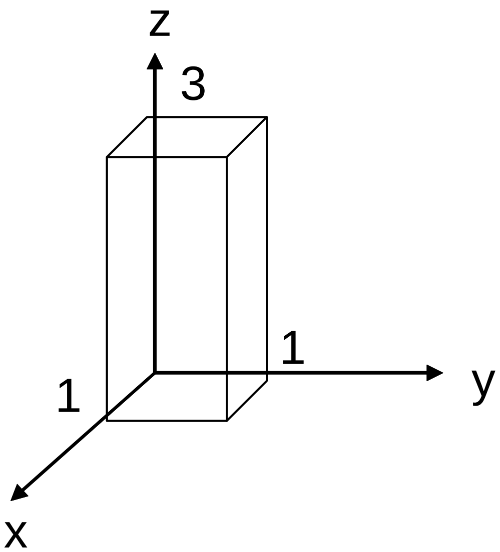
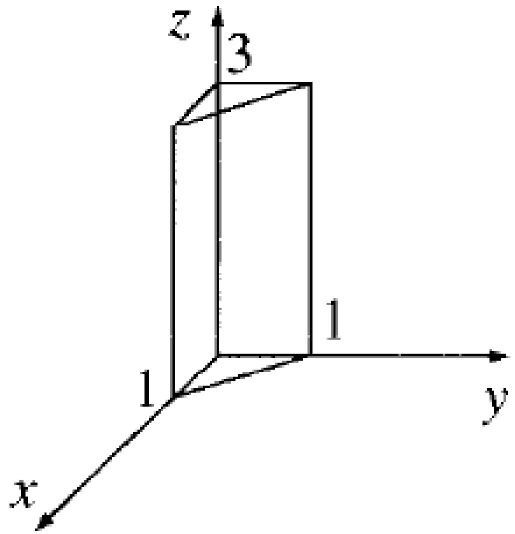
Volume Integral

$$\int_{\mathcal{V}} T\, d\tau \qquad\qquad d\tau = dx\,dy\,dz$$

Calculate the volume integral of $T = xyz^2$ over the prism in Fig.





$$\int T\, d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x\, dx \right] dy \right\} dz =$$

$$\frac{1}{2} \int_0^3 z^2\, dz \int_0^1 (1-y)^2 y\, dy = \tfrac{1}{2}(9)(\tfrac{1}{12}) = \tfrac{3}{8}.$$

The Fundamental Theorem for Gradients

The integral of a derivative over a region is equal to the value of the function at the boundary

$$\boxed{\int_{\substack{\mathbf{a}\\ \mathcal{P}}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).}$$

$dT = (\nabla T) \cdot d\mathbf{l}_1$

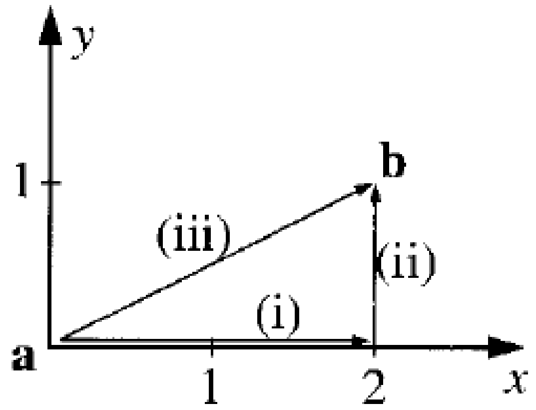
Difference of function's value at b and a

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is independent of path taken from **a** to **b**.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

$\vec{\nabla} \times \vec{\nabla} \boldsymbol{T} = 0$ $\vec{\nabla} \times \vec{\boldsymbol{F}} = 0$ F conservative field

Let $T = xy^2$, and take point **a** to be the origin $(0, 0, 0)$ and **b** the point $(2, 1, 0)$. Check the fundamental theorem for gradients.



Divergence

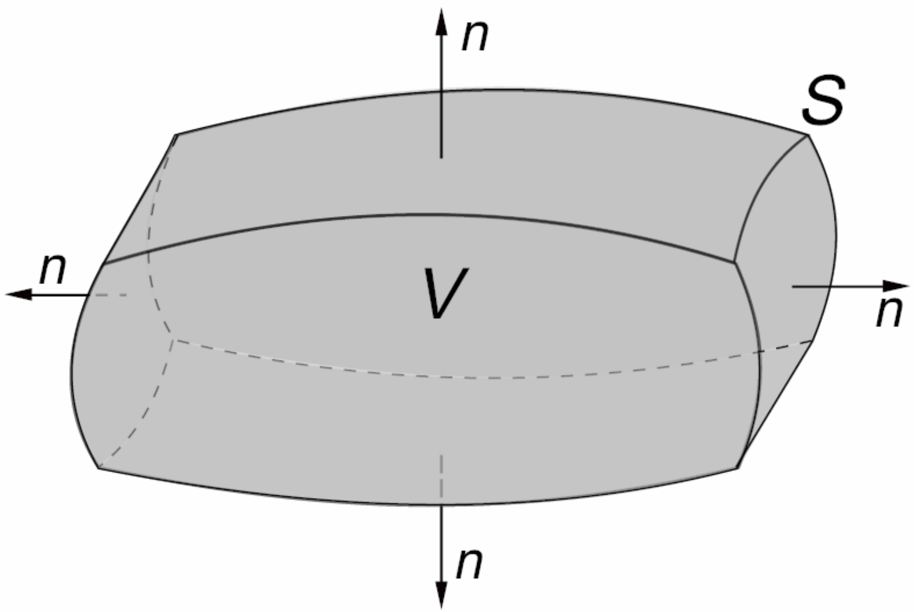
$$div\ \vec{F} = \lim_{\Delta v \to 0} \frac{\displaystyle\oiint_S \vec{F} \cdot d\vec{s}}{\Delta v}$$

Curl

$$\lim_{\Delta S \to 0} \frac{\oint \vec{F} \cdot d\vec{l}}{\Delta S} = (curl\, F) \cdot \hat{n}$$

The Fundamental Theorem for Divergences

$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.}$$



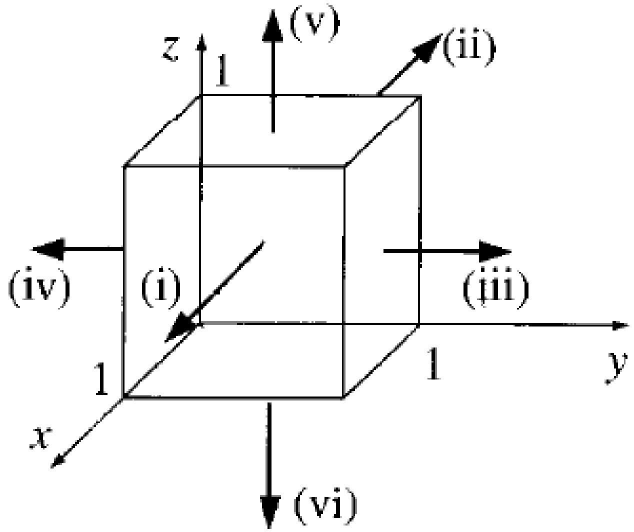
The integral of a derivative over a region is equal to the value of the function at the boundary

$$\nabla \cdot \vec{V} = \text{outflow} - \text{inflow} \longrightarrow$$

+ve (source)
-ve (sink)

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface})$$

Check the divergence theorem using the function $\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$



(iii) $$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^1\int_0^1 (2x+z^2)\,dx\,dz = \tfrac{4}{3}.$$

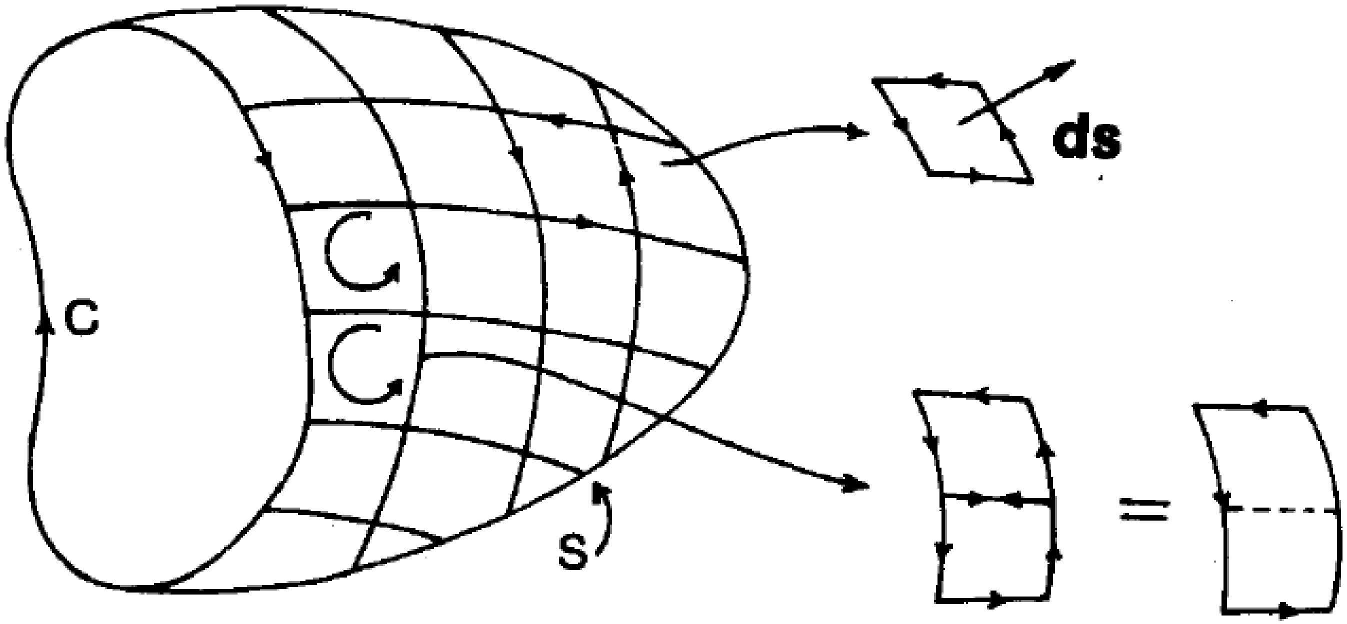
(iv) $$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^1\int_0^1 z^2\,dx\,dz = -\tfrac{1}{3}.$$

(v) $$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^1\int_0^1 2y\,dx\,dy = 1.$$

The Fundamental Theorem for Curls Stokes' theorem

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}.$$

The integral of a derivative over a region is equal to the value of the function at the boundary

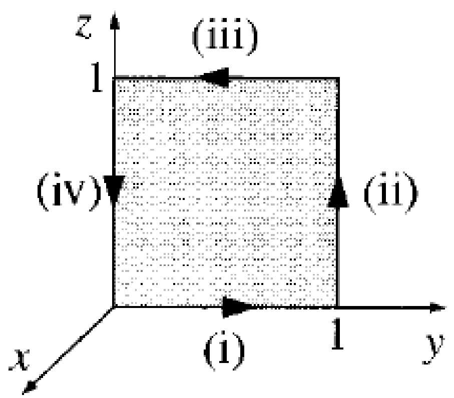


rotational force field / non-conservative force field

Corollary 1: $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface

Suppose $\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$. Check Stokes' theorem for the square surface



$$\nabla \times \mathbf{v} = (4z^2 - 2x)\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{z}} \quad \text{and} \quad d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}.$$

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2\,dy\,dz = \tfrac{4}{3}.$$

(i) $x = 0, \quad z = 0, \quad \mathbf{v} \cdot d\mathbf{l} = 3y^2\,dy, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 3y^2\,dy = 1,$

(ii) $x = 0, \quad y = 1, \quad \mathbf{v} \cdot d\mathbf{l} = 4z^2\,dz, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4z^2\,dz = \tfrac{4}{3},$

(iii) $x = 0, \quad z = 1, \quad \mathbf{v} \cdot d\mathbf{l} = 3y^2\,dy, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 3y^2\,dy = -1,$

(iv) $x = 0, \quad y = 0, \quad \mathbf{v} \cdot d\mathbf{l} = 0, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 0\,dz = 0.$

$$\oint \mathbf{v} \cdot d\mathbf{l} = 1 + \tfrac{4}{3} - 1 + 0 = \tfrac{4}{3}$$

- Concepts of Vector Field
- Gradient [converts scalar field to a vector field]
- Divergence [A measure of change of flux per unit volume]
- Curl [measure of rotational nature of a vector field]
- Line integration
- Surface integration
- Volume integration
- Fundamental theorem for Gradient
- Fundamental theorem for Divergence
- Fundamental theorem for Curl

Cylindrical and spherical co-ordinate system

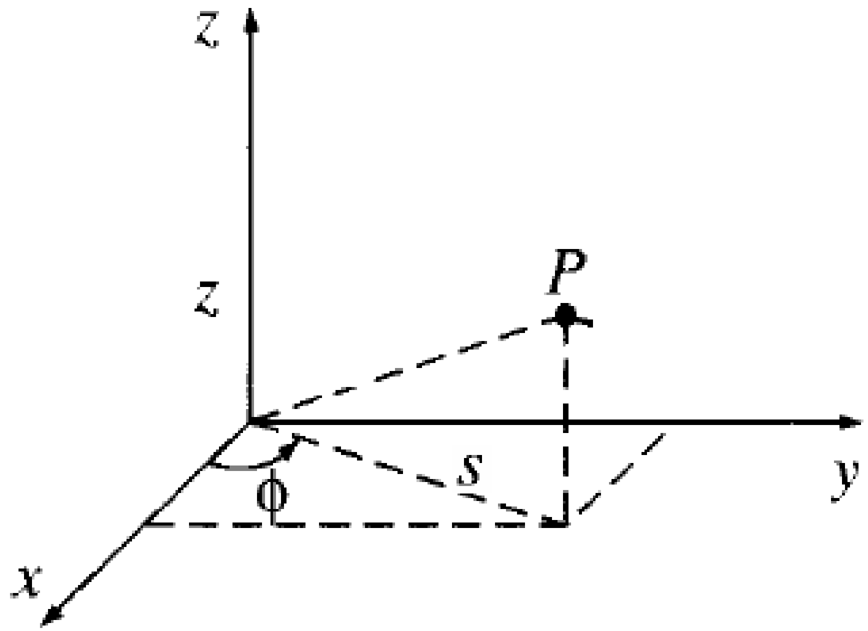
s

Cylindrical Coordinates

$$0 \le s \le \infty$$

$$0 \le \varphi \le 2\pi$$

$$-\infty \le z \le \infty$$



s

φ

z

$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z$$

$$s = \sqrt{x^2 + y^2} \qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right) \qquad z = z$$

Unit Vectors

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\hat{s} = \frac{\frac{\partial \vec{r}}{\partial s}}{\left|\frac{\partial \vec{r}}{\partial s}\right|} \qquad \hat{\varphi} = \frac{\frac{\partial \vec{r}}{\partial \varphi}}{\left|\frac{\partial \vec{r}}{\partial \varphi}\right|} \qquad \hat{z} = \frac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|}$$

$$\vec{r} = s\cos\varphi\,\hat{i} + s\sin\varphi\,\hat{j} + z\hat{k}$$

$$\hat{s} = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}$$

$$\hat{\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$$

$$\hat{z} = \hat{k}$$