Boolean Algebra

- Boolean algebra is a mathematical system for the manipulation of variables that can have one of two values.
  - In formal logic, these values are “true” and “false.”
  - In digital systems, these values are “on” and “off,” 1 and 0, or “high” and “low.”
- Boolean expressions are created by performing operations on Boolean variables.
  - Common Boolean operators include AND, OR, and NOT.

Boolean Algebra

- A Boolean operator can be completely described using a truth table.
- The truth table for the Boolean operators AND and OR are shown at the right.
- The AND operator is also known as a Boolean product. The OR operator is the Boolean sum.

X AND Y

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

X OR Y

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

Boolean Algebra

- The truth table for the Boolean NOT operator is shown at the right.
- The NOT operation is most often designated by an overbar. It is sometimes indicated by a prime mark (‘) or an “elbow” (¬).

NOT X

X	$\overline{X}$
0	1
1	0

Boolean Algebra

- A Boolean function has:
  - At least one Boolean variable,
  - At least one Boolean operator, and
  - At least one input from the set {0,1}.
- It produces an output that is also a member of the set {0,1}.

Now you know why the binary numbering system is so handy in digital systems.

Boolean Algebra

- The truth table for the Boolean function:

  $$F(x,y,z) = x\bar{z}+y$$

  is shown at the right.
- To make evaluation of the Boolean function easier, the truth table contains extra (shaded) columns to hold evaluations of subparts of the function.

$$F(x,y,z) = x\bar{z}+y$$

x	y	z	$\bar{z}$	$x\bar{z}$	$x\bar{z}+y$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

Boolean Algebra

- As with common arithmetic, Boolean operations have rules of precedence.
- The NOT operator has highest priority, followed by AND and then OR.
- This is how we chose the (shaded) function subparts in our table.

$$F(x,y,z) = x\bar{z}+y$$

x	y	z	$\bar{z}$	$x\bar{z}$	$x\bar{z}+y$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

Boolean Algebra

- Digital computers contain circuits that implement Boolean functions.
- The simpler that we can make a Boolean function, the smaller the circuit that will result.
  - Simpler circuits are cheaper to build, consume less power, and run faster than complex circuits.
- With this in mind, we always want to reduce our Boolean functions to their simplest form.
- There are a number of Boolean identities that help us to do this.

Boolean Algebra

- Most Boolean identities have an AND (product) form as well as an OR (sum) form. We give our identities using both forms. Our first group is rather intuitive:

Identity Name	AND Form	OR Form
Identity Law	$1x = x$	$0 + x = x$
Null Law	$0x = 0$	$1 + x = 1$
Idempotent Law	$xx = x$	$x + x = x$
Inverse Law	$x\overline{x} = 0$	$x + \overline{x} = 1$

Boolean Algebra

- Our second group of Boolean identities should be familiar to you from your study of algebra:

Identity Name	AND Form	OR Form
Commutative Law	$xy = yx$	$x+y = y+x$
Associative Law	$(xy)z = x(yz)$	$(x+y)+z = x + (y+z)$
Distributive Law	$x+yz = (x+y)(x+z)$	$x(y+z) = xy+xz$

Boolean Algebra

- Our last group of Boolean identities are perhaps the most useful.
- If you have studied set theory or formal logic, these laws are also familiar to you.

Identity Name	AND Form	OR Form
Absorption Law	$x(x+y) = x$	$x + xy = x$
DeMorgan's Law	$\overline{(xy)} = \bar{x} + \bar{y}$	$\overline{(x+y)} = \bar{x}\bar{y}$
Double Complement Law	$\overline{(\bar{x})} = x$	

Boolean Algebra

- We can use Boolean identities to simplify the function: $F(X,Y,Z) = (X + Y)(X + \overline{Y})\overline{(X\overline{Z})}$ as follows:

$(X + Y)(X + \overline{Y})(\overline{X\overline{Z}})$	Idempotent Law (Rewriting)
$(X + Y)(X + \overline{Y})(\overline{X} + Z)$	DeMorgan's Law
$(XX + X\overline{Y} + XY + Y\overline{Y})(\overline{X} + Z)$	Distributive Law
$((X + Y\overline{Y}) + X(Y + \overline{Y}))(\overline{X} + Z)$	Commutative & Distributive Laws
$((X + 0) + X(1))(\overline{X} + Z)$	Inverse Law
$X(\overline{X} + Z)$	Idempotent Law
$X\overline{X} + XZ$	Distributive Law
$0 + XZ$	Inverse Law
XZ	Idempotent Law

Boolean Algebra

- Sometimes it is more economical to build a circuit using the complement of a function (and complementing its result) than it is to implement the function directly.
- DeMorgan's law provides an easy way of finding the complement of a Boolean function.
- Recall DeMorgan's law states:

$$\overline{(xy)} = \overline{x} + \overline{y} \quad \text{and} \quad \overline{(x+y)} = \overline{x}\,\overline{y}$$

Boolean Algebra

- DeMorgan's law can be extended to any number of variables.
- Replace each variable by its complement and change all ANDs to ORs and all ORs to ANDs.
- Thus, we find the the complement of:

$$F(X,Y,Z) = (XY) + (\overline{X}Z) + (Y\overline{Z})$$

is:

$$\overline{F}(X,Y,Z) = \overline{(XY) + (\overline{X}Z) + (Y\overline{Z})}$$

$$= \overline{(XY)}\,\overline{(\overline{X}Z)}\,\overline{(Y\overline{Z})}$$

$$= (\overline{X}+\overline{Y})(X+\overline{Z})(\overline{Y}+Z)$$

Boolean Algebra

- There are two canonical forms for Boolean expressions: sum-of-products and product-of-sums.
  - Recall the Boolean product is the AND operation and the Boolean sum is the OR operation.
- In the sum-of-products form, ANDed variables are ORed together.
  - For example: `F(x,y,z) = xy + xz + yz`
- In the product-of-sums form, ORed variables are ANDed together:
  - For example: `F(x,y,z) = (x+y)(x+z)(y+z)`

Boolean Algebra

- It is easy to convert a function to sum-of-products form using its truth table.
- We are interested in the values of the variables that make the function true (=1).
- Using the truth table, we list the values of the variables that result in a true function value.
- Each group of variables is then ORed together.

$F(x,y,z) = x\bar{z}+y$

x	y	z	$x\bar{z}+y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Boolean Algebra

- The sum-of-products form for our function is:

$$F(x,y,z) = \bar{x}y\bar{z} + \bar{x}yz + x\bar{y}\bar{z} + xy\bar{z} + xyz$$

We note that this function is not in simplest terms. Our aim is only to rewrite our function in canonical sum-of-products form.

$$F(x,y,z) = x\bar{z} + y$$

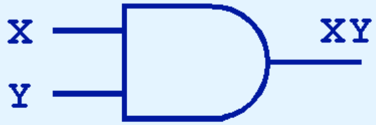
x	y	z	$x\bar{z}+y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

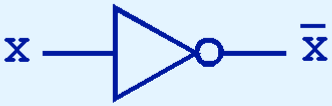
Logic Gates

- We have looked at Boolean functions in abstract terms.
- In this section, we see that Boolean functions are implemented in digital computer circuits called gates.
- A gate is an electronic device that produces a result based on two or more input values.
  – In reality, gates consist of one to six transistors, but digital designers think of them as a single unit.
  – Integrated circuits contain collections of gates suited to a particular purpose.

Logic Gates

- The three simplest gates are the AND, OR, and NOT gates.





X AND Y

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

X OR Y

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

NOT X

X	$\overline{X}$
0	1
1	0

- They correspond directly to their respective Boolean operations, as you can see by their truth tables.

Logic Gates

- Another very useful gate is the exclusive OR (XOR) gate.
- The output of the XOR operation is true only when the values of the inputs differ.

X XOR Y

X	Y	$X \oplus Y$
0	0	0
0	1	1
1	0	1
1	1	0

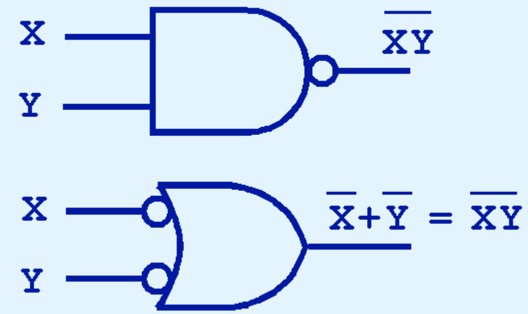
Note the special symbol ⊕ for the XOR operation.

Logic Gates

- NAND and NOR are two very important gates. Their symbols and truth tables are shown at the right.

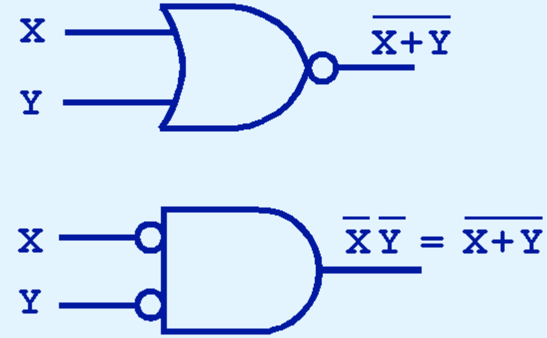
X NAND Y

X	Y	X NAND Y
0	0	1
0	1	1
1	0	1
1	1	0



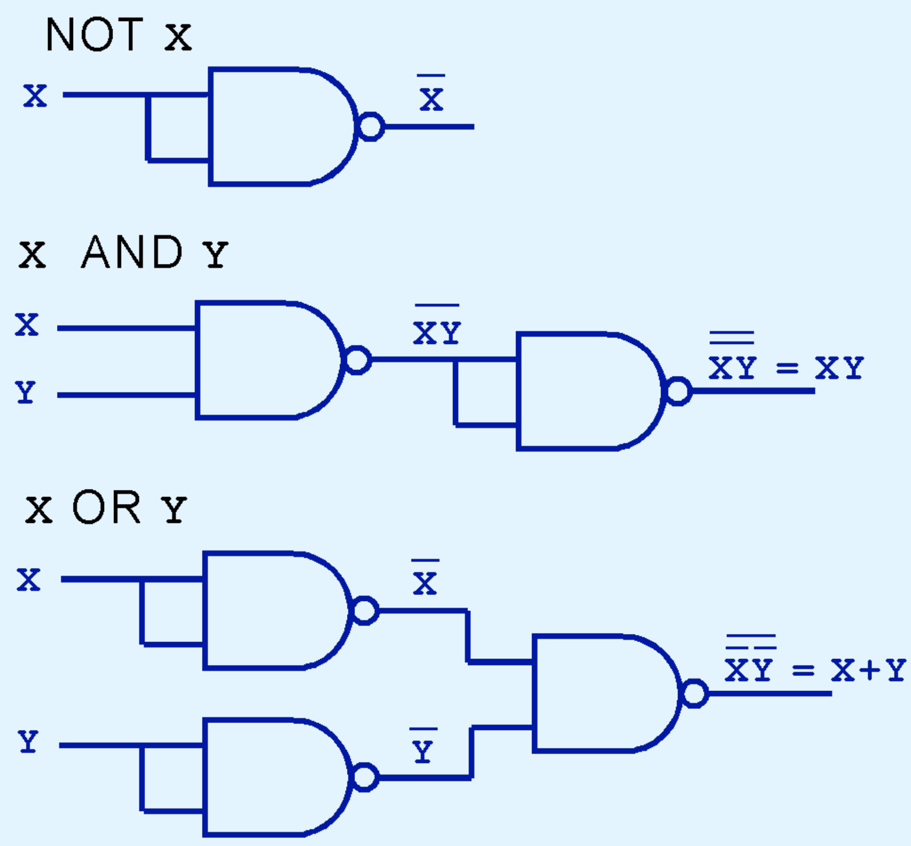
X NOR Y

X	Y	X NOR Y
0	0	1
0	1	0
1	0	0
1	1	0



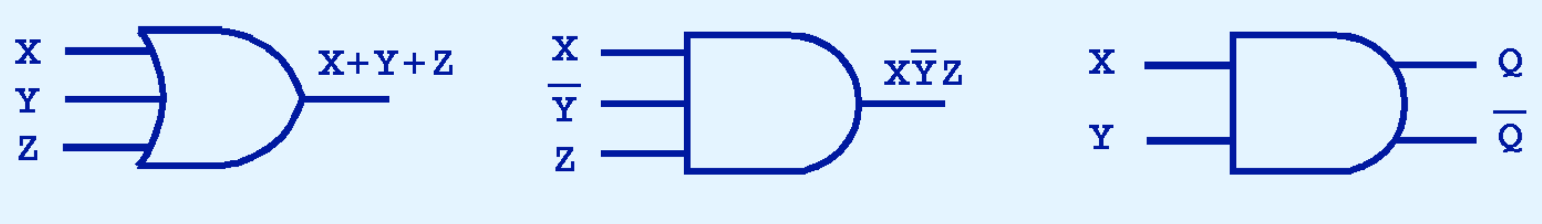
Logic Gates

- NAND and NOR are known as *universal gates* because they are inexpensive to manufacture and any Boolean function can be constructed using only NAND or only NOR gates.



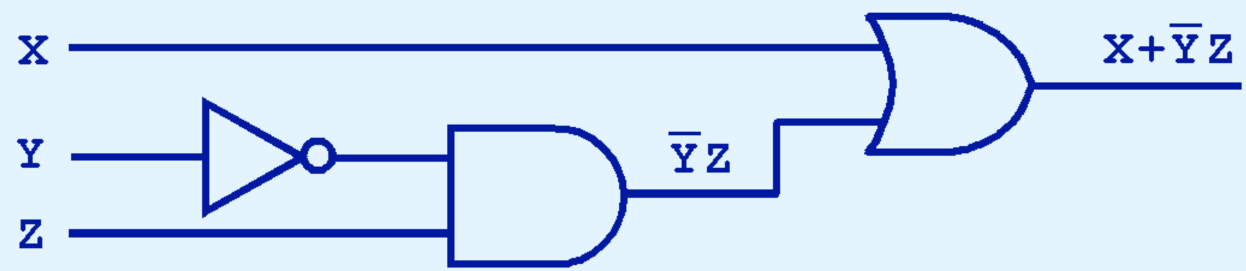
Logic Gates

- Gates can have multiple inputs and more than one output.
  - A second output can be provided for the complement of the operation.
  - We'll see more of this later.



Digital Components

- The main thing to remember is that combinations of gates implement Boolean functions.
- The circuit below implements the Boolean function: $F(X,Y,Z) = X + \overline{Y}Z$



We simplify our Boolean expressions so that we can create simpler circuits.

Combinational Circuits

- We have designed a circuit that implements the Boolean function:

$$F(X,Y,Z) = X+\overline{Y}Z$$

- This circuit is an example of a *combinational logic* circuit.
- Combinational logic circuits produce a specified output (almost) at the instant when input values are applied.
  - In a later section, we will explore circuits where this is not the case.

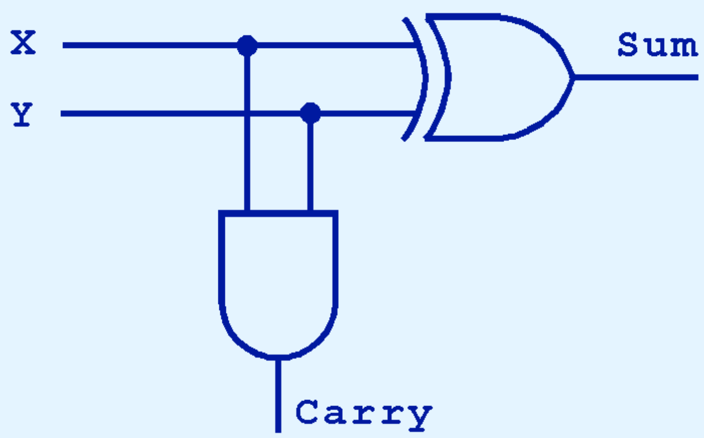
Combinational Circuits

- Combinational logic circuits give us many useful devices.
- One of the simplest is the *half adder*, which finds the sum of two bits.
- We can gain some insight as to the construction of a half adder by looking at its truth table, shown at the right.

Inputs		Outputs	
X	Y	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Combinational Circuits

- As we see, the sum can be found using the XOR operation and the carry using the AND operation.



Inputs		Outputs	
X	Y	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Combinational Circuits

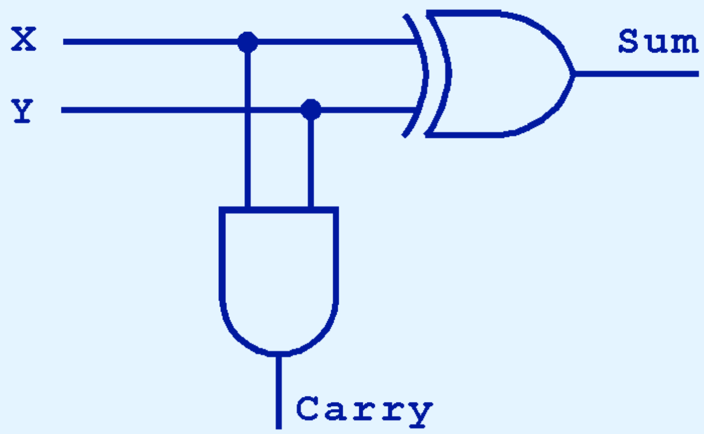
- We can change our half adder into to a full adder by including gates for processing the carry bit.
- The truth table for a full adder is shown at the right.

Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Combinational Circuits

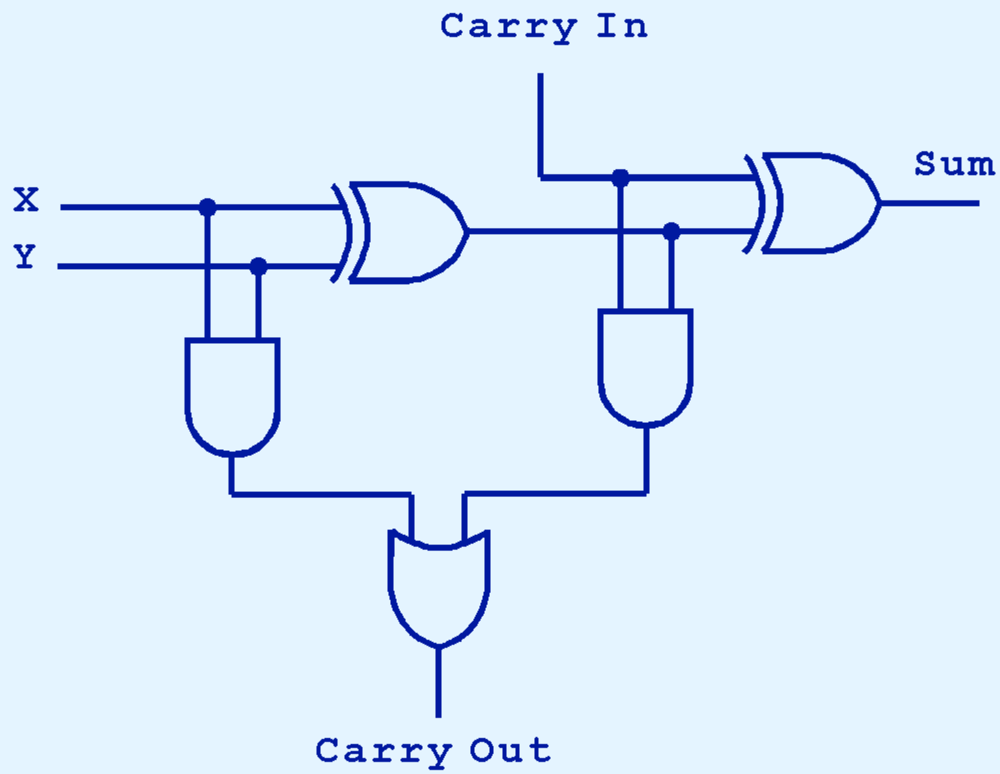
- How can we change the half adder shown below to make it a full adder?

Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1



Combinational Circuits

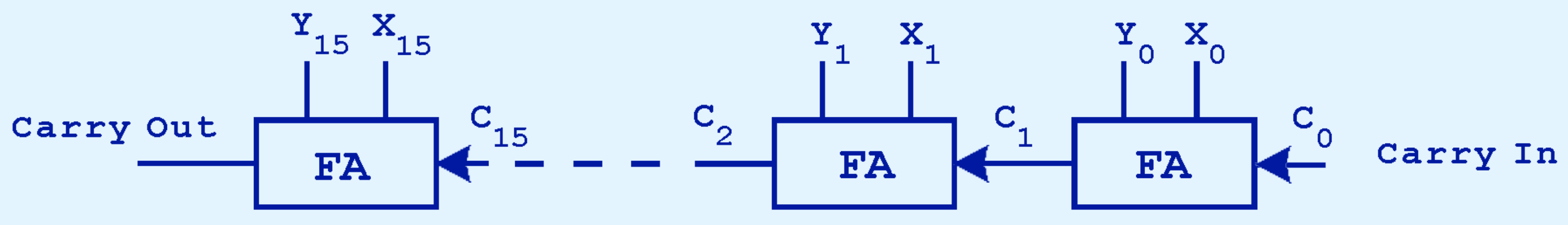
- Here’s our completed full adder.



Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Combinational Circuits

- Just as we combined half adders to make a full adder, full adders can connected in series.
- The carry bit "ripples" from one adder to the next; hence, this configuration is called a *ripple-carry adder*.



Today's systems employ more efficient adders.

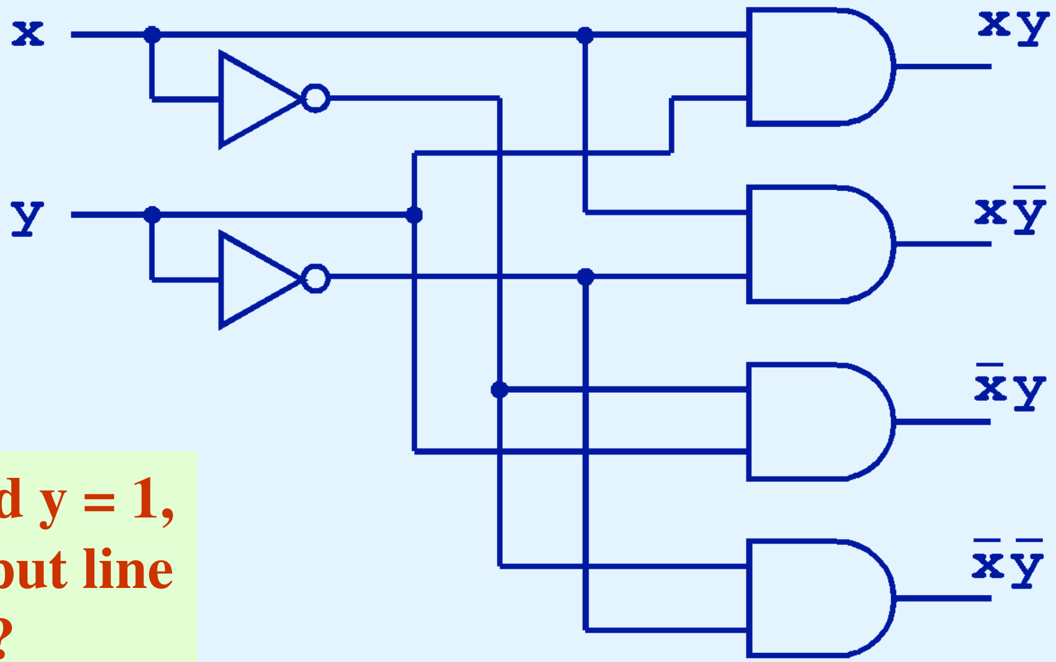
Combinational Circuits

- Decoders are another important type of combinational circuit.
- Among other things, they are useful in selecting a memory location according a binary value placed on the address lines of a memory bus.
- Address decoders with n inputs can select any of 2^n locations.

This is a block diagram for a decoder.

Combinational Circuits

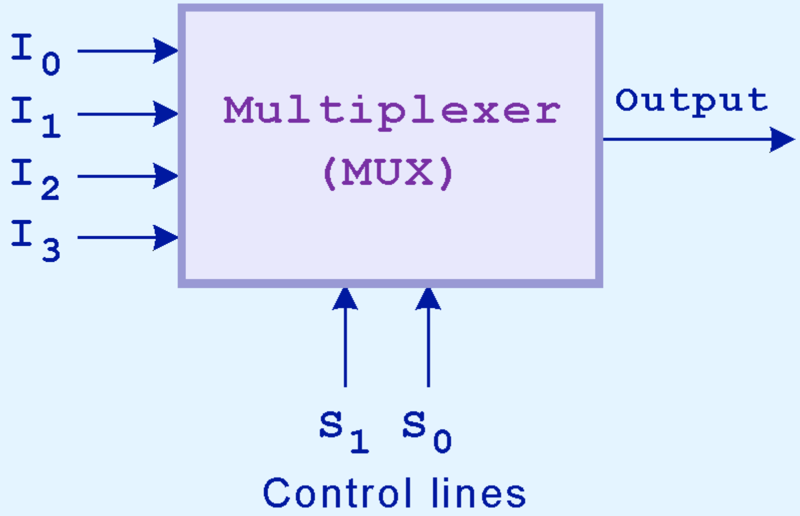
- This is what a 2-to-4 decoder looks like on the inside.



If x = 0 and y = 1, which output line is enabled?

Combinational Circuits

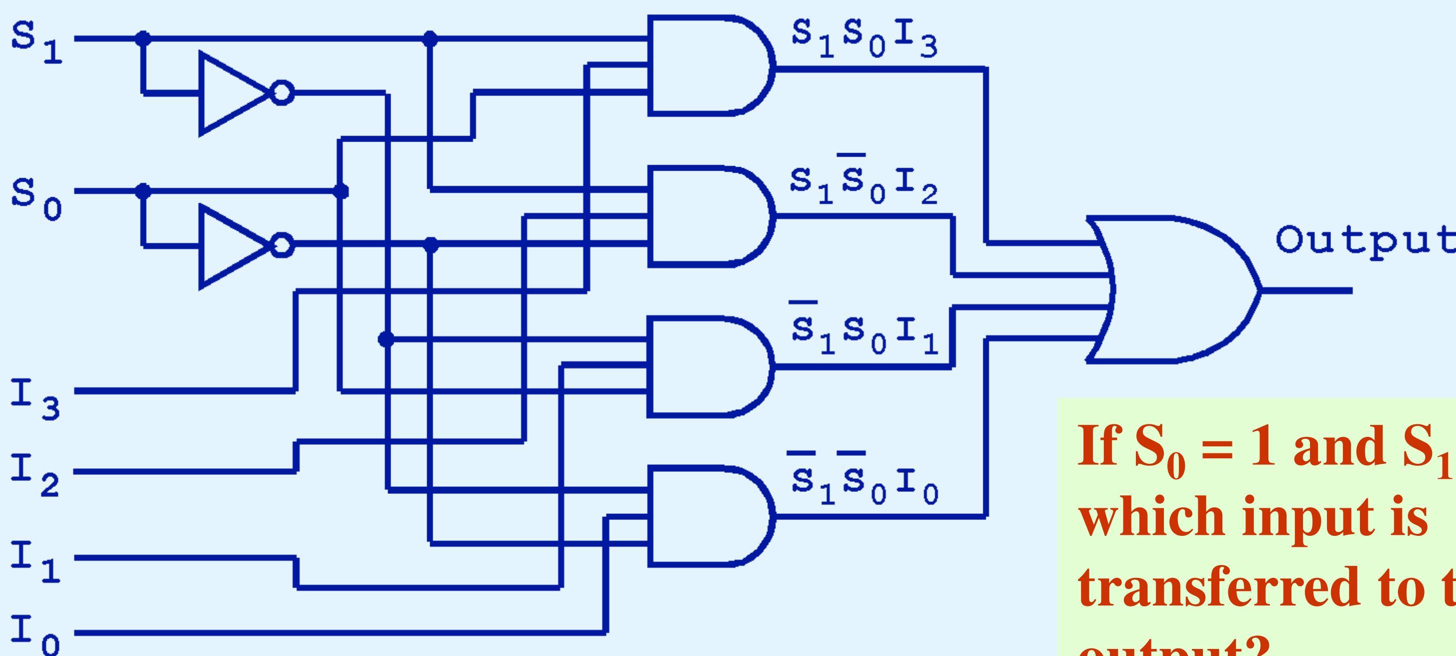
- A multiplexer does just the opposite of a decoder.
- It selects a single output from several inputs.
- The particular input chosen for output is determined by the value of the multiplexer's control lines.
- To be able to select among *n* inputs, $\log_2 n$ control lines are needed.



This is a block diagram for a multiplexer.

Combinational Circuits

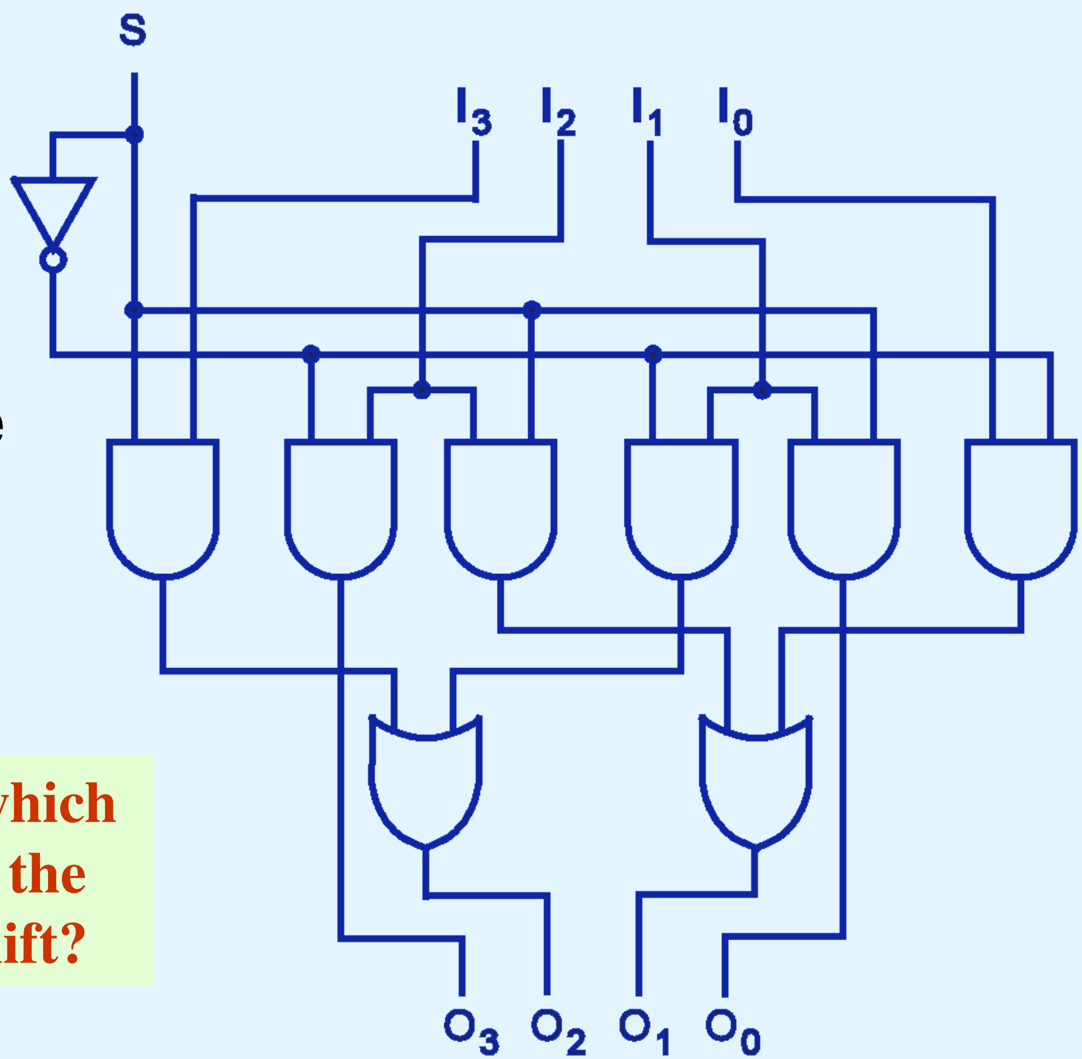
- This is what a 4-to-1 multiplexer looks like on the inside.



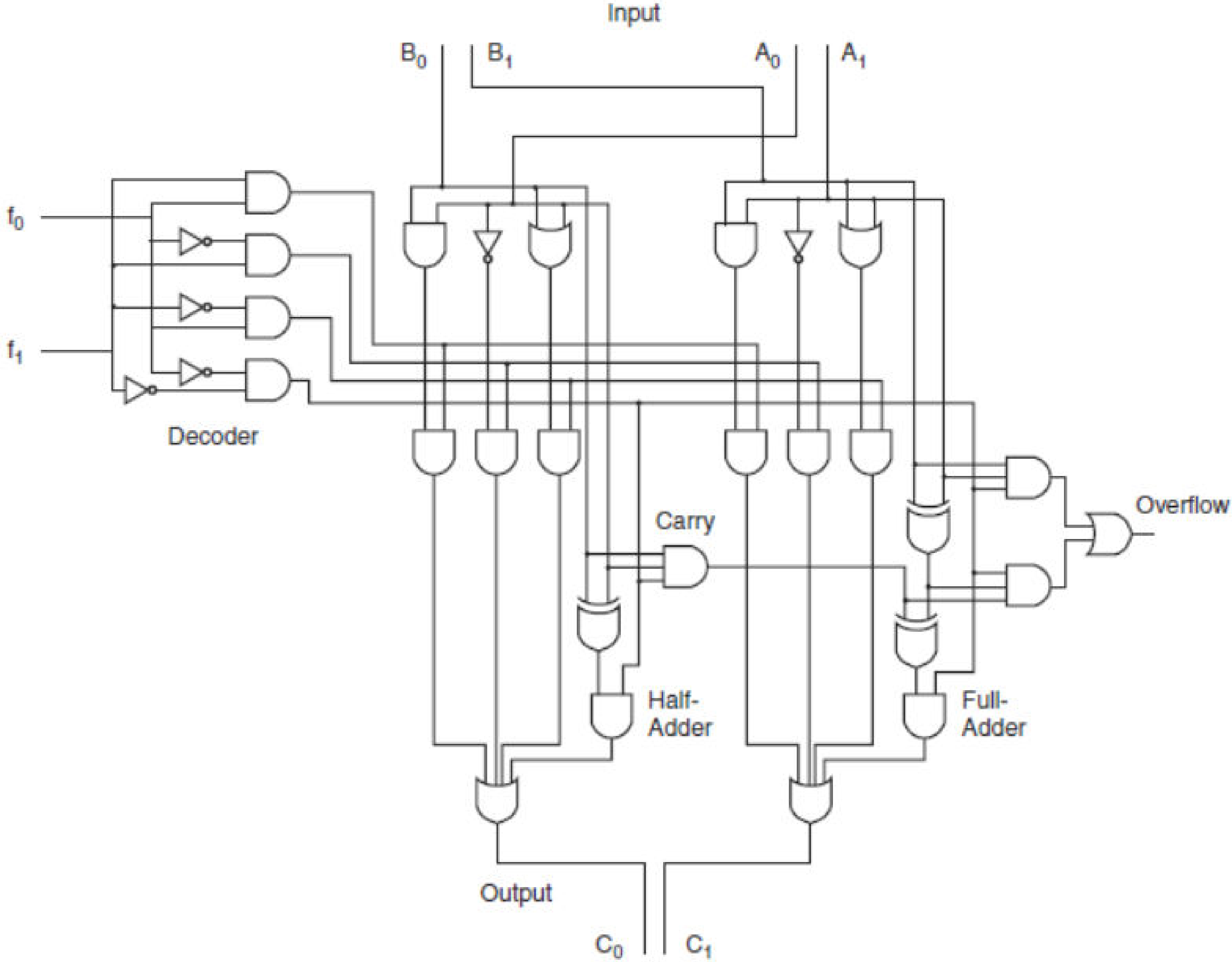
If $S_0 = 1$ and $S_1 = 0$, which input is transferred to the output?

Combinational Circuits

- This shifter moves the bits of a nibble one position to the left or right.



If S = 0, in which direction do the input bits shift?


Input
B0
B1
A0
A1
f0
f1
Decoder
Carry
Overflow
Half-
Adder
Full-
Adder
Output
C0
C1

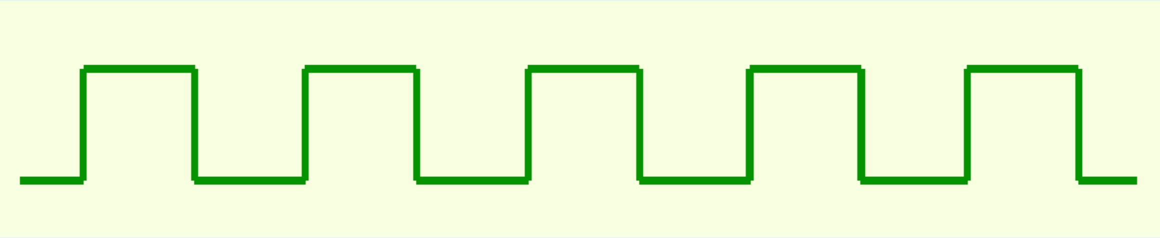
Sequential Circuits

Sequential Circuits

- Combinational logic circuits are perfect for situations when we require the immediate application of a Boolean function to a set of inputs.
- There are other times, however, when we need a circuit to change its value with consideration to its current state as well as its inputs.
  - These circuits have to "remember" their current state.
- *Sequential logic circuits* provide this functionality for us.

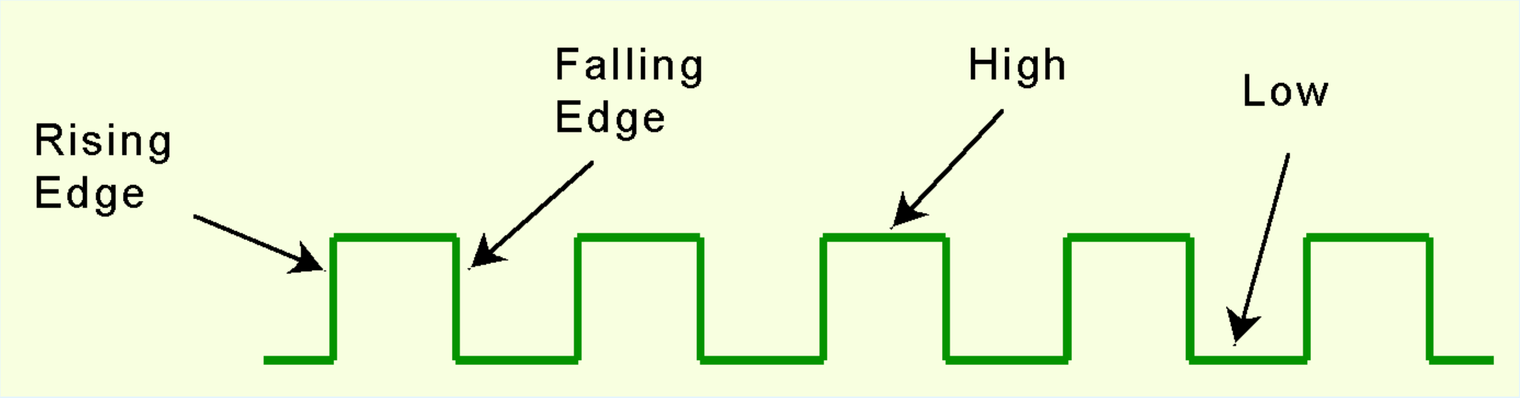
Sequential Circuits

- As the name implies, sequential logic circuits require a means by which events can be sequenced.
- State changes are controlled by clocks.
  - A “clock” is a special circuit that sends electrical pulses through a circuit.
- Clocks produce electrical waveforms such as the one shown below.



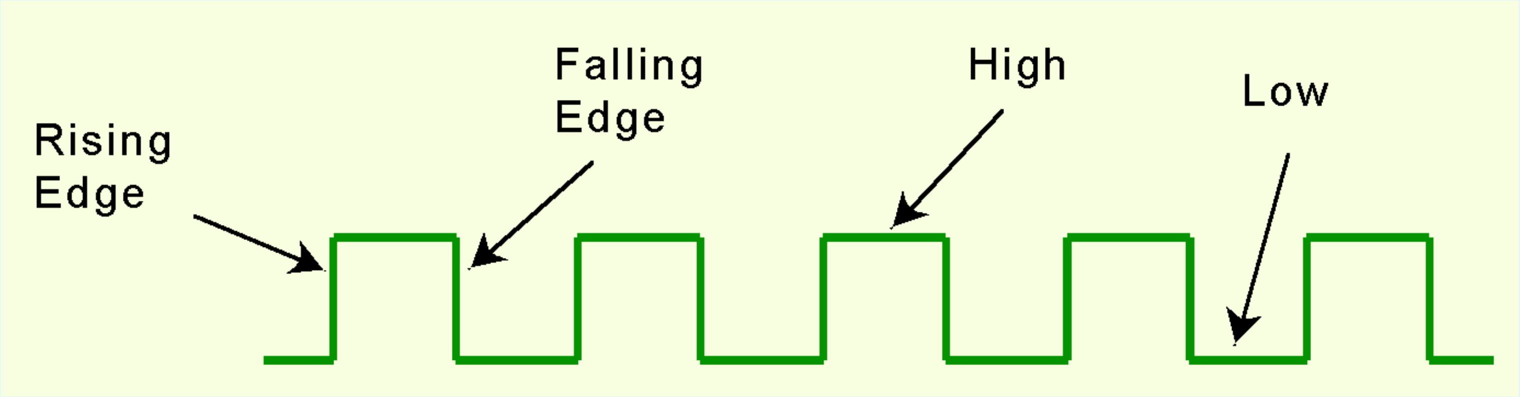
Sequential Circuits

- State changes occur in sequential circuits only when the clock ticks.
- Circuits can change state on the rising edge, falling edge, or when the clock pulse reaches its highest voltage.



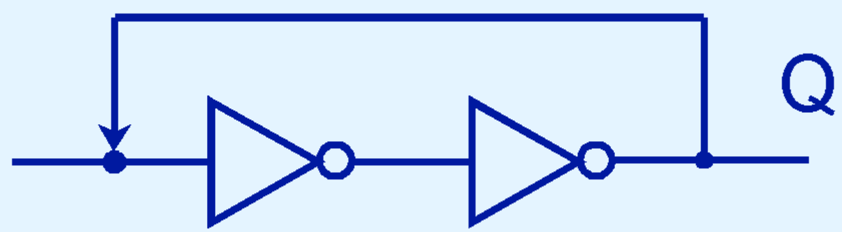
Sequential Circuits

- Circuits that change state on the rising edge, or falling edge of the clock pulse are called *edge-triggered.*
- *Level-triggered circuits* change state when the clock voltage reaches its highest or lowest level.



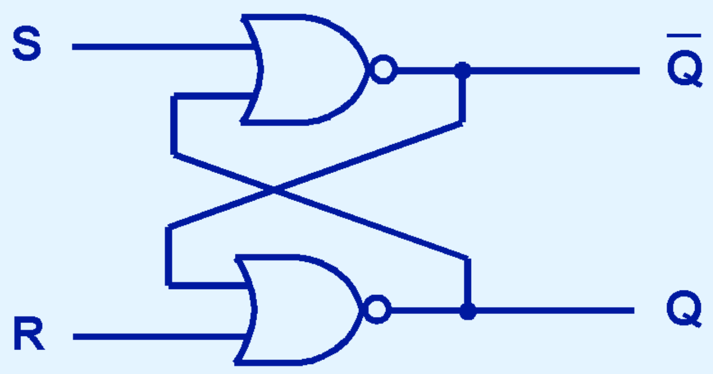
Sequential Circuits

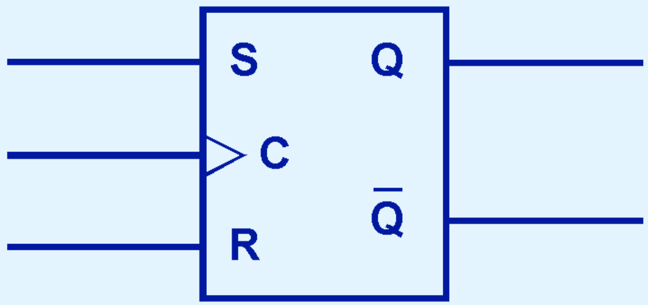
- To retain their state values, sequential circuits rely on *feedback.*
- Feedback in digital circuits occurs when an output is looped back to the input.
- A simple example of this concept is shown below.
  - If Q is 0 it will always be 0, if it is 1, it will always be 1. Why?



Sequential Circuits

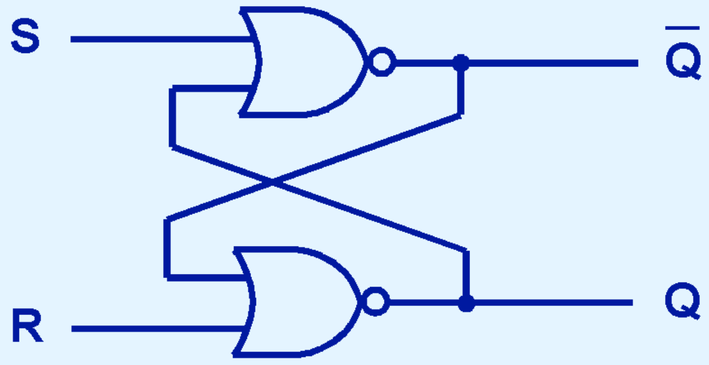
- You can see how feedback works by examining the most basic sequential logic components, the SR flip-flop.
  - The “SR” stands for set/reset.
- The internals of an SR flip-flop are shown below, along with its block diagram.





Sequential Circuits

- The behavior of an SR flip-flop is described by a characteristic table.
- Q(t) means the value of the output at time t. Q(t+1) is the value of Q after the next clock pulse.



S	R	Q(t+1)
0	0	Q(t) (no change)
0	1	0 (reset to 0)
1	0	1 (set to 1)
1	1	undefined

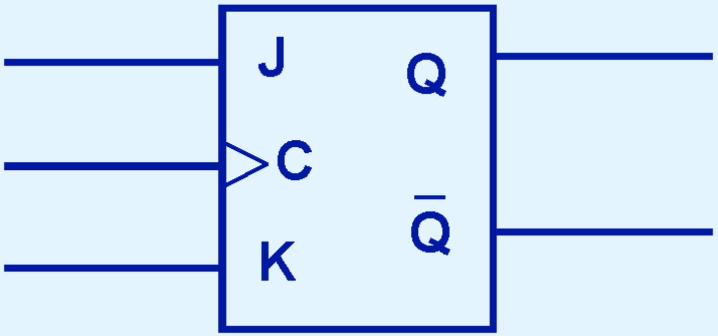
Sequential Circuits

- The SR flip-flop actually has three inputs: S, R, and its current output, Q.
- Thus, we can construct a truth table for this circuit, as shown at the right.
- Notice the two undefined values. When both S and R are 1, the SR flip-flop is unstable.

		Present State	Next State
S	R	Q(t)	Q(t+1)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	undefined
1	1	1	undefined

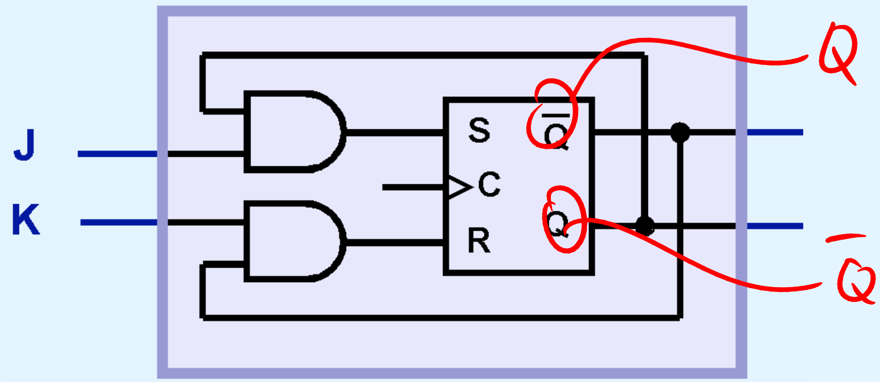
Sequential Circuits

- If we can be sure that the inputs to an SR flip-flop will never both be 1, we will never have an unstable circuit. This may not always be the case.
- The SR flip-flop can be modified to provide a stable state when both inputs are 1.
- This modified flip-flop is called a JK flip-flop, shown at the right.
  - The "JK" is in honor of Jack Kilby.



Sequential Circuits

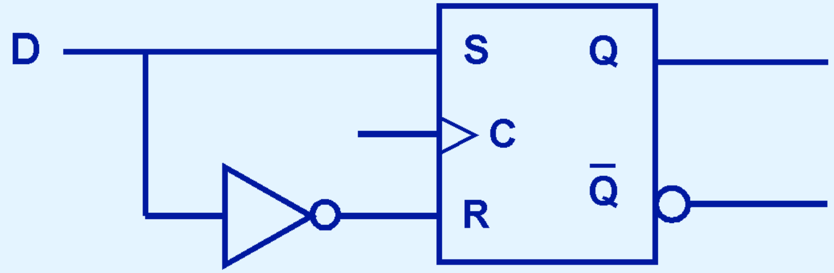
- At the right, we see how an SR flip-flop can be modified to create a JK flip-flop.
- The characteristic table indicates that the flip-flop is stable for all inputs.



J	K	Q(t+1)
0	0	Q(t) (no change)
0	1	0 (reset to 0)
1	0	1 (set to 1)
1	1	$\overline{Q}$(t)

Sequential Circuits

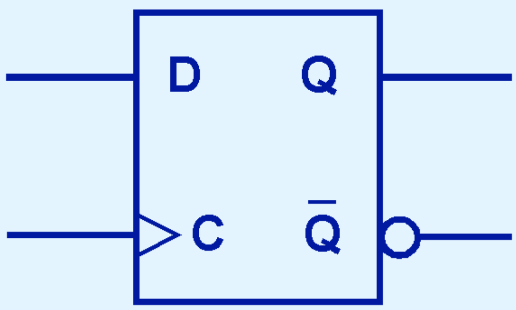
- Another modification of the SR flip-flop is the D flip-flop, shown below with its characteristic table.
- You will notice that the output of the flip-flop remains the same during subsequent clock pulses. The output changes only when the value of D changes.



D	Q(t+1)
0	0
1	1

Sequential Circuits

- The D flip-flop is the fundamental circuit of computer memory.
  - D flip-flops are usually illustrated using the block diagram shown below.
- The characteristic table for the D flip-flop is shown at the right.



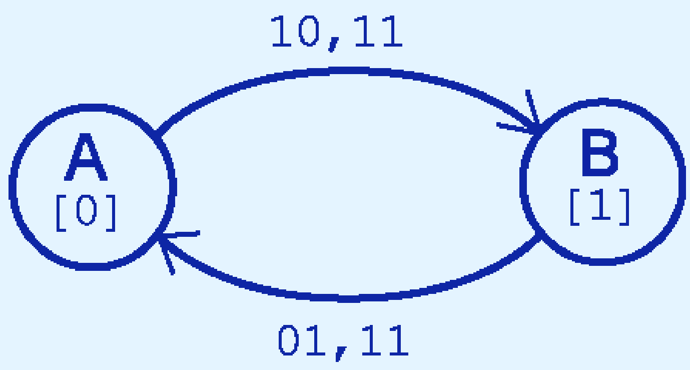
D	Q(t+1)
0	0
1	1

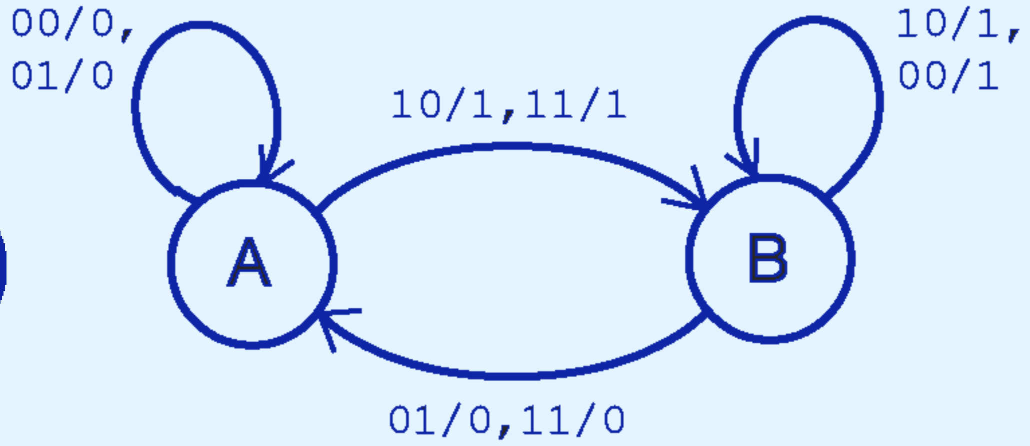
Sequential Circuits

- The behavior of sequential circuits can be expressed using characteristic tables or finite state machines (FSMs).
  - FSMs consist of a set of nodes that hold the states of the machine and a set of arcs that connect the states.
- Moore and Mealy machines are two types of FSMs that are equivalent.
  - They differ only in how they express the outputs of the machine.
- Moore machines place outputs on each node, while Mealy machines present their outputs on the transitions.

Sequential Circuits

- The behavior of a JK flop-flop is depicted below by a Moore machine (left) and a Mealy machine (right).

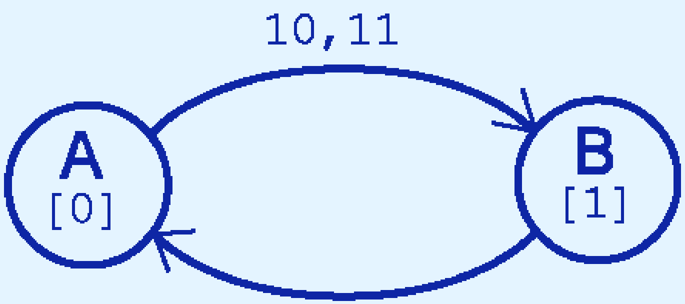


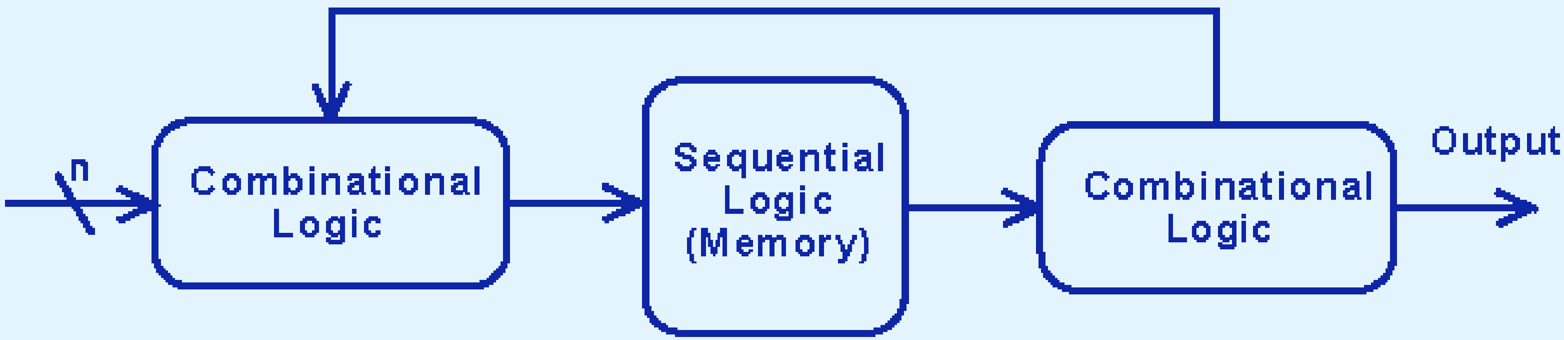


Sequential Circuits

- Although the behavior of Moore and Mealy machines is identical, their implementations differ.

This is our Moore machine.

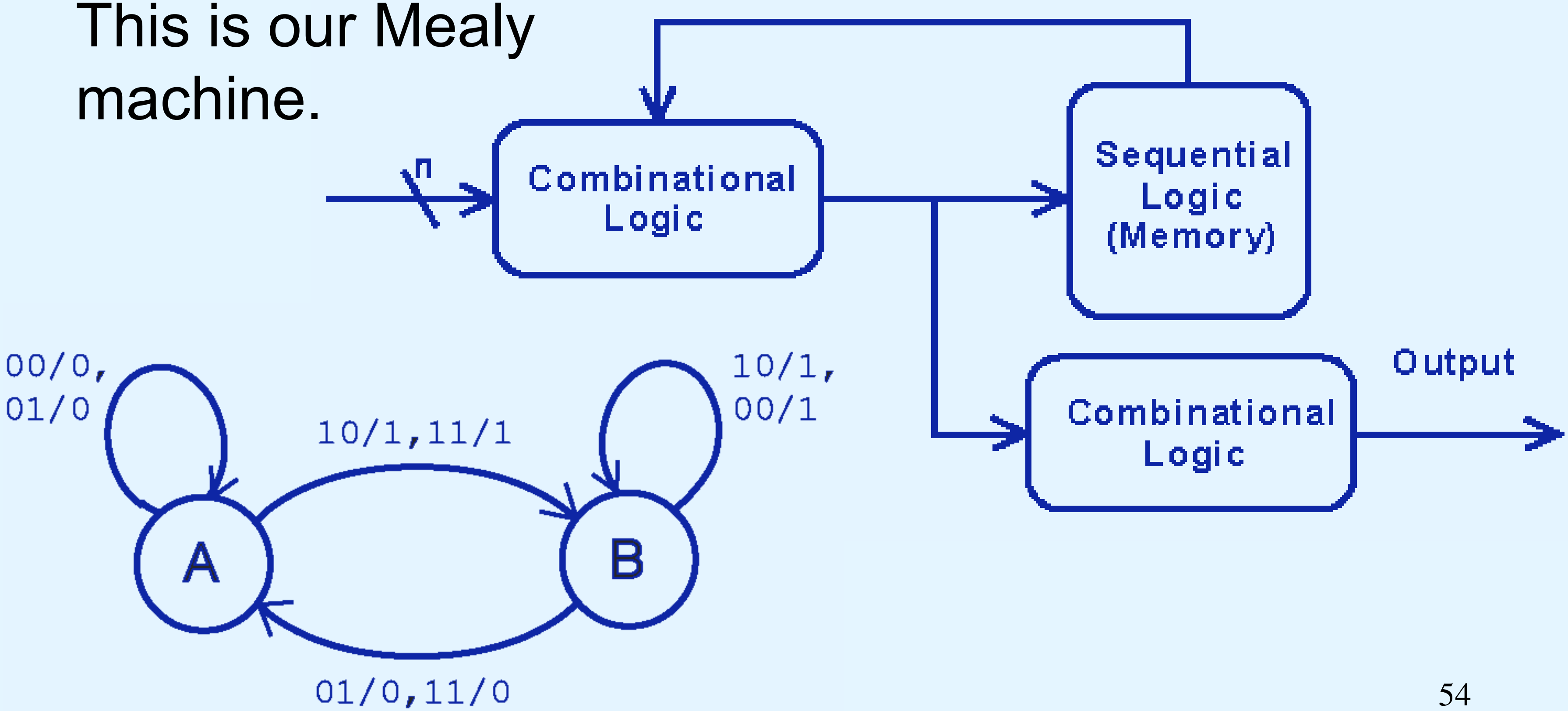




Sequential Circuits

- Although the behavior of Moore and Mealy machines is identical, their implementations differ.

This is our Mealy machine.

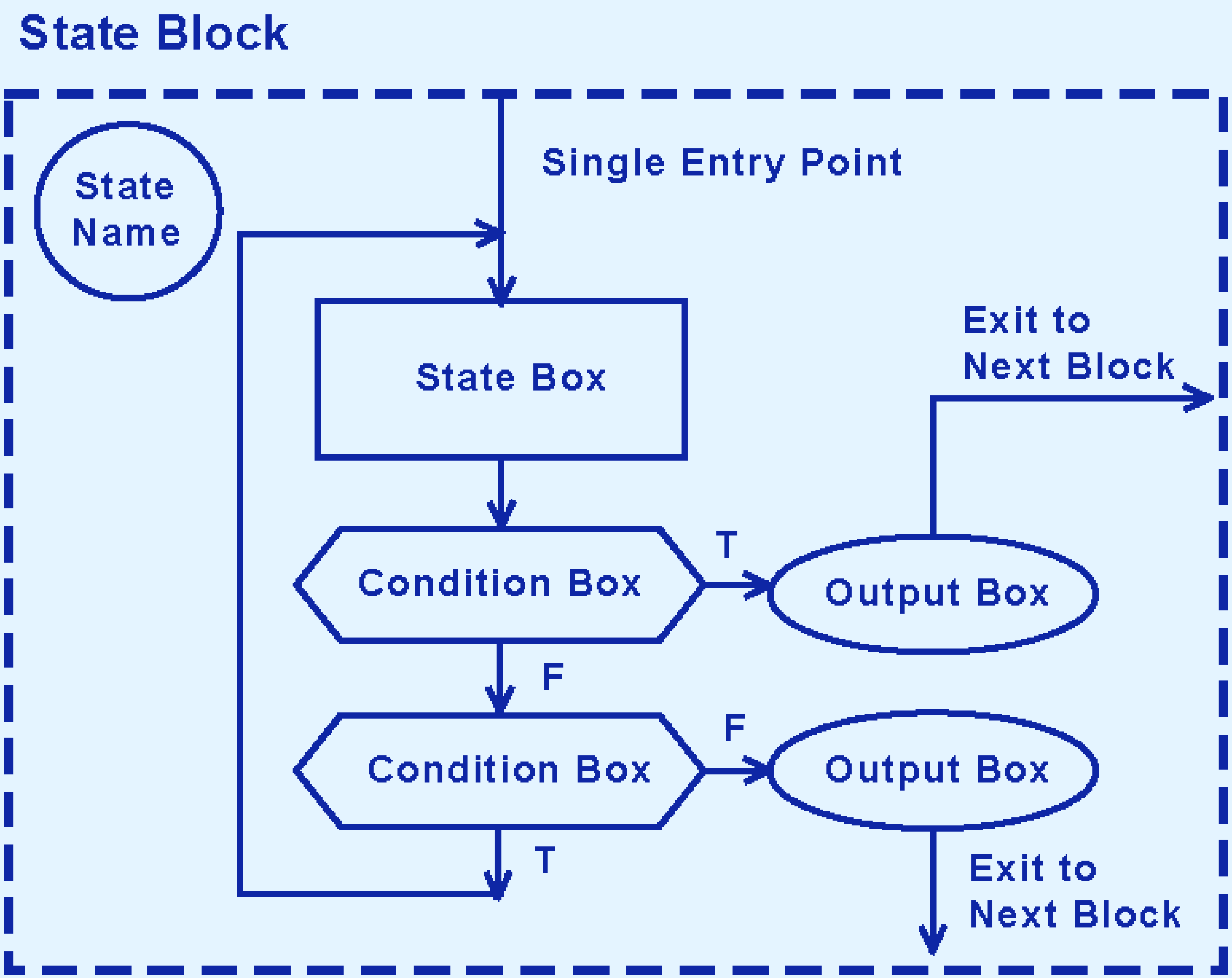


Sequential Circuits

- It is difficult to express the complexities of actual implementations using only Moore and Mealy machines.
  - For one thing, they do not address the intricacies of timing very well.
  - Secondly, it is often the case that an interaction of numerous signals is required to advance a machine from one state to the next.
- For these reasons, Christopher Clare invented the algorithmic state machine (ASM).

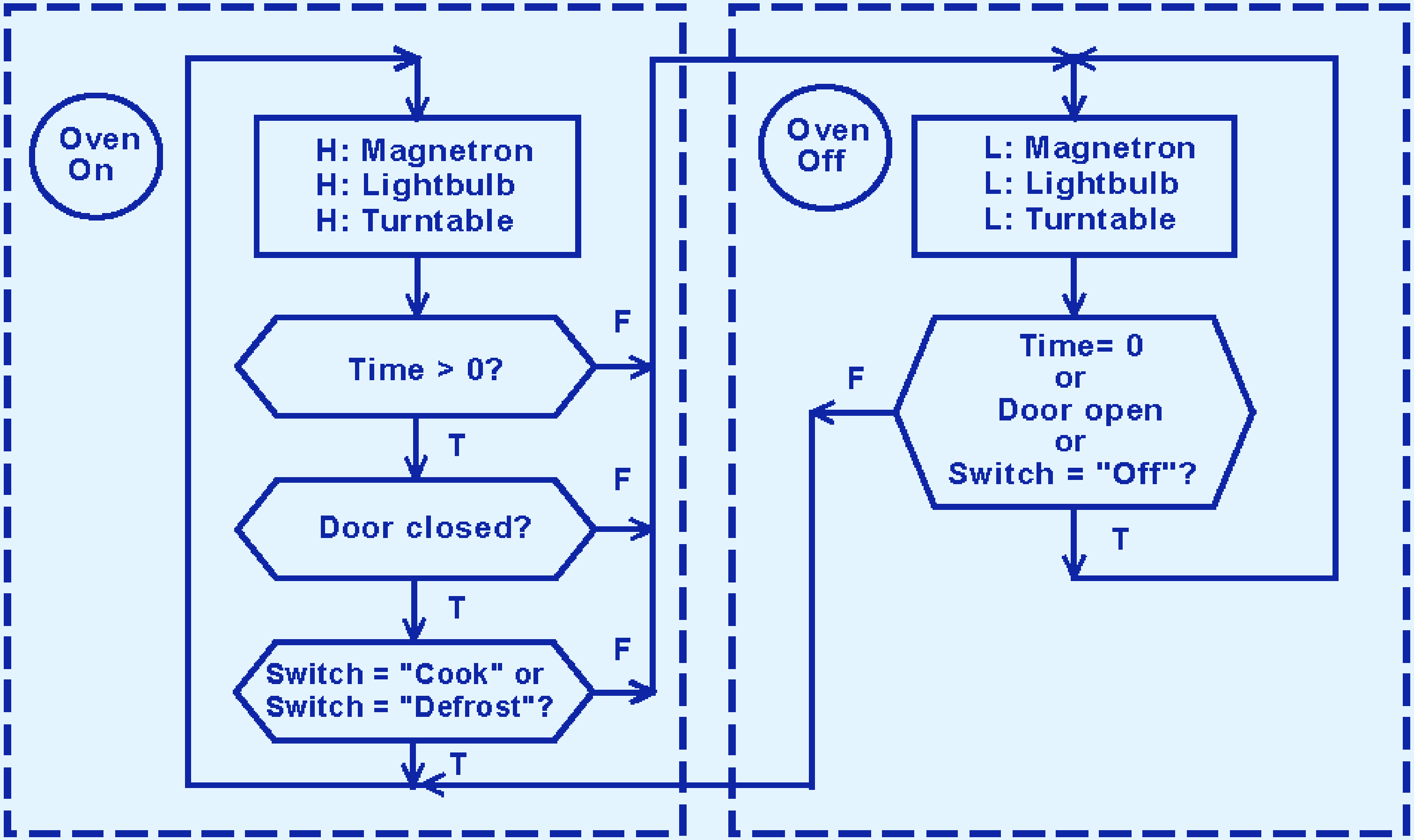
The next slide illustrates the components of an ASM.

Sequential Circuits



Sequential Circuits

- This is an ASM for a microwave oven.

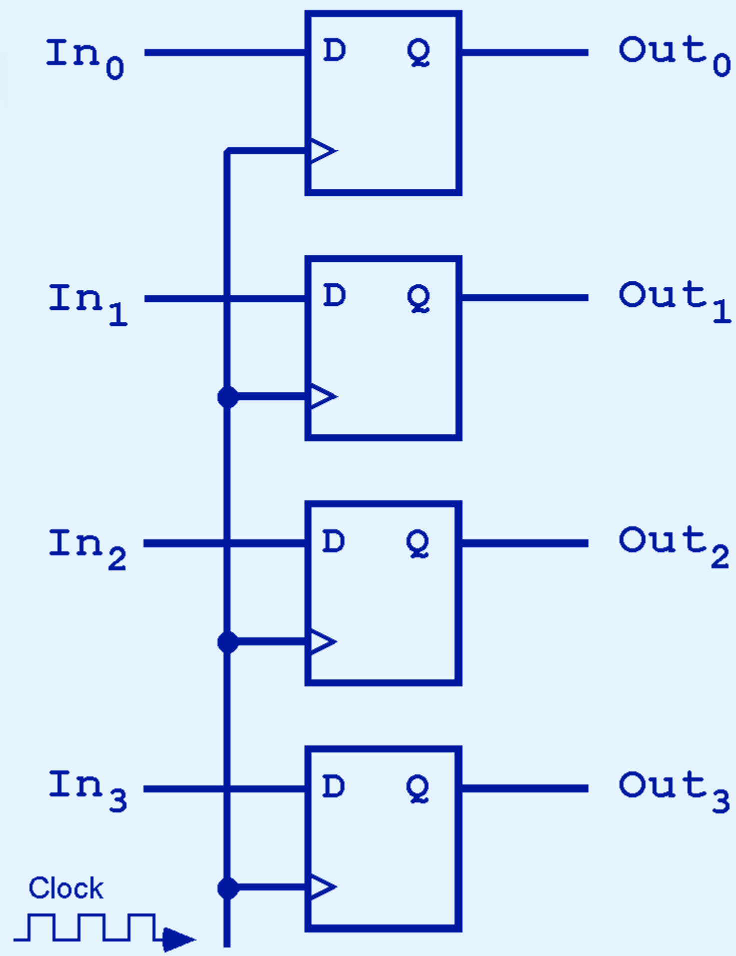


Sequential Circuits

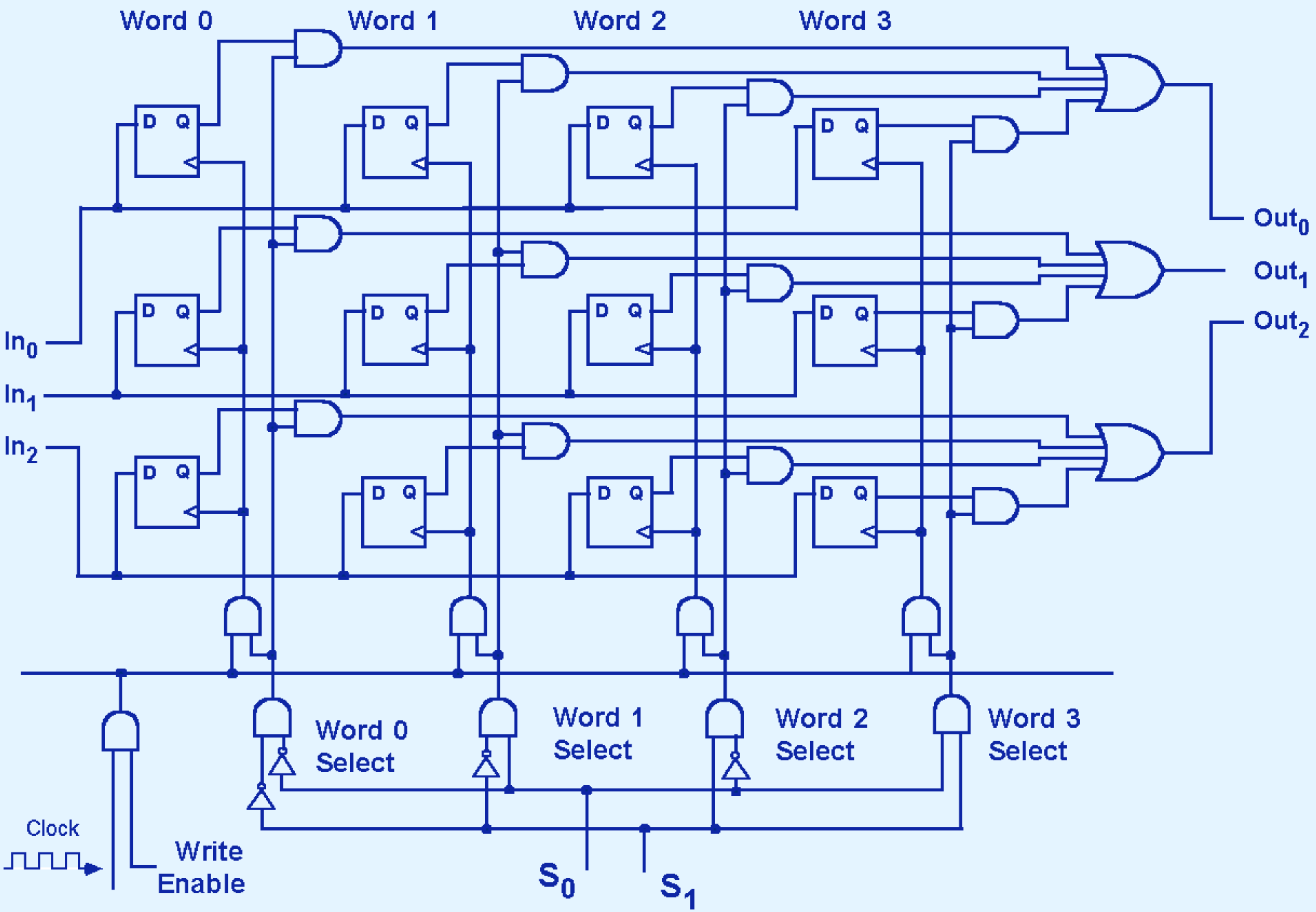
- Sequential circuits are used anytime that we have a "stateful" application.
  - A stateful application is one where the next state of the machine depends on the current state of the machine and the input.
- A stateful application requires both combinational and sequential logic.
- The following slides provide several examples of circuits that fall into this category.

Sequential Circuits

- This illustration shows a 4-bit register consisting of D flip-flops. You will usually see its block diagram (below) instead.



Sequential Circuits



Sequential Circuits

- A binary counter is another example of a sequential circuit.
- The low-order bit is complemented at each clock pulse.
- Whenever it changes from 0 to 1, the next bit is complemented, and so on through the other flip-flops.