PROBABILITY AND STATISTICS

→ Sheldon M. Ross : A 1st Course in Probability

→ " " : Introductory Statistics

→ Relative frequency : If we flip the coin a large no. of times, probability of getting heads ~~is~~ is $\frac{1}{2}$.

→ ϕ : Improper set ($A \subseteq S$)

Proper set : $A \subset S$

Superset : $A \supseteq B$

Universal set : S or Ω

→ Partition : disjoint & union is A.

Infinite set

Countable : finite set

Eg N, Z, Q

If we can place each ele in a set, we say it is countable

Uncountable :

Eg. R

→ Sample Space : ω

$\Downarrow$

all possible outcomes

→ Event : Subset of a sample space.

→ $(0, 5)$: Uncountable set
don't know which will be next element

Countable

L Finite
L It has bijective mapping from N to the set
because in case of sequence, we count Natural no.

Eg. set of all even no.

Defn : A set S is said to be Countably infinite if ∃ a func^n $f : N \to S$ s.t. f is one-one & onto (surjective)

Eg. Let S be the set of all even positive integers. Prove or disprove S is contable.

Sol^n : We have to construct $f : N \to S$

$$f(n) = 2n \quad \forall \quad n \in N$$

One-One : Let $n = m$

$$f(n) = 2(n) = 2(m) = f(m)$$

$\Rightarrow f(n) = f(m)$

$\Rightarrow$ f is One-One func^n

OR

Let $f(m) \neq f(n)$

$\Rightarrow 2m \neq 2n$

$\Rightarrow m \neq n$

$\Rightarrow$ f is One-One func^n

Onto: We've to show that each of element of S has a pre-image in N

Replace n by $n/_2$ for even and $\frac{n-1}{2}$ for odd

It will cover all the elements of S

* One-One, Onto $\not\Rightarrow$ Sets are same

eg. Set of all integers Z is a countable set

$$f : N \longrightarrow Z$$

Let $f(n) = \begin{cases} n/2 & , n \in \text{even no} \\ -(n-1)/_2 & , n \in \text{odd no.} \end{cases}$ $\rightarrow$ It covers all integers

$\rightarrow$ Natural no. just give the position

eg. Set of all positive rational no. is Countable.

We begin here with all positive rational no. using sum of ~~those~~ numerator and denominator.

sum is upto 2 $\quad \frac{1}{1}$

sum is upto 3 $\quad \frac{1}{1} \quad \frac{1}{2} \quad \frac{2}{1}$

sum is upto 4 $\quad \frac{1}{1} \quad \frac{1}{2} \quad \frac{2}{1} \quad \frac{1}{3} \quad \frac{3}{1} \quad \frac{2}{2}$

Pattern :

$\frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \ldots$

$\frac{2}{1} \quad \frac{2}{2} \quad \frac{2}{3} \quad \frac{2}{4} \quad \frac{2}{5} \quad \ldots$

$\frac{3}{1} \quad \frac{3}{2} \quad \frac{3}{3} \quad \frac{3}{4} \quad \frac{3}{5} \quad \ldots$

We are getting all the rationals in a pattern

→ Any small interval over real line is uncountable because irrational no. are uncountable

Countability

→ Multiplication Principle :

$$|A \times B| = |A|\,|B|$$

~~Eg → $^{26}C_3 \; ^{10}C_4$~~

Eg. $(26)^3 \times (10)^4$

↳ Sampling : choosing an element randomly from a set

↳ With Replacement : Repetition is possible : Direct Multiplication

↳ Ordered : sampling in which ordering matters

Ordered : Use Permutation (without Replacement)

Unordered : Use Combination

Theorem → No. of permutations of n distinct objects is n!

→ " taken k at a time : $^nP_k = \dfrac{n!}{(n-k)!}$

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Eg. How many even 3-digit no. using 1, 2, 5, 6, 9 if each digit can be used only once ?

$$4 \times 3 \times 2 = \underline{\underline{24}}$$

Theorem → No of permutations of n distinct objects arranged in a circle : $(n-1)!$

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→ No of ways of partitioning n objects into r cells of sizes $n_1, n_2, \ldots, n_r$: $\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$

$P(\text{top card is an Ace}) = \frac{4 * 51!}{52!} = \frac{4}{52} = \frac{1}{13}$

P(same suit next to each other)

$= \frac{4! \cdot 13!^4}{52!}$

$P(\text{hearts are together}) = \frac{40! \cdot 13!}{52!}$

↳ Unordered sampling without replacement : Combination

→ Difference b/w ^nC_k & ^nP_k is in the ordering

$^nP_k = {}^nC_k * k!$ → ordering is considered]

→ $^5C_2 * {}^5C_3 + {}^5C_2 * {}^5C_2 + {}^5C_2 * {}^5C_2$

$\frac{5\times4}{2} * \left(\frac{5\times4}{2}\right) * 2 + \frac{5\times4}{2} * \frac{5\times4}{2} = \underline{300}$ 7 ⟵⟶ 3

Binomial Theorem :

$$(x+y)^n = \sum_{k=0}^{n} {}^nC_k \, x^k y^{n-k}$$

Proof: Using Mathematical Induction

Let $(x+y)^1 = {}^1C_0 x^0 y^{1-0} + {}^1C_1 x^1 y^{1-1} = \sum_{k=0}^{1} {}^1C_k x^k y^{1-k}$

Let it be true for $n-1$ Then

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^k y^{n-1-k}$$

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Then, prove for $n = n$

$$(x+y)^n = (x+y)(x+y)^{n-1}$$

$$= (x+y) \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^k \, y^{(n-1)-k}$$

$$= \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^k y^{n-k}$$

Let us put $k = i-1$ in 1st term and $k = i$ in 2nd term.

$$= \sum_{i=1}^{n} {}^{n-1}C_{i-1} \, x^i y^{n-i} + \sum_{i=0}^{n-1} {}^{n-1}C_i \, x^i y^{n-i}$$

$$= \underbrace{x^n}_{\text{last term of 1st } \Sigma} + \sum_{i=1}^{n-1} {}^{n-1}C_{i-1} \, x^i y^{n-i} + \underbrace{y^n}_{\text{1st term of 2nd } \Sigma} + \sum_{i=1}^{n-1} {}^{n-1}C_i \, x^i y^{n-i}$$

$$= x^n + y^n + \sum_{i=1}^{n-1} \left(\left({}^{n-1}C_{i-1} + {}^{n-1}C_i \right) x^i y^{n-i} \right)$$

$$= x^n + y^n + \sum_{i=1}^{n-1} {}^{n}C_i \, x^i y^{n-i}$$

$$= \sum_{i=0}^{n} {}^{n}C_i \, x^i y^{n-i}$$

Hence Proved

10/1/18 Relative Frequency: (Axiome / assumptions)

We suppose that an experiment, whose sample space is S, is repeatedly performed under exactly the same condⁿ and if $n(E)$ to be the no. of times in the first n repetition of the exp. that E occurs

$$\lim_{n \to \infty} \frac{n(E)}{n}$$

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$E_1 \cap E_2 = E_1 E_2$

Kolmogorov's Axioms

Axiom 1: $0 \le P(E) \le 1$

Axiom 2: $P(S) = 1$

Axiom 3: For any sequence of mutually exclusive events $E_1, E_2, E_3, \ldots$ we have

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

ie, $E_i E_j = \phi \quad \forall\, i, j$

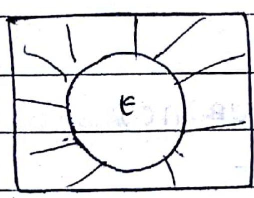
Eg. $P[\{1,3,5\}] = \frac{1}{2}$ (dice)

all are mutually exclusive

$\Rightarrow P[\{1,3,5\}] = P(1) + P(3) + P(5)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

P1 $\rightarrow$



$P(S) = P(E \cup E^c)$

$= P(E) + P(E^c)$ (by Axiom-3)

$S - E = E^c$

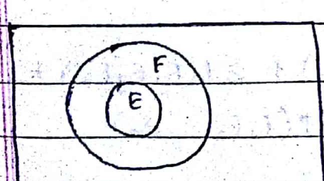
$P(S) = 1$ (by Axiom 2)

$S = E \cup E^c$

$\Rightarrow 1 = P(E) + P(E^c)$

$\Rightarrow \underline{P(E^c) = 1 - P(E)}$

P2 If $E \subseteq F$, then $P(E) \le P(F)$



E^cF

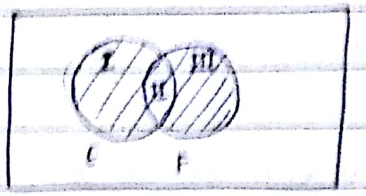
$F = E \cup (E^c \cap F)$ $\rightarrow$ mutually exclusive

$\Rightarrow P(F) = P(E) + \underline{P(E^cF)}$ (by Axiom 3)

$\Rightarrow \underline{P(F) \ge P(E)}$ $[P(E^cF) \ge 0]$

$\hookrightarrow$ by Axiom 1

P3 → $P(E \cup F) = P(E) + P(F) - P(EF)$



$P(I) + P(II) = P(E)$

$P(II) + P(III) = P(F)$

$P(I) + 2P(II) + P(III) = P(E) + P(F)$

$P(E \cup F) + P(II) = P(E) + P(F)$

mutually exclusive

formal proof :- mutually exclusive (Axiom 3)

$E \cup F = E \cup E^c F$

$\Rightarrow P(E \cup F) = P(E) + P(E^c F)$ — (1)

$F = EF \cup E^c F$ → mutually exclusive (Axiom-3)

$\Rightarrow P(F) = P(EF) + P(E^c F)$ — (2)

Using (1) & (2),

$P(E \cup F) = P(E) + P(F) - P(EF)$

Hence Proved.

$P(A \cup B \cup C) = \cancel{P(A) + P(B) + P(C)}$

$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$

$= P(A) + P(B) - P(A \cap B) + P(C) -$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Inclusion - Exclusion Identity

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum P(E_i) - \sum P(E_i E_j) + \sum P(E_i E_j E_k) \pm \dots + (-1)^{n+1} P(E_1 E_2 \dots E_n)$$

Proof : By Mathematical Induction (Try)

Sample Space with equally likely Outcomes

If in a sample space each event is equally likely to occur, then

$$P(E) = \frac{\text{no. of outcomes in } E}{\text{no. of outcomes in } S}$$

Eg. If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the ball is white and other 2 are black

$$= \frac{6}{11} * \frac{5}{10} * \frac{4}{9} * \frac{3!}{2!}$$

→ If order is considered :

$$S = {}^{11}P_3$$

$$= \frac{({}^{6}C_1 \cdot {}^{5}C_2) \cdot 3}{{}^{11}P_3} = \frac{4}{11}$$

~~Without~~

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Eg. Probability of getting Head exactly in n^{th} ~~3rd~~ trial for 1st time :

First $P(A) = \frac{1}{2}$

Second $P(A_2) = P(TH) = \left(1-\frac{1}{2}\right) * \frac{1}{2}$

$P(A_3) = P(TTH) = \left(1-\frac{1}{2}\right) * \left(1-\frac{1}{2}\right) * \frac{1}{2} = \frac{1}{8}$

$P(A_n) = \left(\frac{1}{2}\right)^n$

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→ ~~P(getting a head)~~ :

P(getting a head sometime) =

$$S = \frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^n}$$

(add^n of all we got in previous ques^n) → mutually exclusive

$$= \frac{\frac{1}{2}\cancel{(1-(\frac{1}{2})^n)}}{\frac{1}{2}} = \boxed{1}$$

Eg. Probability that a Projector stops on the n^{th} trial

Let us assume the probability of stopping the system = p

(we don't know exactly)

success in 1^{st} trial : p

2^{nd} trial = $(1-p) * p$

3^{rd} = $(1-p)^2 * p$

⋮

n^{th} trial : $(1-p)^{n-1} * p$

mutually exclusive

→ P(stopping sometime) = $p + (1-p) * p + (1-p)^2 * p \dots = S$

$$= p\,[1 + (1-p) + (1-p)^2 + \dots]$$

$$= p\left[\frac{1}{1-(1-p)}\right] = \frac{p}{p} = \boxed{1} \quad (p \neq 0)$$

$$S = p\,(1 + (1-p) + (1-p)^2 + \dots)$$

$$S = p + (1-p)\,p\,[1 + (1-p) + (1-p)^2 + \dots]$$

$$S = p + (1-p)\,p\left(\frac{S}{p}\right)$$

$$Sp = p$$

$$\boxed{S = 1}$$

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Everyone gets his head : $\frac{1}{n!}$

summation is just possible comb^n

Inclusion - Exclusion Identity

$$P(E_1 \cup E_2 \cup E_3 \cup \ldots E_n) = \sum P(E_i) - \sum_{i<j} P(E_i E_j) \ldots + (-1)^{n+1} P(E_1 E_2 \ldots E_n) \quad (1)$$

atleast 1 of the n-Events is happening

The summation $\sum_{i_1<i_2<i_r} P(E_{i_1} E_{i_2} \ldots E_{i_r})$ is taken over all of the ^nC_r possible subsets of size r of the set $\{1, 2, \ldots, n\}$

The Matching Problem

Suppose that each of N men at a party throws ~~hat~~ his hat into the center of the room. The hats are first mixed up and then each man randomly selects a hat.

① What is the probability that none of the men selects his own hat?

$P(E_1 \cup E_2 \ldots \cup E_n) \rightarrow$ atleast 1 gets his hat

let E_i, $i = 1, 2, 3, \ldots N$ denote the event that i^{th} man selects his own hat.

To address this problem, we will first calculate the probability that atleast 1 man gets correct hat.

Applying ①, we get : P( ) = — — —

If we regard the outcomes of this experiment as a vector of N numbers where the i^{th} element is the no of the hat drawn by the i^{th} man. Now, if $E_1, E_2, \ldots, E_r$ ~~are~~ the events that each of the $1, 2, \ldots, r$ selects his own hat can occur, so possible outcomes : $(N-r)!$

$(N-r)(N-\overline{r+1})(N-\overline{r+2}) \ldots 3 \; 2 \cdot 1$

$$P(E_1 E_2 \ldots E_r) = \frac{(N-r)!}{N!}$$

(other may get / may not get correct hat, we're not bothered about it)

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→ One gets his own hat

then $E_i = (N-1)!$

$$P(E_i) = \frac{(N-1)!}{N!} = \frac{1}{N}$$

→ If 2 get their own hat

$$P(E_{i_1} E_{i_2}) = \frac{(N-2)!}{N!}$$

→ If r get their own hat

$$P(E_1 E_2 \ldots E_r) = \frac{(N-r)!}{N!}$$

Using Inclusion – Exclusion Principle,

$$P\left(\bigcup_{i=1}^{N} E_i\right) = {}^{N}C_1 \cdot \frac{1}{N} - {}^{N}C_2 \frac{(N-2)!}{N!} + \ldots + {}^{N}C_r \frac{(N-r)!}{N!}$$

$$\ldots + {}^{N}C_N \frac{1}{N!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} \ldots + (-1)^{N+1} \frac{1}{N!}$$

$$P(\text{none gets matched}) = P\left(\bigcap E_i^c\right)$$

$$= P\left(\left(\bigcup E_i\right)\right)^c$$

$$= 1 - P\left(\bigcup_{i=1}^{N} E_i\right)$$

$$= 1 - \left(1 - \frac{1}{2!} + \ldots + (-1)^{N+1} \frac{1}{N!}\right)$$

Q. Compute probability that 10 married couples are selected at random at a round table, then such that no wife sits next to her husband. (Solve yourself - in book)

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(getting exactly 1 head) $\subseteq$ (getting atmost 2 heads) : ↑ng sequence of events

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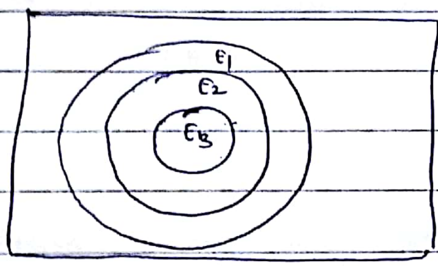
Continuous set functions

A sequence of events $\{E_n : n \geq 1\}$ is said to be increasing sequence if

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots \subseteq E_{r+1} \subseteq \ldots$$

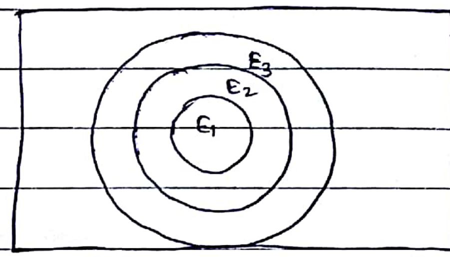
whereas it is said to be decreasing if

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots \supseteq E_{r+1} \supseteq \ldots$$



↓ng sequence

$E_1 \cap E_2 \cap E_3 = E_3$



↑ng sequence

$E_1 \cup E_2 \cup E_3 = E_3$

* If $\{E_n, n \geq 1\}$ is an increasing sequence of events, we denote $\lim_{n \to \infty} E_n$ by

$$\lim_{n \to \infty} E_n = \bigcup_{i=1}^{\infty} E_i$$

* If $\{E_n, n \geq 1\}$ is a decreasing sequence of events, we define

$$\lim_{n \to \infty} E_n = \bigcap_{i=1}^{\infty} E_i$$

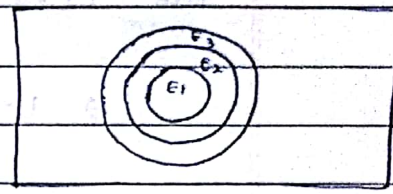
* If we convert set into mutually exclusive sets, it is easy to get probability of union of sets.

Proposition: If $\{E_n, n \geq 1\}$ is either an increasing sequence or a decreasing sequence of events, then

$$\lim_{n\to\infty} P(E_n) = P\left(\lim_{n\to\infty} E_n\right)$$

Proof: If $\{E_n, n \geq 1\}$ is an increasing sequence of events

Let $F_1 = E_1$



$$F_2 = E_2 \cap E_1^c$$

$$F_3 = E_3 \cap (E_1 \cup E_2)^c$$

$$\vdots$$

$$F_n = E_n \cap \left(\bigcup_{i=1}^{n-1} E_i\right)^c .$$

~~Now, given~~ This gives:

we have constructed our event s.t. this cond^n is true

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i \quad \text{and} \quad \bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{\infty} F_i$$

$\Rightarrow$ F_n consists of those outcomes in E_n which are not in any of earlier E_i, $i = 1, 2, \ldots, n-1$.

RHS $= P\left(\lim_{n\to\infty} E_n\right) = P\left(\bigcup_{i=1}^{\infty} E_i\right)$

$$= P\left(\bigcup_{i=1}^{\infty} F_i\right) \longrightarrow \text{mutually exclusive events}$$

$$= \sum_{i=1}^{\infty} P(F_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} F_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} E_i\right) = \lim_{n\to\infty} P(E_n)$$

finite union $= E_n$

$$= \text{LHS}$$

Hence Proved.

→ Even if we have 70 people, P (at least 2 have same birthday) is 99.9 %

→ 23 → 50 %

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If $\{E_n, n \geq 1\}$ is decreasing sequence

then $\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$

Then $\{E_n^c, n \geq 1\}$ becomes increasing sequence

then

$$\lim_{n\to\infty} P(E_n^c) = P(\lim_{n\to\infty} E_n^c) = P\left(\bigcup_{i=1}^{\infty} E_i^c\right) = P\left[\left(\bigcap E_i\right)^c\right]$$

$$\Rightarrow 1 - \lim_{n\to\infty} P(E_n) = 1 - P\left(\bigcap_{i=1}^{\infty} E_i\right)$$

$$\Rightarrow \lim_{n\to\infty} P(E_n) = P\left(\bigcap_{i=1}^{\infty} E_i\right)$$

$$= P(\lim_{n\to\infty} E_n)$$

Birthday Paradox :

At least two people have same birthday

sol^n :- No two people have same birthday;

for $n = 1$ $\quad P(E_1) = \frac{365}{365}$ $\quad P(E_1^c) = 1 - 1 = 0$

for $n = 2$ $\quad P(E_2) = \left(\frac{365 \times 364}{365 \times 365}\right)$ $\quad P(E_2^c) = 1 - \frac{364}{365} = \frac{1}{365}$

for $n = 3$ $\quad P(E_3) = \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right)$

for $n = r+1$ $\quad P(E_{r+1}) = \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \cdots \times \left(\frac{365-r}{365}\right)$

$$= \frac{{}^{365}P_{r+1}}{(365)^{r+1}}$$

* We can't take tossing of coin and throwing a dice in same sample space.

$$\boxed{P(E_{r+1}^c) = 1 - \frac{{}^{365}P_{r+1}}{(365)^{r+1}}}$$

→ for $n = 23$

$P(E_{23}) = 0.507$

→ for $n = 41$

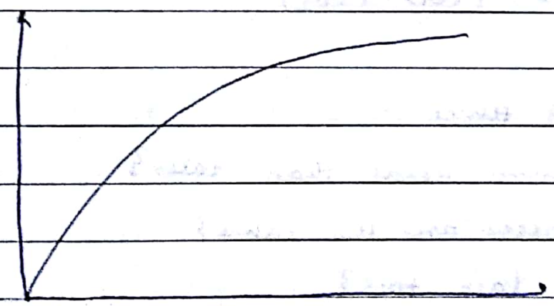
$P(E_{41}) = 0.903$

→ for $n = 70$

$P(E_{70}) = 0.990$

→ for $n = 366$

$P(366) = 1$



Independence of Event

Eg. Let Rahul have 2 Red $(R_1\, R_2)$, 1 Green (G) & 1 Black (B) Pen. We have to draw out 2 pens.

Sample ~~samp~~ space : $^4C_2 = 6$

$R_1 R_2$		
$R_1 G$	$R_2 G$	BG
$R_1 B$	$R_2 B$	

$P(E_1) = 4/6$, $P(E_2) = 3/6$

$P(E_1 \cap E_2) = 2/6$ $\quad P(E_1) \cdot P(E_2) = 2/6$

$\boxed{P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)}$ → Here

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→ 2R, 1G, 1B, 1Y

$P(E_1) = 6/10$ $\quad$ $P(E_2) = 4/10$

1 Red $\quad$ 1 Green

$P(E_1 \cap E_2) = \frac{2}{10}$

$\boxed{P(E_1)\cdot P(E_2) \neq P(E_1 \cap E_2)}$ → Here

⇒ Just by adding 1 pen, the ~~p~~ event no longer remains independent

Eg. Throwing a Dice:

E_1 : In first throw, get 4

E_2 : 2nd, get 5

$P(E_1) = \frac{1}{6}$ $\quad$ $P(E_2) = \frac{1}{6}$

$P(E_1 \cap E_2) = \frac{1}{36} = P(E_1)\cdot P(E_2)$

Ex. Toss a fair coin 3 times

E_1 = { there are more heads than tales }

E_2 = { first 2 tosses are the same }

E_3 = { heads on last toss }

HHH $\quad$ HHT $\quad$ TTH $\quad$ TTT

HTH $\quad$ THT

THH $\quad$ HTT

$P(E_1) = 1/2$ $\quad$ $P(E_2) = 1/2$ $\quad$ $P(E_3) = 1/2$

$P(E_1 \cap E_2) = 1/4 = P(E_1)\cdot P(E_2) = 1/4$ ✓ Independent

$P(E_1 \cap E_3) = 3/8 \neq P(E_1)\cdot P(E_3)$ ✗ Dependent

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$P(E_1 \cap E_2 \cap E_3) = \frac{1}{3}$ [{HHH}]

$P(E_2 \cap E_3) = \frac{1}{4} = P(E_2) \cdot P(E_3)$ ✓ → Independent

Independence :

Two events A and B are said to be independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

Mutual Independence :

For 3 events are mutual independent if

$$P(A \cap B \cap C) = P\{A\}.P\{B\}.P\{C\}$$

Let $E_1, E_2, \ldots, E_n$ ~~are~~ be n events. They are mutually independent if

$$P(E_1 \cap E_2 \ldots \cap E_n) = P(E_1) \cdot P(E_2) \cdot \ldots \cdot P(E_n)$$

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If E_1, E_2, E_3 are independent events iff — ①

$$\left.\begin{aligned} P(E_1 \cap E_2) &= P(E_1) \cdot P(E_2) \\ P(E_1 \cap E_3) &= P(E_1) \cdot P(E_3) \\ P(E_2 \cap E_3) &= P(E_2) \cdot P(E_3) \end{aligned}\right\} ②$$

$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2) \cdot P(E_3)$ — ③

① is true ⇒ ③ is true

but ③ is true ⇏ ① is true

② + ③ is true ⇒ ① is true

→ In previous eg. :

$P(E_1 \cap E_2 \cap E_3) = 1/8$

$= \frac{1}{2} * \frac{1}{2} * \frac{1}{2}$

$= P(E_1) \cdot P(E_2) \cdot P(E_3)$

But we can't say E_1, E_2, E_3 are mutually independent because

$P(E_1 \cap E_3) \neq P(E_1) \cdot P(E_3)$

$2^n - (n+1)$

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Def^n: n events $E_1, E_2, \ldots, E_n$ are mutually independent iff

all these conditions should hold true

$$P(E_i E_j) = P(E_i) \cdot P(E_j) \quad i \neq j \quad i,j = 1,2,\ldots,n$$

$$P(E_i E_j E_k) = P(E_i)\, P(E_j)\, P(E_k) \quad i \neq j \neq k \quad i,j,k = 1,2,\ldots,n$$

$\vdots$

$$P(E_1 E_2 \ldots E_n) = P(E_1) \cdot P(E_2) \ldots P(E_n)$$

Events	No. of trials
2	1 $({}^2C_2)$
3	4 $({}^3C_2 + {}^3C_3)$
$\vdots$	
n	${}^nC_2 + {}^nC_3 + {}^nC_4 + \ldots + {}^nC_n$
	$= 2^n - (n+1)$

$\Rightarrow$ we must check $2^n - n - 1$ relations

$\rightarrow$ If E_1 and E_2 are ~~mutually~~ independent, then:

i) E_1^c & E_2 are also independent

ii) E_1 & E_2^c " " "

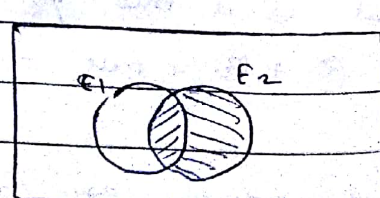
iii) E_1^c & E_2^c " " "

PROOF: i) $E_2 = (E_2 \cap E_1) \cup (E_2 \cap E_1^c)$

ii) $E_1 = (E_1 \cap E_2) \cup (E_1 \cap E_2^c)$

i) $E_2 = \underbrace{(E_2 \cap E_1) \cup (E_2 \cap E_1^c)}_{\text{mutually exclusive}}$

~~already independent~~



$\Rightarrow P(E_2) = P(E_2 \cap E_1) + P(E_2 \cap E_1^c)$

$= P(E_2) \cdot P(E_1) + P(E_2 \cap E_1^c)$

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$P(E_1) \cdot P(E_2)$

$$P(E_2 \cap E_1^c) = P(E_2) - P(E_1 \cap E_2)$$
$$= P(E_2)\,[1 - P(E_1)]$$
$$= P(E_2) \cdot P(E_1^c)$$

$\Rightarrow$ E_2 and E_1^c are independent.

Hence Proved

ii.) $E_1 = (E_1 \cap E_2) \cup (E_1 \cap E_2^c)$

same as above proof.

iii.) To prove : $P(E_1^c \cap E_2^c) = P(E_1^c) \cdot P(E_2^c)$

$$(E_1 \cup E_2)^c = E_1^c \cap E_2^c$$
$$\Rightarrow P(E_1 \cup E_2)^c = P(E_1^c \cap E_2^c)$$
$$\Rightarrow 1 - P(E_1 \cup E_2) = P(E_1^c \cap E_2^c)$$

independent

$$\text{or } P(E_1^c \cap E_2^c) = 1 - [P(E_1) + P(E_2) - P(E_1 \cap E_2)]$$
$$= 1 - [P(E_1) + P(E_2) - P(E_1) \cdot P(E_2)]$$
$$= 1 - P(E_1) - P(E_2)\,[1 - P(E_1)]$$
$$= [1 - P(E_2)]\,[1 - P(E_1)]$$
$$= P(E_2^c) \cdot P(E_1^c)$$

Hence Proved

Eg. (conditional prob) If the result of the 1st toss is head, then E_1: wins the game if 2 heads occur out of 3 trials. E_2 : If this doesn't happen. (need two tails)

$E_1 : \{\underline{H}HT, \underline{H}TH, \underline{H}HH\}$: $P(E_1) = 3/4$

$E_2 : \{HTT\}$ $\quad P(E_2) = 1/4$

$S = \{HTH, HHT, HHH, HTT\}$

Sample Space

Conditional Probability

OB

→ $S = \{ \quad \}_8$ (coin tossed 3 times)

$E_1 = \{HHH, HHT, HTH, THH\}$

$A = \{HHH, HHT, HTH, HTT\}$

E_1, E_1, A are all separ. events

when 1st is fixed

$A \cap E_1 = \{HHH, HTH, HHT\}$

$$\boxed{P(E_1|A) = \frac{P(E_1 \cap A)}{P(A)}} = \frac{3/8}{4/8} = \frac{3}{4}$$

$(P(A) \neq 0)$

→ In case of independent,

$P(E_1|A) = P(E_1)$

$\Rightarrow P(E_1) \cdot P(A) = P(E_1 \cap A)$

* A is going to influence $P(E_1) \Rightarrow$ dependent
A isn't " "
" $P(E_1)$ & $P(E_1|A) = P(E_1)$
⇓
independent

Proposition → Let A and B be events with $P(B) \neq 0$. Then A & B are independent iff (Have to prove from both sides)

$P(A|B) = P(A)$

Proof: Let A and B be two independent events. By defn of conditional Probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A) \cdot P(B)}{P(B)} \quad [\because \text{independent}]$$

$$= P(A)$$

Now, let us assume

$P(A|B) = P(A)$ (have to prove independent)

$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$\Rightarrow$ A & B are independent events Hence Proved.

pairwise & mutually exclusive $\Rightarrow$ all mutually exclus.

" " independent $\not\Rightarrow$ " " independent

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Eg. Throwing a dice

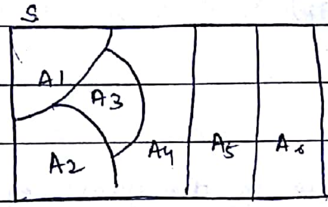
E_1 : getting even no. $P(E_1) = 1/2$

E_2 : getting 2. $P(E_2) = 1/6$

P(getting 2 given even no. has come) : $\frac{1}{3}$

$$P(E_2 | E_1) = 1/3$$

Partition of a set



$$A_i \cap A_j = \phi$$

$$A_1 \cup A_2 \cup A_3 \cdots \cup A_6 = S$$

Partition of a Sample Space

The events $A_1, A_2, \ldots, A_n$ form the partition of the sample space if the following conditions hold :

a) The events are pairwise disjoint

$$A_i \cap A_j = \phi \quad i \neq j$$

b) $A_1 \cup A_2 \cup \cdots \cup A_n = S$

Total Probability Theorem :

Let $A_1, A_2, \ldots, A_n$ form a partition of the sample space with $P(A_i) \neq 0$ for all i and B is any event.

Then,

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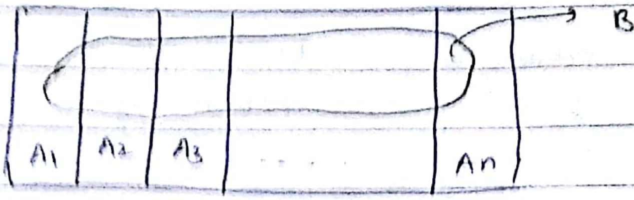
$$P(B) = \sum_{i=1}^{n} P(B | A_i) P(A_i)$$



→ mutually exclusive

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_n)$$
$$= \bigcup_{i=1}^{n} (B \cap A_i)$$

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$
$$= \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

Ex. Two players, A and B, participate in the game of throwing 2 dices. The first player who gets a sum of 7 is awarded the winner. (Assume A throws 1st)

Sol^n $\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$ 6

$$P(\text{getting sum of } 7) = \frac{1}{6} = p$$

$$P(\text{not getting sum of } 7) = 1 - \frac{1}{6} = \frac{5}{6} = q$$

~~P(A wins) = 1/6~~ A could win in 1st, 3rd, 5th ... chances

$$P(A_1) = \frac{1}{6} = p$$

$$P(A_2) = \frac{5}{6} * \frac{5}{6} * \frac{1}{6} = q^2 p$$

$$P(A_3) = q^4 p$$

$\vdots$

$$P(A_n) = q^{n-1} p$$

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$$P(A \text{ wins}) = p + q^2p + q^4p + \cdots + q^{n-1}p + \cdots$$

$$= p\left[1 + q^2 + \cdots\right]$$

$$= \frac{p}{1-q^2}$$

$$= \frac{1/6}{1-\frac{25}{36}} = \frac{6}{11}$$

$$P(B \text{ wins}) = \frac{5}{11}$$

$\Rightarrow$ $P(A \text{ wins}) > P(B \text{ wins})$

Ex An ice cream seller has to decide whether he has to order the stock or not. The chances of selling is 90% on sunny day, 60% on cloudy day, 20% on rainy day.

Wheather forecast :

chances of sunshine is 30%. $P(E_1) = 3/10 = 0.3$

chance of cloud is 45%. $P(E_2) = 0.45$

chance of rain is 25%. $P(E_3) = 0.25$

Mutually disjoint

(Apply Total Probability theorem)

Let A : event that you completely sell all the ice-creams

$P(A|E_1) = 0.9$

$P(A|E_2) = 0.6$

$P(A|E_3) = 0.2$

$$P(A) = \sum_{i=1}^{3} P(A|E_i)\, P(E_i)$$

$$= \left(\frac{3}{10}\right)\left(\frac{9}{10}\right) + \left(\frac{45}{100}\right)\left(\frac{6}{10}\right) + \left(\frac{25}{100}\right)\left(\frac{2}{10}\right) = \frac{270 + 270 + 50}{1000} = \frac{590}{1000}$$

$$= \boxed{0.59}$$

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Can you predict weather if sale is 100% ? → Posterior Probability
↓ (reverse of what we solved)
Using Baye's Theorem
↓
Rel^n b/w $P(A|B)$ & $P(B|A)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\cancel{P(A \cap B)} \; P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow \boxed{P(A|B) \cdot P(B) = P(B|A) \cdot P(A)}$$

$$\Rightarrow P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

$$\boxed{P(A|B) = \frac{P(A) \cdot P(B|A)}{\sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}} \rightarrow \text{Baye's Theorem}$$

<u>Baye's Theorem</u> :

let $B_1, B_2, \ldots, B_n$ be a set of mutually exclusive events of the sample ~~set~~ space S with $P(B_K) \neq 0$, $K = 1, 2, \ldots, n$ and A be any event of S with $P(A) \neq 0$

then → Priori Probababililty

Posterior prob

$$\boxed{P(B_k|A) = \frac{P(B_K)\,P(A|B_K)}{\sum_{i=1}^{n} P(B_K)\,P(A|B_K)}}$$

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If sale is 100%, prob. that it is sunny day

$$P(\quad) = \frac{0.3 * 0.9}{0.59}$$

P (cloudy day) =

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Ex. Suppose a calculator manufacturer purchases his IC's from suppliers S_1, S_2, S_3 with 40% from S_1, 30% from S_2 & 30% from S_3. Let 1% of S_1 supply is defective 5% of S_2 and 10% of S_3 is defective.

1.) what is probability that selected IC is defective

A : IC is defective

S_1, S_2, S_3 :- independent $(\cap = \phi)$ & $\cup = S$ (can form partitions)

$$P(A) = \sum_{i=1}^{3} P(A|S_i)\, P(S_i)$$

$$= 0.01 * 0.4 + 0.05 * 0.3 + 0.1 * 0.3$$

$$= 0.004 + 0.015 + 0.03$$

$$= 0.049$$

2) In stock, what is the percentage of defective item by a particular supplier
(already defected : given)

$$P(S_1|A) = \frac{P(A|S_1)\, P(S_1)}{P(A)}$$

$$= \frac{0.01 * 0.4}{0.049} = \frac{4}{49}$$

$P(S_2|A) = \frac{P(A|S_2)\, P(S_2)}{P(A)}$ $= 15/49$, $P(S_3|A) = \frac{P(A|S_3)\, P(S_3)}{P(A)} = 30/49$ (>60%)

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↳ Out of all defective items, more than 60% are supplied by S_3.

↳ This is an appⁿ of Baye's Theorem.

Random Variable

A random variable is a funcⁿ defined on a sample space

I/P → sample space   o/p → Real no.

Eg 1) Tossing 3 coins,

$f(S) \longrightarrow R$ ~~set~~ is defined by getting head (no.)

$\{0, 1, 2, 3\}$.

$f(TTT) \longrightarrow 0$

$f(TTH) \longrightarrow 1$ and so on.

Eg 2) $f(S) \longrightarrow (4, 7)$

set of students → height interval

Eg 3) $f(i, j) = i + j$

Throw 2 dices

Possible Outcomes : 2 to 12

Eg 1, 3 : can be discrete random variable → range is discrete $(2, 3, \ldots)$

Eg 2 : can be continuous "

Ex. Select a student from class at random and measure his / her height in centimeters.

Sample space : {students in class} = S

$(f(s))$ Output : Real no. (height)

Random variable : mapping b/w these 2.

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Ex. Throwing a six sided dice twice, then sum of the 2 no.

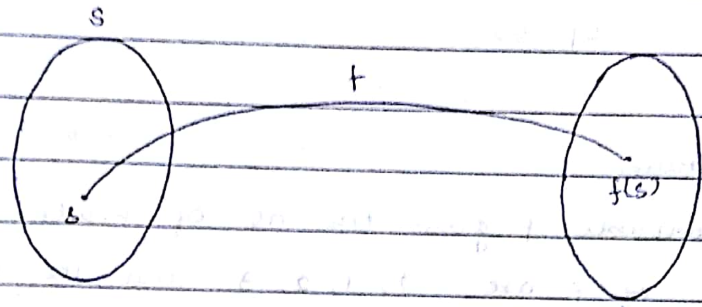
$$S = \{(i,j) ; 1 \le i,j \le 6\}$$

and the random variable f is given by:

$$f(s) = f(i,j) = i+j$$

Then, the target set (range set) is $\{2,3,4, \ldots, 11, 12\}$

↳



$s \in S$

↳ Random variable can be classified in 2 types:

1) Discrete Random Variable

2) Continuous Random variable

1) Discrete: A random variable is discrete if the values it takes are separated by gaps (or Image set is discrete)

2.) Continuous: If there is no gap b/w its possible values.

Random Variable:

A func^n f that associates a real no. with each element in the sample space S.

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→ Random variable can be defined as per your convenience

eg → getting 1st head Look in Book (toss n times)

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Probability Mass Funcⁿ (PMF) :

The PMF of a discrete random variable f is the funcⁿ formula or table which gives the value of probability $P(F=a)$ for each element a in the range set of f.

$F = a \; : \; A = \{x \in S : F(x) = a\}$

thus, we can write

$P(A) = P(f=a)$

Eg. Toss a fair coin 3 times

TTT TTH HTT THT THH HTH HHT HHH

The random variable f gives the no. of heads. The possible values of f are 0, 1, 2, 3. and its PMF is :

a	0	1	2	3
P(F=a)	1/8	3/8	3/8	1/8

* summation = 1

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eg. Three balls are ~~two~~ randomly selected without replacement from an urn containing 20 balls numbered 1 to 20. Probability that atleast 1 of the ball that are drawn has a no. greater than 18. (or as large as)

Let X denote the largest no. selected. Then X is a random variable taking on one of the values 3, 4, 5, ..., 20.

If each of the $^{20}C_3$ possible selection are equally likely to occur, then

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1 select करना हो

$$P(X=i) = \frac{{}^{i-1}C_2}{{}^{20}C_3} \quad [{}^1C_1 \cdot {}^{i-1}C_2]$$

18
19
20

$$P(X>18) = P(X=19) + P(X=20)$$

$$= \frac{{}^{18}C_2}{{}^{20}C_3} + \frac{{}^{19}C_2}{{}^{20}C_3}$$

Eg. Flipping a coin until have head or complete n flips

let X denote the no. of times a coin is flipped.

$X = 1, 2, 3, \ldots$

$P(X) = \frac{1}{2} \cdot (\frac{1}{2})^2 \ldots$

$$P(X=i) = \begin{cases} (1-p)^{i-1} p & i = 1,2,3,\ldots, n-1 \\ (1-p)^{i-1} & i = n \end{cases}$$

n^{th} trial : Either H can appear or T in the n^{th} flip

$P\{TT\ldots TH, TT\ldots T\}$ [mutually exclusive]

$P = (1-p)^{n-1} \cdot p \qquad (1-p)^n$

$$(1-p)^{n-1} [\cancel{p} + 1 - \cancel{p}] = (1-p)^{n-1}.$$

→ If add all probabilities, get sum 1 (check)

$$\sum_{i=1}^{n} P(X=i) = p + (1-p)p + (1-p)^2 p + \ldots + (1-p)^{n-2} p + (1-p)^{n-1}$$

$$= p\,[1 + (1-p) + (1-p)^2 \ldots + (1-p)^{n-2}] + (1-p)^{n-1}$$

$$= p \left[\frac{(1-p)^{n-1} - 1}{(1-p) - 1}\right] + (1-p)^{n-1}$$

$$= -1\cancel{p} \left[\frac{\cancel{(1-p)^{n-1}} - 1}{-\cancel{p}}\right] + \cancel{(1-p)^{n-1}}$$

$$= \boxed{1}$$

$P(X=a) \equiv P(a)$

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Cumulative Distribution Function (CDF)

For a random variable X, the func^n F defined by:

$$F(x) = P\{X \le x\} \qquad -\infty < x < \infty$$

The CDF can be expressed in terms of $P(X=a)$ or $P(a)$ by

$$F(a) = \sum_{\text{all } x \le a} P(x)$$

In previous eg,

$$F(5) = \sum_{i=1}^{5} P(x=i)$$

$$= P + (1-P)P \ \text{---} \ + (1-P)^4 P$$

→ If X is a discrete random variable (DRV) whose possible values are $x_1, x_2, x_3 \ldots$ where $x_1 < x_2 < x_3 < \ldots$ then the CDF F of X is a step func^n.

Ex. If X has probability mass func^n given by:

$P(1) = \frac{1}{4}$ $\quad P(2) = \frac{1}{2}$ $\quad P(3) = \frac{1}{8}$ $\quad P(4) = \frac{1}{8}$

$$F(a) = \begin{cases} 0 & a < 1 \\ 1/4 & 1 \le a < 2 \\ 3/4 \ \left(\frac{1}{4} + \frac{1}{2}\right) & 2 \le a < 3 \\ 7/8 \ \left(P(1)+P(2)+P(3)\right) & 3 \le a < 4 \\ 1 & 4 \le a \end{cases}$$

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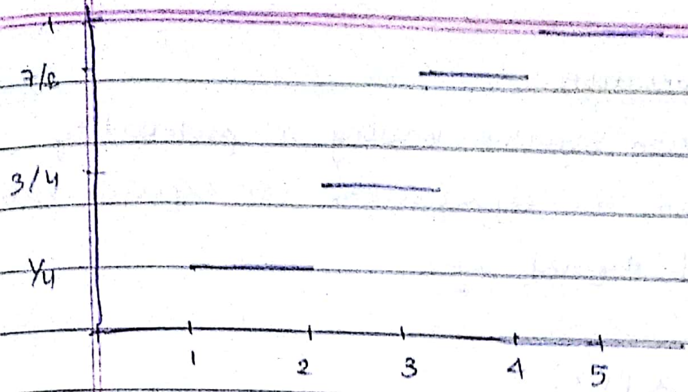

Eg. The PMF of a Random variable x is given by:

$$P(X=i) = \frac{c\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

1.) Find $P(X=0)$

2.) Find $P\{x>2\}$

1.) $\cancel{P(X=0) = c}$

2.) $\cancel{P\{x>2\} = 1 - P\{x \le 2\}}$

Using concept that $\sum_{i=1}^{\infty} P(x=1) = 1$

$$\Rightarrow c \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} = 1$$

$$\Rightarrow c e^{\lambda} = 1$$

$$\Rightarrow \boxed{c = e^{-\lambda}}$$ Now, we can redefine PMF

$$P(X=i) = \frac{e^{-\lambda}\lambda^i}{i!}$$

1.) $P(X=0) = e^{-\lambda}$

2.) $P\{x>2\} = 1 - P\{x \le 2\}$

$$= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$$

$$= 1 - \left\{ e^{-\lambda} + e^{-\lambda}\lambda + \frac{e^{-\lambda}\lambda^2}{2} \right\}$$

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Expected value or Expectation

If X is a discrete random variable having a probability mass func^n $P(X)$. Then, the expectations (or expected value) of X denoted by $E[X]$ defined by:

$$E[X] = \sum_{x : P(x) > 0} x P(x)$$

→ Let $a_1, a_2, \ldots, a_n$ be equally likely events. Then,

$$P(a_i) = \frac{1}{n}$$

$$E[X] = \sum a_i x = a_1 \frac{1}{n} + a_2 \frac{1}{n} + a_3 \frac{1}{n} \ldots$$

$$= \frac{a_1 + a_2 + a_3 \ldots + a_n}{n}$$

(sometimes, it is called mean also)
⇓
weighted sum

Ex. Let I be the indicator variable for the event A if

$$I = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$$

$$P(X = 1) = P(A)$$

$$P(X = 0) = P(A^c) = 1 - P(A)$$

$$E[I] = 1\,P(A) + 0 \cdot P(A^c)$$

$$= \underline{P(A)}$$

Property: If a and b are constant, then

$$E[aX+b] = aE[X]+b$$

Proof: $E[aX+b] = \sum_{x:P(X)>0} (ax+b)\, P(X)$

$$= a \sum x\, P(X) + b \underbrace{\sum P(X)}_{1}$$

$$= aE[X]+b$$

* $E[X^n] = \sum_{P(x)>0} x^n\, p(x)$

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If X is a D.R.V. that takes one of the values x_i, $i \geq 1$ with respective probabilities $P(x_i)$, then for any real valued func^n g.

$$E[g(x)] = \sum_i g(x_i)\, P(x_i)$$

$$E[X^2] = \sum_i x_i^2\, P(x_i)$$

$$E[aX+b] = aE[X]+b$$

$$E[X^n] = \sum_i x_i^n\, P(x_i)$$

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Q.1 Random Variable W, Y, Z as

$W = 0$ with ~~pr~~ prob 1

$$Y = \begin{cases} -1 & \text{with prob } 1/2 \\ +1 & \text{with prob } 1/2 \end{cases}$$

$$Z = \begin{cases} -100 & \text{with prob } 1/2 \\ 100 & \text{with prob } 1/2 \end{cases}$$

$E[W] = 0 * 1 = 0$

$E[Y] = -1 * \frac{1}{2} + 1 * \frac{1}{2} = 0$

$E[Z] = -100 * \frac{1}{2} + 100 * \frac{1}{2} = 0$

We don't get correct information

Variance (Var(X)) :-

If X is a random variable with mean μ, then the variance of X, denoted by Var(X) is defined by :

$$\text{Var}(X) = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] \quad \text{— ①}$$

$$= E[X^2] - 2\mu E[X] + \mu^2 E[1]$$

$$= E[X^2] - 2\mu E[X] + \mu^2$$

$$= E[X^2] - 2\mu^2 + \mu^2 \qquad [\because E[X] = \text{mean}]$$

$$= E[X^2] - \mu^2$$

$$\boxed{\text{Var}(X) = E(X^2) - (E(X))^2}$$

In above ques^n:

$Var(W) = 0^2 * 1 - (0*1)^2 = 0$

$Var(Y) = E(W^2) - (E(W))^2$

$= [1 * 1/2 + 1 * 1/2] - 0$

$= 1$

$Var(Z) = E(Z^2) - (E(Z))^2$

$= [(100)^2 * 1/2 + (100)^2 * 1/2] - 0$

$= (100)^2$

* We can see Expec^n of W, Y, Z are same but Variance of W, Y, Z are different, spread is also more.

Eg. Calculate $E[X]$ and $Var(X)$ where X represents the outcome when a fair dice is rolled.

Sol^n X is a random variable taking values of outcomes which are: 1, 2, 3, 4, 5, 6

X	1	2	3	4	5	6
P(X)	1/6	1/6	1/6	1/6	1/6	1/6

$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \dots + 6 \times \frac{1}{6}$

$= \frac{1}{6} \frac{(6 \times 7)}{2} = 3.5$

~~V(X) =~~ $E(X^2) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + \dots + 6^2 \times \frac{1}{6}$

$= \frac{1}{\cancel{6}} \left[\frac{\cancel{6}(7)(13)}{6} \right] = \frac{91}{6}$

$V(X) = E[X^2] - (E[X])^2$

$= \frac{91}{6}\times 2 - \frac{49}{4} \times 3$

$= \frac{182 - 147}{12} = \boxed{\frac{35}{12}}$

Prop → $\boxed{Var(aX+b) = a^2 Var(X)}$

Using eq^n (1)

$$Var(aX+b) = E[((aX+b) - E(aX+b))^2] \quad [\because \mu = E[aX+b]]$$

$$= E[(aX+b - aE[X] - bE[1])^2]$$

$$= E[(aX+b-a\mu-b)^2]$$

$$= E[a^2(X-\mu)^2]$$

$$= a^2 E[[X-\mu]^2]$$

$$= a^2 Var(X)$$

* adding some const won't affect variance

The Bernoulli Random Variable

A random variable X is said to be Bernoulli random variable if its probability mass function is given by:

$$P(0) = P[X=0] = 1-p$$

$$P(1) = P[X=1] = p$$

where p, $0 \le p \le 1$, is the probability of success

Binomial Random Variable

If X represents the no. of successes that occur in n trials then X is said to be ~~Bon~~ a binomial random variable with parameter (n, p)

The ~~prop~~ probability mass func^n (pmf) of Binomial random variable having parameter (n, p) is given by

$$P(i) = {}^nC_i \, p^i (1-p)^{n-i}$$

* Bernoulli is special case of Binomial (when n = 1)

→ n = 2

SS SF FS FF

P(1S) ⇒ includes $^2C_1$

→ n = 3 SFF , FSF , FFS

P(1S) ⇒ $^3C_1$

Eg: Tossed ~~...~~ a coins 3 times

X : getting head

$p = 1/2$

$q = 1/2$

$P(X=0) = {}^3C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^3 = \frac{1}{8} = P(X=3)$

$P(X=1) = {}^3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = \frac{3}{8} = P(X=2)$

Eg. Two players A and B play tennis. chances of winning by A & B are in the ratio 3 : 2. Find the probability that A wins atleast 2 games out of 4

$p = \frac{3}{5}$, $1-p = \frac{2}{5}$

P(A wins atleast 2) = P(2) + P(3) + P(4)

$= {}^4C_2 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2 + {}^4C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right) + {}^4C_4 \left(\frac{3}{5}\right)^4$

$= 6 * \frac{9}{25} * \frac{4}{25} + 4 * \frac{27}{125} * \frac{2}{5} + \frac{81}{625}$

$= \frac{216 + 216 + 81}{625}$

$= \frac{432 + 81}{625}$

$= 513/625$

→ not applicable when S & F is not case

Teacher's Signature..........

$e \times \frac{3}{4} \times \frac{3}{4}$

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Ques An irregular six faced dice is thrown 12 times. The chances that it 'll give six even no. is twice the chance that it'll get 5 even no.
If 500 people (set), each of exactly 12 trials are made, how many people are expected not to give any even no. ?

let P(getting an even no.) = p

$\Rightarrow \quad {}^{12}C_6\, p^6 (1-p)^6 = 2\ {}^{12}C_5\, p^5 (1-p)^7$

$\Rightarrow \quad \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1}\, p = \cancel{2 \times 12 \times 10 \times 9 \times 8 \times 7}$

$\frac{12!}{6!\cdot 6!}\, p = 2 \times \frac{12!}{5!\ 7!} (1-p)$

$\frac{p}{6} = \frac{2}{7}(1-p)$

$7p = 12 - 12p$

$19p = 12 \Rightarrow \boxed{p = \frac{12}{19}} \Rightarrow \boxed{1-p = \frac{7}{19}}$

P(not any even no.) $= P(X=0)$

$= {}^{12}C_0\, p^0 q^{12}$

$= \left(\frac{7}{19}\right)^{12}$

For 500 people $= 500 \times \left(\frac{7}{19}\right)^{12}$

Q. Min. no. of slips required = ?
chances slip has col^n = 75%.
Atleast 2 quest^{n}s to solve pass exam

$\cancel{{}^nC_2\, p^2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^{n-2} \geq \frac{2}{n}}$

Teacher's Signature

$^nC_2 \left(\frac{1}{4}\right)^{n-2} \geq$

Standard Deviation of a r.v.

Let x be a r.v. then, the square root of the Var (x) is called S.D.

$$S.D.(x) = \sqrt{Var(x)}$$

→ $B(n, P)$: If it is r.v. ⇒ it should satisfy : $\sum (P(i)) = 1$

$$\sum_{i=0}^{n} P\{x=i\} = \sum_{i=0}^{n} P(i)$$

$$= \sum_{i=0}^{n} {}^nC_i \, p^i q^{n-i} = (p+q)^n = 1 \quad \text{—— Eq}^n \, 2$$

Properties of Binomial r.v.

$$E[X] = \sum_{i=0}^{n} i\, P(i) = \sum_{i=1}^{n} i\, P(i) = \sum_{i=1}^{n} i\, {}^nC_i \, p^i q^{n-i} \quad \text{— (1)}$$

$$E[X^k] = \sum_{i=1}^{n} i^k \, {}^nC_i \, p^i q^{n-i} \qquad \left[\because \; i\, {}^nC_i = n\left({}^{n-1}C_{i-1}\right)\right]$$

$$= \sum_{i=1}^{n} p \, \underbrace{i \cdot i^{k-1} \, {}^nC_i}\, p^{i-1} q^{n-i}$$

$$= \sum_{i=1}^{n} p \, i^{k-1} n \, {}^{n-1}C_{i-1} \, p^{i-1} q^{n-i}$$

$$= np \sum_{i=1}^{n} i^{k-1} \, {}^{n-1}C_{i-1} \, p^{i-1} q^{n-i}$$

take $j = i-1$ ⇒ $i = j+1$

$$\boxed{E[X^k] = np \sum_{j=0}^{n-1} (j+1)^{k-1} \, {}^{n-1}C_j \, p^j q^{n-1-j}}$$

$$= np \; E[(Y+1)^{k-1}] \quad : Y : \text{Binomial r.v. with } (n-1) \text{ trials}$$

If we take $k=1$, we get expectation

$$E[X] = np \Big(\underbrace{\sum_{j=0}^{n-1} {}^{n-1}C_j \, p^j q^{n-1-j}}_{\Sigma B(n-1,\, p) = 1} \Big) \text{ — From eq}^n \text{ 2}$$

**

$$\rightarrow \boxed{E[X] = np}$$

$\rightarrow$ $E[X^2]$ (taking $k=2$) $= np \sum_{j=0}^{n-1} (j+1) \, {}^{n-1}C_j \, p^j q^{n-1-j}$

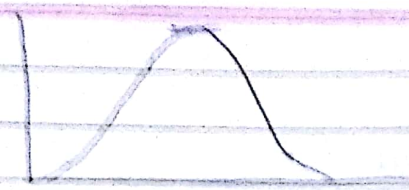
$$= np \Big[\underbrace{\sum_{j=0}^{n-1} j \cdot {}^{n-1}C_j \, p^j q^{n-1-j}}_{\text{formula for } E[X] \text{ with } n-1 \text{ (From eq}^n \text{ 1)}} + \sum_{j=0}^{n-1} \underbrace{{}^{n-1}C_j \, p^j q^{n-1-j}}_{B(n-1,\, p)} \Big]$$

$$= np \left[(n-1)p + 1 \right]$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$
$$= np[(n-1)p+1] - (np)^2$$
$$= np[np-p+1] - (np)^2$$
$$= np[1-p]$$

12

Proposition $\rightarrow$ If X is a Binomial r.v. with parameters (n, p) and $p \in (0,1)$ then as k goes from 0 to n, $P\{X=k\}$ first increases montonically, ~~reaching its large~~ then decreases monotonically, reaching its largest value when k is the largest integer less than or equal to $(n+1)p$.



$$\Rightarrow \frac{P\{X=k\}}{P\{X=k-1\}} = \begin{cases} >1 & \Rightarrow \text{Increasing} \\ <1 & \Rightarrow \text{Decreasing} \end{cases}$$

$$= \frac{{}^nC_k\, p^k q^{n-k}}{{}^nC_{k-1}\, p^{k-1} q^{n-k+1}} = \frac{n!}{(k)!(n-k)!} \times \frac{p\,(k-1)!\,(n-k+1)!}{n!\; q}$$

$$= \frac{(n-k+1)}{k} \cdot \frac{p}{q}$$

$$\Rightarrow P\{X=k\} = \frac{(n-k+1)}{k}\frac{p}{(1-p)} P\{X=k-1\}$$

if $P\{X=k\} \geq P\{X=k-1\}$: Increasing

$$\Rightarrow (n-k+1)p \geq k(1-p)$$

$$\Rightarrow np - kp + p \geq k - kp$$

$$\Rightarrow \boxed{(n+1)p \geq k}$$

if $P\{X=k\} \leq P\{X=k-1\}$: Decreasing

$$\Rightarrow \boxed{k \geq (n+1)p}$$

13/02/18 **Binomial Distribution Func^n**

Let X be a binomial r.v. with parameters (n, p). Then, binomial distribution func^n is defined as:

$$P\{X \leq i\} = \sum_{k=0}^{i} {}^nC_k\, p^k (1-p)^{n-k}$$

where $i = 0, 1, 2, \ldots, n$

We know :

$$\frac{P\{X = k+1\}}{P\{X = k\}} = \frac{p}{1-p} \frac{(n-k)}{(k+1)}$$

$$\Rightarrow P\{X = k+1\} = \frac{p}{1-p} \frac{(n-k)}{(k+1)} P\{X = k\}$$ $\Rightarrow$ Recursively get $P\{X = k+1\}$ — ①

Utilizing the above relationship b/w $P\{X = k+1\}$ and $P\{X = k\}$ to calculate binomial distribution func^n.

Ex. Let X be a binomial r.v. with parameters $n = 6$, $p = 0.4$. Then, if we start with $P\{X = 0\}$ recursively employing in above eq^n ①, we obtain:

$$P\{X = 0\} = {}^{n}C_0 (0.4)^0 (0.6)^6$$

$$P\{X = 1\} = \frac{0.4}{0.6} \left(\frac{6-0}{0+1}\right) (0.6)^6$$

* What happens when $n \to \infty$??

$$P\{X = i\} = {}^{n}C_i \, p^i (1-p)^{n-i}$$

$6C_2$

6×5

$$= \frac{n!}{(n-i)! \, i!} p^i (1-p)^{n-i}$$

$$= \frac{n(n-1)(n-2)\ldots(n-i+1)}{i!} p^i (1-p)^{n-i} \quad \text{— ②}$$

Now we talk about the limiting case of the b.r.v.

$n \to \infty$

$\lambda = np = $ finite (p : very small)

$\Rightarrow p = \frac{\lambda}{n}$

Using ② & taking $n \to \infty$

$$= \frac{\overbrace{n(n-1)\dots(n-i+1)}^{i \text{ terms}}}{i!} \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{1}{i!} \left\{ \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right) \dots \left(\frac{n-i+1}{n}\right) \right\} \lambda^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{1}{i!} \left\{ \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \dots \left(1-\frac{i-1}{n}\right) \right\} \lambda^i \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i} \to \text{finite}$$

$n \to \infty$:

$$= \lim_{n\to\infty} \left(\frac{1}{i!}\right) \{1 . 1 . \dots 1.\} \lambda^i \frac{e^{-\lambda}}{1}$$

$$= \frac{\lambda^i e^{-\lambda}}{i!} \qquad e^{n\left(1-\frac{\lambda}{n}-1\right)}$$

$$\boxed{P\{X=i\} = \frac{\lambda^i e^{-\lambda}}{i!}}$$: Poission Random Variable

$\longrightarrow$ pmf for ↑

* To check whether it's r.v. or not.

$\Sigma P(x_i) = 1$ for r.v.

$$\to \text{Here } \sum_{i=0}^{\infty} P(x_i) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}_{\to e^{\lambda}}$$

$$= e^{-\lambda} \cdot e^{\lambda} = \underline{1}$$

$\Rightarrow$ This is a random variable

* If it is some r.v., it must be some expectation & variance also.

Teacher's Signature..........

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Expected Value $E[X]$ for Poisson r.v.

$$E[X] = \sum_{i=0}^{\infty} i\,P(i)$$

$$= \sum_{i=1}^{\infty} i \frac{\lambda^i}{i!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!}$$

$$= \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}$$ {To get ~~equal~~ $i-1$ in both num. & denom.}

take $i-1 = j$

$$= \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{E[X] = \lambda} = np = \text{finite}$$

Variance :

$$E[X^2] = \sum_{i=0}^{\infty} i^2 P(i)$$

$$= \sum_{i=1}^{\infty} \frac{i^2 \lambda^i}{i!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{i=\phi}^{\infty} \frac{i\,\lambda^{i-1}}{(i-1)!}$$

Let $i-1 = j$

$$= \lambda e^{-\lambda} \sum_{j=0}^{\infty} (j+1) \frac{\lambda^j}{j!}$$

$$= \lambda e^{-\lambda} \left[\underbrace{\sum_{j=0}^{\infty} j \frac{\lambda^j}{j!}} + \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \right]$$

$$= \lambda e^{-\lambda} \left[E[X]\, e^{\lambda} + e^{\lambda} \right]$$

$$= \lambda e^{-\lambda} \left[\lambda e^{\lambda} + e^{\lambda} \right]$$

$$= \lambda(\lambda+1)$$

Teacher's Signature..........

* Poisson → usually gives best sol^n

$$Var(X) = E[X^2] - (E[X])^2$$
$$= \lambda(\lambda+1) - \lambda^2$$
$$\boxed{Var(X) = \lambda} = E[X]$$

* In Poisson r.v.,

$$\boxed{Var(X) = E[X]}$$

Que. A manufacturer of pins knows that 5% of this product is defective. If he sells pins in a boxes of 100 & guarantees that not more than 10 pins will be defective. What is the probability that a box fails to meet the claim?

Ans. probability of success = 0.05 (can use both Binomial / Poisson)

$$n = 100$$
$$p = 5\% = 0.05$$
$$\lambda = np = 5$$
$$P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!}$$

So, P (a box has more than 10 defective pins) =

$$P(X > 10) = 1 - P(X \leq 10)$$
$$= 1 - \left\{ \sum_{x=0}^{10} \frac{e^{-5}\, 5^x}{x!} \right.$$

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→ Binomial (n, p) is app. Poisson (λ)

$n \to \infty$, p is small, λ = finite

Q. If X & Y are independent Poisson variates s.t.

$P\{X=1\} = \underbrace{P\{X=2\}}_{①}$ and $P\{Y=2\} = \underbrace{P\{Y=3\}}_{②}$

Find variance for $X - 2Y$.

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Sol^n: Let X is $P(\lambda)$ and Y is $P(\mu)$, $\lambda, \mu > 0$

Then, we have

$$P\{X = x\} = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2 \ldots$$

$$P\{Y = y\} = \frac{e^{-\mu} \mu^y}{y!}, \quad y = 0, 1, 2, \ldots$$

By (1):

$$P\{X=1\} = P\{X=2\}$$

$$\Rightarrow \frac{e^{-\lambda} \lambda}{1!} = \frac{e^{-\lambda} \lambda^2}{2!}$$

$$\Rightarrow \lambda e^{-\lambda} (2-\lambda) = 0$$

$$\Rightarrow \boxed{\lambda = 2} \qquad [\because \lambda > 0]$$

By (2):

$$P\{Y=2\} = P\{Y=3\}$$

$$\Rightarrow \frac{e^{-\mu} \mu^2}{2!} = \frac{e^{-\mu} \mu^3}{3!}$$

$$\Rightarrow e^{-\mu} \mu^2 [3 - \mu] = 0$$

$$\Rightarrow \boxed{\mu = 3} \qquad [\because \mu > 0]$$

$$Var(X) = \lambda = 2 = E[X]$$

$$Var(Y) = \mu = 3 = E[Y]$$

$$P\{X = x\} = \frac{e^{-2} 2^x}{x!}, \quad x = 0, 1, 2 \ldots$$

$$P\{Y = y\} = \frac{e^{-3} 3^y}{y!}, \quad y = 0, 1, 2, \ldots$$

* $Var(aX+b) = a^2 \, Var(X)$

$$Var(X - 2Y) = \cancel{E[(X-2Y)^2] - (E[X-2Y])^2}$$

$$= Var(X) + (-2)^2 \, Var(Y)$$

$$= 2 + 4 * 3 = \boxed{14}$$

Teacher's Signature.........

Que Two cars are in a car firm which are hired on each day. The no. of demand on each day is distributed as a Poisson distribution with mean 1.5, then calculate the proportion of the days on which :

(a) Neither car is used → $X = 0$

(b) some demand is refused

X : no. of cars ~~asked~~ hired by people

(a) $P\{X = 0\} = e^{-\lambda}$

also, $E[X] = \lambda = 1.5$

$\Rightarrow P\{X = 0\} = e^{-1.5}$

(b) calculate when 3 demands, 4 demands, ... etc.

$= 1 - (P\{X=0\} + P\{X=1\} + P\{X=2\})$

$= 1 - \left\{ e^{-1.5} + \frac{e^{-1.5}(1.5)}{1!} + \frac{e^{-1.5}(1.5)^2}{2!} \right\}$

Geometric Random Variable

suppose that independent trials each having a probability $p \in (0,1)$ of being a success are performed until a success occurs.

If X equals the no. of trials, then

$$\boxed{P\{X = n\} = (1-p)^{n-1} p \quad , \quad n = 1, 2, \ldots}$$

→ If this is a pmf, then $\sum P = 1$

here: $\sum_{i=1}^{\infty} P\{X = i\}$ [atleast 1 trial ⇒ $i = 1$ se start hoga]

$= \sum_{i=1}^{\infty} (1-p)^{i-1} p$ [~~i goes to ∞ as~~ we don't know when will we get success]

$= p \sum_{i=1}^{\infty} (1-p)^{i-1} = p\left[\frac{1}{1-(1-p)}\right] = 1$

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For any r.v. X whose p.m.f is given by:

$P\{X=n\} = (1-p)^{n-1}p$

is said to be a geometric random variable

→ Expectation $E[X]$

$$E[X] = \sum_{i=1}^{\infty} i\,P(i)$$

$$= \sum_{i=1}^{\infty} i\,(1-p)^{i-1}p$$

$$= \sum_{i=1}^{\infty}[(i-1)+1]\,(1-p)^{i-1}p$$

$$= p\left[\sum_{i=1}^{\infty}(i-1)(1-p)^{i-1} + \cancel{\sum_{i=1}^{\infty}[1-p]^{i-1}}\right] \quad 1/p$$

$i = j+1$

$$= p\left[\sum_{j=0}^{\infty} j(1-p)^{j}p\right] + 1 = (1-p)\sum_{j=1}^{\infty}\underbrace{j\,(1-p)^{j-1}p} + 1$$

$$E[X] = (1-p)\,E[X] + 1$$

$$\boxed{E[X] = \frac{1}{p}}$$

→ $$E[X^2] = \sum_{i=1}^{\infty} i^2 P(i) = \sum_{i=1}^{\infty}((i-1)+1)^2(1-p)^{i-1}p$$

$$= \sum_{i=1}^{\infty}(i-1)^2q^{i-1}p + 2\underbrace{\sum(i-1)\,q^{i-1}p}_{E[X]} + \sum \cancel{q^{i-1}p}^{\,1}$$

$$= q\sum_{j=0}^{\infty} j^2q^{j-1}p + 2qE[X] + 1$$

$$E[X^2] = q\,E[X^2] + 2q\left(\frac{1}{p}\right) + 1$$

$$\Rightarrow \boxed{E[X^2] = \frac{q+1}{p^2}}$$

$$\Rightarrow \mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$= \frac{q+1}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2} = \boxed{\frac{1-p}{p^2}}$$

Teacher's Signature............

$S = 1 + 2(1-p) + 3(1-p)^2 + \cdots$

$S(1-p) = 1(1-p) + 2(1-p)^2 + \cdots$

$S[p] = 1 + (1-p) + (1-p)^2 + \cdots$

$pS = \frac{1}{p}$

* $E[X] = Var[X]$ if $\boxed{p = 1-p}$

Eg. Let X be the no. of tosses of a fair coin required to get first head. Then, calculate $E[X]$ & $Var(X)$

$E[X] = \frac{1}{p} = 2$

$Var(X) = \frac{q}{p^2} = \frac{1/2}{(1/2)^2} = 2$

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Ques. A rolls a dice and B tosses a coin. If A rolls 6, he wins. If A doesn't roll 6 & B tosses head, then A looses. Otherwise, the game repeats. How many rounds does the game lasts?

Game is going until one of them wins ⇒ can apply geometric r.v. (atleast 1 round)

Let X be a geometric r.v.

$P(\text{of 1st round}) = \underset{\uparrow \text{ wins}}{\frac{1}{6}} + \underset{\uparrow \text{ loose}}{\frac{5}{6} \times \frac{1}{2}} = \frac{7}{12}$

pmf :

$P\{X = r\} = (1-p)^r p = \left(\frac{5}{12}\right)^r \frac{7}{12}$

$E[X] = 1/p$ → Asked

$= \frac{12}{7}$ [How many rounds are actually required]

↳ Cumulative Distribution Func^n for Geometric r.v. :

$F(r) = P\{X <= r\} = \sum_{i=1}^{r} P\{X = i\}$

Defn: Let F be a distribution func^n of X denoted by F(b), which takes on the values of the random variable X that are less than or equal to b.

$F(b) = P\{X <= b\}$

Discrete → Pmf   Discrete : Σ

Continuous → Pdf   Continuous : $\int$

* We can define this for all r.v.

It has the following properties:

① F is non-decreasing func^n

ie, $F(a) \le F(b)$ if $a < b$

② $\lim_{b \to \infty} F(b) = 1$

③ $\lim_{b \to -\infty} F(b) = 0$

④ F is right continuous, ie, for any b and any decreasing sequence b_n, $n \geq 1$ that converges to b

$$\lim_{n \to \infty} F\{b_n\} = F(b)$$

Continuous Random Variable

We X is said to be a continuous r.v. if there exist a non-negative func^n f, defined for all real $x \in (-\infty, \infty)$ having the property that for any set B of real no:

$$P\{X \in B\} = \int_B f(x)dx$$

$f(x)$: Probability density func^n (Pdf) of r.v. X

→ Since X, we assume, has some value, f must satisfy

$$\int_{-\infty}^{\infty} f(x)\,dx = P\{X \in (-\infty, \infty)\} = 1$$

→ Non - negative r.v (look.

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→ If $B \in [a, b]$, then

$$P\{a \le X \le b\} = \int_a^b f(x)\,dx$$

→ If $a = b$, ie, $B = [a, a]$

$$P\{X = a\} = 0$$

↳ It shows that a continuous random variable will assume any fixed value with probability zero.

↳ Cumulative distribution func^n :

(talking wrt r.v. X)

The func^n $F = F_X$, given by :

$$F(a) = P\{X \le a\} = \int_{-\infty}^{a} f(x)\,dx$$

is called distribution func^n of X

→ On an open interval where f is continuous, the result is :

$$\boxed{F'(a) = f(a)}$$

Expectation :

If X is a continuous r.v. having pdf $f(x)$, then the expectation value of X is given by :

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

Th^m : If X is a cont. r.v. with pdf $f(x)$. Then, for any real valued func^n g,

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\,dx}$$

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→ For a non-negative r.v. y.

$$E[Y] = \int_0^\infty P\{Y > y\}\, dy$$

Thm : If a & b are const., then

$$E[ax+b] = aE[x]+b$$

Variance :

$$Var(X) = E[(X-\mu)^2]$$
$$= E[X^2] - (E[X])^2$$

Thm : $Var(ax+b) = a^2\, Var(x)$

Ex 1 : let Pdf is defined as :

$$f(x) = \begin{cases} cx & \text{if } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Since : $\int_{-\infty}^{\infty} f(x)\, dx = 1 \Rightarrow \int_0^4 cx\, dx = 1$

$$\Rightarrow c\left[\frac{4^2}{2}\right] = 1 \Rightarrow \boxed{c = \frac{1}{8}}$$

then,

Pdf becomes :

$$f(x) = \begin{cases} x/8 & 0 < x < 4 \\ 0 & \text{Otherwise} \end{cases}$$

$$\rightarrow E[x] = \int_0^4 x f(x)\, dx = \frac{1}{8}\int_0^4 x^2\, dx = \frac{4^3}{8 \times 3} = \boxed{\frac{8}{3}}$$

$$\rightarrow E[x^2] = \int_0^4 x^2\left(\frac{x}{8}\right) dx = \frac{1}{8} \times \frac{4 \times 4 \times 4 \times 4}{4}_{2}^{2} = \boxed{8}$$

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→ $var(X) = E[X^2] - (E[X])^2$

$= 8 - \left(\frac{8}{3}\right)^2 = 8 - \frac{64}{9} = \boxed{\frac{8}{9}}$??

→ $E[e^x] = \int_0^4 e^x f(x)\,dx$ (can use below eq. if don't know this identity)

<u>Ex.</u> The pdf for X is given by:

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

construct pdf for Y where $Y = e^x$

* Using cdf, we can get pdf

cdf for Y

$F_Y(x) = P\{Y \le x\} = P\{e^X \le x\}$

$= \int_{-\infty}^{\cancel{a}} P\{X \le \log x\}$

$= \int_{-\infty}^{\log x} f(x)\,dx = \int_0^{\log x} 1 \cdot dx = \log_e x$

* pdf : derivative of cdf

$F_Y = \log_e x$

⇒ $F'_y = \frac{1}{x} = f(y)$ $\quad x \in (0,1)$

⇒ $$f_y(x) = \begin{cases} 1/x & 1 < x < e \\ 0 & \text{Otherwise} \end{cases} \qquad \left[\begin{array}{l} x \to (0,1) \\ e^x \to (1,e) \end{array} \right.$$

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Q let X has the pdf

$$f_x(x) = \begin{cases} 3x^2 & \text{if } x \in [0,1] \\ 0 & \text{Otherwise} \end{cases}$$

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compute pdf f_Y of $Y = 1 - X^4$ $\quad E[Y], Var(Y) = ??$

* If $x \in [2, 100]$ instead of $x \in [0, 1]$

$\Rightarrow \int_{(1-y)^{1/4}}^{100}$

* In order to calculate pdf, first, we've to calculate cdf.

Sol^n: To calculate pdf for Y, we've to calculate cdf for Y 1st.

$$F_Y(y) = P\{Y \le y\} = P\{1 - x^4 \le y\}$$
$$= P\{1 - y \le x^4\}$$
$$= P\{(1-y)^{1/4} \le x\}$$

$$F_Y(y) = P\{(1-y)^{1/4} \le x\}$$

$$= \int_{(1-y)^{1/4}}^{\infty} f(x)\, dx$$

$$= \int_{(1-y)^{1/4}}^{1} 3x^2\, dx = 1 - (1-y)^{3/4}$$

$x \in [0,1] \Rightarrow Y \in [0,1]$

If $y \in [0,1]$, pdf of Y exists s.t.

$$f_Y(y) = \frac{d}{dy}[F_Y]$$

$$= +\frac{3}{4}(1-y)^{-1/4}$$

$$\Rightarrow f_y = \begin{cases} \dfrac{3}{4(1-y)^{1/4}} & y \in [0,1] \\ 0 & \text{Otherwise} \end{cases}$$

in def^n

1) The Uniform Random Variable

X is a uniform random variable on the interval (α, β) if the pdf of X is given by:

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{Otherwise} \end{cases} \quad \text{--- (1)}$$

↳ $(\alpha, \beta) = (0,1)$

$$F(-5) = 0$$

✱ $$F(a) = P\{X \le a\}$$
$$= \int_{\alpha}^{a} \frac{1}{\beta-\alpha}\, dx \quad \text{if } \alpha < a < \beta$$

↳ This is a pdf ⇒ $\int_{-\infty}^{\infty} f(x)\, dx = 1$ (in eq^n (1))

⇒ $\boxed{\text{cdf}}$:

$$F(a) = \begin{cases} 0 & a \le \alpha \\ \dfrac{a-\alpha}{\beta-\alpha} & \alpha < a < \beta \\ 1 & a \geq \beta \end{cases}$$

→ Expectation: & Variance

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha}\, dx = \frac{(\beta+\alpha)}{2}$$

$$E[X^2] = \int_{\alpha}^{\beta} \frac{x^2}{\beta-\alpha}\, dx = \frac{\beta^3-\alpha^3}{(\beta-\alpha)3} = \frac{\beta^2+\alpha^2+\beta\alpha}{3}$$

$$\text{Var}(X) = \frac{\beta^2+\alpha^2+\beta\alpha}{3} - \left(\frac{\beta+\alpha}{2}\right)^2$$
$$= (\beta-\alpha)^2/12$$

Ex. The local train arrives at a specified stop at 15-minute interval starting at 7 a.m. If a passenger arrives at the stop at a time that is uniformaly distributed b/w 7 and 7:30, find the probability that the person will have to:

(a) less than 5 min. wait

(b) more than 10 min. wait

$(\alpha, \beta) = (7, 7{:}30) \Rightarrow$ 30 minutes difference

We are just concerned about trains b/w (α, β)

* if arrives exactly at 5 min. late $\Rightarrow P(\quad) = 0$
(distinct ho jaayega : $\int = 0$)

Trains $7 \rightarrow 7{:}15 \rightarrow 7{:}30$

(a) 7:10 - 7:15 or 7:25 to 7:30

pdf $f(x) = \begin{cases} 1/30 & x \in [7{:}00, 7{:}30] \\ 0 & \text{otherwise} \end{cases}$

$P(x \in (7{:}10, 7{:}15)) + P(x \in (7{:}25, 7{:}30))$

$$= \int_{7:10}^{7:15} \frac{1}{30} dx + \int_{7:25}^{7:30} \frac{1}{30} dx = \boxed{\frac{1}{3}}$$

(b) $7 \rightarrow 7{:}05$ or $7{:}15 \rightarrow 7{:}20$

$P\{x \in (7, 7{:}05)\} + P\{x \in (7{:}15, 7{:}20)\}$

$$= \int_0^5 \frac{1}{30} dx + \int_{15}^{20} \frac{1}{30} dx = \boxed{\frac{1}{3}}$$

2.) Exponential Random Variable

A continuous r.v. whose pdf is given for som $\lambda > 0$ by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \left[\frac{\lambda e^{-\lambda x}}{-\lambda}\right]\Bigg|_{0}^{\infty} = 1$$

↳ C.d.f.:

$$F(a) = P\{X \leq a\}$$

$$= \int_{0}^{a} \lambda e^{-\lambda x}\,dx$$

$$= 1 - e^{-\lambda a} \qquad a \geq 0$$

↳ $$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\infty} \lambda x e^{-\lambda x}\,dx = \frac{1}{\lambda}$$

↳ $$Var(X) = \frac{1}{\lambda^2}$$

* $$E[X^n] = \int_{-\infty}^{\infty} x^n f(x)\,dx = \int_{0}^{\infty} x^n \lambda e^{-\lambda x}\,dx$$

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