MSE

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Continuous Probability distribution

$f(x) \rightarrow$ pdf

iff:

(i.) $f(x) \geq 0 \quad \forall x$

(ii.) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii.) $\int_a^b f(x)\,dx = P(a < x < b)$

$\Sigma f(x) = 1 \rightarrow$ Discrete

→ Cumulative Distrib^n Func^n (cdf)

$F(x) = P(X \leq x)$

$= \int_{-\infty}^{x} f(t)\,dt$

Properties of $F(x)$

(i.) Domain is $(-\infty, \infty)$

Range $[0,1]$

(ii.) $F(x)$ is non-↓ng func^n of x in the right

ie, $F'(x) = f(x) \quad \forall \quad x \geq 0$

(iii.) $F(x)$ is continuous on the right

(iv.) $F(-\infty) = 0$ (Prob. always defined on an interval)

$F(\infty) = 1$

(v) $P(a \leq x \leq b) = P(a \leq x < b) = P(a < x \leq b) = P(a < x < b)$

$= \int_a^b f(x)\,dx$

in discrete → $P(a \leq x \leq b) = F(b) - F(a)$

$P(a < x \leq b) = F(b) - F(a-1)$

Ex. $f(x) = ke^{-3x}, \quad x \geq 0$

$0, \quad x < 0$

Determine k & hence, compute cdf of x

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$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \rightarrow \int_{-\infty}^{0} + \int_{0}^{\infty} k e^{-3x}\,dx = 1$$

$$= k\left(\frac{e^{-3x}}{-3}\right)\Bigg|_{0}^{\infty} = k\left[\frac{+1}{3}\right] = 1 \rightarrow \boxed{k = 3}$$

$$F(x) = \int_{0}^{x} 3e^{-3x}\,dx = \frac{3}{-3}\left[e^{-3x}\right]_{0}^{x} = 1 - e^{-3x},\ x \geq 0$$

0 ew

$$F(2) = P(x \leq 2) = 1 - e^{-3(2)} = 1 - e^{-6}$$

$$F(4) = 1 - e^{-3(4)} = 1 - e^{-12}$$

Ex. Pdf of random variable x :

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x \geq 2 \end{cases}$$

4 − 2

Compute cdf of x :

$$F(x < 1) = \int_{0}^{x} x\,dx = \frac{x^2}{2}\Bigg|_{0} = \frac{x^2}{2}$$

$2 - \frac{1}{2}$

$$F(x < 2) = \int_{0}^{1} x\,dx + \int_{1}^{x} (2 - x)\,dx = \frac{1}{2} + 2x - \frac{x^2}{2}\Bigg|_{1}^{x}$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2} = 2x - \frac{x^2}{2} - 1$$

$$F(x \geq 2) = \int_{0}^{1} x\,dx + \int_{1}^{2} (2 - x)\,dx = \frac{1}{2} + 2 - \frac{3}{2} = \underline{\underline{1}}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2/2 & 0 \leq x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

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(i) $P(-1 \le x \le 3) = F(3) - F(-1) = 1 - 0 = \boxed{1}$

(ii) $P(1 \le x \le 1.5) = 2x - \frac{x^2}{2} - 1 = 2\left(\frac{3}{2}\right) - \left(\frac{9}{8}\right) - 1$

$\Downarrow$

$F(1.5) - F(1) \qquad = 3 - \frac{17}{8} = \frac{7}{8} - \left(\frac{1}{2}\right)$

$= \boxed{\frac{3}{8}}$

Expectation :

$\mu = E(x) = \sum_{\forall x} x\,P(x)$ if x is discrete

$= \int_{-\infty}^{\infty} x\,f(x)\,dx$ if x is continuous

Let $g(x)$ = func^n of x, so,

$E[g(x)] = \sum_{\forall x} g(x)\,p(x)$

$= \int_{-\infty}^{\infty} g(x)\,f(x)\,dx$

Variance : (Always +ive) (unitless)

$\sigma^2 = E[(x-\mu)^2] = \sum_{\forall x} (x-\mu)^2\,p(x)$ discrete

$= \int_{-\infty}^{\infty} (x-\mu)^2\,f(x)\,dx$ cont.

(Has unit) S.D. $(\sigma) = +\sqrt{E(x-\mu)^2}$

→ Computational formula for σ^2 :

$\sigma^2 = \sum x^2\,p(x) - [\sum x\,p(x)]^2$

$\boxed{\sigma^2 = E[x^2] - (E[x])^2}$

$\sigma^2 = E[(x-\mu)^2] = \sum_{\forall x} (x-\mu)^2\,p(x)$

$= \sum_{\forall x} x^2 p(x) - 2\mu x\,p(x) + \mu^2 p(x)$

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$$= \sum_{\forall x} x^2 p(x) - 2\mu \sum_{\forall x} x p(x) + \mu^2 \sum_{\forall x} p(x)$$

$$= \sum x^2 p(x) - 2\mu(\mu) + \mu^2 (1)$$

$$= \sum x^2 p(x) - \mu^2$$

$$= E[X]^2 - (E[X])^2$$

continuous : $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

Properties of Expectations & Variance

(i) $E(c) = c$ where c : constant $= \sum c p(x) = c \sum p(x) = c$

(ii) $E(cX) = c\, E(x)$

(iii) $E(c+X) = c + E(X)$

(iv) $Var(c) = 0 \quad = E[c^2] - (E[c])^2 = c^2 - c^2 = 0$

(v) $Var(cX) = c^2\, Var(X) = E[x^2 c^2] - [E(cX)]^2$

$$= c^2 E[X^2] - (c^2 E(X]^2) = \sigma^2 Var($$

Ex $E[X] = 5$

$var(X) = 2$

$E[X-5] = ?$ $\quad Var(-2X) = ?$

$var(-2X) = 4\, Var(X) = 4 \times 2 = 8$

$E[X-5] = E[x] - 5 = 0$

The r^{th} moment about origin :

$\mu'_r = E[X^r] = \sum_{\forall x} x^r p(x)$: x is discrete

$\int_{-\infty}^{\infty} x^r f(x)\, dx$ ~ cont.

$r=1 \rightarrow E[X]$: 1^{st} moment about origin $= \underline{\underline{\mu}}$.
(μ'_1) is Mean.

r^{th} moment about Mean :

$\mu_r = E[(x-\mu)^r]$: $\sum_{\forall x} (x-\mu)^r p(x)$

$\int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx$

~~$M[X(t)] = E[$~~

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If $r = 2$

$\mu_2 = E[(x-\mu)^2] = Var(x) = \sigma^2$ → 2^{nd} Moment about Mean is Variance

$$\boxed{\sigma^2 = \mu_2' - \mu^2}$$

$$\sigma^2 = E[(x-\mu)^2] = \Sigma (x-\mu)^2 p(x)$$
$$= E[x^2] - (E(X))^2$$
$$= \mu_2' - \mu^2$$

→ Moment Generating Func^n : MGF → func^n is used to generate moments

$$M_x(t) = E[e^{tx}] = \sum_{\forall x} e^{tx} p(x) \quad \text{discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \quad \text{continuous}$$

$$\mu_r' = \frac{d^r}{dt^r} [M_x(t)] \Big|_{t=0}$$

diff. r times

$$\mu = \mu_1' = \frac{d}{dt} [M_x(t)] \Big|_{t=0}$$

$\sigma^2 = \mu_2' - \mu^2$ → calculate from $M_x(t)$

→ Mean Deviation about Mean :

$$M.D. = \sum_{\forall x} |x-\mu|\, p(x)$$

- sometimes, moments generated are complex, but we want only real func^{n}s, so we multiply it with i :

→ Characteristic Func^n :

$$\phi_x(t) = E[e^{itx}]$$

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Probability Distributions

① Bernoulli :

Assump^ns :

→ 2 outcomes of each experiment

→ P(success) is const. (p) , P(failure) = (1-p) = q

$$p(x) = \begin{cases} p & x=1 \rightarrow \text{success} \\ q & x=0 \rightarrow \text{failure} \end{cases}$$

$P(X=1) = p \rightarrow$ success $\quad P(X=0) = q \rightarrow$ failure

$$p(x) = \begin{cases} p^x q^{1-x}, & x = 0,1 \\ 0 & \text{e.w.} \end{cases}$$

$E(X) = p$, $Var(X) = pq$

$M_X(t) = q + pe^t$

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Joint Distributions

~~Discrete~~ Joint Mass Func^n (discrete joint densities)

$f_{XY}(x,y) = P(x,y) = P[X=x \text{ and } Y=y]$

→ $f_{XY}(x,y) \geq 0 \quad \forall (x,y)$

→ $\sum_{\forall x} \sum_{\forall y} f_{XY}(x,y) = 1$

→ Necessary & sufficient

* Necessary / Sufficient / Necessary & sufficient

Marginal distrib^n (individual ki prob.)

$f_X(x) = \sum_{\forall y} f(x,y)$ → Marginal density for x.

$f_Y(y) = \sum_{\forall x} f(x,y)$

Discrete : Random variable takes value countable.
cont : interval

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Continuous Joint Density

1. $f_{XY}(x,y) \geq 0 \quad ; \quad x, y \in \mathbb{R}$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$
3. $P[a \leq X \leq b \text{ and } c \leq Y \leq d] = \int_a^b \int_c^d f(x,y)\, dy\, dx$

↳ Joint density for (X,Y)

Cont. Marginal Densities :

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy \quad , \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Independence :

$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$

$P(X=x, Y=y) = P[X=x] * P[Y=y]$

or $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

Multivariant : $f(x_1, x_2, \ldots x_n) = f(x_1) \cdot f(x_2) \cdots f(x_n)$

Ex-1 1^{st} task → 2 ~~g~~ joints $\quad$ 2^{nd} task → 3 bolts

X → ~~no. of defective welds~~

Y → no. of improperly tightened bolts produced per car

x/y	0	1	2	3	$f_X(x)$
0	.840	.030	.020	.010	.900
1	.060	.010	.008	.002	.080
2	.010	.005	.004	.001	.020
$f_Y(y)$	.910	.045	.032	.013	1.000

$f_X(x=0) = \sum_{\forall y} f(0,y) \Rightarrow 1^{st}$ row sum

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checking Independence here are different (check all 12 values combinations)

→ That's why checking independence in case of discrete is difficult

Question ~~Table~~ 2

(a) $P(X=2, y=1) = 0.005$

(b) $P(X>=1, Y>=1) =$ ~~0.045 + 0.032 + 0.~~ 0.030

(c) $P(X<=1) = 0.980$

(d) $P(Y>=2) = 0.032 + 0.013 = 0.045$

Ques^n . $f_{XY}(x,y) = \frac{1}{n^2}$, $x = 1, 2, \ldots n$

$y = 1, 2 \ldots n$

(a) verify $f(x,y)$ satisfies conditional

(1) $f(x,y) \geq 0$

$n^2 \geq 0$

$\Rightarrow \frac{1}{n^2} \geq 0$

(2) $\sum_{\forall x} \sum_{\forall y} f(x,y) = 1$

$\sum_{\forall x} \sum_{\forall y} f(x,y) = \sum_{x=1,2,\ldots n} \sum_{y=1,2\ldots n} \frac{1}{n^2}$

$= \sum_{x=1,\ldots n} \frac{1}{n^2} + \frac{1}{n^2} + \ldots$ n times $= \sum_{x=1,2..n} n\left(\frac{1}{n^2}\right) = \frac{1}{n}$

$= \boxed{1}$

(b) find marginal density for X & Y

$f_X(x) = \sum_{y=1}^{n} \frac{1}{n^2} = \frac{1}{n}$, $f_Y(y) = \frac{1}{n}$, $y = 1, 2, \ldots n$

$x = 1, 2, \ldots n$

0 , elsewhere

(c) Are X & Y independent?

$f_X(x) \times f_Y(y) = f(x,y) \Rightarrow$ Independent

Ques^n 4 $f_{XY}(x,y) = c(4x + 2y + 1)$ $0 \leq x \leq 40$; $0 \leq y \leq 2$;

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$f(x,y) = c(4x+2y+1) \qquad 0 \le x \le 40 \quad , \quad 0 \le y \le 2$

(a) Find c

$c \int_0^{40} \int_0^2 (4x+2y+1)\,dy\,dx$

$c \int_0^{40} (4xy + y^2 + y)\Big|_0^2 dx = c \int_0^{40} (6+8x)\,dx = 1$

$4+2+8x \Rightarrow c \quad 6x + 4x^2 \Big|_0^{40} = 1$

6400
240

$\Rightarrow \boxed{c = \frac{1}{6640}}$

$\Rightarrow f(x,y) \begin{cases} \frac{1}{6640}(4x+2y+1) & 0 \le x \le 40 \ , \ 0 \le y \le 2 \\ 0 & e.w. \end{cases}$

(b) $f_X(x) = ? \qquad f_Y(y) = ?$

$f_X(x) = c \int_0^2 (4x+2y+1)\,dy = \frac{8x+6}{6640} \quad , \quad 0 \le x \le 40$
$0 \quad e.w.$

3200
+ 40
3240

$f_Y(y) = c \int_0^{40} (4x+2y+1)\,dx = c\left[2x^2 + (2y+1)x\right]\Big|_0^{40} = \frac{80y+3240}{6640}$
$0 \ , \ e.w.$

(c) $f(X>20, Y>1) = c \int_{20}^{40} \int_1^2 (4x+2y+1)\,dy\,dx$

$= c \int_{20}^{40} (4xy+y^2+y)\Big|_1^2 dx = \cancel{c \int_{20}^{40}} (\quad \frac{2480}{6640}$

(d) $P(Y>1) = \cancel{c} \int_1^2 \cancel{\int_0^{40}} \frac{80y+3240}{6640}\,dy = 0.526$

(e) $P(X>20) = c \int_{20}^{40} \frac{8x+6}{c}\,dx = 0.741$

(f) $f_X(x) \cdot f_Y(y) \ne f_{XY}(x,y) \rightarrow$ Not Independent

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Q5. (n- dimensional discrete random variables)

$f(x_1, x_2, \ldots x_n) = P[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n]$

$P(x_1) = 0.9$ $\downarrow$ non-def

$P(x_2) = 0.08$ $\downarrow$ def but salvagable

$P(x_3) = 0.02$ $\downarrow$ neither

$n = 20$ items

(a) $P[X_1 = 15, X_2 = 3, X_3 = 2] = ?$

(b) Find general formula for density for (X_1, X_2, X_3).

sol^n. $P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = f(x_1, x_2, x_3)$

$= \dfrac{n!}{x_1!\, x_2!\, x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$ $\Rightarrow$ $\boxed{\text{Multinomial distrib}^n}$ $\downarrow$ extension of binomial one.

$= \dfrac{20!}{15!\, 3!\, 2!} (0.9)^{15} (0.08)^3 (0.02)^2$

$= \underline{\underline{0.0065}}$

Lecture 2:

Expectation & Covariance

① $E[H(X,Y)] = \sum_{\forall x} \sum_{\forall y} H(x,y) f_{XY}$ ~~provided~~

② $= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x,y) f_{XY}\, dy\, dx$

Ex. $f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & 0 \leq x < 1, \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

$E[X^2 + Y^2]$

$= \int_0^1 \int_0^1 (x^2+y^2) \frac{3}{2}(x^2+y^2)\, dy\, dx$

$= \frac{3}{2} \int_0^1 \left(\int_0^1 x^4 + 2x^2y^2 + y^4 \right) dy\, dx =$

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$$\frac{3}{2}\int_0^1 \left[x^4 y + \frac{2x^2y^3}{3} + \frac{y^5}{5}\right]\Big|_0^1 dx$$

$$= \frac{3}{2}\int_0^1 \left(x^4 + \frac{2x^2}{3} + \frac{1}{5}\right) dx$$

$$= \frac{3}{2}\left[\frac{1}{5} + \frac{2}{3}\times\frac{1}{3} + \frac{1}{5}\right] = \frac{3}{2}\left[\frac{1}{5} + \frac{2}{9} + \frac{1}{5}\right]$$

$$= \frac{3}{2}\left[\frac{2}{5} + \frac{2}{9}\right] = 3\left[\frac{1}{5} + \frac{1}{9}\right] = \frac{\cancel{3}\times 14}{5\times \cancel{9}3} = \frac{14}{15}$$

<u>Univariate Average</u> : calculate $E[X]$ from $f(x,y)$

$$E[X] = \sum_{\forall x} x \underbrace{f_X(x)}_{\text{Marginal density of } x} = \sum_{\forall x}\sum_{\forall y} x\, f(x,y)$$

<u>Covariance</u> : → How X & Y are related

$$\text{cov}(X,Y) = E\left[(X-\mu_x)(Y-\mu_y)\right] \qquad E[X]=\mu_x,\ E[Y]=\mu_y$$

* $X\downarrow$ & $Y\downarrow$ ⇒ $(X-\mu_x)(Y-\mu_y) > 0$ } +tively correlated

$X\uparrow$ & $Y\uparrow$ ⇒ " " .

else < 0

if $\text{Cov}(X,Y) = 0$ → X & Y are independent

$$\boxed{\text{Cov}(X,Y) = E[XY] - E[X]E[Y]}$$

$$\begin{aligned}
\text{cov}(X,Y) &= E[(X-\mu_x)(Y-\mu_y)] \\
&= E[XY - YE[X] - XE[Y] + E[X]E[Y]] \\
&= E[XY] - E[Y]E[X] - E[X]E[Y] + \cancel{E[X]E[Y]} \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

Thm If X & Y are independent :

$$E[XY] = E[X]\,E[Y] \Rightarrow \underline{\mathrm{cov}(X,Y) = 0}$$ But not vice versa

Proof: $$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx \int_{-\infty}^{\infty} y f_Y(y)\,dy = E[X]\,E[Y]$$

Proof

Ex. $p(-1) = p(0) = p(1) = 1/3$

let $Y = X^2$

$E[X] = (-1+0+1)/3 = 0$

$E[Y] = (1+0+1)/3 = 2/3$

$E[XY] = (-1+0+1)/3 = 0$

$E[XY] = E[X]\,E[Y]$, but X & Y are <u>dependent</u>.

Proof. $\mathrm{Cov}(X,Y) = \mathrm{Var}(X) = \mathrm{Var}(Y)$ if $X = Y$

if $X = Y$, $\rightarrow \mathrm{Cov}(X,X) = E[X\cdot X] - E[X]\,E[X]$

$$= E[X^2] - (E[X])^2 = \mathrm{Var}(X)$$

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$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$$

Exercise 19. The joint density is given by

$$f_{XY}(x,y) = \frac{c}{x} \qquad 27 \le y \le x \le 33$$

$$c = \frac{1}{(6 - 27\ln 33/27)} = 1.72$$

(a) Find $E[X]$, $E[Y]$, $E[XY]$, $\mathrm{cov}(X,Y)$

(b) $E[X-Y]$

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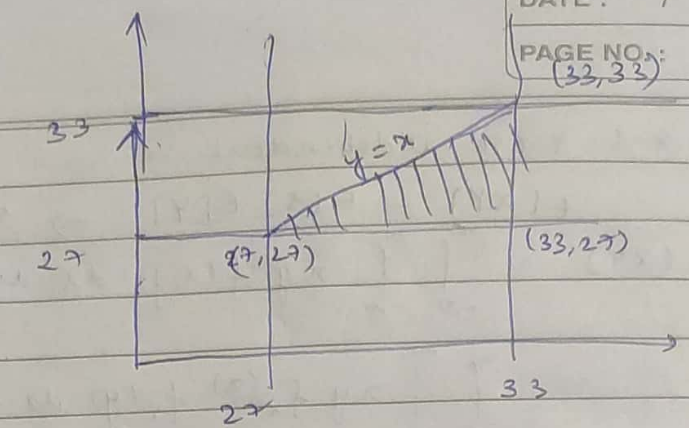
$$\int \frac{c}{x}\,dx$$



To find c :

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$= \int_{27}^{33}\int_{27}^{x} \frac{c}{x}\,dy\,dx \Rightarrow \int_{27}^{33} \frac{c}{x}(x-27)\,dx$$

$$= c\int_{27}^{33}\left(1 - \frac{27}{x}\right)dx = 1 \Rightarrow c\left[x - 27\ln x \Big|_{27}^{33} \right] = 1$$

$$\Rightarrow c = \frac{1}{6 - 27\ln 33/27} = \underline{1.72}$$

(a) $\cancel{E[x]} \Rightarrow$

$$f_X(x) = \int_{27}^{x} \frac{c}{x}\,dx \Rightarrow \frac{(x-27)}{x}c$$

$$f_Y(y) = \int_{y}^{33} \frac{c}{x}\,dx \Rightarrow c(\ln 33 - \ln y)$$

$$E[X] = \int_{27}^{33} x f_X(x)\,dx = \int_{27}^{33} c(x-27)\,dx = \frac{x^2}{2} - 6 \cdot 27$$

$$= 30.96$$

$$E[Y] = \int_{27}^{33} c(\ln 33 - \ln y)\,y\,dy = 28.99$$

$$E[XY] = \int_{27}^{33}\int_{y}^{33} xy\,\frac{c}{x}\,dx\,dy = 897.84$$

$\cancel{E[X-Y]} =$ Cov (XY) = E[XY] − E[X]E[Y] = $\underline{0.31}$

(b) $$E[X-Y] = \int_{27}^{33}\int_{27}^{x} (x-y)\frac{c}{x}\,dy\,dy = 1.97$$

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Prove $Var(X \pm Y) = Var(X) \pm Var(Y) \pm 2\,Cov(X,Y)$

If, X & Y are independent:

$Var(X+Y) = Var(X) + Var(Y)$, $Var(X-Y) = Var(X) + Var(Y)$

$$Var(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$$

$$= E[X^2 + Y^2 + 2XY] - (E[X] + E[Y])^2$$

$$= E(X^2) + E[Y^2] + 2E[XY] - (E[X])^2 - (E[Y])^2 - 2E[X]E[Y]$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

Ex. $f_{XY}(x,y) = \frac{1}{2(e-1)}\left[\frac{1}{x} + \frac{1}{y}\right)$ $\quad 1 \le x \le e$, $1 \le y \le e$

(a) show that it is valid joint density

(b) Find $E[X]$, $E[Y]$, $E[XY]$ $\quad \frac{3e-1}{4}, \frac{3e-1}{4}, \frac{e^2-1}{2}$

(c) X & Y Independent? No

(a) $$\int_1^e \int_1^e \frac{1}{2(e-1)}\left[\frac{1}{x} + \frac{1}{y}\right) dy\, dx$$

$$= \int_1^e \frac{1}{2(e-1)}\left(\frac{y}{x} + \ln y\right)\Big|_1^e dx = \frac{1}{2(e-1)}\int_1^e \left(\frac{(e-1)}{x} + 1\right) dx$$

$$= \frac{1}{2(e-1)}\left((e-1)\ln x + x\right)\Big|_1^e = \frac{1}{2(e-1)}\left[e-1+e-1\right] = 1$$

(b) $E[X]$:

$$f_X(x) = \int_1^e \frac{1}{2(e-1)}\left[\frac{1}{x} + \frac{1}{y}\right) dy = \frac{1}{2(e-1)}\left[\frac{(e-1)}{x} + 1\right]$$

$$E(X) = \int_1^e x f_X(x)\, dx = \int_1^e \left(\frac{x}{2(e-1)}\left[\frac{e-1}{x} + 1\right]\right) dx$$

$$= \int_1^e \frac{1}{2} + \int_1^e \frac{x}{2(e-1)} dx \qquad \frac{x^2}{2}$$

$$= \frac{e-1}{2} + \frac{1}{2(e-1)}\left[\frac{e^2-1}{2}\right] \Rightarrow \frac{e-1 \times 2}{2 \times 2} + \frac{e+1}{4} = \frac{3e-1}{4}$$

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Conditional Distribution

Conditional Densities :

$f_{X|Y} = P(X=x \mid Y=y) = \dfrac{P[X=x \ \& \ Y=y]}{P[Y=y]} = \dfrac{f_{XY}(x,y)}{f_Y(y)}, \quad f_Y(y)>0$

satisfies

1) ≥ 0

2) $\int_{-\infty}^{\infty} f_{X|Y} \, dx = 1$

Cond^n al Mean & variance :

$E(Y|x) = \sum y \, f_{Y|x}(y) = \int y \, f_{Y|x}(y) \, dy$

$E(Y|X=x) = \sum_{\forall y} y \, f_{Y|x}(Y=y|X=x)$

$Var(Y|x) = \sum_y y^2 f_{Y|x}(y) - \mu^2_{Y|x}$

Ex. $f(x,y) = x^2 + \dfrac{xy}{3}, \quad 0 \leq x \leq 1, \ 0 \leq y \leq 2$

$= 0 \quad$ e.w.

sol^n Marginal density for x :

$f_X(x) = \int_0^2 \left(x^2 + \dfrac{xy}{3}\right) dy = x^2 y + \dfrac{x}{6} y^2 \Big|_0^2$

$= 2x^2 + \dfrac{2}{3} x \quad 0 \leq x \leq 1$

$f_Y(y) = \int_0^1 \left(x^2 + \dfrac{xy}{3}\right) dx = \dfrac{1}{3} + \dfrac{y}{6}$

- $f_{X|y}(X|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{x^2 + xy/3}{1/3 + y/6} = \dfrac{6x^2 + 2xy}{2+y}, \quad 0 \leq x \ldots, \ 0 \leq y \ldots$

- $f_{Y|x} = \dfrac{f_{XY}(x,y)}{f_X(x)}$

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Mean & Variance of linear combⁿ of Random variables

Ex.

$$Y = c_1 X_1 + c_2 X_2 + \cdots + c_p X_p$$

$X_i \to$ r.v.
$c_i \to$ consts.

$$E[Y] = c_1 E[X_i] + c_2 E[X_2] + \cdots + c_p E[X_p]$$

$$V(Y) = c_1^2 V(X_1) + c_2^2 V(X_2) + \cdots + c_p^2 V(X_p) + 2 \sum_{i<j} \sum c_i c_j \, cov(X_i, X_j)$$

If $x_1, x_2, \ldots x_p$ are independent

$$V(Y) = c_1^2 V(X_1) + c_2^2 V(X_2) + \cdots + c_p^2 V(X_p)$$

$$\to V(Y) = E[(Y - E[Y])^2]$$

$$= E[(c_1 X_1 + c_2 X_2 + \cdots + c_p X_p - (c_1 \mu_1 + c_2 \mu_2 + \cdots + c_n \mu_n))^2]$$

$$= E[(c_1(X_1 - \mu_1) + c_2(X_2 - \mu_2) + \cdots + c_p(X_p - \mu_p))^2]$$

$$= E[c_1^2 (X_1 - \mu_1)^2 + c_2^2 (X_2 - \mu_2)^2 + \cdots + 2 \cdots$$

Ex. X & Y : independent r.v.

$\mu_x = 7 \qquad \mu_y = -5$
$V(X) = 9 \qquad V(Y) = 3$

(i) $E[4X - 2Y + 6] = 4E[X] - 2E[Y] + 6 = 4(7) - 2(-5) + 6$
$= 28 + 10 + 6$
$= 44$

(ii) $V(4X - 2Y + 6) = 16 V(X) + 4 V(Y) + 36$
$= 16(9) + 4(3) + \cancel{36}^{\,0} = 144 + 12 + 36$
$=$ ~~192~~ 156

Ex. Let X_1 & X_2 be 2 independent r.v. having normal distribⁿ having mean 5 & 6, var 2 & 3. Find mean of $Y = X_1 + X_2$. Find distribⁿ of Y

Solⁿ (Take help of Moment generating funcⁿ)

$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

given $X_1 \sim N(5, 2)$

$M_{X_1}(t) = e^{5t + \frac{1}{2} \cdot 2t^2}$

$M_{X_2}(t) = e^{6t + \frac{1}{2} \cdot 3t^2}$

$M_Y(t) = M_{(X_1+X_2)}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$ [X_1 & X_2 = independent]

$= (e^{5t + t^2}) \cdot e^{(6t + 3/2 t^2)}$

$= e^{11t + \frac{5}{2}t^2} \longrightarrow N(11, 5)$

Y → also a Normal distribⁿ.

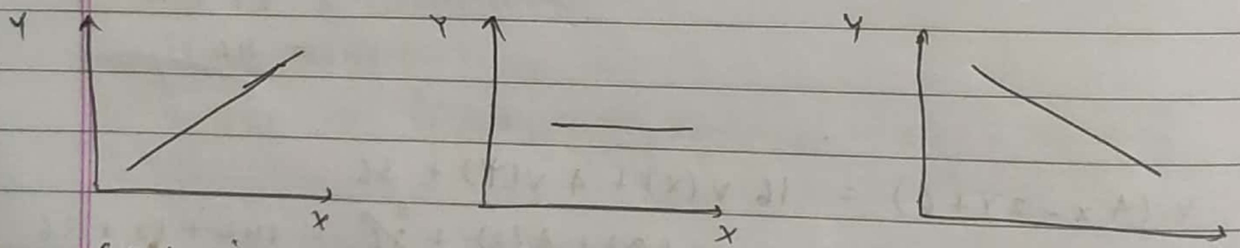
Correlation

ρ → Pearson Coeffecient

$$\boxed{\rho = \frac{cov(X,Y)}{\sqrt{(var(X))(var(Y))}}}$$

⇒ linear relⁿ b/w X & Y

↳ If distribⁿ is given, use this



Cor: +ive   Cor: 0   Cor: -ive

$\rho = -1$ ⇒ Perfect -ive correlation

$\rho = 0$ ⇒ Independent var (Not correlated) → Non-Linear relⁿship

$\rho = 0$ ⇒ Perfect +ve correlⁿ.

→ Not possible for large popⁿ to calculate ρ, taken random sam of size n,

$Var^n\ x = \sum_{i=1}^{n} (x_i - \bar{x})^2 / n = S_{xx}/n$, similarly for y.

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$Cov(X, Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$\Rightarrow \hat{cov}(X, Y) = \Sigma(X_i - \bar{X})(Y_i - \bar{Y})/n = S_{XY}/n$

$\hat{\rho} = R = \frac{S_{XY}}{\sqrt{S_{XX} S_{XY}}}$

Computational formula:

$$\hat{\rho} = \frac{n\Sigma xy - \Sigma x \Sigma y}{\left(\left[(n\Sigma x^2 - (\Sigma x)^2)\right]\left[n\Sigma y^2 - (\Sigma y)^2\right]\right)^{1/2}}$$ $\Rightarrow$ Use when data is given

Q. Find sample correl^n coeff.

1.) X : 1, 2, 3, 4, 5

Y = 2, 4, 6, 8, 10

sol^n: ~~$\hat{\rho}$ = 0 5 (~~

x	y	x^2	y^2	xy
1	2	1	4	2
2	4	4	16	8
3	6	9	36	18
4	8	16	64	~~10~~ 32
5	10	25	100	50
Σ: 15	30	55	220	110

225

1100
-900

$$\hat{\rho} = r = \frac{5(110) - (15)(30)}{\sqrt{[5\times 55 - 225][5\times 220 - 900]}}$$

$$= \frac{550 - 450}{\sqrt{50 * 200}} = \frac{100}{\sqrt{10000}} = \frac{100}{100} = \boxed{1}$$

2.) $f_{XY}(x,y) = \frac{1}{6640}(4x+2y+1)$ $\quad 0 \le x \le 40$, $0 \le y \le 2$

Find $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$, $Var(X)$, $var(Y)$, $Cov(XY)$, ρ

Sol^n $E[X,Y] = \int_0^{40}\int_0^{2} xy \frac{1}{6640}(4x+2y+1)\,dy\,dx = 26.586$

$E[X] = \int_0^{40}\int_0^{2} x f_{X}(x)\,dx\,dy = 26.426$ $\quad E[X^2] = 700.361$

$E[Y^2] = 1.3494$

$E[Y] = 1.008$

$cov(X,Y) = E[XY] - E[X]E[Y] = 26.586 - (26.426)(1.008)$
$= -0.051$

$var(X) = 700.361 - (26.426)^2 = 2.027$

$var(Y) = 1.3494 - (1.008)^2 = 0.333$

$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = -0.\underline{009}$ $\Rightarrow$ -ively correlated

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Chi-square distribution

Let $Z \sim N(0,1)$

$\therefore Z^2 \sim \chi^2$ → chi-square $\quad \chi^2$

Let $x_1, x_2, \dots, x_n$ be n independent variables with mean & variance μ_i & σ_i^2 $(i = 1, 2, \dots n)$ then,

$\sum_{i=1}^{n}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2$ is a χ^2-variate with n.d.f.

~~proto~~ Let x be χ^2 variate with n.d.f. n

$E[X] = n$

$Var(X) = 2n$

χ χ χ

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→ This distribution is used for Goodness of fit

Observed & Expected frequency → calc. chi-square

Chi-square (χ^2) Test of goodness of fit :

check if theoretical value is same as observed

Let O_i $(i = 1, 2, \ldots n)$ be a set of observed frequencies & e_i $(i = 1, 2, \ldots n)$ corresponding set of expected (theoretical) frequencies. Then

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - e_i)^2}{e_i} \text{ has a } \chi^2 \text{ distrib}^n \text{ with } (n-1) \text{ d.f.}$$

degree of freedom

given, all $e_i \geq 5$. (Thumb rule)

if $\chi^2_{calc} < \chi^2_{Tab}$ ⇒ We have made correct assumption. No significant difference. We can accept it.

Ex. The demand of a spare part in a factory was found to vary from day to day. In a sample study, foll. info. was obtained:

Days	M	T	W	Th	F	S.	Total
No. of parts demanded (O_i)	1124	1125	1110	1120	1126	1115	6720 = n

⇒ Observed frequencies

Test the hypothesis, that the no. of parts doesn't depend on the day.

Sol^n ⇓

~~p(demand) is same for all day~~

① find total of all days $= 6720 = n$

H_0 : The demand of no. of parts doesn't depend on day

H_1 : H_0 is not true

$p_i = \frac{1}{6}$, $i = 1, 2, \ldots, n$ $\quad e_i = n * p_i$

No. of parts (e_i) : 1120 1120 1120 1120 1120 1120

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$$\chi^2 = \sum \frac{(o_i - e_i)^2}{e_i}$$

$$= \frac{(1124-1120)^2}{1120} + \frac{(1125-1120)^2}{1120} + \ldots$$

$$= 0.179$$

$\alpha = 0.05$

No. of cells = 6

⇒ Degree of freedom here = 5

$\chi^2_{\text{tab with 5 d.o.f.}} = 11.07$ (from table)

$0.179 < 11.07$ ⇒ ∄ no significant difference
⇒ H_0 is accepted.

⇒ If any $e_i < 5$ → can add it with any preceding or succeeding cell (add both e_i & o_i). Thus, d.o.f. --;

Student t-distribution

→ It is a sample distrib^n

→ Can be used wherever z-distrib^n is used, only cond^n is $n < 30$

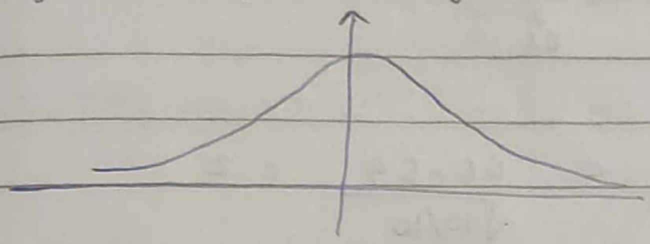
→ ~~If~~ let $x_1, x_2, \ldots x_n$ be random sample of small size n ~~with mean μ_i and variance σ_i^2 respectively. $(i=1,2,\ldots n)$~~ from a normal population with mean μ & variance σ^2. Let $\bar{x}$ be mean of sample & s^2 be variance of sample.

ie, $$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

and $$s^2 = \frac{1}{(n-1)} \sum_{i=1} (x_i - \bar{x})^2$$

In sample, only n−1 samples are linearly independent then $t = \frac{\bar{x}-\mu}{s/\sqrt{n}}$ has a t-distrib^n with $(n-1)$ d.f.

$f(-t) = f(t)$: symmetric



$\Rightarrow$ ~~mean, mode~~ $= 0$

$\mu'_{2k+1} = 0$

$$\mu_{2k} = \frac{n^k (2k-1)(2k-2) \ldots 3.1}{(n-2)(n-4)\ldots(n-2k)}, \quad \frac{n}{2} > k$$

$$\mu_2 \text{ (var)} = \frac{n}{n-2}, \quad n > 2, \quad \boxed{\text{mean} = 0} \; (\mu_1 = 0)$$

① Significance Test for sample Mean

let $x_1, x_2, \ldots x_n$ be a random sample of size n. from a normal population with a specified popul^n mean μ ~~and~~.
To test $\mu = \mu_0$ or there is no significant difference between sample mean $\bar{x}$ and ~~sam~~ popl^n mean μ_0, ~~cat~~ compute

$$t = \frac{\bar{x}-\mu}{s/\sqrt{n}} \quad \text{where} \quad \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} \; \& \; s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

- if $t_{cal} < t_{tab} \rightarrow$ no significant diff. between $\bar{x}$ & μ_0.

~~Note~~ If $n > 30 \rightarrow$ then t will become z.

Quest^n: The heights of 10 men of a locality are found to be:
70, 67, 62, 68, 61, 68, 70, 64, 64, 66
Is it reasonable to believe that avg. height is > 64 inches?
Test at 5% level of significance

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$t > t_\alpha$

$t < t_\alpha$

$|t| < t_{\alpha/2}$

$\alpha = 0.05$

Sol^n $\bar{x} = \frac{\sum x_i}{n} = \frac{660}{10} = 66$

$$s^2 = \frac{1}{10-1} \sum_{i=1}^{10} (x_i - \underset{66}{\bar{x}})^2 = 10$$

$\mu_0 = 64$

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{66-64}{\sqrt{10/10}} = 2.$$

t_{tab} with 9 d.f. $= 2.26$

$t_{calc} < t_{tab}$ ⇒ Avg. height is not greater than 64 inches

F- distribution

Let X & Y be two independent Chi-square variates with ν_1 & ν_2 d.f. Then,

$$F = \frac{X/\nu_1}{Y/\nu_2} = F(\nu_1, \nu_2)$$

App^{n}s :

↳ entire analysis of variance of

↳ Agriculture, Medical

- Let $x_1, x_2 \ldots x_n$ & $y_1, y_2 \ldots y_n$ be 2 independent random samples drawn from same normal population with same variance.

Let $\bar{x}$ & $\bar{y}$ be the sample mean & s_x^2 & s_y^2 be sample variance of 2 samples.

$$\Rightarrow \bar{x} = \frac{1}{n_1} \sum x_i \qquad \bar{y} = \frac{1}{n_2} \sum y_i$$

$$s_x^2 = \frac{1}{n_1-1} \sum (x_i - \bar{x})^2 \qquad s_y^2 = \frac{1}{n_2-1} \sum (y_i - \bar{y})^2$$

Then, $$\boxed{F = \frac{s_x^2}{s_y^2}}$$

Post - Mid sem (1 class Missing)

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Determinants

$a_{11}x_1 + a_{12}x_2 = b_1$ $\Rightarrow$ $x_1 = \dfrac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$; $x_2 = \dfrac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$

$a_{21}x_1 + a_{22}x_2 = b_2$

$a_{11}a_{22} - a_{21}a_{12} \to |D|$

- $|A| = 0$: singular Matrix

Minor

$M_{ij} \Rightarrow$ delete i^{th} row & j^{th} col & get remaining matrix

Cofactor of a_{ij} :

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$|D| = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad i = 1/2/3/\ldots n$$

$$\text{or} \quad \sum_{i=1}^{n} a_{ij} C_{ij} \qquad j = 1/2/3/\ldots n$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

subtract these products

add these products

→ Upper Δgular Matrix : entries below main diagonal are zero

~~Row equivalent forms of a matrix~~

Elementary Row operⁿs of a Matrix : (Same for col)

1. Interchanging 2 rows R_i and R_j. symbollically $R_i \leftrightarrow R_j$
2. Multiply a row R_i by a non-zero no. K, $R_i \to KR_i$
3. Adding a constant k multiple of a row R_i to a row R_j

$$R_j \to R_j + kR_i$$

Equivalent matrix : Order same, not necessary same values

Equal " : same values also

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Matrix A $\xrightarrow{Op^n s}$ Matrix B A & B : equivalent

Ex. $A = \begin{bmatrix} 4 & 8 & 10 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$

~~Applying~~

Op' $R_1 \longleftrightarrow R_2$

$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 10 \\ 3 & 5 & 6 \end{bmatrix}$ $\sim \rightarrow$ equivalent to

Applying $R_2 \rightarrow \frac{R_2}{2}$ $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \xrightarrow[R_2 \rightarrow R_2 - 2R_1,\ R_3 \rightarrow R_3 - 3R_1]{\text{Applying}} A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix}$

- All are equivalent matrix.

Row ~~Equivalent~~ Echelon Form : (REF)

(i.) All the zeros (if any) lie in the bottom

(ii) leading entry (1^{st} non-zero entry) in any row is 1. and the column lies to the right of the columns of leading entry of proceeding row.

Reduced Row Echelon Form (RREF) :

(i) In addition, if all the entries above the leading entries are zero.

Ex. $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ REF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ RREF $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ RREF

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* A matrix can have diff. REF but RREF is unique.

Ex. Find RREF

$$\begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow[\substack{R_2 \to -R_2 \\ R_3 \to -R_3}]{R_2 \leftrightarrow R_3}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{\boxed{REF}} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{R_2 \to R_2 - 3R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boxed{RREF}}$$

→ To get 0 in i^{th} col, we have used i^{th} row.

Inverse of a Matrix

A be a matrix of m rows ~~and n~~ cols.

and B be a matrix of n cols.

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \qquad B = (C_1, C_2, \ldots C_n).$$

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_n \\ R_2C_2 & R_2C_2 & \cdots & R_2C_n \\ \vdots & & & \\ R_nC_1 & R_nC_2 & \cdots & R_nC_n \end{bmatrix}$$

If $R_1 \leftrightarrow R_2$:

$$A = \begin{bmatrix} R_2 \\ R_1 \\ \vdots \end{bmatrix} \qquad AB = \begin{bmatrix} R_2C_1 & R_2C_2 & \cdots & R_2C_n \\ R_1C_1 & \cdots & & \\ & & & \end{bmatrix}$$

if we change R_i in A, it will change R_i in AB also.

$$\boxed{R(AB) = R(A) \cdot B}$$

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→ To find inverse of matrix A:

i.) $|A| \neq 0$

ii.) $A_{n\times n} = I_n A$ → for Row opⁿ ↳ RREF

★ RREF of a square matrix is always Identity Matrix.

$I_n = BA$

$\boxed{A^{-1} = B}$

Extra class (same date) : For col. operⁿ, we've to write:

$A = A I_n$

$I_n = AB = AA^{-1}$

$\Rightarrow \boxed{A^{-1} = B}$

Gauss ~~Gauss~~ - Jordan Method

Ex. $A = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$ Find A^{-1} using Gauss Jordan method.

$A = I_3 A$

$$\begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

→ You want ~~right~~ LHS as I_3.

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix}$ → $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$
$R_2 \to -R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & +1 & +3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 2 & 0 \\ 0 & +3 & -1 \end{bmatrix} A$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & -1 \\ -1 & 2 & 0 \end{bmatrix} A$$

$R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{bmatrix} A$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{bmatrix} A$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -3 & 1 & 2 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

Rank of a Matrix :

$A_{n \times n}$

Rank of A = rank (A) is defined as the no. of non-zero rows in REF of matrix.

Ex. $$A = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Rank $(A) = 3$.

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\begin{bmatrix} 1 & 6 \\ 2 & 12 \end{bmatrix}$

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Solution of a system of linear eq^{n}s

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Matrix form :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\Rightarrow$ $AX = B$

coeff. matrix $\quad A = \{a_{ij}\}_{m \times n}$

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $\quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

if $B \neq 0$ $\rightarrow$ Non-homogenous system of eq^{n}s.

$B = 0$ $\rightarrow$ Homogenous " "

How to check consistency ? $\rightarrow$ sol^n of a problem is unique

$\rightarrow$ Augmented matrix : $[A : B]$ add col. of B to A.

Th^m: The system $AX = B$ of linear eq^{n}s has a

(i) Unique sol^n if rank (A) = rank $(A:B) = n$.

(ii) Infinite many sol^n : rank(A) = rank $(A:B) < n$

(iii) No sol^n.. if rank $(A) \neq$ rank $(A:B)$

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→ if $AB = 0$ → atleast 1 solⁿ

trivial solⁿ : $X = 0$.

Q. Test the consistency of the following systems of eqⁿs & find the solⁿ if exists.

$$2x + 3y + 4z = 11$$
$$x + 5y + 7z = 15$$
$$3x + 11y + 13z = 25$$

(Gauss Elimination Method)

$$\underset{A}{\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{B}{\begin{bmatrix} 11 \\ 15 \\ 25 \end{bmatrix}} \qquad (A:B)\begin{bmatrix} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{bmatrix}$$

$(A:B)$: $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 2 & 3 & 4 & 11 \\ 3 & 11 & 13 & 25 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & -4 & -8 & -20 \end{bmatrix} \xrightarrow{R_3 \to -\frac{1}{4} R_3}$$

$$\begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 1 & 2 & 5 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & -7 & -10 & -19 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 7R_2}$$

$$\begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 4 & 16 \end{bmatrix} \xrightarrow{R_3 \to R_3/4} \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

REF.

⇒ Rank $(A:B) = 3$
Rank $(A) = 3$

⇒ solⁿ is unique

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 4 \end{bmatrix}$$

$\boxed{z = 4}$

$y + 2z = 5$

⇒ $\boxed{y = -3}$

$x - 15 + 28 = 15$ ⇒ $x = 2$

Elementary Linear Algebra
by S. Andrille & D. Hecker
4th edition
Elsevier

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→ if we find RREF of (A : B) : Gauss Jordan Method

$$\begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 5R_2} \begin{bmatrix} 1 & 0 & -3 & -10 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - 2R_1]{R_1 \to R_1 + 3R_3} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

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Ex.
$x + y + 2z + w = 5$
$2x + 3y - z - 2w = 2$
$4x + 5y + 3z = 7$
$AX = B$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

$$(A : B) : \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 5 & 0 \\ 0 & 1 & -5 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank (A : B) = 3
Rank (A) = 2 } ⇒ No solⁿ Inconsistent

$0 = 1$ ✗ (from last row)
⇓
Inconsistent → No solⁿ

Ex.
$-x + 2y + z = 1$
$3x - y + 2z = 1$
$y + \lambda z = 1$, λ : const.

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$$(A:B) = \begin{bmatrix} -1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 1 & \lambda & 1 \end{bmatrix} \underset{\text{Row op}^n}{\overset{\text{applying}}{\sim}} \begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 4/5 \\ 0 & 0 & \lambda-1 & 1/5 \end{bmatrix}$$

rank $(A:B) = 3$ (always)

rank $(A) = 2$ if $\lambda = 1$

3 if $\lambda \neq 1$

system is consistent if $\lambda \neq 1$. $\longrightarrow$

$$x - 2y - z = -1$$
$$y + z = 4/5$$
$$(\lambda - 1)z = 1/5$$

$$x = \frac{3}{5} - \frac{1}{5(\lambda-1)}$$

$$y = \frac{4}{5} - \frac{1}{5(\lambda-1)} \qquad z = \frac{1}{5(\lambda-1)}, \ \lambda \neq 1$$

$\longrightarrow$ upper triangular matrix : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

In all 3 cases, $\boxed{\det |A| = a_{11}a_{22}a_{33}}$

Ex. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$ $|A| = 2 \cdot (-6) = \underline{-12}$

$$\begin{bmatrix} -1 & & & & 0 \\ & +3 & & & \\ & & 2 & & \\ & & & 4 & \\ 0 & & & & -2 \end{bmatrix} \qquad |A| = 48$$

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Row Vectors , Linear Combinations and Row Space :

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix}$$

Row vectors : [2 3 4] [1 5 7] [3 11 13]

Linear combⁿ :

$= \underbrace{a\,[2\ 3\ 4] + b\,[1\ 5\ 7] + c\,[3\ 11\ 13]}_{\text{sum of scalar multiples of row vectors}}$ → a, b, c : scalar

$= \{[2a + b + 3c,\ 3a + 5b + 11c,\ 4a + 7b + 13c] : a, b, c \in \mathbb{R}\}$

Row space → set of linear combⁿ of row vectors

* ↳ Row space for 2 equivalent matrices is same
(applying Row opⁿs)

↓

We get equivalent matrix by applying row opⁿs. So, linear combⁿ will be same in both matrices. Since row space is set of linear combⁿs, it will also be same for both matrices

↳ Can reduce matrix to RREF because RREF is unique
↓ use this to find row space
↓ will be identity matrix

Ex. $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix}$

RREF of A : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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Row space of A :

$\{a(1,0,0) + b(0,1,0) + c(0,0,1)\}$

$\equiv \{ [a,b,c] , \ a,b,c \in \mathbb{R} \}$

Ex. Determine whether the row vector $[5, 17, -20]$ is in the row space of the matrix

$$P = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ -2 & 4 & -3 \end{bmatrix}$$

$(5, 17, -20) = a(3,1,-2) + b(4,0,1) + c(-2,4,-3)$

$= (3a + 4b - 2c, \ a + 4c, \ -2a + b - 3c)$

$$\left.\begin{aligned} 5 &= 3a + 4b - 2c \\ 17 &= a + 4c \\ -20 &= -2a + b - 3c \end{aligned}\right\} \Rightarrow AX = b$$

$$(A:B) \equiv \begin{bmatrix} 3 & 4 & -2 & 5 \\ 1 & 0 & 4 & 17 \\ -2 & 1 & -3 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{aligned} a &= 5 \\ b &= -1 \\ c &= 3 \end{aligned}$$

$\Rightarrow (5, 17, -20) = 5(\quad) - 1(\quad) + 3(\quad)$

can be expressed as linear comb^n of rows of matrix P

$\Rightarrow$ It'll be in Row space of P.

Linear Independent Row Vectors $\rightarrow$ A row vector can't be expressed as linear comb^n of rest of row vectors

$\rightarrow$ Make row space $= 0$ $\rightarrow$ If get non-zero values of a, b, c $\Rightarrow$ Linearly dependent

else, not

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Ex. $P = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ -2 & 4 & -3 \end{bmatrix}$

Find 3 real no. a, b, c s.t. $a(3,1,-2) + b(4,0,1) + c(-2,4,-3) = (0,0,0)$

$3a + 4b - 2c = 0$

$a + 4c = 0$

$-2a + b - 3c = 0$

$(A:B) : \begin{bmatrix} 3 & 4 & -2 & 0 \\ 1 & 0 & 4 & 0 \\ -2 & 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} a = \\ b = \\ c = 0 \end{matrix}}$

⇒ All 3 row vectors are L I.

Ex. $P = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ 7 & 1 & -1 \end{bmatrix}$ ⇒ ~~3a + 4b~~ $R_3 \to R_1 + R_2$

⇓

R_3 is Linearly dependent

If we solve it : $3a + 4b + 7c = 0$

$a + b = 0$

$-2a + b - c = 0$

$(A:B) : \begin{bmatrix} 3 & 4 & 7 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank (A: B) = 2

⇒ $\boxed{a = b = 0}$, $\boxed{\underbrace{c}_{\text{arbitary no.}} \in \mathbb{R}}$

⇒ Rows vectors of this matrix are not L I.

** When Rank (A : B) < no. of given rows ⇒ Linearly Dependent

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Eigen Values and eigen vectors

A real no. λ is an eigen value of an n-square matrix iff $\exists$ a non-zero vector x s.t.

$$AX = \lambda X$$

$$\Rightarrow (A - \lambda I_n) X = 0$$

$X \rightarrow$ eigen vector of A corresponding to eigen value λ.

$\because$ X is non-zero vector,

$$\Rightarrow (A - \lambda I_n) = 0$$

$$\Rightarrow |A - \lambda I_n| = 0 \quad \rightarrow \text{characteristic eq}^n$$

The set $E_\lambda = \{ X : AX = \lambda X \}$ $\rightarrow$ eigen space of λ.

$\rightarrow$ $X = 0$: a trivial sol^n of $AX = \lambda X$.

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2 classes missing

Vector Space

Real Vector Space

Add^n

I. Closure Property of Add^n: Let $a, b \in R \Rightarrow a + b \in \mathbb{R}$.
Or, we can say that R is closed w r t addition or real no. satisfy closure property w r t add^n

II Commutative Property of Add^n:

$$a + b = b + a, \quad a, b \in \mathbb{R}$$

III Associative Property of Add^n

$$(a + b) + c = a + (b + c), \quad \forall a, b, c \in R$$

IV Additive Identity

$$a + 0 = a = 0 + a \quad \forall a \in R$$

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V Additive Inverse : $a \in R$

$a + (-a) = 0 = (-a) + a$

$\downarrow$

additive inverse

Multiplication :

I $a, b \in R \Rightarrow a * b \in R$

II $a \cdot b = b \cdot a \quad \forall a, b \in R$

III $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in R$

IV $a \cdot 1 = a = 1 \cdot a \quad \forall a \in R$

V $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a \quad \forall a \in R$

$\uparrow$

multiplicative inverse

* Multiplicative is distributive ~~of~~ over addition.

$a, b, c \in R$

$\Rightarrow a \cdot (b+c) = a \cdot b + a \cdot c \rightarrow$ Left distribution law

$(b+c) \cdot a = b \cdot a + c \cdot a \rightarrow$ Right " "

* Division only has Right distributive property :

~~Ex.~~ $7 / (4+5) \neq 7/4 + 7/5$

$(4+5)/7 = 4/7 + 5/7$

Real Vector space

A non-empty set V is said to be a real vector space if there are two operations : vector addition $(\oplus)$ and scalar multiplication $(\odot)$ s.t. $\forall$ vectors $u, v, w \in V$ and $a, b \in R$, the following properties are satisfied :

1. $u \oplus v \in V$ (Closure Property)
2. $u \oplus v = v \oplus u$ (Commutative)
3. $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ (Ass.)
4.

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Elements of Vector space are vectors
Real " " scalars

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4. $\exists$ some element $0 \in V$ s.t
$u \oplus 0 = u = 0 \oplus u$

5. $\exists$ $-u \in V$ s.t.
$u \oplus (-u) = 0 = (-u) \oplus u$

6. $a \odot u \in V$

7. $a \odot (u \oplus v) = a \odot u \oplus a \odot v$

8. $(a * b) \odot u = a \odot (b \odot u)$
$\downarrow$
sum of Real no.

9. $1 \odot u = u$

NOTE:

i) Any scalar multiplied with a zero vector gives a zero vector

$a \odot 0 = 0$ $\oplus 0$

$a \odot 0 \oplus 0 = a \odot 0 + (-a \odot 0)$

$= a \odot (0 \oplus (0) \oplus (-a \odot 0) = a \odot 0 + (-a \odot 0) = 0$

For clarity, use $+$ & $\cdot$

$\Rightarrow a \cdot 0 + 0 = a \cdot 0 = a \cdot 0 + a \cdot 0 + (-a \cdot 0)$

$= a \cdot (0 + a) + (-a \cdot 0) = a \cdot 0 + (-a \cdot 0) = 0$

ii) If scalar 0 is multiplied with any vector, it gives zero vector
ie, $0 \odot u = 0$

iii) $(-1) \odot u$ gives additive inverse of u $[-u]$.

Ex. The set $\mathbb{R}$ of all real no. is a vector space wrt the following operations:

(i) $u \oplus v = u + v$ (vector add^n)

(ii) $a \odot u = a u$ (scalar multp^n) $\forall$ $a, u, v \in R$.

Sol^n $V = \mathbb{R}$. $\rightarrow$ Have to verify by all properties of R.

Study R_1, R_2 $\quad R_n \to n$ - dimensional

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Q. Prove / Disprove.

Sub-space : subset of space satisfying all properties of that space

$\to$ If V is a vector space and W is a subset of V s.t. W is also a vector space under the same operations as in V, then W is called a subspace of V.

Ex. The set $W = \{[x_1, 0] : x_1 \in R\}$

$W \in R^2$, $V \in R^2$

Th^m: A subset W of a vector space V is a subspace of V iff W is closed wrt vector addition and scalar multiplication. (and vice versa)

NOTE: It is also easy to show that a subset W of a vector space V is a subspace of V iff

$a \odot u \oplus b \odot v \in W \quad \forall \quad a, b \in R$ and $u, v \in V$

Ex. Show that the set $W = \{[x_1, 0], x_1 \in R\}$ is subspace of R^2.

Sol^n Let $u = [x_1, 0]$, $v = [y_1, 0]$ $\quad u, v \in W$ and $a \in R$

Then

$u \oplus v = [x_1 + y_1, 0] \in W$ (by closer property) vector add^n

$a \cdot u = a \cdot [x_1, 0] = [ax_1, 0] \in W$ scalar multiplic

$\Rightarrow$ W is closed under 2 op^{n}s vector add^n & scalar multip^n.

Hence, W is a subspace in R^2.

Ex. Verify whether $W = \{[x, y], x - y = 0, x, y \in R\}$ is a subspace in R^2.

Sol^n Let $u = [x_1, x_2]$, $v = [y_1, y_2] \in W$ and $a \in R$

$\Rightarrow x_1 - x_2 = 0$ and $y_1 - y_2 = 0$

$u \oplus v = [x_1 + y_1, x_2 + y_2] \in W$ $[\because (x_1 + y_1) - (x_2 + y_2) = 0]$

$a \odot u = [ax_1, ax_2] \in W$ $[\because ax_1 - ax_2 = a(x_1 - x_2) = 0]$

$\Rightarrow$ W is a subspace in R^2

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Ex. $W = \left\{ \begin{vmatrix} p & q \\ r & s \end{vmatrix} : ps \cdot qr \neq 0 \ , \ p, q, r, s \in R \right\}$ is subspace of M_{22} → Matrix of dimension 2x2.

let $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $v = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in W$

$u \oplus v = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$

⇒ W is not a subspace of M_{22}

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Ex. Verify whether

$W = \{ [x, y] : y = x^2, \ x, y \in R \}$ is a subspace in R^2.

~~let $u = [x_1, x_2]$, $v = [y_1, y_2]$ $\in W$~~

⇒

Let $u = [1, 1]$ & $v = [2, 4]$

(i.) $u \oplus v = [1, 1] \oplus [2, 4] = [3, 5] \notin W$

⇒ W is not a subspace in R^2.

→ let W_1 and W_2 be 2 ~~sup~~. sub-spaces of vector space V.

Then,

(i) $W_1 \cap W_2$ is a subspace of V.

(ii.) $W_1 \cup W_2$ need not be a subspace of V.

(iii) $W_1 \cup W_2$ is a ~~sup~~ subspace of V ~~iff~~

either $W_1 \subset W_2$ or ~~$W_2 \subset W_1$~~, $W_2 \subset W_1$

[Prove yourself]

Span of a set

Let S be any subset of vectorspace V. Then, the set of all the linear combinations of finite number of members of S is called span of S. It is denoted by span(S) or L(S)

$\therefore \ L(S) = span(S) = \{ a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ s.t. $a_i \in R$, $v_i \in S$, $i = 1, 2, \dots, n \}$

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Ex. $S = \{(1,0), (0,1)\}$

then $L(S) = \{a(1,0) + b(0,1) : a, b \in R\}$

$= \{(a,b) : a, b \in R\} = \mathbb{R}^2$

Ex. show that span of the set $S = \{[2,3,4], [1,5,7], [3,11,13]\}$ is $\mathbb{R}^3$.

By definition,

$L(S) = \{a[2,3,4] + b[1,5,7] + c[3,11,13] : a, b, c \in \mathbb{R}\}$

$= \{(2a+b+3c, 3a+5b+11c, 4a+7b+13c) : a, b, c \in \mathbb{R}\}$

for simplified span

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore L(S) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$= [(a,b,c) : a, b, c \in \mathbb{R}] = \mathbb{R}^3$

Ex. $\text{Span}(\phi) = ?$

$\text{span}(\phi) = \{0\}$

Thm: Let S be subset of a vector space V. Then,

(i) $L(S)$ is a subset of V

(ii) $L(S)$ is a subspace of V

(iii) $L(S)$ is the minimal subspace of V containing S.

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Dimension :

The number of elements in the basis of a vector space is called its dimensions

→ If a vector space has a finite dimension, then we call it as finite dimensional vector space.

Ex. Let $B = [(1,0), (0,1)]$ of R^2

so, it has 2 vectors $\Rightarrow$ $\dim(R^2) = 2$

Ex. $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

↓

4 vectors of M_{22} $\Rightarrow$ $\dim(M_{22}) = 4$

Thm: The maximal LI subset of a spanning set of a vector space forms a basis of the vector space.

We'll use LI method to find Maximal LI subset :

Method :

- Write the given vectors as the columns in a matrix & find REF.
- The columns carrying the leading entries correspond to the LI vectors

Ex. Find a maximal LI subset of the set

$S = \{(1,0), (2,1), (1,5)\}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 5 \end{bmatrix}$$

maximal LI $\{(1,0), (2,1)\}$

Ex. $S = \{[1,0], [2,1], [1,5]\}$ spans R^2

What is the basis in R^2 from this set?

Sol^n:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

Maximal LI forms a basis

so, $\{[1,0], [2,1]\}$ forms a basis

Linear Transformation

Let V and W be 2 vector spaces. Then, a function $f: V \to W$ is said to be linear transformation (LT) if and only if $\forall\, v_1, v_2 \in V$ and $c \in R$, we have

$$f(v_1 + v_2) = f(v_1) + f(v_2) \text{ and}$$
$$f(cv_1) = c\, f(v_1)$$

Ex. $f: M_{mn} \longrightarrow M_{nm}$ given by:

$f(A) = A^T$ is LT.

Sol^n: Let $A, B \in M_{mn}$ and $c \in R$

$$f(A+B) = (A+B)^T = A^T + B^T = f(A) + f(B)$$

$$f(cA) = (cA)^T = cA^T = c\, f(A)$$

$\Rightarrow$ f is LT.

Ex. $f: R^3 \to R^3$ given by:

$f(x_1, x_2, x_3) = \{x_1, x_2, -x_3\}$ is a LT.

Sol^n Let $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$

$f(X+Y) = \cancel{(X+Y, Y)}\ (x_1 + y_1, x_2 + y_2, -x_3 - y_3)$

$= (x_1, x_2, -x_3) + (\ldots$

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$= f(x) + f(y)$

$f(cX) = \{cx_1, cx_2, -cx_3\}$

$= c\ \{x_1, x_2, -x_3\} = c\ f(X)$

$\Rightarrow f$ is LT.

Ex. let A be a $m \times n$ matrix. ~~and~~ Then $f : R^n \rightarrow R^m$ given by:

$f(X) = AX$ is a LT.

sol^n Let $X_1, X_2 \in R^n$ and $a \in R$. Then,

- $f(X_1 + X_2) = A(X_1 + X_2) = AX_1 + AX_2 = f(X_1) + f(X_2)$
- $f(aX_1) = A(aX_1) = a(AX_1) = af(X_1)$

$\Rightarrow f$ is a LT.

Note: A LT $f : V \rightarrow V$ is called a linear operator on V.

Theorems :

I. If $L : V \rightarrow W$ is a LT ~~and~~, then $L(0) = 0$

and $L(a_1 v_1 + a_2 v_2) = a_1 L(v_1) + a_2 L(v_2)$ $\forall a_1, a_2 \in R$ & $v_1, v_2 \in V$

Proof: Let $u \in V$ then

- $L(0) = L\{(u) + (-u)\} = L(u) + L(-u)$

$= L(u) + (-1)L(u) = 0.$

- $L(a_1 v_1 + a_2 v_2) = L(a_1 v_1) + L(a_2 v_2)$

$= a_1 L(v_1) + a_2 L(v_2)$

II. The composition of two LT is also a LT. , ie,

if $L_1 : V_1 \rightarrow V_2$ & $L_2 : V_2 \rightarrow V_3$ are LT, then

$L_2 \circ L_1 : V_1 \rightarrow V_3$ is a LT

Proof: Let $u, v \in V_1$ and $a \in \mathbb{R}$

Given : $L_1 : V_1 \rightarrow V_2$ & $L_2 : V_2 \rightarrow V_3$ are LT

We have $(L_2 \circ L_1)(u+v) = L_2\left[L_1(u+v)\right]$

$= L_2\left[L_1(u) + L_1(v)\right] = L_2[L_1(u)] + L_2[L_1(v)]$

$= L_2 \circ L_1(u) + L_2 \circ L_1(v)$

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$(L_2 o L_1)(au) = L_2[L_1(au)] = L_2[aL_1(u)] = a L_2[L_1(u)]$

$= a L_2 o L_1(u)$

$\Rightarrow$ $L_2 o L_1$ is a LT.

III. If $L : V \to W$ is a LT and V_1 & W_1 are subspace of V and W resp., then

$L(V_1) = \{L(u) : u \in V_1\}$ is a subspace of W

and $L^{-1}(W_1) = \{v : L(v) \in W_1\}$ is a subspace of V.

Proof:

To prove:

$L(V_1) = \{L(v) : v \in V_1\}$ is a subspace of W

Let $L(u), L(v) \in L(V_1)$ and $a \in \mathbb{R}$.

Then, $u, v \in V_1$ and we have

$L(u) + L(v) = L(u+v) \in L(V_1)$

since $u, v \in V_1$

Also, $aL(u) = L(au) \in L(V_1)$

$\because au \in V_1$ $\Rightarrow$ $L(V_1)$ is a subspace of W.