P(II) (Electrodynamics)

Assignment - 2

Q1 i) $\vec{F} = \rho(2+\sin^2\phi)\hat{\rho} + \rho\sin\phi\cos\phi\,\hat{\phi} + 3z\hat{z}$

$$\nabla\cdot F = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z}$$

$$= \frac{1}{\rho}2\rho(2+\sin^2\phi) + \frac{1}{\rho}\frac{\rho}{2}2\cos2\phi + 3$$

$$= 7+1 = 8$$

$$\nabla\times F = \left(\frac{1}{\rho}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial\rho}\right)\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial(\rho V_\phi)}{\partial\rho} - \frac{\partial V_\rho}{\partial\phi}\right]\hat{z}$$

$$= (0-0) + (0-0) + \frac{1}{\rho}[\rho\sin2\phi - \rho\sin2\phi]\hat{z}$$

$$= 0$$

ii) $\vec{F} = (r\cos\theta)\hat{r} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$

$$\nabla\cdot F = \frac{\partial}{\partial r}(r\cos\theta) + \frac{1}{r}\frac{\partial}{\partial\theta}(r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)$$

$$= \cos\theta + \cos\theta - \sin\phi$$
$$= 2\cos\theta - \sin\phi.$$

$$\nabla\times F = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\hat{\theta}$$
$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi}$$

$$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) - \frac{\partial}{\partial\phi} r\sin\theta\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\cos\theta) - \frac{\partial}{\partial r}\right.$$

$$+\frac{1}{r}\left(\frac{\partial}{\partial r}(r^2\sin\theta) - \frac{\partial}{\partial\theta}(r^2\cos\theta)\right)\hat{\phi}$$

$$= \frac{1}{r\sin\theta}(r\sin 2\theta\cos\phi)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\times 0 - 2r\sin\theta\cos\phi\right)\hat{\theta}$$

$$+\frac{1}{r}(2r\sin\theta + r^2\sin\theta)\hat{\phi}$$ ①

$$= (2\cos\theta\cos\phi)\hat{r} - (2\sin\theta\cos\phi)\hat{\theta} + (2\sin\theta + r\sin\theta)\hat{\phi}$$ ②

) $\vec{A} = xy\hat{i} + (3x^2+y)\hat{j}$

$$\begin{bmatrix} a_s \\ a_\phi \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} xy \\ 3x^2+y \\ 0 \end{bmatrix}$$ ③

$$a_s = xy\cos\phi + 3x^2\sin\phi + y\sin\phi$$
$$a_\phi = -xy\sin\phi + (3x^2+y)\cos\phi$$
$$a_z = 0$$

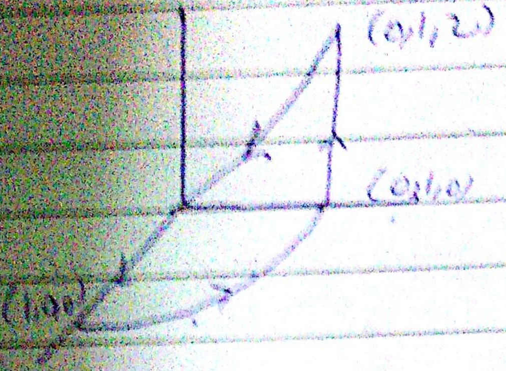
$$a_s = (s\cos\phi)(s\sin\phi)\cos\phi + 3(s\cos\phi)^2\sin\phi + s\sin\phi\sin\phi)$$

$$= s^2\cos^2\phi\sin\phi + 3s^2\cos^2\phi\sin\phi + s\sin^2\phi$$
$$= 4s^2\cos^2\phi\sin\phi + s\sin^2\phi$$

$$a_\phi = -(s\cos\phi)(s\sin\phi)\sin\phi + (3s^2\cos^2\phi + s\sin\phi)\cos\phi$$

$$\vec{A} = (4s^2\cos^2\phi\sin\phi + s\sin^2\phi)\hat{s} + (3s^2\cos^2\phi + s\sin\phi\cos\phi - s^2\cos\phi\sin^2\phi)\hat{\phi}$$

1.38 $v = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$



① $\theta = \frac{\pi}{2}$, $\phi = 0$, $r: 0 \to 1$ $\quad v\cdot dl = (r\cos^2\theta)\,dr = 0$

$\int v\cdot dl = 0$

② $r = 1$, $\theta = \pi/2$, $\phi: 0 \to \pi/2$ $\quad v\cdot dl = (3r)(r\sin\theta\, d\phi) = 3d\phi$

$$\int v\cdot dl = 3\int_0^{\pi/2} d\phi = \frac{3\pi}{2}$$

③ $\phi = \frac{\pi}{2}$; $r\sin\theta = y = 1$ $\quad r = \frac{1}{\sin\theta}$ $\quad dr = \frac{-1}{\sin^2\theta}\cos\theta\, d\theta$

$\theta : \frac{\pi}{2} \to \theta_0 = \tan^{-1}(1/2)$.

$v\cdot dl = (r\cos^2\theta)\,dr - (r\cos\theta\sin\theta)(r\,d\theta)$

$= \frac{\cos^2\theta}{\sin\theta}\left(\frac{-\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}\,d\theta$

$= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = \frac{-\cos\theta}{\sin\theta}\left(\frac{1}{\sin^2\theta}\right)d\theta$

$= \frac{-\cos\theta}{\sin^3\theta}\,d\theta$

Therefore:

$$\int v\cdot dl = \int_{-\pi/2}^{\theta_0} \frac{\cos\theta}{\sin^3\theta}\,d\theta$$

$$= \frac{1}{2\sin^2\theta}\Big|_{\pi/2}^{\theta_0} = \frac{1}{2\cdot(1/5)} - \frac{1}{2\cdot 1}$$
$$= \frac{5}{2} - \frac{1}{2} = 2.$$

4) $\theta = \theta_0 \quad \phi = \frac{\pi}{2}, \; r : \sqrt{5} \to 0 \quad v\cdot dl = (r\cos^2\theta)dr = \frac{4}{5} r\,dr.$

$$\int v\cdot dl = \frac{4}{5}\int_{\sqrt{5}}^{0} r\,dr = \frac{4}{5}\frac{r^2}{2}\Big|_{\sqrt{5}}^{0} = -\frac{4}{5}\cdot\frac{5}{2} = -2.$$

$$\oint v\cdot dl = 0 + \frac{3\pi}{2} + 2 - 2 = \frac{3\pi}{2}$$

<u>Using Stokes Theorem:-</u>

$\int (\nabla\times v)\cdot da$

$$(\nabla\times v) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, 3r) - \frac{\partial}{\partial\phi}(-r\sin\theta\cos\theta)\right]\hat{r}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\cos^2\theta) - \frac{\partial}{\partial r}(r\,3r)\right]\hat{\theta}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(-r^2\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}(r\cos^2\theta)\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}[3r\cos\theta]\hat{r} + \frac{1}{r}[-6r]\hat{\theta} + \frac{1}{r}[-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta]\hat{\phi}$$

$$= 3\cot\theta\,\hat{r} - 6\hat{\theta}$$

1) Backface $da = -r\,dr\,d\theta\,\hat{\phi}$
$(\nabla\times v)\cdot da = 0 \quad \int(\nabla\times v)\,da = 0$

2) Bottom : $da = -r\sin\theta\,dr\,d\phi\,\hat{\theta}$ so $(\nabla\times v)\cdot da = 6r\,dr\,d\phi$
$\int(\nabla\times v)\cdot da = \int_0^1 6r\,dr \int_0^{\pi/2} d\phi = 3\pi/2$

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Q4. $V = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

$$\nabla\cdot V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta\, r^2\cos\phi + \frac{1}{r\sin\theta}(-r^2\cos\theta\cos\phi)$$

$$= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta$$

$$\int\nabla\cdot V\,d\tau = \int(4r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi = 4\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$$

$$= R^4\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi R^4}{4}$$

Surface consists of 4 parts:-

1) curved: $da = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$, $r=R$, $V\cdot da = (R^2\cos\theta)(R^2\sin\theta\,d\theta\,d\phi)$

$$= \int V\cdot da = R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi = \frac{\pi R^4}{4}$$

2) Left: $da = -r\,dr\,d\theta\,\hat{\phi}$, $\phi=0$, $V\cdot da = (r^2\cos\theta\sin\phi)(r\,dr\,d\theta) = 0$

$\int V\cdot da = 0$

3) Back: $da = r\,dr\,d\theta\,\hat{\phi}$, $\phi = \pi/2$, $V\cdot da = (-r^2\cos\theta\sin\phi)(r\,dr\,d\theta)$
$= -r^3\cos\theta\,dr\,d\theta$

$\int V\cdot da = -\frac{R^4}{4}$

4) Bottom: $da = r\sin\theta\,dr\,d\phi\,\hat{\theta}$, $\theta = \pi/2$, $V\cdot da = (r^2\cos\phi)\,r\,dr\,d\phi$

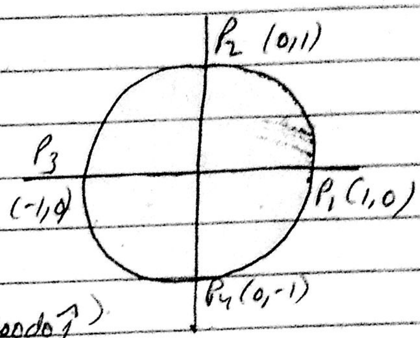
$$\int v\,da = \int_0^R r^3\,dr \int_0^{\pi/2} \cos\phi\,d\phi = \frac{1}{4}R^4.$$

$$\oint v\,da = \frac{\pi R^4}{4} + 0 - \frac{1}{4}R^4 + \frac{R^4}{4} = \frac{\pi R^4}{4}$$

Q5) $\vec{A} = -y\hat{i} + x\hat{j}$

$x = r\cos\theta \qquad dx = -r\sin\theta\,d\theta$

$y = r\sin\theta \qquad dy = r\cos\theta\,d\theta$



$\oint \vec{A}\cdot d\vec{l}$

$\underline{P_1 - P_2} = \int (-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j})(-r\sin\theta\,d\theta\,\hat{i} + r\cos\theta\,d\theta\,\hat{j})$

$$= \int r^2\sin^2\theta + r^2\cos^2\theta\,d\theta$$

$$\int_0^{\pi/2} r^2 \times 1\,d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

$\underline{P_2 - P_3} \qquad \int_{\pi/2}^{\pi} d\theta = \frac{\pi}{2}$

$\underline{P_3 - P_4} \qquad \int_{\pi}^{3\pi/2} d\theta = \frac{\pi}{2}$

$\underline{P_4 - P_1} \qquad \int_{3\pi/2}^{2\pi} d\theta = \frac{\pi}{2}$

$4 \times \frac{\pi}{2} = \underline{2\pi}$