

03/09/19

continuous probability distribution $f(x) \rightarrow \text{pdf}$

if & :

(i) $f(x) \geq 0 \forall x$

(ii) $\int_{-\infty}^{\infty} f(x) dx = 1$

(iii.) $\int_a^b f(x) dx = P(a < x < b)$

$\sum f(x) = 1 \rightarrow \text{discrete}$

→ cumulative Distrib' Func' (cdf)

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^x f(t) dt \end{aligned}$$

Properties of $f(x)$ (i.) Domain is $(-\infty, \infty)$ Range $[0, 1]$ (ii.) $F(x)$ is non-decreasing func' of x in the right

ie, $F'(x) = f(x) \quad \forall x \geq 0$

(iii.) $F(x)$ is continuous on the right

(iv.) $F(-\infty) = 0$ (Prob. always defined on an interval)

$$F(\infty) = 1$$

(v) $P(a \leq x \leq b) = P(a \leq x < b) = P(a < x \leq b) = P(a < x < b)$

$$= \int_a^b f(x) dx$$

in discrete $\rightarrow P(a \leq x \leq b) = F(b) - F(a)$
 $P(a < x \leq b) = F(b) - F(a-1)$

Ex. $f(x) = ke^{-3x}, x \geq 0$
 $0, x < 0$

Determine k & hence, compute cdf of x

$$\int_{-\infty}^{\infty} f(x) dx = 1 \Rightarrow \int_{-\infty}^{0} + \int_{0}^{\infty} k e^{-3x} dx = 1$$

$$= K \left(\frac{e^{-3x}}{-3} \right) \Big|_0^{\infty} = K \left[\frac{-1}{3} \right] = 1 \Rightarrow K = 6 + 3$$

$$F(x) = \int_0^x 3 e^{-3x} dx = \frac{3}{-3} \left[e^{-3x} \right]_0^x = 1 - e^{-3x}, x > 0$$

$$F(2) = P(X \leq 2) = 1 - e^{-3(2)} = 1 - e^{-6}$$

$$F(4) = 1 - e^{-3(4)} = 1 - e^{-12}$$

Ex: Pdf of random variable x :

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & x \geq 2 \end{cases}$$

Compute cdf of x

$$F(x \leq 1) = \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

$$F(x \leq 2) = \int_0^x x dx + \int_1^x (2-x) dx = \frac{x^2}{2} + 2x - \frac{x^2}{2} \Big|_1^x$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2} = 2x - \frac{x^2}{2} - 1$$

$$F(x \geq 2) = \int_0^2 x dx + \int_2^{\infty} (2-x) dx = \frac{1}{2} + 2 - \frac{3}{2} = 1$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \leq x < 1 \\ \frac{2x - x^2 - 1}{2} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$$\begin{aligned}
 \text{(i)} \quad P(-1 \leq X \leq 3) &= F(3) - F(-1) = 1 - 0 = \boxed{1} \\
 \text{(ii.)} \quad P(1 \leq X \leq 1.5) &= 2x - \frac{x^2}{2} - 1 - 2\left(\frac{3}{2}\right) - \left(\frac{9}{8}\right) - 1 \\
 &\downarrow \\
 &F(1.5) - F(1) \\
 &= 3 - \frac{17}{8} = \frac{7}{8} - \left(\frac{1}{2}\right) \\
 &= \boxed{\frac{3}{8}}
 \end{aligned}$$

Expectation :

$$\mu = E(X) = \sum_{x_n} x_n p(x) \quad \text{if } x \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} x f(x) dx \quad \text{if } x \text{ is continuous}$$

Let $g(x)$ = func' of x , so,

$$E[g(x)] = \sum_{x_n} g(x_n) p(x)$$

$$= \int_{-\infty}^{\infty} g(x) f(x) dx$$

(Always
+ive)
(units)

Variance :

$$\sigma^2 = E[(X - \mu)^2] = \sum_{x_n} (x_n - \mu)^2 p(x) \quad \text{discrete}$$

$$= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \quad \text{cont.}$$

$$\text{S.D.} (\sigma) = + \sqrt{E(X - \mu)^2}$$

→ Computational formula for σ^2 :

$$\begin{aligned}
 \sigma^2 &= \sum x^2 p(x) - [\sum x p(x)]^2 \\
 \boxed{\sigma^2} &= E[X^2] - (E[X])^2
 \end{aligned}$$

$$\begin{aligned}
 \sigma^2 &= E[(X - \mu)^2] = \sum_{x_n} (x_n - \mu)^2 p(x) \\
 &= \sum x^2 p(x) - 2\mu x p(x) + \mu^2 p(x)
 \end{aligned}$$

$$= \sum_{x \in A} x^2 p(x) - 2\mu \sum_{x \in A} x p(x) + \mu^2 \sum_{x \in A} p(x)$$

$$= \sum_{x \in A} x^2 p(x) - 2\mu(\mu) + \mu^2 (1)$$

$$= \sum_{x \in A} x^2 p(x) - \mu^2$$

$$= E[X]^2 - (E[X])^2$$

continuous : $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$

Properties of Expectations & Variance

$$(i) E(c) = c \quad \text{where } c: \text{constant} = \sum c p(x) = c \sum p(x) = c$$

$$(ii) E(cx) = c E(x)$$

$$(iii) E(c+x) = c + E(x)$$

$$(iv) \text{Var}(c) = 0 = E[c^2] - (E[c])^2 = c^2 - c^2 = 0$$

$$(v) \text{Var}(cx) = c^2 \text{Var}(x) = E[x^2 c^2] - [E(cx)]^2 \\ = c^2 E[X^2] - [c^2 E(X)^2] = c^2 \text{Var}(x)$$

Ex $E[X] = 5$

$$\text{Var}(X) = 2$$

$$E[X-5] = ? \quad \text{Var}(-2x) = ?$$

$$\text{Var}(-2x) = 4 \text{Var}(x) = 4 \times 2 = 8$$

$$E[X-5] = E[X] - 5 = 0$$

The r th moment about origin :

$$\mu_r' = E[X^r] = \sum_{x \in A} x^r p(x) \quad : x \text{ is discrete}$$

$$\int_{-\infty}^{\infty} x^r f(x) dx \quad : \text{cont.}$$

$$r=1 \rightarrow E[X] : 1^{\text{st}} \text{ moment about origin} = \underline{\underline{\mu}}. \\ (\mu_1') \quad \text{is Mean.}$$

r th moment about mean :

$$\mu_r = E[(x-\mu)^r] = \sum_{x \in A} (x-\mu)^r p(x)$$

$$\int_{-\infty}^{\infty} (x-\mu)^r f(x) dx$$

if $\sigma = 2$

$$\mu_2 = E[(X-\mu)^2] = \text{Var}(X) = \sigma^2 \rightarrow \text{2nd moment about mean}$$

is Variance

$$\boxed{\sigma^2 = \mu'_2 - \mu^2}$$

$$\begin{aligned} \sigma^2 &= E[(X-\mu)^2] \\ &= E[X^2] - (E(X))^2 \\ &= \mu'_2 - \mu^2 \end{aligned}$$

`func` is used to generate
moments

→ Moment Generating Func : MGF

$$M_X(t) = E[e^{tx}] = \sum_{x=n} e^{tx} p(x) \quad \text{discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx \quad \text{continuous}$$

$$\mu'_r = \left. \frac{d^r}{dt^r} [M_X(t)] \right|_{t=0}$$

diff. r times

$$\mu = \mu'_1 = \left. \frac{d}{dt} [M_X(t)] \right|_{t=0} \quad \sigma^2 = \mu'_2 - \mu^2 \rightarrow \text{calculate from } M_X(t)$$

→ Mean Deviation about mean:

$$\text{M.D.} = \sum_{x} |x - \mu| p(x)$$

• sometimes, moments generated are complex, but we want only real funcⁿ, so we multiply it with i :

→ Characteristic Func:

$$\phi_X(t) = E[e^{itx}]$$

Probability Distributions

① Bernoulli :

Assump's:

→ 2 outcomes of each experiment

→ $P(\text{success})$ is const. (p) , $P(\text{failure}) : (1-p) = q$

$$p(x) = \begin{cases} p & x=1 \\ q & x=0 \end{cases}$$

$x=1 \rightarrow \text{success}$
 $x=0 \rightarrow \text{failure}$

$$P(X=1) = p \rightarrow \text{success} \quad P(X=0) = q \rightarrow \text{failure}$$

$$p(x) = \begin{cases} p^x q^{1-x}, & x=0,1 \\ 0 & \text{e.w.} \end{cases}$$

$$E(X) = p, \quad \text{var}(X) = pq$$

$$M_X(t) = q + pe^t$$

13/9/19

Joint DistributionsJoint Discrete Mass Funcⁿ (discrete joint densities)

$$f_{XY}(x,y) = P(X=x \text{ and } Y=y)$$

$$\rightarrow f_{XY}(x,y) \geq 0 \quad \forall (x,y)$$

$$\rightarrow \sum_{\forall x} \sum_{\forall y} f_{XY}(x,y) = 1$$

) → Necessary & sufficient

* Necessary / sufficient / necessary & sufficient

Marginal distribⁿ (individual ki prob.)

$$f_X(x) = \sum_{\forall y} f(x,y) \rightarrow \text{Marginal density for } X$$

$$f_Y(y) = \sum_{\forall x} f(x,y)$$

discrete : Random variable takes value countable.
cont. : Random variable takes value uncountable.

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PAGE NO. :

Continuous Joint Density

$$1. f_{xy}(x,y) \geq 0 ; x, y \in \mathbb{R}$$

$$2. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy dx = 1$$

$$3. P[a \leq x \leq b \text{ and } c \leq y \leq d] = \int_a^b \int_c^d f(x,y) dy dx$$

↳ joint density
for (x,y)

cont. marginal densities :

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y) dy, \quad f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y) dx$$

Independence :

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

$$P(X=x, Y=y) = P(X=x) * P(Y=y)$$

$$\text{or } f_{xy}(x,y) = f_x(x) \cdot f_y(y)$$

Multivariate : $f(x_1, x_2, \dots, x_n) = f(x_1) \cdot f(x_2) \cdots f(x_n)$

Ex - 1

1st task \rightarrow 2 joints

2nd task \rightarrow 3 bolts

$x \rightarrow$ no. of defective welds

$y \rightarrow$ no. of improperly tightened bolts produced per car

x/y	0	1	2	3	$f_x(x)$
0	.840	.030	.020	.010	.900
1	.060	.010	.008	.002	.060
2	.010	.005	.004	.001	.020
$f_y(y)$.910	.045	.032	.013	1.000
$f_x(x=0)$	$\sum_{y=0} f(0,y) \Rightarrow 1^{\text{st}} \text{ row sum}$				

Question Test 2 → checking independence
 here are different (check all 12 values combination) ↳ That's why checking independence in case of discrete is difficult

$$(a) P(X=2, Y=1) = 0.005$$

$$(b) P(X \geq 1, Y \geq 1) = \cancel{0.045} + \cancel{0.032} \rightarrow 0.030$$

$$(c) P(X \leq 1) = 0.980$$

$$(d) P(Y \geq 2) = 0.032 + 0.013 = 0.045$$

Questⁿ $f_{XY}(x,y) = \frac{1}{n^2}, x=1,2,\dots,n$
 $y=1,2,\dots,n$

(a) Verify $f(x,y)$ satisfies conditional.

$$(i) f(x,y) \geq 0$$

$$n^2 \geq 0$$

$$\Rightarrow \frac{1}{n^2} \geq 0$$

$$(ii) \sum_{x,y} f(x,y) = 1$$

$$\begin{aligned} \sum_{x,y} f(x,y) &= \sum_{x=1,2,\dots,n} \sum_{y=1,2,\dots,n} \frac{1}{n^2} \\ &= \sum_{x=1,\dots,n} \frac{1}{n^2} + \frac{1}{n^2} + \dots \text{ n times } = \sum_{x=1,2,\dots,n} n \left(\frac{1}{n^2} \right) = \frac{1}{n} \\ &= \boxed{1} \end{aligned}$$

(b) find Marginal density for $X \& Y$

$$f_X(x) = \sum_{y=1}^n \frac{1}{n^2} = \frac{1}{n}, f_Y(y) = \frac{1}{n}, y=1,2,\dots,n$$

$0, \text{ elsewhere}$

(c) Are $X \& Y$ independent?

$$f_X(x) * f_Y(y) = f(x,y) \Rightarrow \text{Independent}$$

Questⁿ $f_{XY}(x,y) = c(4x+2y+1) \quad 0 \leq x \leq 40; 0 \leq y \leq 2;$

$$f(x,y) = c(4x+2y+1) \quad 0 \leq x \leq 40, \quad 0 \leq y \leq 2$$

(a) Find c

$$c \int_0^{40} \int_0^2 (4x+2y+1) dy dx$$

$$c \int_0^{40} (4xy + y^2 + y) \Big|_0^2 dx = c \int_0^{40} (6 + 8x) dx = 1$$

$$4 + 2 + 8x \Rightarrow c (6x + 4x^2) \Big|_0^{40} = 1$$

 $\frac{6400}{240}$

$$\boxed{c = \frac{1}{6640}}$$

$$\Rightarrow f(x,y) = \begin{cases} \frac{1}{6640} (4x+2y+1) & 0 \leq x \leq 40, 0 \leq y \leq 2 \\ 0 & \text{e.w.} \end{cases}$$

$$(b) f_x(x) = ? \quad f_y(y) = ?$$

$$f_x(x) = c \int_0^2 (4x+2y+1) dy = \frac{8x+6}{6640}, \quad 0 \leq x \leq 40 \quad \text{e.w.}$$

$$f_y(y) = c \int_0^{40} (4x+2y+1) dx = c \left[2x^2 + (2y+1)x \right] \Big|_0^{40} = \frac{80y+3240}{6640} \quad 0, \quad \text{e.w.}$$

$$(c) f(x > 20, y > 1) = c \int_{20}^{40} \int_1^2 (4x+2y+1) dy dx$$

$$= c \int_{20}^{40} (4xy + y^2 + y) \Big|_1^2 dx = c \int_{20}^{40} (2480) dx = \frac{2480}{6640}$$

$$(d) P(y > 1) = \int_1^2 \frac{80y+3240}{6640} dy = 0.526$$

$$(e) P(x > 20) = c \int_{20}^{40} \frac{8x+6}{6640} dx = 0.741$$

$$(f) f_x(x) \cdot f_y(y) \neq f_{xy}(x,y) \rightarrow \text{Not Independent}$$

Q5. (n -dimensional discrete random variables)

$$f(x_1, x_2, \dots, x_n) = P[x_1=x_1, x_2=x_2, \dots, x_n=x_n]$$

$$P(x_1) = 0.9 \quad P(x_2) = 0.08 \quad P(x_3) = 0.02$$

↓
non-def

↓
def but
salvagable

↑
neutri

$n = 20$ items

$$(a) P[x_1=15, x_2=3, x_3=2] = ?$$

(b) Find general formula for density for (x_1, x_2, x_3) .

$$\begin{aligned} \text{sol'n} \quad P(x_1=x_1, x_2=x_2, x_3=x_3) &= f(x_1, x_2, x_3) \\ &= \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \Rightarrow \boxed{\text{Multinomial distib'}} \\ &= \frac{20!}{15! 3! 2!} (0.9)^{15} (0.08)^3 (0.02)^2 \\ &= 0.0065 \end{aligned}$$

↑
extension of
binomial one.

Lecture 2:

Expectation & Covariance

$$(1) E[H(x, y)] = \sum_{\forall x} \sum_{\forall y} H(x, y) f_{x,y} \quad \text{permitted}$$

$$(2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y) f_{x,y} dy dx$$

$$\text{Ex. } f(x, y) = \begin{cases} \frac{3}{2} (x^2 + y^2) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[x^2 + y^2]$$

$$= \int_0^1 \int_0^1 (x^2 + y^2) \frac{3}{2} (x^2 + y^2) dy dx$$

$$= \frac{3}{2} \int_0^1 \left(\int_0^1 (x^4 + 2x^2y^2 + y^4) dy \right) dx$$

$$\begin{aligned}
 & \frac{3}{2} \int_0^1 \left[x^4 y + \frac{2x^2 y^3}{3} + \frac{y^5}{5} \right] \Big|_0^1 dx \\
 &= \frac{3}{2} \int_0^1 \left(x^4 + \frac{2x^2}{3} + \frac{1}{5} \right) dx \\
 &= \frac{3}{2} \left[\frac{1}{5} + \frac{2}{3} + \frac{1}{5} \right] = \frac{3}{2} \left[\frac{1}{5} + \frac{2}{9} + \frac{1}{5} \right] \\
 &= \frac{3}{2} \left[\frac{2}{5} + \frac{2}{9} \right] = 3 \left[\frac{1}{5} + \frac{1}{9} \right] = \frac{3 \times 14}{5 \times 9} = \frac{14}{15}.
 \end{aligned}$$

Univariate Average : calculate $E(x)$ from $f(x, y)$

$$E[x] = \sum_{x \in A} x \underbrace{f_x(x)}_{\substack{\text{Marginal} \\ \text{density of } x}} = \sum_{x \in A} \sum_{y \in Y} x f(x, y)$$

Covariance : → How X & Y are related

$$\text{cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] \quad E[X] = \mu_X, E[Y] = \mu_Y$$

* $X \uparrow$ & $Y \uparrow \Rightarrow (X - \mu_X)(Y - \mu_Y) > 0 \rightarrow$ positively correlated
 $X \uparrow$ & $Y \downarrow \Rightarrow \quad \quad \quad < 0$
 else

if $\text{cov}(X, Y) = 0 \rightarrow X$ & Y are independent

$$\boxed{\text{cov}(X, Y) = E[XY] - E[X]E[Y]}$$

$$\begin{aligned}
 \text{cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\
 &\leftarrow E[XY - YE[X] - XE[Y] + E[X]E[Y]] \\
 &= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\
 &= E[XY] - E[X]E[Y]
 \end{aligned}$$

If X & Y are independent :

$$E[XY] = E[X] E[Y] \Rightarrow \text{cov}(X, Y) = 0 \quad \boxed{\text{But not vice versa}}$$

Proof:

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x, y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_x(x) f_y(y) dx dy$$

$$= \int_{-\infty}^{\infty} x f_x(x) dx \int_{-\infty}^{\infty} y f_y(y) dy = E[X] E[Y]$$

Ex: $p(-1) = p(0) = p(1) = 1/3$

$$\text{Let } Y = X^2$$

$$E[X] = (-1+0+1)/3 = 0$$

$$E[Y] = (1+0+1)/3 = 2/3$$

$$E[XY] = (-1+0+1)/3 = 0$$

$E[XY] = E[X] E[Y]$, but X & Y are dependent.

Proof

$$\text{cov}(X, Y) = \text{Var}(X) = \text{Var}(Y) \quad \boxed{X = Y}$$

$$\text{if } X = Y, \Rightarrow \text{cov}(X, X) = E[X \cdot X] - E[X] \cdot E[X] \\ = E[X^2] - (E[X])^2 = \text{Var}(X)$$

10/9/19
Also $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{cov}(X, Y)$

 \therefore

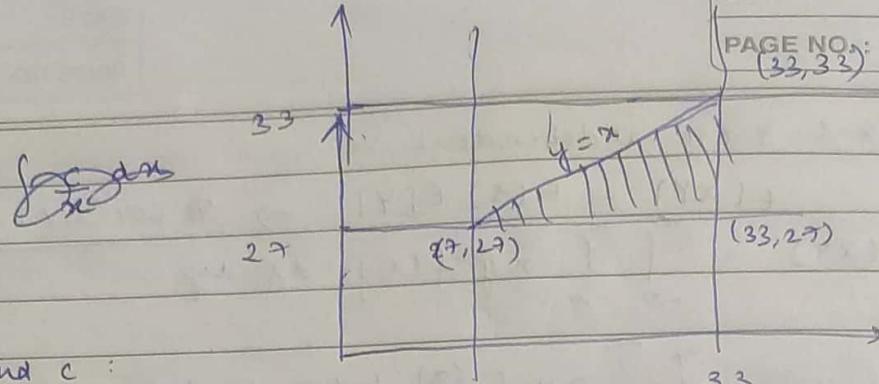
Exercise 19- The joint density is given by

$$f_{XY}(x, y) = \frac{c}{x} \quad 27 \leq y \leq x \leq 33$$

$$c = \frac{1}{(6 - 27 \ln 33 / 27)} = 1.72$$

(a) Find $E[X]$, $E[Y]$, $E[XY]$, $\text{cov}(X, Y)$

(b) $E[X-Y]$



To find c :

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$$

$$= \int_{27}^{33} \int_{27}^{x} \frac{c}{x} dy dx \Rightarrow \int_{27}^{33} \frac{c}{x} (x - 27) dx$$

$$= c \int_{27}^{33} \left(1 - \frac{27}{x}\right) dx = 1 \Rightarrow c \left[x - 27 \ln x \Big|_{27}^{33} \right] = 1$$

$$\therefore c = \frac{1}{6 - 27 \ln 33/27} = 1.72$$

(a) ~~E[X]~~

$$f_x(x) = \int_{27}^x \frac{c}{x} dx \Rightarrow \frac{(x-27)}{x} c$$

$$f_y(y) = \int_y^{33} \frac{c}{x} dx \Rightarrow \frac{(33-y)}{y} c (\ln 33 - \ln y)$$

$$E[X] = \int_{27}^{33} x f_x(x) dx = \int_{27}^{33} c(x-27) dx + 2 \cdot \frac{x^2}{2} - 6 \cdot 27$$

$$E[X] = \int_{27}^{33} c(\ln 33 - \ln y) y dy = 30.96$$

$$E[XY] = \int_{27}^{33} \int_y^{33} xy \frac{c}{x} dx dy = 897.84$$

~~$$E[X-Y] = \text{cov}(XY) = E[XY] - E[X]E[Y] = 0.81$$~~

$$(b) E[X-Y] = \int_{27}^{33} \int_{27}^x (x-y) \frac{c}{x} dy dx = 1.97$$

~~PROVE~~ $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

If, X & Y are independent:

$$\begin{aligned}\text{Var}(X+Y) &= \text{Var}(X) + \text{Var}(Y), \quad \text{Var}(X-Y) = \text{Var}(X) + \text{Var}(Y) \\ \text{Var}(X+Y) &= E[(X+Y)^2] - (E[X+Y])^2 \\ &= E[X^2 + Y^2 + 2XY] - (E[X] + E[Y])^2 \\ &= E[X^2] + E[Y^2] + 2E[XY] - (E[X])^2 - (E[Y])^2 - 2E[X]E[Y] \\ &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)\end{aligned}$$

ex. $f_{XY}(x, y) = \frac{1}{2(e-1)} \left[\frac{1}{x} + \frac{1}{y} \right] \quad 1 \leq x \leq e$
 $1 \leq y \leq e$

(a) show that it is valid joint density

(b) Find $E[X]$, $E[Y]$, $E[XY]$ $\frac{3e-1}{4}$, $\frac{3e-1}{4}$, $\frac{e^2-1}{2}$

(c) X & Y independent? No

$$(a) \int_1^e \int_1^e \frac{1}{2(e-1)} \left[\frac{1}{x} + \frac{1}{y} \right] dy dx$$

$$= \int_1^e \frac{1}{2(e-1)} \left(\frac{y}{x} + \ln y \right) \Big|_1^e dx = \frac{1}{2(e-1)} \int_1^e \left(\frac{(e-1) + \ln 1}{x} \right) dx$$

$$= \frac{1}{2(e-1)} \left((e-1) \ln x + x \right) \Big|_1^e = \frac{1}{2(e-1)} [e-1 + e-1] = 1$$

(b) $E[X]$:

$$f_X(x) = \int_1^e \frac{1}{2(e-1)} \left[\frac{1}{x} + \frac{1}{y} \right] dy = \frac{1}{2(e-1)} \left[\frac{(e-1)}{x} + 1 \right]$$

$$E(X) = \int_1^e x f_X(x) dx = \int_1^e \left(\frac{x}{2(e-1)} \left[\frac{e-1}{x} + 1 \right] \right) dx$$

$$= \int_1^e \frac{1}{2} + \int_1^e \frac{x}{2(e-1)} dx = \frac{x^2}{2} \Big|_1^e$$

$$= \frac{e-1}{2} + \frac{1}{2(e-1)} \left[\frac{e^2-1}{2} \right] \Rightarrow \frac{e-1}{2} \cdot \frac{e+1}{2} = \frac{3e-1}{4}$$

20/9/19

PAGE NO. :

Conditional DistributionConditional densities:

$$f_{X|Y} = \frac{P(X=x \mid Y=y)}{P(Y=y)} = \frac{f_{XY}(x,y)}{f_Y(y)}, f_Y(y) > 0$$

satisfies

- 1) $\int_{-\infty}^{\infty} f_{X|Y} dx \geq 0$
- 2) $\int_{-\infty}^{\infty} f_{X|Y} dx = 1$

condⁿal mean & variance:

$$E(Y|x) = \sum y f_{Y|x}(y) = \int y f_{Y|x}(y) dy$$

$$E(Y|x=x) = \sum y f_{Y|x}(Y=y|x=x)$$

$$\text{var}(Y|x) = \sum y^2 f_{Y|x}(y) - \mu_{Y|x}^2$$

Ex. $f(x,y) = x^2 + \frac{xy}{3}, 0 \leq x \leq 1, 0 \leq y \leq 2$
e.w.

solⁿ Marginal density for x?

$$f_X(x) = \int_0^2 \left(x^2 + \frac{xy}{3} \right) dy = \left. x^2 y + \frac{x}{6} y^2 \right|_0^2 \\ = 2x^2 + \frac{2}{3} x, 0 \leq x \leq 1$$

$$f_Y(y) = \int_0^1 \left(x^2 + \frac{xy}{3} \right) dx = \frac{1}{3} + \frac{y}{6}$$

$$f_{X|Y} = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{x^2 + \frac{xy}{3}}{\frac{1}{3} + \frac{y}{6}} = \frac{6x^2 + 2xy}{2+y}, 0 \leq x \leq 1, 0 \leq y \leq 2$$

$$f_{Y|x} = \frac{f_{XY}(x,y)}{f_X(x)}$$

Mean & Variance of linear combⁿ of Random Variables

Ex.

$$Y = c_1 X_1 + c_2 X_2 + \dots + c_p X_p \quad X_i \rightarrow r.v. \\ c_i \rightarrow \text{consts.}$$

$$E[Y] = c_1 E[X_1] + c_2 E[X_2] + \dots + c_p E[X_p]$$

$$V(Y) = c_1^2 V(X_1) + c_2^2 V(X_2) + \dots + c_p^2 V(X_p) + 2 \sum_{i < j} c_i c_j \text{cov}(X_i, X_j)$$

If X_1, X_2, \dots, X_p are independent

$$V(Y) = c_1^2 V(X_1) + c_2^2 V(X_2) + \dots + c_p^2 V(X_p)$$

$$\begin{aligned} \rightarrow V(Y) &= E[(Y - E[Y])^2] \\ &= E[(c_1 X_1 + c_2 X_2 + \dots + c_p X_p - (c_1 \mu_1 + c_2 \mu_2 + \dots + c_p \mu_p))^2] \\ &= E[(c_1 (X_1 - \mu_1) + c_2 (X_2 - \mu_2) + \dots + c_p (X_p - \mu_p))^2] \\ &= E[c_1^2 (X_1 - \mu_1)^2 + c_2^2 (X_2 - \mu_2)^2 + \dots + c_p^2 (X_p - \mu_p)^2] \end{aligned}$$

Ex. $x \& Y$: independent r.v.

$$\mu_x = 7 \quad \mu_y = -5$$

$$V(x) = 9 \quad V(y) = 3$$

$$\begin{aligned} (i) \quad E[4x - 2y + 6] &= 4E[x] - 2E[y] + 6 = 4(7) - 2(-5) + 6 \\ &= 28 + 10 + 6 \\ &= 44 \end{aligned}$$

$$\begin{aligned} (ii) \quad V(4x - 2y + 6) &= 16V(x) + 4V(y) + 36 \\ &= 16(9) + 4(3) + 36 = 144 + 12 + 36 \\ &= 192 \quad \text{Ans} 156 \end{aligned}$$

Ex. Let X_1 & X_2 be 2 independent r.v. having normal distrib' having mean 5 & 6, var 2 & 3. Find mean of $Y = X_1 + X_2$. Find distrib' of Y

Solⁿ (Take help of moment generating funcⁿ)

$$M_X(t) = e^{5t + \frac{1}{2} \cdot 2t^2}$$

Given $X_1 \sim N(5, 2)$

$$M_{X_1}(t) = e^{5t + \frac{1}{2} \cdot 2t^2}$$

$$M_{X_2}(t) = e^{6t + \frac{1}{2} \cdot 3t^2}$$

$$\begin{aligned} M_Y(t) &= M_{(X_1+X_2)}(t) = M_{X_1}(t) \cdot M_{X_2}(t) && \{ X_1 \& X_2 \text{ are independent} \} \\ &= (e^{5t + t^2}) \cdot e^{(6t + 3/2 t^2)} \\ &= e^{11t + \frac{5}{2} t^2} \rightarrow N(11, 5) \end{aligned}$$

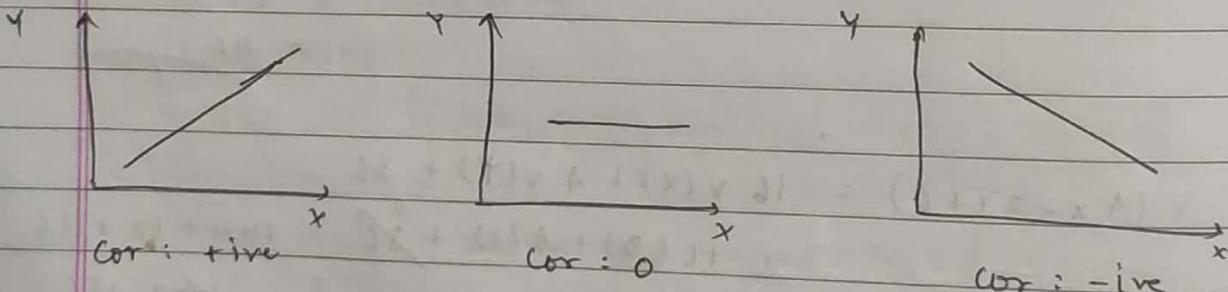
$Y \rightarrow$ also a normal distrib'

Correlation

$\rho \rightarrow$ Pearson Coefficient

~~$\rho = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)}(\text{var}(Y))}$~~ \Rightarrow linear rel' b/w X & Y

If distrib' is given, use this



$\rho = -1 \Rightarrow$ Perfect -ive correlation

$\rho = 0 \Rightarrow$ Independent var (Not correlated) \rightarrow Non-linear rel' ship

$\rho = 1 \Rightarrow$ Perfect +ve correl'.

\Rightarrow Not possible for large popⁿ to calculate ρ , taken random samp of size n .

$$\text{Var}^n X = \sum_{i=1}^n (X_i - \bar{X})^2 / n = S_{xx}/n \text{ similarly for } Y$$

$$\text{Cov}(x, y) = E[(x - \mu_x)(y - \mu_y)]$$

$$\Rightarrow \text{cov}(x, y) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n} = s_{xy}/n$$

$$\hat{r} = R = \frac{s_{xy}}{\sqrt{s_{xx}s_{yy}}}$$

Computational formula :

$$\hat{r} = \frac{n \sum xy - \sum x \sum y}{\left[\left(n \sum x^2 - (\sum x)^2 \right) \left(n \sum y^2 - (\sum y)^2 \right) \right]^{1/2}} \quad \Rightarrow \begin{array}{l} \text{use when} \\ \text{data is} \\ \text{given} \end{array}$$

Q. Find sample correl' coeff.

$$1.) \quad x : 1, 2, 3, 4, 5$$

$$y : 2, 4, 6, 8, 10$$

$$\text{soln: } \hat{r} = ?$$

x	y	x^2	y^2	xy
1	2	1	4	2
2	4	4	16	8
3	6	9	36	18
4	8	16	64	32
5	10	25	100	50
<u>Σ</u>	<u>30</u>	<u>55</u>	<u>220</u>	<u>110</u>
				225
				<u>1100</u> <u>- 900</u>

$$\hat{r} = r = \frac{5(110) - (15)(30)}{\sqrt{[5 \times 55 - 225][5 \times 220 - 900]}}$$

$$= \frac{550 - 450}{\sqrt{50 \times 200}} = \frac{100}{\sqrt{10000}} = \frac{100}{100} = \boxed{1}$$

$$2.) f_{XY}(x,y) = \frac{1}{6640} (4x+2y+1) \quad 0 \leq x \leq 40 \\ 0 \leq y \leq 2$$

Find $E(X)$, $E(Y)$, $E(X^2)$, $E(Y^2)$, $\text{Var}(X)$, $\text{Var}(Y)$, $\text{Cov}(XY)$.

$$\underline{\text{Soln}} \quad E[X,Y] = \iint_0^4 \int_0^2 xy \frac{1}{6640} (4x+2y+1) dy dx = \frac{26.586}{6640} = 0.00854$$

$$E[X] = \int_0^4 x f_{X,Y}(x) dx = 26.426 \quad E[X^2] = 700.361$$

$$E[Y] = 1.008$$

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = 26.586 - (26.426)(1.008) = -0.051$$

$$\text{Var}(X) = 700.361 - (26.426)^2 = 2.027$$

$$\text{Var}(Y) = 1.3494 - (1.008)^2 = 0.333$$

$$\rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = -0.009 \Rightarrow \text{-ively correlated}$$

24/9/19

Chi-square distribution.

Let $Z \sim N(0,1)$

$$\Rightarrow Z^2 \sim \chi^2 \rightarrow \text{chi-square}$$

χ^2

Let X_1, X_2, \dots, X_n be n independent variables with mean μ_i and σ_i^2 ($i=1, 2, \dots, n$) then,

$\sum_{i=1}^n (X_i - \mu_i)^2 / \sigma_i^2$ is a χ^2 -variate with n d.f.

Proof Let X be a χ^2 variate with n d.f. n

$$\text{E}[X] = n$$

$$\text{Var}(X) = 2n$$

→ This distribution is used for Goodness of fit

Chi-square (χ^2) Test Observed & expected frequency \rightarrow calc.
of goodness of fit : check if theoretical
value is same as observed

Let O_i ($i=1, 2, \dots, n$) be a set of observed frequencies &
 e_i ($i=1, 2, \dots, n$) corresponding set of expected (theoretical)
frequencies. Then.

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - e_i)^2}{e_i}$$

has a χ^2 distrib' with $(n-1)$ d.f.

Given, all $e_i \geq 5$. (thumb rule)

\downarrow
degree of freedom

If $\chi^2_{\text{calc}} < \chi^2_{\text{Tab}}$ \Rightarrow We have made correct assumption. \therefore no significant difference was found to vary.

Ex. The demand of a spare part in a factory from day to day.
In a sample study, foll. info. was obtained.

Days	M	T	W	Tn	F	S.	Total
No. of parts demanded	1124	1125	1110	1120	1126	1115	6720
(O_i)							$\frac{6720}{7} = 960$

Observed frequencies

Test the hypothesis, that the no. of parts doesn't depend on the day.

\Downarrow
P(demand) is same for all day

① find total of all days = 6720 = n

H_0 : The demand of no. of parts doesn't depend on day

H_1 : H_0 is not true

$$p_i = \frac{1}{6}, i=1, 2, \dots, n$$

$$e_i = n * p_i$$

② No. of parts : 1120 1120 1120 1120 1120 1120
(e_i)

$$\chi^2 = \sum_{e_i} (o_i - e_i)^2$$

$$= \frac{(1124 - 1120)^2}{1120} + \frac{(1125 - 1120)^2}{1120} + \dots$$

$$= 0.179$$

No. of cells = 6

⇒ Degree of freedom here = 5

$$\chi^2_{\text{tab with } 5 \text{ d.o.f.}} = 11.07 \quad (\text{from table})$$

$0.179 < 11.07 \Rightarrow$ If no significant difference
 $\Rightarrow H_0$ is accepted.

⇒ If any $e_i < 5 \rightarrow$ can add it with any preceding or succeeding cell (add both e_i & o_i). Thus, d.o.f. --;

Student t-distribution

- It is a sample distrib'
- Can be used whenever Z-distrib' is used, only cond' is $n \leq 30$
- Let x_1, x_2, \dots, x_n be random sample of small size n with mean μ and variance σ^2 respectively. ($i=1, 2, \dots, n$) from a normal population with mean μ & variance σ^2 . Let \bar{x} be mean of sample & s^2 be variance of sample.

$$\text{i.e., } \bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$$

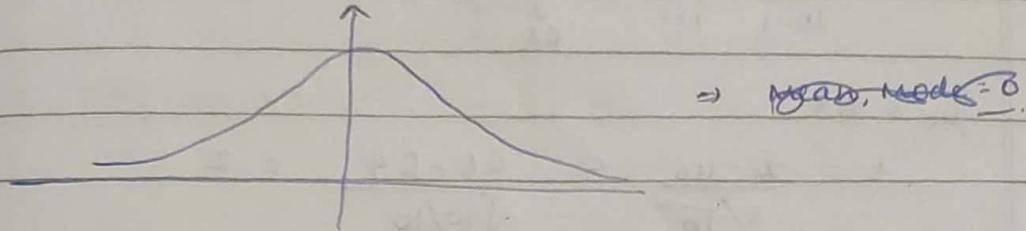
$$\text{and } s^2 = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2$$

↓

In sample, only $n-1$ samples are linearly independent

then $t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$ has a t-distribⁿ with $(n-1)$ d.f.

$$f(-t) = f(t) : \text{symmetric}$$



$$\mu_{2k+1}^* = 0$$

$$\mu_{2k} = \frac{n k (2k-1)(2k-2) \dots 3 \cdot 1}{(n-2)(n-4) \dots (n-2k)}, \quad \frac{n}{2} > k$$

$$\mu_2 (\text{Var}) = \frac{n}{n-2}, \quad n > 2 \quad , \quad \boxed{\text{mean}=0} \quad (\mu_1=0)$$

④ Significance Test for sample Mean

Let x_1, x_2, \dots, x_n be a random sample of size n from a normal population with a specified populⁿ mean μ .

To test $\mu = \mu_0$ or there is no significant difference between sample mean \bar{x} and samⁿ populⁿ mean μ_0 , we compute

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \quad \text{where} \quad \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad \& \quad s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

If $|t|_{\text{cal}} < |t|_{\text{tab}} \Rightarrow$ no significant diff. between \bar{x} & μ_0 .

~~If~~ If $n > 30 \rightarrow$ then t will become z .

Ques: The heights of 10 men of a locality are found to be:

70, 67, 62, 68, 61, 68, 70, 64, 64, 66

Is it reasonable to believe that avg. height is > 64 inches?

Test at 5% level of significance

$$\begin{aligned} t &> t_{\alpha} \\ t &< t_{\alpha} \\ |t| &< t_{\alpha/2} \end{aligned}$$

Soln. $\bar{x} = \frac{\sum x_i}{n} = \frac{660}{10} = 66$

$\alpha = 0.05$

$$s^2 = \frac{1}{10-1} \sum_{i=1}^{10} (x_i - \bar{x})^2 = 10$$

$$\mu_0 = 64$$

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{66 - 64}{\sqrt{10/10}} = 2$$

$$t_{tab} \text{ with } 9 \text{ d.f.} = 2.26$$

$t_{calc} < t_{tab} \Rightarrow$ Avg. height is not greater than 64 inches

F-distribution

Let X & Y be two independent Chi-square variates with ν_1 & ν_2 d.f. Then,

$$F = \frac{X/\nu_1}{Y/\nu_2} = F(\nu_1, \nu_2)$$

App's:

- entire analysis of variance of
- Agriculture, Medical

- Let x_1, x_2, \dots, x_n & y_1, y_2, \dots, y_n be 2 independent random samples drawn from same normal population with same variance. Let \bar{x} & \bar{y} be the sample mean & s_x^2 & s_y^2 be sample variance of 2 samples.

$$\Rightarrow \bar{x} = \frac{1}{n_1} \sum x_i \quad \bar{y} = \frac{1}{n_2} \sum y_i$$

$$s_x^2 = \frac{1}{n_1-1} \sum (x_i - \bar{x})^2 \quad s_y^2 = \frac{1}{n_2-1} \sum (y_i - \bar{y})^2$$

Then,
$$\boxed{F = \frac{s_x^2}{s_y^2}}$$

Determinants

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \Rightarrow x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{21} a_{12}} \\ a_{21}x_1 + a_{22}x_2 &= b_2 \quad \text{or} \quad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{21} a_{12}} \end{aligned}$$

↓
|D|

$|A|=0$: singular Matrix

Minor

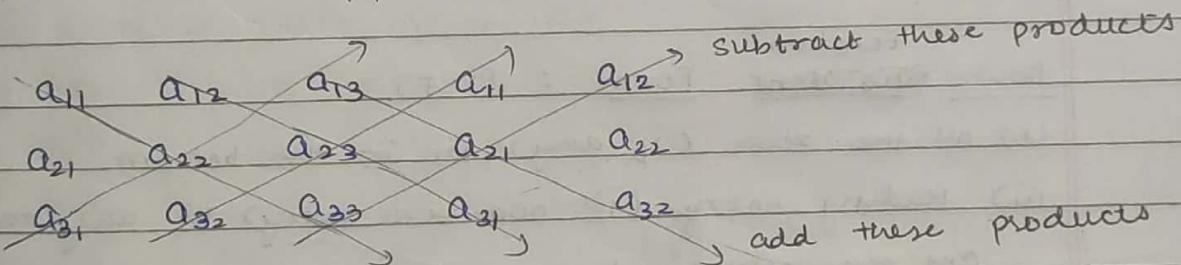
$M_{ij} \Rightarrow$ delete i^{th} row & j^{th} col & get remaining matrix

Cofactor of a_{ij} :

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$|D| = \sum_{j=1}^n a_{ij} C_{ij} \quad i = 1/2/3/ \dots /n$$

$$\text{or } \sum_{i=1}^n a_{ij} C_{ij} \quad j = 1/2/3/ \dots /n$$



→ Upper triangular Matrix : entries below main diagonal are zero

Equivalent forms of a Matrix

Elementary Row ops of a Matrix : (Same for col)

- ① Interchanging 2 rows R_i and R_j . symbolically $R_i \leftrightarrow R_j$
- ② Multiply a row R_i by a non-zero no. K , $R_i \rightarrow KR_i$
- ③ Adding a constant K multiple of a row R_i to a row R_j
 $R_j \rightarrow R_j + KR_i$

Equivalent matrix : Order same, not necessary same values

Equal " : Same values also

Matrix A $\xrightarrow{\text{Opns}}$ Matrix B A & B = equivalent

Ex. $A = \begin{bmatrix} 4 & 8 & 10 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$

Applying,

Op¹ $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 10 \\ 3 & 5 & 6 \end{bmatrix}$$

$\sim \rightarrow$ equivalent to

Applying
 $R_2 \rightarrow R_2 - \frac{1}{2}R_1$ $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ Applying
 $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$ $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix}$

All are equivalent matrix.

Echelon

Row Equivalent Form : (REF)

(i) All the zeros (if any) lie in the bottom

(ii) leading entry (1^{st} non-zero entry) in any row is 1. and the column lies to the right of the column of leading entry of proceeding row.

Reduced Row Echelon Form (RREF) :

(i) In addition, if all the entries above the leading entries are zero.

Ex. $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ $\xrightarrow{\text{REF}}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\xrightarrow{\text{RREF}}$ $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ $\xrightarrow{\text{RREF}}$

* A matrix can have diff. REF but RREF is unique.

Ex. Find RREF

$$\left[\begin{array}{ccc} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{array} \right] \xrightarrow{\begin{matrix} R_2 \leftrightarrow R_3 \\ R_2 \rightarrow -R_2 \\ R_3 \rightarrow -R_2 \end{matrix}} \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 - R_2} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{matrix} R_2 \rightarrow R_2 - 3R_3 \\ R_1 \rightarrow R_1 - R_2 \end{matrix}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

REF

RREF

→ To get 0 in i^{th} col, we have used i^{th} row.

Inverse of a Matrix

A be a matrix of m rows ~~and n cols~~ and n cols.

and B be a matrix of n cols.

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \quad B = (C_1, C_2, \dots, C_n).$$

$$AB = \begin{bmatrix} R_1 C_1 & R_1 C_2 & \dots & R_1 C_n \\ R_2 C_1 & R_2 C_2 & \dots & R_2 C_n \\ \vdots & \vdots & \ddots & \vdots \\ R_n C_1 & R_n C_2 & \dots & R_n C_n \end{bmatrix}$$

$$\text{If } R_1 \leftrightarrow R_2 : AB = \begin{bmatrix} R_2 C_1 & R_2 C_2 & \dots & R_2 C_n \\ R_1 C_1 & \dots & \dots & \dots \end{bmatrix}$$

$$A = \begin{bmatrix} R_2 \\ R_1 \\ \vdots \end{bmatrix}$$

If we change R_j in A, it will change R_i in AB also.

$$R(AB) = R(A) \cdot B$$

→ To find inverse of matrix A:

i) $|A| \neq 0$

ii) $A_{n \times n} = I_n A \rightarrow$ for row opn
 \hookrightarrow RREF *

$I_n = BA$

$\boxed{A^{-1} = B}$

RREF of a square matrix is always Identity Matrix.

Extra class : For col.- oper, we've to write:

(same date)

$A = A I_n$

$I_n = A B = A A^{-1}$

$\Rightarrow \boxed{A^{-1} = B}$

Gauss

~~Gauss~~ - Jordan

Method

Ex. $A = \begin{pmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{pmatrix}$

Find A^{-1} using

Gauss Jordan method.

$A = I_3 A$

$$\left[\begin{array}{ccc} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{array} \right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] A$$

You want right LHS as 1s.

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc} 2 & 3 & 1 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{array} \right]$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc} 2 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] A$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{array} \right]$$

$$\left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{array} \right] A$$

$$\begin{array}{l} R_1 \rightarrow R_1 + R_3 \\ R_2 \rightarrow R_2 - R_3 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & +1 & +3 \end{array} \right] = \left[\begin{array}{ccc} 0 & -2 & 1 \\ -1 & 2 & 0 \\ 0 & +3 & -1 \end{array} \right] A$$

$$\begin{array}{l} R_1 \rightarrow R_1 - R_3 \\ R_2 \rightarrow R_2 - R_3 \end{array} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc} 0 & -2 & 1 \\ 0 & 3 & -1 \\ -1 & 2 & 0 \end{array} \right] A$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 3R_3 \\ R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc} 0 & -2 & 1 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{array} \right] A$$

$$\begin{array}{l} R_1 \rightarrow R_1 - R_2 \\ R_2 \rightarrow R_2 \end{array} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc} -3 & 1 & 2 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{array} \right] A$$

$$\Rightarrow A^{-1} = \left[\begin{array}{ccc} -3 & 1 & 2 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{array} \right]$$

Rank of a Matrix :

$A_{n \times n}$

Rank of $A = \text{rank}(A)$ is defined as the no. of non-zero rows in REF of matrix.

$$\text{Ex. } A = \left[\begin{array}{ccc} 2 & 4 & 5 \\ * & 2 & 3 \\ 3 & 5 & 6 \end{array} \right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right]$$

$$\Rightarrow \text{Rank}(A) = 3$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 6 \\ 2 & 12 \end{pmatrix}$$

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Solution of a System of Linear Eq's

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Matrix form :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Rightarrow AX = B$$

$$A = \{a_{ij}\}_{m \times n}$$

coeff. matrix

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

If $B \neq 0 \rightarrow$ Non-homogeneous system of eq's.

$B = 0 \rightarrow$ Homogeneous

How to check consistency? \rightarrow sol'n of a problem is unique

Augmented matrix : $[A : B]$ add col. of B to A

- Thm: The system $AX = B$ of linear eq's has a
- Unique sol'n if $\text{rank}(A) = \text{rank}(A : B) = n$.
 - Infinite many sol'n: $\text{rank}(A) = \text{rank}(A : B) < n$
 - No sol'n.. if $\text{rank}(A) \neq \text{rank}(A : B)$

→ if $AB = 0 \rightarrow$ atleast 1 soln
trivial soln: $x = 0$

Q. Test the consistency of the following systems of eqns & find the soln if exists.

$$2x + 3y + 4z = 11$$

$$x + 5y + 7z = 15$$

$$3x + 11y + 13z = 25$$

(Gauss
Elimination
Method)

$$\begin{array}{c|c|c} A & X & B \\ \left[\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{array} \right] & \left(\begin{array}{c} x \\ y \\ z \end{array} \right) & = \left(\begin{array}{c} 11 \\ 15 \\ 25 \end{array} \right) \\ & & (A:B) \\ & & \left[\begin{array}{cccc} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{array} \right] \end{array}$$

(A:B): $R_1 \leftrightarrow R_2$

$$A \sim \left[\begin{array}{cccc} 1 & 5 & 7 & 15 \\ 2 & 3 & 4 & 11 \\ 3 & 11 & 13 & 25 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1}} \left[\begin{array}{cccc} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & -4 & -8 & -20 \end{array} \right] \xrightarrow{R_3 \rightarrow -\frac{1}{4}R_3}$$

$$\left[\begin{array}{cccc} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 1 & 2 & 5 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & -7 & -10 & -19 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 + 7R_2}$$

$$\left[\begin{array}{cccc} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 4 & 16 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3/4} \left[\begin{array}{cccc} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{array} \right] \Rightarrow \text{Rank}(A:B) = 3 \\ \text{Rank}(A) = 3$$

∴ soln is unique

$$\left[\begin{array}{ccc} 1 & 5 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right] \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = \left(\begin{array}{c} 15 \\ 5 \\ 4 \end{array} \right)$$

$\boxed{x_1 = 4}$
 $y + 2z = 5$
 $\Rightarrow \boxed{y = 3}$
 $x - 15 + 2y = 15 \Rightarrow \boxed{x = 2}$

→ if we find RREF of $(A:B)$: Gauss Jordan Method

$$\left[\begin{array}{cccc} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 - 5R_2} \left[\begin{array}{cccc} 1 & 0 & -3 & -10 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

$$\xrightarrow{\substack{+3R_3 \\ R_1 \rightarrow R_1 + 3R_3 \\ R_2 \rightarrow R_2 - 2R_1}} \left[\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

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$$\text{Ex. } x + y + 2z + w = 5$$

$$2x + 3y - z - 2w = 2$$

$$4x + 5y + 3z = 7$$

$$Ax = b$$

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{array} \right] \left[\begin{array}{c} x \\ y \\ z \\ w \end{array} \right] = \left[\begin{array}{c} 5 \\ 2 \\ 7 \end{array} \right]$$

$$(A:B) : \left[\begin{array}{ccccc} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{array} \right] \sim \left[\begin{array}{ccccc} 1 & 0 & 7 & 5 & 0 \\ 0 & 1 & -5 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$0 = 1 \times$ (from last row)
↓

Inconsistent \rightarrow NO soln

Rank $(A:B) = 3 \Rightarrow$ No soln
Rank $(A) = 2 \Rightarrow$ Inconsistent

$$\text{Ex. } -x + 2y + z = 1$$

$$3x - y + 2z = 1$$

$$y + 2z = 1, \lambda: \text{const.}$$

$$(A:B) = \left[\begin{array}{cccc|c} -1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array} \right] \xrightarrow{\text{Row Opns}} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 4/5 \\ 0 & 0 & 2-1 & 1/5 \end{array} \right]$$

rank (A:B) = 3 (always)

rank (A) = 2 if $\lambda = 1$

3 if $\lambda \neq 1$

System is consistent if $\lambda \neq 1$.

$$\rightarrow x - 2y - z = -1$$

$$y + z = 4/5$$

$$(A-1)z = 1/5$$

$$x = \frac{3}{5} - \frac{1}{5(\lambda-1)}$$

$$y = \frac{4}{5} - \frac{1}{5(\lambda-1)} \quad z = \frac{1}{5(\lambda-1)}, \lambda \neq 1$$

→ Upper triangular matrix :

$$\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{array} \right]$$

$$\left[\begin{array}{ccc} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{array} \right] \rightarrow \left[\begin{array}{ccc} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{array} \right]$$

In all 3 cases, $\det |A| = a_{11}a_{22}a_{33}$

Ex. $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{array} \right] \quad |A| = 2 \quad \left\{ (-6) = -12 \right.$

$$\left[\begin{array}{cccc} -1 & & & \\ & +3 & & \\ & & 0 & \\ & 2 & & \\ & & 4 & \\ & & & -2 \end{array} \right] \quad |A| = 48$$

Row Vectors, Linear Combinations and Row Space :

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix}$$

Row Vectors : $[2 \ 3 \ 4]$ $[1 \ 5 \ 7]$ $[3 \ 11 \ 13]$

Linear combⁿ :

$$= a [2 \ 3 \ 4] + b [1 \ 5 \ 7] + c [3 \ 11 \ 13] \rightarrow a, b, c : \text{scalar}$$

↓
sum of scalar multiples of
row vectors

$$= \left\{ [2a + b + 3c, 3a + 5b + 11c, 4a + 7b + 13c] : a, b, c \in \mathbb{R} \right\}$$

row
space

→ set of linear combⁿ of row vectors

* ↳ Row space for 2 equivalent matrices is same.
↓ (applying Row opⁿs)

We get equivalent matrix by applying row opⁿs. So, linear combⁿ will be same in both matrices. Since row space is set of linear combⁿs, it will also be same for both matrices.

↳ Can reduce matrix to RREF because RREF is unique

Ex. $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix}$

↓ will be identity
use this matrix
to find row space

RREF of A : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Row space of A :

$$\{ a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) \} \\ \equiv \{ [a, b, c], a, b, c \in \mathbb{R} \}$$

Ex. Determine whether the row vector $[5, 17, -20]$ is in the row space of the matrix

$$P = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ -2 & 4 & -3 \end{bmatrix}$$

$$(5, 17, -20) = a(3, 1, -2) + b(4, 0, 1) + c(-2, 4, -3) \\ = (3a + 4b - 2c, a + 4c, -2a + b - 3c)$$

$$\left. \begin{array}{l} 5 = 3a + 4b - 2c \\ 17 = a + 4c \\ -20 = -2a + b - 3c \end{array} \right\} \Rightarrow AX = b$$

$$(A:B) \equiv \left[\begin{array}{cccc} 3 & 4 & -2 & 5 \\ 1 & 0 & 4 & 17 \\ -2 & 1 & -3 & -20 \end{array} \right] \sim \left[\begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right] \Rightarrow \begin{array}{l} a=5 \\ b=-1 \\ c=3 \end{array}$$

$$\therefore (5, 17, -20) = 5(1, 0, 0) + (-1)(0, 1, 0) + 3(-2, 4, -3)$$

can be expressed as
linear combⁿ of rows of matrix P

\Rightarrow It'll be in Row space of P.

Linear Independent Row Vectors \rightarrow A row vector can't be expressed as linear combⁿ of rest of row vectors

\Rightarrow Make row space = 0 \rightarrow If get values of a, b, c $\stackrel{\text{non-zero}}{\Rightarrow}$ Linearly dependent else, not

$$\text{Ex. } P = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ -2 & 4 & -3 \end{bmatrix}$$

Find 3 real no. a, b, c s.t.
 $a(3, 1, -2) + b(4, 0, 1) + c(-2, 4, -3) = (0, 0, 0)$

$$3a + 4b - 2c = 0$$

$$a + 4c = 0$$

$$-2a + b - 3c = 0$$

$$(A:B) : \begin{bmatrix} 3 & 1 & -2 & 0 \\ 1 & 0 & 4 & 0 \\ -2 & 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$$

\Rightarrow All 3 row vectors are LI.

$$\text{Ex. } P = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ 7 & 1 & -1 \end{bmatrix} \Rightarrow 3a + 4b + 7c = 0 \quad R_3 \rightarrow R_1 + R_2$$

R_3 is linearly dependent

$$\text{If we solve it: } 3a + 4b + 7c = 0$$

$$a + b = 0$$

$$-2a + b - c = 0$$

$$(A:B) = \begin{bmatrix} 3 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank (A:B) = 2

$$\Rightarrow \boxed{a=b=0}, \boxed{c \in \mathbb{R}}$$

arbitrary no.

\Rightarrow Row vectors of this matrix are not LI.

** When Rank (A:B) < no. of given rows \Rightarrow Linearly Dependent