computational methods is it's use in analyzing the real world physical structures and changes in it's behavior depending upon the change in it's surrounding and/or change in the parameters effecting it's properties. The problem is approached by numerically discretizing the governing equations using second order accurate approximation finite difference schemes. The temperature variation along the radius and height as the time proceeds has been analyzed by this discretization. The transient temperature change part of the problem is solved using BTCS method which is unconditionally stable.

Keywords:

Thermal flux, Finite difference methods, Implicit method, Successive Over-Relaxation (SOR), Convective boundary condition, Steady state temperature .

Introduction:

The analysis in various domains (physical ,chemical, etc.) of definite structures using methods in computational engineering gives us an idea of the behavior of the material over a pre-determined time period which otherwise would not be possible in physical practices. However, the accuracy of the result depends on the methods choose to solve the problem and can be increased at the cost of increased computational time and effort.

Thermal Flux:

Thermal flux is defined as the amount of heat transferred per unit area per unit time from or to a surface. In a basic sense it is a derived quantity since it involves, in principle, two quantities viz. the amount of heat transfer per unit time and the area from/to which this heat transfer takes place. In practice, the heat flux is measured by the change in temperature brought about by its effect on a sensor of known area. The temperature field set up may either be perpendicular to the direction of heat flux or parallel to the direction of heat flux. Unit of thermal flux is W/m^2 .

System Description:

The system is cylindrical pressure vessel consists of two material, Tungsten Carbide and Steel. The inner radius of cylinder is 70 mm and outer radius is 200 mm. Tungsten Carbide lengths from 70 mm to 80 mm and Steel lengths from 80 mm to 200 mm. The height of the cylinder is 1.2 m. The bottom of the cylinder is insulated. The initial temperature of vessel is 20°C and the ambient temperature is 30°C.

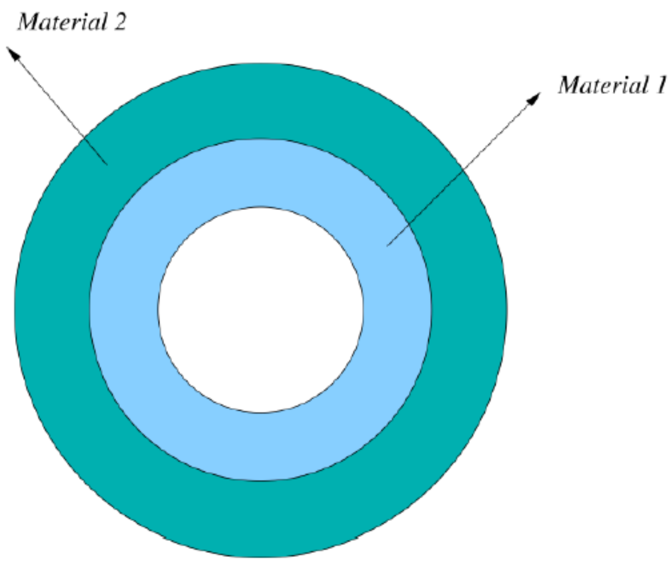

Fig 1: Composite Cylindrical Pressure Vessel

Physical Properties	**Tungsten Carbide**	**Steel**
Thermal Conductivity (k) (W/(m-K)	110	45
Density (ρ) (kg/m^3)	15800	7800
Specific heat at constant pressure (C_P) (J/(kg-K)	250	450

Governing Equations:

At t=0, the temperature at the inner surface increased according to,

Equation 1:

$$T\,(a, t) = T_0 + (T_f - T_0)\,(1 - e^{-\beta t})$$

Where, Final Temperature $= T_f = 300°C$ and $\beta = 10\ s^{-1}$

The convective heat transfer coefficient, $\lambda = 200$ W/ (m^2-K).

The governing equation for the heat transfer in two materials is given by,

Equation 2:

$$\frac{1}{D}\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial^2 z}$$

Where, D = k/ (ρ Cp)

D = thermal diffusivity of the material.

Computational Methodology:

The given example is solved by second order accurate finite difference scheme. The time and space are discretized into small elements.

Let, dt = length of discretized element in time.

dr = length of discretized element in radial direction.

dz = length of discretized element in axial direction.

Formation of 'A' matrix:

By applying 'Backward in Time and Center in Space (BTCS)' method to equation (2),

$$\frac{1}{D}\,\frac{T_{(i,j)}^{n+1} - T_{(i,j)}^{n}}{\partial t} = \frac{1}{r}\,\left(\frac{T_{(i+1,j)}^{n+1} - T_{(i-1,j)}^{n+1}}{2\partial r}\right) + \left(\frac{T_{(i+1,j)}^{n+1} - 2T_{(i,j)}^{n+1} + T_{(i-1,j)}^{n+1}}{\partial r^2}\right) + \left(\frac{T_{(i,j+1)}^{n+1} - 2T_{(i,j)}^{n+1} + T_{(i,j-1)}^{n+1}}{\partial z^2}\right)$$

On solving the equation (2), we get,

$$T_{(i,j)}^{n} = -\left(\frac{D\alpha}{2r} + D\beta\right) T_{(i+1,j)}^{n+1} + \left(\frac{D\alpha}{2r} - D\beta\right) T_{(i-1,j)}^{n+1} + (1 + 2\mathrm{D}\beta + 2\mathrm{Dy})\, T_{(i,j)}^{n+1} - (\mathrm{D\,y})\, T_{(i,j+1)}^{n+1} - (\mathrm{D\,y})\, T_{(i,j-1)}^{n+1}$$

Where,

$\alpha = \frac{dt}{dr}$, $\beta = \frac{dt}{dr^2}$,   y= $\frac{dt}{dz^2}$

Equation for the interface:

Equation 3:

$$-k1\ (\frac{\partial T}{\partial r}) = -k2\ (\frac{\partial T}{\partial r})$$

Applying backward difference scheme for 1st material and forward difference scheme for 2nd material,

$$-k_1\ (\frac{3T_{(i,j)}^{n+1} - 4T_{(i-1,j)}^{n+1} + T_{(i-2,j)}^{n+1}}{2dr}) = -k2\ (\frac{-T_{(i+2,j)}^{n+1} + 4T_{(i+1,j)}^{n+1} - 3T_{(i,j)}^{n+1}}{2dr})$$

On solving, we get,

$$(3k1+3k2)\ T_{(i,j)}^{n+1} - (4k1)\ T_{(i-1,j)}^{n+1} + (k1)\ T_{(i-2,j)}^{n+1} + (k2)\ T_{(i+2,j)}^{n+1} - (4k2)\ T_{(i+1,j)}^{n+1} = 0$$

Boundary Conditions:

For convective heat transfer on the lateral surface:

Equation 4

$$-k2\ \frac{\partial T}{\partial r} = h(T - T\infty) \quad at\ r = r_{\text{outer}}$$

On applying backward difference scheme in second order, equation 4 becomes,

$$-k2\ (\frac{3T_{(i,j)}^{n+1} - 4T_{(i-1,j)}^{n+1} + T_{(i-2,j)}^{n+1}}{2dr}) = h\ (T_{(i,j)}^{n+1} - T\infty)$$

On solving the above equation we get,

$$-(2h\ \partial r + 3k2)T_{(i,j)} + (4k2)\ T_{(i-1,j)} - (k2)\ T_{(i-2,j)} = -2\ \partial r h\ T\infty$$

For convective heat transfer on the top surface:

Equation 5

$$- k2 \frac{\partial T}{\partial z} = h(T - T\infty) \quad at\ z = h$$

On applying backward difference scheme in second order, equation 5 becomes,

$$-k2 \left(\frac{3T_{(i,j)}^{n+1} - 4T_{(i-1,j)}^{n+1} + T_{(i-2,j)}^{n+1}}{2\partial z}\right) = h\ (T_{(i,j)}^{n+1} - T\infty)$$

On solving the above equation, we get,

$-(2h\ \partial z + 3k2)T_{(i,j)} + (4k2)\ T_{(i,j-2)} - (k2)\ T_{(i,j-2)} = -2\ \partial z h\ T\infty$

The condition for the insulated bottom of the cylinder is,

Equation (6)

$$\frac{\partial T}{\partial z} = 0$$

After applying forward scheme and discretizing,

$$\frac{-T_{(i,j+2)}^{n+1} + 4T_{(i,j+1)}^{n+1} - 3T_{(i,j)}^{n+1}}{2\partial z} = 0$$

Thermal Flux:

Equation for the heat flux in radial direction is given as,

Equation (7)

$$-k \left(\frac{\partial T}{\partial r}\right) = q_r \quad , \quad q_r = \text{Thermal flux}$$

After discretization and applying forward scheme for extreme left boundary,

$$q_r(i,j) = -k\left(\frac{-T_{(i,j+2)} + 4T_{(i,j+1)} - 3T_{(i,j)}}{2\partial r}\right)$$

Similarly,

For lateral surface after applying backward scheme,

$$q_r(i,j) = -k(\frac{T_{(i,j-2)} - 4T_{(i,j-1)} + 3T_{(i,j)}}{2\partial r})$$

For inside nodes, after applying center scheme,

$$q_r(i,j) = \frac{-k((T_{(i,j+1)} - T_{(i,j-1)}))}{2\partial r}$$

Equation for the heat flux in axial direction is given as,

Equation (8)

$$-k(\frac{\partial T}{\partial z}) = q_z \ , q_z = \text{Thermal flux}$$

For top surface, after applying forward scheme,

$$q_z(i,j) = -k(\frac{-T_{(i+2,j)} + 4T_{(i+1,j)} - 3T_{(i,j)}}{2\partial z}$$

For bottom surface, after applying backward scheme,

$$q_z(i,j) = -k(\frac{T_{(i-2,j)} - 4T_{(i-1,j)} + 3T_{(i,j)}}{2\partial z}$$

For center points, after applying center scheme,

$$q_z(i,j) = -k(\frac{T_{(i+1,j)} - T_{(i-1,j)}}{2\partial z})$$

Mapping:

By discretizing the given governing equations, we formed the system as equations as,

$$Ax=b$$

Where,

A= Matrix of co- efficient of unknown temperatures at all nodes. (Order: $n\times n$)

x= Vector of unknown temperatures at all nodes. (Order: $n\times 1$)

b= Vector of RHS of governing equations after discretization. (Order: $n\times 1$)

The elements of matrix x have two subscripts, to depict the position of node in radial and axial direction. Since the components usually has single subscript, we need to convert given two subscripts in single subscripts by mapping. Mapping is done by giving equation number to each node. Since we have the grid of total $(I+2) \times (J+2)$ points, the total number of equations will be $(I+2) \times (J+2)$.Let 'P' denotes the equation number for the node (i, j), then,

$$P= (I+2)\, j+ (i+1)$$

The equations at inner nodes:

$P= j\times (I+2) + (i+1)$

The equations at interface nodes:

$P= j\times (I+2) + (i+1)$

The equations at top surface nodes:

$P= j\times (I+2) + (i+1)$

The equations at lateral nodes:

$P = (I+2) \times (j+1)$

Successive Over -Relaxation Method (SOR):

In numerical linear algebra, the method of successive over-relaxation (SOR) is a variant of the Gauss–Seidel method for solving a linear system of equations, resulting in faster convergence. A similar method can be used for any slowly converging iterative process.

Consider the system of 'n' linear equations as follows,

$$AX=b$$

By SOR method,

$$X_i^{(k+1)} = X_i^k + w * R_i^k$$

Where,

$$R_i^k = \frac{1}{a_{ii}}(-\sum_{j=1}^{i-1} a_{ij} * X_j^k - \sum_{j=1}^{n} a_{ij} * X_j^k + b_i)$$

- A = Square matrix of order 'n'.
- b= Column matrix of order n×1.
- w = constant (for SOR, 1< w < 2)
- X = Column matrix of order n×1.
- $a_{ij} = element\ of\ A\ in\ i^{th}\ row\ and\ j^{th}\ column.$
- bi = element of b in 'i'th row.
- X_i^k = Value of i^th variable X after k^th iteration.
- X_i^{k+1} = Value of i^th variable X after (k+1)^th iteration.
- $R_i^k = Residual\ for\ k^{th}\ iteration\ of\ i^{th} variable\ X$

Example:

Consider the following equations.

2X1-X2 = -1

-X1+2 X2-X3 =7

-X2+2X3=-7

The above system can be represented as,

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} X1 \\ X2 \\ X3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix}$$

After putting the above equations in SOR solver, following plot obtained,

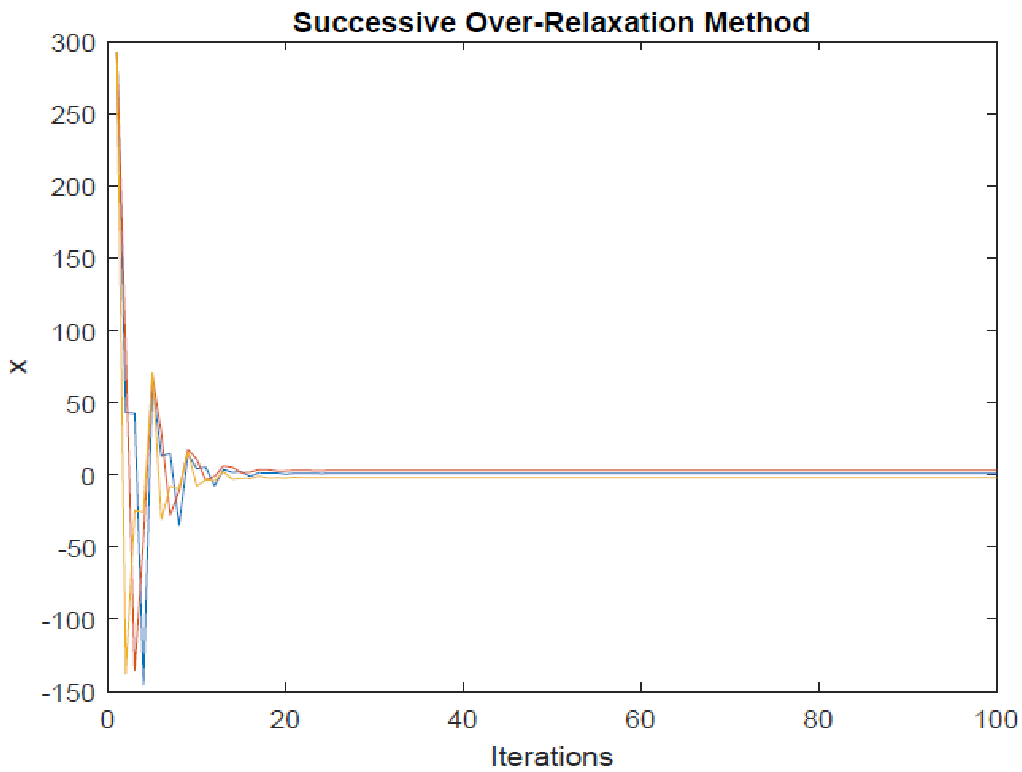

Fig 2: Plot of iterations Vs. X for SOR method

For the given system of equations exact solution is,

X1= 1, X2=3, X3= -2.

which is same as obtained after solving the given system by SOR.

From figure 2, this solution is obtained after 42 iterations for w = 1.7.

The sufficient condition for using SOR, matrix A should be symmetric, positive definite.

For different matrices, ‘w’ will be different in order to converge the solution in minimum iterations.

Validation of Code:

To check the accuracy of the code, we considered the different set of temperature conditions as,

Initial Temperature = 293K

Ambient Temperature = 293 K

Final Temperature =293 K

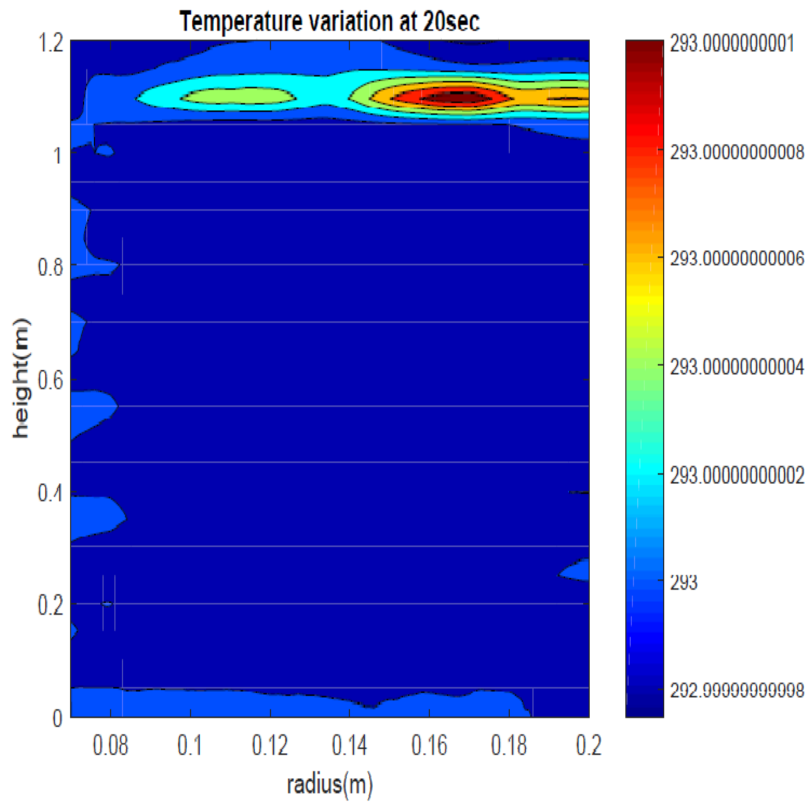

Fig 3: Temperature Variation at 20 seconds

As the system and the surround are at thermal equilibrium, the temperature should be the same over the period . This can be seen in fig 3 ,which shows the temperature variation after 20 seconds. The maximum deviation for the given temperature is of the order 10^{-6}, which is negligible.

As the temperature of the system is same throughout, the thermal flux should be equal to zero at every node for the entire system. Figure 4 shows the similar results for the thermal flux after 20 seconds. The maximum flux in the system is 1.5×10^{-7} W/m^2,which is negligible. This shows the accuracy of the reading over the mesh grid .

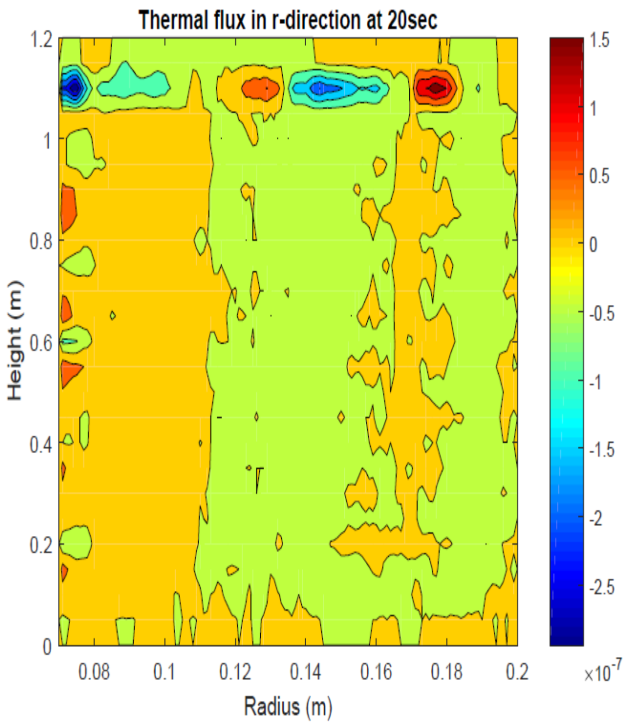

Fig 4: Thermal flux in r- direction at 20 seconds

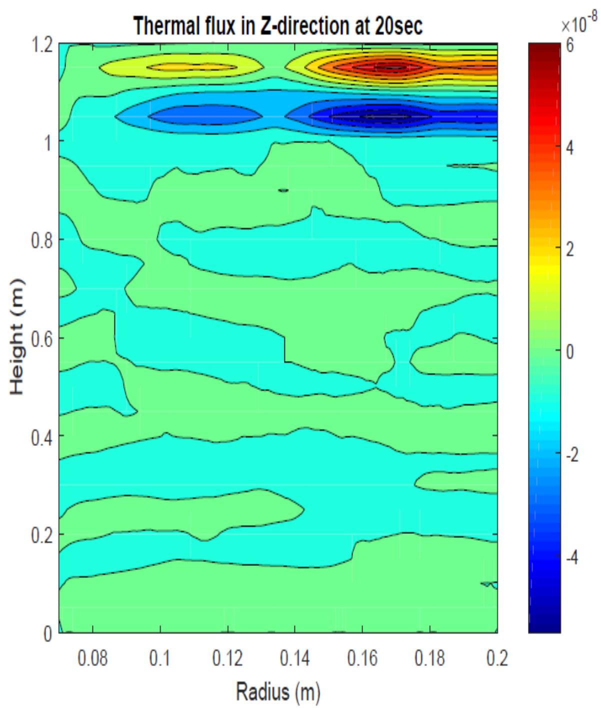

Fig 5: Thermal flux in z- direction at 20 seconds

To validate the consistency of this accuracy over the time, temperature and thermal flux at 100 seconds has been calculated and plotted as follows.

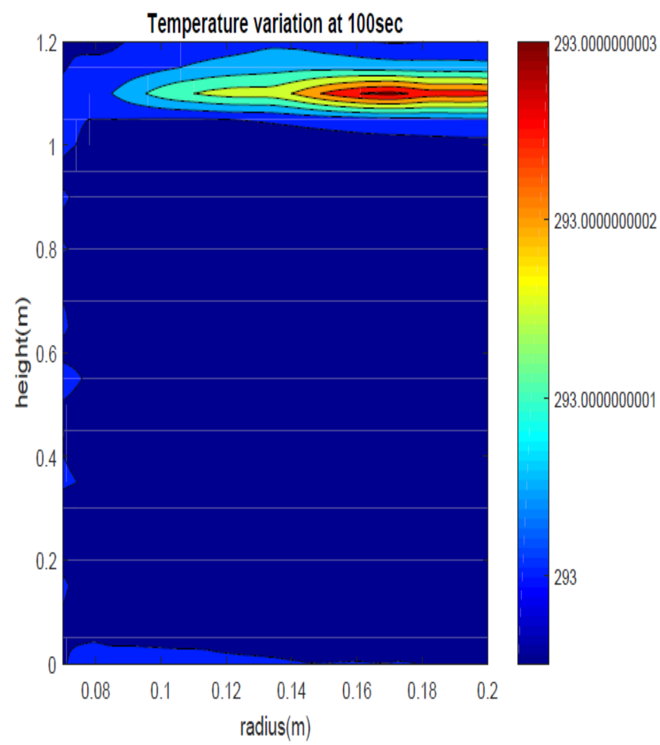

Fig 6: Temperature variation at 100 seconds

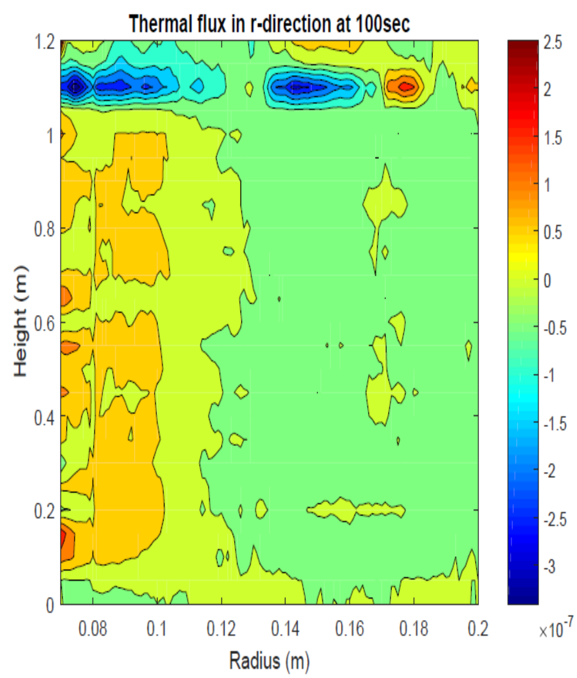

Fig 7: Thermal flux in r-direction at 100 seconds

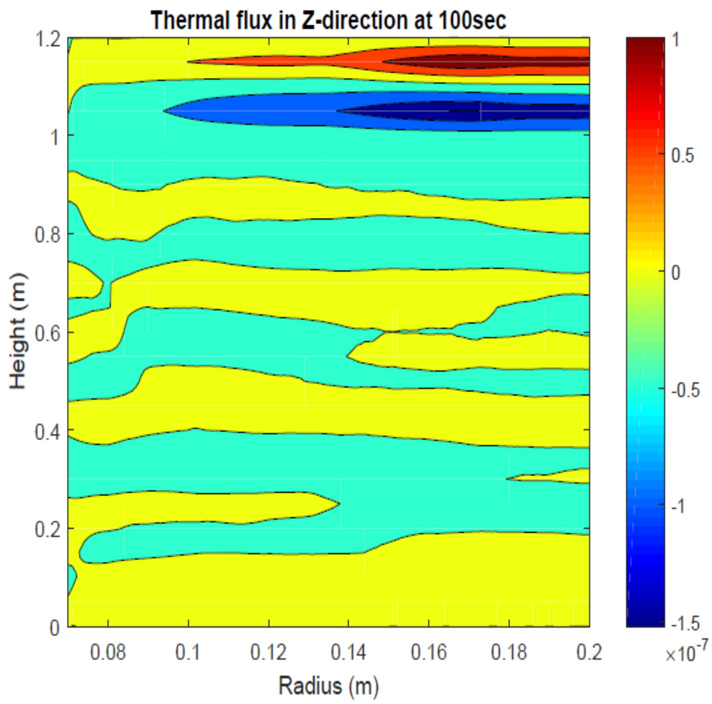

Fig 8: Thermal flux in z-direction at 100 seconds

The results for the 100 seconds are similar to that for 20 seconds. This shows the consistency of accuracy of code over the time.

Results: Temperature Variation Contour Plots

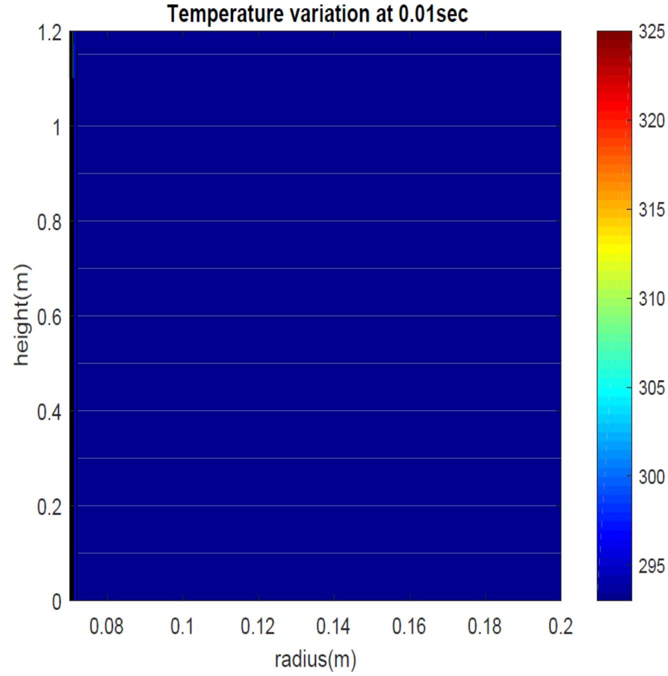

Fig 9: Temperature variation at 0.01 second

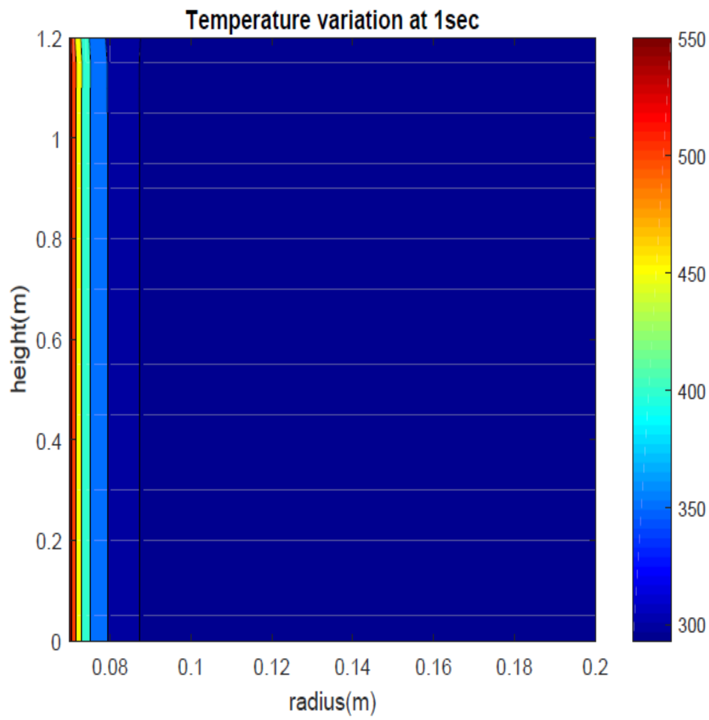

Fig 10: Temperature variation at 1 second

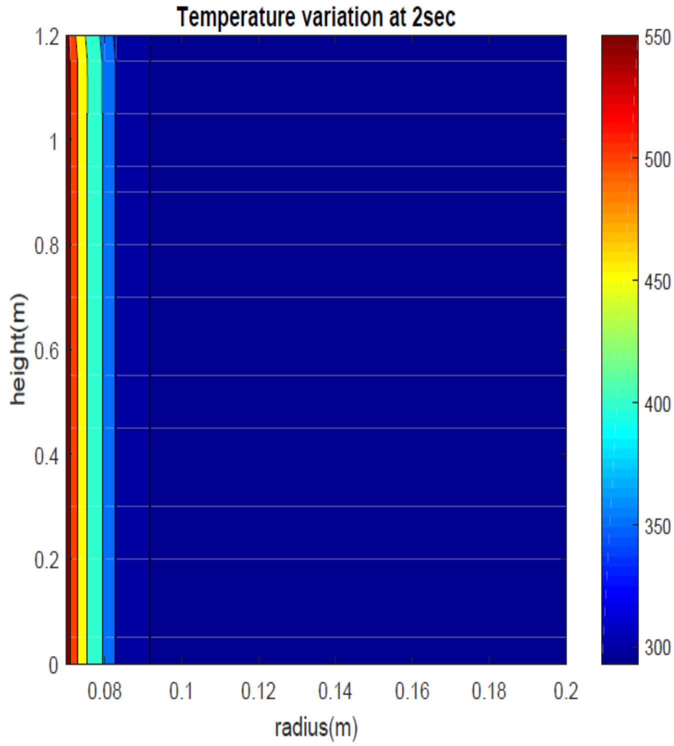

Fig 11: Temperature variation at 2 seconds

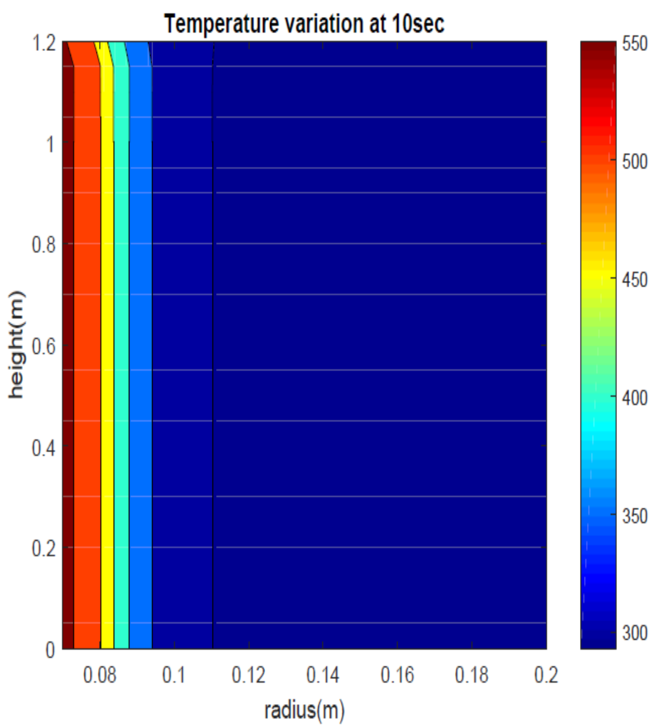

Fig 12: Temperature variation at 10 seconds

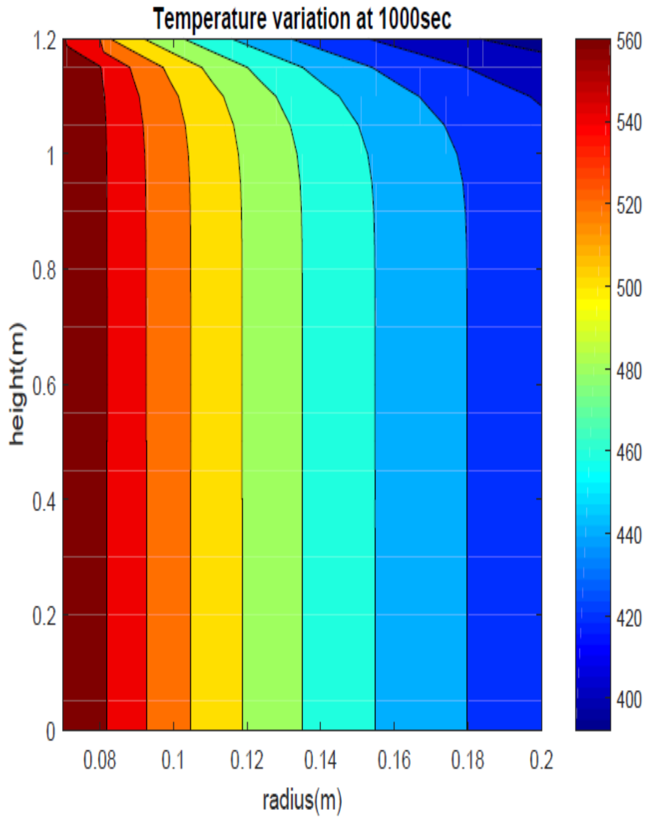

Fig 13: Temperature variation at 1000 seconds

Thermal Flux Contour Plots

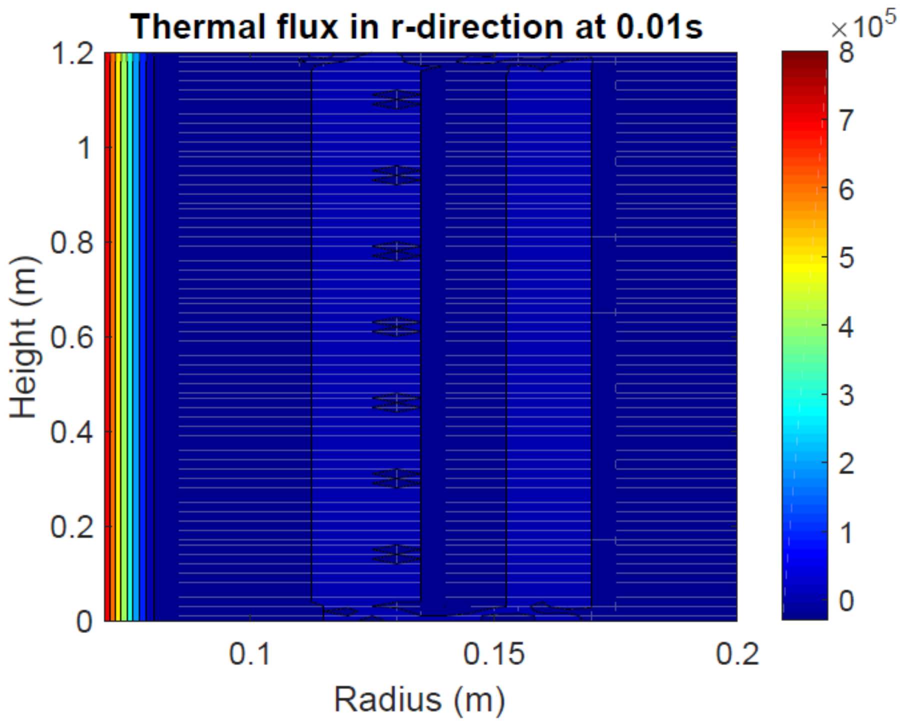

Fig 14: Thermal flux in r-direction at 0.01 second

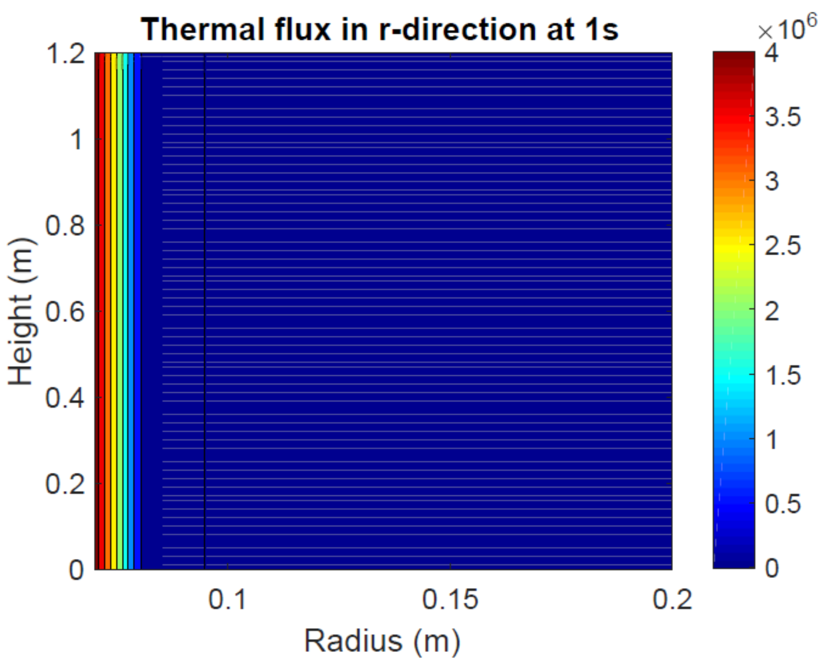

Fig 15: Thermal flux in r-direction at 1 second

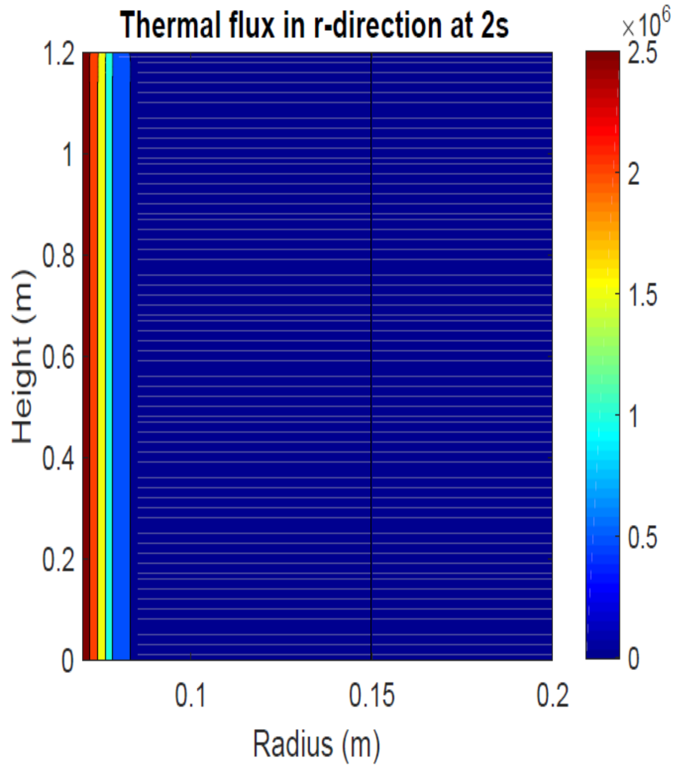

Fig 16: Thermal flux in r-direction at 2 seconds

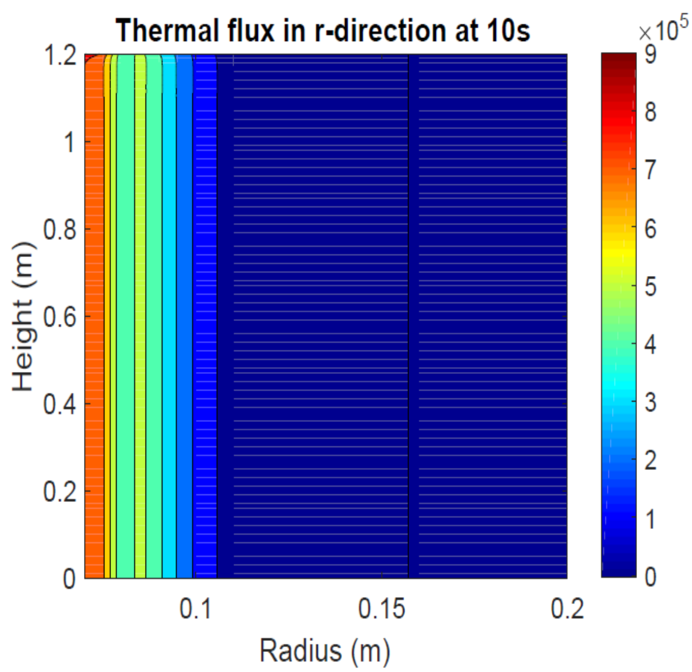

Fig 17: Thermal flux in r-direction at 10 seconds

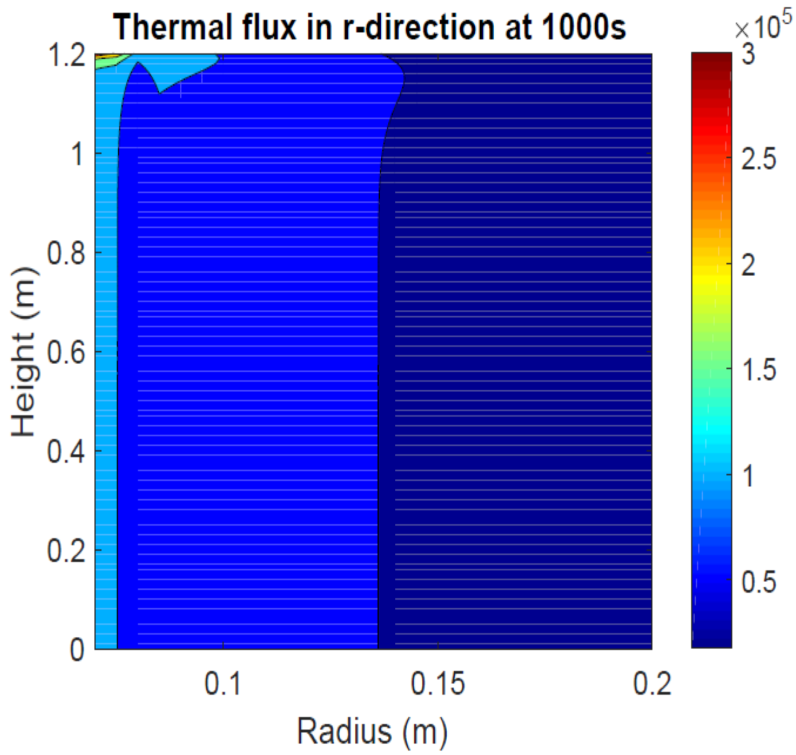

Fig 18: Thermal flux in r-direction at 1000 seconds

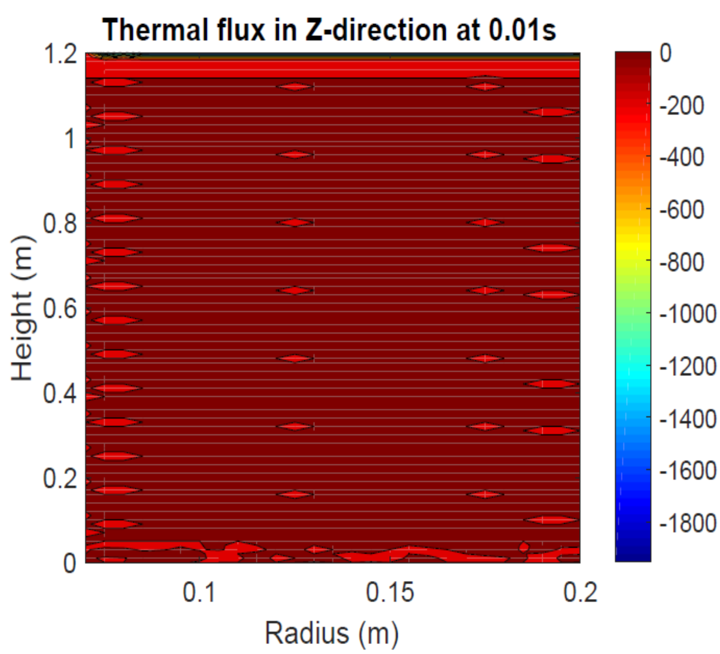

Fig 19: Thermal flux in z-direction at 0.01second

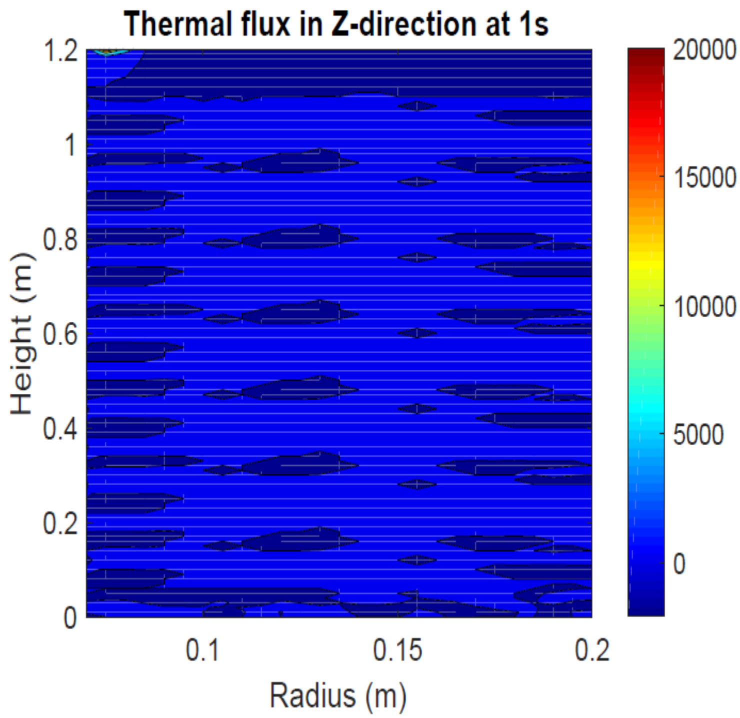

Fig 20: Thermal flux in z-direction at 1 second

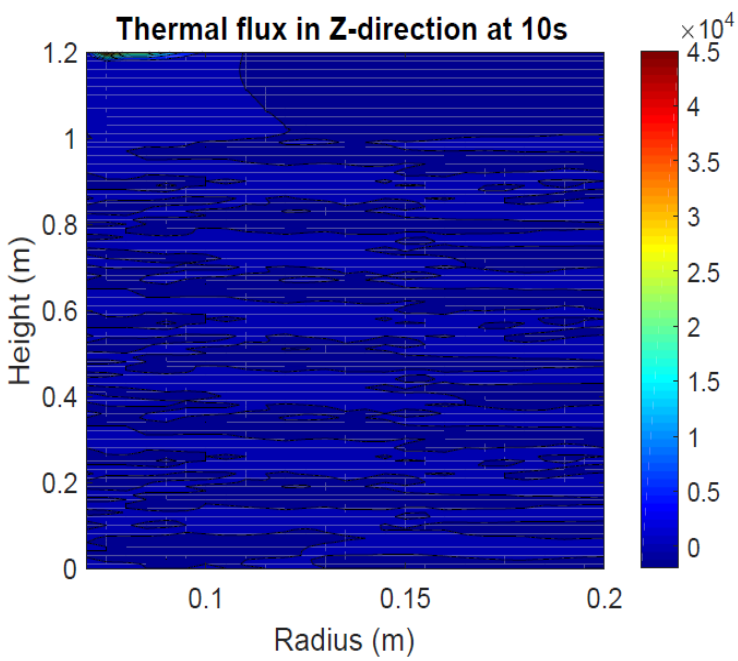

Fig 21: Thermal flux in z-direction at 10 seconds

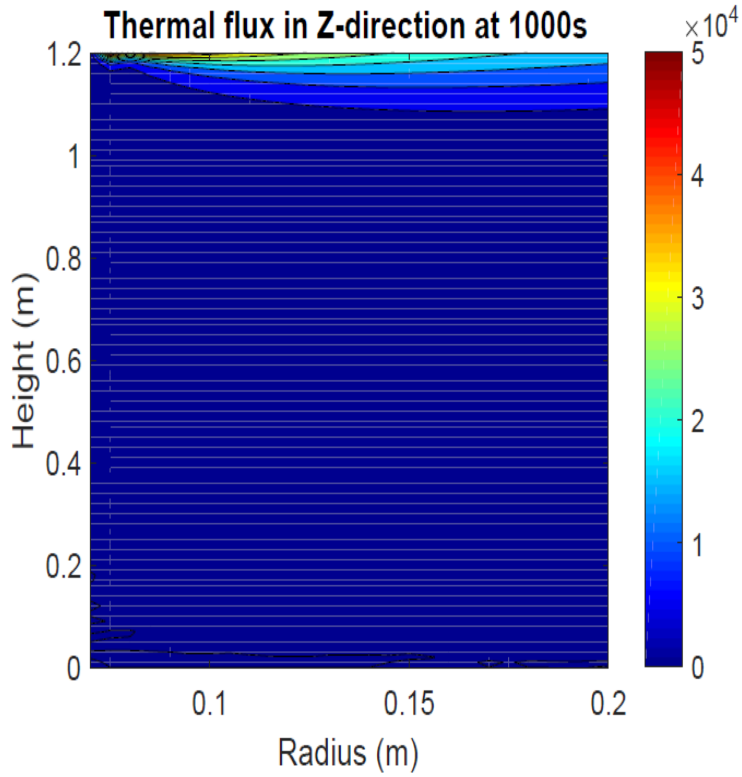

Fig 22: Thermal flux in z-direction at 1000 seconds

Vector Representation of Thermal Flux (In Radial and Axial direction combined)

Analysis:

Temperature Variation:

At initial time the cylinder is at the uniform temperature of 20℃ (293K).It is observed that the innermost layer of the cylinder reaches the steady state temperature of 573K after 1.56 seconds. The rate of temperature increase in Tungsten Carbide is more than that of Steel, this trend is physically supported by thermal conductivity of Tungsten Carbide which is nearly 2.5 times of that of Steel. The outermost layer reaches the steady state temperature of 446.6K after 3165 seconds. At this point of time the entire system reaches the steady state temperature.

As the bottom of the cylinder is insulated, heat loss from bottom is nearly zero. But significant temperature decrease has been observed along the top because of the convective heat transfer to the ambient air which is at 30℃ (303K).So the bottom of the cylinder is at higher temperature as compared to top.

Thermal Flux:

In Radial Direction:

In radial direction, initially the temperature of the layer near the heat source increases rapidly, which leads to the higher thermal flux. As the time processed, system gradually reached to the steady state condition, at this point heat flux gradually decreases.

In Axial Direction:

In axial direction, temperature at all nodes nearly remains the through the period of observation. This leads to the constant heat flux in axial direction, which is less than value of flux in radial direction for respective node. However at 1000 seconds increase in flux at top surface can be seen because of convective heat loss to the ambient air.

Conclusion:

The MATLAB code developed to solve this problem provides a strong stable solver, which ensures the stability of the system for wide range of values. At the same time, the post processing and plots for variation in temperature and thermal flux with time provides excellent visualization of the results. The finite difference method acts as a better approximation to this problem as it has shown promising results.

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2. Finite Difference Methods for Ordinary and Partial Differential Equations: Steady State and Time Dependent Problems – Randall J. LeVeque