5/Sep/2017

Assignment-1

Q1.

(i) $(1+i)^{-1} = \frac{1}{1+i} = \frac{1-i}{2} = \frac{1}{2} - i\frac{1}{2}$

(ii) $\frac{i}{1+i} = \frac{i(1-i)}{2} = \frac{i+1}{2} = \frac{1}{2} + i\frac{1}{2}$

(iii) $\frac{1}{-1+i} = \frac{-1-i}{2} = \frac{-1}{2} - i\frac{1}{2}$

(iv) $\frac{5+5i}{3-4i} = \frac{(5+5i)(3+4i)}{25} = \frac{(1+i)(3+4i)}{5}$

$= \frac{3+4i+3i-4}{5} = \frac{-1+7i}{5} = \frac{-1}{5} + i\frac{7}{5}$

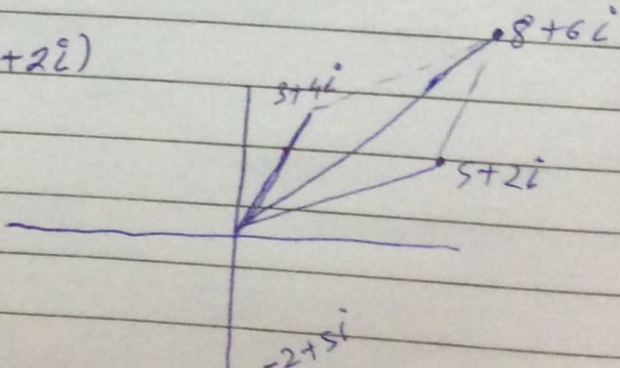
(v) $\frac{3i^{30} - i^{19}}{2i-1} = \frac{3i^{2} - i^{3}}{2i-1} = \frac{-3+i}{2i-1} = \frac{(i-3)(-1-2i)}{5}$

$= \frac{-i+2+3+6i}{5} = \frac{5+5i}{5} = 1+i$

Q2.

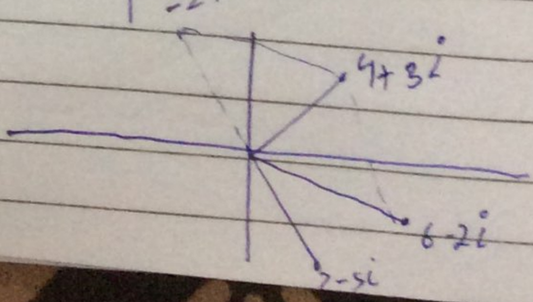
(i) $(3+4i) + (5+2i)$

$8+6i$



(ii) $(6-2i) - (2-5i)$

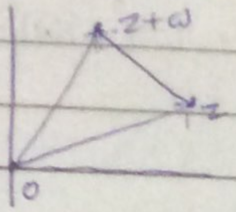
$4+3i$



Q3

(i) $|z+\omega| \leq |z| + |\omega|$

Graphically Proof: Let us consider three points $0, z, z+\omega$



According to triangle inequality sum of two side should be greater than third side.

Three sides of this triangle are

$|z|$ $|\omega|$ and $|\omega+z|$ thus by applying triangle inequality $|z|+|\omega| \geq |z+\omega|$

and equality holds when ~~|z| and |ω| and |z+ω|~~ z and ω ~~are~~ are in straight line Thus

$$|z+\omega| \leq |z| + |\omega|$$

(ii) $|z+\omega| \geq ||z|-|\omega||$

Let points be

$[z+\omega-z]$, $(-z)$, $(z+\omega)$

Thus by triangle inequality ~~and we know that equality holds when~~

$$|z+\omega-z| \leq |z+\omega| + |-z|$$

$$|\omega| - |z| \leq |z+\omega|$$

for. $|\omega| > |z|$

for $|\omega| < |z|$

$$|z| - |\omega| \leq |\omega + z|$$

Thus combining we get $||z| - |\omega|| \leq |z+\omega|$

(iii) $|z+\omega|^2 + |z-\omega|^2 = 2(|z|^2+|\omega|^2)$

$= |z|^2 + |\omega|^2 + 2|z||\omega|\cos\theta + |z|^2+|\omega|^2 - \cancel{2|z||\omega|}\cos\theta$

$= 2(|z|^2+|\omega|^2)$

04.

(i) $z = \dfrac{-2}{1+\sqrt{3}i} = \dfrac{-2(1-\sqrt{3}i)}{\cancel{4}\,2} = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}i$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$2\pi/3$

(ii) $z = -2-2i$

$= 2(-1-i) = 2\sqrt{2}\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$-3\pi/4$

(iii) $z = 2-2i$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$-\pi/4$

(iv) $z = -2+2i$

$= 2\sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

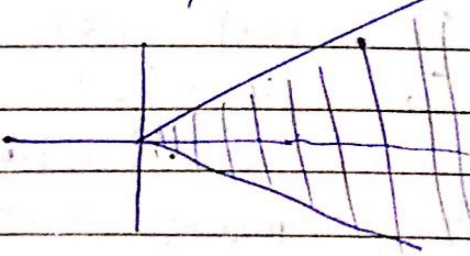
$3\pi/4$

(v) $z = 2+2i$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$\pi/4$

(i) $|\text{Arg } z| \le \pi/4$



ii $\operatorname{Re}\left(\frac{1}{z}\right) < 1$

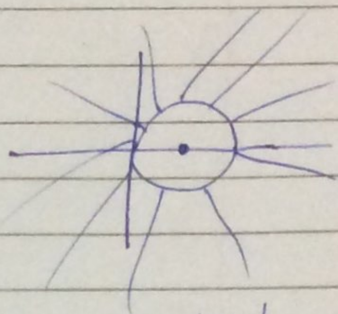
$$\operatorname{Re}\left(\frac{\bar{Z}}{|Z|^2}\right) \qquad \cancel{x+}\ \frac{x - iy}{x^2+y^2}$$

$$\frac{x}{x^2+y^2} < 1$$

$$0 < x^2+y^2-x$$

$$\frac{1}{4} < x^2 - x + \frac{1}{4} + y^2$$

$$\frac{1}{4} < \left(x-\frac{1}{2}\right)^2 + y^2$$



$\frac{1}{16} + \frac{1}{16}$

$\frac{1}{8}$

Q5.

(i) $-5-5i = 5\sqrt{2}\left(\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$$= 5\sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$$

(ii) $\left(\frac{-\sqrt{6} - \sqrt{2}i}{2\sqrt{2}}\right) 2\sqrt{2} = \left(-\frac{\sqrt{3}}{2} - \frac{i\,1}{2}\right) 2\sqrt{2}$

$$= 2\sqrt{2}\left(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}\right)$$

(iii) $2-2\sqrt{3}i \qquad \sqrt{12+4} = \sqrt{16} = 4$

$$4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 4\left(\cos(-\pi/3) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

Q6.

(i) $3\times4\ (\cos 120 + i\sin 120)$

$$12\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) = -6 + i6\sqrt{3}$$

ii) $\cancel{\sqrt{3}-i}$ $2(\cos(105-135)+i\sin(105-135))$

$2(\cos(-30)+i\sin(-30))$

$= \sqrt{3}-i$

$$\begin{array}{r} 113 \\ -45 \\ \hline 10 \end{array}$$

iii) $\left(\frac{2e^{\pi/3 i}}{2e^{-\frac{\pi}{3}i}}\right)^{10} = e^{\frac{20\pi}{3}i}$

$$\begin{array}{r} 15 \\ \times 7 \\ \hline 105 \end{array}$$

$$\begin{array}{r} 45 \\ \times 3 \\ \hline 135 \end{array}$$

$= e^{\left(6\pi+\frac{2\pi}{3}\right)i}$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$\frac{-1}{2} + i\frac{\sqrt{3}}{2}$

Q7. i) $\frac{5i}{2+i} = \cancel{5e^{i\pi/2}}\ \frac{5i(2-i)}{\cancel{4+1}} = 1+2i$

$\frac{5e^{i\pi/2}}{\sqrt{5}\left(\frac{2}{\sqrt{5}}+i\frac{1}{\sqrt{5}}\right)}$

$\frac{\sqrt{5}\,e^{i\pi/2}}{e^{i\phi}} = \sqrt{5}\,e^{i(\pi/2-\phi)} = \sqrt{5}(\sin\phi + i\cos\phi)$

$= 1+2i$

$\cos\phi = \frac{2}{\sqrt{5}}$ $\quad \sin\phi = \frac{1}{\sqrt{5}}$

ii) $(-1+i)^7 = (\sqrt{2})^7\ e^{i\frac{3\pi}{4}\cdot 7} = (\sqrt{2})^7 e^{i\left(5\pi+\frac{\pi}{4}\right)}$

$(\sqrt{2})^7\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$(\sqrt{2})^6(-1-i) = -8(1+i)$

iii) $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left(\frac{\cos\pi/3 + i\sin\pi/3}{\cos -\pi/3 + i\sin -\frac{\pi}{3}}\right)^{10}$

$= \left(\frac{e^{i\pi/3}}{e^{-\pi/3 i}}\right)^{10} = e^{i\frac{20\pi}{3}} = e^{\left(6\pi + \frac{2\pi}{3}\right)i}$

$= e^{\frac{i2\pi}{3}}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Q8.

(i) $(-1+i)^{\frac{1}{3}} = (\sqrt{2})^{\frac{1}{3}}\left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^{\frac{1}{3}}$

$\cancel{(\sqrt{2})^{\frac{1}{3}} e^{i3\pi/4}}$

$\cancel{(\sqrt{2})^{\frac{1}{3}} e^{\frac{i2n\pi}{3} + \frac{i3\pi}{4}}}$

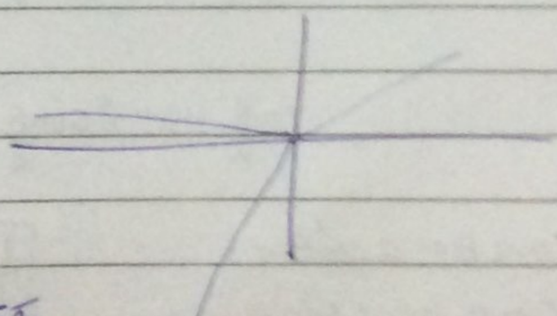
$(\sqrt{2})^{\frac{1}{3}} e^{\frac{i3\pi}{4}}$, $\cancel{(\sqrt{2})^{\frac{1}{3}} e^{\frac{(2\pi + 3\pi)i}{4}}}$, $(\sqrt{2})^{\frac{1}{3}} e$

$(\sqrt{2})^{\frac{1}{3}} e^{i\left(2n\pi + \frac{3\pi}{4}\right)\frac{1}{3}}$

$(\sqrt{2})^{\frac{1}{3}} e^{\frac{i\pi}{4}}$ ✓, $(\sqrt{2})^{\frac{1}{3}} e^{\frac{i11\pi}{12}}$ ✓, $(\sqrt{2})^{\frac{1}{3}} e^{\frac{i19\pi}{12}}$

$e^{\frac{i5\pi}{12}}$,

$\sqrt[6]{2}\, e^{i\left(-\frac{5\pi}{12}\right)}$



$2\pi - \frac{5\pi}{12}$.

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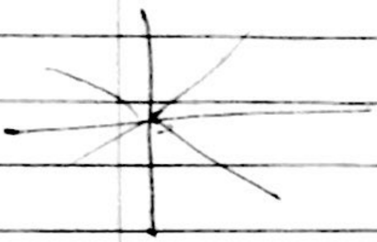
ii) $(4)^{\frac{1}{4}}\left(\frac{-2\sqrt{3}-2i}{4}\right)^{\frac{1}{4}} = \sqrt{2}\left(\frac{-\sqrt{3}}{2} - i\frac{1}{2}\right)^{\frac{1}{4}}$



$= \sqrt{2}\left(\cos\frac{-5\pi}{6} + \sin\frac{-5\pi}{6}\right)^{\frac{1}{4}}$

$= \sqrt{2}\; e^{\left(-\frac{5\pi}{6} + 2n\pi\right)\frac{i}{4}}$

$n = 0, 1, 2, 3$

$\sqrt{2}\, e^{\frac{-5\pi}{6\times4}i}$ ✓, $\sqrt{2}\, e^{\frac{7\pi}{6\times4}i}$ ✓, $\sqrt{2}\, e^{\frac{19\pi}{24}i}$ ✓, $\sqrt{2}\, e^{\frac{31\pi}{24}i}$ ✓

$\Rightarrow \sqrt{z} = \pm\left[\sqrt{\frac{1}{2}(|z| + x)} + sgn(y)\, i\sqrt{\frac{1}{2}(|z| - x)}\right]$

$\sqrt{x + iy}$

$\left(\sqrt{|z|}\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{iy}{\sqrt{x^2+y^2}}\right)\right)^{\frac{1}{2}}$

$(|z|)^{\frac{1}{4}}(\cos\theta + i\sin\theta)^{\frac{1}{2}}$

$\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$ ← Roughly.

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\sqrt{\frac{1+\cos\theta}{2}} + \overset{sgn(y)}{i}\sqrt{\frac{1-\cos\theta}{2}}$

$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$$\pm |z|^{\frac{1}{4}}\left(\sqrt{\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)^{\frac{1}{2}}} + \text{sgn}(y)\, i\sqrt{\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)^{\frac{1}{2}}}\right)$$

$$\sqrt{(|z|+x)^{\frac{1}{2}}} + \text{sgn}(y)\, i\sqrt{(|z|-x)^{\frac{1}{2}}}$$

Q9. a) $z = \dfrac{i}{-2-2i} = \dfrac{-1}{2}\left(\dfrac{i}{1+i}\right) = \dfrac{-1}{2}\;\dfrac{i(1-i)}{2}$

$= \dfrac{-i+1}{2} = \dfrac{1-i}{2}$

$= \dfrac{1}{2}\, e^{-i\pi/4}$

b) $\dfrac{1}{2}\, e^{-\frac{i5\pi}{4}}$

c) $\left(\dfrac{1}{2}\right)^{1/5} e^{(2k\pi - i\pi/4)\frac{1}{5}}$ $\quad k = 0, 1, \ldots 4$

$$\frac{e^{-i\pi/4 \times \frac{1}{5}}}{(2)^{\frac{1}{5}}}, \quad \frac{e^{i\pi/4 \times \frac{7}{5}}}{(2)^{\frac{1}{5}}}, \quad \frac{e^{\frac{15\pi}{20} i}}{(2)^{\frac{1}{5}}}, \quad \frac{e^{\frac{23\pi}{20} i}}{(2)^{\frac{1}{5}}}, \quad \frac{e^{\frac{31\pi i}{20}}}{(2)^{\frac{1}{5}}}$$

Q10. $z^6 + 1 = \sqrt{3}i$

$z^6 = -1 + \sqrt{3}i$

$z^6 = 2\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)$

$z^6 = 2e^{i\frac{2\pi}{3}}$

$= (2^{\frac{1}{6}})\, e^{\left(2n\pi + i\frac{2\pi}{3}\right)\frac{1}{6}}$ $\quad n = 0, 1, \ldots 5$

$= (2^{\frac{1}{6}})\, e^{i20^\circ}, \ (2^{\frac{1}{6}})\, e^{i50^\circ}$

$\frac{\cancel{120}^{20}}{\cancel{6}}$

$\frac{120+180}{6}$

$\frac{\cancel{300}^{50}}{\cancel{6}}$

Q11. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

$\cos 5\theta + i\sin 5\theta = (\cos 3\theta + i\sin 3\theta)(\cos 2\theta + i\sin 2\theta)$

$(4\cos^3\theta - 3\cos\theta + i\sin 3\theta)(2\cos^2\theta - 1 + i\sin 2\theta)$

$8\cos^5\theta - 4\cos^3\theta - 6\cos^3\theta + 3\cos\theta - \sin 3\theta\sin 2\theta$

$-\frac{1}{2}\cos\theta + \frac{1}{2}\cos 5\theta$

$\frac{\cos 5\theta}{2} = 8\cos^5\theta - 4\cos^3\theta - 6\cos^3\theta + 3\cos\theta - \frac{1}{2}\cos\theta$.

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

b) $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

$\sin 5\theta = \sin 3\theta\cos 2\theta + \sin 2\theta\cos 3\theta$

$= \sin(2\theta+\theta)\cos 2\theta + \sin 2\theta\cos 3\theta$

$= (\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)\cos 2\theta + 2\sin\theta\cos 3\theta\cos\theta$

$= \sin 2\theta\cos\theta\cos 2\theta + \cos^2 2\theta\sin\theta + 2\sin\theta\cos 3\theta\cos\theta$.

$= 2\sin\theta\cos^2\theta\cos 2\theta + \cos^2 2\theta\sin\theta + 2\sin\theta\cos 3\theta\cos\theta$.

$\Rightarrow \frac{\sin 5\theta}{\sin\theta} = 2\cos^2\theta(2\cos^2\theta - 1) + (2\cos^2\theta - 1)^2 + 2\cos 3\theta\cos\theta$

$= 4\cos^4\theta - 2\cos^2\theta + 4\cos^4\theta + 1 - 4\cos^2\theta + \cos 4\theta + \cos 2\theta$

$= 8\cos^4\theta - 6\cos^2\theta + 1 + 2\cos^2 2\theta - 1 + 2\cos^2\theta - 1$

$= 8\cos^4\theta - 4\cos^2\theta - 1 + 2(2\cos^2\theta - 1)^2$

$= 8\cos^4\theta - 4\cos^2\theta - 1 + 2(4\cos^4\theta + 1 - 4\cos^2\theta)$

$= 16\cos^4\theta \cancel{+ 4\cos^2\theta} - 12\cos^2\theta + 1$

Q11(c) $\cos^4\theta = \frac{1}{3}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$\cos 4\theta = (2\cos^2 2\theta - 1)$

$= \cancel{4}(2(2\cos^2\theta - 1)^2 - 1)$

$= (2(4\cos^4\theta + 1 - 4\cos^2\theta) - 1)$

$= (8\cos^4\theta + 1 - 8\cos^2\theta)$

$\frac{\cos 4\theta - 1 + 8\cos^2\theta}{8} = \cos^4\theta$

$2\cos^2\theta = 1 + \cos 2\theta$

$\frac{\cos 4\theta - 1 + 4 + 4\cos 2\theta}{8}$

$\frac{\cos 4\theta}{8} + \frac{3}{8} + \frac{\cos 2\theta}{2} = \cos^4\theta$