The LNM Institute of Information Technology
Jaipur, Rajasthan
MATH-III
Assignment #1

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1. Let $z, w \in \mathbb{C}$. Show that
(a) $\overline{z+w} = \overline{z} + \overline{w}$, (b) $\overline{zw} = \overline{z}\,\overline{w}$, (c) $\overline{\overline{z}} = z$, (d) $|\overline{z}| = |z|$ and (e) $|\overline{zw}| = |z||w|$.

2. Show that
(a) $|z+w|^2 = |z|^2 + |w|^2 + 2Re(z\overline{w})$
(b) $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$
(c) $|z+w| = |z| + |w|$ if and only if either $zw = 0$ or $z = cw$ for some positive real number c.

3. Give a geometric description of the following sets:
(a) $\{z \in \mathbb{C} : |z-i| > |z+i|\}$ (b) $\{z \in \mathbb{C} : |z-i| + |z+i| = 2\}$

4. Determine the values of the following:
(a) $(1+i)^{20} - (1-i)^{20}$.
(b) $\cos\frac{1}{4}\pi + i\cos\frac{3}{4}\pi + ... + i^n \cos\frac{(2n+1)}{4}\pi + ... + i^{40}\cos\frac{81}{4}\pi$.

5. Let z be a non zero complex number and n a positive integer. If $z = r(\cos\theta + i\sin\theta)$ show that $z^{-n} = r^{-n}(\cos n\theta - i\sin\theta)$.

6. Let α be any of the n^{th} roots of 1 except 1. Show that $1 + \alpha + \alpha^2 + \cdots + \alpha^n = 0$. Find all the possible values of i^i. Express your answer in the form $x + iy$.

7. Find the roots of each of the following in the form $x + iy$. Indicate the principal root.
(a) $\sqrt{2i}$ (b) $(-1)^{\frac{1}{3}}$ and (c) $(-16)^{\frac{1}{4}}$.

8. Find all the values in $[0, 2\pi)$ where $lim_{r\to\infty} e^{re^{i\theta}}$ exists.

9. Find the roots of $z^4 + 4 = 0$. Use these roots to factor $z^4 + 4$ as a product of two quadratics with real coefficients.

10. Discuss the convergence of the following sequences:
(a) z^n (b) $\dfrac{z^n}{n!}$ (c) $i^n \sin\dfrac{n\pi}{4}$ and (d) $\dfrac{1}{n} + i^n$.

11. Discuss the behavior of $e^{1/z}$ as z approaches 0.

12. Verify if each of the following functions can be given a value at $z = 0$ so that they become continuous.
(a) $f(z) = \dfrac{|z|^2}{z}$, (b) $f(z) = \dfrac{z+1}{|z|-1}$ (c) $f(z) = \dfrac{\overline{z}}{z}$.

13. Let $f(z) = z^3, z_1 = 1$ and $z_2 = i$. Show that there exists no c on the line segment joining z_1 and z_2 such that $\dfrac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$. What can you infer from this?

14. Determine whether the following regions in C are domains:
(a) $Re(z) > 1$ (b) $0 \leq \text{Arg } z \leq \dfrac{\pi}{4}$ (c) $Im(z) = 1$
(d) $|z - 2 + i| < 1$ (e) $|2z + 3| > 4$.