ODE $\rightarrow$ 1 independent variable

PDE $\rightarrow$ ≥ 1 " "

PARTIAL DIFFERENTIAL EQUATIONS

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- App^{n}s of PDE :

1. Black - Scholes Eq^n : used in mathematical finance to predict value of given stock.

$$f_t + r S f_s + \sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} = rf$$

f : $f(s)$

r : ~~rate~~ risk free rate of return

σ = Volatility const.

2. Navier - stoke's Eq^n : used in Fluid mechanics

3D

$$u_t + (u_0 \nabla) u = -\nabla_p + \nu \nabla u$$

$$\nabla u = 0 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

PDE :-

1st order PDE is of the form :

① — $f(x, y, z, z_x, z_y) = 0$

$\hookrightarrow$ independent variable

$z = z(x, y) \leftarrow$ dependent variable

$z_x = \frac{\partial z}{\partial x}$, $z_y = \frac{\partial z}{\partial y}$

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2nd Order PDE :

$f(x, y, z(x,y), z_{xx}, z_{xy}, z_{yy}, z_x, z_y) = 0$

- PDE with more than 2 independent variables :

$f(x, y, t, \ldots, z, z_{xx}, z_{yy}, z_{tt}, \ldots) = 0$

$z = z(x, y, t, \ldots)$

Order of PDE : Highest order partial derivative which appears in the PDE.

Eg. $z_{xx} + 2x z_x^3 + z_y = 0$ Order : 2

$\longrightarrow$ Quasilinear & semi linear

Classification of PDEs

1.) Quasilinear PDE

PDE is said to be Quasilinear if the highest order derivatives are linear

2) Semi-linear PDE

A quasilinear PDE is semi-linear if the coefficients of highest order derivatives do not contain dependent variable or its derivative.

- All ~~some~~ semi-linear are ~~so~~ quasilinear

3.) Linear Eqn

A semilinear PDE is said to be linear, if it is linear in the ~~linear~~ dependent variable & its derivatives

4) Non-Linear

If a PDE is not quasilinear, then it is non-linear.

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All linear are semilinear

All semilinear are quasilinear

eg. $\underbrace{u u_{xx}}_{\text{highest order}} + u_t = xt \Rightarrow 2^{nd}$ Order quasilinear PDE

$u = u(x,t)$: Quasi linear ✓

Not semilinear

Not linear

eg. $x u_{xx} + u u_t = xt$

not linear (product of dependent variables)

2^{nd} Order semi linear PDE

eg. $x u_{xx} + u_t = xt$

2^{nd} Order Linear PDE.

eg. $(u_{xx})^2 + u_t = xt$: Not quasilinear $\Rightarrow$ Non-linear 2^{nd} order PDE

eg. $u_{xx} + (u_t)^2 = xt$: 2^{nd} order semi linear PDE.

$\hookrightarrow$ Not linear

<u>Classification</u> of 1^{st} <u>order PDE</u> :

1.) Quasilinear PDE $\rightarrow$ p & q have to be linear

$f(x, y, z, p, q)$

$p = z_x \quad q = z_y \quad z = z(x,y)$

$z^2 x p + e^z y q = x^2 \sin y$

$P(x,y,z)\, p + Q(x,y,z)\, q = R(x,y,z) \Rightarrow$

2.) semi linear PDE : $P(x,y)\, p + \underset{\text{not func}^n \text{ of } z}{Q(x,y)}\, q = R(x,y,z) \Rightarrow$

eg. $x y^2 p + e^x \sin y\, q = z^3 \sin y$

3.) linear PDE : diff. b/w 2.) & 3.) $\Rightarrow$ $R(x,y,z)$ is allowed to be non-linear

$P(x,y) p + Q(x,y) q = R(x,y) + z\, S(x,y)$

won't contain $z^2, z^3 \ldots$

4.) Non-linear PDE

$$f(x, y, z, p, q) = 0$$

Eg. $pq = 0$.

Originating 1^{st} Order PDE

x, y : independent

Consider an eq^n :

$$x^2 + y^2 + (z-c)^2 = a^2 \quad : \text{eq}^n \text{ of sphere}$$

diff. wrt x :

$$2x + \cancel{2y y_x}^{0} + 2(z-c)z_x = 0 \quad \text{—— (1)}$$

Similarly

$$\cancel{2x z_y}^{0} + 2y + 2(z-c)z_y = 0 \quad \text{—— (2)}$$

$$2x + 2(z-c)p = 0$$

$$2y + 2(z-c)q = 0$$

$$\Rightarrow \quad qx + (z-c)pq = 0$$

$$py + (z-c)pq = 0$$

$$\boxed{qx - py = 0}$$ — 1^{st} order linear PDE characterizes the eq^n of spheres with center at z-axis.

Ex. → Consider an eq^n of the form :

(1) —— $F(x, y, z, a, b) = 0$ $a, b \rightarrow$ const.

$x, y \rightarrow$ independent variable

diff. wrt. x :

$$\frac{\partial F}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y} \underbrace{\frac{\partial y}{\partial x}}_{=2} + \frac{\partial F}{\partial z} \underbrace{\frac{\partial z}{\partial x}}_{p} = 0$$

$$\Rightarrow \quad \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \quad \text{—— (2)}$$

diff. wrt. y :

$$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \quad \text{—— (3)}$$

F, a & b need to be eliminated from these 3 eq^{n}s.
we get a PDE : (Use above eq. here)

$$f(x, y, z, p, q) = 0 \quad (\text{non-linear PDE})$$

Surface of Revolution

All the surfaces with z-axis as the axis of revolution are of the form:

(4) $z = F(r) \qquad r = \sqrt{x^2+y^2}$

diff wrt x,

$$p = z_x = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} = F'(r) \cdot \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x$$

$$p = \left(\frac{x}{r}\right) F'(r) \quad \text{—— (5)}$$

diff wrt y,

$$q = z_y = \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial y} = F'(r) \cdot \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y$$

$$q = \left(\frac{y}{r}\right) F'(r) \quad \text{—— (6)}$$

$\Rightarrow$ $\boxed{py - xq = 0}$ again, linear 1st order PDE

20/10/17 In general, consider the surface of the form

$$F(u, v) = 0 \quad \text{—— (1)}$$

where $u = u(x, y, z)$ and $v = v(x, y, z)$ are two known func^n of x, y & z.

$x, y \rightarrow$ ~~independent variable~~

Diff eq^n (1) wrt x

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\cancelto{0}{\frac{\partial y}{\partial x}} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] +$$

$$\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\cdot\cancelto{0}{\frac{\partial y}{\partial x}} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

→ In ODE

1st order ⇒ 1-D sol^n

2nd order ⇒ 2-D sol^n

→ But in PDE, it is not so

$$\Rightarrow \frac{\partial F}{\partial u}\left[\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\right] = 0 \quad \text{—— (2)}$$

Similarly, diff. wrt y, we get:

$$\frac{\partial F}{\partial v}\left[\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\right] = 0 \quad \text{—— (3)}$$

Eliminate $\frac{\partial F}{\partial u}$ & $\frac{\partial F}{\partial v}$ from eq^n (2) & (3).

$$\boxed{\frac{\partial(u,v)}{\partial(y,z)}p + \frac{\partial(u,v)}{\partial(z,x)}q = \frac{\partial(u,v)}{\partial(x,y)}}$$

— ~~Not Non~~ Linear P.D.E. (could be Quasilinear / Semi-linear / linear)

where $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

(Jacobian of u, v w.r.t. x & y)

Ex. 1) consider : $(x-a)^2 + (y-b)^2 + z^2 = 1$ [1st type] Sphere

leads to non-linear PDE.

Verify it.

2) $z = x^n f(y/x)$ [2nd type] ↳ Rotation

leads to a 1st order PDE.

Verify it.

Find the PDE which characterizes the above surfaces?

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System of Surfaces

$$f(x, y, z, c) = 0 \quad \text{--- (4)}$$

$c \longrightarrow$ parameter

$$f(x, y, z, a, b) = 0 \quad : 2 \text{ parameter} \quad \text{--- (5)}$$

Def^n: Envelope of one-parameter system :

The surface determined by eliminating the parameter 'C' between the eq^n :

$$f(x, y, z, c) = 0 \quad \& \quad \frac{\partial}{\partial c} f(x, y, z, c) = 0$$

is called the envelope of one-parameter system (4)

Eg. $x^2 + y^2 + (z-c)^2 = 1$

diff. w.r.t. c

$$F(x, y, z, c) = x^2 + y^2 + (z-c)^2 - 1 = 0$$

$$\frac{\partial}{\partial c} F(x, y, z, c) = -2(z-c) = 0 \Rightarrow \boxed{z = c}$$

Substituting $z = c$, envelope of above eq^n is :

$$x^2 + y^2 = 1$$

(unit circle on x-y plane)

$\hookrightarrow$ Cross-section of system of surfaces (here, spheres)

Def^n Envelope of two-parameter system :

Consider system of surfaces. (5)

The surface obtained by eliminating a, and b from eq^ns

$$f(x, y, z, a, b) = 0, \quad \frac{\partial f}{\partial a} = 0 \text{ and } \frac{\partial f}{\partial b} = 0$$

is called the envelope of two-parameter system. (5)

eg. $(x-c)^2 + (y-d)^2 + z^2 = 1$

$f(x,y,z,c,d) = (x-c)^2 + (y-d)^2 + z^2 - 1 = 0$

$\frac{\partial f}{\partial c} = -2(x-c) = 0 \Rightarrow \boxed{x=c}$

$\frac{\partial f}{\partial d} = -2(y-d) = 0 \Rightarrow \boxed{y=d}$

$\Rightarrow$ The envelope is : $z^2 = 1$ or $z = \pm 1$ [2 parallel planes]

Solution of 1st Order PDE

Consider

$f(x, y, z, p, q) = 0$ where $p = z_x$, $q = z_y$ —(6)

(i) A func^n $z = z(x,y)$ should satisfy eq^n (6)

(ii) The func^n z is continuously differentiable on $(x,y) \in D \subset R \times R$ (because p & q must exist)

A sol^n $z = z(x,y)$ exists in 3D space (ie, $(x,y,z(x,y)) \in R^3$) can be interpreted as surface & hence is called integral surface of PDE (6)

Classification based on solutions :-

1. Complete Integral or Complete sol^n :
   $f(x,y,z,a,b) = 0$ lead to PDE of 1st order. Any such relation which contain 2 arbitary const. a & b, and is a sol^n of a 1st order PDE is said to be complete integral

Ex. $x^2 + y^2 + (z-c)^2 = a^2$

sol^n : $qx - py = 0$: 1st order

2. General Integral or General sol^n :
   $f(u,v) = 0$ provides a sol^n of 1st order PDE, known as General Integral

Ex. $\frac{\partial(u,v)}{\partial(z,x)} + p\frac{\partial(u,v)}{\partial(y,z)} = \frac{\partial(u,v)}{\partial(x,y)}$

will give rise to $f(u,v) = 0$

Singular Integral :

The sol^n obtained from the envelope of the two-parameter family is known as singular Point. ~~or~~ This is obtained by eliminating a & b from

$$z = F(x, y, a, b), \quad \frac{\partial F}{\partial a} = 0, \quad \frac{\partial F}{\partial b} = 0$$

Ex $z - px - qy - p^2 - q^2 = 0$

$$z = p^2 + q^2 + px + qy$$

Check this, $ax + by + a^2 + b^2$ is a complete integral.

$$z = F(x, y, a, b)$$

Find singular sol^{n}s :

$$\frac{\partial F}{\partial a} = \cancel{x} + 2a = 0, \qquad \frac{\partial F}{\partial b} = y + 2b = 0$$

$$\Rightarrow \underline{x = -2a} \qquad y = -2b$$

$$\Rightarrow 4z = -(x^2 + y^2) \quad \text{— singular sol}^n.$$

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PDE

Cauchy value Problem (or Initial value Problem)

Objective : To find an integral surface of the given PDE

$$f(x, y, z, p, q) = 0 \quad \text{--- (1)}$$

which contain an initial curve

$$C : x = x_0(s), \; y = y_0(s), \; z = z_0(s), \; s \in I$$

The cauchy value problem is to find a solⁿ $z(x,y)$ of the PDE (1) s.t.

$$z_0(s) = z(x_0(s), y_0(s)) \quad \forall \; s \in I$$

If we can solve quasilinear then we can solve for linear & semi-linear also.

Langrage Method (for solving Quasilinear Problem)

$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z) \quad \text{--- (2)}$$

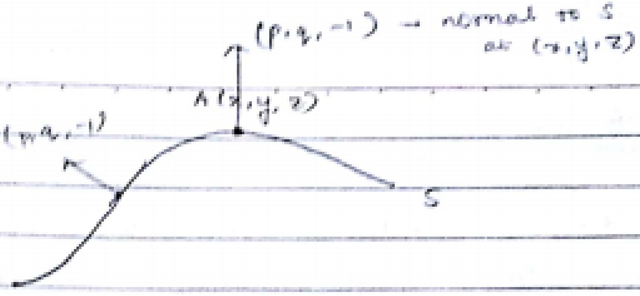
P, Q, R are smooth funcⁿ in $D \subset R^3$

$P, Q, R \in C^1(D) \Rightarrow$ (derivatives of P, Q, R are continuously differentiable)

$P, Q, R : D \to R$ don't vanish simultaneously

$z = z(x,y)$ is an integral surface in xyz - space

$$S : \{ z = z(x,y) : (x,y) \in D' \subset R \times R \}$$



Eq^n ① becomes:

$$Pp + Qq - R = 0$$

We can say that (P, Q, R) are orthogonal to $(p, q, -1)$

eq^n ② is equivalent to say that vector $(p, q, -1)$ and (P, Q, R) are orthogonal at each point $A \in S$

$P\hat{i} + Q\hat{j} + R\hat{k}$ lies on the tangent plane at A.
(then only, they'll be orthogonal)

For a curve $C: x = x(t), y = y(t), z = z(t)$ on S

we have $(P\hat{i} + Q\hat{j} + R\hat{k}) \parallel (\underbrace{\dot{x}}_{\text{diff. wrt } t}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$

equivalently,

$$\boxed{\frac{\dot{x}}{P} = \frac{\dot{y}}{Q} = \frac{\dot{z}}{R}} \quad \text{— characteristic eq}^n \text{ of PDE – ②} \quad \text{— ③}$$

or $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

Th^m: The general sol^n of the PDE ② is

Implicit form ← $F(u, v) = 0$ (or $u = G(v)$ or $v = H(u)$) ← explicit form

where F is an arbitrary smooth func^n of u and v.

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent sol^n of characteristic eq^n ③

Proof: $\underbrace{u(x, y, z) = c_1 \;\&\; v(x, y, z) = c_2}_{\text{called characteristic curves (here)}}$ are two sol^{n}s of characteristic eq^n ③

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$\Rightarrow du = 0$ and $dv = 0$

$\Rightarrow u_x dx + u_y dy + u_z dz = 0$ and $v_x dx + v_y dy + v_z dz = 0$

But $\dfrac{\overset{dx}{\dot{x}}}{P} = \dfrac{\overset{dy}{\dot{y}}}{Q} = \dfrac{\overset{dz}{\dot{z}}}{R} = k$ (let)

$\Rightarrow P u_x + Q u_y + R u_z = 0$

In the same way,

$P v_x + Q v_y + R v_z = 0$

Solving for P, Q and R, we get

$$k = \frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \quad \text{--- (4)}$$

$(u_y v_z - v_y u_z)$

Since $F(u,v) = 0$ leads to PDE of form

$$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)}$$ [seen in Revolution of surface Part] --- (5)

Comparing eq^n (4) & (5), we get

$$\boxed{P(x,y,z)\, p + Q(x,y,z)\, q = R(x,y,z)}$$

Eg. $x^2 p + y^2 q - (x+y) z = 0$

Sol^n: The characteristic is given by:

take on RHS first

$$\underset{\text{coeff of } p}{\frac{dx}{x^2}} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$\Rightarrow \dfrac{dx}{x^2} = \dfrac{dy}{y^2} \Rightarrow -\dfrac{1}{x} = -\dfrac{1}{y} + C$

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$\Rightarrow \frac{1}{x} - \frac{1}{y} = c_1 = u(x,y)$ —— ①

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{dx-dy}{x^2-y^2}$$

$$\frac{dx-dy}{x^2-y^2} = \frac{dz}{(x+y)z}$$

$$\Rightarrow \frac{dx-dy}{x-y} = \frac{dz}{z}$$

$$\Rightarrow \frac{d(x-y)}{x-y} = \frac{dz}{z}$$

$$\Rightarrow \log|x-y| = \log|z| + \log k$$

$$\Rightarrow (x-y) = kz = c_2 z$$

$\Rightarrow \frac{x-y}{z} = c_2 = v(x,y)$ —— ②

Now, we have a sol^{n}s $u(x,y)$ & $v(x,y)$

$$F(u(x,y), v(x,y)) = F(c_1, c_2)$$

$$= \boxed{F\left(\frac{1}{x}-\frac{1}{y}, \frac{x-y}{z}\right) = 0}$$

This will give general sol^n for given PDE

Only need is it should be ~~cont~~ diff.
To get exact sol^n, we must eliminate F using Cauchy value Problem.

$$F(\quad) \equiv \phi\left(\frac{1}{x}, \frac{-1}{y}\right) = G\left(\frac{x-y}{z}\right)$$

$$\text{or} \quad \frac{x-y}{z} = H\left(\frac{1}{x}, \frac{-1}{y}\right)$$

Ex: $xp + yq = z$ containing the curve
$C: x_0 = s^2, \; y_0 = s+1, \; z_0 = 1$

Sol^n Characteristic eq^n:

$$\frac{dx}{x} + \frac{dy}{y} = \frac{dz}{z}$$

$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log|y| = \log|z| + \log c_1$

$\Rightarrow \underline{\frac{y}{z} = c_1} = u(x,y,z)$

$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \underline{\frac{y}{x} = c_2} = v(x,y,z)$

Two independent solⁿ for this characteristic eqⁿ.

General solⁿ : $F(u,v) = 0$

$\Rightarrow F\left(\frac{y}{z}, \frac{y}{x}\right) = 0$

Option 1

$\frac{y}{z} = c_1 \Rightarrow \frac{s+1}{s} = c_1$

$\frac{y}{x} = c_2 \Rightarrow \frac{s+1}{s^2} = c_2$

~~Relation b/w~~ c_1 & c_2 (find)

$\Rightarrow$ we get : $(c_1 - 1)c_1 = c_2$

$\Rightarrow \left(\frac{y}{z} - 1\right)\frac{y}{z} = \frac{y}{x} \Rightarrow \frac{(y-z)y}{z^2} = \frac{y}{x}$

$\Rightarrow \boxed{(y-z)x = z^2}$

Option 2.

$\frac{y}{x} = G\left(\frac{y}{z}\right) \rightarrow$ we have to determine G

$G(t)$: in terms of t

$\Rightarrow \frac{s+1}{s^2} = G\left(\frac{s+1}{s}\right)$

$\frac{s+1}{s} = t \Rightarrow 1 + \frac{1}{s} = t \Rightarrow \frac{1}{s} = t-1$

$\Rightarrow s = \frac{1}{t-1}$

$\Rightarrow \frac{\frac{1}{t-1} + 1}{\frac{1}{(t-1)^2}} = G(t)$

$\Rightarrow t(t-1) = G(t)$

put $t = y/z$

$\Rightarrow G(y/z) = (y/z)(y/z - 1) = \frac{y}{x}$

$\Rightarrow \boxed{(y-z)x = z^2}$

→ $\left.\begin{aligned} u(x,y,z) &= c_1 \\ v(x,y,z) &= c_2 \end{aligned}\right\} \rightarrow$ characteristic curve

Independent solⁿ :

$$\nabla u \times \nabla v \neq 0$$

The general solⁿ : $F(u,v) = 0$

$(p,q,-1)$ is normal ?? ∵

$F(x,y,z)$: $z = z(x,y)$ — Integral surface (Explicit form)

$F(x,y,z) = z(x,y) - z = 0$ — Implicit solⁿ

$\nabla F = (z_x, z_y, -1) = (p,q,-1)$: normal derivative to S at (x,y,z)

⇒ $P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$

is equivalent to

$$(P, Q, R) \cdot \nabla F = 0$$

$$(P, Q, R) \parallel (\dot{x}, \dot{y}, \dot{z})$$

Eg. $yzp + xzq = xy$

Solⁿ Characteristic eqⁿ :

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

$\frac{dx}{yz} = \frac{dy}{xz} \Rightarrow x\,dx - y\,dy = 0 \rightarrow x^2 - y^2 = c_1 = u(x,y,z)$

$\frac{dy}{xz} = \frac{dz}{xy} \Rightarrow y\,dy - z\,dz = 0$

$\Rightarrow y^2 - z^2 = c_2 = v(x,y,z)$

$F(x^2-y^2,\ y^2-z^2) = 0 \longrightarrow$ Implicit form

$\Rightarrow x^2 - y^2 = G(y^2 - z^2)$ or $y^2 - z^2 = H(x^2 - y^2) \rightarrow$ Explicit form

→ Not every implicit form can be converted into explicit form

Eg.: $(z^2 - 2yz - y^2)p + x(y+z)q = x(y-z)$

Sol^n: Characteristic eq^n:

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} \quad \text{—— (1)}$$

$$\frac{dy}{y+z} = \frac{dz}{y-z} \Rightarrow y\,dy - z\,dy = y\,dz + z\,dz$$

$$\Rightarrow \cancel{\frac{y^2}{2} - yz} = \cancel{yz + \frac{z^2}{2}} + c$$

$$\Rightarrow y\,dy - z\,dz - \underbrace{z\,dy - y\,dz}_{C_1} \quad d((yz)'$$

$$\Rightarrow \frac{y^2}{2} - \frac{z^2}{2} - yz = C_1$$

or $\quad y^2 - z^2 - 2yz = C_1 = u(x,y,z)$

Modifying (1) : (Componendo and Dividendo)

$$\frac{x\,dx + y\,dy + z\,dz}{\cancel{xz^2} - \cancel{2xyz} - \cancel{xy^2} + \cancel{xy^2} + \cancel{xyz} + \cancel{xzy} - \cancel{xz^2}} = \frac{}{0} = K$$

0 also makes sense here.

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$ { we are not dividing here,

$\Rightarrow x^2 + y^2 + z^2 = C_2 = v(x,y,z)$ it's like some notation. }

$$F(y^2 - z^2 - 2yz,\ x^2 + y^2 + z^2) = 0$$

Eg. $2p + 3q + 8z = 0$

Find the integral curve containing the following curves:

i) σ : $z = 1 - 3x$ on line $y = 0$

ii) σ : $z = x^2$ on line $2y = 1 + 3x$

iii) σ : $z = e^{-4x}$ on line $2y = 3x$

1st, find general sol^n, for this, find characteristic eq^n,

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{-8z}$$

$2dy - 3dx = 0$

$\Rightarrow 2y - 3x = C_1 = u(x,y,z)$

$\frac{dx}{2} = \frac{dz}{-8z}$ $\Rightarrow$ $-4x = \ln z + \ln c_2$

$\Rightarrow$ $e^{-4x} = z c_2$

$\Rightarrow$ $\frac{e^{-4x}}{z} = c_2 = v(x,y,z)$

or $z e^{4x} = c_2 = v(x,y,z)$

general sol^n :

$F(2y-3x, z e^{4x}) = 0$

or $z e^{4x} = G(2y-3x)$ — ①

eg. i) σ : $z = 1-3x$ on $y=0$

$(1-3x) e^{4x} = G(-3x)$

put $-3x = s$ $\Rightarrow$ $x = -s/3$

$\Rightarrow$ ~~(1+s) e~~

$G(s) = (1+s) e^{-4s/3}$

From ①

~~$(1+3x)\ldots$~~ $\Rightarrow$ ~~$z e^{4x} = G(1-3x)$~~

From ①,

$z e^{4x} = (1 + 2y - 3x) e^{-4(2y-3x)/3}$

$+4x$

$\Rightarrow$ $z = (1 + 2y - 3x) e^{-8/3 y}$

ii) σ : $z = x^2$ on line $2y = 1 + 3x$

$(x^2) e^{4x} = G(1 + 3x - 3x) = G(1)$

or $\underset{\text{polynomial}}{x^2} = \underset{\text{const}}{G(1)}\ \underset{\text{exponential}}{e^{-4x}}$ $\Rightarrow$ no possible value of x

$\Rightarrow$ NO SOL^n

iii) $z = e^{-4x}$ on $2y = 3x$

$\Rightarrow$ $e^{-4x} \cdot e^{4x} = G(0)$

$\Rightarrow$ $G(0) = 1$ $\Rightarrow$ func^n takes value 1 at $x=0$

$G(t)$: can be: e^t, $\cos t$, $1+t$, $1+t^5$, ...

$\boxed{\text{Infinite many sol}^n\text{s}}$

→ Cauchy problem may have:
unique solⁿ
no solⁿ
infinitely many solⁿ

eg: $(2xy-1)p + (z-2x^2)q = 2(x-yz)$

Find Integral surface containing the curve

$\gamma : x_0(s) = 1, \; y_0(s) = 0, \; z_0(s) = s$

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)} \left(\frac{x}{x}\right)$$

try to make denominator $= 0$ & exact diff eqⁿ in numerator

① $\Rightarrow \dfrac{z\,dx + dy + x\,dz}{2xyz - \cancel{z} + \cancel{z} - \cancel{2x^2} + \cancel{2x^2} - 2xyz} = k$

$\Rightarrow z\,dx + x\,dz + dy = 0$

$\Rightarrow xz + y = c_1 = u(x,y,z)$

② $\dfrac{2x\,dx + 2y\,dy + dz}{4x^2y - \cancel{2x} + \cancel{2yz} - 4x^2y + \cancel{2x} - \cancel{2yz}}$

$\Rightarrow x^2 + y^2 + z = c_2 = v(x,y,z)$

~~For~~ Verify c_1 & c_2 are independent solⁿs.

$F(u,v) = 0$

$\Rightarrow xz + y = G(x^2+y^2+z)$
or $x^2+y^2+z = H(xz+y)$

} both may not help to get exact ⇒ need to take care while choosing.

$xz + y = G(x^2+y^2+z)$
$s = G(1+s)$
$1+s = t$
$G(t) = t-1$

$\Rightarrow xz + y = (x^2+y^2+z) - 1$

$1 + s = H(s)$

$\Rightarrow x^2+y^2+z = 1 + (xz+y)$

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Ex. $x^3 p + y(3x^2+y) q = z(2x^2+y)$

$$\frac{dx}{x^3} + \frac{dy}{y(3x^2+y)} = \frac{dz}{2zx^2+zy}$$

① $$\frac{-x^{-1}dx + y^{-1}dy - z^{-1}dz}{-x^2+3x^2+y-2x^2-y} = k$$

$$\Rightarrow -\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0$$

② $$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} \Rightarrow \left(\frac{3x^2+y}{x^3}\right) dx = \frac{dy}{y}$$

$$\Rightarrow \frac{(3x^2+y)\,dx + dy}{x^3+y} = \frac{(3x^2+y)\,dx + dy + x\,dy}{x^3+y+xy} = \frac{dy}{y}$$

Exact

$$\Rightarrow \frac{d(x^3+y+xy)}{x^3+y+xy} = \frac{dy}{y}$$

$$\Rightarrow \frac{x^3+y+xy}{y} = c_2$$

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Second Order PDE

The std. second order PDE in 2 independent variable

$f(x, y, z, z_{xx}, z_{yy}, z_{xy}, z_x, z_y) = 0$

(Semi-linear PDE)

↓

Can be represented as:

$u \rightarrow$ dependent variable

$u = u(x,y)$

① —— $$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

A, B, C, D, E, F, G : func^n of independent variables x & y

(D, E, F, G may also be func^n of u)

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same eqⁿ may be classified in diff. form depending on domain

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$A^2 + B^2 + C^2 \neq 0$, A, B, C are continuous as possess continuous partial derivative of as high order as necessary

Defⁿ: A funcⁿ $u(x,y)$ is said to be regular solⁿ of

(2) —— $Au_{xx} + Bu_{xy} + Cu_{yy} + g(x, y, u, u_x, u_y) = 0$

in $D \subset R \times R$ if $u \in C^2(D)$ (upto 2nd order derivative are continuous)

and the funcⁿ u & it's derivatives satisfies (2) for all $x, y \in D$

Genesis of 2nd Order PDE

$f \in C^2(D)$ & $u = f(x+at)$

$u_x = f'(x+at)$ $\quad u_t = a f'(x+at)$

$u_{xx} = f''(x+at)$ $\quad u_{tt} = a^2 f''(x+at)$

$\Rightarrow \boxed{u_{tt} = a^2 u_{xx}}$: leads to 2nd order PDE

Classification of 2nd Order PDE (Parabola, Ellipse, Hyperbola)

(3) —— $\underbrace{ax^2 + bxy + cy^2}_{\text{Principal part}} + dx + ey + f = 0$

Principal part (~~d~~ classifications depends on these variables only)

Normal form

→ $b^2 - 4ac > 0$: Hyperbola $\left(x^2/a^2 - y^2/b^2 = 1\right)$

→ $b^2 - 4ac = 0$: Parabola $\left(x^2 = y\right)$

→ $b^2 - 4ac < 0$: Ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$

For PDE :

Principal part of (2) :

$Lu = A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy}$

i) $B^2(x,y) - 4A(x,y)C(x,y) > 0 \Rightarrow (x,y)$ — hyperbolic PDE

ii) $= 0 \Rightarrow$ — parabolic PDE

iii) — elliptic PDE

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(1)

eg. $u_{xx} - x^2 u_{yy} = 0$

$A(x,y) = 1$ $\quad C(x,y) = -x^2$

$B(x,y) = 0$

$B^2 - 4AC = 0 - 4(1)(-x^2) = 4x^2 \geq 0$

$\Rightarrow u_{xx} - x^2 u_{yy} = 0$ → hyperbolic $x \neq 0$; → Parabolic $x = 0$

eg. $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = \underbrace{\frac{y^2}{x} u_x + \frac{x^2}{y} u_y}$

not included in classification but should be defined $(x \neq 0)$

$A = y^2 \quad B = -2xy \quad C = x^2$

$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0 \Rightarrow$ Parabolic PDE

(But we need to take care of u_x & u_y, they should be defined, so $x \neq 0$)

eg. $u_{xx} + x^2 u_{yy} = 0$

$B^2 - 4AC = 0 - 4(1)(x^2) = -4x^2$

$\Rightarrow$ ellipse for $x \neq 0$

parabolic for $x = 0$

eg. $u_{xx} + x u_{yy} = 0$

$\Rightarrow -4x$

$$\begin{cases} \text{Parabolic} & : x = 0 \\ \text{Elliptic} & : x > 0 \\ \text{hyperbolic} & : x < 0 \end{cases}$$

Here, we will study 2^{nd} order Semi-Linear PDE with 2 independent variables

<u>Cannonical (Normal) form of 2^{nd} Order PDE</u> : good for hyperbolic & ~~elli~~ parabolic (may not help in case of elliptic)

(1) — $A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0$

$A^2 + B^2 + C^2 \neq 0$ (all A, B, C can't be 0 at same time)
(otherwise, it won't be 2^{nd} order)

Our aim : $(x, y) \rightsquigarrow (\xi, \eta)$, $\xi = \xi(x, y)$

$u(x, y) \rightsquigarrow u(\xi, \eta)$ $\quad \eta = \eta(x, y)$

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \quad \text{(Assume)}$$

Thus, transformation is invertible

$u_\xi = u_\xi(\xi, \eta)$

$u_x = u_\xi \xi_x + u_\eta \eta_x$ (Chain Rule)

~~$u_{xx} = u_{\xi\xi}(\xi_x)^2 \pm u_\xi \xi_{xx} + 2 u_\eta (\eta_x)^2$~~

$u_{xx} = (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) \xi_x + u_\xi \eta_{xx}$

$\quad + (u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x) \eta_x + u_\eta \eta_{xx}$

$u_{xx} = u_{\xi\xi} \xi_x^2 + 2 u_{\xi\eta} \eta_x \xi_x + u_\xi \xi_{xx} + u_{\eta\eta} \eta_x^2 + u_\eta \eta_{xx}$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_{xy} = (u_{\xi\xi} \xi_y + u_{\xi\eta} \eta_y) \xi_x + u_\xi \xi_{xy}$

$\quad + (u_{\eta\xi} \xi_y + u_{\eta\eta} \eta_y) \eta_x + u_\eta \eta_{xy}$

$\quad = u_{\xi\xi} \xi_y \xi_x + u_\xi \xi_{xy} + u_{\xi\eta} \eta_y \xi_x + u_{\eta\xi} \xi_y \eta_x$

$\quad + u_{\eta\eta} \eta_y \eta_x + u_\eta \eta_{xy}$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{yy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_y + u_\xi \xi_{yy}$$
$$+ (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_y + u_\eta \eta_{yy}$$
$$= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_\xi \xi_{yy} + u_{\eta\eta}\eta_y^2 + u_\eta \eta_{yy}$$

Principal part:

$$A u_{xx} + B u_{xy} + C u_{yy} = u_{\xi\xi}(A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2)$$
$$+ u_{\xi\eta}(2A\xi_x\xi_y + B(\xi_x\eta_y + \eta_x\xi_y) + 2\xi_x\eta_y)$$
$$+ u_{\eta\eta}(A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2) + h(\xi, \eta, u, u_\xi$$

We choose ξ and η such that Jacobian $\neq 0$.

$\Rightarrow$ Eqn (1) becomes:

(2) — $\underbrace{\bar{A}(\xi_x ; \xi_y)}_{\text{coeff of } u_{\xi\xi} \text{ (func}^n\text{ of something)}} u_{\xi\xi} + 2\bar{B}(\xi_x, \xi_y ; \eta_x, \eta_y)u_{\xi\eta} + \bar{A}(\eta_x ; \eta_y)u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$ (3)

$$\bar{A}(u ; v) = Au^2 + Buv + Cv^2$$
$$\bar{B}(u_1, v_1 ; u_2, v_2) = Au_1u_2 + \frac{1}{2}B(u_1v_2 + u_2v_1) + Cv_1v_2$$

Eg. $\bar{B}^2(\xi_x, \xi_y ; \eta_x, \eta_y) - 4\bar{A}(\xi_x ; \xi_y)\bar{A}(\eta_x ; \eta_y) \; (=0)$

$$= (B^2 - 4AC)(\xi_x\eta_y - \xi_y\eta_x)^2$$

$(B^2-4AC) > 0$ (hyperbola) — (*)

$(\xi_x\eta_y - \xi_y\eta_x)$ ↳ Jacobian ↓ > 0 ($\neq 0$)

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1) Hyperbolic PDE : $B^2 - 4AC > 0$ (Wave eqⁿ)

consider the quadratic eqⁿ

$$A\alpha^2 + B\alpha + C = 0$$

We've 2 real and distinct roots : $\lambda_1(x,y)$, $\lambda_2(x,y)$ (say)

We choose $\xi(x,y)$ & $\eta(x,y)$ s.t.

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y} \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y} \quad \text{— (3)}$$

$$\Rightarrow \xi_x = \lambda_1 \xi_y \quad \text{and} \quad \eta_x = \lambda_2 \eta_y$$

$$\begin{aligned}\bar{A}(\xi_x; \xi_y) &= A\xi_x^2 + B\xi_x \xi_y + C\xi_y^2 \\ &= A\lambda_1^2 \xi_y^2 + B\lambda_1 \xi_y^2 + C\xi_y^2 \\ &= (A\lambda_1^2 + B\lambda_1 + C)\xi_y^2 \qquad (\lambda_1 : \text{root of eq}^n) \\ &= 0\end{aligned}$$

Similarly, $\bar{A}(\eta_x, \eta_y) = 0$

~~$2\bar{B}(\xi_x, \xi_y; u$~~

$$2\bar{B}(\xi_x, \xi_y; \eta_x, \eta_y)\, u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$$

since $\bar{B} > 0$

$$\boxed{u_{\xi\eta} = Q(\xi, \eta, u, u_\xi, \eta)}$$ canonical form for Hyperbolic case.

Eg. $u_{\xi\eta} = k$

$$\Rightarrow u_\xi = f(\xi)$$

$$\begin{aligned} u &= \int f(\xi)\, d\xi + g(\eta) \\ &= F(\xi) + g(\eta)\end{aligned}$$

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ξ & η are sol^{n}s of eq^{n}s ③

→ $\xi_x - \lambda_1 \xi_y = 0$

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0}$$

$d\xi = 0$

$\Rightarrow \xi = C_2$

$$\frac{dy}{dx} + \lambda_1 = 0$$

Assume $f_1(x,y) = c_1$ satisfies above eq^n

General sol^n

$F(c_1, c_2) = 0 \Rightarrow$

$F(f_1(x,y), \xi) = 0$

$\Rightarrow \xi = G_1(f_1(x,y))$

In particular, the simplest one is

$$\boxed{\xi = f_1(x,y)}$$

$\eta_x - \lambda_2 \eta_y = 0$

The sol^n of this PDE is :

$$\boxed{\eta = f_2(x,y)}$$

where $f_2(x,y)$ is sol^n of $\frac{dy}{dx} + \lambda_2 = 0$

Now, η & ξ are known, so we can find the canonical form.

Ex. $u_{xx} = x^2 u_{yy}$ (Find canonical form)

$A = 1 \quad B = 0 \quad C = -x^2$

$B^2 - 4AC = 0 + 4x^2 > 0 \quad \forall \quad x \neq 0$

$\Rightarrow$ Hyperbolic Type

Consider eq^n :

$$1\alpha^2 + 0 + (-x^2) = 0$$

$$\alpha^2 - x^2 = 0$$

$\Rightarrow$ $\cancel{\alpha = \pm 1}$ $\qquad \lambda_1(x,y) = +x$

$\Rightarrow$ $\cancel{\lambda_1, \lambda_2 = \pm 1}$ $\qquad \lambda_2(x,y) = -x$

Choose ξ & η s.t.

$$\xi_x = \lambda_1 \xi_y \qquad \& \qquad \eta_x = \lambda_2 \eta_y$$

$\Rightarrow$ $\xi_x - x\,\xi_y = 0$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{-x} = \frac{d\xi}{0}$$

$+\lambda_2$

$$\Rightarrow \frac{dy}{dx} + x = 0 \qquad \& \qquad \frac{dy}{dx} - x = 0$$

$$\Rightarrow \underbrace{y + \frac{x^2}{2}}_{f_1} = c_1 \qquad \& \qquad \underbrace{y - \frac{x^2}{2}}_{f_2} = c_2$$

$$\xi(x,y) = y + \frac{x^2}{2} \qquad \eta(x,y) = \frac{y - x^2}{2}$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$= u_\xi(x) + u_\eta(-x) = x(u_\xi - u_\eta)$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\eta_x\xi_x + u_\xi \xi_{xx} + u_{\eta\eta}\eta_x^2 + u_\eta \eta_{xx}$$

$$= u_{\xi\xi}(x^2) + 2u_{\xi\eta}(-x^2) + u_\xi(1) + u_{\eta\eta}(x^2) + u_\eta(-1)$$

$$u_{xx} = x^2\left[u_{\xi\xi} + u_{\eta\eta} - 2u_{\xi\eta}\right] + \underline{\left[u_\xi - u_\eta\right]}$$

Similarly,

$$\boxed{u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}}$$

Put in eq^n ①

$A \quad u_{xx} = x^2 u_{yy}$

$\Rightarrow \quad x^2 \left[\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}} \right] + \left[u_\xi - u_\eta \right] = x^2 \left[\cancel{u_{\xi\xi}} + 2u_{\eta\xi} + \cancel{u_{\eta\eta}} \right]$

Checkpoint : coeff. of $u_{\xi\xi}$ & $u_{\eta\eta}$ should be 0 in case of hyperbolic PDE.

$$\Rightarrow \quad 4u_{\xi\eta}x^2 = u_\xi - u_\eta$$

$$\Rightarrow \quad u_{\xi\eta} = \frac{u_\xi - u_\eta}{4x^2} = \boxed{\frac{u_\xi - u_\eta}{4(\xi - \eta)}}$$

canonical form.

$$= Q(\xi, \eta, u, u_\xi, u_\eta)$$

2) Parabolic PDE : $B^2 - 4AC = 0$ (Heat Eq^n)

$A\alpha^2 + B\alpha + C = 0$: Repeated real roots $= \lambda(x,y)$

Choose $\xi(x,y)$ s.t.

$$\frac{\partial \xi}{\partial x} = \lambda \frac{\partial \xi}{\partial y} \quad \left[\text{makes } \bar{A} = 0 \right]$$

This choice of ξ makes the coeff. of $u_{\xi\xi}$ as 0.

$$\bar{A}(\xi_x ; \xi_y) = \xi_y^2 \underbrace{(A\lambda^2 + B\lambda + C)}_{=0}$$

Choose $\eta(x,y)$ s.t. (ξ & η should be independent func^{n}s)

$$\frac{\partial(\xi, \eta)}{\partial(x,y)} \neq 0 \quad \text{OR} \quad \nabla\xi \times \nabla\eta \neq 0$$

here $\bar{A}(\eta_x, \eta_y)$ may not be equal to 0

From eq^n ⊛,

$(B^2 - 4AC) = 0$

$\bar{B}^2(\quad) - 4\bar{A}\overset{0}{(\quad)} \dots = 0 \Rightarrow \bar{B}^2 = 0$

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$\Rightarrow \bar{B} = 0$

Using ②, the canonical form is reduced to
$\xrightarrow{\text{non-0}}$

$\bar{A}(\eta_x, \eta_y) u_{\eta\eta} = G(\xi, \eta, \ldots)$

$\Rightarrow \boxed{u_{\eta\eta} = Q(\xi, \eta, u, u_\xi, u_\eta)}$

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Ques. Find canonical form of (Solve it iff possible)

$u_{xx} + 2u_{xy} + u_{yy} = 0 \quad A = 1 \quad B = 2 \quad C = 1$

$B^2 - 4AC = 4 - 4 = 0 \Rightarrow$ parabolic for all x and y

$\alpha^2 + 2\alpha + 1 = 0$

$\Rightarrow (\alpha+1)^2 = 0$

$\Rightarrow \alpha = -1$

Choose $\xi(x,y)$ s.t.

$\Rightarrow \frac{\partial \xi}{\partial x} = (-1) \frac{\partial \xi}{\partial y}$

$\Rightarrow \xi_x + \xi_y = 0$: Linear 1st order PDE

$\frac{dy}{dx} + \lambda = 0$

$\Rightarrow \frac{dy}{dx} - 1 = 0$

$\Rightarrow \boxed{y - x = c_1}$

$\Rightarrow \xi(x,y) = y - x$

$\frac{dx}{1} = \frac{dy}{1}$

$x - y = c$.

$\eta(x,y) = ??$

$\eta_x = \lambda \eta_y - X$

Choose $\eta(x,y)$ s.t.

$\frac{\partial(\xi, \eta)}{\partial(x,y)} \neq 0$

Let $\eta(x,y) = x + y$ (can choose any value)

$\xi = y - x$

$\xi_x = -1 \qquad \xi_{xx} = 0 \qquad \xi_{xy} = 0$

$\xi_y = 1 \qquad \xi_{yy} = 0$

$\eta(x,y) = x + y$

$\eta_x = 1$ all other $= 0$

$\eta_y = 1$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\eta_x\xi_x + u_{\eta\eta}\eta_x^2 + \cancel{u_\eta \eta_{xx}} + \cancel{u_\xi \xi_{xx}}$$

$$= u_{\xi\xi}(1) + 2u_{\xi\eta}(-1) + u_{\eta\eta}$$

$$u_{xy} = u_{\xi\xi}(-1) + 0 + u_{\xi\eta}(-1) + u_{\eta\xi}(1) + u_{\eta\eta}$$

$$= -u_{\xi\xi} - \cancel{u_{\xi\eta}} + \cancel{u_{\eta\xi}} + u_{\eta\eta}$$

$$= u_{\eta\eta} - u_{\xi\xi}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

substitute in given problem:

$$\Rightarrow u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} + 2\left[u_{\eta\eta} - u_{\xi\xi}\right] + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} = 0$$

$$\Rightarrow 4u_{\eta\eta} + 0\,(u_{\xi\xi}) + 0\,(u_{\eta\xi}) = 0$$

In case of parabola, these should be 0. (checkpoint)

$\eta = x+y$

$\xi = y-x$

or $\boxed{u_{\eta\eta} = 0}$ → Canonical form

$\Rightarrow u_\eta = f(\xi)$

$\Rightarrow u(\xi, \eta) = \int f(\xi)\, d\eta + g(\xi)$

$u(\xi, \eta) = f(\xi)\,\eta + g(\xi)$

$$\boxed{u(x,y) = f(y-x)\,(x+y) + g(y-x)}$$

Here, we are able to get explicit form.

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If we check from here,

$u_x =$

$u_{yy} =$

$\vdots$

this will satisfy parabolic eq^n.

For same eq, take $\eta(x,y) = x$

$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0 \Rightarrow$ find corresponding sol^n.

3.) **Elliptic PDE** : $B^2 - 4AC < 0$

Consider : $A\alpha^2 + B\alpha + C = 0$

Suppose $\lambda_1(x,y)$ and $\lambda_2(x,y)$: ~~two~~ distinct imaginary roots

$\Rightarrow \quad \lambda_1 = \bar{\lambda}_2$

Choose $\xi(x,y)$ and $\eta(x,y)$ s.t.

$\xi_x = \lambda_1 \xi_y$ and $\eta_x = \lambda_2 \eta_y$

$\Rightarrow \quad \bar{A}(\xi_x, \xi_y) = 0 = \bar{A}(\eta_x, \eta_y)$

$\Rightarrow$ Canonical form is :

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{—— (1)}$$

complex canonical form

Since $\xi(x,y)$ and $\eta(x,y)$ are complex characteristic curves and we want a real ~~form~~ curve
(we can add and subtract to get real curve using the superposition principle)

$\xi(x,y) = \alpha + i\beta$ and $\eta(x,y) = \alpha - i\beta$ [ξ & η are conjugate bcz λ_1 & λ_2 are conjugate]

$\alpha(\xi,\eta)$: $\alpha = \frac{1}{2}(\xi + \eta)$ $\quad \beta = \frac{1}{2i}(\xi - \eta) = \beta(\xi, \eta)$

$\downarrow$

α & β are two real characteristic curves

We've to get real canonical form in (α, β) form

$$u(x,y) \longrightarrow \underbrace{u(\xi,\eta) \longrightarrow u(\alpha,\beta)}_{\text{using this part:}}$$

$$u_\xi = u_\alpha \alpha_\xi + u_\beta \beta_\xi = \frac{1}{2} u_\alpha + \frac{1}{2i} u_\beta$$

$$u_\eta = u_\alpha \alpha_\eta + u_\beta \beta_\eta = \frac{1}{2} u_\alpha - \frac{1}{2i} u_\beta$$

$$u_{\xi\eta} = \frac{1}{2}\left(u_{\alpha\alpha}\alpha_\eta + u_{\alpha\beta}\beta_\eta\right) + \frac{1}{2i}\left(u_{\beta\alpha}\alpha_\eta + u_{\beta\beta}\beta_\eta\right)$$

$$= \frac{1}{2}\left(u_{\alpha\alpha}\cdot\frac{1}{2} + u_{\alpha\beta}\left(\frac{-1}{2i}\right)\right) + \frac{1}{2i}\left(u_{\beta\alpha}\cdot\frac{1}{2} + u_{\beta\beta}\left(\frac{-1}{2i}\right)\right)$$

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$= \frac{1}{4} u_{\alpha\alpha} - \frac{1}{4i} \cancel{u_{\alpha\beta}} + \frac{1}{4i} \cancel{u_{\beta\alpha}} + \frac{1}{4} u_{\beta\beta}$

$\Rightarrow \boxed{u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} + u_{\beta\beta})}$

The real canonical form is :

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)}$$

Ex. $u_{xx} + x^2 u_{yy} = 0$

$A = 1 \quad B = 0 \quad C = x^2$

$B^2 - 4AC = -4x^2 < 0 \quad \forall \; x \neq 0 \Rightarrow$ Elliptic for all $x \neq 0$

$\lambda^2 + x^2 = 0$

$\lambda = \pm ix \quad \lambda_1 = ix \quad \lambda_2 = -ix$

Choose $\xi(x,y)$ and $\eta(x,y)$ s.t.

~~$\xi_x = ix\, \xi_y$~~

The 2 associated ODE are:

$\frac{dy}{dx} + ix = 0 \quad \& \quad \frac{dy}{dx} - ix = 0$

$\Rightarrow \quad y + i\frac{x^2}{2} = c_1 \quad \& \quad y - i\frac{x^2}{2} = c_2$

$\xi(x,y) = y + i\frac{x^2}{2} \quad \& \quad \eta(x,y) = y - i\frac{x^2}{2}$

$\alpha(\xi,\eta) = \frac{1}{2}(2y) = y$ (Real part) $\quad \beta(\xi,\eta) = \frac{x^2}{2}$ (Imaginary part)

directly get $u(x,y) \rightarrow u(\alpha,\beta)$

$u_x = u_\alpha \overset{0}{\cancel{\alpha_x}} + u_\beta \beta_x = u_\beta x$

$u_{xx} = 1 \cdot u_\beta + x(\cancel{u_{\beta\alpha}\alpha_x} + u_{\beta\beta}\beta_x)$

$= u_\beta + x^2 u_{\beta\beta}$

$u_y = u_\alpha \overset{1}{\alpha_y} + \overset{0}{\cancel{u_\beta \beta_y}} = u_\alpha$

$u_{yy} = u_{\alpha\alpha}\alpha_y + \overset{0}{\cancel{u_{\alpha\beta}\beta_y}} = u_{\alpha\alpha}$

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substitute :

$\Rightarrow \quad u_\beta + x^2 u_{\beta\beta} + x^2 u_{\alpha\alpha} = 0$

$\Rightarrow \quad \boxed{x^2 (u_{\beta\beta} + u_{\alpha\alpha}) + u_\beta = 0}$ ~~Canonical~~ form.

$x^2 = 2\beta$ (write all x and y in terms of α & β)

$\Rightarrow \quad 2\beta (u_{\alpha\alpha} + u_{\beta\beta}) + u_\beta = 0$

$\Rightarrow \quad \boxed{u_{\alpha\alpha} + u_{\beta\beta} = -\dfrac{u_\beta}{2\beta}}$ canonical

* If $u_{xx} - 4u_{yy} = 0$ is given, it is already in canonical form. so, no need to proceed further.

Same has to be applied in case of hyperbolic & parabolic

* In elliptic, we can't get solⁿ (can't integrate $u_{\alpha\alpha}$ & $u_{\beta\beta}$) from

Canonical form. So, they are of not help.

eg. $u_{xx} - 2(\sin x) u_{xy} - (\cos^2 x) u_{yy} - u_y \cos x = 0$

$B = -2\sin x \quad A = 1 \quad C = -\cos x$

$B^2 - 4AC = 4\sin^2 x + 4\cos^2 x = 4 > 0 \Rightarrow$ Hyperbolic

$\alpha^2 - (2\sin x)\alpha + (-\cos^2 x) = 0$

$\alpha = \sin x \pm 1$

$\lambda_1 = \sin x + 1 \qquad \lambda_2 = \sin x - 1$

$\dfrac{dy}{dx} + (\sin x + 1) = 0$ & $\dfrac{dy}{dx} + (\sin x - 1) = 0$

$\Rightarrow \quad y - \cos x + x = C_1$ & $y - \cos x - x = C_2$

$\xi = y - \cos x + x \qquad \eta = y - \cos x - x$

$u(x,y) \to u(\xi, \eta)$

Find $u_{xx}, u_{yy}, u_{xy}, u_y$, will get $\boxed{u_{\eta\xi} = 0}$

$u_y = u_\xi \xi_y + u_\eta \eta_y$

$= u_\xi(1) + u_\eta(1) = u_\xi + u_\eta$

$\Rightarrow u_\eta = f(\eta)$

$u_\xi = \eta f(\xi) + g(\eta)$

$u_y = u_\xi + u_\eta$

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y$

$= \underline{u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}}$

$= u_\xi \sin x + \eta \sin x$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi(\sin x + 1) + u_\eta(\sin x - 1)$

$u_{xx} =$