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Q1 Form the partial differential -----
---- from the following:

a) $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$

differentiating wrt x

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{dz}{dx} = 0 \quad \text{-(i)}$$

differenciating wrt x again

$$\frac{2}{a^2} + \frac{2z}{c^2}\frac{d^2z}{dx^2} + \frac{2}{c^2}\left(\frac{dz}{dx}\right)^2$$

$$z\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 = \frac{-c^2}{a^2} \quad \text{(ii)}$$

Solving (i)

$$\frac{z\,dz}{x\,dx} = \frac{-c^2}{a^2} \quad \text{(iii)}$$

using (ii) & (iii)

$$\boxed{z\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 = \frac{z\,dz}{x\,dx}}$$

b) $(x-h)^2 + (y-k)^2 + z^2 = c^2$

differentiating wrt x

$2(x-h) + 2zp = 0$

$(x-h) = -zp \quad \text{-(i)}$

differentiating wrt y

$2(y-k) + 2zq = 0$

$(y-k) = -zq \quad \text{-(ii)}$

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putting value of $(x-h)$ & $(y-k)$ in original eq^n

$$(-zp)^2 + (-zq)^2 + z^2 = c^2$$

$$\boxed{z^2(1+p^2+q^2) = c^2}$$

c) $z = (x^2+a)(y^2+b)$

differentiating wrt x

$p = 2x(y^2+b)$ — (i)

differentiating wrt y

$q = 2y(x^2+a)$ — (ii)

(i) × (ii)

$pq = 4xy(x^2+a)(y^2+b)$

$$\boxed{pq = 4xyz}$$

Q2 → Form partial ------ following

a) $z = e^{ny} f(x-y)$

differentiating wrt x

$p = e^{ny} f'(x-y)$ — (i)

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differentiating wrt y

$q = ne^{ny} f(x-y) - e^{ny} f'(x-y)$ —(ii)

(i) + (ii)

$p+q = ne^{ny} f(x-y)$

$\boxed{p+q = nz}$

b) $z = f(x+ay) + g(x-ay)$

differentiating wrt x

$p = f'(x+ay) + g'(x-ay)$

differentiating again wrt x

$\frac{d^2z}{dx^2} = f''(x+ay) + g''(x-ay)$ —(i)

differentiating original eqⁿ wrt y twice

$\frac{d^2z}{dy^2} = a^2 f''(x+ay) + a^2 g''(x-ay)$

$\frac{d^2z}{dy^2} = a^2(f''(x+ay) + g''(x-ay)) = a^2 \frac{d^2z}{dx^2}$ (using (i))

$\boxed{\frac{d^2z}{dy^2} = a^2 \frac{d^2z}{dx^2}}$

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c) $z = f(x+iy) + F(x-iy)$

differentiating wrt x twice

$$\frac{d^2z}{dx^2} = f''(x+iy) + F''(x-iy) \quad \text{—(i)}$$

differentiating wrt y twice

$$\frac{d^2z}{dy^2} = i^2 f''(x+iy) + i^2 F''(x-iy)$$

$$\frac{d^2z}{dy^2} = i^2\left(f''(x+iy) + F''(x-iy)\right) = i^2 \frac{d^2z}{dx^2}$$

$$\boxed{\frac{d^2z}{dy^2} + \frac{d^2z}{dx^2} = 0}$$

d) $f(x+y+z,\ x^2+y^2-z^2) = 0$

$u = x+y+z$ $\qquad$ $v = x^2+y^2-z^2$

$\frac{du}{dx} = 1+p$ $\qquad$ $\frac{du}{dy} = 1+q$

$\frac{dv}{dx} = 2x-2zp$ $\qquad$ $\frac{dv}{dy} = 2y-2zq$

$f(u,v) = 0$

differentiating f wrt x

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx} + \frac{df}{dv}\cdot\frac{dv}{dx} = 0$$

$$\frac{df}{dx} = (1+p)\frac{df}{du} + (2x-2zp)\frac{df}{dv} \quad \text{—(i)}$$

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diff f wrt y

$$\frac{df}{dy} = \left(\frac{df}{du}\cdot\frac{du}{dy}\right) + \left(\frac{df}{dv}\cdot\frac{dv}{dy}\right)$$

$$\frac{df}{dy} = (1+q)\frac{df}{du} + (2y-2zq)\frac{df}{dv} \quad \text{—(ii)}$$

from (i) & (ii)

$$\begin{vmatrix} 1+p & 2x-2zp \\ 1+q & 2y-2zq \end{vmatrix} = 0$$

$$(1+p)(2y-2zq) = (1+q)(2x-2zp)$$

$$\boxed{p(y+z) - q(x+z) = x-y}$$ //

Q3→ Solve the following PDE by Lagrange's method

a) $p\cos(x+y) + q\sin(x+y) = z$

$$\underset{(i)}{\frac{dx}{\cos(x+y)}} = \underset{(ii)}{\frac{dy}{\sin(x+y)}} = \underset{(iii)}{\frac{dz}{z}} = \underset{(iv)}{\frac{dx+dy}{\cos(x+y)+\sin(x+y)}}$$

(iii) = (iv)

$$\frac{dz}{z} = \frac{dx+dy}{\sqrt{2}\{\sin(x+y+\pi/4)\}}$$

$$\int\frac{dz}{z} = \frac{1}{\sqrt{2}}\int \operatorname{cosec}(x+y+\pi/4)\,d(x+y)$$

$$\Rightarrow z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right) = C_1$$

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$$\frac{dx-dy}{\cos(x+y)-\sin(x+y)} = \frac{dx+dy}{\cos(x+y)+\sin(x+y)}$$

$$\frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}\, d(x+y) = d(x-y)$$

$$\log(\sin(x+y)+\cos(x+y)) = x-y+C_2$$

$$\phi(C_1, C_2)$$

$$= \boxed{\phi\left(z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right),\ \log[\cos(x+y)+\sin(x+y)]-x+y\right)=0}$$

b) $yzp + xzq = xy$

$$\underset{(i)}{\frac{dx}{yz}} = \underset{(ii)}{\frac{dy}{xz}} = \underset{(iii)}{\frac{dz}{xy}}$$

(i) = (ii)

$$\frac{dx}{yz} = \frac{dy}{xz}$$

$$x^2 - y^2 = C_1 \quad \text{—(a)}$$

(ii) = (iii)

$$\frac{dy}{xz} = \frac{dz}{xy}$$

$$y^2 - z^2 = C_2 \quad \text{—(b)}$$

$$\boxed{\phi(x^2-y^2,\ y^2-z^2)=0}$$

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c) $x(y^2-z)p - y(x^2+z)q = z(x^2-y^2)$

$$\frac{dx}{x(y^2-z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

taking multiplier $x, y, -1$

$$\frac{dx}{x(y^2-z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx + y\,dy - dz}{0}$$

$$C_1 = \frac{x^2}{2} + \frac{y^2}{2} - z$$

$$C_1' = x^2 + y^2 - 2z \quad -ⓐ$$

taking multiplier $1/x \quad 1/y \quad 1/z$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \Rightarrow \log x + \log y + \log z = C_2$$

$$xyz = C_2' \quad -ⓑ$$

$$\boxed{\phi(x^2+y^2-2z,\ xyz) = 0}$$

d) $(z^2 - 2yz - y^2)p + (xy + xz)q = xy - xz$

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+xz} = \frac{dz}{xy-xz}$$

using multiplier (x, y, z)

$$x\,dx + y\,dy + z\,dz = 0 \Rightarrow x^2 + y^2 + z^2 = C_1 \quad -ⓘ$$

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$$\frac{dy}{xy+xz} = \frac{dz}{xy-xz}$$

$$\frac{y^2}{2} - (zdy + ydz) \mp \frac{z^2}{2} = C_2$$

$$C_2 = \frac{y^2}{2} - \frac{z^2}{2} - yz$$

$$\boxed{\phi\left(x^2+y^2+z^2, \frac{y^2}{2} - \frac{z^2}{2} - yz\right) = 0}$$

e) $y^2p - xyq = x(z-2y)$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow -xdx = ydy$$

$$\frac{-x^2}{2} - \frac{y^2}{2} = C_1 \Rightarrow x^2+y^2 = C_1 \quad \text{—(a)}$$

$$\Rightarrow \frac{dz}{x(z-2y)} = \frac{dy}{-xy} \Rightarrow ydz - 2ydy + \cancel{ydz}\ zdy = 0$$

$$\Rightarrow zy - y^2 = C_2 \quad \text{—(b)}$$

$$\boxed{\phi(x^2+y^2, zy-y^2) = 0}$$

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f) $p + 3q = 5z + \tan(y-3x)$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$$

$$\frac{dx}{1} = \frac{dy}{3} \Rightarrow 3x - y = C_1 \quad -ⓐ$$

$$\frac{dx}{1} = \frac{dz}{5z+\tan(y-3x)}$$

$$x = \frac{1}{5}\ln(5z+\tan(y-3x)) + C_2$$

$$e^{5x} = C_2 \cdot [5z+\tan(y-3x)]$$

$$C_2 = e^{-5x}[5z+\tan(y-3x)]$$

$$\boxed{\phi\left(3x-y,\ e^{-5x}[5z+\tan(y-3x)]\right) = 0}$$

g) $x\frac{du}{dx} + y\frac{du}{dy} + z\frac{du}{dz} = xyz$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{xyz}$$

(i) (ii) (iii) (iv)

(i) & (ii)

$\log x = \log y + C_1$

$C_1 = x/y \quad -ⓐ$

(ii) & (iii)

$\log y = \log z + C_2$

$C_2 = y/z \quad -ⓑ$

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$$\frac{du}{xyz} = \frac{yz\,dx + xz\,dy + xy\,dz}{3xyz}$$

$3\int du = xyz + C_3$

$3u - xyz = C_3$

$$\boxed{\phi\left(\frac{x}{y}, \frac{y}{z}, 3u - xyz\right) = 0}$$

Q4. Obtain the complete solⁿ of the following PDEs using standard form

a) $p + q = pq$

$z = ax + by + c$ (let)

$p = a$ & $q = b$

$a + b = ab$ $[p + q = pq]$

$a = b(a-1)$

$b = \frac{a}{(a-1)}$

$$\boxed{z = ax + \left(\frac{a}{a-b}\right) y + c}$$

b) $x^2 p^2 + y^2 q^2 = z^2$

$p = (dz/dx)$ $q = (dz/dy)$

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$$\frac{x^2}{z^2} p^2 + \frac{y^2}{z^2} q^2 = 1$$

$$\frac{(dz/z)^2}{(dx/x)^2} + \frac{(dz/z)^2}{(dy/y)^2} = 1$$

Let $Z = \log z \quad X = \log x \quad Y = \log y$

$$\left(\frac{dZ}{dX}\right)^2 + \left(\frac{dZ}{dY}\right)^2 = 1$$

$$P^2 + Q^2 = 1$$

If $Z = AX + BY + C \leftarrow B = \sqrt{1-A^2}$

$$\log z = A \log x + \sqrt{1-A^2} \log y + C$$

$$\boxed{z = C x^A y^{\sqrt{1-A^2}}}$$

c) $p^2 + q^2 = z$

$X = x + ay + b$

$z = f(X)$

$$p = \frac{dz}{dx} = \frac{dz}{dX} \cdot \frac{dX}{dx} = \frac{dz}{dX}$$

$$q = \frac{dz}{dy} = \frac{dz}{dX} \cdot \frac{dX}{dy} = \frac{a\,dz}{dX}$$

$$z = p^2 + q^2 = \left(\frac{dz}{dX}\right)^2 + a^2\left(\frac{dz}{dX}\right)^2$$

$$\frac{dz}{dX} = \sqrt{\frac{z}{1+a^2}}$$

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$$\frac{dX}{\sqrt{1+a^2}} = \frac{dz}{\sqrt{z}} \Rightarrow \frac{X}{\sqrt{1+a^2}} = 2\sqrt{z}$$

$$\boxed{(x+ay+b)^2 = 4(1+a^2)z}$$

d) $p^3+q^3 = 27z$

$X = x+ay$

$$\left(\frac{dz}{dX}\right)^3 + a^3\left(\frac{dz}{dX}\right)^3 = 3a\left(\frac{dz}{dX}\right)^2 z$$

$$\frac{dz}{dX} = \frac{3az}{(1+a^3)}$$

$$\int\frac{dz}{z} = \frac{3aX}{(1+a^3)} \Rightarrow \ln z = \frac{3aX}{1+a^3}+c$$

$$\boxed{z = e^{\frac{3a(x+ay)}{1+a^3}} + c}$$

e) $p^2+q^2 = x+y$

$p^2-x = -(q^2-y)$

$p = \pm\sqrt{x+a} \quad q = \pm\sqrt{y-a}$

$dz = p\,dx + q\,dy \Rightarrow \pm\int\sqrt{x+a}\,dx \pm \int\sqrt{y-a}\,dy$

$$\boxed{z = \frac{2}{3}\left(\pm(x+a)^{3/2} \pm (y-a)^{3/2}\right)}$$

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f) $px + qy + \log pq = z$

$z = ax + by + c$

$p = a \quad q = b \quad c = \log pq$

$$\boxed{z = ax + by + \log(ab)}$$

Q5 Solve the following PDEs by Charpit's method.

a) $z = pq$

let $f = z - pq = 0$ — (i)

$f_p = -q \quad f_x = 0 \quad f_y = 0 \quad f_z = 1$

$$\frac{dx}{-q} = \frac{dy}{-p} = \frac{dz}{-2pq} = \frac{dp}{p} = \frac{dq}{q}$$

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log q + \log c$$

$$\boxed{p = cq} \text{ — (ii)}$$

$z - pq = 0 \Rightarrow z - cq^2 = 0$

$q = \sqrt{z/c} \qquad p = \sqrt{cz}$

$dz = p\,dx + q\,dy \Rightarrow dz = \sqrt{cz}\,dx + \sqrt{\frac{z}{c}}\,dy$

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$\cancel{2\sqrt{az}} \cancel{=} \cancel{ax}$

$$\boxed{2\sqrt{cz} = cx + y + c'}$$

b) $px + qy = pq$

$f = px - qy - pq$ — (i)

$$\frac{dx}{x-q} = \frac{dy}{y-p} = \frac{dz}{p(x-q)+q(y-p)} = \frac{dp}{-p} = \frac{dq}{-q}$$

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow p = qc \quad \text{— (ii)}$$

$$f = qcx - qy - cq^2 = 0$$

$$q = \frac{cx+y}{c} \qquad p = cx + y$$

$$dz = p\,dx + q\,dy \Rightarrow \boxed{2cz = (cx+y)^2 + 2c'} \text{ //}$$

c) $(p^2+q^2)y = qz$

$$f = (p^2+q^2)y - qz = 0$$

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{p(2yp) + q(2qy - z)} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow p^2 + q^2 = c^2 \quad \text{(ii)}$$

$$c^2 y = qz -$$

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$q = \frac{c^2 y}{z}$

$p = \frac{c}{z}\sqrt{z^2 - a^2 y^2}$

$dz = px + q\,dy$

$z\,dz - c^2 y\,dy = c\sqrt{z^2 - a^2y^2}\,dx$

~~adx = ...~~ $c\,dx = \frac{1}{b}\int \left(\frac{z\,dz - a^2 y\,dy}{\sqrt{z^2 - c^2y^2}} \right)$

$\boxed{z^2 = (cx+b)^2 + c^2y^2}$

d) $z^2 = pqxy$

$f = z^2 - pqxy = 0$

$\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-2pqxy} = \frac{dp}{-(-pqy + 2pz)} = \frac{dq}{-(-pqx + 2pz)}$

$\int \frac{p\,dx + x\,dp}{px} = \int \frac{q\,dy + y\,dq}{yq}$

$\int \frac{d(px)}{px} = \int \frac{d(qy)}{qy} \Rightarrow px = a^2 qy$ — (ii)

$z^2 = \frac{a^2 qy}{x} qyx \Rightarrow z^2 = a^2 q^2 y^2$

$q = \frac{z}{ay}$ $\quad p = \frac{a^2 y}{x}$

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$$dz = \frac{az}{x}\,dx + \frac{z}{ay}\,dy$$

$$\ln z = \ln(x^a y^{1/a} b)$$

$$\boxed{z = x^a y^{1/a} b}$$

e) $z = px + qy + p^2 + q^2$

$f = px + qy + p^2 + q^2 - z = 0$

$$\frac{dx}{x+2p} = \frac{dy}{y+2q} = \frac{dz}{p(x+2p)+q(y+2q)} = \frac{dp}{0} = \frac{dq}{0}$$

$dp = 0 \Rightarrow p = a$
$dq = 0 \Rightarrow q = b$

$$\boxed{z = ax + by + a^2 + b^2}$$

(f) $(p+q)(xp+yq) = 1$

Let $z = px + qy$

$(p+q)z = 1$

$X = x + ay$

$$p = \frac{dz}{dX}\cdot\frac{dX}{dx} = \frac{dz}{dX}$$

$$q = \frac{dz}{dX}\cdot\frac{dX}{dy} = a\frac{dz}{dX}$$

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$$\left(\frac{dz}{dx} + a\frac{dz}{dx}\right)z = 1$$

$$z\,dz = \frac{1}{1+a}dx$$

$$(1+a)\frac{z^2}{2} = \frac{x+c}{\cancel{1+a}}$$

$$\boxed{z = \frac{\sqrt{2}\sqrt{x+ay+c}}{\sqrt{1+a}}}$$

Q6 → Obtain the general sol^n ------ $z=1,\ y=x$

$$(2y^2+z)p + (y+2x)q = 4xy - z$$

$$\underset{(i)}{\frac{dx}{2y^2+z}} = \underset{(ii)}{\frac{dy}{y+2x}} = \underset{(iii)}{\frac{dz}{2yx-z}} = \underset{(iv)}{\frac{dx+dz}{2y^2+z+2yx-z}}$$

(ii) & (iv)

$$\frac{dy}{y+2x} = \frac{d(x+z)}{2y(y+2x)} \Rightarrow d(x+z) = 2y\,dy$$

$$\Rightarrow x+z = y^2 + C_1$$

$$C_1 = x - y^2 + z \quad \text{-(i)}$$

using multipier $2x, -z, -y$

$$2x\,dx - z\,dy - y\,dz = 0$$

$$x^2 - yz = C_2$$

$$\cancel{\phi(x^2-yz,\ x^2-y^2)}\quad \boxed{\phi(x-y^2+z,\ x^2-yz) = 0}$$

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for particular Solution →

$\cancel{C_1 \Rightarrow}$ $C_1 + \lambda C_2 = a$

$(x - y^2 + z) + \lambda(x^2 - yz) = \text{const.}$

for $(x=y)$ & $(z=1)$

$(x - x^2 + 1) + \lambda(x^2 - x) = \text{const.}$

$\therefore \lambda = 1$ for a to be const.

for $\lambda = 1$, $a = 1$

$\therefore (x - y^2 + z) + (x^2 - yz) = 1$

$$\boxed{z(1-y) + x - y^2 + x^2 = 1}$$

Q7 → Find the equation of the Surface Satisfying $t = 6x^3y$ & containing the two lines $y=0, z=0$ & $y=1, z=1$

$t = 6x^3y \Rightarrow \frac{dq}{dy} = 6x^3y$

$q = 3x^3y^2 + f(x) \leftarrow \left(q = \frac{dz}{dy}\right)$

$z = x^3y^3 + \int f(x)\,dy$

$z = x^3y^3 + y\,f(x) + g(x)$

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for $y=z=0$

$$0 = 0+0+g(x)$$
$$g(x)=0 \quad \text{(i)}$$

for $y=z=1$

$$1 = x^3 + f(x)$$

$$f(x) = 1-x^3 \quad \text{(ii)}$$

$$\therefore \quad \boxed{z = x^3+y^3+ y(1-x^3)}$$