1. $$\vec{\nabla} \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\Rightarrow \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial v_x} - \frac{\partial v_x}{\partial y}\right)$$

$$\Rightarrow \left(\frac{\partial^2 v_z}{\partial x\,\partial y} - \frac{\partial^2 v_z}{\partial y\,\partial x}\right) + \left(\frac{\partial^2 v_x}{\partial y\,\partial x} - \frac{\partial^2 v_x}{\partial z\,\partial y}\right) + \left(\frac{\partial^2 v_y}{\partial z\,\partial x} - \frac{\partial^2 v_y}{\partial x\,\partial z}\right) = 0$$

$$\vec{v_a} = -y\hat{x} + x\hat{y}$$

$$\vec{\nabla} \times \vec{V_a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\hat{k}$$

$$\vec{\nabla}\cdot(2\hat{k}) = 0$$

2. $$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$\vec{\nabla} \times \vec{\nabla} T = \begin{vmatrix} i & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix} \Rightarrow$$

$$\Rightarrow \hat{i}\left(\frac{\partial^2 T}{\partial y \partial z} - \frac{\partial^2 T}{\partial z \partial y}\right) + \hat{j}\left(\frac{\partial^2 T}{\partial z \partial x} - \frac{\partial^2 T}{\partial x \partial z}\right)$$

$$+ \hat{k}\left(\frac{\partial^2 T}{\partial x \partial y} - \frac{\partial^2 T}{\partial y \partial x}\right) = 0$$

$$f(x, y, z) = x^2 y^3 z^4$$

$$\nabla f(x, y, z) = (2xy^3z^4)\hat{i} + (3x^2y^2z^4)\hat{j} + (4x^2y^3z^3)\hat{k}$$

$$\vec{\nabla} \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & 3y^3x^2z^4 & 4x^2y^3z^3 \end{vmatrix}$$

$$\Rightarrow \hat{i}\,(3 \times 4 \times x^2y^2z^3 - 4 \times 3x^2y^2z^3) + \hat{j}\,(4 \times 2xy^3z^3 - 2 \times 4xy^3z^3)$$

$$+ \hat{k}\,(2 \times 3xy^2z^4 - 3 \times 2xy^2z^4)$$

$$= 0$$

(3) $A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{A} \cdot \vec{\nabla} = A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}$$ $\longrightarrow$ Scalar operator.

$$B = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$(\vec{A}\cdot\vec{\nabla})\vec{B} = \left(A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_x}{\partial z}\right)\hat{i}$$

$$+ \left(A_y \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z}\right)\hat{j}$$

$$+ \left(A_z \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z}\right)\hat{k}$$

This is a Vector field Is there any deeper meaning ?

(b) $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, Let's just calculate the X component we have just derived the expression for $(\vec{A}\cdot\vec{\nabla})\vec{B}$. Comparing the expression of $[(\vec{A}\cdot\vec{\nabla})\vec{B}]_x$.

$$[(\hat{r}\cdot\vec{\nabla})\hat{r}]_x = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \frac{1}{r}\left[x\left[\frac{1}{\sqrt{x^2+y^2+z^2}} + x\left(-\frac{1}{2}\right)\frac{2x}{(x^2+y^2+z^2)^{3/2}}\right]\right.$$

$$+ yx\left[-\frac{1}{2}\cdot\frac{2y}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$\left. + zx\left[-\frac{1}{2}\,\frac{1\times 2z}{(x^2+y^2+z^2)^{3/2}}\right]\right\}$$

$$\Rightarrow \frac{1}{r}\left\{\frac{x}{r} - \frac{1}{r^3}(x^3 + xy^2 + xz^2)\right\}$$

$$\Rightarrow \frac{1}{r}\left\{\frac{x}{r} - \frac{x}{r^3} \times \underbrace{(x^2+y^2+z^2)}_{\Rightarrow r^2}\right\}$$

$$\Rightarrow \frac{1}{r}\left[\frac{x}{r} - \frac{x}{r}\right] = 0$$

(c) $$(\vec{V}a \cdot \vec{\nabla}) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + 0$$

$$(\vec{V}a \cdot \vec{\nabla}) x\hat{y} = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)(x\hat{y}) = -y\hat{y}$$

4. $$(\vec{A} \times (\vec{\nabla} \times \vec{B}))_x = A_y(\vec{\nabla} \times \vec{B})_z - A_z(\vec{\nabla} \times \vec{B})_y$$

$$\Rightarrow A_y\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - A_z\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)$$

$$[\vec{B} \times (\vec{\nabla} \times A)]_x = B_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - B_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$[(\vec{A} \cdot \vec{\nabla})B]_x = \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_x$$

$$\Rightarrow A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}$$

$$[\vec{\nabla}(\vec{A} \cdot \vec{B})]_x \Rightarrow [\vec{B} \times (\vec{\nabla} \times \vec{A})]_x + [(\vec{A} \cdot \vec{\nabla})B]_x$$

$$+ [(\vec{B} \cdot \vec{\nabla})A]_x + [\vec{\nabla}(\vec{A} \cdot \vec{B})]_x$$

By the Similar tecnique one Can prove the 2nd identity.

5(a) $(0,0,0) \to (1,0,0)$, x goes from $0 \to 1$

$y = z = 0$

$d\vec{l} = dx\,\hat{x}$

$\vec{V}\cdot d\vec{l} = x^2 dx$ $\int V \cdot dl = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$(1,0,0) \to (1,1,0)$ $x = 1$

$y =$ goes from 0 to 1

$z = 0$ $d\vec{l} = dy\,\hat{y}$

$\vec{V}\cdot d\vec{l} = 2yz\,dy$

$2 \times 0 \int_0^1 dy = 0$

$(1,1,0) \to (1,1,1)$, $x = y = 1$, z goes to 1.

$dl = dz\,\hat{z}$, $\vec{V}\cdot d\vec{l} = y^2 dz$

$\int V \cdot dl = \int_0^1 V \cdot dl = \int_0^1 dz = 1$

Total (a) $\int V \cdot dl = \frac{4}{3}$

(b) $(0,0,0) \to (0,0,1)$, $x = y = 0$, $z: 0 \to 1$

$$d\vec{l} = dz\,\hat{z}, \quad \vec{v}\cdot d\vec{l} = y^2 dz = 0, \quad \int \vec{v}\cdot d\vec{l} = 0$$

$(0,0,1) \to (0,1,1)$, $x = 0$, $y \to 1$, $z = 1$, $d\vec{l} = dy\,\hat{y}$

$$\vec{v}\cdot d\vec{l} = 2yz\,dy = 2y\,dy, \quad \int \vec{v}\cdot d\vec{l} = \int_0^1 2y\,dy$$
$$= y^2\Big|_0^1 = 1$$

$(0,1,1) \to (1,1,1)$, for $x: 0 \to 1$, $y = z = 1$, $d\vec{l} = dx\,\hat{x}$

$$\vec{v}\cdot d\vec{l} = x^2 dx, \quad \int \vec{v}\cdot d\vec{l} = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\text{Total}_{(b)} \int \vec{v}\cdot d\vec{l} = 4/3$$

(c) for direct straight line

$$x = y = z \longrightarrow 0 \to 1, \quad dx = dy = dz$$

$$\vec{v}\cdot d\vec{l} = x^2 dx + 2yz\,dy + y^2 dz$$
$$= x^2 dx + 2x^2 dx + x^2 dx = 4x^2 dx$$

(d) Around the close loop

$$\oint \vec{v}\cdot d\vec{l} = \frac{4}{3} - \frac{4}{3} = 0$$

$\vec{v}(x, y, t)$ is a Conservative field

6(a) 1st Segment

$\Rightarrow (2x+4y)\hat{x} + (4x+2z^3)\hat{y} + 6yz^2\hat{z}$

$x: 0 \to 1$, $\int_0^1 (2x)\,dx = x^2\Big|_0^1 = 1$

2nd segment - $y: 0 \to 1$, $x=1, z=0$, $dx = dz = 0$, $\int_0^1 4\,dy = 4$

3rd segment - $z: 0 \to 1$, $x=y=1$, $dx = dy = 0$ $\int_0^1 6z^2 dz = 2$

Total = 7

6(b) Segment 1: $z: 0 \to 1$, $x = y = dx = dy = 0$, $\int_0^1 (0)\,dz = 0$

2nd segment 2: $y: 0 \to 1$, $x=0$, $z=1$, $dx = dz = 0$.

$\int_0^1 (2)\,dy = 2y\Big|_0^1 = 2$

3rd segment - $x: 0 \to 1$, $y=1, z=1$, $dy = dz = 0$

$\int_0^1 (2x+4)\,dx = \frac{2x^2}{2} + 4x\Big|_0^1 = 5$

Total = 7

(c) $\int [(2x+4y)\hat{x} + (4x+2z^3)\hat{y} + 6yz^2\hat{z}] \cdot [dx\hat{x} + dy\hat{y} + dz\hat{z}]$

$\Rightarrow \int (2x+4y)\,dx + (4x+2z^3)\,dy + (6yz^2)\,dz$

$x=y$, $z=x^2$, $dx=dy$, $dz = 2x\,dx$.

$\int_0^1 (10x + 14x^6]\,dx = 5x^2 + 14\frac{x^7}{7}\Big|_0^1 = 7$

6(c)

$T = x^2 + 4xy + 2yz^3$

$$\int \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$$

$T(b) = 1 + 4 + 2 = 7$

$T(a) = 0$

$T(b) - T(a) = 7$

7. $V = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$

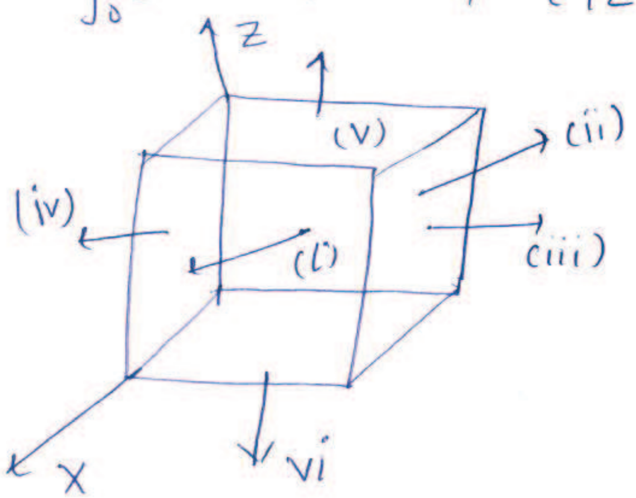
$$\int_V (\vec{\nabla}\cdot\vec{V})\,d\tau \Rightarrow \int_0^2\int_0^2\int_0^2 (y+2z+3x)\,dx\,dy\,dz$$

$$\int_0^2 (y+2z+3x)\,dx \Rightarrow yx + 2zx + \frac{3x^2}{2}\Big|_0^2$$

$$= 2y + 4z + 6$$

$$\int_0^2 (2y+4z+6)\,dy = y^2 + (4z+6)\,y\Big]_0^2 = 8z+16$$

$$\int_0^2 (8z+16)\,dz \Rightarrow (4z^2+16z)\Big|_0^2 = 16+32 = 48.$$



(ii) $da = -dy\,dz\,\hat{x}$ since

$V.da = 0$

$\int v.da = 0$, similarly $\int v.da$ for (iv) & (vi)

will be zero for surface (i) $da = dy\,dz\,\hat{x}$, $x=2$

$$\int v\,da = \int_0^2\int_0^2 2y\,dy\,dz = \int_0^2 \frac{2y^2}{2}\Big|_0^2 dz$$

$$\Rightarrow 4z\Big|_0^2 = 8$$

(iii) $da = dx\,dz\,\hat{y}$, $y=2$, $\int v \cdot da = \int_0^2 \int_0^2 yz\,dx\,dz$

$$= \int_0^2 \frac{4z^2}{2}\Big|_0^2 dx = 16$$

(v) $da = dx\,dy\,\hat{z}$, $z=2$, $\int v \cdot da = 2\int_0^2 \int_0^2 6x\,dx\,dy$

$$= \int_0^2 \frac{6x^2}{2}\Big|_0^2 dy = 12\int_0^2 dy$$

$$= 24$$

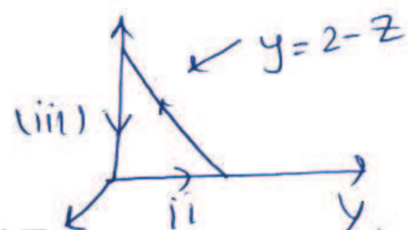
Total $= 48$

8. $V = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$

$$\vec{\nabla} \times \vec{V} = \hat{x}(0-2y) + \hat{y}(0-3z) + \hat{z}(0-x)$$

$$= -2y\hat{x} - 3z\hat{y} - x\hat{z}$$

$da = dy\,dz\,\hat{x}$



$$\int_0^2 \int_0^{2-z} (-2y\,dy)\,dz = \int_0^2 -2\frac{(2-z)^2}{2}\,dz$$

$$= -\int_0^2 (4 - 4z + z^2)\,dz$$

$$= -\left(4z - 2z^2 + \frac{z^3}{3}\right)\Big|_0^2$$

$$= -8 + 8 - \frac{8}{3} = \frac{-8}{3}$$

$\vec{V}.d\vec{l} = (xy)dx + (2yz)dy + (3zx)dz$

(i) $x = 0 = z$, $dx = dz = 0$ $\quad \int \vec{V}.dl = 0$

(ii) $x = 0$, $z = 2-y$, $dx = 0$, $dz = -dy$ $\quad$ y goes from $2 \to 0$

$V.dl = 2yz\,dy$

$V.dl = \int_2^0 2y(2-y)\,dy = -\left(2y^2 - \frac{2}{3}y^3\right)\Big|_0^2$

$= -8 + \frac{2}{3}\times 8 = -\frac{8}{3}$

(iii) $x = y = 0$, $dx = dy = 0$, $\int V.dl = 0$

total $= -8/3$