1) $V_\rho = \rho(2+\sin^2\phi)$, $V_\phi = \rho\sin\phi\cos\phi$, $V_z = 3z$

$$\bar{\nabla}\cdot\bar{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot\rho(2+\sin^2\phi)\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\rho\sin\phi\cos\phi\right) + \frac{\partial}{\partial z}(3z)$$

$$= \frac{1}{\rho}\cdot 2\rho(2+\sin^2\phi) + \frac{1}{\rho}\rho\left(\cos^2\phi - \sin^2\phi\right) + 3$$

$$= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3$$

$$= 4 + 1 + 3 = 8$$

2) $v_r = r\cos\theta$, $v_\theta = r\sin\theta$, $v_\phi = r\sin\theta\cos\phi$

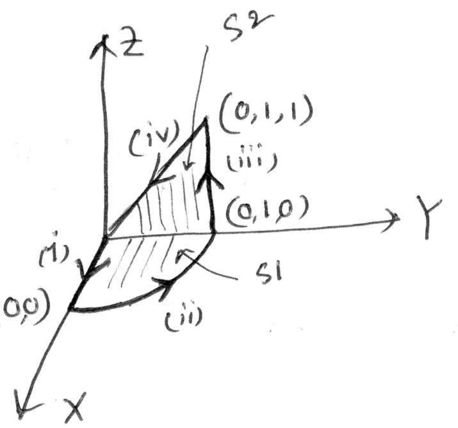
$$\bar{\nabla}\cdot\bar{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot r\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r\,\cancel{\cos\theta}\sin\theta\cos\phi\right)$$

$$= \frac{1}{r^2}3r^2\cos\theta + \frac{r}{r\sin\theta}\times 2\sin\theta\cos\theta + \frac{\cancel{r\sin\theta}(-\sin\phi)}{\cancel{r\sin\theta}}$$

$$= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi$$

2.

$$\iint (\bar\nabla \times \bar V)\cdot d\bar s = \oint \bar V \cdot d\bar l$$



R.H.S

Segment (i)

$$\int_{(i)} \bar V\cdot d\bar l = \int \bar V\cdot(dr\hat r + r d\theta\hat\theta + r\sin\theta d\phi\hat\phi)$$

0

$$= \int \bar V\cdot dr\hat r$$

$$= \int (r\cos^2\theta)\hat r\cdot dr\hat r = \int r(\cos^2\theta)\,dr$$

For segment (i) $\theta = \frac{\pi}{2}$, $\phi = 0$, $r \to 0 \to 1$

So, $\int_{(i)} \bar V\cdot d\bar l = 0$

Segment (ii)

$$\int_{(ii)} \bar V\cdot d\bar l = \int_{(ii)} \bar V\cdot(dr\hat r + r d\theta\hat\theta + r\sin\theta d\phi\hat\phi)$$

$r=1$ and $\theta = \pi/2$, only ϕ varies.

So. $$= \int_{(ii)} \bar V\cdot r\sin\theta d\phi\hat\phi = \int_{(ii)} \bar V\cdot r d\phi\hat\phi$$

$$= \int_{(ii)} (3r\hat\phi)\cdot r d\phi\hat\phi$$

$$= 3\int_0^{\pi/2} r^2 d\phi = \frac{3\pi}{2}$$

$r=1$

Segment (iii)

$$\int_{(iii)} \bar V\cdot d\bar l = \int \bar V\cdot(dr\hat r + r d\theta\hat\theta + r\sin\theta d\phi\hat\phi)$$

In this particular segment both θ and r vary.
However, ϕ is constant $(\pi/2)$

$$= \int_{(iii)} \bar V\cdot(dr\hat r + r d\theta\hat\theta)$$

Line integration should be written in terms of one variable. r and θ are related by the following relation

$$r = \frac{1}{\sin\theta}, \quad dr = -\frac{1}{\sin^2\theta}\cos\theta\, d\theta$$

$$\bar V\cdot d\bar l = (r\cos^2\theta)dr - (r\cos\theta\sin\theta)(r d\theta)$$

$$= \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}d\theta$$

$$= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)d\theta$$

$$= -\frac{\cos\theta}{\sin^3\theta}\, d\theta$$

$$\int_{(iii)} \bar{V}\cdot d\bar{l} = -\int_{\pi/2}^{\pi/4} \frac{\cos\theta}{\sin^3\theta}\, d\theta = \frac{1}{2\sin^2\theta}\Bigg|_{\pi/2}^{\pi/4} = \frac{1}{2\cdot(1/2)} - \frac{1}{2}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

Segment (iv)

$$\int_{iv} \bar{V}\cdot d\bar{l} = \int_{iv} \bar{V}\cdot(dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi})$$

$$0$$

$$= \int \bar{V}\cdot dr\,\hat{r} = \int (r\cos^2\theta)\, dr$$

$$\theta = \pi/4$$

$$= \frac{1}{2}\int_{\sqrt{2}}^{0} r\,dr = -\frac{1}{2}\frac{r^2}{2}\Bigg|_{\sqrt{2}}^{0} = -\frac{1}{4}\cdot 2 = -\frac{1}{2}$$

$$\oint \bar{V}\cdot d\bar{l} = 0 + \frac{3\pi}{2} + \frac{1}{2} - \frac{1}{2} = \frac{3\pi}{2}$$

L.H.S

$$\bar{\nabla}\times\bar{V} = 3\cot\theta\,\hat{r} - 6\hat{\theta}$$

Surface 1 (θ reemains constant) $= \frac{\pi}{2}$

$$d\bar{s}_1 = r\sin\theta\, dr\, d\phi\,(-\hat{\theta}) = r\,dr\,d\phi\,(-\hat{\theta})$$

$$\iint_{S_1} (\bar{\nabla}\times\bar{V})\cdot d\vec{s}_1 = \iint 6r\,dr\,d\phi$$

$$= \int_0^1 6r\,dr \int_0^{\pi/2} d\phi = \frac{3\pi}{2}$$

Surface 2 (ϕ reemains constant) $= \pi/2$

$$d\vec{s}_2 = -r\,dr\,d\theta\,\hat{\phi}$$

$$\iint (\bar{\nabla}\times\bar{V})\cdot d\bar{s}_2 = 0$$

L.H.S = R.H.S