After Mid-Sem

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Sequence and Convergence Series

$\{z_1, z_2, \ldots, z_n, \ldots\}$, $z_i \in C$

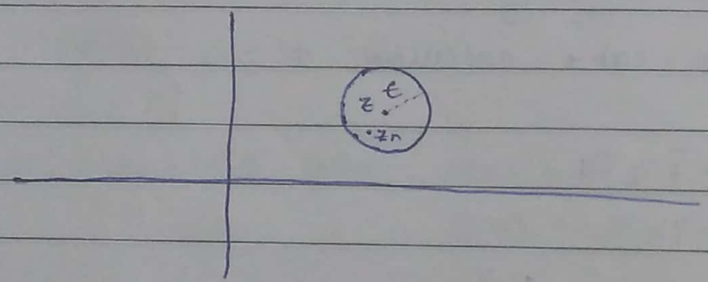
$N = \;\; 1 \;\; 2$

(One - one correspondence with natural no.)

Represented as : $\{z_n\}_{n=1}^{\infty}$

Def^n: An infinite sequence $\{z_n\}_{n=1}^{\infty}$ of complex no. has a limit z if for every $\epsilon > 0$ $\exists$ a positive no. n_0 s.t. $|z_n - z| < \epsilon$ whenever $n > n_0$

Geometrical Interpretation



If the limit exists and is unique, then we say z_n converges to limit z or

$$\lim_{n\to\infty} z_n = z$$

Eg. $z_n = i^n$, $\{z_n\}_{n=1}^{\infty}$

$z_n = \{i, -1, -i, 1, i, \ldots\}$: It is oscillating (non-unique)

$\rightarrow$ It diverges

Eg: $z_n = 1 + i^n$

$\{z_n\}$: diverges

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Churchill & Kreyzig

eg. $z_n = \frac{i^n}{n}$

$z_n = \left\{ i, \frac{-1}{2}, \frac{-i}{3}, \frac{1}{4}, \ldots \right\}$

$\lim_{n\to\infty} z_n = 0 \Rightarrow$ Converges

Theorem: Suppose $z_n = x_n + iy_n$, $z = x + iy$

Then,

$$\lim_{n\to\infty} z_n = z \quad \text{——— (1)}$$

iff $\lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} y_n = y$ ——— (2)

* $\lim_{n\to\infty} z_n = \lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$

(provided x_n & y_n are convergent)

PROOF: Let us assume that eq^n (2) holds true.

Claim : eq^n (1) holds true

$\Rightarrow |z_n - z| < \epsilon$ whenever $n > n_0$

(no needs to be determined)

Given $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$

$\Rightarrow$ For every $\epsilon > 0$, $\exists$ positive no. n_1 & n_2 s.t.

$|x_n - x| < \epsilon/2$ whenever $n > n_1$

$|y_n - y| < \epsilon/2$ whenever $n > n_2$

$|z_n - z| = |x_n + iy_n - x - iy_n| = |(x_n - x) + i(y_n - y)|$

$\leq |x_n - x| + |y_n - y| = \epsilon$ whenever $n > \underset{n_0}{\underset{\parallel}{\max(n_1, n_2)}}$

Hence Proved.

* ~~Sum~~ of two convergent ~~series~~ sequence is also convergent

Comb^n

Converse :- Given $\lim_{n\to\infty} z_n = z$

For every $\epsilon > 0$, $\exists$ $n_0 > 0$ s.t.
$|z_n - z| < \epsilon$ whenever $n > n_0$

$|x_n - x| \leq |(x_n - x) + i(y_n - y)| = |x_n + iy_n - (x+iy)|$
$= |z_n - z| < \epsilon$ whenever $n > n_0$

Similarly, $|y_n - y| \leq |(y_n - y)i + (x_n - x)| < \epsilon$ whenever $n > n_0$
Hence
$\lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} y_n = y$

Hence Proved.

Ex. $z_n = x_n + iy_n = \left(1 - \frac{1}{n^2}\right) + i\left(2 + \frac{4}{n}\right)$
(Assuming z_n is convergent)
$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$

$= \boxed{1 + 2i}$

Ex. (How to find n_0) [depends on ϵ]
$z_n = \frac{1}{n^3} + i$, $n = 1$ to ∞

$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{1}{n^3} + i \lim_{n\to\infty}(1)$

$= 0 + i = i$

To get n_0 :-
$|z_n - z| = \left|\frac{1}{n^3}\right| < \epsilon$ whenever $n > \frac{1}{(\epsilon)^{1/3}}$

Let $\epsilon = 0.001$ $\Rightarrow$ $n_0 = 1/(0.001)^{1/3}$

Series :

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \dots + z_n + \dots$$

$$S_N = \sum_{n=1}^{N} z_n \longrightarrow N^{th} \text{ partial sol}^n$$

Def^n: A complex series $\sum_{n=1}^{\infty} z_n$ converges to the sum S if the sequence of partial sum S_N, $N = 1, 2, \dots$ converges to S

$$\sum_{n=1}^{\infty} z_n = S \quad , \quad \lim_{N \to \infty} S_N = S$$

$|S_N - S| < \epsilon$ whenever $N > N_0$

Theorem : Suppose $z_n = x_n + i y_n$, $n = 1, 2, 3, \dots$

$S = X + iY$

Then $\sum_{n=1}^{\infty} z_n = S$

if and only if

$$\underbrace{\sum_{n=1}^{\infty} x_n}_{\text{real}} = X \quad \text{and} \quad \underbrace{\sum_{n=1}^{\infty} y_n}_{\text{real}} = Y$$

$$\cancel{\lim_{n \to \infty} \sum_{n=1}^{\infty} z_n} = \cancel{\lim_{n \to \infty} \sum_{n=1}^{\infty} x_n} + i \cancel{\lim_{n \to \infty} \sum_{n=1}^{\infty} y_n}$$

$$\sum_{n=1}^{\infty} (x_n + i y_n) = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

{ Proof is same as that in sequences }

Theorem : If a series $\sum_{n=1}^{\infty} z_n$ of complex no. converges, then the n^{th} term converges to zero as $n \longrightarrow \infty$

If n^{th} term of series doesn't goes to zero, then the given series is divergent.

However, this is necessary cond^n ~~to~~ for convergence.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n} \longrightarrow$ divergent eventhough $\frac{1}{n} \to 0$ as $n \to \infty$

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PROPERTIES :

① The terms of a convergent series are bounded
i.e. $\exists$ $M > 0$ s.t.

$$|z_n| \le M \quad \forall n$$

② A series is called absolutely convergent if the series $\sum_{n=1}^{\infty} |z_n|$ converges
↓
Real series

$$\left(\sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} \right)$$

③ Corollary : The absolute convergence of a series of complex no. implies the convergence of that series.

Ex. $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$ (convergent) (cauchy's Principle)

$\sum_{n=1}^{\infty} |z_n| = \sum \frac{1}{n}$: not convergent

This type of series is called conditionally convergent.

→ $\sum_{n=1}^{\infty} z_n = S$

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \ldots + z_N$$

define $R_N = \sum_{n=N+1}^{\infty} z_n = z_{N+1} + z_{N+2} \ldots + \ldots$
↓
Remainder term

From here, $R_N = S - S_N$, $S_N = S - R_N$

$$S = S_N + R_N$$

$|R_N| \rightarrow$ Error term

We can observe that $S_N \rightarrow S$ when $R_N \rightarrow 0$ (Take ϵ very small) (N sufficiently large)

$|s_n - s_{n+1}| < \epsilon$

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Test for Convergence (cauchy)

Theorem : For every $\epsilon > 0$, $\exists N > 0$ s.t.

$|z_{n+1} + z_{n+2} \dots + z_{n+p}| < \epsilon \quad \forall \quad n > N$

and $p = 1, 2, \dots$

•) Comparison Test

$\sum_{n=1}^{\infty} z_n$

If we can find another convergent series $\sum b_n$ with non-negative real terms s.t.

$|z_n| \le b_n \quad \forall \quad n = 1, 2, \dots$

then $\sum z_n$ converges absolutely.

•) Ratio Test

$\{z_n\}_{n=0}^{\infty}$ with $z_n \neq 0 \; \forall n$ $\lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = L$

z_n converges ~~absolute~~ if $L < 1$

diverges if $L > 1$

Test fails if $L = 1$

•) Root test

$\lim_{n \to \infty} |z_n|^{1/n} = L$

same results as above.

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POWER SERIES

A power series in $(z - z_0)$ is of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots \quad \text{— (1)}$$

a_n's $\rightarrow$ complex (or real) const.

$z \rightarrow$ complex variable

$z_0 \rightarrow$ complex (or real) const.

Centre of series (1)

If $z_0 = 0$ $\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots$

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→ Every power series converges at its centre

Convergence of Power Series

Ex. 1 Convergence in a ~~series~~ disk.

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

→ disk

It converges to 0 $(z_0 = 0)$ for $|z| < 1$

It diverges for $|z| \geq 1$

Ex. 2 $$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

It converges ~~to~~ $\forall\, z$

Proof (by Ratio Test):

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$$

$L < 1 \Rightarrow$ it converges for all z.

Ex. 3 $$\sum_{n=0}^{\infty} n!\, z^n$$

Ratio test :

$$\lim_{n\to\infty} \left| \frac{(n+1)!\; z^{n+1}}{n!\; z^n} \right| = \lim_{n\to\infty} (n+1)\,|z| \; : \; \text{~~divergent~~}$$

converges at $z = 0$ $(z_0 = 0)$

diverges at all other points.

Radius of Convergence :

Smallest circle with centre at z_0 that includes all the points at which power series converges.

$|z - z_0| = R$ → circle of convergence.

R : radius of convergence

$R \to \infty \Rightarrow$ series converges for all z.

$R = 0 \Rightarrow$ series converges only at $z = z_0$.

Earlier :

Ex 1 : $R = 1$

Ex 2 : $R = \infty$

Ex 3 : $R = 0$

How to find this radius of convergence ?

Cauchy - Hardamard :-

we only need to include coeff, not z (proof in Kreyzig)

(i) (Ratio Test) $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{L} = R$

$L = 0 \longrightarrow R = \infty$: Series converges everywhere

$L = \infty \longrightarrow R = 0$: series converges only at one point

(ii.) (Root Test) $\lim_{n\to\infty} \left| \frac{1}{\sqrt[n]{a_n}} \right| = \frac{1}{L} = R$

Ex. $\sum_{n=0}^{\infty} \frac{2n!}{n!^2} (z-3i)^n$. Find R ?

$$R = \lim_{n\to\infty} \left| \frac{2n! \cdot (n+1)!^2}{n!^2 \cdot (2n+2)!} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} =$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4}$$

$\Rightarrow$ The series converges for $|z-3i| < \frac{1}{4}$.

diverges outside this disk $|z-3i| > 1/4$

$\Rightarrow$ But, nothing can be concluded at point $|z-3i| = 1/4$

We have to calculate it separately.

How ??

Remark : $\sum_{n=0}^{\infty} a_n z^n$ — non-zero radius of convergence $R > 0$

then the sum func^n is a func^n of z

$\sum z^n = 1 + z + z^2 + \dots = \frac{1}{1-z}$, $|z| < 1$

Here, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is continuous in $|z| < R$ (if convergent in disk, it is cont. also)

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→ Every

Properties

1.) We can differentiate and integrate the power series term wise and the resulting series has same radius of convergence as your original series
(Apply Ratio Test to prove)

2.) A power series with $R > 0$ represents an analytic func^n at every point interior to its circle of convergence, ie, $|z - z_0| < R$.

Taylor series :

Theorem : Suppose $f(z)$ is analytic in $|z - z_0| < R_0$.
Then, $f(z)$ has a power series representation

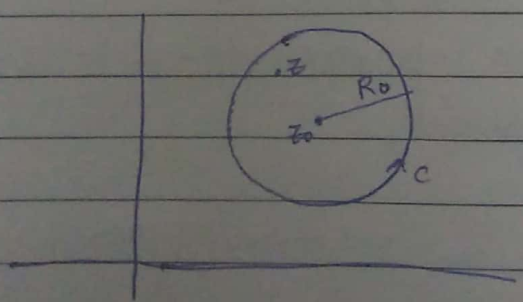
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0) \quad \text{— ②}$$

where, $a_n = \dfrac{f^{(n)}(z_0)}{n!}$, $n = 0, 1, 2, \ldots$

ie, the series ② converges to $f(z)$ when z lies in $|z - z_0| < R_0$.

By Cauchy Integral Formula,

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^*$$



Here, $f(z) = f(z_0) + (z - z_0) f'(z_0) + \dfrac{(z - z_0)^2}{2} f^2(z_0)$

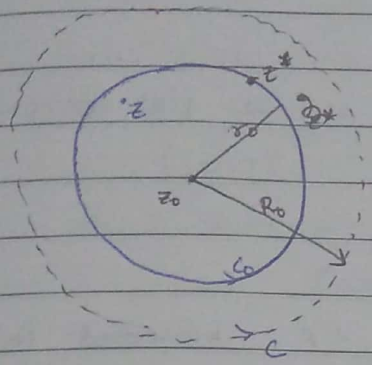
$\ldots + \dfrac{(z - z_0)^n}{n!} f^n(z_0) + R_n(z)$

↓
Remainder

remainder term is considered as error term & $\to 0$

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} \frac{dz^*}{(z^*-z)}$$

Proof :



we have 2 discs here :

$|z-z_0| < R_0$: C

$|z-z_0| < r_0$: C_0

$f(z)$ is analytic inside $|z-z_0| < R_0$

We can see that f is analytic at z

$$f(z) = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z^*)\, dz^*}{(z^*-z)} \quad \text{— (4)}$$

write $\frac{1}{z^*-z}$ in power of $(z-z_0)$

$$\frac{1}{z^*-z} = \frac{1}{(z^*-z_0)-(z-z_0)} = \frac{1}{(z^*-z_0)}\left[\frac{1}{1-\left(\frac{z-z_0}{z^*-z_0}\right)}\right]$$

always less than 1 (from curve)

$$1+q+q^2+q^3 \dots = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}$$

that is why, sum formula is used.

$$\frac{1}{1-q} = 1+q+q^2+\dots+q^n+\frac{q^{n+1}}{1-q}$$

$$\frac{1}{z^*-z} = \frac{1}{(z^*-z_0)}\left[1+\left(\frac{z-z_0}{z^*-z_0}\right)+\left(\frac{z-z_0}{z^*-z_0}\right)^2+\dots+\left(\frac{z-z_0}{z^*-z_0}\right)^n+\frac{\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}}{1-\left(\frac{z-z_0}{z^*-z_0}\right)}\right]$$

$$= \frac{1}{(z^*-z_0)}\left[1+\dots+\left(\frac{z-z_0}{z^*-z_0}\right)^n\right] + \frac{1}{z^*-z}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}$$

From (4) :

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(z^*)}{(z^*-z_0)}dz^* + \frac{(z-z_0)}{2\pi i}\oint_{C_0}\frac{f(z^*)\,dz^*}{(z^*-z_0)^2} + \dots$$

$$+ \frac{(z-z_0)^n}{2\pi i}\oint_{C_0}\frac{f(z^*)}{(z^*-z_0)^n}dz^* + \underbrace{\frac{(z-z_0)^{n+1}}{2\pi i}\oint_{C_0}\frac{f(z^*)}{(z^*-z_0)^{n+1}}\frac{dz}{(z^*-z)}}_{R_n(z)}$$

~~If we have~~

$\rightarrow f(z) = f(z_0) + (z-z_0) f'(z_0) + \dots + \frac{(z-z_0)^n}{n!} f^n(z_0) + R_n(z)$

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If we can prove that $R_n \rightarrow 0$, $\Rightarrow$ $f(z)$ converges to $\sum_{n=0}^{\infty} a_n z^n$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{(series)} \qquad \left(a_n = \frac{f^{(n)}(z_0)}{n!}\right)$$

Proof: (Consider previous figure)

$f(z)$ is analytic in $|z-z_0| < R$, bounded in C

we can observe from fig. $|z^* - z| > 0$

$\frac{f(z^*)}{z^*-z}$ is analytic & bounded in C

Assume $\left|\frac{f(z^*)}{z^*-z}\right| \le \tilde{M}$

C_0 with radius r_0 ie $|z^*-z| = r_0$ & length $= 2\pi r_0$ (length of circle)

$$|R_n| = \frac{|z-z_0|^{n+1}}{|2\pi i|} \left| \oint \frac{f(z^*)\, dz^*}{\underbrace{(z^*-z_0)^{n+1}}_{r_0}(z^*-z)} \right| \le \frac{|z-z_0|^{n+1}}{2\pi} \tilde{M} \frac{1}{r_0^{n+1}} (2\pi r_0)$$

$$\Rightarrow |R_n| \le \tilde{M} \frac{|z-z_0|^{n+1}}{r_0^n} = \tilde{M} r_0 \left| \frac{z-z_0}{\underbrace{r_0}_{(z^*-z_0)}} \right|^{n+1}$$

always less than 1

$\left|\frac{z-z_0}{r_0}\right| < 1$ (from figure)

$\Rightarrow |R_n| \rightarrow 0$ as $n \rightarrow \infty$

* Taylor's series is unique (proof in Kreyzig)

* By Cauchy's inequality,

$$|f^{(n)}(z_0)| = \left| \frac{1}{2\pi i} \oint_{C_0} \frac{f(z^*)\, dz^*}{(z^*-z_0)^{n+1}} \right|$$

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$$|a_n| \leq \frac{n!\,M}{r_0^n}, \quad \forall\ n = 0, 1, 2, \ldots$$

Corollary:

A power series with non-zero radius of convergence is the Taylor series of its sum $\underline{\hphantom{its sum}}$ ⟷ $f(z)$

Proof: $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots + a_n(z-z_0)^n$

{We've to prove $a_n = \frac{f^{(n)}(z_0)}{n!}$)

$z = z_0$ $f(z_0) = a_0$

diff. wrt z

$$f'(z) = \cancel{\text{const}}\ a_1 + 2a_2(z-z_0) \ldots + n a_n (z-z_0)^{n-1}$$

$f'(z_0) = a_1$

$f''(z_0) = 2a_2$

$f^n(z_0) = n!\, a_n$

$$\Rightarrow \boxed{a_n = \frac{f^{(n)}(z_0)}{n!}}$$

Comparison with Real Series

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f(x)$ is differentiable for all orders

$f^{(n)}(0) = 0 \ \forall\ n$

$f(x)$ doesn't have Maclaurin Series

↓

(where $z_0 = 0$)

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$\cancel{1+x+x^2}$ $\quad x - \frac{x^3}{3!}$

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Ex. $f(z) = e^z$

$f^n(z) = e^z \quad \forall\ n = 0, 1, 2, \ldots$

$f^n(0) = 1 \quad \forall\ n$

$\Rightarrow \quad e^z = \sum_{z=0} \frac{z^n}{n!} \qquad (|z| < \infty)$

If $z = x + i.0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

Ex. $f(z) = z^2 e^{3z}$

We can use above eq. here.

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{(3z)^n}{n!} \cdot z^2 = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{(n+2)} \qquad m = n+2$$

$$= \sum_{m=2}^{\infty} \frac{3^{m-2}}{(m-2)!} z^m$$

Ex. $e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \cancel{\sum_{n=0}^{\infty} i}$

$$= \underbrace{\sum_{k=0} \frac{(-1)^k y^{2k}}{(2k)!}}_{\text{even}} + i \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}}_{\text{odd}}$$

$$e^{iy} = \cos y + i \sin y$$

Euler's formula

Ex. $f(z) = \frac{1}{1-z}$

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}, \quad n = 0, 1, 2$$

$f(z)$ is non analytic at $z = 1$

we can find $f^{(n)}(0)$ (analytic at $z = 0$)

$f^{(n)}(0) = n!$

$\frac{1}{1-(1-z)}$

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Maclaurin Series of $f(z)$:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 \quad \ldots \qquad |z| < 1$$

Binomial Expansions

↳ Replacing z by $-z$, we get

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 \ldots \qquad |z| < 1$$

↳ Replacing z by $1-z$, we get

$$f(z) = \frac{1}{z}$$

$$\frac{1}{z} = 1 + (1-z) + (1-z)^2 + \ldots$$

$$= \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad \forall \quad |z-1| < 1$$

Ex. $f(z) = \frac{1+2z^2}{z^3+z^5}$: Expand into a series involving power of z.

$$f(z) = \frac{1}{z^3}\left(\frac{1+2z^2}{1+z^2}\right) \quad : \text{non-analytic at } z=0$$

$$= \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$: ~~not~~ whole is non-analytic, but we can find exp^n of $\frac{1}{1+z^2}$ at $z=0$.

Putting in previous ex. → deleted nbd. of 0.

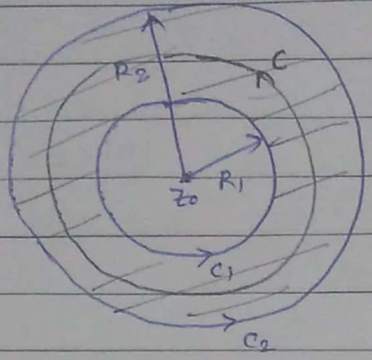
$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 \ldots \qquad |z| < 1$$

$$f(z) = \frac{1}{z^3}\left(2 - 1 + z^2 - z^4 + z^6 \ldots\right) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 \ldots$$

$f(z)$ is analytic in $\boxed{0 < |z| < 1}$

→ How to find out power series rep^n ~~for~~ ~~som~~ for some region when $f(z)$ is non-analytic ? :- Laurent Series

Laurent Series



analytic in

$$\boxed{R_1 < |z-z_0| < R_2}$$

$\Downarrow$

known as Annulus Region
(we don't know anything about other regions)

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$D: R_1 < |z-z_0| < R_2$

$f(z)$ is analytic throughout the region D.

Let C be any positively oriented simple closed contour around z_0 and lying inside C.

Then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{principal part (negative power of } z)}$, $\quad (R_1 < |z-z_0| < R_2)$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-z_0)^{n+1}}, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-z_0)^{-n+1}}, \quad n = 0, 1, 2 \ldots$$

Replacing n by $-n$ in principal part, we get

$$\sum_{n=-\infty}^{-1} b_{-n} (z-z_0)^n$$

$$b_{-n} = \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-z_0)^{n+1}}, \quad n = -1, -2, \ldots$$

$$\Rightarrow f(z) = \sum_{n=-\infty}^{-1} b_{-n} (z-z_0)^n + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$\boxed{f(z) = \sum_{-\infty}^{\infty} c_n (z-z_0)^n} \qquad R_1 < |z-z_0| < R_2$$

→ we can find power series everywhere inside the annulus. This is laurent series

$\rightarrow b_{-n}???$

$$c_n = \begin{cases} b_n & n \le -1 \\ a_n & n \geq 0 \end{cases}$$

$$\boxed{c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-z_0)^{n+1}}} \quad (n = 0, \pm 1, \ldots)$$

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Special cases :

(i) $f(z)$ is analytic throughout $|z-z_0| < R_2$
(2^{nd} term in laurent series is also valid)

$f(z)\ (z-z_0)^{n-1}$ is also analytic

$\underset{\text{analytic}}{f(z)}\ \underset{\text{analytic}}{(z-z_0)}$ (also at $z = z_0$)

analytic

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz = 0 \qquad a_n = \frac{1}{2\pi i} \oint_C f(z)\,(z-z_0)^{n+1} dz$$

$$\Rightarrow f(z) = \sum a_n (z-z_0)^n$$

(ii.) $f(z)$ is not analytic at $z = z_0$. Otherwise, analytic everywhere
(R_1 you can choose so small so that it only includes z_0)

→ $0 < |z-z_0| < R_2$

$f(z)$ has laurent series in $0 < |z-z_0| < R_2$

(iii.) $f(z)$ is analytic at each point in the finite plane exterior to the circle $|z-z_0| = R_1$

$f(z)$ has laurent series in $R_1 < |z-z_0| < \infty$

(iv.) $0 < |z-z_0| < \infty$ (Extending (ii.))

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Ex. $f(z) = z^2 e^{1/z}$; problem at $z=0$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad , |z| < \infty$$

$$z^2 e^{1/z} = \sum_{n=0}^{\infty} z^2 \left(\frac{1}{z^n \cdot n!} \right) \quad , \left|\frac{1}{z}\right| < \infty \Rightarrow |z| > 0$$

$$z^2 e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! \; z^{n-2}} \quad , |z| > 0$$

Laurent series $= z^2 + z + \frac{1}{2!} + \underbrace{\frac{1}{3!\,z} + \frac{1}{4!\,z^2} + \dots}_{\text{principal part}} \quad , |z|>0$

Ex. $f(z) = \frac{1}{1-z}$

i.) Develop a power series with non-negative power of z

ii.) — negative —

i) $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 \dots \quad , |z| < 1$

using above expansion, find its exp^n

ii) $f(z) = \frac{-1}{z(1-\frac{1}{z})} = \frac{-1}{z} \left(\frac{1}{1-1/z} \right)$

$$= \frac{-1}{z} \sum_{n=0}^{\infty} \frac{1}{(z)^n} = - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \quad , \left|\frac{1}{z}\right| < 1 \Rightarrow |z| > 1$$

$$= -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \dots \right) \quad , |z| > 1$$

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Uniqueness :

The Laurent series of a given analytic func^n $f(z)$ in its specified annulus of convergence is unique.

$f(z)$ may have different Laurent series in 2 different annuli with same centre (Eg: Prev example)

Ex. $f(z) = \dfrac{-1}{(z-1)(z-2)} = \dfrac{1}{z-1} - \dfrac{1}{z-2}$

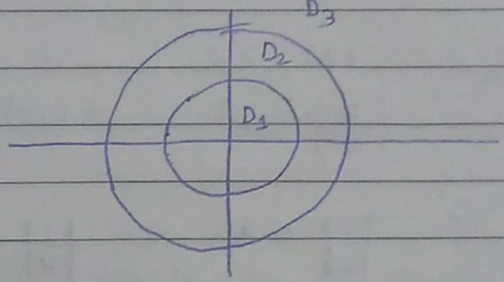
Singular at $z=1$, $z=2$

$f(z)$ is analytic in following domains:

1.) $|z| < 1$

2.) $1 < |z| < 2$

3.) $|z| > 2$



1.) $|z| < 1$: Find Maclauren Series

$|z| < 1 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{-1}{1-z} + \frac{1/2}{1-z/2}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n , \quad |z|<1$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \quad (|z| < 1)$$

* It doesn't contain negative power ($b_n = 0$) as func^n is analytic everywhere in given domain D_1.

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(ii) $D_2 : 1 < |z| < 2$

$$\left|\frac{1}{z}\right| < 1 \quad \& \quad \left|\frac{z}{2}\right| < 1$$

take care of this while writing $f(z)$ in next step.

$$f(z) = \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{z^n}$$

~~(iii)~~

(iii) $D_3 : |z| > 2$

$$\cancel{\left|\frac{z}{2}\right| > 2} \qquad \cancel{\left|\frac{1}{z}\right| > 1}$$

$$\left|\frac{2}{z}\right| < 1 \Rightarrow \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}}$$ ⇒ we are only getting negative part.

$$= \sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}$$

laurent series exist for ~~this region~~

Eg. $f(z) = e^{1/z}$, $\left|\frac{1}{z}\right| < \infty$. $|z| > 0$

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{n!\,z^n} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \dots \qquad |z| > 0$$

① diff. from ②

Coeff of $\frac{1}{z-z_0}$: Residue (b_1)

you will

From $b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$

$$b_1 = \frac{1}{2\pi i} \oint_C \underset{\to e^{1/z}}{f(z)}\, dz = \frac{1}{2\pi i} \oint_C e^z dz = 1 \text{ (from expn)}$$

$\Rightarrow \boxed{\oint_C e^{1/z} dz = 2\pi i}$: Can find integral of ~~complex funct's using~~ this method

Singularity of $f(z)$ at $z = z_0$ — ①

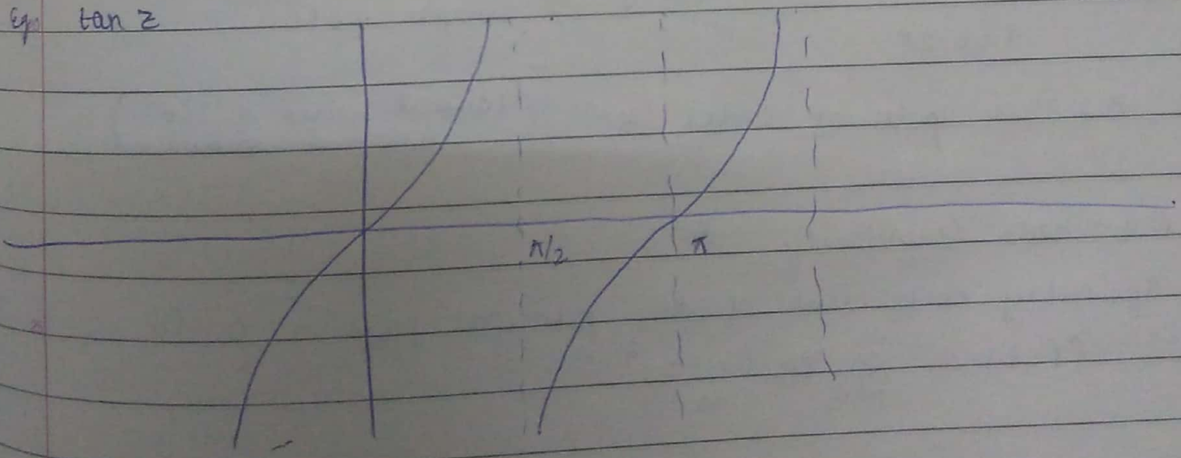
(when $f(z)$ is non-analytic at $z = z_0$)

— ②

Isolated Singularity :

We call $z = z_0$ as an isolated singularity if $z = z_0$ has a nbd without furthur singularity of $f(z)$.

Eg. tan z



Isolated Singularity at $z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$

* See, why we are getting negative |

(i) If the principal part contains only finitely many terms of the form

$$= \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

then, the singularity of $f(z)$ at $z = z_0$ is called **Pole** of **order m**.

(ii) If principal part contains infinite terms, then $z = z_0$ is an **Isolated Essential Singularity** of $f(z)$.

Eg. $f(z) = e^{1/z}$: Isolated essential singularity $(z=0)$

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Revise

Let $f(z)$ is analytic in
$0 < |z - z_0| < R$

$f(z)$ has laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$\frac{1}{(z-2)^5}$ $\frac{1}{(z-2)+2}$ $\frac{1}{(z-2)^5} \frac{1}{2}$

Eg $f(z) = \frac{1}{z(z-2)^5}$ → laurent series ??? $m = 1 \Rightarrow$ simple pole

$z = 2$ is pole of order 5. (laurent series of $f(z)$ around $z = 2$)

(iii) Removable Singularity

If every coefficient of the principal part is 0 $(b_n = 0 \ \forall n)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$z = z_0$ is a removal singularity.

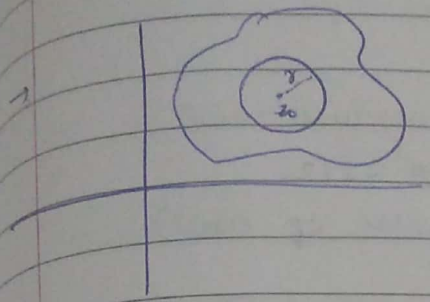
define $f(z_0) = a_0$ to remove singularity
then $f(z)$ is analytic in entire disk $|z - z_0| < R$

$\sum_{n=1}^{\infty} 1/n$ diverges

* $f(z)$ has power series only if $f(z)$ is analytic

$f(z) = \frac{\sin z}{z}$ find laurent series about $z=0$, you will get $a_0 = 1$

$z=0$ is a removable singularity.



$f(z)$ is analytic everywhere except at z_0 $(|z-z_0|<r)$

Residue :-

$f(z)$ has singularity at $z = z_0$ inside C (counter-clockwise)

C is inside the region $|z-z_0| < R$

$n=1$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} \, dz$$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z) \, dz$$

$$\boxed{\oint_C f(z) \, dz = 2\pi i \, b_1}$$

b_1 - ~~Radius~~ Residue of $f(z)$ at $z = z_0$

$$\operatorname{Res}_{z=z_0} f(z) = b_1$$

Ex. $f(z) = \frac{\sin z}{z^4}$, $C : |z| = 1$ with positive orientation.

has singularity at $z=0$

$\oint f(z) \, dz = ??$

$2n+1-4$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad |z| < \infty$$

$$\frac{1}{z^4} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-3}}{(2n+1)!} = \frac{1}{z^3} - \frac{1}{3! \, z} + \frac{z}{5!} + \cdots \quad \underset{\text{deleted nbd.}}{0 < |z| < \infty}$$

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From expansion, $b_1 = \frac{-1}{3!}$

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{-1}{3!}\right) = -\frac{\pi i}{3}$$

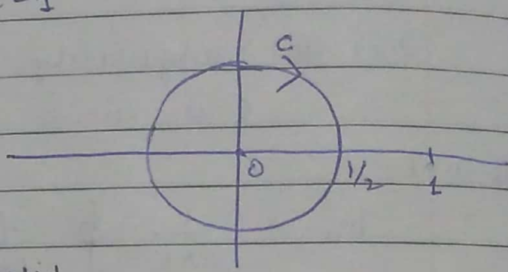
Ex. $f(z) = \frac{1}{z^3 - z^4}$ C is with clockwise - dir^n

C: $|z| = 1/2$

(different centre of series)

$$= \frac{1}{z^3(1-z)}$$

Singularities are : $z = 0$ and $z = 1$



Only $z = 0$ is included in curve C.

So, laurent series about $z = 0$:

$$\frac{1}{z^3} * \frac{1}{1-z}$$ → This will be valid throughout the region.

$$= \frac{1}{z^3}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-3}$$ $0 < |z| < 1$ ↳ deleted nbd $\left\{ \begin{array}{l} |z|<1 \\ \& \\ C: |z|=1 \end{array} \right.$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \dots$$

$b_1 = 1$

$$\boxed{\oint f(z)\,dz = -2\pi i}$$

Other possible way :-

Ex. $f(z) = \frac{1}{z^3 - z^4}$

$$= \frac{-1}{z^4\left(1 - \frac{1}{z}\right)}$$

$$= \frac{-1}{z^4}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \qquad |z| > 1$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+4}} = -\left(\frac{1}{z^4} + \frac{1}{z^5} + \frac{1}{z^6} + \dots\right)$$

Residue at $z = 0 = 0$

$\boxed{\operatorname{res}_{z=0} f(z) = 0}$ $\boxed{\oint_C f(z)\,dz = 0}$ this is wrong, because we have chosen wrong region

So, it is important to choose the right region.

→ $f(z)$ has a simple pole at $z = z_0$

⇒ $m = 1$ = Only 1 negative term

⇒ $f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$

$(z - z_0) f(z) = b_1 + a_0(z - z_0) + a_1(z - z_0)^2 + \ldots$

$$\boxed{b_1 = \lim_{z \to z_0} (z - z_0) f(z)}$$

we can find residue this way without expanding series if given it has simple pole

Ex. $f(z) = \frac{9z + i}{z(z^2+1)}$

$z = i$: simple pole

$b_1 = \lim_{z \to i} \frac{\cancel{(z-i)}(9z + i)}{z(z+i)\cancel{(z-i)}}$

$= \lim_{z \to i} \frac{\overset{5}{\cancel{10}}\cancel{i}}{\cancel{i}(2i)} = -5i$

~~$\oint f(z)\,dz$~~

Zero :-

$f(z)$ has a zero of order n at $z = z_0$ if

$f(z_0) = f'(z_0) = f''(z_0) \ldots = f^{(n-1)}(z_0) = 0$

and $f^n(z_0) \neq 0$

→ $f(z)$ is analytic & z_0 is a zero of $f(z)$ then $\frac{1}{f(z)}$ has a pole at $z = z_0$

(valid for rational func^n $\frac{p(x)}{q(x)}$ also)

$(|z|>1)$ (not same as c)

Suppose $f(z) = \frac{p(z)}{q(z)}$,

$p(z)$ is analytic ; $p(z_0) \neq 0$

let $q(z)$ has a simple 0 at $z = z_0$.

Then, $q(z)$ has taylor series representation at $z = z_0$.

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}} \quad \frac{p(z_0)}{q'(z_0)}$$

- In earlier problem,

$$\operatorname*{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \frac{9+1}{(z=i)\ 3z^2+1} = \frac{(9+1)i}{-2} = \boxed{-5i}$$

→ $f(z)$ has a pole of order $m > 1$ at $z = z_0$

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

$$① - (z-z_0)^m f(z) = \underbrace{b_m + (b_{m-1})(z-z_0) + \dots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + \dots}_{\Downarrow}$$

Let ① $= g(z)$ taylor series of $g(z)$ at $z = z_0$

Our aim is to get b_1.

$$g(z) = \sum_{m=0}^{\infty} c_m (z-z_0)^m = \sum_{m=0}^{\infty} \frac{g^{(m)}(z_0)}{m!} (z-z_0)^m \quad — ②$$

b_1 : coeff. of $(z-z_0)^{m-1}$ {Comparing ① & ②}

$$\Rightarrow b_1 = \frac{g^{(m-1)}(z_0)}{(m-1)!}$$

$$\boxed{b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-z_0)^m f(z) \right\}}$$

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→ Residue of $f(z)$: (One way is using laurent series)

(i) At simple pole:

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z)$$

(ii) At a pole of order m

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \{(z-z_0)^m f(z)\}$$

Ex $f(z) = \dfrac{50z}{(z+4)(z-1)^2}$

→ order 2

$$\operatorname{Res}_{z=1} f(z) = \frac{1}{1!} \lim_{z\to z_0} \frac{d}{dz}\left\{\cancel{(z-1)^2} \frac{50z}{(z+4)\cancel{(z-1)^2}}\right\}$$

$$= \lim_{z\to 1} \frac{(z+4)50 - 50z}{(z+4)^2} = \frac{200}{25} = 8$$

Ex. $f(z) = e^{1/z^2}$ → singularity at $z=0$

$I = \oint_C f(z)\,dz$ $\quad$ $C: |z| = 1$ in counterclockwise

Laurent series

$$e^z = \sum \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!}, \quad |z| < \infty$$

$$e^{1/z^2} = \sum \left(\frac{1}{z^2}\right)^n \frac{1}{n!} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \dots \qquad \left|\frac{1}{z^2}\right| < \infty$$

$\Rightarrow |z^2| > 0$

$z=0$: Isolated singularity (Infinite terms in laurent series) $\Rightarrow 0 < |z| < \infty$

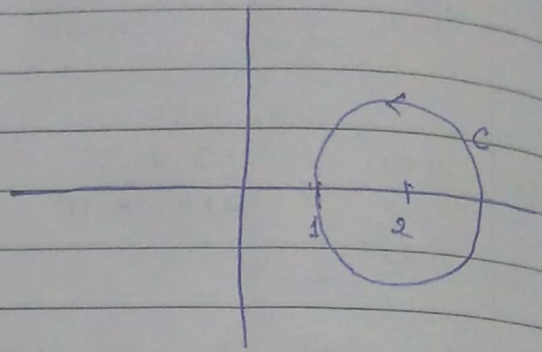
$$\operatorname{Res}_{z=0} f(z) = 0 \quad \{\text{coeff. of } 1/z = 0\}$$

$$\oint_C f(z)\,dz = \underline{0}$$

* Even though $f(z)$ is non analytic, we are getting $I = 0$

* Analyticity of $f(z)$ within and on a simple closed contour C is a sufficient condition for the value of integral around C to be zero but it is not a necessary cond^n.

Ex. $\oint \frac{dz}{z(z-2)^4} dz$ $\qquad C: |z-2| = 1 \rightarrow \left|\frac{z-2}{2}\right| < 1$



Singularity at $\underset{\substack{\downarrow \\ \text{Not included}}}{z=0}$ & $z=2$

Laurent series in the region

$$\underset{\substack{\downarrow \\ \text{larger region than } C}}{0 < |z-2| < 2}$$

$$\frac{1}{z(z-2)^4} = \frac{1}{(z-2)^4}\left[\frac{1}{2+(z-2)}\right]$$

$$= \frac{1}{2(z-2)^4}\left[\frac{1}{1+\left(\frac{z-2}{2}\right)}\right]$$

$$= \frac{1}{2(z-2)^4}\left[1 - \left(\frac{z-2}{2}\right) + \left(\frac{z-2}{2}\right)^2 - \cdots\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^{n-4}}{2^{n+1}}$$

$$\operatorname*{Res}_{z=2} f(z) = \text{Coeff. of } \frac{1}{z-2} = \frac{-1}{16} \quad (n=3)$$

$$\int \frac{dz}{z(z-2)^2} dz = 2\pi i\left(\frac{-1}{16}\right) = -\frac{\pi}{8} i$$

Alternative

$$\operatorname*{Res}_{z=2} f(z) = \frac{1}{3!} \lim_{z\to 2} \frac{d^3}{dz^3}\left\{(z-2)^4 \frac{dz}{z(z-2)^4}\right\}$$

$$= \frac{1}{3!} \lim_{z\to 2} \frac{d^3}{dz^3}\left\{\frac{1}{z}\right\}$$

$\frac{1}{z} \rightarrow \frac{-1}{z^2} \rightarrow \frac{-2}{z^3} \rightarrow -$

$= \frac{1}{3!} \lim_{z \to 2} \left[\frac{-3!}{z^4} \right] = \frac{-1}{16}$

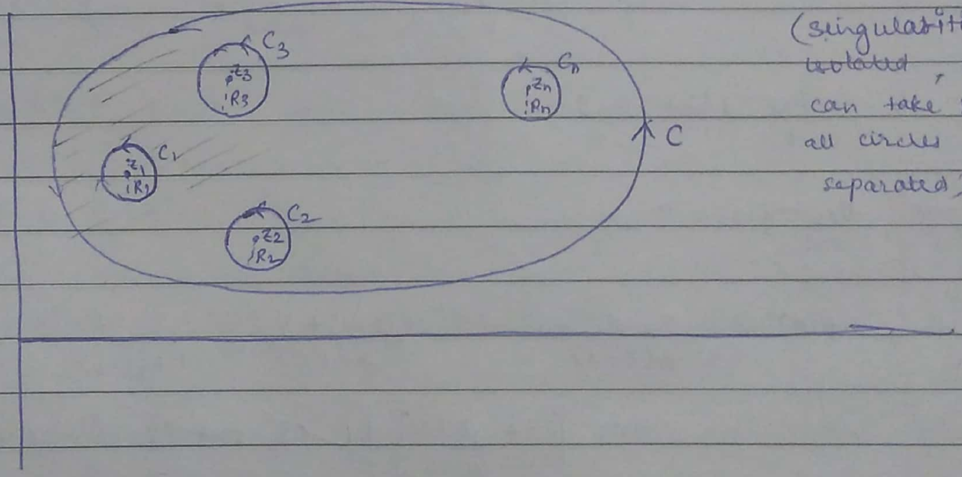
→ But, what if C includes finite no. of singularities ?

Cauchy's Residue Theorem :

f(z) has more than one singularity within a simple closed curve

Thm: Let C be simple closed contour, described in +ve sense. If a func^n f is analytic inside and on C except for a finite no. of singularities z_k, $k = 1, 2, \ldots, n$ inside C, then
(Assuming that singularities are isolated)

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z = z_k}{\mathrm{Res}}\, f(z)}$$



(singularities are ~~isolated~~, so we can take R s.t all circles are separated)

PROOF :

f(z) is analytic in the multiply connected closed domain D bounded by $C_1, C_2, \ldots \& C_n$

Using Cauchy's Integral Formula,

$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \ldots \oint_{C_n} f(z)\,dz$

$\oint_C f(z)\,dz - \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz = 0$ $\left\{ \begin{array}{l} I = 0 \text{ in region where} \\ f(z) \text{ is analytic} \\ \text{in closed domain } D \end{array} \right\}$

↳ We are making closed domain D.
(by making clockwise $C_1, C_2, \cdots C_n$)

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} 2\pi i \operatorname*{Res}_{z=z_k} f(z)$$

Ex $\displaystyle\oint_C \frac{5z-2}{z(z-1)}\,dz$ C : $|z| = 2$ described in counter-clockwise dir^n

singularities are : 0, 1 (Both included in C)

$\operatorname*{Res}_{z=0} f(z) = ?$

$$f(z) = \frac{5z-2}{z}\left(\frac{-1}{1-z}\right) = \frac{5z-2}{z}\left(-1 - z - z^2 - z^3 \dots\right)$$

$$= \left(5 - \frac{2}{z}\right)\left(-1 - z - z^2 - z^3 \dots\right)$$

$\operatorname*{Res}_{z=0} f(z) = 2$

$\operatorname*{Res}_{z=1} f(z) = ?$

need in power of $z-1$

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1} \quad \frac{1}{1+(z-1)}$$

$$= \left[5 + \frac{3}{(z-1)}\right]\left[1 - (z-1) + (z-1)^2 - (z-1)^3 \dots\right]$$

$\operatorname*{Res}_{z=1} f(z) = +3$

$\Rightarrow \oint_C f(z)\,dz = 2\pi i\,(2+3) = 10\pi i$

OR

already Laurent series in z

$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{z-1}$$ already Laurent series in $(z-1)$

~~Res f~~

$\oint f(z)\,dz = 2 + 3 = 5$

clockwise

OR

$z = 0, 1 \rightarrow$ simple pole

$\oint \operatorname{Res}_{z=0} f(z) = \lim_{z \to 0} (z) f(z) = 2$

$\operatorname{Res}_{z \to 1} \oint (z-1) f(z) = \lim_{z \to 1} \frac{5z-2}{z} = 3$

$, 2 + 3 = 5$

$\rightarrow$ if $C_1 : |z| = 1$, we only need to calculate for $z = 0$

$$\oint_{C_1} f(z)\, dz = 2\pi i (2)$$

13/10/17 Applications of Cauchy - Residue Th^m

① $\int_0^{\infty} f(x)\, dx = \lim_{R \to \infty} \int_0^R f(x)\, dx$

If the R.H.S. limit exist, then the improper integral converges

$f \in C(R)$

② $\int_{-\infty}^{\infty} f(x) dx = \lim_{R_1 \to \infty} \int_{-R_1}^{0} f(x)\, dx + \lim_{R_2 \to \infty} \int_0^{R_2} f(x) dx$

If the R.H.S. limit exist, then the L.H.S. improper integral converges to the sum

Def^n. Cauchy Principal value of ② is the no.

$$PV \int_{-\infty}^{\infty} f(x)\, dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx$$

provided RHS limit exists.

* when the integral ② converges, i.e. cauchy principal value ③ exists and that value is the no. to which the integral ② converges

* CONVERSE IS NOT TRUE

Eg. (Example for converse)

$$PV \int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx = 0$$

Acc. to defn ② →

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} x\,dx + \lim_{R_2\to\infty} \int_{0}^{R_2} x\,dx$$

$$= \lim_{R_1\to\infty} \frac{-R_1^2}{2} + \lim_{R_2\to\infty} \frac{R_2^2}{2}$$

⇒ limit D.N.E.

⇒ This improper integral doesn't converge (although principal value exists)

→ If $f(x)$ is an even funcn, i.e., $f(x) = f(-x)\ \forall x$

Suppose cauchy principal value ③ exists, then

$$\int_{-\infty}^{\infty} f(x)\,dx = PV \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\left[PV \int_{-\infty}^{\infty} f(x)\,dx\right]$$ (Try it yourself)

→ $f(x) = \dfrac{p(x)}{q(x)}$ (rational funcn) {zeroes of $q(x)$: poles of $f(x)$}

(i) $p(x)$ and $q(x)$ are polynomials of x with real coefficients & no factors in common

(ii) $q(z)$ has no real zeroes and atleast one zero above the real axis

(iii) degree of $q(x)$ is atleast 2 units more than degree of $p(x)$.

Our aim is to calculate : PV $\int_{-\infty}^{\infty} f(x)\,dx$

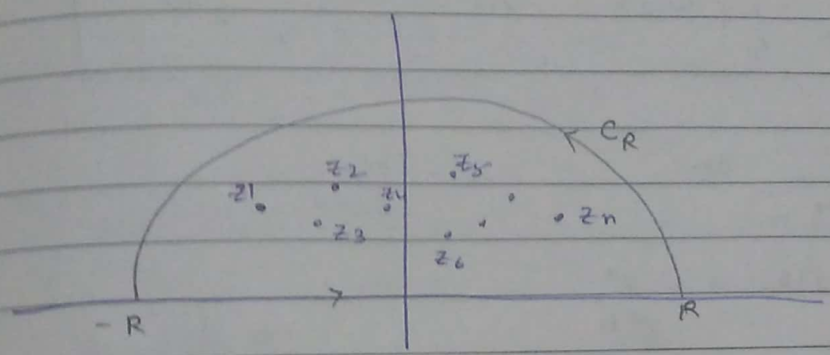
Steps :

consider $f(z) = \dfrac{p(z)}{q(z)}$

(i) Find all zeroes of $q(z)$ above the real axis (finite no. of 0s)

let us assume $q(z)$ has n no. of distinct zeroes above the real axis

$z_1, z_2, \ldots z_n \longrightarrow$ zeroes of $q(z)$



Enclose all zeroes in a simple closed contour with a R

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$ [Cauchy's Residue Thm]

Here, $C \equiv \{ |z| = R$ and $z = x, -R \le x \le R\}$

$$\int_{-R}^{R} f(x)\,dx + \oint_{C_R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

$\Downarrow$ on real axis

taking limit ~~both sides~~,

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) - \lim_{R\to\infty} \oint_{C_R} f(z)\,dz$$

let $z = Re^{i\theta}$

C_R represented by R : const

θ ranges from 0 to π

{ using point (iii) }

$$|f(z)| < \frac{k}{|z|^2}$$

using M-L Inequality :

$$\left| \oint_{C_R} f(z)\,dz \right| < \underbrace{\frac{k}{R^2}}_{M} \underbrace{(\pi R)}_{L} = \frac{k\pi}{R}$$

$$\lim_{R \to \infty} \left| \oint_{C_R} f(z)\,dz \right| < 0$$

$$\Rightarrow \boxed{\lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z = z_k} f(z)}$$

Ex. $\displaystyle\int_0^\infty \frac{x^2}{x^6+1}\,dx \qquad = \frac{p(z)}{q(z)}$

consider $f(z) = \dfrac{z^2}{z^6+1}$

zeroes of $z^6 + 1 = 0 \Rightarrow z = (-1)^{1/6}$

$$z_k = \exp\left[i \left(\frac{\pi}{6} + \frac{2k\pi}{6} \right) \right], \quad k = 0, 1, 2, 3, 4, 5$$

zeroes : (above real axis)

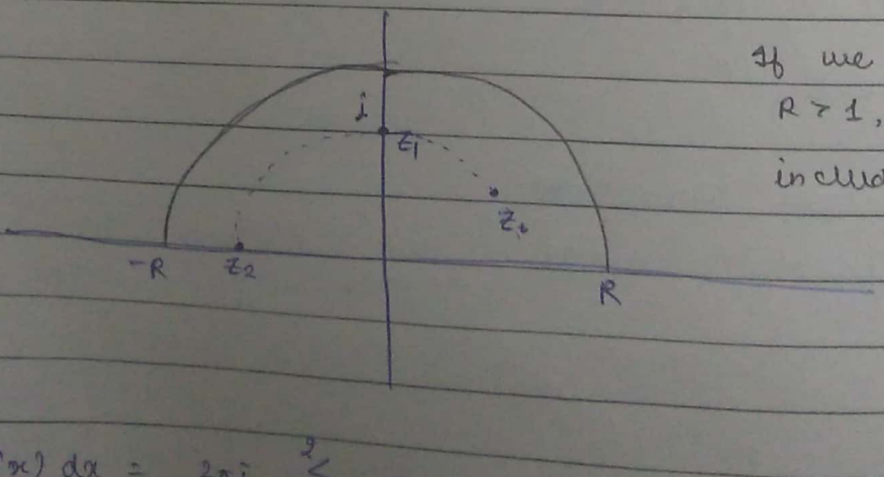
$z_0 = e^{i\pi/6}$ ✓ (30°)

$z_1 = e^{i\pi/2}$ ✓ (90°)

$z_2 = e^{i5\pi/6}$ ✓ (150°)

$\frac{\pi}{6} + \frac{4\pi}{6}$

other 3 will be below the real axis.



If we choose $R > 1$, then it'll include all the roots

$$\lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=0}^{2} \operatorname{Res}_{z = z_k} f(z) - \lim_{R \to \infty} \oint_{C_R} f(z)\,dz$$

z_k are simple poles of $f(z)$

$$\operatorname*{Res}_{z=z_k} f(z) = \lim_{z\to z_k} (z-z_k)\, f(z) = \frac{p(z_k)}{q'(z_k)}$$

$$\Rightarrow \operatorname*{Res}_{z=z_0} f(z_0) = \operatorname*{Res}_{z=z_0} \frac{z^2}{6z^5} = \operatorname*{Res}_{z=z_0} \frac{1}{6z^3}$$

$$= \frac{1}{6} e^{-i\pi/2} = \boxed{\frac{-i}{6}}$$

$$\Rightarrow \operatorname*{Res}_{z=z_1} = \frac{1}{6z_1^3} = \frac{1}{6} e^{-3i\pi/2} = \boxed{\frac{i}{6}}$$

$$\Rightarrow \operatorname*{Res}_{z=z_2} = \frac{1}{6z_2^3} = \frac{1}{6} e^{-i15\pi/6} = \frac{1}{6} e^{-i5\pi/2} = \boxed{-\frac{i}{6}}$$

$$\cancel{\Rightarrow \int_{-R}^{R} f(x)\,dx = 2\pi i\left(\frac{-i}{6}\right) = \boxed{\frac{\pi}{3}}} \text{ Ans.}$$

for $\oint_{C_R} f(z)\,dz$:

with $|z| = R$, $\quad |z|^2 = |z^2| = R^2$

$$|z^6+1| \geq \big||z|^6 - 1\big| = R^6 - 1$$

$$|f(z)| = \frac{|z^2|}{|z^6+1|} \leq M_R$$

$$\therefore \left|\oint_{C_R} f(z)\,dz\right| \leq M_R\, \pi R = \frac{R^3\pi}{R^6-1}$$

$$\lim_{R\to\infty} \left|\oint_{C_R} f(z)\,dz\right| = 0$$

$$\Rightarrow \int_{-R}^{R} f(x)\,dx = 2\pi i\left(\frac{-i}{6}\right) = \boxed{\frac{\pi}{3}} \text{ Ans.}$$

since $f(x)$ is an even func^n

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2\,dx}{x^6+1} = \frac{\pi}{3} = PV\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx$$

also, $$\int_{0}^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{6} \quad \{\text{Even func}^n\}$$

Improper ~~analysis~~ integral from Fourier Analysis

$$\int_{-\infty}^{\infty} f(x) \sin ax \, dx \quad , \quad \int_{-\infty}^{\infty} f(x) \cos ax \, dx$$

$$|\sin az|^2 = \sin^2 ax + \sinh^2 ay$$
$$|\cos az|^2 = \cos^2 ax + \sinh^2 ay$$

$\sinh ay = \dfrac{e^{ay} - e^{-ay}}{2i}$: gets problem when $y \to \infty$

$\Rightarrow$ can't apply what we used in previous section for $\int_{-\infty}^{\infty} \dfrac{p(x)}{q(x)} dx$.

$$\Rightarrow \int_{-R}^{R} f(x) \cos ax \, dx + i \int_{-R}^{R} f(x) \sin ax \, dx$$
$$= \int_{-R}^{R} f(x) e^{iax} dx$$

$$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{-ay} \cdot e^{iax}| = e^{-ay}$$

$|e^{iaz}|$ is bounded in the upper half plane $y \geq 0$

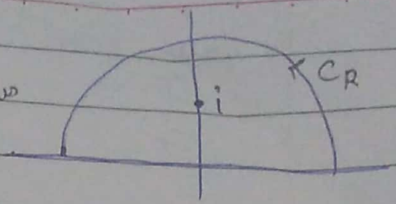
eg. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx$

Consider $f(z) = \dfrac{1}{(z^2+1)^2}$ singularities : $z = \pm i$

e^{i3z} $f(z)$ is analytic everywhere on and above real axis except at $z = i$

Applying cauchy-Residue theorem,

choose $R > 1$



cauchy Principle value (we don't want it in general but since it is an even func^n, it gives the ans)

$$\lim_{R\to\infty} \int_{-R}^{R} f(x) \cdot e^{i3x} = 2\pi i \left(\operatorname{Res}_{z=i} \left(f(z)\cdot e^{i3z}\right)\right) - \oint_{C_R} e^{i3z} f(z)\,dz$$

$\Downarrow$ already real (on real axis)

$$\cancel{\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2} = 2\pi i} \Big[$$

$m = 2$

$$\operatorname{Res}_{z=i} \left(f(z)\cdot e^{i3z}\right) = \frac{1}{1!} \lim_{z\to i} \frac{d}{dz} \left\{ \cancel{(z-i)^2} \cdot \frac{e^{i3z}}{\cancel{(z-i)^2}\cdot(z+i)^2} \right\}$$

$$= \lim_{z\to i} \frac{d}{dz} \left\{ \frac{e^{i3z}}{(z+i)^2} \right\}$$

$$= \lim_{z\to i} \left\{ \frac{3i\, e^{i3z}(z+i)^2 - e^{i3z}\cdot 2\,(z+i)}{(z+i)^4} \right\}$$

$$= \frac{3i\,e^{-3}(2i)^2 - e^{-3}\cdot 2\cdot 2i}{16}$$

$$= \frac{3i\,e^{-3}(-4) - e^{-3}\cdot 4i}{16} = \frac{1}{i e^3}$$

$$\int_{-R}^{R} f(x)\cdot e^{i3x}\,dx = \frac{2\pi}{e^3} - \operatorname{Re}\left[\oint_{C_R} e^{i3z} f(z)\,dz \right]$$

z is a point on C_R $\qquad |z| = R$

$$|f(z)| = \frac{1}{(z^2+1)^2} \le \frac{1}{(R^2-1)^2}$$

$$\therefore |e^{i3z}| = e^{-3}$$

Using ML inequality,

$$\left|\operatorname{Re} \oint e^{i3z} f(z)\,dz\right| \le \left| \oint e^{i3z} f(z)\,dz \right|$$

$\to$ max. value $= 1$

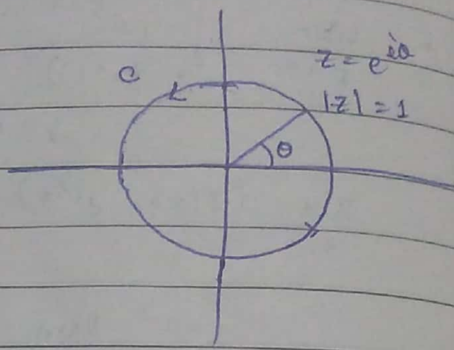
$$\le \frac{\pi R}{(R^2-1)^2} \longrightarrow 0 \text{ as } R\to\infty$$

given func^n is even func^n

$$\Rightarrow \int_{-\infty}^{\infty} \text{---} = PV \int_{-\infty}^{\infty} \text{---}$$

$$= \boxed{\frac{2\pi}{e^3}}$$

→ <u>Some Definite Integrals</u> :-



$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$$

$z = e^{i\theta}$, $r = 1$

$0 \le \theta \le 2\pi$, $R = 1$

$$\frac{dz}{d\theta} = i e^{i\theta} = iz$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

$$d\theta = \frac{dz}{iz}$$

⇒ the integral becomes :

$$\oint_C F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{iz}$$

: If it is rational func^n, we can apply Cauchy-Residue Theorem here.

Ex. $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta}$ $\qquad z = e^{i\theta}$

$$\oint_C \frac{dz}{iz\left\{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)\right\}} = \oint_C \frac{2 \cdot dz}{i\left(2\sqrt{2}z - z^2 - 1\right)}$$

$\frac{\sqrt{8-4}}{2} = \frac{2}{2} \pm 1 \quad \frac{-2\sqrt{2} \pm 1}{2}$

$$= \oint_c \frac{2}{i} \frac{dz}{((2\sqrt{2}z - 1) - z^2)} = \frac{-2}{i} \oint \frac{dz}{z^2 - 2\sqrt{2}z + 1}$$

$$= \frac{-2}{i} \oint \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}$$

$z_0 = \sqrt{2} + 1$ × — lies outside the unit circle

$z_1 = \sqrt{2} - 1$ ✓ — lies inside the unit circle

z_1 is simple pole

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z \to z_1} \cancel{(z - z_1)} \cdot \frac{1}{\cancel{(z - z_1)}(z - \sqrt{2} - 1)}$$

$$= \frac{1}{\sqrt{2} - 1 - \sqrt{2} - 1} = \frac{-1}{2}$$

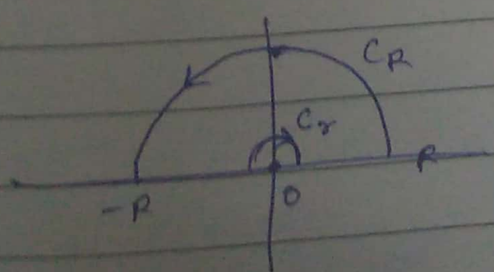
$$\Rightarrow \oint \frac{dz}{(\quad)(\quad)} = 2\pi i \left(\frac{-1}{2}\right)$$

$$\Rightarrow \frac{-2}{i} \oint \frac{dz}{(\quad)(\quad)} = -\pi i \times \frac{-2}{i} = \boxed{2\pi}$$ Ans.

eg. If we have $\cos 2\theta$

$$\cos 2\theta = \frac{e^{i2\theta} - e^{-i2\theta}}{2} = \frac{z^2 - z^{-2}}{2}$$

Intended path : if singularity at real axis $(z=0)$



We can handle this by excluding 0 point.

Cauchy principal value:

$$PV\left(\int_{-\infty}^{\infty} f(x)\,dx\right) = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1\to 0} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty} \int_{0}^{R_2} f(x)\,dx$$

Both of these are equal when $f(x)$ is even func^n.

Proof: $f(x) = f(-x)$

$$\lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\,dx \quad , \quad \text{let } x = -y, \; dx = -dy$$

$$= \lim_{R_1\to\infty} -\int_{R_1}^{0} f(-y)\,dy = \lim_{R_1\to\infty} \int_{0}^{R_1} f(-y)\,dy$$

$$= \lim_{R_1\to\infty} \int_{0}^{R_1} \underbrace{f(-x)}_{\text{even func}^n}\,dx = \lim_{R_1\to\infty} \int_{0}^{R_1} f(x)\,dx$$

$$= \lim_{R\to\infty} \frac{1}{2} \int_{-R}^{R} f(x)\,dx \quad \text{—①}$$

Similarly, $$\lim_{R_2\to\infty} \int_{0}^{R_2} f(x)\,dx = \lim_{R_2\to\infty} \frac{1}{2} \int_{-R}^{R} f(x)\,dx \quad \text{—②}$$

$$\therefore \; ① + ② = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = PV \int_{-\infty}^{\infty} f(x)\,dx$$

Hence proved