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→ Normal : limiting case of Bin...
\+ symmetry around mean

Moment generating func^n (mgf) :

$$M_x(t) = E[e^{tx}] = \begin{cases} \int e^{tx} f(x)\, dx & \text{in continuous} \\ \sum_i e^{tx} f(x) & \text{in discrete} \end{cases}$$

$$E[e^{tx}] = E\left[1 + tx + \frac{t^2 x^2}{2!} + \dots\right]$$

$$= E[1] + tE[x] + \frac{t^2}{2!} E[x^2] + \dots$$

3) Normal Random Variable

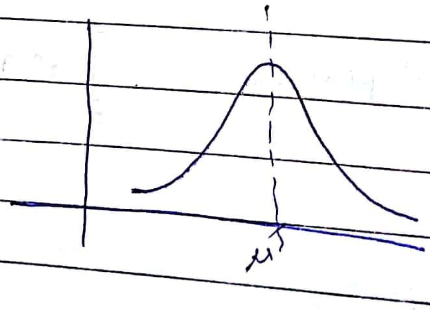
x is said to be Normal r.v. with parameter μ and σ^2 if the pdf of x is defined as follows :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left[\frac{(x-\mu)^2}{2\sigma^2}\right]}$$

where $-\infty < x < \infty$

μ : mean, σ^2 : variance

graph :



To show $f(x)$ is pdf, we show that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx$$

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$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

let $z = \frac{x-\mu}{\sqrt{2}\sigma}$

$$\Rightarrow dz = \frac{dx}{\sqrt{2}\sigma}$$

$$I = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \, dz$$

$$I_1 = \int_{-\infty}^{\infty} e^{-z^2/2} \, dz$$

$$I_1 * I_2 = \int_{-\infty}^{\infty} e^{-z^2/2} \, dz \int_{-\infty}^{\infty} e^{-y^2/2} \, dy$$

$$(I_2 = I_1) \quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2+y^2}{2}\right)} \, dy \, dz \quad \rightarrow \text{changed to } \underline{2D}$$

$y = r\cos\theta \quad z = r\sin\theta \quad r : [0, \infty) \rightarrow$ complete plane

$\theta : [0, 2\pi)$

$$I_1^2 = \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{2}} r \, dr \, d\theta$$

$$= \int_0^{\infty} e^{-\frac{r^2}{2}} r \, dr$$

$$= 2\pi$$

$$\Rightarrow I_1 = \sqrt{2\pi}$$

$$\Rightarrow \boxed{I = 1}$$

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Q. Show that the constant in the Normal Distribution must be $\frac{1}{\sqrt{2\pi}}$

let constant be : a

$$\frac{a}{\sigma} e^{-(x-\mu)^2/2\sigma^2} = 1$$

$$\Rightarrow a = \frac{1}{\sqrt{2\pi}}$$

* cdf : $F_X(a) = \int_{-\infty}^{a} f(x)\,dx$

Q. Prove that if x is normally distributed with parameter μ and σ^2, then $y = ax+b$ is normally distributed with parameters $a\mu+b$ and $a^2\sigma^2$.

$$F_y(x) = P\{y \le x\} = P\{ax+b \le x\}$$

$$P\{x \le (x-b)/a\}$$

$$F_y(x) = F_x\left(\frac{x-b}{a}\right) \rightarrow \text{if it is a cdf}$$

Relation b/w both cdf's

$$f_y(x) = \frac{d(F_x)}{dx}$$

$$= \frac{1}{a} f_x\left(\frac{x-b}{a}\right)$$

$$= \frac{1}{a}\left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-b}{a}-\mu\right)^2/2\sigma^2}\right]$$

$$f_y(x) = \frac{1}{a\sqrt{2\pi}\,\sigma} e^{-(x-(a\mu+b))^2/2\sigma^2 a^2}$$

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→ Expectation and Variance

$$E[X] = \int_{-\infty}^{\infty} \ldots$$

↳ pdf for $x \sim N(\mu, \sigma^2)$ let $z = \frac{x-\mu}{\sigma}$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-z^2/2}$$

$\boxed{x = \sigma z + \mu}$

similar to $\underline{ax+b}$

⇓

can calculate $E[x]$ if we calculate $E[z]$

$$E[z] = \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}\,\sigma} e^{-z^2/2} dz$$

$$= 0$$

$-z^2/2 = t$ $-z\,dz = dt$

$$-\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{t} dt$$

$$E[z^2] = \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}\sigma} e^{-z^2/2} dz = 1$$

$$Var(z) = 1 - (0)^2 = \underline{\underline{1}}.$$

If we compare ; $a = \sigma$, $b = \mu$

$$E[X] = a\mu + b$$

$$= \sigma \cancel{E[z]}^{0} + \mu$$

$$\Rightarrow \boxed{E[X] = \mu}$$

$$Var(X) = a^2 \sigma^2 \quad \rightarrow Var(z)$$

$$= \sigma^2 (1)$$

$$\Rightarrow \boxed{Var(x) = \sigma^2}$$

$*$ $F_X(2) = \int_{-\infty}^{2} \ldots = \phi(2)$

* Applicable when n is very large.

→ cdf of std. normal r.v. is given as :

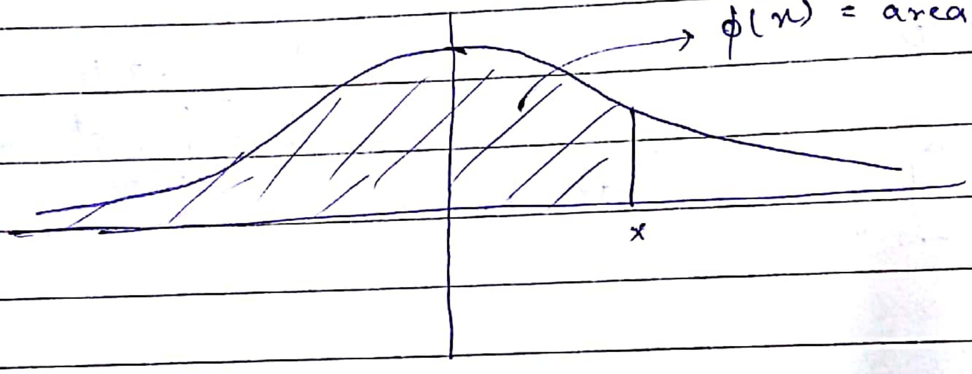
$$\phi(x) = P\{Z \le x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \, dz \quad , \quad Z = \frac{X-\mu}{\sigma}$$

↓ this represents ~~std~~ std. normal r.v.

$$\lim_{x \to \infty} \phi(x) = 1 \qquad \left\{ \begin{array}{l} \text{value of } I = \sqrt{2\pi} \\ \text{result} = 0 \end{array} \right\}$$

$$\lim_{x \to -\infty} \phi(x) = 0$$

$$\phi(0) = \frac{1}{2} \qquad \left[\text{half of complete portion symmetrical around mean} \right]$$



→ $\phi(0.25)$: follow row of 0.2 & column of 0.05.

→ $$F_X(a) = P\{X \le a\} = P\left\{ \frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma} \right\}$$

$$= \phi\left(\frac{a-\mu}{\sigma}\right) = \phi(z)$$

• values used are for z, not for x, so, 1st we've to convert into $\phi(z)$.

~~std~~ std.

Normal
general cdf : $F(\ )$
std. cdf : ϕ

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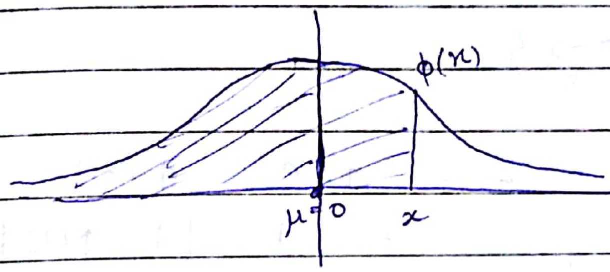
Standard cdf :

The cumulative distribution func^n (cdf) of a standard normal random variable is denoted by $\phi(x)$ and is given by :

$$\boxed{\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy}$$

$\boxed{\begin{matrix}\sigma = 1 \\ \mu = 0\end{matrix}}$ ⇒ Centers y axis



Properties :

→ $\lim_{x \to \infty} \phi(x) = 1$ $\quad \lim_{x \to -\infty} \phi(x) = 0$ $\quad \lim_{x} \phi(0) = \frac{1}{2}$

→ $\boxed{\phi(-x) = 1 - \phi(x)}$ where $-\infty < x < \infty$

*** → $\boxed{F_X(a) = \phi\left(\frac{a-\mu}{\sigma}\right)}$

Eg. If X is $N(3, 9)$, find : $\qquad \mu = 3$, $\sigma = 3$ (μ, σ^2)

a) $P\{2 < X < 5\}$

$P\left\{\frac{2-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right\}$

$P\left\{\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right\}$

$P\left\{\frac{-1}{3} < \underbrace{\frac{X-3}{3}}_{Z} < \frac{2}{3}\right\}$

⇓

$P\left\{\frac{X-3}{3} < \frac{2}{3}\right\} - P\left\{\frac{X-3}{3} \geq \frac{-1}{3}\right\}$

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$= \phi\left(\frac{2}{3}\right) - \phi\left(\frac{-1}{3}\right)$

$= \phi(0.66) - \left[1 - \phi\left(\overset{0.33}{\frac{1}{3}}\right)\right]$ $\quad \left\{\because \phi(x) = 1 - \phi(-x)\right\}$

from table — row of 0.6, col of 0.06

$\approx (0.7454) - (1 - 0.06293)$

≈ 0.37

b.) $P\{X > 0\}$

$P\left\{\frac{X-3}{3} > \frac{0-3}{3}\right\}$

$P\{\infty > Z > -1\} = \phi(\infty) - \phi(-1)$

$= 1 - [1 - \phi(1)]$

$= \phi(1)$

$= \cancel{-\phi(-1)}$

$= \cancel{-(1 - \phi(1))}$

$= \cancel{\phi(1) - 1}$

c.) $P\{|X-3| > 6\}$

$P\{X-3 < -6\} + P\{\infty > X-3 > 6\}$

$P\left\{\frac{X-3}{3} < -2\right\} + P\left\{\infty > \frac{X-3}{3} > 2\right\}$

$\phi(-2) + [1 - \phi(2)]$

$2[1 - \phi(2)]$

Q. $X \sim N(-5, 4)$

$P\{X > -3 \mid X > -5\} = \frac{P\{X > -3 \cap X > -5\}}{P\{X > -5\}}$

$= \frac{P\{\infty > X > -3\}}{P\{\infty > X > -5\}} \Rightarrow \frac{\phi(3)}{\phi(5)}$

$= \frac{P\left(\frac{X+5}{2} > +1\right)}{P\left(\frac{X+5}{2} > 0\right)} = \boxed{\frac{1 - \phi(1)}{1 - \phi(0)}}$

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Q :→ $P(X > 2 \mid Y < 1)$ $\qquad X \sim N(2, 4)$

$Y = 3 - 2X$

$\cancel{Y(3-2X)} \to Y(a\mu + b, \sigma^2 a^2)$

$\Rightarrow Y(-1, 16)$

OR

$-2X < -2$

$X > 1$

$$\frac{P(X > 2 \cap 3 - 2X < 1)}{P(3 - 2X < 1)} = \frac{P(X > 2 \cap X > 1)}{P(X > 1)}$$

$$= \frac{P(X > 2)}{P(X > 1)} = \frac{P\left(\frac{X-2}{2} > 0\right)}{P\left(\frac{X-2}{2} > \frac{-1}{2}\right)} = \frac{P(\infty > Z > 0)}{P(\infty > Z > -\frac{1}{2})}$$

$$= \frac{1 - \Phi(0)}{1 - \Phi(-\frac{1}{2})} = \boxed{\frac{1 - \Phi(0)}{\Phi(\frac{1}{2})}}$$

Jointly ~~Distributeds~~ Distributed Random Variable

Let X & Y be ~~a~~ r.v. having pmf as

$P_X(x) = P\{X = x\}$ & $P_Y(y) = P\{Y = y\}$

the pmf of two random variables X and Y is defined as:

$$P_{XY}(x, y) = P(X = x, Y = y)$$

usually,

$$P_{XY}(x, y) = P((X = x) \text{ and } (Y = y))$$

→ If we have P_{XY}, we can get P_X & P_Y

We can also define the range for X and Y as,

$$R_{XY} = \{(x_i, y_j) \text{ s.t. } P_{XY}(x_i, y_j) > 0\}$$

In particular,

if $R_X = \{x_1, x_2, \ldots\}$

$R_Y = \{y_1, y_2, y_3, \ldots\}$

then we can write

$R_{XY} \subset R_X \times R_Y = \{(x_i, y_i) : x_i \in R_X, y_i \in R_Y\}$

where X & Y are both random variables

→ as P_{XY} is pmf, so

$$\sum_{(x_i, y_j) \in R_{XY}} P_{XY}(x, y) = 1$$

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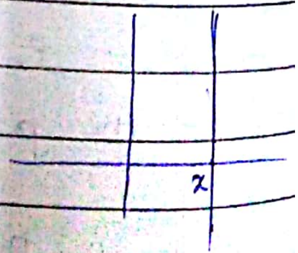
* $P(X \leq 1 \mid Y \leq 2)$ & don't have any relation b/w them :- use Jointly Distributed Random Variable in that case.

→ The event $X = x$ can be written as

$\{(x_i, y_j) : x_i = x, y_j \in R_Y\}$



→ The event $Y = y$ can be written as

$\{(x_i, y_j) : x_i \in R_X, y_j = y\}$

→ $P_{XY}(x, y) = P\{X = x, Y = y\}$

$= P\{(X = x) \cap (Y = y)\}$

Marginal pmf (separate X & Y pmf from Joint pmf)

Joint pmf contains the information regarding the distribution of X and Y. Thus, we can obtain pmf of X from its Joint pmf with Y as :

$$P_X(x) = P\{X = x\} = \sum_{y_j \in R_Y} P\{X = x, Y = y_j\} = \sum_{y_j \in R_Y} P_{XY}(x, y_j)$$

pmf of Y :

$$P_Y(y) = P\{Y = y\} = \sum_{x_i \in R_X} P\{X = x_i, Y = y\} = \sum_{x_i \in R_X} P_{XY}(x_i, y)$$

Marginal pmf of X and Y

$$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j) \quad \text{for any } x \in R_X$$

$$P_Y(y) = \sum_{x_i \in R_X} P_{XY}(x_i, y) \quad \text{for any } y \in R_Y$$

Que Consider 2 r.v. X and Y with their joint pmf as :

X \ Y	Y = 0	Y = 1	Y = 2
X = 0	1/6	1/4	1/8
X = 1	1/8	1/6	1/6

a.) Find $P(X=0, Y \le 1)$

b.) Find the marginal pmf of X and Y

c.) Find $P(Y=1 \mid X=0)$

d.) Are X & Y independent ?

a.) $P(X=0, Y \le 1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$

$R_X = \{0,1\}$
$R_Y = \{0,1,2\}$

b.) $P_X(0) = \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$

$P_X(1) = \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}$

$P_Y(0) = \frac{1}{6} + \frac{1}{8} = \frac{7}{24}$

$P_Y(1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

$P_Y(2) = \frac{1}{8} + \frac{1}{6} = \frac{7}{24}$

$$P_X = \begin{cases} \frac{13}{24} & X=0 \\ \frac{11}{24} & X=1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y = \begin{cases} \frac{7}{24} & Y=0 \\ \frac{5}{12} & Y=1 \\ \frac{7}{24} & Y=2 \\ 0 & \text{otherwise} \end{cases}$$

c.) $P(Y=1 \mid X=0) = \frac{P((Y=1) \cap (X=0))}{P(X=0)}$

$= \frac{1/4}{13/24} = \frac{6}{13}$

d.) if independent,

$P(Y=1 \mid X=0) = P(Y=1)$ (can take any 2 values)

$\frac{6}{13} \neq \frac{5}{12}$

$\Rightarrow$ they are not independent

Ques. Suppose a car showroom has 10 cars of a particular brand out of which 5 are good, 2 are having defective transmission, 3 have defective steering. 2 cars are selected at random. Discuss the joint pmf.

let X = no. of ~~defective~~ cars having defective transmission $\rightarrow 0,1,2$

Y = " " " steering $\rightarrow 0,1,2$

X \ Y	Y = 0	Y = 1	Y = 2
X = 0	${}^5C_2/{}^{10}C_2$	${}^3C_1 \cdot {}^5C_1 / {}^{10}C_2$	${}^3C_2/{}^{10}C_2$
X = 1	${}^2C_1 \cdot {}^5C_1/{}^{10}C_2$	${}^3C_1 \, {}^2C_1 / {}^{10}C_2$	0
X = 2	${}^2C_2/{}^{10}C_2$	0	0

10/45	15/45	3/45
10/45	6/45	0
1/45	0	0

$P_{XY}(0,0) = \frac{{}^5C_2}{{}^{10}C_2} = \frac{\cancel{5}\times\cancel{4}^{2}}{{}_{2}\cancel{10}\times 9} = \frac{2}{9}$

$P_{XY}(0,1) = \frac{{}^3C_1 \cdot {}^5C_1}{{}^{10}C_2}$

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→ $P\{X \le 1, Y \le 1\} = \frac{10+10+15+6}{45} = \frac{41}{45}$.

→ $P\{X \le 2\} = \frac{10+10+15+6+3}{45} = \frac{44}{45}$

Joint Cumulative Distribution Function

The joint cdf of X and Y is defined as

$F_X(a,b) = F(a,b) = P\{X \le a, Y \le b\} \quad -\infty < a,b < \infty$

(prev eg) :-

↳ $F(0,0) = \frac{10}{45} = P\{X \le 0, Y \le 0\}$

↳ $F(0,1) = P\{X \le 0, Y \le 1\} = \frac{10+15}{45} = \frac{25}{45}$

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Marginal cdf :-

Now, the distribution of X can be derived as :

$F_X(a) = P\{X \le a\} = P\{X \le a, Y < \infty\} = F(a, \infty)$

$= \lim_{b\to\infty} F(a,b)$

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Similarly,

$F_Y(b) = P\{Y \le b\} = P\{X < \infty, Y \le b\} = \lim_{a \to \infty} F\{a, b\}$

→ $P\{X > a, Y > b\}$

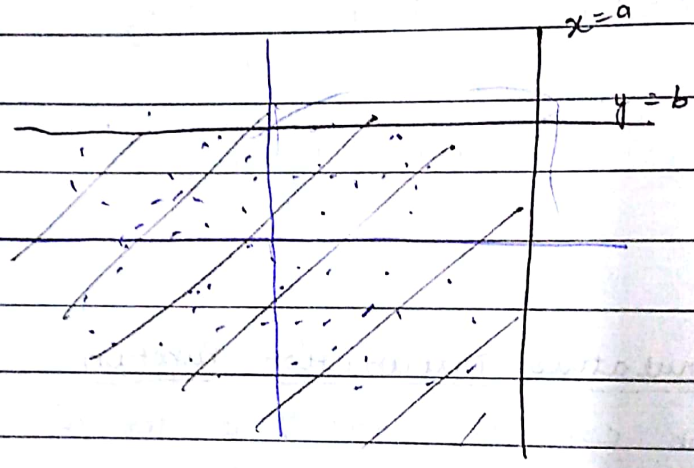
$= 1 - P\{(X > a, Y > b)^c\}$

$= 1 - [P\{(X > a)^c \cup (Y > b)^c\}]$

$= 1 - P\{X \le a \cup Y \le b\}$

$= 1 - [P\{X \le a\} + P\{Y \le b\} - P\{X \le a, Y \le b\}]$

$= 1 - F_X(a) - F_Y(b) - F(a, b)$



→ $\lim_{a,b \to \infty} F(a,b) = 1$ (cover all points)

→ $\lim_{a,b \to -\infty} F(a,b) = 0$ (covers no point)

→ $F(-\infty, Y) = 0$ for any Y

→ $F(X, -\infty) = 0$ for any X

Ex. Let $X \sim$ Bernoulli (P) and Y is Bernoulli (q) be independent where $0 < p, q < 1$

Find PMF and joint CDF of X and Y.

$R_x = \{0, 1\}$
$R_y = \{0, 1\}$
could be 0 or 1

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$R_{XY} = \{(0,0), (0,1), (1,0), (1,1)\}$

$P(0,0) = P\{X=0, Y=0\} = P\{X=0\} \cdot P\{Y=0\}$ (Independent)

$\boxed{\text{Joint Prob. (for every } x \text{ \& } y) = \text{Product of Marginal Prob.}}$

$= (1-p)(1-q)$

$$f_{XY}(x,y) = \begin{cases} (1-p)(1-q) & (x,y) = (0,0) \\ (1-p)\,q & = (0,1) \\ p\,(1-q) & = (1,0) \\ pq & = (1,1) \\ 0 & \text{Otherwise} \end{cases}$$

→ CDF :

•) $x \not< 0$ & $y \not< 0$

$F(x,y) = 0$

•) $x \le 0$, $y \le 0$

$F(x,y) = (1-p)(1-q)$

$\boxed{* \; x < 0, \; y \le 1 \Downarrow F(x,y) = 0 \text{ because } x \text{ is not in the range}}$

•) $0 \le x$, $y \le 1$

$F(x,y) = P$

•) $x \le 1$, $y \le 0$

$F(x,y) = (1-p)(1-q) + p(1-q)$

$= \underline{1-q}$

•) $F(1,1) = P\{x \le 1, y \le 1\} = 1$

•) $F(2,2) = 1$

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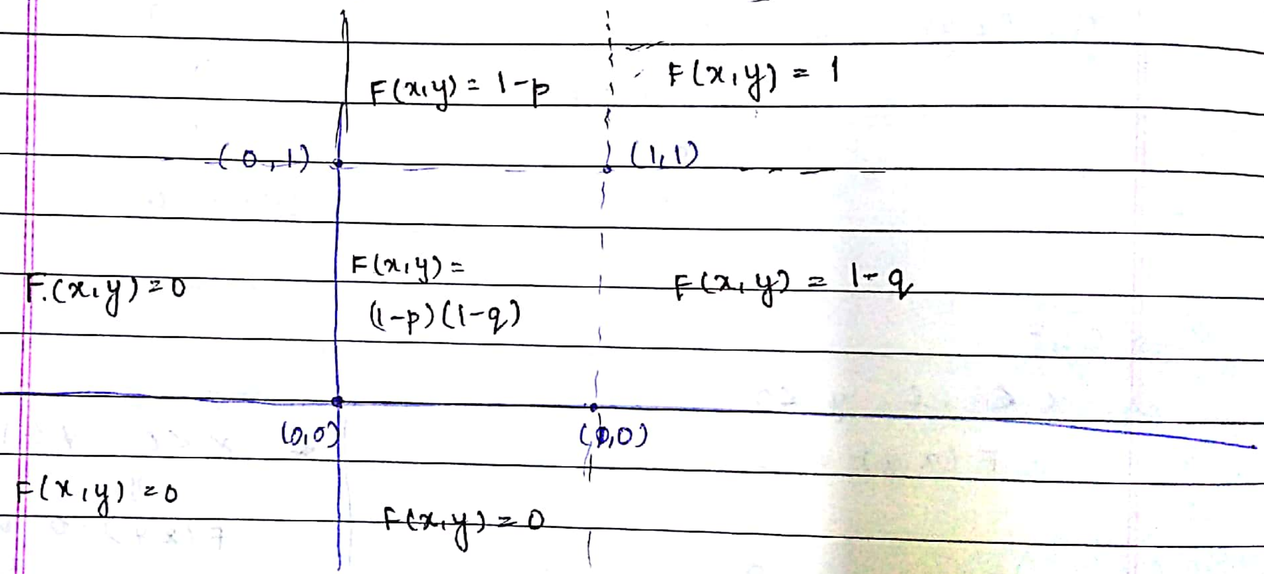
→ $x > 1 \quad y > 1$

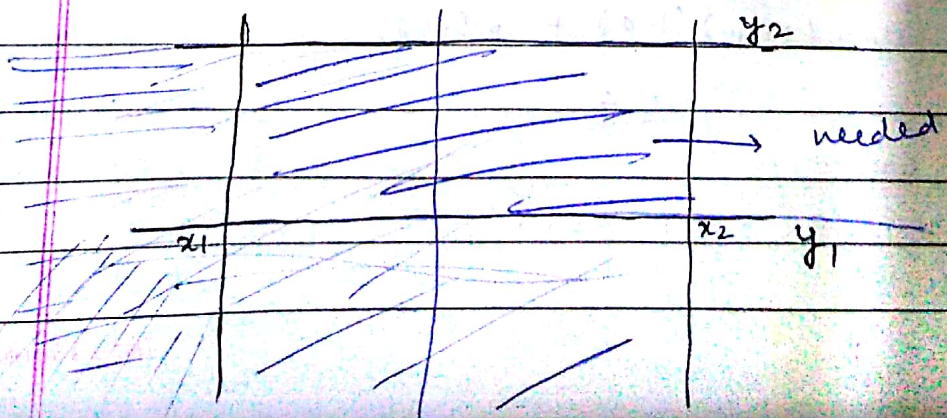
$F(1,1) = 1$

$$F(x,y) = \begin{cases} 0 & \text{if } x < 0 \\ 0 & \text{if } y < 0 \\ 1 & \text{if } x \geq 1,\ y \geq 1 \\ 1-p & 0 \leq x < 1 \ \& \ y \geq 1 \\ 1-q & 0 \leq y < 1 \ \& \ x \geq 1 \\ (1-p)(1-q) & 0 \leq y < 1 \ \& \ 0 \leq x < 1 \end{cases}$$



Note : For 2 random variables X and Y and real no. $x_1 \leq x_2$, $y_1 \leq y_2$

$$P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2)$$
$$= F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1)$$



$= P\{X \le x_2, Y \le y_2\} + P\{X \le x_1, Y \le y_1\}$
$- P\{X \le x_1, Y \le y_2\} - P\{X \le x_2, Y \le y_1\}$

Ques An urn 2 red, 5 white, 3 green balls. Select 3 balls at random and let X be a random variable of red balls & Y be the no. of white balls. Then, calculate ~~prob~~

a) $P(X \ge Y)$ $R_X = \{0, 1, 2\}$

b) $P(X = 2 \mid X \ge Y)$ $R_Y = \{0, 1, 2, 3, \cancel{\dots}\}$

$$P(X \ge Y) = P(X=0, Y=0) + P(X=1, Y=0) + P(X=2, Y=0) + P(X=1, Y=1) + P(X=2, Y=1)$$

$$= \frac{{}^3C_3}{{}^{10}C_3} + \frac{{}^2C_1 \cdot {}^3C_2}{{}^{10}C_3} + \frac{{}^2C_2 \cdot {}^3C_1}{{}^{10}C_3} + \frac{{}^2C_1 \cdot {}^5C_1 \cdot {}^3C_1}{{}^{10}C_3} + \frac{{}^2C_2 \cdot {}^5C_1}{{}^{10}C_3}$$

$$= \frac{1 + 6 + 3 + 30 + 5}{120} = \frac{3}{8}$$

b) $$P(X = 2 \mid X \ge Y) = \frac{P(X = 2, X \ge Y)}{P(X \ge Y)}$$

$$= \frac{P(X=2, Y=0) + P(X=2, Y=1)}{P(X \ge Y)}$$

$$= \frac{(3+5)/120}{3/8} = \frac{\cancel{8}^{2}/120}{3/8}$$

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Independent Random Variable

2 random variable X & Y are independent if :

$$P_{XY}(x,y) = P_X(x) \cdot P_Y(y) \quad \forall x,y$$

Equivalently, X and Y are independent if

$$F_{XY}(x,y) = F_X(x) \cdot F_Y(y) \quad \forall x,y$$

→ We always see that dependence is something like

$$Y = g(X)$$

for $g: R \to R$

Is it true ?

Eg. Let X & Y be 2 r.v. and joint pdf is given by :

X \ Y	1	2	3	4
1	1/4	1/8	1/16	1/16
2	1/16	1/16	1/4	1/8

$R_X = \{1, 2\}$

$R_Y = \{1, 2, 3, 4\}$

$$f_X(x) = \sum_{y_j \in R_Y} f(x, y_j)$$

$$f_X(1) = \frac{1\times4}{4\times4} + \frac{1\times2}{8\times2} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}$$

$$f_X(2) = \frac{1}{2} = 1 - f_X(1)$$

$f_Y(1) = \frac{5}{16}$ $\quad f_Y(2) = \frac{3}{16}$ $\quad f_Y(3) = \frac{5}{16}$ $\quad f_Y(4) = \frac{3}{16}$

→ For Independence

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_{XY}(1,1) = \frac{1}{4} \neq f_X(1) \cdot f_Y(1) = \frac{1}{2} \times \frac{5}{16}$$

⇒ They are dependent.

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* R.V. dependence $\not\Rightarrow$ we can predict func^n (In general)

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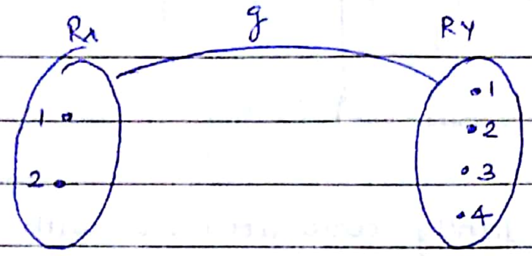
↳ can we have func^n g s.t. $Y = g(X)$?

$g : R_X \to R_Y$



Not possible
because max^n you
(It is generally one-to-one)

$$f_X(x) = \begin{cases} 1/2 & x = 1, 2 \\ 0 & \text{Otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 5/16 & y = 1, 3 \\ 3/16 & y = 2, 4 \\ 0 & \text{Otherwise} \end{cases}$$

* Joint pmf is equal to the product of marginal pmf in case of independent r.v.

<u>Joint for continuous Random Variable</u>

2 continuous r.v. are jointly continuous if there exists a non-negative func^n $f_{XY} : R^2 \to R$

s.t. ~~for~~ for any set $A \in R^2$, we have

$$\underbrace{P((X,Y) \in A)}_{\text{Prob. in area of A}} = \iint_A f_{XY}(x,y)\, dx\, dy$$

where the func^n $f_{XY}(x,y)$ is called the joint probability density func^n (pdf) of X and Y

- As $P(X=1)$ was 0 in continuous earlier $P(X=1)$ is 0 here
  (Point) → 1D (Line) → 2D

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$f_{XY}(x,y)$

since it's pdf,

→ The range of $f_{XY}(x,y)$ in entire R^2 is

$R_{XY} = \{(x,y) : f_{XY} > 0\}$ {Prob. is +ve)

→ $$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1}$$

Q. Let X and Y be two jointly continuous r.v. with joint pdf:

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

a.) Find c

b.) Find $P\left(0 \le X \le \frac{1}{2},\ 0 \le Y \le \frac{1}{2}\right)$

c.) $P(Y < X)$

d.) Marginal pdf of X and Y

e.) $P\left(Y \le \frac{X}{4} \mid Y \le \frac{X}{2}\right)$

f.) $P(Y = X) = \underline{\underline{0}}$

Marginal pdf's

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \quad \forall x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \quad \forall y$$

---

a.) $$\int_0^1\int_0^1 (x + cy^2)\,dx\,dy = 1$$

$$\int_0^1 \frac{1}{2} + cy^2\,dy = \frac{1}{2} + c\left[\frac{1}{3}\right] = 1$$

$$\Rightarrow \frac{c}{3} = \frac{1}{2} \Rightarrow \boxed{c = \frac{3}{2}}$$

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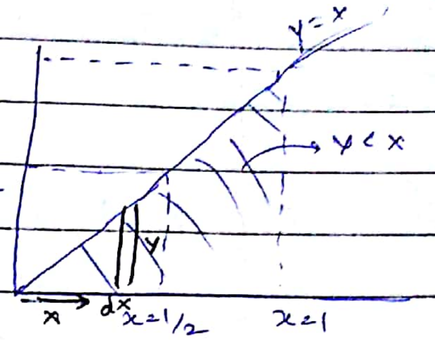
$R_{XY} = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$

b) $\displaystyle\int_0^{1/2}\int_0^{1/2} \left(x + \frac{3}{2}y^2\right) dx\, dy$



$\Rightarrow \displaystyle\int_0^{1/2} \left(\frac{1}{8} + \frac{3}{4}y^2\right) dy$

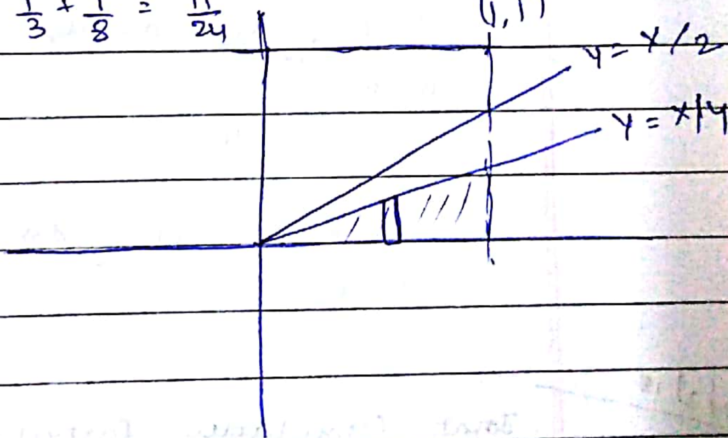
$= \frac{1}{8}\left(\frac{1}{2}\right) + \frac{3}{4}\left(\frac{1}{3} * \frac{1}{8}\right)$

$= \frac{1}{16} + \frac{1}{32} = \frac{3}{32}$

c) $P(Y < X)$

$\Rightarrow \displaystyle\int_0^1\int_0^x \left(x + \frac{3}{2}y^2\right) dy\, dx = \int_0^1 \left(x^2 + \frac{3}{2}\frac{x^3}{3}\right) dx = \frac{1}{3} + \frac{1}{2} * \frac{1}{4}$

$= \frac{1}{3} + \frac{1}{8} = \frac{11}{24}$



e) $P\left(Y \le \frac{X}{4} \,\middle|\, Y \le \frac{X}{2}\right)$

$= \dfrac{P\left(Y \le \frac{X}{4} \cap Y \le \frac{X}{2}\right)}{P\left(Y \le \frac{X}{2}\right)}$

$= \dfrac{P(Y \le X/4)}{P(Y \le X/2)}$

* here, $P(Y \le X/4) \equiv P(Y < X/4)$

because '=' doesn't contribute.

$P(Y \le X/4) = \displaystyle\int_0^1\int_0^{x/4} \left(x + \frac{3}{2}y^2\right) dy\, dx$

$P(Y \le X/2) = \displaystyle\int_0^1\int_0^{x/2} \left(x + \frac{3}{2}y^2\right) dy\, dx$

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$$= \int_0^1 \left\{ \left(xy + \frac{1}{2} y^3 \right) \Big|_0^{x/2} \right\} dx$$

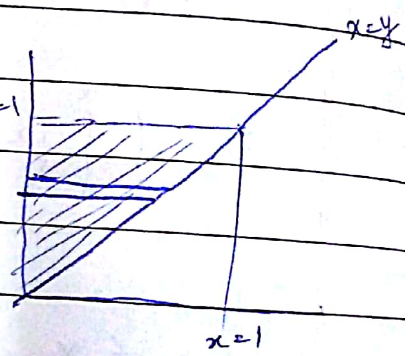
d.) $$f_X(x) = \int_0^1 \left(x + \frac{3}{2} y^2 \right) dy$$

$$= x + \frac{1}{2}$$

$$f_Y(y) = \int_0^1 \left(x + \frac{3}{2} y^2 \right) dx$$

$$= \frac{1}{2} + \frac{3}{2} y^2$$

→ $P(Y > X)$



$$\int_0^1 \int_0^x f_{XY}(x,y)\, dx\, dy$$

$$\int_0^1 \int_x^1 f_{XY}(x,y)\, dy\, dx$$

13/3/18

Joint Cumulative Distribution Func^n (CDF)

Joint CDF of 2 r.v. X & Y is defined as

$$F_{XY}(a,b) = P\{X \le a, Y \le b\}$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{b} f_{XY}(u,v)\, du\, dv$$

$$f_{XY} = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

→ $F_{XY}(\infty, \infty) = 1$

→ $F_{XY}(a, -\infty) = F_{XY}(-\infty, b) = 0$

Marginal CDF :-

$F_X(x) = F_{XY}(x, \infty)$ for any x

$F_Y(y) = F_{XY}(\infty, y)$ for any y

→ If X & Y are independent, then

$F_{XY}(x, y) = F_X(x) \cdot F_Y(y)$

Eg. Let X and Y be 2 independent uniform (0,1) random variables. Calculate Joint cdf and pdf

$$f_X(x) = \begin{cases} \frac{1}{\beta - \alpha} & x \in (\alpha, \beta) \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} \frac{a-\alpha}{\beta-\alpha} & \alpha < a < \beta \\ 0 & a \leq \alpha \\ 1 & a \geq \beta \end{cases}$$

Here, $\alpha = 0$, $\beta = 1$

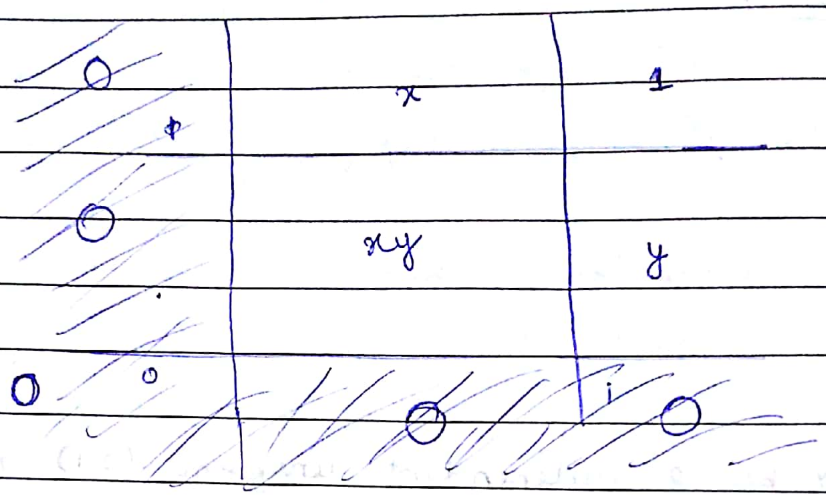
On uniform (0,1)

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

Independent →

$F_{XY}(x,y) = F_X(x) \cdot F_Y(y)$



or

$$F_{XY}(x,y) = \begin{cases} 0 & x \le 0 \wedge y \le 0 \\ x & 0 < x \cancel{\le} 1,\ y \ge 1 \\ y & x \ge 1,\ 0 < y < 1 \quad \rightarrow \text{depends on } y \\ xy & 0 < x, y < 1 \\ 1 & x \ge 1,\ y \ge 1 \end{cases}$$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

$$f_{XY} = \begin{cases} 1 & (x,y) \in [0,1]^2 \\ 0 & \text{Otherwise} \end{cases}$$

→ From def^n

$$f_X(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{Otherwise} \end{cases}$$

$$f_Y(x) = \begin{cases} 1 & y \in (0,1) \\ 0 & \text{Otherwise} \end{cases}$$

Again, we'll get same answer (by taking product)

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$\boxed{F_Y(y) = F_{XY}(\infty, y)}$ → to get Marginal cdf

Q. (1) $P(X+Y \le 1)$

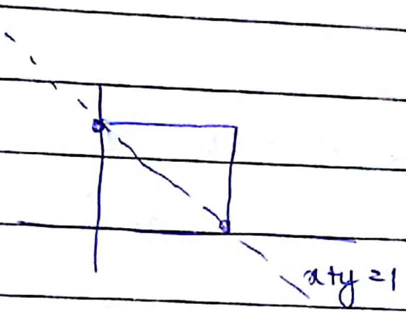
(2) $F_{XYZ}(x,y,z)$

(3) $f_{XYZ}(x,y,z)$

(4) $P(X+Y+Z \le 1)$

(5) $P(X_1 + X_2 + \dots + X_n \le 1)$

(1) : $\int_0^1 \left(\int_0^{1-x} dy \right) dx$



(2) $F_{XYZ}(x,y,z) = F_X(x) \cdot F_Y(y) \cdot F_Z(z)$

(3) $= f_x(x) \cdot f_y(y) \cdot f_z(z)$ $\begin{cases} 1 & (x,y,z) \in [0,1]^3 \\ 0 & \text{Otherwise} \end{cases}$

(4) $\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz$

(5) Similar like above.

Q. $f_{XY}(x,y) = \begin{cases} x + \frac{3}{2} y^2 & 0 \le x, y \le 1 \\ 0 & \text{Otherwise} \end{cases}$

$F_{XY}(x,y) = ?$

$R_{XY} = \{ (x,y) : 0 \le x \le 1, 0 \le y \le 1 \}$

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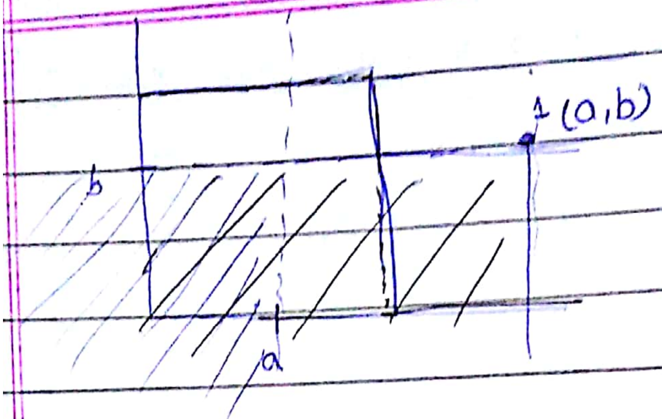
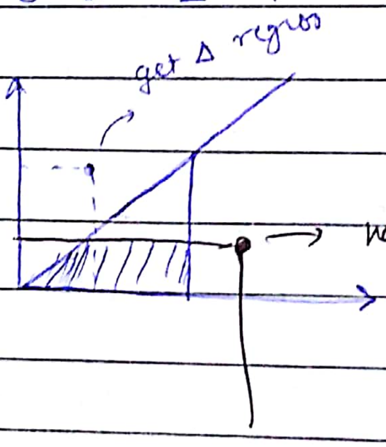


→ $a > 1, \quad b < 1$

If it's a Δ



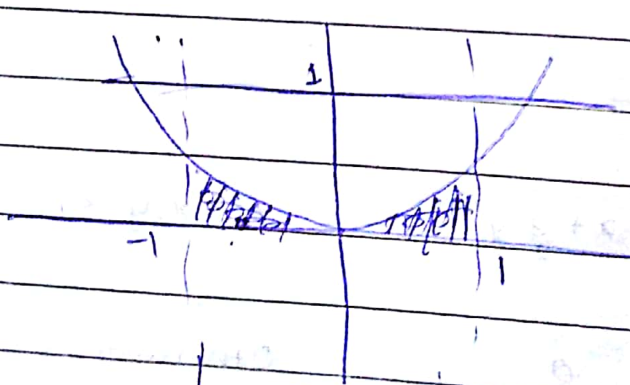
→ have to do double integration

$$f_{xy}(x,y) = \begin{cases} 2 & 0 < y < x,\ 0 < x \\ 0 \end{cases}$$

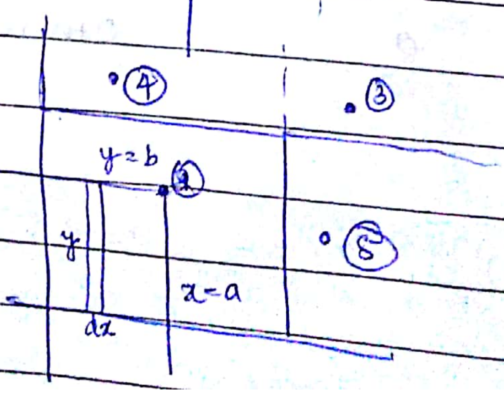
eg. $\begin{cases} x^2 y & y < x^2 < 1 \\ 0 & \text{otherwise} \end{cases}$ → Parabola

$x^2 < 1$

$y < 1$



In eg.



→ ② :

$$= \int_0^x \left(\int_0^y \left(x + \frac{3}{2} y^2\right) dy \right) dx$$

$$= \frac{1}{2} x^2 y + \frac{1}{2} x y^3$$

→ ③ : $\int_0^1 \int_0^1 \text{——} \; dy \; dx = 1$

→ ④ : $\int_0^x \left(\int_0^1 dy \right) dx$ ⊙

$$= \frac{1}{2} x^2 + \frac{1}{2} x \qquad (y = 1)$$

→ ⑤ $\int_0^1 \left(\int_0^y (\;) dy \right) dx$

$$= \frac{1}{2} y + \frac{1}{2} y^3 \qquad (x = 1)$$

$$\Rightarrow F_{XY}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ 1 & x \geq 1 \;\&\; y \geq 1 \\ \frac{1}{2}(x^2 y + x y^3) & (x,y) \in [0,1]^2 \\ \frac{1}{2}(x^2 + x) & 0 < x \leq 1, \; y \geq 1 \\ \frac{1}{2}(y + y^3) & x > 1, \; 0 < y < 1 \end{cases}$$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

14-3-18

→ If range of a r.v. is atmost countable ⇒ discrete r.v.

Functions of Random Vectors

$X : \Omega \to \mathbb{R}$

$Y : \Omega \to \mathbb{R}$

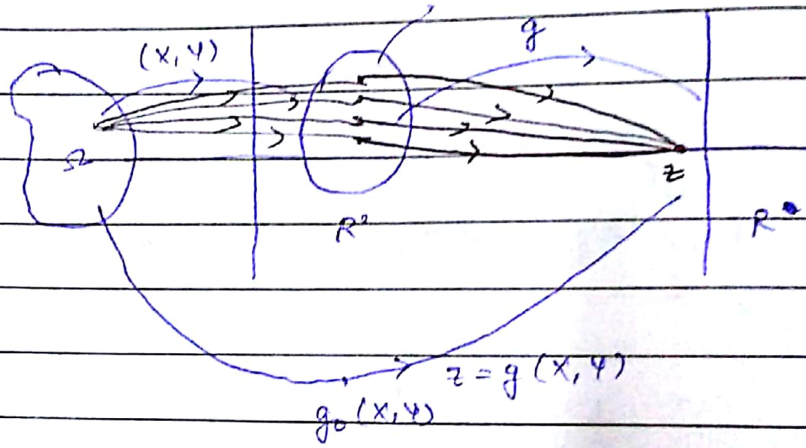
$(X, Y) \to$ random vector

X & Y : both discrete variables (ranges of r.v. X & Y ~~are atmost countable~~)

Suppose $g : \mathbb{R}^2 \to \mathbb{R}$

$g(X, Y)$

only interested in points which are in range of X & Y



$Z = g(X, Y)$

X, Y : take atmost countable values → g can also take atmost countable value

⇒ Z : discrete r.v.

Q. What is the pmf of r.v. Z ?

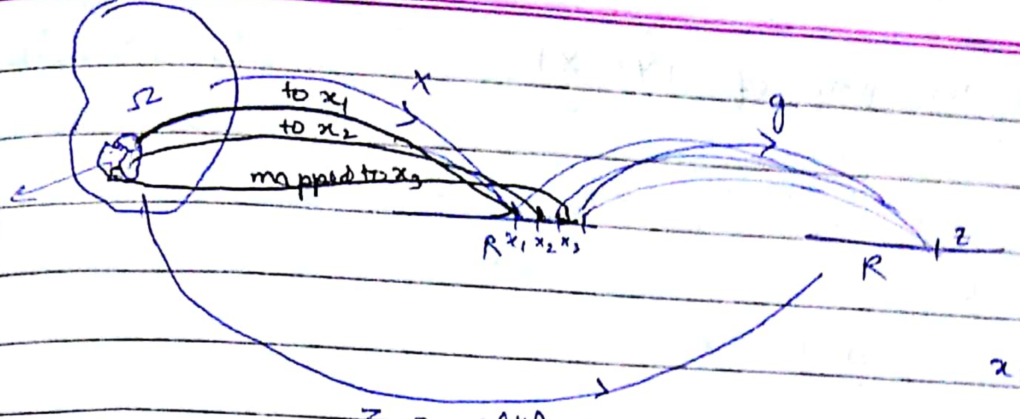
Let $X : \Omega \to \mathbb{R}$ be r.v.

$g : \mathbb{R} \to \mathbb{R}$

$Z = g(X)$

$$\boxed{P(Z = z) = \sum_{x : g(x) = z} P(X = x)}$$ (For single r.v.)

Proof :



(x_1, x_2, x_3 are mapped to z)

$x_1, x_2, x_3 \in$ Range (X)

$Z = g(X)$

$$\{Z = z\} = \{\omega : Z(\omega) = z\}$$

$$= \bigcup_{x : g(x) = z} \{X = x\}$$

$$\Rightarrow P\{Z = z\} = P\left(\bigcup_{x : g(x) = z} \{X = x\}\right) \quad (x : \text{uncountable})$$

$$= \sum_{x : g(x) = z} P\{X = x\}$$

In original problem :

$$P(Z = z) = P\left(\bigcup_{(x,y) : g(x,y) = z} \{X = x, Y = y\}\right)$$

$$\boxed{P(Z = z) = \sum_{(x,y) : g(x,y) = z} f(x,y)}$$

$f(x,y)$ → joint pmf of r.v. X & Y

Ex. Let X & Y be two r.v. having joint pmf given as :

X \ Y	-1	0	2	6
-2	$1/9$ ^{3}	$1/27$ ^{1}	$1/27$ ^{1}	$1/9$ ^{3}
1	$2/9$ ^{6}	0	$1/9$ ^{3}	$1/9$ ^{3}
3	0	0	$1/9$ ^{3}	$4/27$

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find the pmf of $|Y-X|$

Sol^n $g(x,y) = |y-x|$

have to find pmf of Z s.t.

$Z = \cancel{g}\ |Y-X|$

$x=-2$

$\cancel{R_Z = \{1\}}$

$X=-2 \Rightarrow \cancel{Z} \ |y-(-2)| = |y+2|$

$\Rightarrow 1, 2, 4, 8$

$X = 1 \Rightarrow |y-1|$

$\Rightarrow 2, 1, 1, 5$

$X = 3 \Rightarrow |y-3|$

$\Rightarrow 4, 3, 1, 3$

$R_Z = \{1, 2, 3, 4, 5, 8\}$

$\rightarrow P(Z=1) = \cancel{\Phi(-3,5)} \displaystyle\sum_{(x,y):|y-x|=1} f(x,y)$

$= f(-2,-1) + f(1,0) + f(1,2) + f(3,2)$

$= \frac{1}{9} + 0 + \frac{1}{9} + \frac{1}{9} = \boxed{\frac{1}{3}}$

$\rightarrow P(Z=2) = f(-2,0) + f(1,-1)$

$= \frac{1}{27} + \frac{3\times 2}{3\times 9} = \frac{7}{27}$

$\rightarrow P(Z=3) = f(3,0) + f(3,6)$

$= 0 + \frac{4}{27} = \frac{4}{27}$

$3+3+4+5+3$

$P(Z=4)$: $(-2,2)+(3,-1)$

$9+7+4+3+3$

$$f(Z) = \begin{cases} 1/3 & Z=1 \\ 7/27 & Z=2 \\ 4/27 & Z=3 \\ 1/27 & Z=4 \\ 1/9 & Z=5 \\ 1/9 & Z=8 \end{cases}$$

Eg. let X & Y be r.v. with joint pmf

$X \backslash Y$	1	2	...	i	...	N
1	$1/N^2$	$1/N^2$		$1/N^2$		$1/N^2$
2	$1/N^2$	...		...		
⋮	⋮	⋮				
j	⋮	⋮				
⋮						
N						

Find the pmf of $\min\{X, Y\}$

$g(x,y) = \min\{X, Y\}$

$\{(i,j) \; ; \; i = 1,2,\ldots,n \quad j = 1,2,\ldots,n\}$

$\min(i,j)$

$R_Z : \{1, 2, 3, \ldots, n\}$

~~$P\{Z = z\}$~~ :

Fix some $i \in \{1, 2, \ldots, N\}$

$$P\{Z=i\} = \sum_{(x,y):\, \min(x,y)=i} f(x,y)$$

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$x \to i$ to $N \Rightarrow (N-i+1)$

$\Rightarrow y \to i$ to $N \Rightarrow (N-i+1)$

$x \geq i \;,\; y \geq i$

$x = i \Rightarrow y = i, \ldots\ldots, N \Rightarrow N-i+1$

$y = i \Rightarrow x = i+1, \ldots\ldots, N \Rightarrow N-i$

~~delete case $(x=i, y=i)$~~

Add $\cancel{2(N-i+1)}-1$

$\Rightarrow 2(N-i)+1$

$$P(Z=i) = \sum_{y=i}^{N} f(i,y) + \sum_{x=i+1}^{N} f(x,i)$$

$$= (N-i+1)\frac{1}{N^2} + (N-i)\frac{1}{N^2}$$

$$\boxed{P(Z=i) = (2N-2i+1)\frac{1}{N^2}}$$

$$\sum = 1 \qquad \sum_{i=1}^{N} \frac{2N-2i+1}{N^2}$$

$$\Rightarrow \sum_{i=1}^{N} \frac{2}{N} - \frac{2i}{N^2} + \frac{1}{N^2}$$

$$= \left(\frac{2N}{N^2} + \frac{1}{N^2}\right)N - \frac{\cancel{2}}{N^2}\frac{N(N+1)}{\cancel{2}}$$

$$= \frac{2N^2+N-N^2-N}{N^2}$$

$$= \boxed{1}$$

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(learn)
* Ex. Show that the sum of two independent Poisson r.v with parameter $\{\lambda \ \& \ \mu\}$ is a Poisson r.v. with parameter $(\lambda + \mu)$

X is Poission (λ)
Y is Poission (μ)
prove : $Z = X + Y$ is Poission $(\lambda + \mu)$

Soln : * Earlier joint pmf was given. Here, it's not given

Using independence,

$$f(x, y) = f_X(x) \cdot f_Y(y)$$

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$P(Y = k) = \frac{\mu^k}{k!} e^{-\mu} \quad \text{for } k = 0, 1, \ldots$$

$$f(x, y) = P(X = x, Y = y)$$

6/3/18

$$Z = X + Y$$

Fix
$x = 0$ $\quad Z = 0 + y \Rightarrow z = 0, 1, 2, 3 \ldots$
$x = 1$ $\quad Z = 1 + y \Rightarrow z = 1, 2, 3 \ldots$
$x = 2$ $\quad Z = 2 + y \Rightarrow z = 2, 3, \ldots$
$x =$

$R_Z = \{0, 1, 2, \ldots\}$ (Max of above found)

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Fix $k \in \{0,1,2,\ldots\}$

$$P(Z=k) = \sum_{(x,y):\,x+y=k} f(x;y)$$

$$= \sum_{(x,y):\,x+y=k} f_X(x)\cdot f_Y(y)$$

$x \in \{0,1,2,\ldots\}$

$y \in \{0,1,2,\ldots\}$

$\Rightarrow \quad y = k-i \qquad i = 0,1,2,\ldots k$

$$= \sum_{i=0}^{k} f_X(X=i)\, f_Y(Y=k-i)$$

$$= \sum_{i=0}^{k} \frac{\mu^i e^{-\mu}}{i!}\cdot\frac{\lambda^{k-i} e^{-\lambda}}{(k-i)!}$$

$$= e^{-(\mu+\lambda)} \sum_{i=0}^{k} \frac{\mu^i \lambda^{k-i}}{(i)!\,(k-i)!} * \frac{k!}{k!}$$

$$= \frac{e^{-(\mu+\lambda)}}{k!} \sum_{i=0}^{k} {}^kC_i\, \mu^i \lambda^{k-i}$$

$$P(Z=k) = \frac{e^{-(\mu+\lambda)}}{k!}(\mu+\lambda)^k$$

* $P(Z)$ is poission with parameter $(\mu+\lambda)$.

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range isn't countable ~~#~~ $\Rightarrow$ wrong def^n

continuous

Suppose X and Y have joint pdf as $f(x,y)$

$g: \mathbb{R}^2 \to \mathbb{R}$ be any func^n.

Then $Z = g(X, Y)$

we wish to determine ~~a~~ pdf of Z.

Q. if X is a r.v. having pdf and $g: \mathbb{R}^2 \to \mathbb{R}$ be any func^n then can we say that $Z = g(X)$ ~~is~~ necessarily has a pdf?

~~Nord Kis kross~~ :- A r.v. which takes values over an interval is cont. r.v. Moreover, it has a pdf $\longrightarrow$ Wrong def^n

$\hookrightarrow$ should have a pdf to be cont. r.v.

Eg. Let $X \sim \exp(5)$

$$g(x) = \min\{x, 10\} \quad \forall \; x \in \mathbb{R}$$

$$= \begin{cases} x & x \le 10 \\ 10 & x > 10 \end{cases}$$

$$Z = g(X)$$

$$F_Z(z) = P\{Z \le z\}$$

Let $z \in \mathbb{R}$ be given

$$\{Z \le z\} = \{\omega \in \Omega \mid Z(\omega) \le z\}$$

$Z(\omega) \le z \iff$ either $X(\omega) \le z$

$\Updownarrow$ or

$\min(X(\omega), 10) \le z$ $\quad 10 \le z$ $\quad$ ~~union~~

or

both

subset of Ω

$$\Rightarrow \{Z \le z\} = \{X \le z\} \cup \{\omega : 10 \le z\}$$

for 2nd term $\Rightarrow$

if $10 \le z \Rightarrow \{10 \le z\} = \Omega$

if $10 > z \Rightarrow \Omega = \phi$ ($\{10 \le z\}$)

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$$\{Z \le z\} = \begin{cases} \{X \le x\} & z < 10 \\ \Omega & z \ge 10 \end{cases}$$

$$F_Z(z) = \begin{cases} P\{X \le z\} = F_X(z) & z < 10 \\ P(\Omega) = 1 & z \ge 10 \end{cases}$$

$$F_X(z) = \begin{cases} 1 - e^{-5z} & ; \ z \ge 0 \\ 0 & ; \ z < 0 \end{cases}$$

$$\Rightarrow F_Z(z) = \begin{cases} 1 - e^{-5z} & 0 \le z < 10 \\ 0 & z < 0 \\ 1 & z \ge 10 \end{cases}$$

to get $f(z)$, diff. wrt z

$$f_Z(z) = \begin{cases} 5e^{-5z} & 0 \le z < 10 \\ 0 & \text{otherwise} \end{cases}$$

→ non-negative

$$\int_0^{10} 5e^{-5z}\,dz = \frac{5}{-5}\left[e^{-5z}\right]\Big|_0^{10} = 1 - e^{-50} \underline{< 1}$$

⇓

we are getting discrete ~~valu~~ pmf here.

* If a r.v. has pdf, its distribution func^n is continuous on R

* If $F_X(x)$ is not continuous on R ⇒ r.v. X doesn't have a pdf.

→ Here, at $z = 10$, $F_Z(z)$ is not cont.

$\Downarrow$

Z is not a r.v. having pdf

→ X is exponential func^n

→ $R_X = [0, \infty)$

$Z = \min\{X, 10\}$

$\Rightarrow R_Z = [0, 10]$

Z takes a value ~~of~~ over interval but it doesn't have pdf

<u>Example</u> : Let X and Y be r.v. having joint pdf $f(x,y)$. Find the pdf of $X+Y$ (if it exists)

<u>Solution</u> : Let $Z = X + Y$

If $F_Z(\;)$ can be written as

$$F_Z(z) = \int_{-\infty}^{z} g(t)\,dt \quad \forall\ z \in \mathbb{R}$$

then $g(t)$ is the pdf. of Z.

Let $z \in \mathbb{R}$ be given

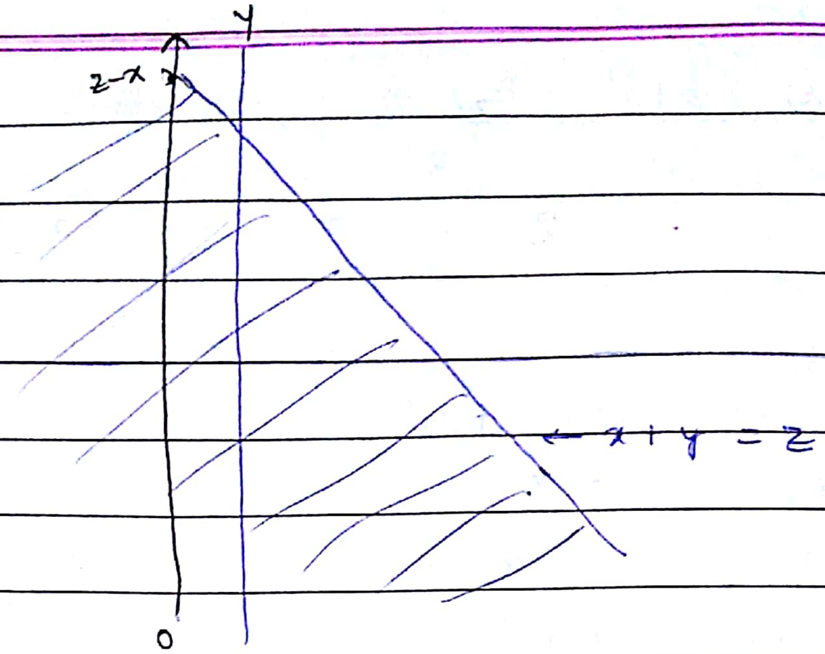
$F_Z(z) = P\{X + Y \le z\}$

$$\{X + Y \le z\} = \{\omega \in \Omega \mid X(\omega) + Y(\omega) \le z\}$$

$$= \{\omega \in \Omega \mid (\underset{X(\omega)}{X}, \underset{Y(\omega)}{Y}) \in A_z\}$$

where $A_z = \{(x,y) \in \mathbb{R}^2 \mid x + y \le z\}$

$$P(X+Y \le z) = P((X,Y) \in A_z) = \iint_{A_z} f(x,y)\,dx\,dy$$



$x+y=z$
$y=$ ...

Z : positive

$$F_Z(z) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-x} f(x,y)\,dy\right)dx$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z} f(x, s-x)\,ds\right)dx \qquad y = s-x$$

$$= \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty} f(x, s-x)\,dx\right)ds$$

$$\boxed{f_Z(s) = \int_{-\infty}^{\infty} f(x, s-x)\,dx}$$

$Z : \Omega \to \mathbb{R}$ (1-D)

that's why we diff. ~~just once to get $f_Z(z)$.~~

17/3/18

Example (Tutorial) The joint density func^n of X & Y is given by :

$$f(x,y) = \begin{cases} e^{-(x+y)} & ; 0 < x, y < \infty \\ 0 & ; \text{otherwise} \end{cases}$$ → 1st Quadrant

Find the density of r.v. $\frac{X}{Y}$.

$$g(x,y) = \frac{x}{y}$$

$$Z = \frac{X}{Y}$$

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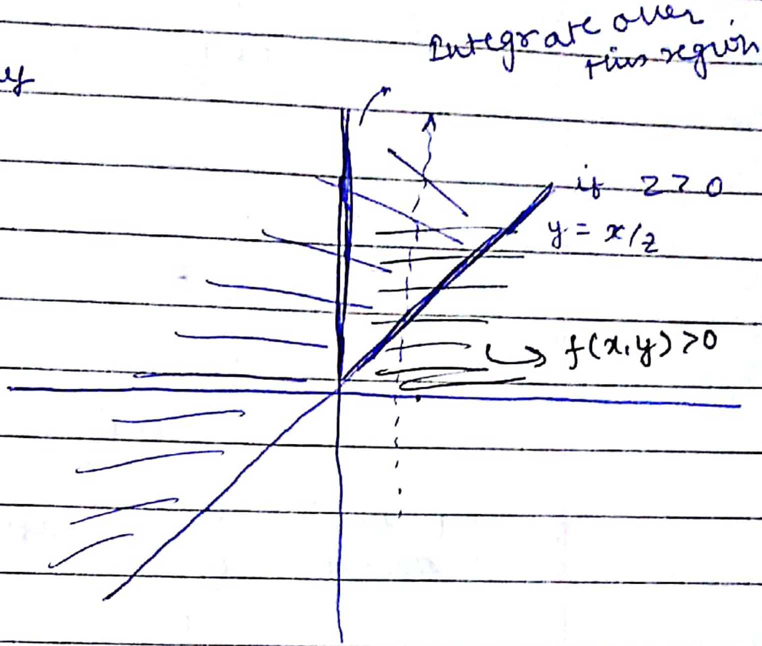
$F_Z(z) = P\{Z \le z\} = P\{(X,Y) \in A_z\}$

where $A_z = \{(x,y) \in \mathbb{R}^2 \mid x/y \le z\}$

~~$\{(x,y) \in \mathbb{R}^2 \mid x \le zy\}$~~

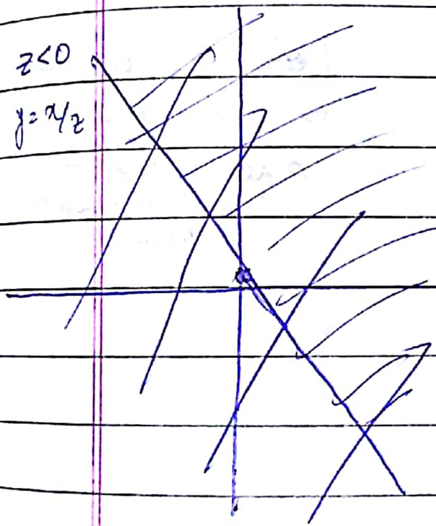
$F_Z(z) = \iint_{A_z} f(x,y)\,dx\,dy$

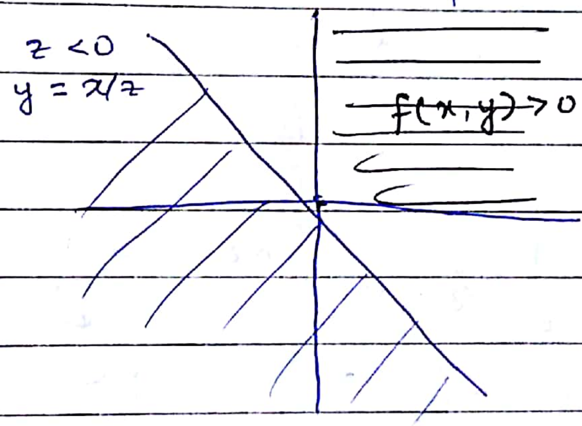


Let $z \in \mathbb{R}$ be given

$\frac{x}{y} \le z$

$\boxed{y \ge \frac{x}{z}}$





$\Rightarrow$ won't consider this case

If $z < 0$ then $A_z \cap \{x>0, y>0\} = \phi$

Therefore $F_Z(z) = 0$

if $z \geq 0$

$$F_Z(z) = \int_0^\infty \int_{x/z}^\infty f(x,y)\,dy\,dx$$

$$= \int_0^\infty \int_{x/z}^\infty \left(e^{-(x+y)}\,dy\right)dx = \int_0^\infty e^{-x}\left[e^{-y}\right]_{\infty}^{x/z} dx$$

$$= \int^\infty e^{-x}\left[e^{-x/z}\right] dx = \int_0^\infty e^{-x(1+\frac{1}{z})}\,dx$$

Teacher's Signature..................

→ cdf are right continuous always.
Just 'ue to check for left continuity

$$= \left\{ \left(\frac{e^{-x(1+1/z)}}{1+\frac{1}{z}} \right) \right|_{\infty}^{0}$$

$$= \frac{1}{1+\frac{1}{z}} = \boxed{\frac{z}{1+z}}$$

$$F_Z(z) = \begin{cases} 0 & ; \quad z < 0 \\ \frac{z}{1+z} & ; \quad z \geq 0 \end{cases}$$

→ continuous everywhere ~~rational~~. (check at z=0)
⇓
can get pdf

LHL (z=0) = 0
f(z)(z=0) = 0 } → left hand continuous.

$$f_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{(z+1)^2} & z \geq 0 \end{cases}$$

(not necessary to put equality)
(don't have to check differentiability here)

Check $\int_0^\infty \frac{1}{(z+1)^2} dz = \underline{\underline{1}}$ ✓ z : Real no.

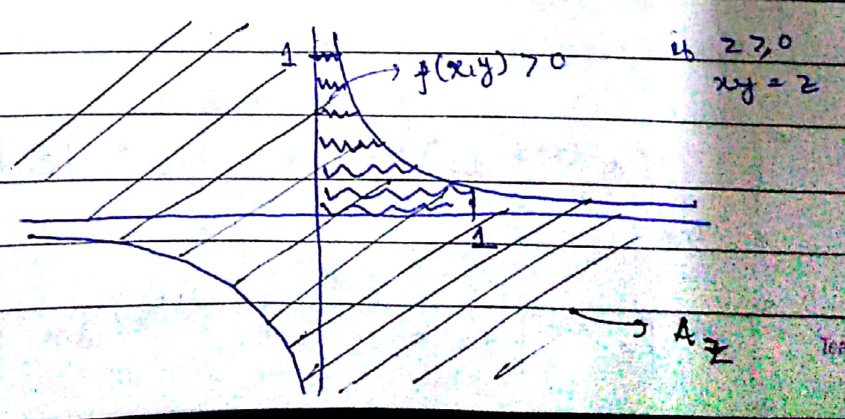
$$-\left(\frac{1}{1+z}\right)\Big|_0^\infty = 1 ✓$$

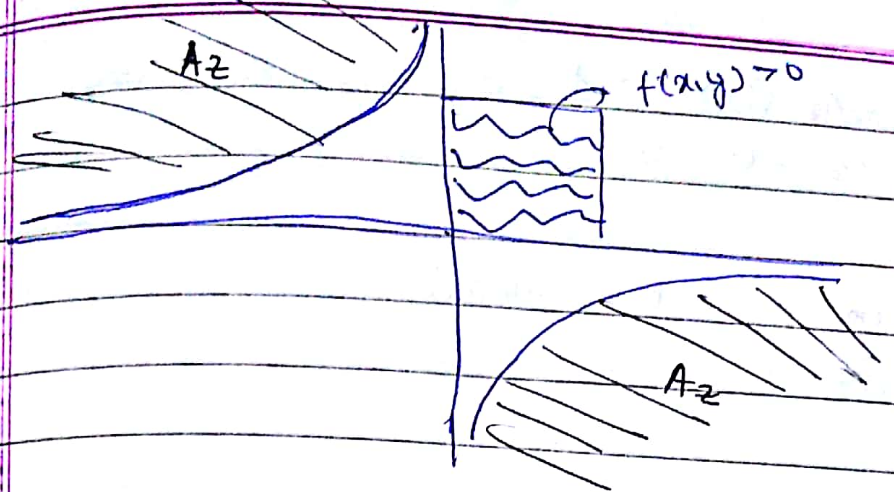
example : Let X and Y be 2 independent uniform (0, 1) r.v. Find pdf of XY if it exists.

soln : $A_z : \{ (x,y) \in \mathbb{R}^2 \mid xy \leq z \}$

$Z = XY$

$F_Z(z) = P\{(X,Y) \in A_z\}$





< 0
$xy = z$

$\Rightarrow$ again if $z < 0$
$F_Z(z) = 0$

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

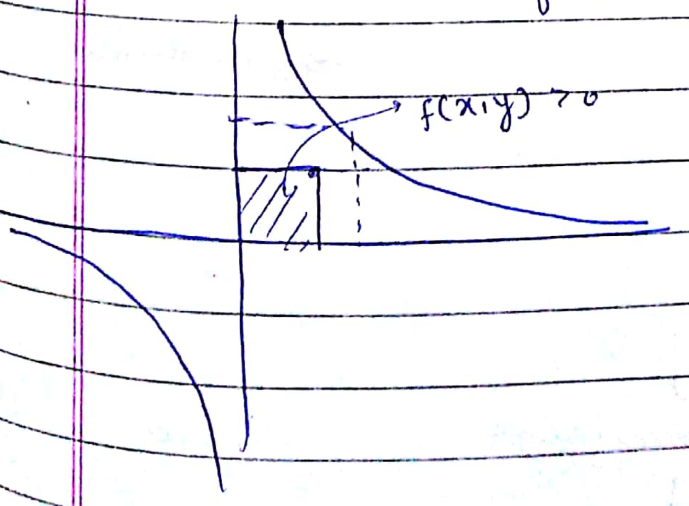
$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Since X & Y are independent

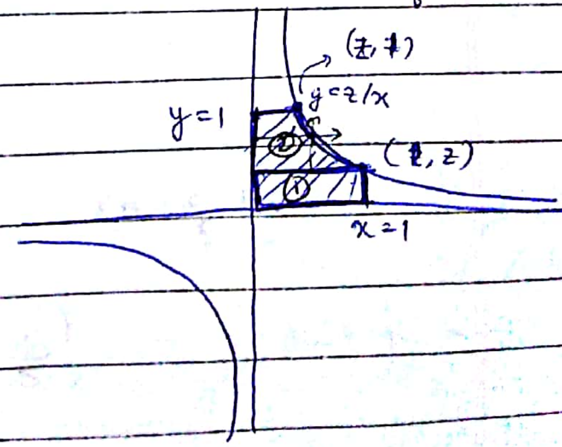
$$f_{XY}(x,y) = \begin{cases} 1 & 0 < x,y < 1 \\ 0 & \text{Otherwise} \end{cases}$$

$\rightarrow$ if $z > 0$

if $z \geq 1$



if $0 < z < 1$



$D = (0,a1) \times (0,1)$

If $z < 0$ then $A_z \cap D = \phi$

Hence, $F_Z(z) = 0$

If $z \geq 1$ then $A_z \supset D$, Hence,

$F_Z(z) = 1$

If $0 \leq z < 1$ then,

$f_X = 1 \Rightarrow$ just calculating area of region

$$= \int_0^1 \int_0^z f(x,y)\, dy\, dx + \int_z^1 \int_0^{z/y} 1 \cdot dx\, dy$$

① ②

$$= z + \int_z^1 \frac{z}{y} dy$$

$$= z + z \ln y \Big|_z^1$$

$$= z - z \ln z$$

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ z - z\ln z & 0 \leq z < 1 \\ 1 & z \geq 1 \end{cases}$$

cont. at 0 & 1

$$\Rightarrow f_Z(z) = \begin{cases} 1 - \ln z - 1 = -\ln z & 0 < z < 1 \\ 0 & \text{Otherwise} \end{cases}$$

check $\int_0^1 -\ln z \, dz = 1$

② ①

$$- \left[\ln z \cdot z - \int \frac{1}{z} \cdot z\, dz \right]$$

$$= z - z\ln z \Big|_0^1 = 1$$

Teacher's Signature

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Expectation of ~~real-valued~~ function of two Random Variables

$g(X, Y) : \Omega \to \mathbb{R}$

- Let $X : \Omega \to R$ be discrete r.v. then

$$E[X] = \sum_{x \in R_X} x P(X = x) \quad -①$$

(discrete r.v. can take ~~inf~~ countably infinite values → becomes " set)

provided $\sum_{x \in R_x} |x| P[X = x] < \infty \quad -②$

→ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$: converges but $\sum_{n=1}^{\infty} \frac{1^n}{n}$: diverges

→ If ① converges & ② diverges, if you calculate sum in ① in diff. order ⇒ ① may get divergent

but if ② is convergent ⇒ any order in which ① is calculated ① as we always get convergent

→ If ① : finite ② : non-definite ⇒ Expec^n not defined

-- continued

& $g : R \to R$, then

$$E[g(X)] = \sum_{x \in R_X} g(x) P(X = x)$$

provided $\sum_{x \in R_X} |g(x)| P(X = x) < \infty$

- let $X : \Omega \to R$ be r.v. with pdf $f(x)$. Then,

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

provided $\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$

↳ Improper Integral

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Theorem (I) Let X and Y be 2 discrete r.v. with joint pmf $f(x,y)$. Let $g: \mathbb{R}^2 \to \mathbb{R}$ be any func^n. Then,

$$E[g(X,Y)] = \sum_{x \in R_X} \sum_{y \in R_Y} g(x,y) f(x,y)$$

provided $\sum_{x \in R_X} \sum_{y \in R_Y} |g(x,y)| f(x,y) < \infty$

(II) Let X and Y be 2 continuous r.v. with joint pdf $f(x,y)$. Let $g: \mathbb{R}^2 \to \mathbb{R}$ be Borel func^n. Then,

(Piecewise cont., cont., cont. except across some smooth curve)

very large class (whatever func^n we may think of)

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy$$

provided $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x,y)| f(x,y)\, dy\, dx < \infty$

* If X & Y : discrete r.v. : we can find pmf of $g(X,Y)$

If X & Y : cont. r.v. : we may (may not find pdf of g

But still, we can talk about its Exp^n value without talking of pdf.

Eg. From 1-D case :

$X \sim \exp(5)$

$g(x) = \min\{x, 10\}$

We've shown that $g(x)$ doesn't have pdf. But still, by Th

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$[0,10]$

$$= \int_0^{\infty} g(x) f_X(x)\, dx \quad f_X(x) = 5e^{-5x}$$

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$$\int_0^{10} x \; 5e^{-5x}\,dx + \int_{10}^{\infty} 10 \times 5 \times e^{-5x}\,dx$$

$$= 5\left[x\frac{e^{-5x}}{-5} - \int \frac{e^{-5x}}{-5}\,dx \right]\Bigg|_0^{10} + 50\,\frac{e^{-5x}}{-5}\Bigg|_{10}^{\infty}$$

$$= \overset{-1}{\cancel{5}}\left[xe^{-5x} + \frac{e^{-5x}}{5} \right]\Bigg|_0^{10} \cancel{+ 5\,50} - 10\left[- e^{-50} \right]$$

$=$

* For discrete, we may 1st find pmf & then find expec^n. But it's not needed.

Ex.

$x \backslash y$	-1	0	2	$\cancel{8}\,6$
-2	$1/9$	$1/27$	$1/27$	$1/9$
1	$2/9$	0	$1/9$	$1/9$
3	0	0	$1/9$	$4/27$

Find $E[Z]$ s.t. $Z = |Y - X|$

$P(Z=1) = \frac{1}{3}$ $\quad P(Z=2) = \frac{7}{27}$ $\quad P(Z=3) = \frac{4}{27}$

$P(Z=4) = \frac{1}{27}$ $\quad P(Z=5) = \frac{1}{9}$ $\quad P(Z=8) = \frac{8}{9}$

1st $\longrightarrow$ $E[Z] = \sum_{z \in R_Z} z\,P(Z=z) = \frac{1}{3} + \frac{14}{27} + \overset{10/9}{\frac{12}{27}} + \frac{4}{27} + \overset{14/9}{\frac{5}{9}} + \frac{8}{9}$

$= \boxed{\frac{26}{9}}$

2nd $\longrightarrow$ from th^m : $E[|Y-X|] = \sum_{x \in R_x} \sum_{y \in R_y} |y-x|\, f(x,y)$ $\quad$ finite sum

$= \sum_{y \in R_y} |y+2|\, f(-2,y) + \sum_{y \in R_y} |y-1|\, f(1,y) + \sum_{y \in R_y} |y-3|\, f(3,y)$

$= \left\{ 1\left(\frac{1}{9}\right) + (2)\left(\frac{1}{27}\right) + (4)\left(\frac{1}{27}\right) + (8)\left(\frac{1}{9}\right) \right\} + \left\{ -2 \times \frac{2}{9} + 1 \times \frac{1}{9} + 5 \times \frac{1}{9} \right\}$

$+ \left\{ -1 \times \frac{1}{9} + 3 \times \frac{4}{27} \right\} \quad = \boxed{\frac{26}{9}}$

Example Let X and Y be independent and identically distributed (exponential (λ)) r.v. Find the mean of max. {X, Y}

$X \sim \exp(\lambda)$ &

$Y \sim \exp(\lambda)$

① can find pdf & then find mean

② By Theorem

$\therefore$ X & Y are independent

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max\{x,y\}\, f_X(x)\, f_Y(y)\, dx\, dy$$

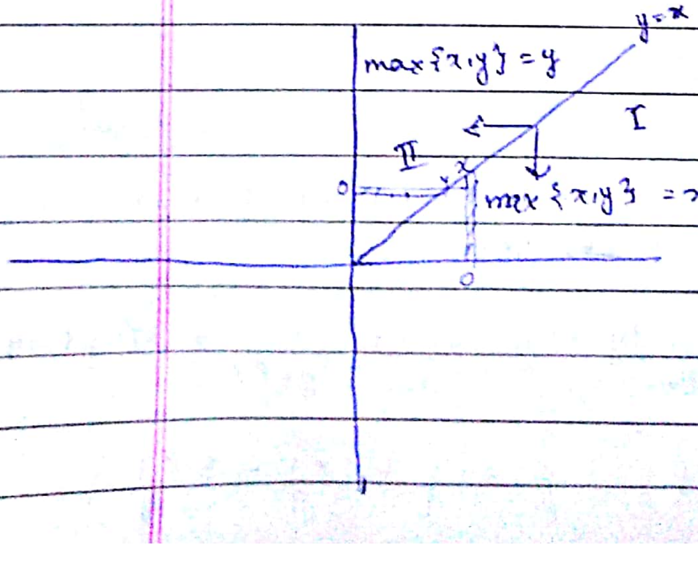
$$f_X(x) = \begin{cases} 0 & x<0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases} \qquad f_Y(y) = \begin{cases} 0 & y<0 \\ \lambda e^{-\lambda y} & y \geq 0 \end{cases}$$

Product will be non-zero in 1st quadrant.

$$f_{XY}(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & x,y \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$\Rightarrow E[g(X,Y)] = \int_0^{\infty}\int_0^{\infty} \underbrace{\max\{x,y\}}\ \lambda^2 e^{-\lambda(x+y)}\, dx\, dy$$

$x>0$ & $y>0 \Rightarrow$ always non-negative, don't have to worry about absolute value.



$$= \int_0^{\infty}\left(\int_0^{x} x\, \lambda^2 e^{-\lambda(x+y)}\, dy\right) dx \ + \quad \text{I}$$

$$\int_0^{\infty}\left(\int_0^{y} \lambda^2 y\, e^{-\lambda(x+y)}\, \cancel{dy}\, dx\right) dy \quad \text{II}$$

$$= \lambda^2 \left[\int_0^\infty x e^{-\lambda x} \left(\int_0^x e^{-\lambda y} dy \right) dx \right]$$

$$= \lambda^2 \left[\int_0^\infty x e^{-\lambda x} \left(\frac{e^{-\lambda y}}{-\lambda} \right) \Big|_0^x \right] = +\lambda^2 \left[\int_0^\infty x e^{-\lambda x} (1 - e^{-\lambda x})\, dx \right]$$

$$I \to \quad = \frac{3}{4\lambda}$$

$$II \to \quad = \frac{3}{4\lambda}$$

$$\Rightarrow \boxed{I = \frac{3}{2\lambda}}$$

if it d.n.e. $\Rightarrow$ have to go back to thm.

Verify :- By 1st finding pdf & then finding $E[g(X,Y)]$

$$Z = \max\{X, Y\}$$

$$\{Z \le z\} = \{X \le z \cap Y \le z\}$$

both x & y have to be less than equal to z.

$$F_Z(z) = P\{X \le z \cap Y \le z\}$$

$$= F_X(z) \cdot F_Y(z) \quad [\text{Independent}]$$

$$F_X(z) = \begin{cases} 0 & z < 0 \\ 1 - e^{-\lambda z} & z \ge 0 \end{cases} \qquad F_Y(z) = \begin{cases} 0 & z < 0 \\ 1 - e^{-\lambda z} & z \ge 0 \end{cases}$$

$$F_Z(z) = \begin{cases} (1 - e^{-\lambda z})^2 & z \ge 0 \\ 0 & z < 0 \end{cases}$$ Left hand continuous at $z = 0$

$$\Rightarrow f_Z(z) = \begin{cases} 2(1 - e^{-\lambda z})\lambda e^{-\lambda z} & z \ge 0 \\ 0 & z < 0 \end{cases}$$

$\ge 0 \Rightarrow$ don't bother about absolute value as func^n is already positive

$$E[Z] = \int_0^\infty z f(z)\, dz = \int_0^\infty z\, 2\lambda e^{-\lambda z} (1 - e^{-\lambda z})\, dz$$

$$= \boxed{\frac{3}{2}\lambda}$$

Conditional Distributions :-

$P(A|B) = \dfrac{P(A\cap B)}{P(B)}$ provided $P(B) > 0$

Conditioning of 1 r.v. w.r.t. another r.v.

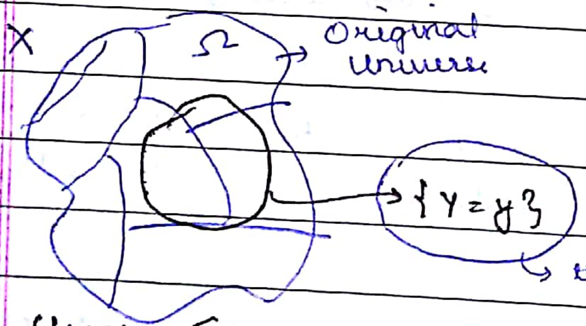
Def^n : Let X and Y be 2 discrete r.v. with the joint pmf $f(x,y)$. Then the conditional pmf of X given $Y=y$ is defined as :

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} & ; \text{ if } f_Y(y) \neq 0 \\ 0 & ; f_Y(y) = 0 \end{cases}$$

→ $f(x,y) = P(X=x, Y=y)$

$f_Y(y) = P(Y=y)$

$f_{X|Y}(x|y) = P(X=x \mid Y=y)$



It's just like simple pmf in new universe

⇒ Now, universe is changed which is determined by conditional event.

this is the modified sample space

→ check : $\sum pmf = 1$

Fix $y \in R_Y$ s.t. $P(Y=y) > 0$

Then

$$\sum_{x\in R_X} f_X(x|y) = \sum_{x \in R_X} P(X=x \mid Y=y)$$

$$P(\Omega \mid Y=y) = P\left(\bigcup_{x\in R_X} \{X=x \mid Y=y\}\right) \quad [3^{rd} \text{ Axiom}]$$

ranging over all R_X

$$\frac{P(\Omega \cap \{Y=y\})}{P(Y=y)} = \frac{P(Y=y)}{P(Y=y)} = 1$$

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→ conditional Prob. also satisfies all 3 axioms of Probability func^n.
↓
is also a probability func^n in new universe

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Example: let the joint pmf of X & Y is given by:

X \ Y	-1	0	1
-1	0	1/4	0
0	1/4	0	1/4
1	0	1/4	0

compute the conditional pmf of X given Y = 0

sol^n

$$f_{X|Y}(x|0) = \begin{cases} \dfrac{f(x,0)}{f_Y(0)} & f_Y(0) \neq 0 \\ 0 & f_Y(0) = 0 \end{cases}$$

$$f_Y(0) = P(Y=0) = \sum_{x \in R_X} f(x,0) = \frac{1}{4} + 0 + \frac{1}{4} = \frac{1}{2}$$

$$f_{X|Y}(-1|0) = \frac{f(-1,0)}{1/2} = \frac{1/4}{1/2} = \frac{1}{2} = f_{XY}(1)$$

$$f_{X|Y}(0|0) = 0$$

$$\Rightarrow f_{X|Y}(x|0) = \begin{cases} 1/2 & x = \pm 1 \\ 0 & x = 0 \end{cases}$$

⇒ check sum = 1
⇓
it's a pmf.

Condition distribution func^n:

$$F_X(x) = P(X \le x) = \sum_{t \in R_X;\, t \le x} f_X(t)$$

$$F_{X|Y}(x|y) = P(X \le x \mid Y \le y)$$
$$= \sum_{t \in R_X;\, t \le x} f_{X|Y}(t|y)$$

⇒ For same eg. (above), CDF Conditional Distribution func^n of X given Y = 0 is:

$$F_{X|Y}(x|0) = \begin{cases} 0 & x < -1 \\ 1/2 & -1 \le x < 0 \\ 1/2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Teacher's Signature..............

→ All ~~four~~ prop. of cdf are satisfied by cont. cdf

1) It is non-decreasing

2) $\lim_{x \to -\infty} cdf = 0$

3) $\lim_{x \to \infty} cdf = 1$

4) It is right cont.

~~(getting unconditional pmf using conditional pmf)~~

Fact: X, Y : 2 discrete r.v. with conditional pmf $f_{X|Y}(x|y)$

$$f_X(x) = \sum_y f(x,y)$$

$$= \sum_y f_{X|Y}(x|y) \cdot f_Y(y)$$

Example: Suppose $f_Y(y) = \begin{cases} 5/6 & y = 10^2 \\ 1/6 & y = 10^4 \end{cases}$

$$f_{X|Y}(x|10^2) = \begin{cases} 1/2 & x = 10^{-2} \\ 1/3 & x = 10^{-1} \\ 1/6 & x = 1 \end{cases}$$

$$f_{X|Y}(x|10^4) = \begin{cases} 1/2 & x = 1 \\ 1/3 & x = 10^1 \\ 1/6 & x = 100 \end{cases}$$

Compute pmf of X

sol^n

$R_X = \{10^{-2}, 10^{-1}, 1, 10, 10^2\}$

$f_X(1) = f_{X|Y}(1|10^2) \frac{5}{6}$ $\frac{1}{6} \times \frac{5}{6} + \frac{1 \times 3}{2 \times 3} \times \frac{1}{6} = \frac{8}{36} = \frac{2}{9}$

$$f_X(x) = \begin{cases} 5/12 & x = 10^{-2} \\ 5/18 & x = 10^{-1} \\ 2/9 & x = 1 \\ 1/18 & x = 10 \\ 1/36 & x = 100 \end{cases}$$

Continuous :

Conditional Density :

let X and Y be 2 r.v. with joint pdf $f(x,y)$. The conditional density of X given $Y=y$ is defined as :

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} & f_Y(y) \neq 0 \\ 0 & f_Y(y) = 0 \end{cases}$$ (same as in discrete)

* If 2 r.v. have joint pdf, then they do've their own pdf.
* If y is continuous r.v.
  $\Rightarrow P\{Y=y\} = 0 \quad \forall \; y \in \mathbb{R}$
* Here, $(f_Y(y) \neq P\{Y=y\})$

If $f_Y(y) > 0$, then

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = 1$$ (should be equal to 1)

integrating w.r.t x

$$\int_{-\infty}^{\infty} \frac{f(x,y)\,dx}{f_Y(y)} = \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{f_Y(y)}{f_Y(y)} = \underline{1}$$

→ $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

If f_X is the pdf of r.v. X then for any 'Boral' subset A of $\mathbb{R}$

$$P(X \in A) = \int_A f_X(x)\,dx$$

Definition $\quad P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx$

$P(Y=y) = 0$ but if we can integrate, we'll get answer.

** "conditional density func"

→ It allows to define conditional prob. even when the conditioned event has prob. 0.

Teacher's Signature..................

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