Matrices

* Elementary Row operations on a matrix A:

(1) Multiplication of one row of a matrix A by a non-zero constant. The resultant matrix we get is called Equivalent matrix.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \xleftarrow{\times 3} \quad B = \begin{bmatrix} 3 & 6 \\ 3 & 1 \end{bmatrix}$ = eqv. matrix

(2) Addition of a constant multiple of one row to another row

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \end{matrix} — R_2 \leftarrow R_2 - 3R_1 \longrightarrow \sim \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$

(3) Interchange of any 2 rows:

Ex: $R_2 \leftrightarrow R_1 \Rightarrow \begin{bmatrix} 0 & -5 \\ 1 & 2 \end{bmatrix}$

Q $x_1 + 2x_2 = 1$
$3x_1 + x_2 = 0$

Sol^n: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad x_2 = X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$[A \mid B] = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & 0 \end{array}\right] \quad R_2 \leftarrow R_2 - 3R_1$

$\sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & -3 \end{array}\right] \quad R_1 \leftarrow R_1 + \frac{2}{5} R_2$

$x_1 + 2x_2 = 1$
$0x_1 - 5x_2 = -3$

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$$\sim \left[\begin{array}{cc|c} 1 & 0 & -1/5 \\ 0 & -5 & -3 \end{array}\right] \overset{-\frac{1}{5}R_2}{\sim} \left[\begin{array}{cc|c} 1 & 0 & +1/5 \\ 0 & 1 & +3/5 \end{array}\right]$$

$\therefore$,

$$x_1 + 0x_2 = -1/5 \Rightarrow x_1 = -1/5$$
$$0x_1 - 5x_2 = -3 \Rightarrow x_2 = 3/5$$

* Row Equivalent:

If A and B are $m \times n$ matrices over the field F. We say that B is a row equivalent to A if B can be obtained from A by a finite sequence of elementary row operations

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

* Echelon form:

(1) All non-zero rows are one above any row of all zeroes.

(2) Each leading entry of a row is in a column to the right of the leading entry of the row above it

(3) All entities in a column below a leading entity are zero.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{\text{Rule (1)}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ leading entries

this should appear in right of leading entry ...

Ex: $A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

Sol^n: $R_1 \longleftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \quad \begin{array}{l} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 + 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 8 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \quad \begin{array}{l} R_3 \leftarrow R_3 - \frac{5}{2} R_2 \\ \\ R_4 \leftarrow R_4 + \frac{3}{2} R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & \cancel{5/2}\,0 & 0 & 0 \\ 0 & 0 & \cancel{5}0 & -5 & 0 \end{bmatrix} \quad R_3 \longleftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{Echelon form}$$

* Row Reduce Echelon form:

If a matrix in Echelon form satisfies the following additional conditions, then it is said to be raw reduce Echelon form.

inverse

ve

First ③ rules of echelon form remains same

(4) Leading entry in each non-zero row is one.

(5) Each leading '1' is the only ~~o~~non-zero entry in its

In above example:

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \leftarrow R_1 - 4R_2$$

$$\sim \begin{bmatrix} 1 & \cancel{4}0 & -3 & 3 & 5 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow R_1 - 3R_3 \\ R_2 \leftarrow R_2 + 3R_3 \end{matrix}$$

→ Pivot Element

$$\sim \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→ Pivot column = $\{1, 2, 4\}$

* System of Linear Equations:

A linear system of ~~eq~~ m eq^n in n known $x_1, x_2, \ldots, x_n$ etc. of the form

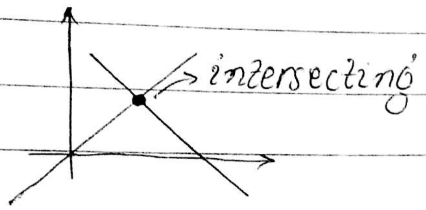
$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \quad \Bigg\} \text{——} (1)$$

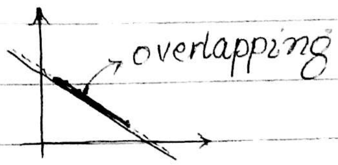
where,

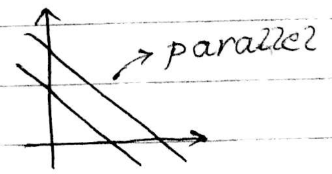
$a_{11}, \ldots, a_{mn}$ are called the coeff. of the sys.

$b_1, b_2, \ldots, b_n$ are given no

Q ① $\left.\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 0 \end{aligned}\right\}$ Q ② $\left.\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6 \end{aligned}\right\}$ Q ③ $\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned}$







One solution, Infinite sol^n → Consistent System

No solution → Inconsistent System

- Homogenous linear System: $(AX = 0)$
  If all b_j are zero
  Ex: $x_1 + 2x_2 = 0$
  $2x_1 + 3x_2 = 0$

- Non-homogenous linear System: $(AX = B)$
  If atleast one of the b_j is non-zero.
  Ex: $x_1 + 2x_2 + x_3 = 0$
  $x_1 - x_2 + x_3 = 0$
  $2x_1 + 3x_3 = 1$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Rank of a matrix: $\rho(A)$
No. of non-zero rows in Echelon form of a matrix A.

- Augmented matrix:

$$[A|B] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{m1} & b_1 \\ a_{21} & & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Ex1: $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Pivot

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ \boxed{1} & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right] \longrightarrow \text{GAUSS ELIMINATION METHOD}$$

To be eliminated

$R_2 \leftarrow R_2 - R_1$

$x_1 + x_2 = 2$

$-2x_2 = -2 \Rightarrow x_2 = 1$ and $x_1 = 1$.

Pivot

Q2: $[A|B] = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ \boxed{2} & 4 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$

eliminate

$R_2 \leftarrow R_2 - 2R_1$ $\qquad x_1 + 2x_2 = 3$

Pivot

Q3: $[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ \boxed{1} & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & -1 \end{array}\right]$

eliminate

$R_2 \leftarrow R_2 - R_1$ $\qquad x_1 + x_2 = 2$

$-0 = -1$

- Unique sol^n:

$$\boxed{\rho(A) = \rho(A|B) = n}$$ $\rightarrow$ no. of unknowns

- Infinite sol^n:

$$\boxed{\rho(A) = \rho(A|B) < n}$$

- No Sol^n:

$$\boxed{\rho(A) \neq \rho(A|B)}$$

<u>Ex:</u> $x_1 + x_2 = 2$

$x_1 + x_2 = k$

Q No. of values of k for which sol^n exist:

(A) 1 (B) 2 (C) 3 (D) DNE

Pivot

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ \boxed{1} & 1 & k \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & k-2 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$x_1 + x_2 = 2$

$0 = k - 2$

$\therefore$ If $k = 2$, sol^n exist and infinitely many sol^n

If $k \neq 2$, No sol^n

Hence, Ans = (B) = 2

* GAUSS ELIMINATION METHOD:

<u>Ex:</u> $x_1 + x_2 + x_3 = 2$

$x_1 + x_2 + x_3 = 2$

$2x_2 + x_3 = 1$

$x_1 - 2x_3 = 0$

Pivot

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ \boxed{1} & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ \boxed{1} & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & -1 & -3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & \boxed{-1} & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(2^{nd}) Pivot

$R_2 \leftarrow R_2 - R_1$

$R_4 \leftarrow R_4 - R_1$

$\because$ no non-zero pivot we interchange to get non-zero

$R_3 \leftarrow R_3 + \frac{R_2}{2}$

{ identity property }

inverse

Methods

Gauss elimin^n → Echelon form

Gauss-Jordan elimin^n → Row-reduced Echelon form.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -5/2 & -3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, Back Substitution,

$$\left.\begin{array}{l} -5/2\, x_3 = -3/2 \\ 2x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \end{array}\right\} \begin{array}{l} x_3 = 3/5 \\ x_2 = 1/5 \\ x_1 = 6/5 \end{array}$$

If asked, we can further apply, (instead of back su

* GAUSS – JORDAN ELIMINATION METHOD:

$R_2 \leftarrow R_2 \times \frac{1}{2}$, $R_3 \leftarrow R_3 \times \left(\frac{-2}{5}\right)$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 1 & 3/5 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 \leftarrow R_2 - \frac{1}{2}R_3 \\ R_1 \leftarrow R_1 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 3/5 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \leftarrow R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 3/5 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x_1 \\ \to x_2 \\ \to x_3 \end{array}$$

Sol^n: $x_1 = 6/5$, $x_2 = 1/5$, $x_3 = 3/5$.

- S+8   ...ions holds..

3.

* Vector Space:

Operation

(P1) * Mathematical Structure: $\langle F,+\rangle, \langle F,-\rangle, \langle F,\cdot\rangle$
Clouse on F: if $a,b \in F$

addition: $a+b \in F$
multiplication: $a \times b \in F$

Ex: $\langle N, +\rangle$ — ①
$\langle N, -\rangle$ — ②
$\langle Z, -\rangle$ — ③

(P2) * Associative: on F for $a,b,c \in F$
Addition $a+(b+c) = (a+b)+c$ (Yes, NO, NO)
multiplication: $a\cdot(b\cdot c) = (a\cdot b)\cdot c$

(P3) * Existence of Identity:
an element e is said to be identity if:
Addition: $a+e = e+a = a$ (NO, NO, NO)
Multiplication: $a\cdot e = e\cdot a = a$

(P4) * Existence of inverse:
in F if $\forall a \ \exists \ a^{-1}$ s.t.
Addition: $a + a^{-1} = e = a^{-1} + a$
Multiplication: $aa^{-1} = e = a^{-1}a$

Ex. $(R, +)$ satisfy all above properties.
$(R-\{0\}, *)$ " " " "

* Group: is a mathematical structures $\langle F, * \rangle$ that satisfies P_1, P_2, P_3 and P_4

Ex: $(Z, +)$ ✓

$(Z, *)$ ×

$(R, +)$ ✓

$(R-\{0\}, *)$ ✓

$(Q, +)$ ✓

$(C-\{0\}, *)$ ✓

$(IR, +)$ × —— $(1-\sqrt{2}) + \sqrt{2} = 1$ — not even P1

$(IR, *)$ × —— $\sqrt{2} * \sqrt{2} = 2$

(P5) * Commutative:

for any $a, b \in F$

Addition: $a + b = b + a$

Multiplication: $a * b = b * a$

* Field: $\langle F, +, \cdot \rangle$

① $(F, +)$ commutative group

② $(F-\{0\}, \cdot)$ commutative group

③ Distributive: $a \cdot (b+c) = ab + ac$ where $a, b, c \in F$

Ex (a) $(R, +, \cdot)$ ✓

(b) $(C, +, \cdot)$ ✓

(c) $(Z, +, \cdot)$ × — inverse (P4) holds false $\therefore$ not field

(d) $(Q, +, \cdot)$ ✓ (To justify in exam show only P_4 is false rather showing P_1, P_2, P_3 holds true & P_4 false)

∘ Subspace:

~~Set~~ If subset of vector space satisfies the properties, then it's called subspace

$R^2 = \{(a,b\}; a,b \in R\}$
$R^n = \{(a_1, a_2, ---, a_n) : a_i \in R\}$

* Vector Space:

Def# Let V be a non-empty set with two operation

Vector addition: This assigns to any $\alpha, \beta \in V$, then a sum $\alpha + \beta \in V$

scalar multiplication: This assigns to any $\alpha \in V$, $a \in F$ then a product $\alpha a \in V$
field

then V is called a vector space over the field F if for any $\alpha, \beta, \gamma, \in V$ and $a, b \in F$, following are true:

[A1] $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ = associative   V, F = 2 sets

[A2] There is vector 0 in V s.t.
$\alpha + 0 = 0 + \alpha = \alpha \quad \forall \alpha \in V$ = identity

[A3] For each $\alpha \in V$, $\exists (-\alpha) \in V$ s.t.
$\alpha + (-\alpha) = (-\alpha) + \alpha = 0$

[A4] $\alpha + \beta = \beta + \alpha$

[M1] $a(\alpha + \beta) = a\alpha + a\beta$, $a \in F$, $\alpha, \beta \in V$

[M2] $(a+b)\alpha = \alpha a + \alpha b$

[M3] $(ab)\alpha = a(b\alpha)$

[M4] $1 \cdot \alpha = \alpha$, $1 \in F$

Elements of V are said to be vectors.
Elements of F are said to be scalars.

To prove vector space, prove all prop.

Ex: $V = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \; a_{ij} \in R \right\}$ ↗ if additional condit^n $|A| \neq$ added, then its not va ∵ closer or identity itself

Check V is vector space over R?

Sol^n: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

(i) Closer:

$(A+B) = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ \underbrace{a_{21}+b_{21}}_{} & a_{22}+b_{22} \end{bmatrix} \in V$ ✓

all are real elements i.e $\in R$ ∴ $(A+B) \in V$

(ii) [A1] $(A+B)+C = \begin{bmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} \end{bmatrix}$ ∵ Real no. are associative

∴ $= \begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) \\ a_{21}+(b_{21}+c_{21}) & (a_{22}+(b_{22}+c_{22}) \end{bmatrix}$ associative ✓

(iii) [A2] = $I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

$A + I = I + A$

(iv) [A3] : $-A = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}$ ✓

∴ $A + (-A) = (-A) + A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(v) [A4] = $A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{bmatrix} = B+A$ ✓

∵ real entries are commutat

(vi) [M1] = $(a+b)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (a+b)a_{11} & (a+b)a_{12} \\ (a+b)a_{21} & (a+b)a_{22} \end{bmatrix}$

$= \begin{bmatrix} aa_{11}+ba_{11} & aa_{12}+ba_{12} \\ aa_{21}+ba_{21} & aa_{22}+ba_{22} \end{bmatrix} = \begin{bmatrix} aa_{11} & aa_{12} \\ aa_{21} & aa_{22} \end{bmatrix} + \begin{bmatrix} ba_{11} & ba_{12} \\ ba_{21} & ba_{22} \end{bmatrix}$

$= aA + bA$ ✓

Ex: (1) Set of all polynomial of degree two
$V = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in R, a_2 \neq 0\}$

Sol^n: Let $\alpha = 1 + 3x - 4x^2$
$\beta = 1 - 2x + 4x^2$
$\alpha + \beta = 2 + x$ = degree 1 polynomial
$\therefore$ not a vector space.

$= n$ ×
for any general n,
< 2 ✓ } vector
≤ 2 ✓ } space

(2) Set of all polynomial of degree ≤ 2
$V = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in R\}$ → 0 also $\in V$
$\because 0 = 0 + 0x + 0x^2$
Its a vector space.

(3) $V = \{(a,b) : a, b \in R\}$; $(0,0) \in V$
operations are defined in ordered pair as:
- addition $= (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$
- scalar multiplic^n $= (a_1, b_1) * (a_2, b_2) = (a_1 a_2, b_1 b_2)$

Check whether V is vector space?

(4) $V_1 = \{(a,b) : a, b \in R\}$
with
$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$
$a(a_1, b_1) = (aa_1, 0)$
Is V_1 a vector space?

Sol^n: identity $= 0$ ✓
Inverse ✓
[A1] to [M3] ✓
but [M4] $\Rightarrow 1 \cdot (a_1, b_1) = (1 \cdot a_1, 0) = (a_1, 0) \neq (a_1, b_1)$
$\therefore$ [M4] holds false.
Hence, not a vector space.

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(5) V_2 is same as (4) with
just addition as: $(a_1,b_1)+(a_2,b_2) = (a_1+b_1, a_2+b_2)$
and multiplication same
Is V_2 a vector space
Not a vector space.

(6) same as (4), with just mult. defined as:
$k(a,b) = (|k|a, |k|b)$

[M2] $(a+b)(a_1,b_1) = (|a+b|\,a_1, |a+b|\,b_1)$
now RHS, $a(a_1,b_1) + b(a_1,b_1) = (|a|+|b|\,a_1, |a|+|b|\,b_1)$
$\therefore$ not a vector space.

(7) $V = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} ; a_{11}, a_{12}, a_{21}, a_{22} \in R \right\}$
under the word addition & scalar multiplicatio

(8) ω_1 = {set of all matrices of order 2} $\subseteq V$
ω_1 is V over $a_{ij} \in R$ as well as $a_{ij} \in C$
(9) ω_2 = {collection of all skew-symmetric matrices of order 2}

(10) ω_3 = {collection of all triangular matrices of orders

(11) ω_4 = {collection of all hermitian matrices of order 2}
over $a_{ij} \in R$, ω_4 is vector space
over $a_{ij} \in R, C$, ω_4 is not vector space since scalar multiplication holds false as shown in below eg.

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- Hermitian matrix:

$$(\bar{A})^T = A$$

- Skew Hermitian matrix:

$$(\bar{A})^T = -A$$

These matrices are complex matrices only, and $(\bar{A})$ is conjugate of A.

Ex: $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \Rightarrow iA = \begin{bmatrix} i & i-1 \\ i+1 & 2i \end{bmatrix}$

$(\overline{iA}) = \begin{bmatrix} -i & -i-1 \\ -i+1 & -2i \end{bmatrix} \Rightarrow (\overline{iA})^T = \begin{bmatrix} -i & -i+1 \\ -i-1 & -2i \end{bmatrix} \neq iA$

* Subspace:

Let V be vector space over F, then any non-empty set ω is said to be a subspace if ω is also a vector space over F

Ex: $\omega_6 = \left\{ \begin{bmatrix} ai & b+di \\ -b+di & ci \end{bmatrix}, a, b, c, d \in R \right\}$

(1) Is ω_6 a subset of V?
   ω_6 is a subset of V.

(2) Is ω_6 ~~is~~ a vector space over R?

(3) Is ω_6 a vector space over ~~R~~ C?

If diagonal entries are 0 or completely imagina
can be skew hermitian matrix.

- Necessary & Sufficient condition for a su
to be a subspace.

$a\alpha + b\beta \in V \quad \forall\ \alpha, \beta \in W,\ a, b \in F$

Only check:

$\left.\begin{array}{l} \alpha + \beta \in W \\ a\alpha \in W \end{array}\right\} a\alpha + \beta \in W$

Ex: $i\underbrace{\begin{bmatrix} i & 1+i \\ -1+i & 2i \end{bmatrix}}_{A} = \begin{bmatrix} -1 & i-1 \\ -i-1 & -2 \end{bmatrix}$ × skew-herm matrix

∴ given matrix is subspace over R but not over C

Ex: ① $V = \{(a,b),\ a, b \in R\}$

$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2,\ b_1 + b_2)$

$a(a_1 + b_1) = (aa_1 + ab_1)$

✓ vector space.

② $\omega_1 = \{(a_1, 0) : a \in R\} \subseteq V$

∵ $(a_1, 0) + (a_2, 0) = (a_1 + a_2, 0)$

③ $\omega_2 = \{(a,b) ; a > 0\} \subseteq V$

On multiplying by negative real, the property f
false (Even identity) ∴ × subspace / vector space

④ $\omega_3 = \{(a,b) : b \geq 0\}$

$\omega_1 \subseteq \omega_3$

$c) = ab$

$2 \times \left(\frac{1}{2}\right)$

Date / /
Page

* Linear Combination:

Let $V(F)$ is a vector spce. Then, an $\alpha \in V$ is said to be a linear combination of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ if there exist some scalar $a_1, a_2, \ldots, a_n \in F$ such that
$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n$

<u>Ex:</u> $(1,4) \in V$

$S_1 = \{(1,0), (0,1)\}$

$(1,4) = (1,0) + 4(0,1)$, so its linear combinat^n

Now, if $S_2 = \{(1,1), (1,2)\}$

$(1,4) = x(1,1) + y(1,2) \Rightarrow \begin{array}{l} x+y=1 \\ x+2y=4 \end{array}$, $y = 3, x = -2$

$\therefore$ $(1,4)$ is a linear combin^n over S_2 also.

$S_3 = \{(-1,3), (2,-6)\}$

$(1,4) = x(-1,3) + y(2,-6)$

$\therefore$ $2y - x = 1$

$3x - 6y = 4$

$\Rightarrow \left[\begin{array}{cc|c} -1 & 2 & 1 \\ 3 & -6 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow$ NO sol^n

$\therefore$ $(1,4)$ can't be written in linear combin^n over S_3.

* Span (Linear Span):

S a non-empty set is said to linear span of V if every element of V can be written in the linear combination of element of S.
denoted by $L(S) = V$

$\therefore$ S_1 & S_2 are linear span of V but not S_3 (in above example).

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Subspace $\rightarrow \begin{matrix} a\alpha \in W \\ \alpha+\beta \in W \end{matrix} \Big\} \; a\alpha+\beta \in W$

$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n \mid a_1, \dots a_n \in F\}$
$\alpha_1, \dots \alpha_n \in S, \; S \subseteq V$

Ex: $R^2 = \{(a,b) : a, b \in R\}$
under additn & scalar multiplication is vector space

(i) $S_1 = \{(1,1), (2,3)\}$

$L(S) \overset{?}{=} R^2 \quad (1,5)$
$(x,y) \in R^2$

$a(1,1) + b(2,3) = (x,y) \Rightarrow \begin{matrix} a + 2b = x \\ a + 3b = y \end{matrix}$

$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & x \\ 1 & 3 & y \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & y-x \end{array}\right]$

$b = y - x$
$a = 3x - 2y$

$\therefore (x,y) = (3x-2y)(1,1) + (y-x)(2,3)$ = co-ordinate
$(x,y) = (1,0)$
co-ordinate $a = 3$
$b = -1$

(ii) $S_2 = \{(-1,2), (2,-4)\}$

$(x,y) = a(-1,2) + b(2,-4) \; : \; x = 2b - a$
$y = 2a - 4b$

$[A|B] \; \left[\begin{array}{cc|c} -1 & 2 & x \\ 2 & -4 & y \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 2 & x \\ 0 & 0 & y+2x \end{array}\right]$ No soln

* Linearly Dependent:

Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a non-empty subset of $V(F)$. Then elements of V are said to be linearly dependent if $\exists$ some scalar $a_1, a_2 \ldots a_n$ (not all zero) such that

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$$

Otherwise these vectors are said to be linearly independent

Ex: $a(1,1) + b(2,3) = (0,0) \;—\; S_1$

$$\left.\begin{aligned} a + 2b &= 0 \\ a + 3b &= 0 \end{aligned}\right\} a, b = 0$$

Ex: $S_2 = \{(-1,2), (2,-4)\}$

$a(-1,2) + b(2,-4) = 0$

$$\left.\begin{aligned} -a + 2b &= 0 \\ 2a - 4b &= 0 \end{aligned}\right\} \left.\begin{aligned} a &= 2b \\ a &= 2b \end{aligned}\right\}$$ infi. sol^n but $a = 2, b = 1$ such non-zero sol^n are possible.

α_2 is lin. comb^n of α_1 $\therefore$ $\alpha_2 = (2)\alpha_1$

* Basis:

Let $V(F)$ be a vector space over F & S is non-empty subset of V, then S is said to be a basis if:

(1) Elements of S are linearly indep.

(2) S span V

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$S_4((1,1,1),(0,0,0))$ — L.D. $\because$ 1 entire element is zero.

$S_1 = \{(1,1),(2,3\}$ — Indep. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$S_1 \to B$

$S_2 = \{(-1,2),(2,-4)\}$ — Dep. $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$

$\times$

$S_3 = \{(1,0),(0,1),(1,1)$ — Dep. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\times$

$S_4 = \{(1,1)\}$ — I

$S_5 = \{(0,0)\}$ — D
$S_6 = \{(1,0),(0,0)\}$ — D
} sub-space with (0,0) vector is D

- Dimension of vector space:
  No. of element in basis.

Q Whether collectⁿ: $R(R)$, $R(Q)$, ~~R~~ $Q(R)$ is a vector space

Ex: $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; a,b,c,d \in R \right\}$

$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(i) $S \subseteq V$

(ii) Linearly Ind

Q Find coordinates of vector (2,3,5) w.r.t $S_1 = \{(1,1,1),(0,1,1),(0,0,$

Solⁿ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow (a,b,c) = (-1,-2,5)$

$S_1 \subseteq V$ and S_1 is LI.

S_1 span V $\therefore$ S_1 is basis of V

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if Rank = no. of rows, its always invertible
: determinant is non-zero.

dimension = no. of element in basis.

Date / /
Page

Ex $S_1 = \{(1,0,0), (0,1,0)(0,0,1)(1,1,1)\}$ — L.D.

→ matrix form = ?

$$S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\therefore$ matrix = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ → echelon form

* Innert product Space:

Let F be a finite field of real numbers or the field of complex numbers, and V a vector space over F. An inner product on V is a function, which assigns to each order pair of vectors α, β in V a scalar $\langle \alpha, \beta \rangle$ in F such a way that for all α, β, γ in V and all scalar c in F.

○ ~~Linear~~ property:

[I] $\langle \alpha+\beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$

[II] $\langle c\alpha, \gamma \rangle = c\langle \alpha, \gamma \rangle$

linear property $\equiv \langle a\alpha+\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$

conjugate symmetry property

[III] $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$

[IV] $\langle \alpha, \alpha \rangle \geq 0$, (Non-negativity property)
and $\langle \alpha, \alpha \rangle = 0$ if and only if $\alpha = 0$

$\overline{(a_1+a_2)} = \bar{a}_1 + \bar{a}_2$

$\overline{(a_1 a_2)} = \bar{a}_1 \bar{a}_2$

$\overline{(\bar{a}_1)} = a_1$

$z\bar{z} = |z|^2$

$(\alpha|\beta) \equiv (\alpha, \beta)$

Associative

Vector Space $V(F)$ $\begin{cases} \alpha+\beta \in V \ ; \ \alpha,\beta \in V \\ \alpha\beta \in V \ ; \ \alpha,\beta \in V \end{cases}$

$\downarrow$ subspace (W) $\quad a\alpha+\beta \in W$

Basis ($S \subseteq V$)

L.I. $\qquad$ S span V

Syst. of homogenous eq^n $\qquad$ Syst. of non-homogenous eq^n

$\downarrow$

Row Echelon form

Q On $V_n(C)$ [→ complex no. field] there is an inner product which we call standard inner product if $\alpha = (a_1, a_2, \dots, a_n)$; $\beta = (b_1, b_2, \dots, b_n) \in V_n(C)$, then we defined $\gamma = (c_1, \dots, c_n)$;

$\langle \alpha, \beta \rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + a_3\bar{b}_3 + \dots + a_n\bar{b}_n = \sum a_i \bar{b}_i$

Sol^n: Linear: $k\alpha+\beta = k(a_1, a_2, \dots, a_n) + (b_1, \dots, b_n)$

$= (ka_1+b_1, \dots, ka_n+b_n)$

[I] $\langle k\alpha+\beta, \gamma \rangle = (ka_1+b_1)\bar{c}_1 + \dots + (ka_n+b_n)\bar{c}_n$

$= ka_1\bar{c}_1 + b_1\bar{c}_1 + \dots + ka_n\bar{c}_n + b_n\bar{c}_n$

$= k(a_1\bar{c}_1 + \dots + a_n\bar{c}_n) + (b_1\bar{c}_1 + \dots + b_n\bar{c}_n)$

$= k\langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$

[II] $\langle \beta, \alpha \rangle = (b_1\bar{a}_1 + b_2\bar{a}_2 + \dots + b_n\bar{a}_n)$

$\overline{\langle \beta, \alpha \rangle} = \bar{b}_1(\bar{\bar{a}}_1) + \dots + \bar{b}_n(\bar{\bar{a}}_n)$

$= \bar{b}_1 a_1 + \dots + \bar{b}_n a_n$

$= a_1\bar{b}_1 + \dots + a_n\bar{b}_n$; as complex no. commute

$= \langle \alpha, \beta \rangle$ = conj. symmetry

[III] Non-(-ve): $\langle \alpha, \alpha \rangle = a_1\bar{a}_1 + a_2\bar{a}_2 + \dots + a_n\bar{a}_n = |a_1|^2 + |a_2|^2 + \dots + |a_n|^2$

$\Rightarrow a_1 = a_2 = \dots = 0$

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Date
Page

Q In above Q's, $V_2(R)$, $\alpha=(a_1,a_2)$, $\beta=(b_1,b_2)$, $c=(\gamma_1,\gamma_2)\in V_2(R)$

$\langle\alpha,\beta\rangle = 2a_1b_1 + 4a_2b_2$

For symmetry,

$\langle\beta,\alpha\rangle = 2b_1a_1 + 4b_2a_2 = 2a_1b_1 + 4a_2b_2 = \langle\alpha,\beta\rangle$ ✓

linear:

$\langle a\alpha,\beta,\gamma\rangle = \langle(aa_1+b_1, aa_2+b_2),(c_1,c_2)\rangle$

$= 2(aa_1+b_1)c_1 + 4(aa_2+b_2)c_2$

$= a(2a_1c_1 + 4a_2c_2) + (2b_1c_1 + 4b_2c_2)$

$= a\langle\alpha,\gamma\rangle + \langle\beta,\gamma\rangle$

Non-neg. $= \langle\alpha,\alpha\rangle = 2a_1^2 + 4a_2^2 \geq 0$

Q $\alpha_1, a_2 \in V(R)$ & $\beta_1, \beta_2 \in V(R)$

$\langle 2\alpha_1 + 3\alpha_2, 4\beta_1 - \beta_2\rangle$

Solⁿ: Linearity prop: $2\langle\alpha_1, 4\beta_1-\beta_2\rangle + 3\langle\alpha_2, 4\beta_1-\beta_2\rangle$

$= 2\langle 4\beta_1 - \beta_2, \alpha_1\rangle + 3\langle 4\beta_1 - \beta_2, \alpha_2\rangle$

$= 8\langle\beta_1,\alpha_1\rangle - 2\langle\beta_2,\alpha_1\rangle + 12\langle\beta_1,\alpha_2\rangle - 3\langle\beta_2,\alpha_2\rangle$

$= 8\langle\alpha_1,\beta_1\rangle - 2\langle\alpha_1,\beta_2\rangle + 12\langle\alpha_2,\beta_1\rangle - 3\langle\alpha_2,\beta_2\rangle$

* Norm of a vector (or length of a vector): $u_1, u_2 \in V$
$v_1, v_2 \in V$

Let $V(F)$ be a vector space, then the norm of a vector $u \in V$ is defined as:

$$\|u\| = \sqrt{\langle u,u\rangle}$$

Ex: Let $V_2(R) = \{(a_1,a_2) : a_1,a_2 \in R\}$. Also $\alpha=(a_1,a_2)$, $\beta=(b_1,b_2) \in R$ and $\gamma=(c_1,c_2)\in R$. Now define inner prod space on $V_2(R)$ as:

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[1] $\langle \alpha, \beta \rangle = a_1 a_2 + 2b_1 b_2$ × No symmetry

[2] $\langle \alpha, \beta \rangle = a_1^2 + b_1^2$ × No linearity

[3] $\langle \alpha, \beta \rangle = a_1 a_2 + b_1 b_2 + 5$ × No linear

[4] $\langle \alpha, \beta \rangle = a_1 b_1 + a_2 b_1 + a_1 b_2 + 3a_2 b_2$

[5] $\langle \alpha, \beta \rangle = a_1 b_1 - a_2 b_2$ × $(a_1 b_1 + a_2 b_2)$ ✓

Solⁿ: [3] $\langle a\alpha + \beta, \gamma \rangle = \langle (aa_1 + b_1, aa_2 + b_2), (c_1, c_2) \rangle$

$= (aa_1 + b_1)(aa_2 + b_2) + c_1 c_2 + 5$

$a\langle \alpha, \gamma \rangle = a(a_1 a_2 + c_1 c_2 + 5) +$

$+ \langle \beta, \gamma \rangle = (b_1 b_2 + c_1 c_2 + 5)$

Q Let V be a vector space of all complex valued continuous functions on the interval $[a,b]$ if $f(t), g(t) \in V$ define

(i) $\langle f, g \rangle = \int_a^b f(t)\, \overline{g(t)}\, dt$

Solⁿ: $\alpha = f(t)$, $\beta = g(t)$ & $\gamma = h(t) \in V$

$a\alpha + \beta = af(t) + g(t)$

$\langle af + g, h \rangle = \int_a^b (af + g)(t)\, \overline{h(t)}\, dt$

$= a\int_a^b f(t)\overline{h(t)}\,dt + \int_a^b g(t)\overline{h(t)}\,dt$

$= a\langle f, h \rangle + \langle g, h \rangle$

$\langle g, f \rangle = \int_a^b g(t)\, \overline{f(t)}\, dt$

$\overline{\langle g, f \rangle} = \int_a^b \overline{\left(g(t)\, \overline{f(t)}\right)} dt = \int_a^b \overline{g(t)} f(t)\, dt = \int_a^b f(t) \overline{g(t)}\, dt$

$\langle f, g \rangle$

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$$\langle f, f\rangle = \int_a^b f(t)\cdot\overline{f(t)}\,dt = \int_a^b |f(t)|^2\,dt \geq 0$$

$$\text{If, } \int_a^b |f(t)|^2\,dt = 0 \Rightarrow f(t) = 0.$$

(ii) $\langle f, g\rangle = \int_a^1 f(t)g(t)\,dt$; $f = 2$, $g = 3t$, $h = t+1$. Find:

(a) $\langle f, g\rangle$ (b) $\|f\|$ (c) $\|g\|$ (d) $\langle f, h\rangle$

Solⁿ: (a) $\langle f, g\rangle = \int_0^1 2(3t)\,dt = \int_0^1 6t\,dt = 3$

(b) $\langle f, f\rangle = \int_0^1 4\,dt = 4 \quad \therefore \|f\| = \sqrt{4} = 2$

(c) $\sqrt{\langle g, g\rangle} = \sqrt{\int_0^1 9t^2\,dt} = \sqrt{3}$

(d) $\langle f, h\rangle = \int_0^1 (2t+2)\,dt = 3$

$R^2 = \{(a, b)\ a, b \in R\}$, $\alpha = (a_1, a_2)$
$\beta = (b_1, b_2)$

$\langle \alpha, \beta\rangle = a_1 b_1 + a_2 b_2$
$\alpha = (1, 0)$, $\beta = (0, 1)$

$\langle \alpha, \beta\rangle = 1\cdot 0 + 0\cdot 1 = 0$
$\therefore$ α is orthogonal to β.

* Orthogonal vectors:
Let $V(F)$ be a vector space and $\alpha, \beta \in V(F)$
if $\langle \alpha, \beta\rangle = 0$ we say α is orthogonal to β.
(if IPS = 0)

$\langle (0,0), (a_1, a_2) \rangle = 0$ — always orthogonal

$S_1 = \{(1,0), (2,2), (0,0)\}$ × not orthogonal ∵ all pairs are

$S_2 = \{(2,0), (0,3), (0,0)\}$ ✓ orthogonal.