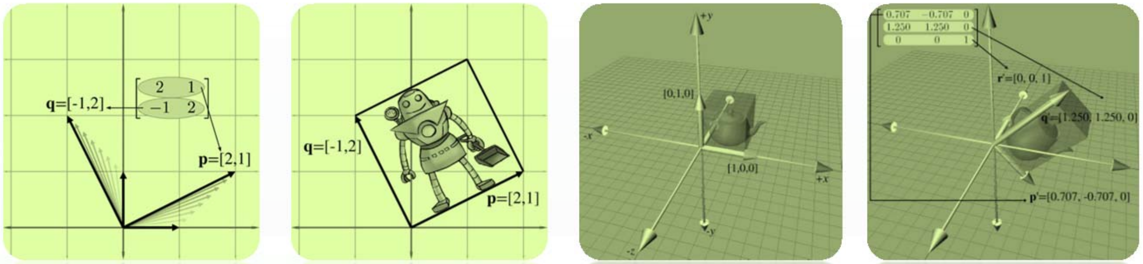


$$\begin{bmatrix} 0.707 & -0.707 & 0 \\ 1.250 & 1.250 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mathematics - II

What You'll See in This Course

This course has two modules.

- Module 1: Linear Algebra

- Module 2: Differential Equation

LINEAR ALGEBRA

- Matrices: Elementary operations, Row reduced Echelon form, Rank of matrix, Special matrices, Matrix Inversion, Determinant, and properties. System of linear equations and equivalent systems.
- Vector spaces, sub-spaces, Linear Dependence and Independence; linear span, Basis; Dimension; Co-ordinates with respect to a basis.
- Inner Products; Norm of a vector, Cauchy-Schwarz Inequality; Orthonormal basis, Gram-Schimdt process.
- Eigen Values/Eigen Vectors, Characteristic Polynomial, Diagonalisable matrices , Similarity of matrices.

Differential Equations

- Introduction to Differential Equations., First Order ODE y'=f(x,y), geometrical interpretation of solutions, Separable forms, Exact Equations, integrating factor, Linear Equations, Orthogonal Trajectories.
- Picard's Theorem, Qualitative properties and Theoretical aspects, Euler's Method, Elementary classifications of equations F(x,y,y')=0.
- Second Order Linear differential equations: fundamental system and general solution of homogeneous equation, reduction of order.
- Existence and uniqueness of solution for second order IVP, Wronskian and general solution of non-homogeneous equations .

Differential Equations

- Euler-Cauchy Equations, extensions of the results to higher order linear equations, Higher order Differential Equations.
- Power series method.
- Legendre Polynomials, Frobenius Method.
- Bessel equation , Properties of Bessel functions.
- Sturm Liouville BVP, Orthogonal functions, Qualitative behaviour of solutions of second order ODE, Sturm comparison Theorem.
- Laplace transform, Fourier Series and Integrals.

Text and Reference books

Text Book:

- ✓ **David C. Lay, Linear Algebra and its Applications, Pearson Education 3rd Ed, 2003.**
- ✓ Erwin Kreyszig, Advanced Engineering Mathematics, 8th edition, Wiley publishers.

Reference books:

- ✓ George F. Simmons, Steven G. Krantz, Differential Equations: Theory, Technique And Practice, Tata McGraw-Hill Education.
- ✓ Coddington, An Introduction to Ordinary Differential Equations.
- ✓ G. Strang, Linear Algebra and Its Applications, Thomson Brooks/Cole, 2007.
- ➢ S. Kumaresan, Linear Algebra, A Geometric Approach, Prentice Hall India, 2008.
- ➢ Kenneth Hoffman & R. Kunze, Linear Algebra, Prentice Hall 2nd Ed, 1971.
- ➢ Additional Resource: NPTEL, MIT Video Lectures

Evaluation Methods:

Methods	Weightage
Quizzes	20%
Midterm Exam	30%
Final Examination	50%

Introduction to Matrix

Definitions - Matrix

- A matrix is an rectangular array of numbers enclosed in brackets

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- Denoted with a **bold Capital letter**

Example:

$$\begin{array}{cc} \begin{bmatrix} 6 & 2 & -1 \\ -2 & 0 & 5 \end{bmatrix} & \text{2 rows} \\ \text{3 columns} & \end{array}$$

What is the order?

All matrices have an order (or **dimension**): the number of rows $\times$ the number of columns.

$$\begin{bmatrix} 8 & -1 & 3 \\ 0 & 0 & 2 \\ 10 & 4 & -3 \end{bmatrix}$$

3 x 3

(square matrix)

$$\begin{bmatrix} 9 & -5 & 7 & 0 \end{bmatrix}$$

1 x 4

(Also called a row matrix)

$$\begin{bmatrix} -2 & 0 & 4 & 6 & 3 \\ 1 & 1 & -5 & -9 & 8 \\ 7 & 3 & 2 & 7 & 6 \end{bmatrix}$$

3 x 5

$$\begin{bmatrix} -9 \\ 7 \\ 0 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

2 x 2

(square matrix)

4 x 1

(Also called a column matrix)

Definitions: square matrix

A square matrix is a matrix that has the same number of rows and columns ($n \times n$)

4 × 4 Square Matix

$$\begin{bmatrix} -2 & 4 & 7 & 31 \\ 6 & 9 & 12 & 6 \\ 12 & 11 & 0 & 1 \\ 9 & 10 & 2 & 3 \end{bmatrix}$$

Square Matrices: *diagonal entries*

- In a square matrix, entries $\boldsymbol{m_{ii}}$ are called the *diagonal* entries. The others are called *non-diagonal* entries

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Diagonal Matrices

A **diagonal matrix** is a square matrix whose non-diagonal elements are zero.

$$A = diag(d_1, d_2, \cdots, d_n) = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in M_{n \times n}$$

A = diag(3, 1, -5, 2) = $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

- Diagonal matrix:

$$A = diag(d_1, d_2, \cdots, d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in M_{n\times n}$$

- Trace: If $A = [a_{ij}]_{n\times n}$

Then $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Tr(A) = 3 + 1 + (-5) + 2 = 1

Vectors as Matrices

- A row vector is a **1 x *n*** matrix.
- Example: **1 x 5** $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$
- A column vector is an *n* x 1 matrix. Eg. 5 x 1 $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

TYPES OF MATRICES

NAME	DESCRIPTION	EXAMPLE
Row matrix	A matrix with only 1 row	$\begin{bmatrix} 3 & 2 & 1 - 4 \end{bmatrix}$
Column matrix	A matrix with only 1 column	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
Square matrix	A matrix with same number of rows and columns	$\begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}$
Zero matrix	A matrix with all zero entries	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Matrix Equality

- Two matrices are equal if and only if
  - ✓ they both have the same number of rows and the same number of columns
  - ✓ their corresponding elements are equal

$$\#1 \quad \begin{bmatrix} 5 & 3 & 2 \\ 33 & -5 & 11 \\ -2 & 13 & 6 \end{bmatrix} \qquad \#2 \quad \begin{bmatrix} 5 & 3 & 2 \\ 33 & 5 & 11 \\ -2 & 13 & 6 \end{bmatrix} \qquad \#3 \quad \begin{bmatrix} 5 & 3 & 2 \\ 33 & -5 & 11 \\ -2 & 13 & 6 \end{bmatrix}$$

Upper triangular matrix

- **Upper triangular matrix:** A square matrix in which all the elements below the diagonal are zero i.e. a matrix of type:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

Lower triangular matrix

- **Lower triangular matrix:** A square matrix in which all the elements above the diagonal are zero i.e. a matrix of type

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Scalar matrix

Scalar matrix: A diagonal matrix in which all of the diagonal elements are equal to some constant "k", i.e., a matrix of type

$$\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & k \end{bmatrix}$$

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Identity matrix

Identity matrix: A diagonal matrix in which all of the diagonal elements are equal to “1" i.e. a matrix of type

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

An identity matrix of order nxn is denoted by I_n .

Transpose of a Matrix

- The transpose of an $r \times c$ matrix **M** is a $c \times r$ matrix called $\mathbf{M}^T$.
- Take every row and rewrite it as a column.
- Equivalently, flip about the diagonal

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

Transpose of a Matrix: Example

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{b}^T = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$$

column ⟶ row

$$\mathbf{d} = \begin{bmatrix} 3 & 4 & 9 \end{bmatrix} \qquad \mathbf{d}^T = \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix}$$

row ⟶ column

$$\mathbf{A} = \begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{5 \quad 4 \quad 1} \\ \boxed{6 \quad 7 \quad 4} \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 7 \\ 3 & 1 & 4 \end{bmatrix}$$

Facts About Transpose

- Transpose is its own inverse: $(\mathbf{M}^T)^T = \mathbf{M}$ for all matrices **M**.

- $\mathbf{D}^T = \mathbf{D}$ for all diagonal matrices **D** (including the identity matrix **I**).

Symmetric matrix

- **Symmetric matrix:** A square matrix in which corresponding elements with respect to the diagonal are equal; a matrix in which $a_{ij} = a_{ji}$ where a_{ij} is the element in the i-th row and j-th column; a matrix which is equal to its transpose; a square matrix in which a flip about the diagonal leaves it unchanged. Example:

$$\begin{bmatrix} 9 & 13 & 5 & 2 \\ 1 & 11 & 7 & 6 \\ 3 & 7 & 4 & 1 \\ 6 & 0 & 7 & 10 \end{bmatrix}$$

Not Symmetrical

$$\begin{bmatrix} 9 & 13 & 3 & 6 \\ 13 & 11 & 7 & 6 \\ 3 & 7 & 4 & 7 \\ 6 & 6 & 7 & 10 \end{bmatrix}$$

Symmetrical

Skew-symmetric matrix

- **Skew-symmetric matrix:** A square matrix in which corresponding elements with respect to the diagonal are negatives of each other; a matrix in which $a_{ij} = -a_{ji}$ where a_{ij} is the element in the i-th row and j-th column; a matrix which is equal to the negative of its transpose. The diagonal elements are always zeros. Example:

$$\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$$

If $A = \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix}$ and $-A = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix}$

Orthogonal Matrix

A n×n matrix **A** is an orthogonal matrix if $\mathbf{A}\mathbf{A}^{T} = \mathbf{I}$, where $\mathbf{A}^{T}$ is the transpose of **A** and **I** is the identity matrix.

In particular, an orthogonal matrix is always invertible, and $\mathbf{A}^{-1} = \mathbf{A}^{T}$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$ are orthogonal.

Periodic matrix

Periodic matrix: A matrix A for which $\mathbf{A^{k+1} = A}$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then A is said to be of **period k**. If k = 1, so that $A^2 = A$, then A is called **idempotent**.

Show that $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is periodic with period 4.

Solution

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix};$$

$$A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = I;$$

$$A^5 = A^4 \cdot A = IA = A$$

Hence A is periodic and $\mathcal{P}(A) = 4$.

An idempotent matrix is a matrix $n \times n$ (square matrix A) which : $A^2 = A$

Example :

$$A = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 - 1\cdot 2 & 2\cdot(-1) - 1\cdot(-1) \\ 2\cdot 2 - 1\cdot 2 & 2\cdot(-1) - 1\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4-2 & -2+1 \\ 4-2 & -2+1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} = A$$

Nilpotent matrix and Unipotent Matrix

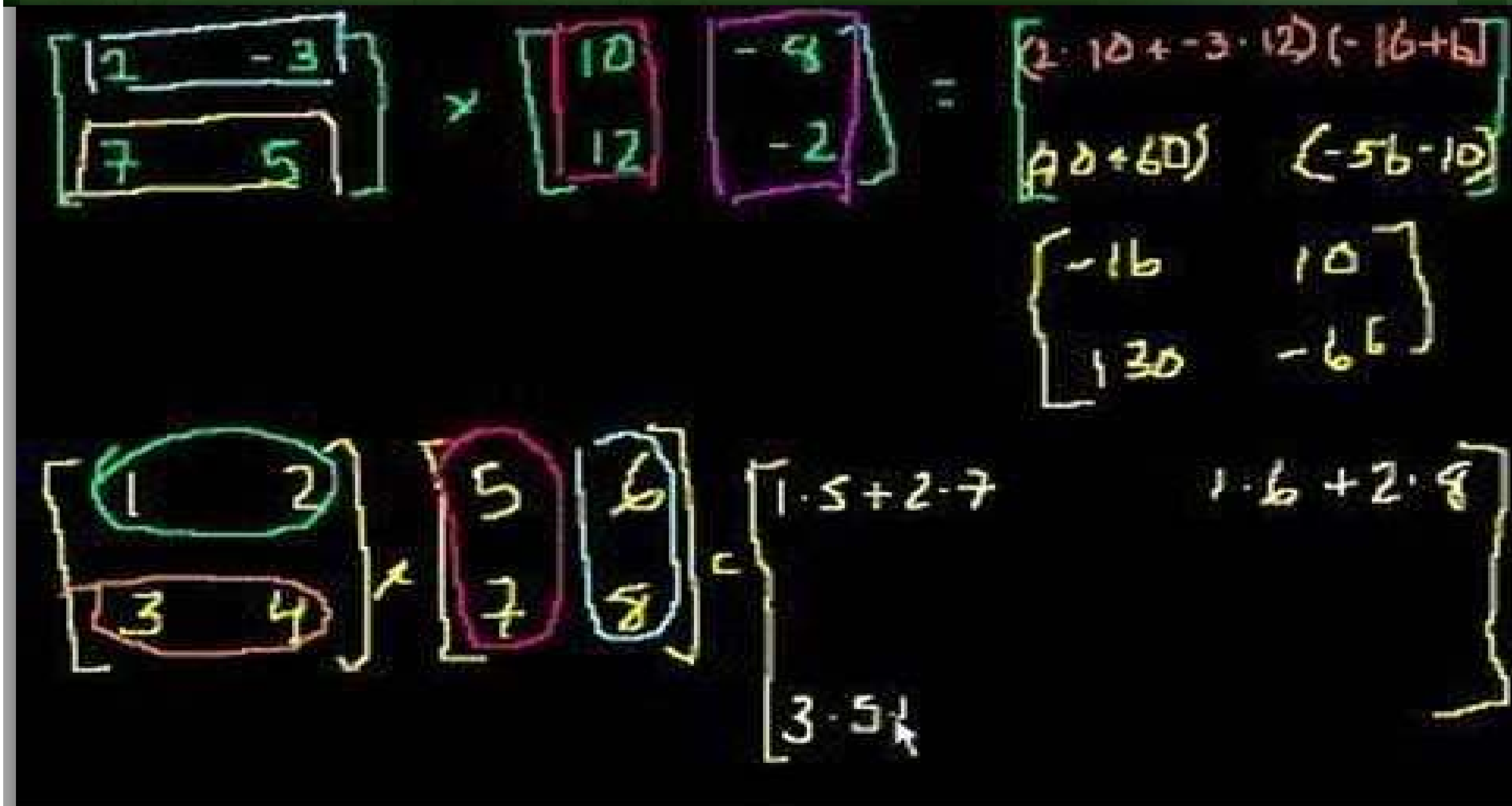
Nilpotent matrix: A matrix A for which $\mathbf{A^p = 0}$, where p is some positive integer. If p is the least positive integer for which $A^p = 0$, then A is said to be **nilpotent of index p**. A is said to be unipotent if A-I, where I is an identity matrix, is a nilpotent matrix

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\Rightarrow A^3 = A^2.A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix Operations



Adding Two Matrices

To add two matrices, they must have the same order. To add, you simply add corresponding entries.

$$3+4=7$$

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$$

Adding Two Matrices

To add two matrices, they must have the same order. To add, you simply add corresponding entries.

If $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$ Then $A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$

$$\begin{bmatrix} 5 & -3 \\ -3 & 4 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 5+(-2) & -3+1 \\ -3+3 & 4+0 \\ 0+4 & 7+(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ 0 & 4 \\ 4 & 4 \end{bmatrix}$$

Adding Two Matrices

$$\begin{bmatrix} 8 & 0 & -1 & 3 \\ -5 & 4 & 2 & 9 \end{bmatrix} + \begin{bmatrix} -1 & 7 & 5 & 2 \\ 5 & 3 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 8+(-1) & 0+7 & -1+5 & 3+2 \\ -5+5 & 4+3 & 2+3 & 9+(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 7 & 4 & 5 \\ 0 & 7 & 5 & 7 \end{bmatrix}$$

Subtracting Two Matrices

To subtract two matrices, they must have the same order. You simply subtract corresponding entries.

$$\begin{bmatrix} 9 & -2 & 4 \\ 5 & 0 & 6 \\ 1 & 3 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 7 \\ 1 & 5 & -4 \\ -2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 9-4 & -2-0 & 4-7 \\ 5-1 & 0-5 & 6-(-4) \\ 1-(-2) & 3-3 & 8-2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & -3 \\ 4 & -5 & 10 \\ 3 & 0 & 6 \end{bmatrix}$$

Subtracting Two Matrices

$$\begin{bmatrix} 2 & -4 & 3 \\ 8 & 0 & -7 \\ 1 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 8 \\ 3 & -1 & 1 \\ -4 & 2 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0 & -4-1 & 3-8 \\ 8-3 & 0-(-1) & -7-1 \\ 1-(-4) & 5-2 & 0-7 \end{bmatrix} = \begin{bmatrix} 2 & -5 & -5 \\ 5 & 1 & -8 \\ 5 & 3 & -7 \end{bmatrix}$$

Multiplying By a Scalar

- Can multiply a matrix by a scalar.
- Result is a matrix of the same dimension.
- To multiply a matrix by a scalar, multiply each component by the scalar.

$$k\mathbf{M} = k\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix} = \begin{bmatrix} km_{11} & km_{12} & km_{13} \\ km_{21} & km_{22} & km_{23} \\ km_{31} & km_{32} & km_{33} \\ km_{41} & km_{42} & km_{43} \end{bmatrix}$$

Multiplying a Matrix by a Scalar

- In matrix algebra, a real number is often called a **SCALAR**. To multiply a matrix by a scalar, you multiply each entry in the matrix by that scalar.

$$4\begin{bmatrix} -2 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 4(-2) & 4(0) \\ 4(4) & 4(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 0 \\ 16 & -4 \end{bmatrix}$$

Matrix Multiplication

Multiplying an r x n matrix **A** by an n x c matrix **B** gives an r x c result **AB**.

$$\underset{\substack{r \times n \\ 4 \times 2}}{\overset{\mathbf{A}}{\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}}} \quad \underset{\substack{n \times c \\ 2 \times 5}}{\overset{\mathbf{B}}{\begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix}}} = \underset{\substack{r \times c \\ 4 \times 5}}{\overset{\mathbf{AB}}{\begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix}}}$$

Multiplication: Result

- Multiply an $r \times n$ matrix **A** by an $n \times c$ matrix **B** to give an $r \times c$ result **C** = **AB**.
- Then **C** = $[c_{ij}]$, where c_{ij} is the dot product of the ith row of **A** with the jth column of **B**.
- That is:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Example

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{bmatrix}$$

$$c_{24} = a_{21}b_{14} + a_{22}b_{24}$$

Another Way of Looking at It

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \quad \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \end{bmatrix}$$

$$c_{43} = a_{41}b_{13} + a_{42}b_{23}$$

2 x 2 Case

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

2 x 2 Example

$$\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 5 & 1/2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -7 & 2 \\ 4 & 6 \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} -3 & 0 \\ 5 & 1/2 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 4 & 6 \end{bmatrix} \\ &= \begin{bmatrix} (-3)(-7) + (0)(4) & (-3)(2) + (0)(6) \\ (5)(-7) + (1/2)(4) & (5)(2) + (1/2)(6) \end{bmatrix} \\ &= \begin{bmatrix} 21 & -6 \\ -33 & 13 \end{bmatrix} \end{aligned}$$

3 x 3 Case

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

3 x 3 Example

$$\mathbf{A} = \begin{bmatrix} 1 & -5 & 3 \\ 0 & -2 & 6 \\ 7 & 2 & -4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -8 & 6 & 1 \\ 7 & 0 & -3 \\ 2 & 4 & 5 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -5 & 3 \\ 0 & -2 & 6 \\ 7 & 2 & -4 \end{bmatrix} \begin{bmatrix} -8 & 6 & 1 \\ 7 & 0 & -3 \\ 2 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot(-8) + (-5)\cdot 7 + 3\cdot 2 & 1\cdot 6 + (-5)\cdot 0 + 3\cdot 4 & 1\cdot 1 + (-5)\cdot(-3) + 3\cdot 5 \\ 0\cdot(-8) + (-2)\cdot 7 + 6\cdot 2 & 0\cdot 6 + (-2)\cdot 0 + 6\cdot 4 & 0\cdot 1 + (-2)\cdot(-3) + 6\cdot 5 \\ 7\cdot(-8) + 2\cdot 7 + (-4)\cdot 2 & 7\cdot 6 + 2\cdot 0 + (-4)\cdot 4 & 7\cdot 1 + 2\cdot(-3) + (-4)\cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} -37 & 18 & 31 \\ -2 & 24 & 36 \\ -50 & 26 & -19 \end{bmatrix}$$

Common Mistakes

- Does AB = BA (In general)?

- Whether Matrix multiplication is commutative?
- Statement is not true in general, see example:
- Example: A = $\begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}$

Caution!

- If AB = 0 does that mean A = 0, B = 0 or AB = 0 ?
- Statement is not true in general, see example:

- Example:
- A = $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

Caution!

- If AC = AD does that mean C = D (when A is non zero matrix) ?
- Statement is not true in general, see example:

Example:

A = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, C= $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$

Matrix Multiplication Facts

- Not commutative: in general $\mathbf{AB} \neq \mathbf{BA}$.
- Associative:

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

- Associates with scalar multiplication:

$$k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B})$$

- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
- $(\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3 \ldots \mathbf{M}_n)^T = \mathbf{M}_n^T \ldots \mathbf{M}_3^T\mathbf{M}_2^T\mathbf{M}_1^T$
- **Does AB = BA?**
- **If AB = 0, then A = ? Or B = ?**
- **If AB = AC, Then C = ? B**

Matrix

Matrix Operations

Properties of Matrix Multiplication:

- In general, $AB \neq BA$ if both exist, but there are special cases that this property is not true.
- If I is an identity matrix $IB = BI = B$.
- $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$
- $A(BC) = (AB)C$
- If AB exist then $(AB)' = B'A'$ (this can be extended to more than 2 matrices, i.e.: $(ABC)' = C'B'A'$
- From $AB = \underline{0}$ we cannot conclude necessarily that $A = \underline{0}$ or $B = \underline{0}$.*
- From $AB = AC$ we cannot conclude necessarily that $B = C$.**

Row Vector Times Matrix Multiplication

Can multiply a row vector times a matrix

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} =$$

$$\begin{bmatrix} xm_{11} + ym_{21} + zm_{31} & xm_{12} + ym_{22} + zm_{32} & xm_{13} + ym_{23} + zm_{33} \end{bmatrix}$$

Matrix Times Column Vector Multiplication

Can multiply a matrix times a column vector.

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xm_{11} + ym_{12} + zm_{13} \\ xm_{21} + ym_{22} + zm_{23} \\ xm_{31} + ym_{32} + zm_{33} \end{bmatrix}$$

Common Mistake

$\mathbf{Mv}^T \neq (\mathbf{vM})^T$, but $\mathbf{Mv}^T = (\mathbf{vM}^T)^T$ – compare the following two results:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$= \begin{bmatrix} xm_{11} + ym_{21} + zm_{31} & xm_{12} + ym_{22} + zm_{32} & xm_{13} + ym_{23} + zm_{33} \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xm_{11} + ym_{12} + zm_{13} \\ xm_{21} + ym_{22} + zm_{23} \\ xm_{31} + ym_{32} + zm_{33} \end{bmatrix}$$

Vector-Matrix Multiplication Facts 1

Associates with vector multiplication.

- Let **v** be a row vector:

$$\mathbf{v}(\mathbf{AB}) = (\mathbf{vA})\mathbf{B}$$

- Let **v** be a column vector:

$$(\mathbf{AB})\mathbf{v} = \mathbf{A}(\mathbf{Bv})$$

Vector-Matrix Multiplication Facts 2

- Vector-matrix multiplication distributes over vector addition:

$$(\mathbf{v} + \mathbf{w})\mathbf{M} = \mathbf{v}\mathbf{M} + \mathbf{w}\mathbf{M}$$

- That was for row vectors **v**, **w**. Similarly for column vectors.