Assignment - 2nd

Q3

$$\int \vec{\nabla} \cdot \vec{V} \, dV = \oint_s \vec{V} \cdot \vec{ds}$$

In spherical Coordinate System.

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_r) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, V_\theta)$$

$$+ \frac{1}{r\sin\theta} \frac{\partial V_\phi}{\partial \phi}$$

The given Vector funtion is

$$V = r^2 \cos\theta \, \hat{r} + r^2 \cos\phi \, \hat{\theta} - r^2 \cos\theta \sin\phi \, \hat{\phi}$$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^2 \cos\theta) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, r^2 \cos\phi)$$

$$+ \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} (-r^2 \cos\theta \sin\phi)$$

$$\frac{4r^{\cancel{3}}}{\cancel{r^2}} \cos\theta + \cancel{\frac{1}{r\sin\theta} (\cos\theta \, r^2 \cos\phi)} + \cancel{\frac{1}{r\sin\theta} - r^2 \cos\theta \cos\phi}$$

$$\Rightarrow \quad 4r\cos\theta$$

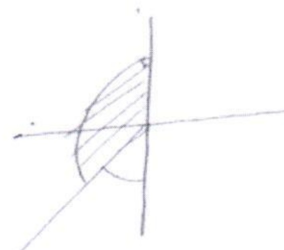
c

$$\int_V \nabla \cdot \vec{V}\, d\tau = \int (4r\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\Rightarrow 4\int_0^R r^3\, dr \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{\pi/2} d\phi$$

$$\Rightarrow R^4 \times \left(\frac{1}{2}\right) \times \left(\pi/2\right) = \frac{\pi R^4}{4}$$

Surface 1 = (X Z) plane

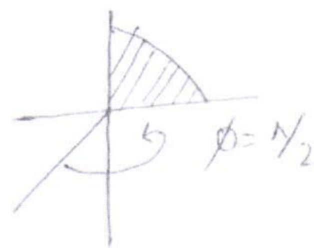


$da = r\, d\theta\, dr\, \hat{\phi}$; $\phi = 0$, $\vec{V} \cdot da = 0$

, $\int V \cdot da = 0$

$da = r\, d\theta\, dr\, \hat{\phi}$, $\phi = \pi/2$

$V \cdot da = (-r^2\cos\theta \sin\phi)(r\, d\theta\, dr)$



$da = -r^3 \cos\theta\, dr\, d\theta$

$$\int_S V\, da = -\int_0^R r^3 dr \int_0^{\pi/2} \cos\theta\, d\theta$$

$$= -\frac{R^4}{4} \times \sin\theta \Big|_0^{\pi/2} = -\frac{R^4}{4}$$

Bottom $\Rightarrow$ $da = r\sin\theta\, d\phi\, dr\, \hat{\theta}$

$\theta = \pi/2$, $\vec{V} \cdot \vec{da} = (r^2 \cos\phi)(r\, dr \sin\theta\, d\phi)$

$\Rightarrow (r^2\cos\phi)(r\, dr\, d\phi)$

$$\int \vec{V} \cdot da = \int_0^R r^3 dr \int_0^{\pi/2} \cos\phi\, d\phi \Rightarrow \frac{R^4}{4}$$

Curved Surface :

$$da = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}, \quad r = R$$

$$\vec{V} \cdot \vec{da} = (R^2 \cos\theta)(R^2 \sin\theta \, d\theta \, d\phi)$$

$$\int \vec{V} \cdot \vec{da} = R^4 \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{\pi/2} d\phi$$

$$\Rightarrow R^4 \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{\pi/2} d\phi$$

$$\Rightarrow R^4 \left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{4} R^4$$

Total $\displaystyle\int_S V \cdot da = \frac{\pi}{4} R^4$.