The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-II ■ Assignment 4

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1. Determine which of the following are linear transformations $T : V \to W$, where the vector spaces V,W are given:
   $(a) V = W = \mathbb{R}^3;\ T(x, y, z) = (2x + y, z, |x|)$
   $(b) V = W = M_2(\mathbb{R})$, the space of all 2×2 real matrices; (i) $T(A) = A^t$, (ii) $T(A) = I + A$, (iii) $T(A) = A^2$, (iv) $T(A) = BAB^{-1}$, where B is some fixed 2×2 matrix.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1, 0, 0) = (1, 0, 0), T(1, 1, 0) = (1, 1, 1)$ and $T(1, 1, 1) = (1, 1, 0)$. Find (a) $T(x, y, z)$, (b) $\ker(T)$, (c) $R(T)$. Also show that $T^3 = T$.

3. Find all linear transformations from $\mathbb{R}^n \to \mathbb{R}$.

4. Let C be an $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by C. Show that the matrix of T with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is C.

5. Determine $N(T)$ and $R(T)$ for each of the following linear transformations:
   a) $T : P_2 \to P_3,\ T(f)(x) = xf(x)$
   b) $T : P_4 \to P_3,\ T(f)(x) = f'(x)$.

6. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T(x_1, x_2, ..., x_n) = (x_2 + x_3, x_3, ..., x_n, 0)$ then write down the matrix of T w.r.t. the standard basis of $\mathbb{R}^n$.

7. Does there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that $Range(T) = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 0\}$?

8. Let V be a vector space of dimension n and let $A = \{v_1, ..., v_n\}$ be an ordered basis of V . Suppose $w_1, ..., w_n \in V$ and let $(a_{1j}, ..., a_{nj})^t$ denote the coordinates of w_j with respect to A. Put $C = [a_{ij}]$. Then show that $w_1, ..., w_n$ is a basis of V if and only if C is invertible.

9. Let T be a linear transformation from an n dimensional vector space V to an m dimensional vector space W and let C be the matrix of T with respect to a basis A of V and B of W. Show that
   (a) $rank(T) = rank(C)$;
   (b) T is one-one if and only if $rank(C) = n$;
   (c) T is onto if and only if $rank(C) = m$;

10. Let $<, >$ be any inner product on $\mathbb{R}^n$. Show that $< x, y >= x^t Ay$ for all vectors $x, y \in \mathbb{R}^n$ where A is the symmetric $n \times n$ matrix whose (i, j) th entry is $< e_i, e_j >$, the vector e_i being the standard basis vectors of $\mathbb{R}^n$.

11. Show that the norm of a vector in a vector space V has the following three properties
(a) $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.
(b) $\|\lambda v\| = |\lambda|\|v\|$ for all $\lambda \in \mathbb{R}$ and $v \in V$.
(c) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.
Note that $\| \cdot \| =< \cdot, \cdot >^{1/2}$, where $< \cdot, \cdot >$ is an inner product defined on V.

12. Use Gram-Schmidt process to transform each of the following into an orthonormal basis; (a) $\{(1,1,1), (1,0,1), (0,1,2)\}$ for $\mathbb{R}^3$ with dot product. (b) Same set as in (i) but using the inner product defined by $< (x, y, z), (x', y', z') >= xx' + 2yy' + 3zz'$.

13. Let $T : V \to V$ be a linear map such that $Ker(T) = Range(T)$. What can you say about T^2. On $\mathbb{R}^2$ can you give example of such a map?

14. Let U be a proper subspace of the inner product space V. Let $U^\perp = \{v \in V :< v, u >= 0, \ \forall u \in U\}$. Show that $U^\perp$ is a subspace of V (it is called orthogonal complement of U). Let $U = \alpha(1, 2, 3) : \alpha \in \mathbb{R}$ be a subspace of $\mathbb{R}^3$ with scalar product. Find $U^\perp$. Also, show that $S^\perp$ is a subspace of V for any arbitrary subset S of V.