/Nov/2017

M-III Assignment-5.

Q1.

a) $\sum \frac{z^n}{n^2}$

~~Root Test~~ Ratio Test.

$\frac{z^{n+1}}{(n+1)^2} \cdot \frac{n^2}{z^n} = \left|\frac{z\, n^2}{(n+1)^2}\right|$

$|z| < 1$

b) $\sum \frac{z^n}{n!}$

$\left|\frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right|$

$z \in C$

c) $\sum \frac{z^n}{2^n}$

$\frac{z^{n+1}}{2^{n+1}} \times \frac{2^n}{z^n} = \left|\frac{z}{2}\right| < 1$

$|z| < 2.$

d) $\sum \frac{1}{2^n}\left(\frac{1}{z}\right)^n$

$= \frac{2^n}{2^{n+1}} \frac{z^n}{z^{n+1}} = \left|\frac{1}{2z}\right| < 1$

$\frac{1}{2} < |z|$

Q2. $a_n = \frac{(-1)^n}{\sqrt{n}} + i\frac{1}{n^2}$

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{\frac{(-1)^{2n+2}}{n+1} + \frac{1}{(n+1)^4}}}{\sqrt{\frac{(-1)^{2n}}{n} + \frac{1}{n^4}}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\sqrt{\frac{((n+1)^3+1)\,n^4}{(n+1)^4\,(n^3+1)}}$$

$$\cancel{1 + \frac{1}{(n+1)^3}}$$

$$L = \lim_{n\to\infty}\sqrt{\frac{\left(1+\frac{1}{n+1}\right)}{\left(\frac{n+1}{n}\right)\left(1+\frac{1}{n^3}\right)}} = 1$$

Test Fails.

For $a_n = \frac{(-1)^n}{n} + i\frac{1}{n^2}$ ~~to be con~~

$\Sigma a_n \to$ convergent Conditionally.

$\Downarrow$ Need to proof.

need to proof $\frac{(-1)^n}{\sqrt{n}} \to$ convergent

and $\frac{1}{n^2} \to$ convergent

For $\Sigma a_n \to$ Absolute convergence

$\Downarrow$ Need to proof

$\frac{1}{\sqrt{n}} \to$ converges $\qquad \frac{1}{n^2} \to$ converges.

$\frac{1}{n^p}$ $\quad p>1$ convergent

$\quad p\le 1$ diverdent thus $\frac{1}{\sqrt{n}}$ diverges $\to$ X Absolute

$\frac{1}{n} + \frac{1}{n^2}$

$$\frac{(n+1)^3 + (n+1)}{(n+1)^4}$$

$$\frac{n^3+1}{n^4}$$

$$\frac{n^3\left[\left(1+\frac{1}{n}\right)^3 + \left(\frac{1}{n^2}+\frac{1}{n^3}\right)\right]}{n^4\left(1+\frac{1}{n^4}\right)}$$

$n^3($

For $\frac{(-1)^n}{a_n}$

By alternating series test

* $\frac{1}{n}$ decreases * $\lim_{n\to\infty} \frac{1}{n} = 0$

Thus converges.

converges but not absolutely. $\frac{1}{n^2} \to$ converges thus

Q3. $\left|\frac{a_n}{a_{n+1}}\right| = R$

a) $\lim_{n\to\infty} \left(\frac{\ln(n)}{\ln(n+1)}\right)^2 = \left(\frac{\frac{1}{n}}{\frac{1}{n+1}}\right)^2 = 1 = R$

b) $\lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \frac{1}{n+1} = 0 = R.$

c) $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} \frac{4^n + 3n}{4^{n+1} + 3(n+1)} = \frac{1}{(1+\frac{1}{n})^2} \frac{1 + \frac{3n}{4^n}}{4 + \frac{3(n+1)}{4^n}} = \frac{1}{4} = R$

d) $\lim_{n\to\infty} \frac{(n!)^3}{\cancel{3n!}} \times \frac{3(n+1)!}{((n+1)!)^3} = \frac{1}{(n+1)^2} = 0 = R.$

e) $\lim_{n\to\infty} \left(\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}}\right) \frac{\sqrt{(n+1)^2+(n+1)}}{\sqrt{n+2} - \sqrt{n+1}}$

$\lim_{n\to\infty} \left(\frac{\sqrt{n+2} + \sqrt{n+1}}{\sqrt{n+1} + \sqrt{n}}\right)\left(\sqrt{\frac{\cancel{(n+1)}(n+2)}{n\cancel{(n+1)}}}\right) = 1 = R$

Q4. $$\sum_{n=0}^{\infty} (n+1)^2 z^n = \frac{1+z}{(1-z)^3}$$

$\cancel{\frac{1}{(1-z)^3}}$ $\cancel{\frac{z}{(1-z)^3}}$

$\cancel{\sum_{n=1}^{\infty} ((n+1))^2 z^n}$

$\cancel{\sum_{n=0}^{\infty} (n^2 + 2n + 1) z^n}$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} n z^{(n-1)}$$

$$\frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} n(n-1) z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) z^n$$

$$\frac{2z}{(1-z)^3} = \sum_{n=0}^{\infty} (n+1)(n) z^n$$

$$\frac{2+2z}{1-z^3} = 2\sum_{n=0}^{\infty} (n+1)^2 z^n = \frac{1+z}{1-z^3}$$

5. a) $f(z) = \frac{1}{z^2}$

$$\frac{1}{a+z-a} = \frac{1}{a} \frac{1}{1+\left(\frac{z-a}{a}\right)} \qquad \left|\frac{z-a}{a}\right| < 1$$

$$\frac{1}{z} = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-a}{a}\right)^n$$

$$\frac{-1}{z^2} = \frac{1}{a}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n (z-a)^{n-1}}{a^n}$$

$$\frac{1}{z^2} = \sum_{n=0} \frac{(-1)^{n+1} n (z-a)^{n-1}}{a^n}$$

$$f(z) = \frac{6z+8}{(2z+3)(4z+5)} \quad \text{at } z=1.$$

$$= \frac{1}{2z+3} + \frac{1}{4z+5}$$

$$= \frac{1}{2z-2+5} + \frac{1}{4z-4+9}$$

$$= \frac{1}{5}\frac{1}{\frac{2}{5}(z-1)+1} + \frac{1}{9}\frac{1}{\frac{4}{9}(z-1)+1}$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n(-1)^n(z-1)^n + \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n(-1)^n(z-1)^n$$

c)

$$f(z) = \frac{e^{z-1}}{z+1} \quad e \qquad |z-1|<1.$$

$$= \cancel{\frac{1+1}{z}} \cancel{\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\right)} \frac{1}{2+(z-1)}$$

$$= \left(\frac{1}{2}\frac{1}{1+(z-1)}\right)\left(1+z-1+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\dots\right)$$

$$= \frac{1}{2}\left(1+(z-1)+(z-1)^2+\dots\right)\left(1+(z-1)+\frac{(z-1)^2}{2!}+\right)$$

06.

a) $f(z) = \frac{1}{z-3}$ $\qquad |z| > 3$

$\frac{1}{|z|} < \frac{1}{3} < 1$

$= \left(\frac{1}{z}\right)\left(\frac{1}{1-\left(\frac{3}{z}\right)}\right)$

$\left|\frac{3}{z}\right| < 1$

$= \frac{1}{z}\left(1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \text{---}\right)$

$= \frac{1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \text{---}$

b) $f(z) = \frac{1}{z(z-1)}$ $\qquad 0 < |z| < 1$

$= \frac{-1}{z}\left(1 + z + z^2 + \text{---}\right) = \frac{-1}{z} - 1 - z - z^2 + \text{---}$

c) $f(z) = z^3 e^{\frac{1}{z}}$ for $|z| > 0$

Case 1. $0 < |z| < 1$

$f(z) = z^3\left(1 + \frac{1}{z} + \frac{1}{z^2} + \text{---}\right)$

$= z^3 + z^2 + z + 1 + \frac{1}{z} + \text{---}$

Case 2 $|z| > 1$ $\qquad \left|\frac{1}{z}\right| < 1$

$f(z) = z^3 + z^2 + z + 1 + \frac{1}{z} + \text{--}$

Q7. $f(z) = \frac{1}{z(1+z^2)}$ $|z| < 1$

$$\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^n = \sum_{n=0}^{\infty}(-1)^n z^{n-1}$$

$$\frac{1}{|z|^3}\left(\frac{1}{1+\frac{1}{|z|^2}}\right)$$ $|z| > 1$

$\left|\frac{1}{z}\right| < 1$

$$\frac{1}{|z|^3}\left(1+\frac{(-1)}{|z|^2}+\frac{1}{|z|^4}+\frac{(-1)}{|z|^6}+\cdots\right)$$

$$\frac{1}{|z|^3}-\frac{1}{|z|^5}+\frac{1}{|z|^7}-\frac{1}{|z|^9}+\cdots$$

Q8. $|f(z)| - |g(z)| < 0$

$|g(z)| - |f(z)| > 0$

$\cancel{|g(z)|}$ $\cancel{|g(z) - f(z)| > |g(z)| - |f(z)| > 0}$

$\cancel{|f(z) - g(z)| > 0}$

$\left|\frac{f(z)}{g(z)}\right| < 1$ by Liouville Theorem

$\left|\frac{f(z)}{g(z)}\right|$ is constant say λ

$f(z) = \lambda g(z)$

$f'(z) = kf(z)$ $\ln f(z) = k + C$

$f(z) = e^{(k+c)z} = e^{pz}$.

Q9

a) $ze^{1/z}$ $\quad z=0$ Essential Singular Point.

b) $\dfrac{\sin z}{z}$ $\quad z=0$ Removable Singular Point.

c) $\dfrac{1-\cos z}{z^2}$ $\quad z=0$ Removable Singular Point.

d) $\dfrac{\pi \cot \pi z}{z^2}$

$$\frac{1}{z} - \frac{z}{3} + \frac{z^3}{245} + \cdots$$

Principal part has finite no. of terms thus removable singularity.

e) $\dfrac{z-\sin(z-1)}{z-1}$

$$= \frac{z - \left((z-1) - \frac{(z-1)^3}{3!} + \cdots \right)}{z-1}$$

$$= \frac{1 + \frac{(z-1)^3}{3!} - \frac{(z-1)^5}{5!} + \cdots}{(z-1)}$$

Non Removable

Essential Singularity.

f) $\dfrac{z^2+\sin z}{\cos(z)-1} = \dfrac{z^2 + z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}{\cancel{1} - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots - \cancel{1}}$

Essential Singularity.

Q10. a) $\dfrac{\sin z}{z^2-\pi^2}$ $\quad$ $z = \pm\pi$ Removable Singularity.

$$\frac{\sin(\pi - z)}{-(z+\pi)(z-\pi)} \qquad \frac{\sin(z+\pi)}{(z-\pi)(z+\pi)}$$

Scanned by CamScanner

Date
Page

$z=\pi$ is pole of order 1.

ii) $\dfrac{z\cos z}{1-\sin z}$ $\quad z=\pi/2$

$\dfrac{z^2\sin(\pi/2-z)}{(1-\cos z)\,z}$ Removable Singularity

Q11.

a) $\dfrac{1}{z(z+1)} = \dfrac{1}{z}\left(1-z+z^2-z^3+\ldots\right)$

$b_1=1$

b) $z\cos\dfrac{1}{z} = z\left(1-\left(\dfrac{1}{z}\right)^2\dfrac{1}{2!}+\left(\dfrac{1}{z}\right)^4\dfrac{1}{4!}+\ldots\right)$

$= z-\dfrac{1}{2z}+\dfrac{1}{4!z^3}+\ldots$

$b_1=-\dfrac{1}{2}$

c) $\dfrac{z-\sin z}{z} = \dfrac{z-\left(z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}+\ldots\right)}{z}$

$= \dfrac{\dfrac{z^3}{3!}-\dfrac{z^5}{5!}+\ldots}{z} = \dfrac{z^2}{3!}-\dfrac{z^4}{5!}+\ldots$

$b_1=0$

d) $\dfrac{\cot z}{z^4} = \dfrac{1}{z}-\dfrac{z}{3}+\dfrac{z^3}{245}+\dfrac{z^5}{(\quad)}+\ldots$

$b_1=-\dfrac{1}{3}$

Scanned by CamScanner

Q12.

a) $\frac{e^{-z}}{z^2}$ $z=0$

$\frac{1}{z^2}\left(1 - z + \frac{z^2}{2!} + \text{---}\right)$

$b_1 = \frac{1}{2}$

$2\pi i\left(\frac{1}{2}\right) = \pi i$

b) $\frac{e^{-z+1}}{\cancel{e}(z-1)^2}$ $\frac{1}{e}$

$\frac{1}{e}\frac{1}{(z-1)^2}\left(1 - (z-1) + \frac{(z-1)^2}{2!} + \text{---}\right)$

πi

c) $z^2 e^{\frac{1}{z}}$

$z^2\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \text{---}\right)$

$2\pi i$

e) $\frac{e^z}{(z^2-1)^2}$

$\operatorname{Res} f(z) = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$

$2e\pi i$

$\frac{z+1}{z^2-2z}$ $z=0$ $z=2$

$\underset{z=0}{\text{Res}} f(z) = -3/4$ $\cancel{z+1}$ $\underset{z=1}{\text{Res}} f(z) = -1/4$

$\frac{d}{dz}\Big|_{z=0} \frac{z+1}{z-2} = \frac{(z-2)-(z+1)}{(z-2)^2} = \frac{-3}{(z-2)^2}$

$= \frac{-3}{4}$

$\frac{d}{dz}\Big|_{z=2} \frac{z+1}{z} = \frac{-1}{z^2} = \frac{-1}{4}$

$-2\pi i$

$\frac{\pi \cot \pi z}{(z+\frac{1}{2})^2}$

$\cancel{\frac{d^2 \pi \cot \pi z}{dz^2}}$ $\cancel{\frac{d}{dz} = \pi^2 \operatorname{cosec}^2 \pi z}$

$-\pi^3 \, 2\operatorname{cosec}^2 \pi z \cot \pi z$

$+ \cancel{2}4\pi i \left(\cancel{\operatorname{cosec}^2 \pi} \; \cancel{\cot \pi}\right)$

Scanned by CamScanner

Q13

a) $\int_{0}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\,dx$ — Roots should be imaginary.

Application of Cauchy Residue Theorem

$\cancel{\frac{(2x^2-1)\,dx}{x^4+4x^2+x^2+4}}$

$\cancel{\frac{(2x^2-1)\,dx}{x^2(x^2+4)+1(x^2+4)}} = \cancel{\frac{(2x^2-1)\,dx}{(x^2+4)^2}}$

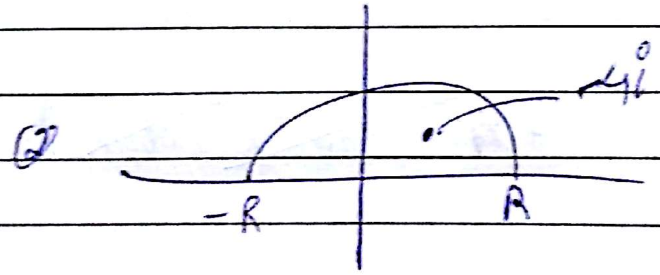
$\cancel{\frac{d(2z^2-1)\,dz}{dz}} - \cancel{4z} = \cancel{16i}$

$\frac{2x^2-1}{x^4+5x^2+4}$ $\;-\; \frac{x^2}{x^2}\left(\frac{(2x^2-1)}{(x^2+4)^2}\right)$

$\Downarrow$

$\frac{2-\frac{1}{x^2}}{x^2+5+\frac{4}{x^2}}$

$f(z) < \frac{K}{|z|^2}$ Numerator 2 power less than denominator



$\lim_{R\to\infty}\int_{-R}^{\infty} f(z)\,dz + \lim_{R\to\infty}\int_{C_R} f(z)\,dx = 2\pi i \sum \text{Res } f(z)$

$\lim_{R\to\infty}\int_{0}^{\infty} f(z)\,dz = \pi i \underset{z=2i}{\text{Res}} f(z) + \pi i \underset{z=i}{\text{Res}} f(z)$

$$\cancel{\frac{d^2}{dx^2}\frac{(2x^2-1)(x^2+4)^2}{(x^2+1)^2}} = 4$$

$$\boxed{4\pi i}$$

$$\frac{2x^2-1}{(x^2+1)(x^2+4)}$$

$$\cancel{\frac{d}{dx}\frac{2x^2-1}{x^2+1}} = \cancel{\frac{4x(x^2+1)-(2x^2-1)2x}{(x^2+1)^2}}$$

$x = 2i$

$$= \cancel{\frac{8i(-3)+18i\times 2}{9}}$$

$$= \cancel{\frac{(-24+36)i}{9}} = \cancel{\frac{6i}{9}} = \cancel{-\frac{4i}{3}}$$

$$\cancel{\frac{2x^2-1}{x^2+4}} = \cancel{\frac{4x(x^2+4)-(2x^2-1)(2x)}{(x^2+4)^2}}$$

$$= \cancel{4i}\ \cancel{\frac{4i(3)+6i}{9}} = \cancel{\frac{18i}{9}} = \cancel{2i}$$

$$\frac{2x^2-1}{4x^3+10x}$$

$$\frac{-9}{-32i+20i} = \frac{9^3}{12i_4} = \frac{-3i}{4}$$

$$\frac{-3}{-4i+10i} = \frac{-3^1}{6i_2} = \frac{+i}{2}$$

$$\frac{1}{4}i \times \pi i = \boxed{-\pi/4}$$

Scanned by CamScanner

Q13. b)

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

x^4

$x^4+1 \longrightarrow$ Roots $\longrightarrow \sqrt{i}, -\sqrt{i}, \sqrt{-i}$

4π

$(-1)^{1/4} = (i)^{1/2}$

2π

$= e$

$(e^{-\pi i})^{1/4} = e^{(-\pi+2k\pi)i)\,1/4}$

$e^{-i\pi/4} \quad e^{i\pi/4} \quad e^{i3\pi/4} \quad e^{5\pi i/4}$

$\cos(-\pi/4) + i\sin(-\pi/4) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$Q\pi - \pi/4$

$\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

Scanned by CamScanner