test

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1 Solution of the initial boundary value problem with the wave equation

1.1 Write down the general analytical solution of the IVP. Note it is composed of two *elementary waves*

Given:

$$\partial_{tt}\phi - c^2\partial_{xx}\phi = 0 \ , \tag{1}$$

Solution:

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)\phi(x,t) = 0 \tag{2}$$

Chose new variable:

$$\begin{aligned} \lambda &= x + ct \\ \eta &= x - ct \end{aligned} \tag{3}$$

That relate to old ones as,

$$\begin{aligned} \lambda + \eta &= 2x \\ \lambda - \eta &= 2ct \end{aligned} \tag{4}$$

$$\begin{aligned} x &= \frac{\lambda + \eta}{2} \\ t &= \frac{\lambda - \eta}{2t} \end{aligned} \tag{5}$$

Computing derivatives,

$$\begin{aligned} \partial_x &= \frac{\partial\lambda}{\partial x}\frac{\partial}{\partial\lambda} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta} = \frac{\partial}{\partial\lambda} + \frac{\partial}{\partial\eta} \\ \partial_t &= \frac{\partial\lambda}{\partial t}\frac{\partial}{\partial\lambda} + \frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta} = c\frac{\partial}{\partial\lambda} - c\frac{\partial}{\partial\eta} \end{aligned} \tag{6}$$

Putting them back into the equation 1,

$$(c\partial_\lambda - c\partial_\eta - c\partial_\lambda - c\partial_\eta)(c\partial_\lambda - c\partial_\eta + c\partial_\lambda + c\partial_\eta)\phi(\lambda,\eta) = 0 \tag{7}$$

simplifying,

$$\partial_\eta\partial_\lambda\phi(\lambda,\eta) = 0 \tag{8}$$

This is a well known equation, that has a general solution of form:

$$\phi(\lambda,\eta) = F(\lambda) + G(\eta) \tag{9}$$

or

$$\phi(x,t) = F(x - ct) + G(x + ct) \tag{10}$$

that can be interpreted as two *elementary waves.*

1.2 Rewrite the wave equation from the second order form above to a first-order in time and second-order in space system

The goal of this exercise is to recognize that the differential equation that is second order in time and space, e.g. Eq. 1 can be reduced to a system of coupled equation, first order in time, using the substitution:

$$\begin{aligned}\partial_t \phi &= \Pi \\ \partial_t \Pi &= c^2 \partial_{xx} \phi\end{aligned} \tag{11}$$

It is also possible to express the solution as a function of the initial data, to show that the solution of the wave equation is the advection equation, that describes a propagation of the initial data.
Given also the initial data as $\phi(x, 0) = \theta(x)$, which putting into the solution above 10, obtaining: $\theta(x) = F(x) + G(x)$, derivative of which is:

$$\partial_x \theta(x) = cF'(x) - cG'(x) = \psi(x) \tag{12}$$

where we defined $\psi(x)$ at the last step.
Integrating, we obtain:

$$cF(x) - cG(x) = \frac{1}{c}\int_{x_0}^{x} \psi(x')dx' + K \tag{13}$$

where K is a constant of integration.
Substitute this into the $F(x) = \theta(x) - G(x)$:

$$\theta(x) - G(x) - G(x) = \frac{1}{c}\int_{x_0}^{x} \psi(x')dx' + K \tag{14}$$

$$G(x) = \frac{1}{2} - \frac{1}{2c}\int_{x_0}^{x} \psi(x')dx' + K \tag{15}$$

Using $F(x) = \theta(x) - G(x)$ again, we obtain,

$$F(x) = \frac{1}{2}\theta(x) + \frac{1}{2c}\int_{x_0}^{x} \psi(x')dx' + K \tag{16}$$

Substituting $x \to x + ct$ in F and $x \to x - ct$ in G, obtain

$$\begin{aligned}\phi(x,t) &= F(x+ct) + G(x-ct) = \\ &= \frac{1}{2}\theta(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} \psi(x')dx' - K + \frac{1}{2}\theta(x-ct) + \frac{1}{2c}\int_{x_0}^{x-ct} \psi(x')dx' = \\ &= \frac{1}{2}\Big(\theta(x+ct) + \theta(x-ct)\Big) + \frac{1}{2c}\int_{x_0}^{x+ct} \psi(x')dx' + \frac{1}{2c}\int_{x-ct}^{x_0} \psi(x')dx' = \\ &= \frac{1}{2}\Big(\theta(x+ct) + \theta(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'\end{aligned} \tag{17}$$

This is the generic solution, if the initial profile is given.
■ TODO: What a student should discuss here? What are the 2 choses for $\Pi(t=0,x)$ most likely the two choices are related to the one- or two-sided wave propagation. You either prescribe the velocity in one direction, or both and the wave sort of dissipates in space.

1.3 Write the wave equation as a fully first order system by defining the new variables

Given:

$$\begin{aligned}\partial_{tt}\phi - c^2\partial_{xx}\phi = 0 \\ \partial_t \Pi = c^2 \partial_x\end{aligned} \tag{18}$$

Introduce $\partial_t \phi = \chi$, thus $\partial_t \partial_x \phi = \partial_t \chi$,assuming that ∂_t and ∂_x commute, and $\partial_x = \partial_t \chi$.
Together, ϕ, Π, χ form a special state-vector.

$$\begin{aligned}\partial_t \phi &= \Pi \\ \partial_t \Pi &= c^2 \partial_x \chi \\ \partial_t \chi &= \partial_x \Pi\end{aligned} \tag{19}$$

which can be expressed in a matrix form

$$\begin{pmatrix} \partial_t\phi \\ \partial_t\Pi \\ \partial_t\chi \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -c^2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x\phi \\ \partial_x\Pi \\ \partial_x\chi \end{pmatrix} = \begin{pmatrix} \Pi \\ 0 \\ 0 \end{pmatrix} \tag{20}$$

defining the state vector

$$u = \begin{pmatrix} \phi \\ \Pi \\ \chi \end{pmatrix} \tag{21}$$

, matrix as A and rhs of the equation as S, we obtain a system that is equivalent to wave equation

$$\partial_t u + A\partial_x u = S \tag{22}$$

First note, that the matrix A is degenerate, containing row of 0s. Taking the principle part of wave equation, sub-block of A:

$$\tilde{A} = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \tag{23}$$

we can obtain eigenvalues and convectors,

$$|\tilde{A} - \bar{E}\lambda| = 0 \tag{24}$$

$$\begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0, & \lambda \end{pmatrix} = 0 \tag{25}$$

$$\begin{pmatrix} -\lambda & -c^2 \\ -1 & -\lambda \end{pmatrix} = 0 \tag{26}$$

$$\lambda^2 - c^2 = 0 \to \lambda = \pm c \tag{27}$$

Eigenvectors are found as

$$A\upsilon = \lambda\upsilon \tag{28}$$

let

$$\upsilon = \begin{pmatrix} x \\ y \end{pmatrix} \tag{29}$$

then

$$\begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \tag{30}$$

or equivalently,

$$\begin{aligned} -c^2 y &= \lambda x \\ -x &= \lambda y \end{aligned} \tag{31}$$

As there is a freedom of choice, we can set $x = c$ and $y = 1$ for $\lambda = -c$, and $x = -c$, $y = 1$ for $\lambda = c$. this the matrix of eigenvectors is

$$R = (\upsilon_1, \upsilon_2) = \begin{pmatrix} c & -c \\ 1 & 1 \end{pmatrix} \tag{32}$$

which can be verified by performing

$$AR = R\Lambda \tag{33}$$

The Λ is a diagonal matrix. Thus, overall, using $w := Ru$, the wave-equation 1 was reduced to a diagonal (decoupled) system. Equivalently, this is a set of advection, transport equations.

■ TODO: Is this all that a student should present for analytical part?

2 Finite differentiating approximation

Principle: derivatives in the partial differential equations are approximated by linear combinations of function values at the grid points.
In 1D we define set of equally spaced grid points $x_i = i\Delta x$ with $i = 0...n-1$ and $\Delta x = 1/(n-1)$ that discretizes the domain $\Omega = (0, 1)$. To compute the derivatives in this framework, consider the definition of a first order derivative

$$\frac{\partial u}{\partial x}(x) = \lim_{\Delta x \to 0} \frac{u(\bar{x} + \Delta x) - u(\bar{x})}{\Delta x} = \lim_{\Delta x \to 0} \frac{\bar{u}(x) - u(\bar{x} - \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(\bar{x} + \Delta x) - u(\bar{x} - \Delta x)}{2\Delta x} \tag{34}$$

Geometrically, the

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1}-u_i}{\Delta x} \tag{35}$$

is a forward difference,

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_i-u_{i-1}}{\Delta x} \tag{36}$$

– backward difference and

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1}-u_{i-1}}{2\Delta x} \tag{37}$$

is a central difference. To asses the level of approximation, consider a Taylor series:

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty} = \frac{(x-x_i)^n}{n!}\left(\frac{\partial^n u}{\partial x^n}\right)_i, \qquad u \in (|0,1|) \\ u_{i+1} &= u_i{+}\Delta x\left(\frac{\partial u}{\partial x}\right)_i + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{(\Delta x)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots \\ u_{i-1} &= u_i{-}\Delta x\left(\frac{\partial u}{\partial x}\right)_i + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{(\Delta x)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots \end{aligned} \tag{38}$$

Analysis of a truncation errors

$$\begin{aligned} T_1 &\Rightarrow \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1}-u_i}{\Delta x} - \frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots \\ T_2 &\Rightarrow \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_i-u_{i-1}}{\Delta x} + \frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots \\ T_1 - T_2 &\Rightarrow \left(\frac{\partial u}{\partial x}\right)_i + \frac{u_{i+1}-u_{i-1}}{2\Delta x} - \frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \dots \end{aligned} \tag{39}$$

with the first term on the right hand side being the central difference and the second, red, being the truncation error $O(\Delta x)^2$ Leading truncation error can be expressed as

$$\epsilon_r = \alpha_m(\Delta x)^m + \alpha_{m+1}(\Delta x)^{m+1} + \dots \approx \alpha_m(\Delta x)^m \tag{40}$$

Consider second derivative, the finite differencing formulation can be similarly obtained,

$$T_1 + T_2 \Rightarrow \left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta x)^2} - \frac{(\Delta x)^2}{24}\left(\frac{\partial^4 u}{\partial x^4}\right) + \dots \tag{41}$$

with the first term on the right hand side being the central difference and the second, red, being the truncation error $O(\Delta x)^2$ Alternatively, the second derivative can be derived as,

$$\begin{aligned} \left(\frac{\partial^2 u}{\partial x^2}\right)_i &= \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right]_i = \lim_{\Delta x\to 0}\left[\left(\frac{\partial u}{\partial x}\right)_{i+1/2} - \left(\frac{\partial u}{\partial x}\right)_{i-1/2}\right]\Big/\Delta x \\ &\approx \left[\frac{u_{i+1}-u_i}{\Delta x} - \frac{u_i-u_{i-1}}{\Delta x}\right]\Big/\Delta x = \frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta x)^2} \end{aligned} \tag{42}$$

Examples. Consider an analytical faction and its derivatives:

$$f(x) = \left(x-\frac{1}{2}\right)^2 + x \qquad f'(x) = 2x \qquad f''(x) = 2 \tag{43}$$

$$f(x) = \left(x-\frac{1}{2}\right)^3 + \left(x-\frac{1}{2}\right)^2 + x \qquad f'(x) = 3x^2 - x + \frac{3}{4} \qquad f''(x) = 6x - 1 \tag{44}$$

$$f(x) = \sqrt{x} \qquad f'(x) = \frac{1}{2\sqrt{x}} \qquad f''(x) = -\frac{1}{4x^{3/2}} \tag{45}$$

$$f(x) = \sin(12\pi x) \qquad f'(x) = 12\pi\cos(12\pi x) \qquad f''(x) = 1 = -144\pi\sin(12\pi x) \tag{46}$$

$$\begin{aligned} f(x) &= \sin^4(12\pi x) \\ f'(x) &= 48\pi\cos(12\pi x)\sin^3(12\pi x) \\ f''(x) &= -576\pi^2\sin^2(12\pi x)(-3\cos^2(12\pi x) + \sin^2(12\pi x)) \end{aligned} \tag{47}$$

$$\begin{aligned} f(x) &= \exp(-ax^2) \\ f'(x) &= -2a\exp(-ax^2)x \\ f''(x) &= 2a\exp(-ax^2)(-1+2ax^2) \end{aligned} \tag{48}$$

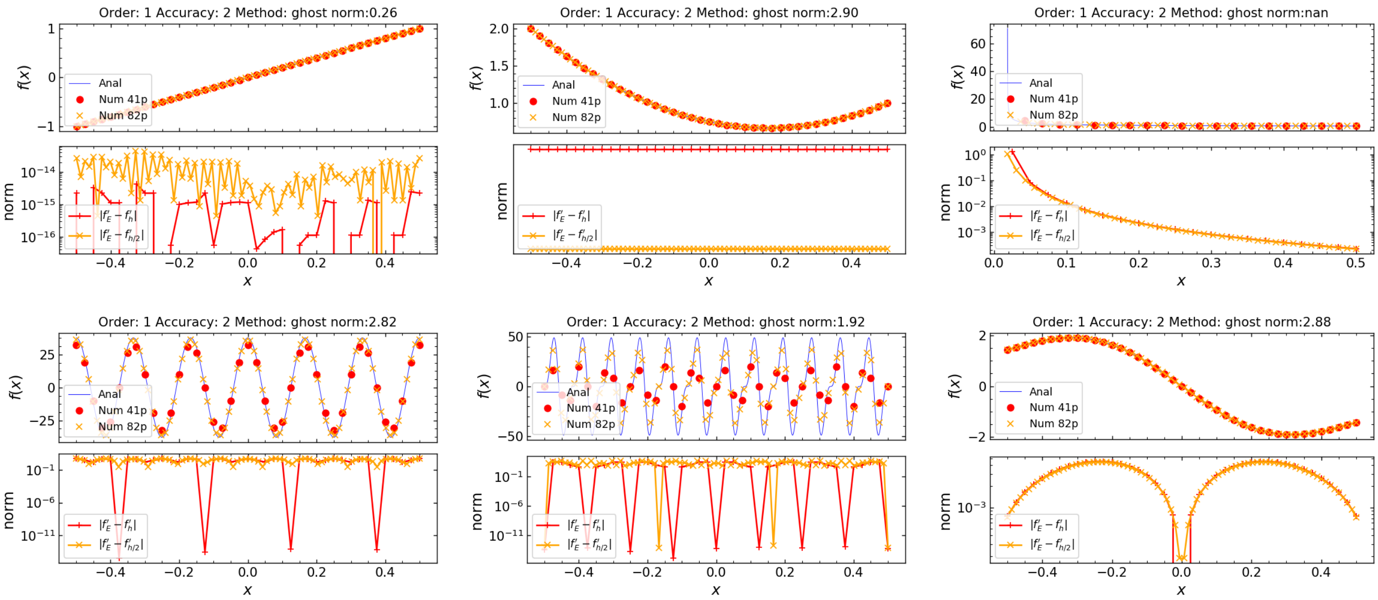

Figure 1: First derivatives, using ghosts and convergence tests

2.1 Numerical implementation

See script `ex1.py` for the code.
In this code see the function `findif(f, n, x1, x2, ord, acc, method)`, that performs the finite differencing of a function $f(x)$, using n equally spaced points between $x1$ and $x2$, using the stencil of order *ord* and accuracy *acc* with method *method*.
Possible orders: $[1, 2]$ for the first and second order derivative
Possible accuracies: $[2, 4]$ for the second and 4th order accuracy stencils
Possible methods: *ghost*, *onesided*. The first one will extend the x to 1 (2) points in left and right for second (forth) accuracies and computes the central finite differencing for points between 0 to n, using added points as ghosts. Second method uses one-sided stencils for 1 (2) border points, where central stencil cannot be used.
To write stencils the coefficients from `https://en.wikipedia.org/wiki/Finite_difference_coefficient` were used.
To study the convergence consider norm:

$$||x||_p = \sqrt[p]{\sum x_i^p} \tag{49}$$

From the previous analysis,

$$f'_E(x) - f'_h(x) = Ch^2 + O(h^3) \tag{50}$$

where $f'_E(x)$ is an analytical derivative, exact solution, and $f'_h(x)$ is a numerical derivative with spacing h. Thus,

$$\frac{||f'_E(x) - f'_h(x)||_2}{||f'_E - f'_{h/2}(x)||_2} = \frac{Ch^\theta}{C(h/2)^\theta} = \frac{h^\theta}{h^\theta} \times 2^\theta = 2^\theta \tag{51}$$

if this equation holds, the convergence is achieved.
If the analytic solution $f'_E(x)$ is not available, the self-convergence test can be performed, using 3 (or more) resolutions in a following manner:

$$\frac{||f'_h(x) - f'_{h/2}(x)||_2}{||f'_{h/2} - f'_{h/4}(x)||_2} = 2^\theta \tag{52}$$

Using special function for the test `convergence(f, df, ddf)`, that takes the actual function $f(x)$, its first analitical derivative $df(x)$ and second $d(df(x))$ perform the convergence test.
Fig. 1 shows the first derivative computed using 2order accuracy stencil (with 2 ghosts) and the convergence for all 6 functions 43, 44, 45, 46, 47, 48. For second order derivatives, 4th order stencil and self-convergence see plots in `/WaveProject/fig/`.

2.2 Runge-Kutta time-integration

Consider the harmonic oscillator. Its Hamiltonian is

$$H(q.p) = \frac{1}{2}\Big(\frac{p^2}{m} - q^2 m\Big), \qquad q = \omega x \tag{53}$$

One can also introduce the constant k such that, $\omega = \sqrt{k/m}$. The period of oscillation is $T = 2\pi/\omega$.
Fo simplicity consider $\omega = 1$, thus, $T = 2\pi$.
This transforms the equation 53 to:

$$2H = p^2 + x^2 \qquad \text{(a const.)} \tag{54}$$

Hence, a trajectory in the "phase space" plot, i.e. trajectory in the $x - p$ plane, should be a circle of radius $\sqrt{2E}$.
The Newton's equation of motion are is I strongly believe that this equation has to be given in the excessive. I did not get, that I have to solve this one

$$\frac{\partial^2 x}{\partial t^2} = -x \tag{55}$$

which can be written as a two first order differential equations

$$\begin{aligned} \frac{\partial x}{\partial t} &= p, \\ \frac{\partial p}{\partial t} &= -x \end{aligned} \tag{56}$$

These equations can be solved analytically as well as numerically.
First consider an analytical solution, (where we restored ω) which is

$$x(t) = C_1 \exp(-t\sqrt{-\omega}) + C_2 \exp(t\sqrt{-\omega}) \tag{57}$$

setting initial conditions allows to obtain constants C_1, C_2,

$$\begin{aligned} C_1 + C_2 &= x_0, \\ -C_1\sqrt{-\omega} + C_2\sqrt{-\omega} &= v_0 \end{aligned} \tag{58}$$

where v_0 and x_0 are initial position and coordinate. Solving this system, gives

$$\begin{aligned} C_1 &= -\frac{v_0}{2\sqrt{-\omega}} + \frac{x_0}{2}, \\ C_2 &= \frac{v_0}{2\sqrt{-\omega}} + \frac{x_0}{2} \end{aligned} \tag{59}$$

and thus, the overall solution is

$$x(t) = \Big(-\frac{v_0}{2\sqrt{-\omega}} + \frac{x_0}{2} \Big) \exp(-t\sqrt{-\omega}) + \Big(\frac{v_0}{2\sqrt{-\omega}} + \frac{x_0}{2} \Big) \exp(t\sqrt{-\omega}) \tag{60}$$

This generic solution can be simplified for two cases.
First. Initial position is $x_0 = 0$ and initial velocity $v_0 \neq 0$. This reduces the Eq. 60 to

$$x(t) = \cos(t) \tag{61}$$

Second. initial position is zero and initial velocity is nonzero:

$$x(t) = -\frac{i}{2}e^{it} + \frac{i}{2}e^{-it} \tag{62}$$

which is equivalent to $\sin(x)$. Proceeding to the numerical solution, see the implementation in script `ex2.py`
The 4-th order Runge-Kutta time-integration is implemented following following `https://www.youtube.com/watch?v=mqoqAovXxWA&t=4s`. There is not need in finite-differencing techniques in the exercise. The result of this implementation is shown in Fig. 2.2

Parameters used are $x_0 = 0$ $p_0 = 1$, $t_0 = 0$, $t_{max} = 20$, $h = 0.2$ and constants were set to $m = 1$, $\omega = 1$.
Convergence and energy conservation are showen in 2.2.

Overall, as expected, the convergence of the problem is > 16 (for $4th$ order Runge-Kutta time-integration due to symmetries. However, self-convergence is lower and is 12. This might be due to low resolution h. The energy conservation is present to a high order, but the system is loosing energy with time, and the energy loss is the highest for low resolution h.

3 Wave Equation

Consider the reduced to a 1st order in time wave equation:

$$\begin{aligned} \partial_t \phi &= \Pi \\ \partial_t \Pi &= c^2 \partial_{xx} \phi \end{aligned} \tag{63}$$

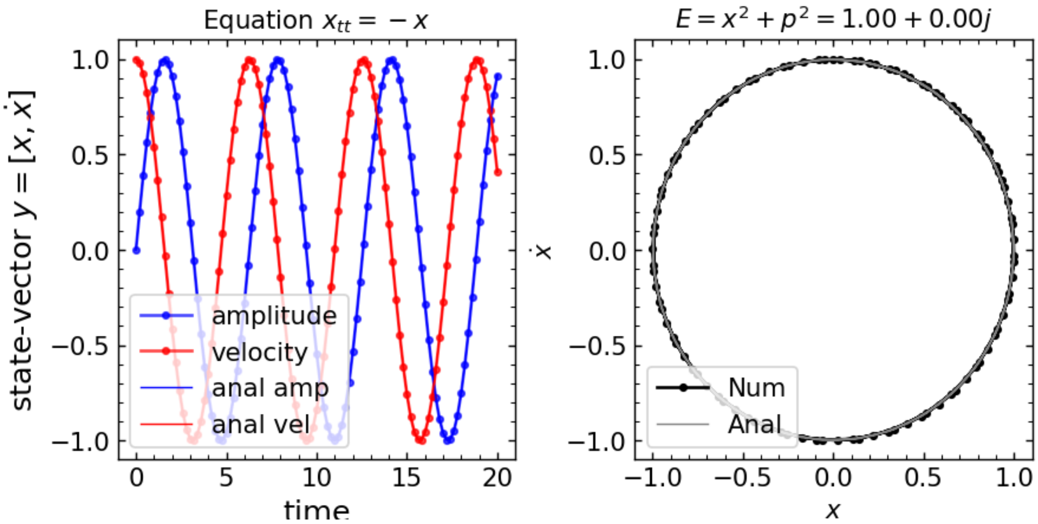

Figure 2: Numerical and analytical solutions to the harmonic oscillator for the coordinate and its velocity on the left and for energy on the right.

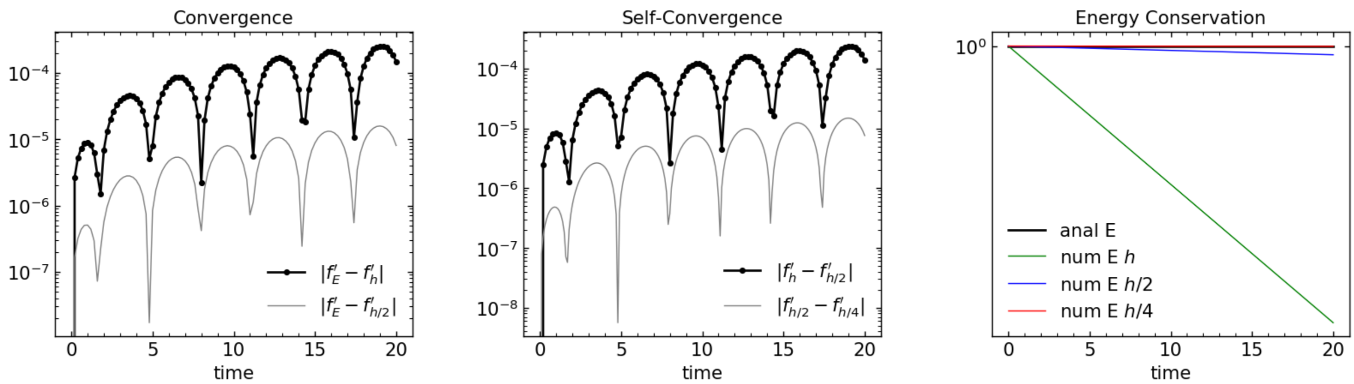

Figure 3: Convergence, self-convergence and energy conservation of the numerical implementation of the harmonic oscillator

For the time derivative the Runge-Kutta scheme can be implemented in accordance with the previous exercise. For the spatial derivative $\partial_{xx}\phi$ the finite diferencing can be used in accordance with the section 2, which will transform the system to a semi-analytic system:

$$\begin{aligned} \partial_t \phi &= \Pi \\ \partial_t \Pi &= c^2 \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta x)^2} \end{aligned} \tag{64}$$

where ϕ_i, ϕ_{i+1}, ϕ_{i-1} are the values of the function at a grid point i, $i+1$ and $i-1$, Δx is the grid spacing. This is the *method of lines*.

As was discussed in the section 2, for the central stencil of the finite diferencing method, points around the i are required. At the boundaries of the grid, for example from $i = 0$ to $i = n$, method requires points at $i = -1$ and $i = n+1$, that are not part of physical domain. The solution here is to use so-called *ghost points*, extending the actual domain of the solution to include the required points. Thus the physical domain would lay in range $i \in [ng, N - ng]$, where ng is the number of ghost points (1 for the first order accuracy schemes and 2 for the second), and N is the total number of points.

In the case of the second order PDE, the boundary condition has to be specified. Here the *periodic boundary conditions* are implemented as a following.

In its core, the periodic boundary conditions imply that at the boundaries of the domain that condition in imposed such that $\phi[0] = \phi[n]$. However, implementing such a condition would lead to numerical complications, such as the difficulty with filling the ghost points. Another approach lays in a considering the grid as a *staggered grid* (see fig. **??**). This leads to the boundary conditions beiing realized as shown in the left panel of the fig. 3. there, the condition that $f(p_1) = f(p_{n-2})$, the function is the same at the last physical points of the grid on the left and on the right is assured by setting ghosts p_0 and p_{n-1} to their respective values on the other side, namely:

$$\begin{aligned} f(p_{n-1}) &= f(p_2) \\ f(p_0) &= f(p_{n-3}) \end{aligned} \tag{65}$$

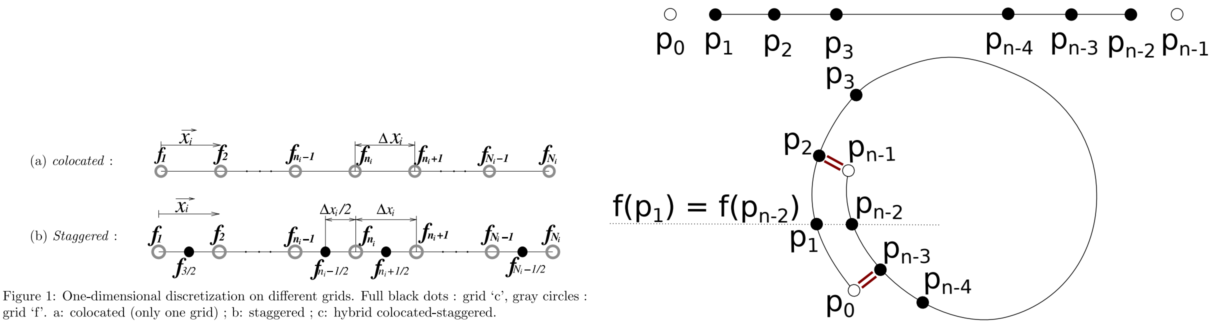

Figure 4: Different grid types, staggered grid periodic boundary conditions for the staggered grid on the right. Here, filled dots are of physical domain. Unfilled – ghost points.

Now, having the equations, methods to solve (Runge-Kutta and finite differencing) and methods to implement boundary conditions, the numerical implementation can be realized. The last thing is to fix the initial profile, the ϕ_0 and $\Pi_0 = \partial/\partial_t \phi_0$. Consider first the simple periodic function:

$$\phi = \cos(2\pi(x - t)) \tag{66}$$

, The initial profile is

$$\begin{aligned} \phi_0 &= \cos(2\pi x) \\ \Pi_0 &= 2\pi \sin(2\pi x) \end{aligned} \tag{67}$$

, With initial profile, the numerical implementation yields the following results (see `ex3.py` for the code):

4 Convergence of the numerical implementation of the Wave Equation

Fig. 4 shows that with time the solution changes due to numerical dissipation. The effect intensifies for higher dt/dx. At dt/dx the evolution becomes unstable. See `https://hplgit.github.io/fdm-book/doc/pub/wave/pdf/wave-4print.pdf` for a very good description of the process. Assessing convergence of the numerical scheme, the familiar formula is utilized:

$$\frac{||f'_E(x) - f'_h(x)||_2}{||f'_E - f'_{h/2}(x)||_2} = \frac{Ch^\theta}{C(h/2)^\theta} = \frac{h^\theta}{h^\theta} \times 2^\theta = 2^\theta \tag{68}$$

In the numerical implementation (see script `ex4.py`) the grid density is set as n, which in turn sets the spacing dx. The evolution timestep is set via the $dt = \xi dx$. Equivalently, in terms of spacing is $\Delta t = (\alpha/|c|)h = \xi h$.
Taking for the convergence study $n_1 = 101$ and $n_2 = 201$, twice as much, and the $\xi = 1$. results not only in having twice as dens spatial grid, but also in having twice as many time steps for the evolution, to preserve the $\xi = 1$. This is important fact not to overlook.
To compute convergence, the follwoing quantities are to be estimated $f'_E(x)$, $f'_h(x)$, $f'_{h/2}(x)$ for the equation

$$norm = \log_2\left(\frac{||f'_E(x) - f'_h(x)||_2}{||f'_E(x) - f'_{h/2}(x)||_2}\right) \tag{69}$$

or in terms of n: $f'_{E,n}(x)$, $f'_n(x)$, $f'_{2n}(x)$. However, to compute the denominator, $f'_{E,n}(x) - f'_{2n}(x)$, either analytic solution has to be recomputed with the $2n$ grid or take half of the array of numerical solution $f'_{2n}(x)$. Taking the first approach the, numerically the $norm$ is estimated as

$$norm = \log_2\left(\frac{\sqrt{\sum(f'_{E;n}(x) - f'_n(x))^2}}{\sqrt{\sum(f'_{E;2n}(x) - f'_{2n}(x))^2}}\right) \tag{70}$$

Computing this norm for every timespte (of the solution with n grid and every second timestep for the solution with $2n$ points), result is shown in the fig Similarly, the self-convergence is computed (Fig.4, left panel), realizing numerically the equation:

$$norm = \log_2\left(\frac{\sqrt{\sum(f'_n(x) - f'_{2n}(x))^2}}{\sqrt{\sum(f'_{2n}(x) - f'_{4n}(x))^2}}\right) \tag{71}$$

with 3 different resolutions and taking every second time step for the solution with $2n$ grid points and every fourth – for the solution with $4n$ points.
Since the accuracy of the finite-differencing scheme is 2 this order of convergence is expected and obtained. The last $dt/dx = 1.5$ is unstable and shows that convergence quickly drops with time.

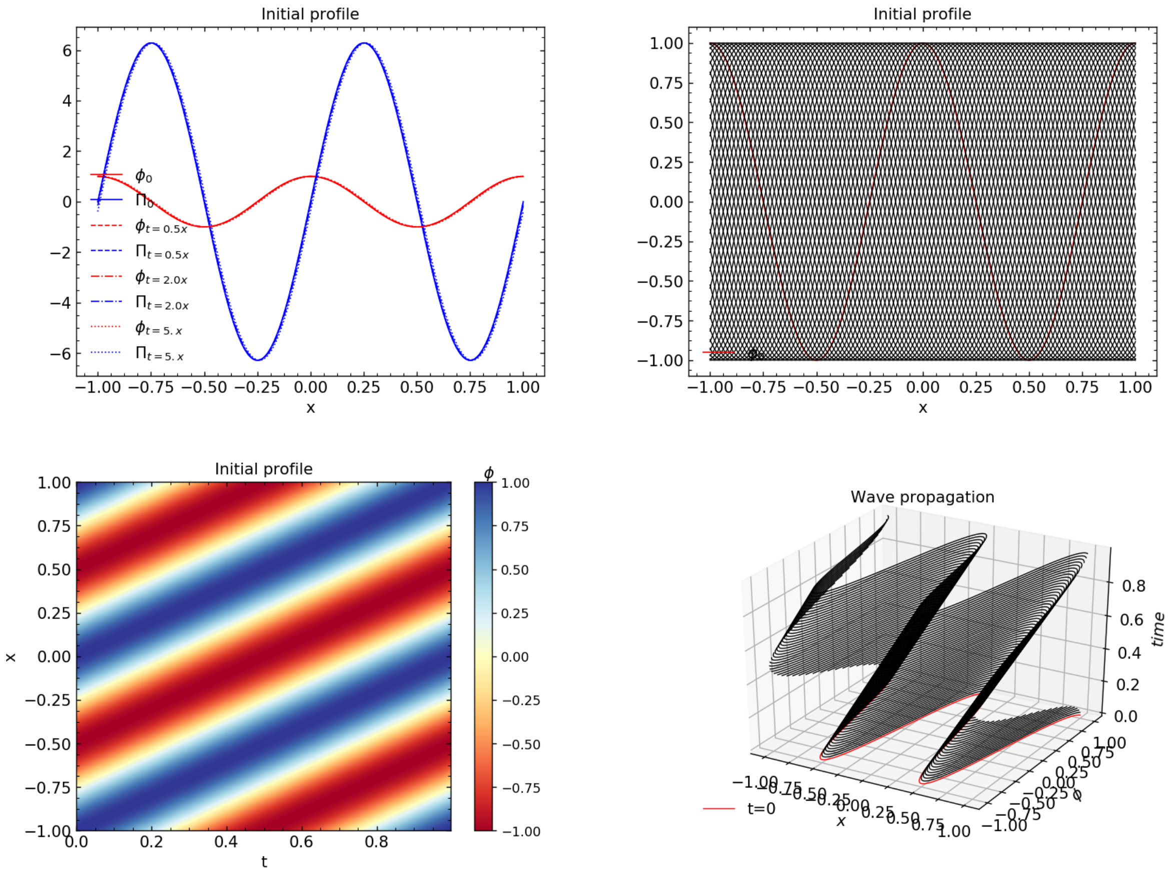

Figure 5: Right top panel – initial profile with different dt steps. It shows that with current N or dx settings, the effect of higher timestep is small. Top left panel – initial profile for ϕ_0 and its time evolution untill $t = 1.$ with timestep $dt = 0.5dx$ to assure stability. Bottom left panel – time evolution of the ϕ colorcoded as a function of x and t. See peaks of the waves moving along x with time. Bottom left – same, but a 3D plot.

4.1 Open boundary conditions

4.1.1 Extrapolating ghosts

Assuming that the x grid consists of $n = N + 2ng$ points, where N are physical points and $ng = 1$ are ghost points on the left and right, with total number of points being n. Points $i = 0$ and $i = n - 1$ are ghosts. To fill them, it is suggested to use linear extrapolation. Meaning, to employ the following formula for $i = n - 1$

$$\phi_{n-1} = \phi_{n-2} + \frac{x_{n-1} - x_{n-3}}{x_{n-2} - x_{n-3}}(\phi_{n-2} - \phi_{n-3}) \tag{72}$$

and for the $i = 0$,

$$\phi_0 = \phi_1 + \frac{x_0 - x_2}{x_1 - x_2}(\phi_1 - \phi_2) \tag{73}$$

5 On Scalar wave equation and numerical solutions in black hole space times

Consider the hyperbolic equation, the $1D$ equation

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{74}$$

which is second order in time and in space. CIntroduce the variable Π that converts the eqiation into

$$\begin{cases} \frac{\partial \psi}{\partial t} & = \Pi \\ \frac{\partial \Pi}{\partial t} & = c^2 \frac{\partial^2 \psi}{\partial x^2} \end{cases} \tag{75}$$

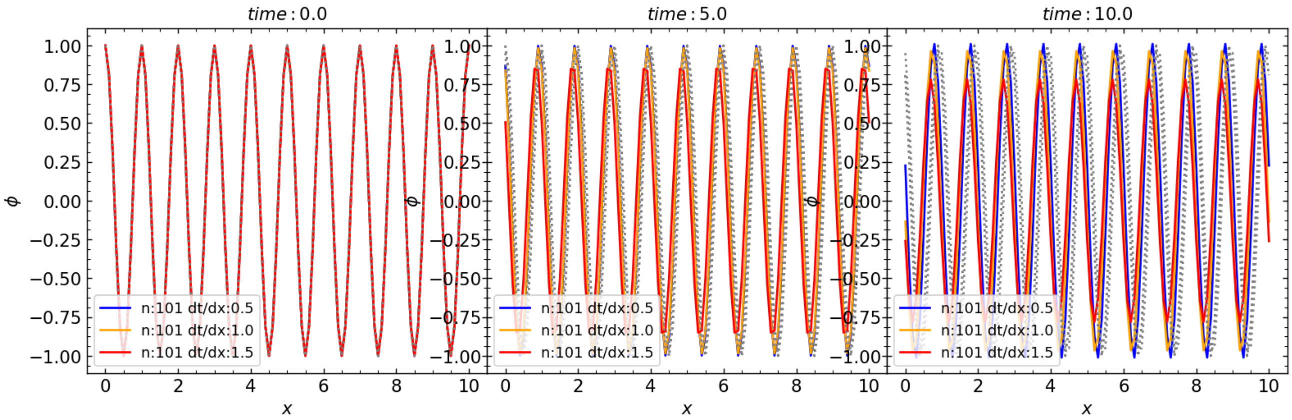

Figure 6: Comparison of the solution at three timepsteps and three values of dt/dx. Used coursed grid of 101 points

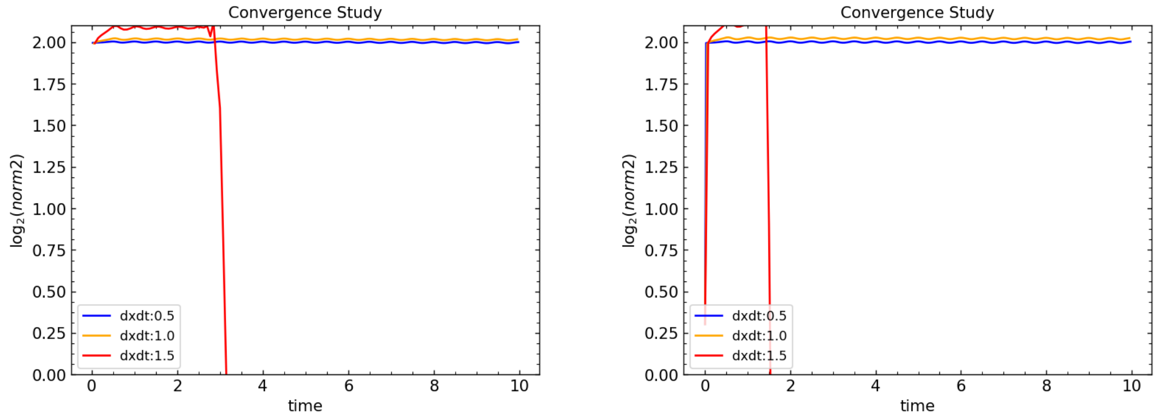

Figure 7: Convergence (left) and self-convergence(right) study. Plotted $\log_2(norm2)$ from eq. 70 as a function of time. Three dx/dt are shown.

which are both first order in time, which simplify the numerical solution
Numerically, the second order derivative (central difference) is

$$\left.\frac{\partial^2 \psi_i}{\partial x^2}\right|_t \approx \frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{\Delta x^2} \tag{76}$$

The solution can be integrated forward in time (with *e.g.,* RK4 scheme)

$$\begin{cases} \psi_i^{next} & = \int \Pi_i dt \\ \Pi_i^{next} & = \int c^2 \frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{\Delta x^2} dt \end{cases} \tag{77}$$

The grid has the following
For the result of this implemnetation, – see the previous sections (for numerical tests and convergence).
See also periodic and open boudaries and the necessiry of the staggared grid.
See the example of the dissipation of the initial gaussian wave.

5.1 General wave equation [Beyond the exercise]

This section is based on the semenar talk of Adam Kiddle. Firther reading: lectured 2 of Emanuele Berti *Black hole perturbation theory.*

Consider a general 4-dimensional spacetime, the scalar wave equation is:

$$\Box_g \Psi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = 0 \tag{78}$$

Take the Schwarzschild metric in Schwarzschild coordinates. Here $(c, M = 1)$.

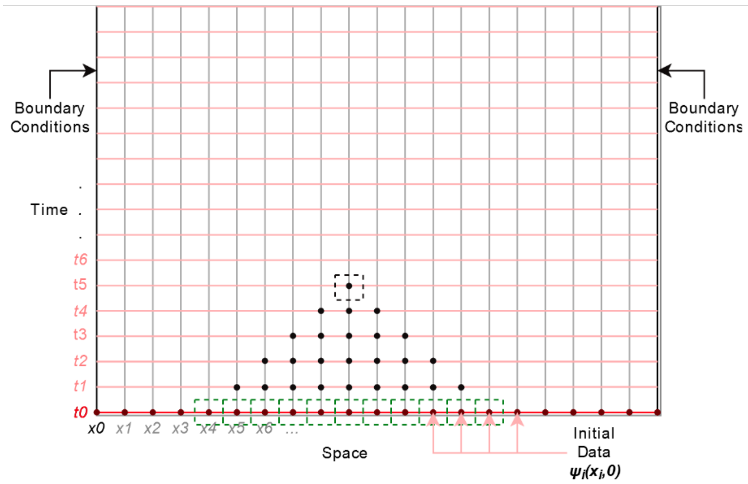

Figure 8: Representation of the grid for 1D wave Eq. (Courtesy of Adam Kiddle)

$$dx^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2 \text{ where } f = 1 - \frac{2}{r} \tag{79}$$

Expand solution into scalar harmonics:

$$\Psi = \sum_{l,m} \frac{\psi_{lm}}{r} P_{lm}(\theta) e^{im\phi} \tag{80}$$

Expand solution into scalar harmonics:

$$\Psi(t,r,\theta,\phi) = \sum_{lm} \frac{\psi_{lm}(t,r)}{r} P_{lm}(\theta) e^{im\phi} \tag{81}$$

Put this solution into the wave equation to obtain a PDE for any given mode $\psi_{lm}(r,t)$

$$f^2 \frac{\partial \psi_{lm}}{\partial r^2} - \frac{\partial^2 \psi_{lm}}{\partial t^2} + f \frac{\partial f}{\partial r} \frac{\partial \psi_{lm}}{\partial r} - \frac{f}{r} \Big(\frac{\partial f}{\partial r} + \frac{l(l+1)}{r} \Big) \psi_{lm} = 0 \tag{82}$$

using a new radial coordiante r^* which statisfies $f = \frac{dr}{dr^*}$ in which the equations are

$$\frac{\partial^2 \psi_{lm}}{\partial r^{*2}} - \frac{\partial^2 \psi_{lm}}{\partial t^2} - \frac{f}{r} \left(\frac{\partial f}{\partial r} + \frac{l(l+1)}{r} \right) \psi_{lm} = 0 \tag{83}$$

See here the potential plotted as $V(l=1)$ as a function of r. It has a sharp peak and decay. The $V \to -\infty$ at $r \to 0$ is the event horizon. The maximum of the potential near the event horizon is the photon ring.

There is a discussion of potential parities.

This modifies the wave equation (initially implemented as)

$$\begin{cases} \psi_i^{next} & = \int \Pi_i dt \\ \Pi_i^{next} & = \int c^2 \frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{\Delta x^2} - \frac{f}{r_i} \left(\left(\frac{\partial f}{\partial r} \right)_i + \frac{l(l+1)}{r_i} \right) \psi_i \end{cases} \tag{84}$$

using the dissipative boundaries at $r^* = \pm\infty$, assuring that no waves are coming in or out (form event horizeon or from spatial infinity)

Here see the results and discussion on the numerical implementation of the potential.

The solution is composed of exponential and sumped sinusoidal parts. The latter is the quasinormal mode.

More to be red in literature.