Discrete Mathematical Structures
(DMS)

Unit: 1 - Set theory, Functions and Relations
Unit: 2 - Combinatorics and Matrix Algebra
Unit: 3 - Mathematical Logic
Unit: 4 - Number and Graph theory

* Book: Discrete mathematics with Applications
(Author: Thomas Koshy)

* Assessment Criteria:

(1) Attendance: 6%
- Above 95% - 6 marks
- 90 - 95% - 5 marks
- 85 - 90% - 4 marks
- 80 - 85% - 3 marks

⋮

- Less than 70% - 0 marks

(2) Quizzes : 24% (2 + 1 Quizzes)
(3) Mid-term: 30%
(4) End-term: 40%

- Cartesian product:

$A = \{a_1, a_2\} \Rightarrow |A| = 2$

$B = \{b_1, b_2, b_3\} \rightarrow |B| = 3$

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_2, b_3)\}$

$|A \times B| = |B \times A| = 6$

$\therefore R \subseteq A \times B$

- Total no. of subsets : $A = \{a_1, a_2, \dots a_n\}$
$= 2^n$
$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$
OR
each element has 2 choices : either to be part of subset or not to be
$= (2 \times 2 \times 2 \dots n \text{ times}) = 2^n$

- How many relation exist on A?
$\therefore$ Relat^n $R \subseteq A \times A$
$\therefore$ no. of $R = |A \times A| = (n \times n) = n^2$

- All fun^n are rel^n but converse is false.
Ex: In pr. ex. of $A \times B$, we take only elements $\{(a_1, b_1), (a_1, b_2)\}$ as a set, then its a relat^n but not fun^n.

* Set theory:
(Georg Cantor)

(1) Null set or Empty set
$\phi = \{\} \neq \{\phi\}$

(2) Singleton set (1 element)

→ Set representation:

(i) Set-Builder form:

$A = \{x \mid P(x)\}$
↓
predicate / cond^n on x

(ii) List method / Roaster form:
listing elements with { }

Ex: $A = \{a, e, i, o, u\}$

In set builder, $A = \{x \mid x \text{ is a vowel of eng. alphabet}\}$

Ex: $A = \{x \mid x^2 - 3x + 2 = 0\}$
$\therefore A = \{1, 2\}$

→ subset:

$A \subseteq B$
$\Leftrightarrow$ if $x \in A \Rightarrow x \in B \;\forall x$

$A \subseteq B \longrightarrow$ improper subset ($A = B$) may be
$A \subset B \longrightarrow$ proper subset ($A \neq B$)

Ex: $A = \{a, b\}$
subsets of $A = \phi, \{a\}, \{b\}, \{a, b\}$

Power set of given set is collection/set of all the subsets of given set.
$\therefore P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

No. of subsets $= 2^n$; n = cardinality of a set
$= |A| = 2$

Ex: $S = \{Y \mid Y \notin Y\}$
S is a collection of all those elements which do not belong to themselves.

- Does $S \in S$?

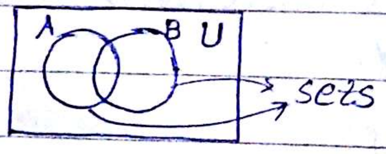
Let
(i) $S \in S \Rightarrow$ through def. $S \notin S$ but if
(ii) $S \notin S \Rightarrow S \in S$ or S should be element of S

$\therefore$ Answer ~~is Yes~~. can be neither Yes/No since there's contradication, being created

Hence, $S = \{Y \mid Y \notin Y\}$ is called Russel's paradox.
$\Rightarrow$ No set contains all sets.
So, acc. to this paradox, U doesn't exist however we chose U in the domain of our question.

Ex: $A = \{a\}$
$B = \{a, c, d\}$
$C = \{b, e, f, g, h, i\}$
$D = \{f, i, j\}$
So, $U = \{a, b, c, d, e, f, g, h, i, j\}$ OR $U = \{a \text{ to } z\}$ & so, on.

→ Venn Diagrams:



Finite – Able to count no. of elements.

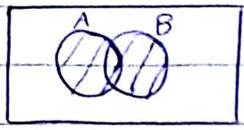
* Operations on Set:
(i) Union
(ii) Intersection
(iii) Difference
(iv) Symmetric diff.
(v) Cartesian product.
(vi) Complement.

if $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

All operations are considering U.

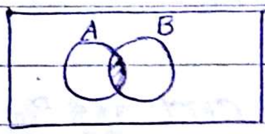
$\{a, a, b, c, c\} = \{a, b, c\} \Rightarrow$ repetition counted only onc

(i) $A \cup B \Rightarrow x \in A$ or $x \in B$ or $x \in A$ & B both



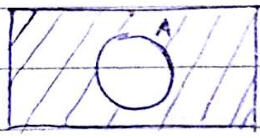
$A \cup B$

(ii) $A \cap B \Rightarrow x \in A$ and $x \in B$



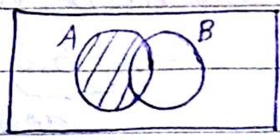
$A \cap B$

(iii) A^c or $A' \Rightarrow$ if $x \notin A \Rightarrow x \in A^c = U - A$



A^c

(iv) $A - B \Rightarrow$ if $x \in A \xrightarrow{\&} x \notin B$



$A - B$

(v) Symmetric diff. ($\oplus$ or Δ)

$$A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Proof: Let $x \in (A \cup B) - (A \cap B)$, then

$\Rightarrow (x \in A$ or $x \in B)$ and $x \notin (A \cap B)$

$A \times B \neq B \times A$
$A - B \neq B - A$
$A \oplus B = B \oplus A$

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$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B)$
$\Rightarrow (x \in A$ or $x \notin B)$ and $(x \in B$ or $x \notin A)$
$\Rightarrow x \in (A-B) \cup (B-A) \Rightarrow (A \cup B) - (A \cap B) \subseteq (A-B) \cup (B-A)$
Similiarly $(A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B)$ $\therefore$, They are equ

* Cartesian product:

Application:

$(J) \xrightarrow{B,T,P} (D) \xrightarrow{T,P} (R)$
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$\therefore$, $J \to D = \{B, T, P\}$
$D \to R = \{T, P\}$

$\therefore$, No. of ways to go $J \to R = (J \to D) \times (D \to R)$ = cartes. prod

$\rightarrow$ If $(x, y) \in A \times B \Rightarrow x \in A$ and $y \in B$.

$(x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \ldots \times A_n \Rightarrow x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n$

* Partition sets:

$U = \{a, b, c, d, e, f, g, h, i\}$
$A = \{a, b, c\}$
$B = \{d, e, f\}$
$C = \{g, h, i\}$

- A, B, C = disjoint Sets &
- $A \cup B \cup C = U$

$\therefore$, $\underbrace{A, B, C}_{\text{each should be non-emp}}$ = partition sets of U.

* Recursively Defined sets:

$S = \{2, 2^2, 2^{2^2}, 2^{2^{2^2}}, 2^{2^{2^{2^2}}}, \ldots\}$

$\downarrow$
primitive/basic clause or fund. ele. = which can't be generated by a def. property.

- Properties:

(i) primitive element existence
(ii) $x \in S \Rightarrow 2^x \in S$
(iii) Termination law: (i) and (ii) are only way to define set.

* Fuzzy Sets

$A = \{x, \mu_x(A)\} \longrightarrow \{0 \le \mu_x(A) \le 1\}$

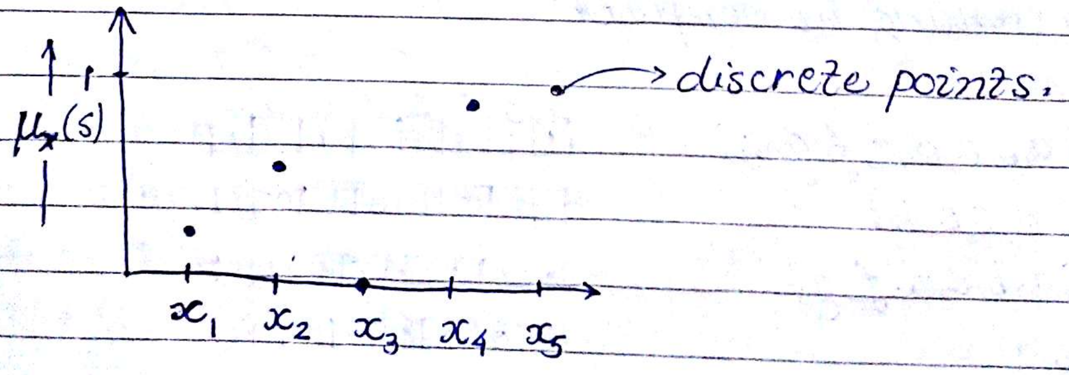
$\mu_x(A)$ ↓ degree of belonginginess of x to A

Ex: $X = \{x_1, x_2, x_3, x_4, x_5\}$

$S = \{(x_1, 0.2), (x_2, 0.5), (x_4, 0.9), (x_5, 1)\}$; $x_3 \notin S$ i.e $(x_3, 0)$
$T = \{(x_1, 0.6), (x_2, 0.8), (x_3, 0.9), (x_4, 1)\}$

$S' = \{(x_1, 0.8), (x_2, 0.5), (x_4, 0.1), (x_3, 1)\}$
$T' = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0.1), (x_5, 1)\}$

→ max. membership of x_1 from S & T
$S \cup T = \{(x_1, 0.6), (x_2, 0.8), (x_3, 0.9), (x_4, 1), (x_5, 1)\}$
$S \cap T = \{(x_1, 0.2), \ldots\ldots (x_5, 0)\}$
↳ min. membership



- Partition set of an ∞ set:

Ex: Z – set of integers.

Partit^n sets: base = remainder left when $\div$ by 5

$Z_0 = \{ \cdots, -10, -5, 0, 5, 10, \cdots \}$

$Z_1 = \{ \cdots -9, -6, -1, \cdots 1, 6, 9, \cdots \}$

$Z_2 = \cdots$

$Z_3 = \cdots$

$Z_4 = \cdots$

$\Sigma = \{a, b, c\}$ = alphabets

Σ^* = words using Σ

$= \{\lambda, a, b, c, aa, ab, \cdots\}$

$\|a\|$ = length of $a = 1$

$\|aa\| = 2$

$\|\lambda\| = 0$

Subset of Σ^* = language.

Ex: $S = \{a, \overset{\wedge}{} ba, ca\}$ in set-builder
aa

$S = \{ x \in \Sigma^* \mid 1 \le |x| \le 2$ & x ends by $a \}$

Ex: Representing in computer:

$U = \{a, b, c, d, e, f, g, h\}$

$A = \{a, c, e, h\}$

$B = \{a, b, c, d, f, g\}$

$A \cap B$

h	g	f	e	d	c	b	a
1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	1
0	1	1	0	1	1	1	1
0	0	0	0	0	1	0	1

... [illegible] ...

$x \in (A \cup B)' \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B \Rightarrow x \in A'$ and $x \in B' \Rightarrow x \in A' \cap B'$

$\therefore (A \cup B)' \subseteq A' \cap B'$

Similiarly, $A' \cap B' \subseteq (A \cup B)' \Rightarrow (A \cup B)' = A' \cap B'$

$A \cap B = A \overset{\text{and}}{\wedge} B$

$A \cup B = A \overset{\text{OR}}{\vee} B$

$A \oplus B = A$ XOR B (single 1)

$A^c = \sim A$

$\boxed{A - B = A \cap B'}$ *

* De-morgan's laws:

*$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

* Distributive law:

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Q Prove using Set laws:

$(X - Y) - Z = X - (Y \cup Z)$

Sol^n: $(X \cap Y') - Z = \overset{\text{RHS}}{X \cap (Y \cup Z)'}$

$(X \cap Y') \cap Z' = X \cap (Y' \cap Z') = \text{LHS}$

$= X \cap (Y \cup Z)' = \text{LHS} = \text{RHS}$

$= X - (Y \cup Z)$

(P1) $|A \cup B| = |A| + |B| - |A \cap B|$

(P2) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

Proof: Let $X = A$ and $Y = (B \cup C)$, then,

$(X \cup Y)$ from (P1), $= |X| + |Y| - |X \cap Y|$

$= |A| + |B \cup C| - |A \cap (B \cup C)|$

Now, using (P1) on $B \cup C$ and distributive law on $A \cap (B \cup C)$

$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$

Now, using (P1) on last term,

$= |A| + |B| + |C| - |B \cap C| - \{|A \cap B| + |A \cap C| - |A \cap B \cap A \cap C|\}$

$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Functions

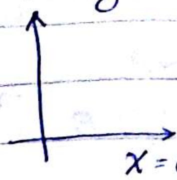
$f : x \longrightarrow y$

$f(x) = y \curvearrowright$ output

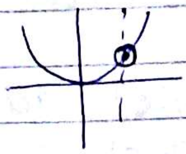
$\downarrow$ input/argument

$x \in$ Domain(f)

y = range

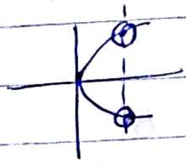


x = domain



fun^n ✓

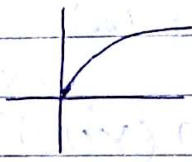
$x^2 = 4ay$



not unique image
$\therefore$, not fun^n

$y^2 = 4ax$

Ex: $y = 2\sqrt{ax}$; $a \neq 0$
$D(y) = R^+ \cup \{0\} = [0, \infty)$



Ex: $(x-a)(x-b) \leq 0$; $a < b$
$x \in [a, b]$

$(x-a)(x-b) \geq 0$
$x \in (-\infty, a] \cup [b, \infty)$

$\rightarrow D(f+g) = D_1(f) + D_2(g)$
$= D_1 \cap D_2$

$\rightarrow D(f \cdot g) = D_1 \cap D_2$

- Composition of fun^n:

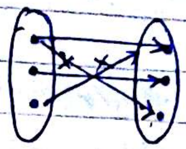
$f : X \longrightarrow Y$
$g : Y \longrightarrow Z$

$g \circ f : X \longrightarrow Z$ Range$(f) \subseteq$ Domain(g)

$g \circ f(x) = g(f(x))$

- Types of functions:

(1) One-One (Injective):



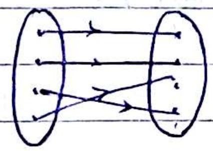
f is one-one if
$x_1 \neq x_2 \in X \Longrightarrow f(x_1) \neq f(x_2)$

(i.e each element in domain has unique image)

OR

f is one-one if
$x_1 = x_2 \in X \Rightarrow f(x_1) = f(x_2)$

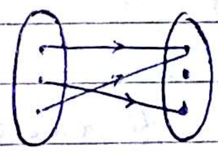
(2) Onto (surjective):



$f: X \longrightarrow Y$

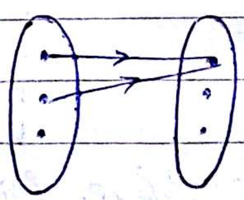
$Range(f) = Y$

(3) Into:



$Range(f) \subset Y$

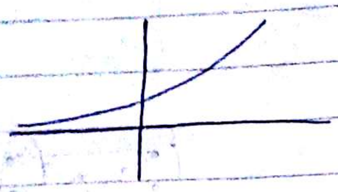
(4) Many-One:

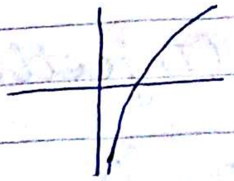


→ One-one correspondence / Bijective:

Both one-one and onto.
If f is bijective inverse / inverse mapping exists.

- e^x, a^x ; $a \neq 1$
- $\log_a x$; $a \neq 1, x > 0$





- Floor/greatest int. funⁿ:

greatest left side int.

$[3.14] = 3$ $[3] = 3$

$[-3.14] = -4$ $[-3] = -3$

$[2.9] = 2$

- Ceiling funⁿ:

least int. to right

$[3.14] = 4$

$[-3.14] = -3$

- Characteristic funⁿ:

$S \subseteq U$

$f : U \to \{0,1\}$

$f_S(x) = \begin{cases} 1 & , x \in S \\ 0 & , \text{otherwise} \end{cases}$

Q Prove: $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$

where, $A, B \subseteq U$

$\rightarrow f_A(x) = 1$; $x \in A$

$\rightarrow f_B(x) = 1$; $x \in B$

$f_{A \cap B}(x) = f_A(x) \wedge f_B(x) = 1 \times 1$

$\rightarrow = f_A(x) \cdot f_B(x)$

- If a and b are 2 positive integers then the no. of positive integers $\leq a$ and divisible by b is given by:

$$\lfloor a/b \rfloor$$

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Q Find the no. of +ve integers ≤ 1776 and divisible by 13.

Sol^n: $\left\lfloor 1776/13 \right\rfloor = 136$

Q Find the no. of +ve int. ≤ 3000 and not divisible by 7 and 8.

Sol^n: $\left\lfloor 3000/7 \right\rfloor = 428 = |A|$ $\quad \left\lfloor 3000/8 \right\rfloor = 375 = |B|$ $\quad \left\lfloor 3000/56 \right\rfloor = 53 = |A \cap B|$

$A = \{x \in Z^+ \mid x$ is divis. by 7 & $x \leq 3000\}$
$B = \{x \in Z^+ \mid x$ is divis. by 8 & $x \leq 3000\}$

$= |A' \cap B'| = |(A \cup B)'| = |U - (A \cup B)| = |U| - \{|A| + |B| - |A \cap B|\}$
$= 3000 - [(428 + 375) - 53] = (3000 - 750) = 2250.$

Q The no of leap year (l) after 1600 and not exceeding a given year $y = ?$ Let $y = 2017$.
(Hint: if century, divis. by 400, otherwise divis. by 4)

Sol^n: $l =$ (no. of l.y till 2017 − no. of l.y till 1600)

no. of l.y till ~~2016~~ 1600 = (total l.y − no. of century + no. of cent. l.y.)

$= \left\lfloor 1600/4 \right\rfloor - 16 + \left\lfloor 1600/400 \right\rfloor = 388$

Similiarly till 2017 =
$\left\lfloor 2017/4 \right\rfloor - 20 + \left\lfloor 2017/400 \right\rfloor = 489$

$\therefore l = (489 - 388) = 101$

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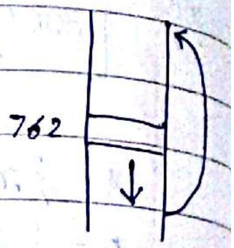
Q Suppose today is Friday. What day will it be 99 days after from today.

Sol^n: $99 \% 7 = 1$ ∴ (Fri+1) = saturday.

○ Hashing:

$h(x) = x \bmod m$
(m = size of array)
= prime



Ex A/c: $207630764 = x$
$x \% 1209 = 762$
Now, if there is another no. which also gives 762, then this is called collision. In such case linearly 763, 764, ... will be checked if empty and if not, then assignment of A/c no. will occur from beginning. This is called Linear probing.

- If X and Y are finite sets of same cardinality and f is a mapping from X to Y, which is one-one if ~~or~~ and only if it is onto.

Proof: $f: X \to Y$, then let
$X = \{x_1, x_2, \cdots, x_n\}$
$Y = \{y_1, y_2, \cdots, y_n\}$ ∴ $|X| = |Y|$
Suppose f is one-one, then
$f(x_1), f(x_2), \cdots$ are different
Hence, all elements of X are uniquely mapped with all elem. of Y ∴, its possible ∴, only if f is onto.

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Cantor: $|N| = \aleph_0$ = eliphnot

- If X and Y are 2 finite sets having the same cardinality, then there must be mapping which is one-one and onto.

Proof: For finite sets only,
if $A \subseteq B \Rightarrow |A| \leq |B|$

* Countable & Uncountable Sets:

↳ which can't be counted.

A set S is called countable if it is either finite or countably ~~finite~~ infinite.

→ A set S is countably infinite if there exists a bijection between S and N.

→ Cardinality of every countably infinite set is eliphnot ($\aleph_0$).
($Ex: |N| = |Z^+$ which are even$| = \aleph_0$)

Ex: N×N is countably ∞ $\because$ it can be written in definite pattern

* Pigeon-hole principle:

If S is a mapping from X to Y, where X and Y are finite sets and $|X| = m$, $|Y| = n$, with $m > n$, then there exists atleast 2 distinct elements x_1, x_2 such that $f(x_1) = f(x_2)$.

\# m # pigeons
n # holes
$m > n$

If m pigeons are assigned to n-pigeon-holes, then there must be a hole containing atleast $\lfloor (m-1)/n \rfloor + 1$ pigeons.

Pf: No pigeon-hole contains atmost $\lfloor (m-1)/n \rfloor$ – suppose

max. no. of pigeons $= n\lfloor (m-1)/n \rfloor$

$$\leq \frac{\cancel{n}(m-1)}{\cancel{n}}$$

$\leq (m-1)$ pigeons but its given m pigeons.

Q If we select 1,000 students from campus, then atleast 3 of them must have same birthday.

Sol^n: $m = 1000$ $\therefore \lfloor (m-1)/n \rfloor + 1 = \lfloor 999/366 \rfloor + 1 = 3$

$n = 366$

* Proof Techniques:

(1) Trivial technique:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad \text{— induction}$$

Check for $n=1$, If its true (i.e. LHS = RHS), its true for other no.

(2) Direct proof:

Product of 2 odd integers is also an odd integer.

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$x = 2m+1$
$y = 2n+1$

then, $x*y = 4mn + 2m + 2n + 1$
$= 2(mn + m + n) + 1 =$ odd no.

(3) Indirect proof technique:

(i) Contrapositive
(ii) Contradiction

(i) Contrapositive:
If p then q $\equiv p \rightarrow q$
then $\sim q \rightarrow \sim p$

(ii) Contradiction:
If p then q $\equiv p \rightarrow q$
then $\sim q$

Q Prove that there exists largest prime no. ✗
There's no largest prime no.

Pf by contradiction:
Let there be infinitely many prime no.

$\exists$ a largest prime p_k (say)
$p_1, p_2, \ldots\ldots, p_k$ are k prime no.

Let $x = (p_1 \cdot p_2 \cdots\cdots p_k) + 1 \Rightarrow x$ is not divisible by any prime. $\forall p_i$

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x must be either prime or composite

Case: 1 : x is a prime $> p_K$

Case 2: x must be divisible by a prime no. $\neq p_i$ (say q)

$\therefore$, p_K ~~is~~ = largest prime doesn't exist.

Proof by Cases:

KNIGHT Always speaks the truth
KNAVE Always tell lie.

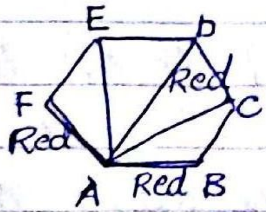
Suppose, A, B & C are only 3 on an island.
Statement: A says B is knave & B says that A & C have same type then proof that C is knave

∘ Case : 1
If A is knight $\Rightarrow$ B is knave
$\because$ A & C are same $\Rightarrow$ C is knight but C & A are diff.
$\Rightarrow$ C = knave

∘ Case : 2
If A is knave $\Rightarrow$ B is knight and C = knave ✓

Suppose we are joining every pair of vertices of hexagon by line segment either red/blue. Then prove ~~each~~ the line segment will form monochromatic Δ.

Sol^n $m = 5$ = no. of line segment to other vertex.
n = no. of colors = 2



$\therefore \lfloor (m-1)/n \rfloor + 1 = (2+1) = 3$

Let AF, AD and AB be Red.
Now, considering ΔADF,
Case: 1 : FD is Red $\Rightarrow$ ΔADF is monochromatic
Case: 2 : FD is blue, then if BD is Red $\Rightarrow$ ΔABD = mono.
BD is blue $\Rightarrow$ ΔABD = non-mono.

Now,
If BF is Red, ΔAFB is mono.
If BF is blue, ΔBDF is mono.

* Logic:
It specifies the meaning of mathematical statement

* Proposition: Declarative statement.

Ex: Delhi is capital of India ✓
$2+2=5$ ✓
$x+1=4$ ×

* Division Algorithm:
$a \in Z, b \in Z^+, q, r \in Z$ s.t. $0 \le r < b$ and $a = bq + r$

Q What is the remainder & the qotient when -23 is divided by 5?
Careful: quotient = -5, rem. = 2 $\longrightarrow$ is true ✓
" = -4, rem. = -3 $\longrightarrow$ its false ×

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Q Let b is an integer such that $b \geq 2$ are randomly selected, then prove that the difference of some of the them must be divisible by 3.

Sol^n: $a = bq + r$; $0 \leq r < b$

b is divisible by a if $r = 0$

$a_i = bq + r_i$; $0 \leq r_i < b$ $\forall\ i = 1, 2, \ldots (b+1)$

$\downarrow$

$a_1, a_2, \ldots, a_{b+1}$

How many (b+1) remainder ——— 'b'

$\therefore$ applying pigeon-hole principle.

#

① If a & b are +ve integers and if (a divides b) & (b divides a) $\Rightarrow a = b$

$\Downarrow a/b$ $\Downarrow b/a$

② If $a, b, c \in Z$, $a/b, b/c \Rightarrow a/c$

③ If $a/b \Rightarrow a/bc$

Euclidean Algorithm (Division method)

④ Relatively prime if HCF = 1

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* mathematical Induction:

$$P(n) \quad ; \quad n \geq n_0$$

(1) Basis step: $P(n_0)$ should be ~~s~~ true/satisfy
(2) Inductive step: If $P(k)$ is true for any $k \geq n_0$
$\Rightarrow P(k+1)$ must be true

$\therefore$, $P(n)$ is always true.

Q Prove that: $2n^3+3n^2+n$ is divisible by 6
Sol^n: Let $P(n) = 2n^3+3n^2+n$ is divisible by 6 ; $n \geq 1$

(1) $P(1) = 2(1)^3+3(1)^2+1 = 6$ which is divisible by 6 ✓
(2) $P(k) = 2k^3+3k^2+k$ is if divisible by 6
$\therefore$, $2k^3+3k^2+k = 6m$; m is any integer
$P(k+1) = 2(k+1)^3+3(k+1)^2+(k+1)$
$= (k+1)[2k^2+2+4k+3k+3+1] = (k+1)[2k^2+7k+6]$
$= 6[m+(k+1)^2]$ ✓

Q A set containing n elements has 2^n subsets. Prove.
Sol^n: (1) Basis: If $n=0$, that ϕ is subset.
$\therefore$, $2^0 = 1$ subset $= \{\phi\}$ ✓
(2) Inductive: Let $|A| = k$ has 2^k subsets, then
$|B| = k+1$ and constructing, $B = A \cup \{x\}$
$\because A \subseteq B \Rightarrow$ every subset of A is a subset of B.
(i) $\therefore$, say $A_1, A_2, \ldots, A_k$ are 2^k subsets of A $\forall A_i \subseteq B$
(ii) $\therefore$, $A_1 \cup \{x\}, A_2 \cup \{x\}, \ldots, A_k \cup \{x\}$ — 2^k
$\therefore$, (i) and (ii) are all subsets of B
$\therefore$, Total subsets of $B = (2^k + 2^k) = 2 \cdot 2^k = 2^{k+1}$ ✓

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* Bernoulli's Inequality:

$(1+x)^n \geq (1+nx)$; $x > -1$

$P(n) : n \geq n_0$

- Basic step: To check the validity of $P(n_0)$
  (truthfulness of $P(n_0)$)
- Inductive step: for $k \geq n_0$
  Assumptions are $P(n_0)$, $P(n_0+1)$, ..., $P(k)$ all are true
  $\Rightarrow$ $P(k+1)$ is also true.

Q. Every +ve integer ≥ 2 is either a prime or can be represented as a product of prime.

Sol^n^ $P(n)$ = Every +ve $n \geq 2$ is either a prime / product of primes
Basic step: $P(n = n_0 = 2)$: 2
$\because$ 2 is prime, $P(n_0)$ is true
Inductive step: any $k \geq 2$
Assumptions: $P(2)$, $P(3)$, ... $P(k)$ all are true
$\therefore$ We will prove $p(k+1)$: $k+1$ ~~is p~~
Suppose, $k+1$ = prime —— case-1
$\Rightarrow$ $P(k+1)$ is true
Suppose $k+1$ = not prime —— case-2
$\therefore$ $(k+1) = xy$ $\therefore$ $2 \leq x, y < (k+1)$
$\therefore$ $2 \leq x, y \leq \frac{(k+1)}{2} < (k+1)$ x, y = prime / prod. of prime
$\therefore$ $P(x)$, $P(y)$ are true.

Every non-empty set of non-negative integers has a least element.

start algo
product = 0,
$i = 0$,
while $(i < x)$
$\{ product_{k+1} = product_k + y;$
$i_{k+1} = i_k + 1; \}$

7. print product
8. Terminate

$Product_n$ is the value of product after n^{th} iteration.
i_n is value of i after i^{th} iterat^n.

Mathematical Induc^n:
$P(n)$: $product_n = i_n \cdot y$, $0 \leq n < x$
Basis step : $p(0)$
$product_0 = 0$, $i_0 = 0$
$LHS = RHS \Rightarrow P(0)$ is true
Inductive step: If $p(k)$ is true ; $k \geq 0$
$\Rightarrow P(k)$: $Product_k = i_k y$ —— (A)
$product_{k+1} = product_k + y$ —— from line (5)
$= \cancel{ik} y i_k + y = (i_k + 1) y$ —— (A)
$= i_{k+1} \cdot y$ —— from line (6)

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* Congruence:

$a, b \in Z$, $m \in N$, then

$a \equiv b \mod m$

if $m \mid (a-b)$

Pf: From division algorithm, $\Rightarrow a = mq_i + b$

- properties of congruency.

① Suppose $a \equiv b \mod m$ & $c \equiv d \mod m$ then

$(a+c) \equiv (b+d) \mod m$?

$ac \equiv bd \mod m$?

* Application in computer science:

→ Pseudo-random numbers

Linear congruence method

you input 4 int. $2 \leq a < m$

$0 \leq c < m$; $0 \leq x_0 < m$

$x_n = (ax_0 + c) \mod m$

$x_1 = (ax_0 + c) \mod m$

$x_2 = (ax_1 + c) \mod m$

Find random no.'s betn 0 & 1?

* Recursion:

$a \in W$
$x = \{a, a+1, a+2, \dots\}$

(i) Basic clause: some initial funⁿ values should be defined
(ii) Recursive clause: $f(n)$ should be defined for k previous values from $f(n-1)$ to $f(n-k)$.
(iii) Termination clause:

Ex: No. of handshakes that can be done in a party with n guests (Let this recursive f^n be $h(n)$)?

Sol: If only 1 person is in party: $h(1) = 0$ handshakes
similiarly for n persons : $h(n) = (n-1)$ " + pr. "
only

$$\therefore h(n) = \begin{cases} h(n-1) + (n-1) & ; n \geq 2 \\ 0 & ; n = 1 \end{cases}$$

Ex: factorial of no.:

$$Fact(n) = \begin{cases} 1 & ; n = 0 \\ n\,fact(n-1) & ; n > 0 ; n \in N \end{cases}$$

Q Mr. x deposits ₹1000 at a bank at an interest rate 8% which is compounded annualy. Define recursively the compound amount: $A(n)$, he will have in his account at the end of n years.

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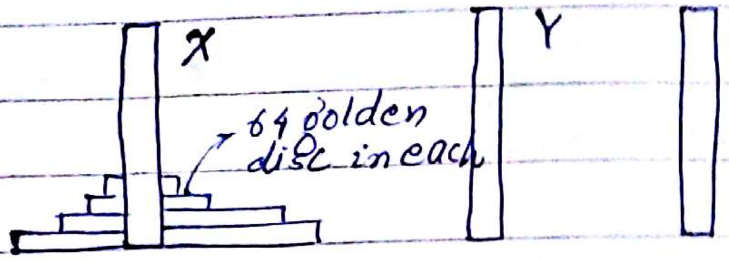
Tower of Brahma



Condit^n:

Only 1 disc can be moved at a time.
No disc can be placed on the top of smaller disc when transferring.

h(n) = no. of moves in transferring disc from 1 to another using/with the help of n auxillary peg.

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Ex: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$; $a_0, a_1, \dots a_k$

Its characteristic eq^n:

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \dots - c_k = 0$$

- If α is one of the roots of characteristic eq^n of multiplicity m

→ Basic sol^n are:

$\alpha^n, n\alpha^n, n^2\alpha^n, \dots, n^{m-1}\alpha^n$

General Sol^n: $A\alpha^n + B \cdot n\alpha^n + \dots + C n^{m-1}\alpha^n$

Q $a_n = 5a_{n-1} - 6a_{n-2}$; $a_0 = 4$, $a_1 = 7$

Sol^n: $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Characteristic eq^n: $x^2 - 5x^1 + 6x^0 = 0$

$= x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

All roots have multiplicity $m = 1$

∴ Basic sol^n: $(2)^n$ & $(3)^n$

General sol^n: $A(2)^n + B(3)^n = a_n$

∴ $a_n = A \cdot 2^n + B \cdot 3^n$

Now to find A, B use initial $cond^n$:

$a_0 = 4 = A \cdot 2^0 + B \cdot 3^0 \Rightarrow A + B = 4$

$a_1 = 7 = A \cdot 2^1 + B \cdot 3^1 \Rightarrow 2A + 3B = 7$

$\Rightarrow A = 5$, $B = -1$

Hence: $a_n = 5 \cdot 2^n - 3^n$

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Q $a_n = 7a_{n-1} - 13a_{n-2} - 3a_{n-3} + 18a_{n-4}$
where, $a_0 = 5, a_1 = 1, a_2 = 6, a_3 = -21$

Solⁿ: It is linear & homogenous of order 4 ✓
∴, Its characteristic eqⁿ:
$x^4 - 7x^3 + 13x^2 + 3x - \underline{18} = 0$
(±)
factors: 1, 2, 3, 6, 9; 18 = possible roots
∴, roots: $x = -1, 2, \underbrace{3, 3}_{} \longrightarrow$ multiplicity = 2
multiplicity = 1
∴, $(-1)^n, 2^n, 3^n, n3^n$ — Basic solⁿ
General solⁿ: $A(-1)^n + B\cdot 2^n + C\cdot 3^n + nD\cdot 3^n = a_n$

Now, using a_0, a_1, a_2, a_3 find A, B, C, D

* Linear Non-Homogenous Recurrence Relation with constant coefficient of (any order k):

(A) $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} + f(n) \longrightarrow$ Non-homogenous

Associated LHRRWCC is $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k}$ (B)

Suppose the solⁿ of B is a_n^*
We will choose another particular solⁿ of given eqⁿ (A) a_n^p
∴, general solⁿ: $a_n = a_n^h + a_n^p$

Suppose: $f(n) = [b_k n^k + b_{k-1} n^{k-1} + \dots + b_1 n + b_0]\alpha^n$, then

Rule: 1: If α is not the root of A LHRRWCC (not characteristic root)

then, $a_n^P = [C_k n^k + C_{k-1} n^{k-1} + \dots + C_1 n + C_0] \alpha^n$

Rule: 2: If α is the root of A LHRRWCC [characteristic multiplicity m]

then, $a_n^P = n^m [\alpha_k n^k + \alpha_{k-1} n^{k-1} + \dots + \alpha_1 n + \alpha_0]$

$a_n = 5a_{n-1} - 6a_{n-2} + 8n^2$ ——— (1) $a_0 = 4, a_1 = 7$

→ Its associated LHRRWCC is $a_n = 5a_{n-1} - 6a_{n-2}$

$a_n = a_n^{(h)} + a_n^{P}$ - general sol^n $\qquad a_n - 5a_{n-1} + 6a_{n-2} = 0$

→ 1 particular sol^n of given eq^n

$\therefore$ Its charac. eq^n = $x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

$(2)^n, (3)^n$ - basic sol^n

$\therefore a_n^{(h)} = A(2)^n + B(3)^n$ ——— (A)

Now,

for particular sol^n,

$f(n) = 8n^2 = \underbrace{[C_K n^K + C_{K-1} n^{K-1} + \dots + C_0]}_{8n^2 \cdot 1^n} \cdot \alpha^n$ (Here, $\alpha = 1$) } Rough part

Applying Rule 1: $\qquad = [8n^2 + 0n + 0] \cdot 1^n$

Let us assume a particular sol^n of eq(1) is $a_n^P = [d_2 n^2 + d_1 n + d_0]$

put in (1)

$a_n^P = a_n = [d_2 n^2 + d_1 n + d_0] = 5[d_2(n-1)^2 + d_1(n-1) + d_0]$
$\qquad - 6[d_2(n-2)^2 + d_1(n-2) + d_0] + 8n^2$

Comparing LHS & RHS coeff:

$d_2 = 5d_2 - 6d_2 + 8 \Rightarrow d_2 = 4, d_1 = 28, d_0 = 60$

$\therefore a_n^P = [d_2 n^2 + d_1 n + d_0] = [4n^2 + 28n + 60]$

Now, general solⁿ of eqⁿ(1) is:

$a_n = a_n^h + a_n^p = A\cdot 2^n + B\cdot 3^n + 4n^2 + 28n + 60$

$A = -83,\ B = 27$

Q $\underline{a_n = 6a_{n-1} - 9a_{n-2}} + 4(n+1)\cdot 3^n$ ——(1), $a_0 = 2,\ a_1 = 3$

Its associated LHRRWCC is $a_n - 6a_{n-1} + 9a_{n-2} = 0$ ——(2)

Its characteristic eqⁿ is:

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0 \Rightarrow x = 3, 3\ (m = 2)$

$\therefore$

Basic solⁿ = $3^n, n\cdot 3^n$

$\therefore a_n^{(h)} = A\cdot 3^n + B\cdot n\cdot 3^n$ ——(3)

$f(n) = 4[n+1]\cdot 3^n \quad \therefore C_K = 4, K = 1, C_0 = 4$

Lets assume a particular solⁿ:

$a_n^p = n^2[d_1 n + d_0]\cdot 3^n$

Ans = $d_1 = 2/3,\ d_0 = 4$

* Generating functions:

Ex: $1 + x + x^2 + x^3 + x^4 + x^5 = \left(\frac{x^6 - 1}{x - 1}\right)$ is a generating fⁿ for coeff: 1, 1, 1, 1, 1, 1

Ex: $1 + 2x + 3x^2 + 4x^3 + \cdots$ is generating fⁿ for natural no.

Ex: $g(x) = a_0 + a_1 x + a_2 x^2 + \dots$

$$= \sum_{n=0}^{\infty} a_n x^n$$

$$f(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

$$= \sum_{n=0}^{\infty} b_n x^n$$

$$f(x) + g(x) = \sum_{n=0}^{\infty} [a_n + b_n] x^n$$

$$f(x) \cdot g(x) = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} a_i b_{n-i} \right] \cdot x^n$$

Ex: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ (∞ GP)

$$= \sum_{n=0}^{\infty} x^n$$

Now, $\frac{1}{(1-x)^2} = \frac{1}{(1-x)} \cdot \frac{1}{(1-x)} = \left[\sum_{n=0}^{\infty} x^n \right] \cdot \left[\sum_{n=0}^{\infty} x^n \right]$

$$= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} 1 \times 1 \right] x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots + (n+1)x^n$$

Ex: $\frac{1}{(1-x)^3} = \frac{1}{(1-x)} \times \frac{1}{(1-x^2)}$

$$= \left[\sum_{n=0}^{\infty} x^n \right] \left[\sum_{n=0}^{\infty} (n+1) x^n \right] = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} 1 \times (n+1-i) \right] \cdot x^n$$

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Q. Use generating f^n to solve tower of Henoi.
$b_n = 2b_{n-1} + 1$, $b_1 = 1$

Sol^n: Let $g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n + \cdots$

put $n=1$ in $b_n \Rightarrow$ $\overset{1}{\cancel{b_1}} = 2b_0 + 1 \Rightarrow b_0 = 0$

$2x\,g(x) = +2b_0 x + 2b_1 x^2 + 2b_2 x^3 + \cdots + 2b_{n-1} x^n + 2b_n x^{n+1} + \cdots$

$$\frac{1}{(1-x)} = 1 + x + x^2 + \cdots$$

$$\frac{1}{(1-ax)} = 1 + ax + a^2x^2 + \cdots$$