Let S be a sample space, a map $X: S \to R^n$, $X(\omega) = \{ X_1(\omega), X_2(\omega), \ldots X_n(\omega) \}$ where ω is an outcome called n-dimensional Random-vector or Random variable if each X_i is a Random Variable on S.

Two-dimensional RV :-

Let X_1 and X_2 be two Random variable then a function $Z = (X_1, X_2)$ that assigns a point in R^2 is called two dimensional Random Variable.

$Z(\omega) = [X_1(\omega), X_2(\omega)]$

$[X_1 \leq a, X_2 \leq b]$ denotes the event of all elements of S. s.t. $X_1(\omega) \leq a$, $X_2(\omega) \leq b$

The probability of event $[X_1 \leq a, X_2 \leq b]$ is $= P[X_1 \leq a, X_2 \leq b]$.

Two dimensional RV X and Y is called discrete if Both X and Y are individually discrete Random Variable.

called continous if X and Y are continious R.V.

Two Dimensional RV, X and Y are called jointly distributed if they are defined on same probability space.

$(\Omega, F, P) \to$ These three should be same

Ω → Sample space
F → event
P → Prob.
(Ω, F, P) → Prob space

Joint Probability Mass Function :- (Joint PMF)

When PMF = RV (discrete)

→ Let $Z = (X, Y)$ be a discrete RV then we define a Mapping $f_{XY} : R^2 \to R$ $\forall x, y \in R$

$p(x, y) = f_{XY}(x, y) = P[X = x, Y = y]$

This f_{XY} is called Joint Prob. Mass function of $Z = (X, Y)$.

- For one D RV we have $f_X(x) = P[X = x]$

→ f_{XY} satisfies following Properties :-

(i) $f_{XY}(x, y) \geq 0$, $\forall (x, y) \in R^2$

(ii) set $[(x, y) \in R^2 : f_{XY}(x, y) \neq 0]$ contains at most countably Infinite point in R^2.

(iii) $\sum_x \sum_y f_{XY}(x, y) = 1$

Any Real valued functions defined on R^2 having above properties is called joint PMF of RV (X and Y).

- **Marginal Probability Mass Function :-**

When we calculate f_X and f_Y from the joint PMF f_{XY}. (Then are Marginal PMF)

Let (X, Y) be a discrete RV

$f_X(x_i) = P[X = x_i]$

$= P[X = x_i \cap Y = y_1] + P[X = x_i \cap Y = y_2]$

$+ \ldots\ P[X = x_i \cap Y = y_m]$

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ASHOKA
Date:
Page No.

· we are taking Intersection of x_i with all y, then at some pt. we will get x_i because all are mutually exclusive events they can be added up.

$$f_X(x_i) = \sum_{j=1}^{m} P[X = x_i \cap Y = y_j]$$

$$= \sum_{j=1}^{m} f_{XY}(x_i, y_j)$$

similarly

$$f_Y(y_j) = \sum_{i=1} f_{XY}(x_i, y_j)$$

$f_X(x_i)$, $f_Y(y_j)$ are called marginal pmf of $Z = (X, Y)$.

ex Suppose 3 balls are randomly selected from a bag containing 3R, 4W, 5 Blue balls. If we let X, Y denotes respectively the no. of Red balls and no. of white balls. Find the joint PMF and marginal PMF.

$X \to 0, 1, 2, 3$ (no. of Red Balls)
$Y \to 0, 1, 2, 3$ (no. of white Balls)

$$p(x,y) = f_{XY}(x,y) = p(0,0) = \frac{5C_3}{12C_3}$$

$$p(0,1) = \frac{4C_1 \cdot 5C_2}{12C_3}$$

$$p(0,2) = \frac{4C_2 \cdot 5C_1}{12C_3}$$

$$p(0,3) = \frac{4C_3}{12C_3}$$

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Date:
Page No.

$p(1,0) = \dfrac{3C_1\ 5C_2}{12C_3}$

$p(1,1) = \dfrac{3C_1 . 4C_1 . 5C_1}{12C_3}$

$p(1,2) = \dfrac{3C_1 . 4C_2}{12C_3}$

$p(2,0) = \dfrac{3C_2 . 5C_1}{12C_3}$

$p(2,1) = \dfrac{3C_2 . 4C_1}{12C_3}$

$p(3,0) = \dfrac{3C_3}{12C_3}$.

Joint PMF

Marginal PMF :-

$X=i$ \ $Y=j$	0	1	2	3	4	
0	10/220	40/220	30/220	4/220	0	$84/220 = f_X(0)$
1	$\frac{30}{220}$	60/220	18/220	0	0	$\frac{108}{220} = f_X(1)$
2	15/220	12/220	0	0	0	$\frac{27}{220} = f_X(2)$
3	1/220	0	0	0	0	$1/220 = f_X(3)$

On adding Rows we get Marginal PMF of 1st RV

$f_Y(0) = \frac{56}{220}$ $\frac{112}{220}$ $\frac{48}{220}$ $\frac{4}{220}$ $0 \rightarrow f_Y(4)$

$f_Y(1)$ $f_Y(2)$ $f_Y(3)$

On adding columns Marginal PMF of 2nd RV.

Date: 7/3/18

Page No.

Joint Distribution function :-

Let (X, Y) be a random variable. Joint Distribution function,

$$F_{XY}(x,y) = P\{X \le a, Y \le b\}$$
$$-\infty < a, b < \infty$$

$$F_X(a) = P\{X \le a\} = P\{X \le a, Y < \infty\}$$
$$= P\left[\lim_{b \to \infty} \{X \le a, Y \le b\}\right]$$
$$= \lim_{b \to \infty} P[X \le a, Y \le b]$$

$$F_X(a) = \lim_{b \to \infty} F_{XY}(a, b) = F_{XY}(a, \infty)$$

Similarly

$$F_Y(b) = \lim_{a \to \infty} F_{XY}(a, b) = F_{XY}(\infty, b)$$

F_X, F_Y are Marginal Distribution function of X and Y wrt (X, Y)

→ $P\{a_1 \le X \le a_2, b_1 \le Y \le b_2\} = F_{XY}(a_2\ b_2) + F(a_1 b_1) - F(a_1 b_2) - F(a_2 b_1)$

$$F(-\infty, y) = 0 = F(x, -\infty)$$
$$F(-\infty, +\infty) = 1$$

* $P\{X > a, Y > b\} = 1 - P(\{X > a, Y > b\}^c)$

$$= 1 - P[\{X > a\}^c \cup \{Y > b\}^c]$$
$$= 1 - P[\{X \le a\} \cup \{Y \le b\}]$$
$$= 1 - P[\{X \le a\}$$

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Date:
Page No.

$$= 1 - P\{X \le a\} - P\{X \le b\} + P\{[X \le a], [Y \le b]\}$$

$$P\{X > a, Y > b\} = \underline{1 - F_X(a)} - \underline{F_X(b) +} F_{X\underline{Y}}\underline{[a\ b]}$$

\# $(X, Y) \rightarrow$ discrete Random Variable

→ $F_X(x) = \sum_y P\{X \le x, Y = y\}$

$F_Y(y) = \sum_x P\{X = x, Y \le y\}$

Marginal distribution f^n

→ summation taking over x

\# if $(X, Y) \rightarrow$ continous Random variable

$$F_X(x) = \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \right) dx$$

$$F_Y(y) = \int_{-\infty}^{y} \left(\int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \right) dy$$

Distribution f^n in terms of joint distribution f^n.

→ **Joint Density function :-**

Let $F_{XY}(x, y)$ be a joint distribution function of continous RV (x, y)

→ $$\boxed{f_{XY}(x, y) = \frac{\partial^2 F(x, y)}{\partial x\, \partial y}}$$

→ **Marginal Probability function of X and Y :-**

$$f_X(x) = \begin{cases} \sum_y f_{XY}(x, y) & \text{discrete} \\ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy & \text{continous} \end{cases}$$

ASHOKA Date: Page No.

discrete → PMF
continuous → PDF

Joint PMF

$$f_Y(y) = \begin{cases} \sum_x f_{XY}(x,y) & \text{discrete} \\ \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx & \text{Continuous} \end{cases}$$

Joint Probability Density function

Result: Let (X,Y) be a random variable with pdf 'f' for any set $A \subseteq R^2$

$$P\{(X,Y) \in A\} = \iint_A f(x,y)\,dx\,dy$$

integeration over A

Question: Joint pdf of X and Y is

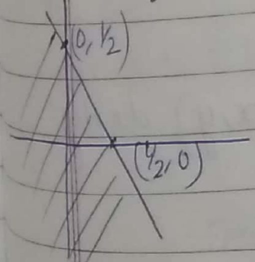
$$f(x,y) = \begin{cases} 2 & \text{if } x>0,\ y>0,\ 0<x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X+Y<\frac{1}{2}\}$

$A = \{(x,y) \in R^2;\ x+y<½\}$

$$P\{X+Y<\frac{1}{2}\} = \iint_A f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{1/2}\int_{-\infty}^{1/2-x} \frac{f(x,y)}{2}\,dx\,dy$$



$\boxed{\text{Ans} = 1/4}$

Question Joint pdf $f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0<x,y<\infty \\ 0 & \text{otherwise} \end{cases}$

(a) $P\{X>1,\ Y<1\}$

(b) $P\{X<Y\}$

(c) $P\{X<a\}$

ASHOKA

Date:

Page No.

solution :-

$$\int_{1}^{\infty}\int_{-\infty}^{1} \cancel{2}\, dy \, \cancel{dy}\, dx \qquad 2e^{-x}e^{-2y}$$

$$\left[2y\right]_{\cancel{-\infty}}^{\cancel{1}}.$$

$\rightarrow$

(b) $P(X<y) = \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{y} f(x,y)\,dx\,dy$

not Resistriction on y

$$= \int_{y=0}^{\infty}\int_{x=0}^{y} 2e^{-x}e^{-2y}\,dx\,dy$$

$$\rightarrow -2\int_{y=0}^{\infty}\left[e^{-x}\right]_{0}^{y} e^{-2y}\,dy$$

$$\rightarrow -2\int_{0}^{\infty} \cancel{e^{-3y}}\,(e^{-y}-1)\,e^{-2y}\,dy \Rightarrow \cancel{\frac{-2}{3}\left[e^{-3y}\right]_{0}^{\infty}}$$

$$\Rightarrow \cancel{-\frac{2}{3}\left[0-1\right] = \frac{2}{3}\text{ Ans}}$$

$\Rightarrow$ 1/3 Ans

(c) $P\{X<a\} = \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{a} f(x,y)\,dx\,dy$

$$\Rightarrow \int_{0}^{\infty}\int_{0}^{a} 2e^{-x}e^{-2y}\,dx\,dy$$

$$\Rightarrow -2\int_{0}^{\infty}\left[e^{-x}\right]_{0}^{a} e^{-2y}\,dy$$

$$\Rightarrow -2\int\left[e^{-a}-1\right]e^{-2y}\,dy$$

$$\Rightarrow -2\int\left(e^{-(2y+a)} - e^{-2y}\right)dy \Rightarrow \boxed{1-e^{-a}}$$

Date: 9 March 2018
Page No.

Independent Random Variable :-

Two RV X and Y are called Independent if for any $A \subseteq R$, $B \subseteq R$.

$$P[X \in A, Y \in R] = P[X \in A] \; P[Y \in B]$$

or

$$P[X \leq a, Y \leq b] = P[X \leq a] \; P[Y \leq b]$$

for all $a, b \in R$.

$$F_{XY}(a,b) = F_X(a) \; F_Y(b)$$

Now, $(X,Y) \rightarrow$ discrete RV, Independent if

- $p_{XY}(x,y) = p_X(x) \; p_Y(y)$

joint Probability Mass function

$(X,Y) \rightarrow$ continous RV,

$$f_{XY}(a,b) = f_X(a) \; f_Y(b)$$

for all points at which f_X, f_Y, f_{XY} are continous.

- $X=1$; heads $\quad X=0$; tails
  $Y \rightarrow 1, 2, 3, 4, 5, 6$

- $X, Y \rightarrow$ for any two $A, B \subseteq R$
  $E_A = [X \in A]$
  $E_B = [X \in B]$
  event corresponding to this Interval
  Then X & Y are called Independent if E_A and E_B are Independent for and 2 subsets A and B

ex

X \ Y	-1	0	1
-1	0	1/4	0
0	1/4	0	1/4
1	0	1/4	0

$\rightarrow$ Joint Probability Mass function

Are X and Y Independent? $P[X=0] = 1/2$

Sol. $P[X=-1, Y=-1] = 0$

$P[X=-1] \cdot P[Y=-1] = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

dependent

ex $\rightarrow$ Joint PDF of (X, Y)

$$f(x,y) = \begin{cases} 6(1-x), & 0<y<x \quad 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

Are X and Y Independent?

Sol

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^1 \cancel{6(1-x)}\; f(x,y)\,dy$$

$$= \int_{y=0}^{x} 6(1-x) + \int_{y=x}^{1} 0$$

$$f_X(x) = \begin{cases} 6x(1-x); & 0<x<1 \\ 0; & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx = \int_{x=0}^{1} f(x,y)\,dx$$

$$= \underbrace{\int_{x=0}^{y} 0}_{x<y} + \underbrace{\int_{x=y}^{1} 6(1-x)\,dx}_{x>y}$$

$$f_{XY}(a,b) = f_X(a) \cdot f_Y(b)$$

for all points at which

$$f_Y(y) = \left[-\frac{6(1-x)^2}{2} \right]_y^1 + 3(1-y)^2$$

$$f_Y(y) = \begin{cases} 3(y-1)^2 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$\left(\frac{1}{2}, \frac{1}{4}\right)$ $\quad f(x,y) = 6 \cdot \frac{1}{2} = 3$

$$f_X f_Y = 6 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 3\left(\frac{3}{4}\right)^2$$

Dependent

→ <u>Necessary and sufficient condition for the RV X and Y to be Independent</u> :-

- Two RV X and Y are independent iff joint pdf (pmf) can be expressed as →

$$f(x,y) = h(x)\, g(y) \qquad \begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \end{matrix}$$

joint pdf (pmf) can be factored into 2 terms, one depending only on x and other depending only on y.

Let x and y be independent variables →

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) \qquad \begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \end{matrix}$$

$$= h(x)\, g(y)$$

Conversely, let $f(x,y) = h(x)\, g(y)$

Date:
Page No.

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)\, g(y)\, dx\, dy$$

$$I = \int_{-\infty}^{\infty} h(x)\,dx \int_{-\infty}^{\infty} g(y)\,dy$$

$$\boxed{I = C_1 C_2}$$

where $C_1 = \int_{-\infty}^{\infty} h(x)\,dx$

$C_2 = \int_{-\infty}^{\infty} g(y)\,dy$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} h(x)\, g(y)\, dy$$

$$\boxed{f_X(x) = h(x)\, C_2} \quad — (2)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$f_X(x) = \int_{-\infty}^{\infty} h(x)\, g(y)\, dx \quad — (3)$$

$$\boxed{f_Y(y) = C_1\, g(y)}$$

$$f_X(x)\, f_Y(y) = C_1 C_2\, h(x)\, g(x)$$

$$= h(x)\, g(x)$$

$$= f_{XY}(x,y)$$

Hence They are independent.

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Date:
Page No.

1) joint pdf $f(x,y) = \begin{cases} 6e^{-2x}e^{-2y} & 0<x<\infty \\ & 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$

Independent

2) $f(x,y) = \begin{cases} 24\, xy & 0<x<1,\ 0<y<1 \\ & 0<x+y<1 \\ 0 & \text{otherwise} \end{cases}$

Dependent

→ $f_X(x) = \int_{y=-\infty}^{\infty}$

→ $X_1, X_2 \ldots X_n$ be a RV
for any $A_1, A_2 \ldots A_n \subseteq R$

$$P[X_1 \in A_1, X_2 \in A_2 \ldots X_n \in A_n] = \prod_{i=1}^{n} P[x_i \in A_i]$$

$$P[X_1 \leq a, X_2 \leq a_2 \ldots X_n \leq a_n] = \prod_{i=1}^{n} P[X_i \leq a_i]$$

→ Infinite collection of RV S are independent if every finite sub collection is independent.

3/18

Sum of Independent RV's :-

Suppose X and Y are two independent continous RV with joint pdf $f(x,y)$.

$$F_{X+Y}(a) = P\{X+Y \leq a\}$$

$$= \iint_{x+y\leq a} f(x,y)\,dx\,dy$$

$$= \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{a-y} f_X(x)\, f_Y(y)\,dx\,dy$$

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$$F_{X+Y}(a) = \int_{y=-\infty}^{\infty} \left[\int_{x=-\infty}^{a-y} f_X(x)\,dx \right] f_Y(y)\,dy$$

distribution function $\boxed{F_{X+Y}(a) = \int_{y=-\infty}^{\infty} F_X(a-y)\, f_Y(y)\,dy}$ This is also called convolution distribution of X and Y.

$$* \boxed{f_{X+Y}(a) = \frac{d}{da}\left[F_{X+Y}(a)\right]} *$$

$$= \int_{y=-\infty}^{\infty} \left[\frac{d}{da} F_X(a-y)\right] f_Y(y)\,dy$$

density function $f_{X+Y}(a) = \int_{y=-\infty}^{\infty} f_X(a-y)\, f_Y(y)\,dy$

ex if X, Y are two Independent continous RV distributed uniformly (0,1)

$$f_X(a) = f_Y(a) = \begin{cases} 1 & \text{if } 0<a<1 \\ 0 & ; \text{ others} \end{cases}$$

$$F_X(a) = \begin{cases} 0 & \text{if } a<0 \\ a & 0 \le a \le 1 \\ 1 & a>1 \end{cases}$$

$$f_{X+Y}(a) = \int_{y=-\infty}^{\infty} f_X(a-y)\, f_Y(y)\,dy$$

limits only 0 to 1 $\int_0^1$

$$= \int_{y=0}^{1} f_X(a-y)\,dy$$

Now if $0 < a < 1$

$$= \int_{y=0}^{a} 1\,dy + \int_{y=a}^{1} 0$$

$y < a$ $\qquad\qquad\qquad\qquad$ $a - y < 0$

$a - y > 0$

$$\boxed{f_{X+Y}(a) = a}$$

if $1 < a < 2$

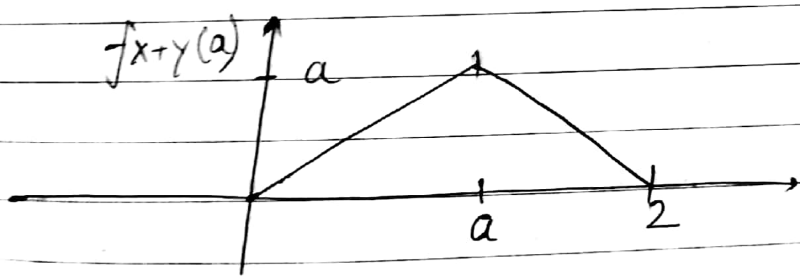
$$f_{X+Y}(a) = \int_{y=0}^{a-1} 0 + \int_{y=a-1}^{1} 1\,dy$$

$y < a-1$ $\qquad\qquad\qquad\qquad$ $a - y < 1$

$a - y > 1$

$\Rightarrow 1 - a + 1 = \underline{\underline{2 - a}}$ ans

$$f_{X+Y}(a) = \begin{cases} a & 0 < a < 1 \\ 2 - a & 1 \leq a < 2 \\ 0 & a \geq 2 \end{cases}$$



Triangular Distribution

$X_1, X_2 \ldots X_n$, independent RV distributed uniformly on $(0,1)$

$$F_n(x) = P[X_1 + X_2 \cdots + X_n \leq x]$$

if $x \leq 1$

$$F_n(x) = \frac{x^n}{n!}$$

Proving it by Mathematical Induction:

for $n=1$ $\qquad F_{X_1}(x) = \frac{x^1}{1!} = x$

This Result is true for $n=1$; suppose true for $n-1$

Suppose $f_{n-1}(x) = \frac{x^{n-1}}{(n-1)!}$

$$\sum_{i=1}^{n} X_i = \underbrace{\sum_{i=1}^{n-1} X_i}_{\text{Independent Random Variable}} + X_n$$

X_n → Random Variable

$$\boxed{F_n(x) = \int_{-\infty}^{\infty} f_{n-1}(x-y) \cdot f_{X_n}(y)\, dy}$$

$$\rightarrow F_n(x) = \int_0^n \frac{(x-y)^{n-1}}{(n-1)!} dy = \frac{1}{(n-1)!}\int_0^n (x-y)^{n-1} dy$$

Sum of discrete Independent Random Variable :-

(i) X, Y are two Independent poission RV with parameter $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively.

we are finding $\{X+Y=n\}$ = union of disjoint event $\{X=k ; Y=n-k\}$ $k=0,1,2\ldots$

$$f_{X+Y}(n) = P[X+Y=n] = \sum_{k=0}^{n} P[X=k, Y=n-k]$$

$$= \sum_{k=0}^{n} P[X=k] \cdot P[Y=n-k]$$

$$P[X+Y=n] = \sum_{k=0}^{n} \frac{e^{-\lambda_1}\lambda_1^k}{(k)!} \frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}$$

$$\Rightarrow \frac{e^{-\lambda_1}e^{-\lambda_2}}{(n!)} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k} (n)!}{(k)!(n-k)!}$$

$$\boxed{P[X+Y=n] \Rightarrow \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n}$$

→ Poission Distribution f^n with Par. $\lambda_1+\lambda_2$

ASHOKA
Date:
Page No.

→ distribution of $X+Y$ is also poission with parameter $\underline{\lambda_1 + \lambda_2}$.

$X \longrightarrow (\mu_1, \sigma_1^2)$

independent $Y \rightarrow (\mu_2, \sigma_2^2)$

$X+Y$ is also Normal Random variable with $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

13/11 Sum of Independent Random Variable

(i) Suppose X and Y are Independent Binomial RV with parameters (n, p) and (m, p)

$$P\{X+Y=k\} = \sum_{i=0}^{n} P\{X=i, Y=k-i\}$$

$$= \sum_{i=0}^{n} P\{X=i\} \, P\{Y=k-i\}$$

$$= \sum_{i=0}^{n} {}^nC_i (p)^i (q)^{n-i} \times {}^mC_{k-i} (p)^{k-i} (q)^{m-k+i}$$

$$\rightarrow \sum_{i=0}^{n} (p)^k (q)^{m+n-k} \underline{{}^nC_i \, {}^mC_{k-i}}$$

$$\rightarrow \sum_{i=0}^{n} {}^{m+n}C_k (p)^k (q)^{m+n-k}$$

$$\sum_{i=0}^{n} {}^nC_i \, {}^mC_{k-i} = {}^nC_0 \, {}^mC_k + {}^nC_1 \, {}^mC_{k-1} + \cdots {}^nC_n \, {}^mC_{k-n}$$

$$(x+y)^n = {}^nC_0 \, x^n + {}^nC_1 \, x^{n-1} y + \cdots {}^nC_n \, y^n$$

$$(x+y)^m = {}^mC_0 \, x^m + {}^mC_1 \, x^{m-1} y + \cdots {}^mC_m \, y^m$$

$${}^mC_{k-1} \, x^{m-k+1} y^{k-1} + {}^mC_k \, x^{m-k} y^k$$

- $^nC_0 \, ^mC_k =$ coeff of $x^n x^{m-k} y^k$

$= x^{m+m-k} y^k$

- $^nC_1 \, ^mC_{k-1} =$ coeff of $y x^{n-1} y^{k-1} x^{m-k+1}$

$= \underline{x^{m+n-k} y^k}$

from RHS we have:-

$\underline{^{m+n}C_k \, x^{m+n-k} y^k} = \underline{RHS}$

Theorem :- Let X and Y be two Independent Random Variables & g & h be two functions defined on R. Then g(X) and h(Y) are also Independent Random variables.

X & Y Independent →

1) |X| and |Y| are also Independent
2) X and sin Y are also independent
3) X^2 and Y^2 are also Independent.

(Proof)

Ques Let X and Y are two poission identically distributed ~~form~~ RV. find PMF of minimum of X and Y

Sol PMF of Min [X, Y].

Suppose $Z =$ Min [X, Y]

$X, Y = 0, 1, 2 \ldots$

$Z = 0, 1, 2 \ldots$

for PMF we are interested in calculating probability at any value k.

$\{Z = k\} = \{X = k, Y \geq k\} \cup \{X \geq k, Y = k\}$

ASHOKA

Date:

Page No.

$P\{Z=k\} = P\{(X=k, Y\geq k) \cup (X\geq k, Y=k)\}$

$= P\{(X=k, Y\geq k)\} + P\{(X\geq k, Y=k)\} -$

$P(\{X=k, Y\geq k\} \cap \{X\geq k, Y=k\})$

$\Rightarrow P\{(X=k, Y\geq k)\} + P\{(X\geq k, Y=k)\} - P\{X=k, Y=k\}$

$\Rightarrow \underline{P\{(X=k, Y\geq k)\}} + \underline{P\{X\geq k, Y=k\}} - P\{X=k\}\, P\{Y=k\}$

Because X and Y are Independent
If X and Y are Identically distributed
Then $P\{(X=k, Y\geq k)\}$ and $P\{X\geq k, Y=k\}$
will be same.

when Identically Distributed we have

$$\boxed{2P\{X=k, Y\geq k\} - P\{X=k\}^2}$$

Q X, Y are Independent RV uniformly distributed over (0,1). find the pdf of Y/X.

* 1st we find the distribution f^n of this. $Z = Y/X$.

Case 1) if $z<0$ $\quad F_Z(z) = 0$ (defined only on (0,1))

Case 2 if $z>0$ $\quad F_Z(z) = P\{Z\leq z\}$

$\Rightarrow P\left[\frac{Y}{X}\leq z\right]$

$\Rightarrow \iint_A f(x,y)\,dx\,dy$

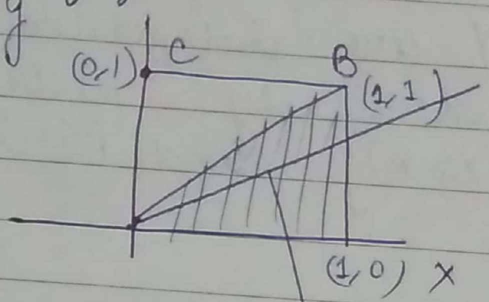
Over the Region A, we integrate

$$A = \left[(x,y) \in \cancel{(0,1)} R^2 \quad \frac{y}{x} < z \right]$$

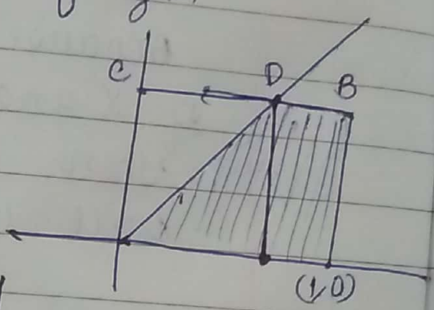
$$f(x,y) = \begin{cases} 1, & 0<x<1 \quad 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

$f(x,y) = \iint_S dx\, dy =$ $\quad S = \left[(x,y) \in (0,1) \times (0,1) \quad \frac{y}{x} \le z \right]$

if $z<1$



if $z>1$



$y \le xz$

for $z=1$

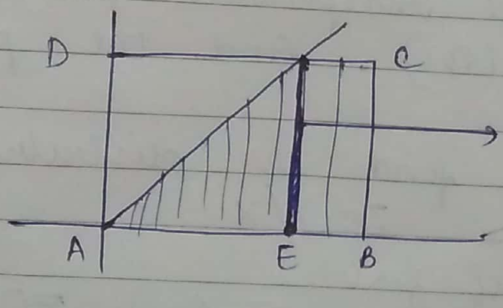
$\boxed{y=x}$

for $z>1$ this line leaves the region from corner

for $z>1$

← this side shift

for $z>1$



perpendicular line

$y = xz$

$1 = xz$

$x = \frac{1}{z}$

coordinate of $E\left(\frac{1}{z}, 0\right)$

$F_Z(z) = \frac{1}{2} \times z \times 1 = z/2 \quad$ for $0<z<1$

$F_Z(z) = \frac{1}{2} \times \frac{1}{z} \times 1 + \left(1 - \frac{1}{z}\right) \times 1$

$\Rightarrow \frac{1}{2z} - \frac{1}{z} + 1 = 1 - \frac{1}{2z} \quad z>1$

ASHOKA
Date:
Page No.

$$F_Z(z) = \begin{cases} z/2 & ; \quad 0<z<1 \\ 1-\frac{1}{2z} & ; \quad z>1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$f_X(z) = \begin{cases} 1/2 & 0<z<1 \\ 1/2z^2 & z>1 \\ 0 & \text{otherwise} \end{cases}$$

16/3/18

<u>conditional</u> <u>distribution</u> :-

E, F are events

$$P[E/F] = \frac{P[E\cap F]}{P[F]}, \text{ provided } P(F)>0$$

<u>Discrete</u>: suppose X and Y are two discrete Random variable with pmf $p(x,y)$ = conditional prob Mass function of X Given that $Y=y$.

$$p_{X/Y}\left(\frac{x}{y}\right) = P[X=x/Y=y]$$

$$= \frac{P[X=x, Y=y]}{P[Y=y]}$$

$$p_{X/Y}(x/y) = \frac{p(x,y)}{p(y)}$$

commulative conditional distrubution of X given that $Y=y$.

$$F_{X/Y}(x/y) = \sum_{a\le x} p_{X/Y}\left(\frac{a}{y}\right) =$$

$$P[X\le x \mid Y\le y]$$

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Date:
Page No.

$X = a_1, a_2, a_3 \ldots a_n$

$$F_{X/Y}\left(\frac{a_i}{y}\right) = P[X \leq a_i \mid Y = y]$$

$$F_{X/Y}\left(\frac{a_i}{y}\right) = P[X = a_1 \mid Y = y] + P[X = a_2 \mid Y = y] - \cdots P[X = a_{i-1} \mid Y = y] + P[X = a_i \mid Y = y]$$

When X, Y are Independent

$$p_{X/Y}(x/y) = \frac{P[X = x]\ \cancel{P[Y = y]}}{\cancel{P[Y = y]}} = P[X = x]$$

Ques pmf is given

$p(0,0) = 0.4 \qquad p(1,0) = 0.1$

$p(0,1) = 0.2 \qquad p(1,1) = 0.3$

$$p_{X/Y}\left(\frac{x}{y=1}\right) = \frac{p(x,1)}{p_Y(1)} \quad \text{for all values of } x$$

$$p_{X/Y}(0/1) = \frac{p(0,1)}{p_Y(1)} = 0.2 / 0.5$$

$p_Y(1) =$ column sum

$$p_{X/Y}(1/1) \Rightarrow \frac{p(1,1)}{p(1)} \rightarrow \frac{0.3}{0.5}$$

Ques If X and Y are Independent poission RV with parameters λ_1 and λ_2. Find the conditional probability of X given that $X + Y = n$.

Sol :-

$$P[X = k \mid Y = n - k] = \frac{P[X = k,\ X + Y = n]}{P[X + Y = n]}$$

Date:
Page No.

$$\cdot\ P\left[\frac{X=k,\quad Y=n-k}{P[X+Y=n]}\right]$$

$$* \quad \frac{P[X=k]\ P[Y=n-k]}{P[X+Y=n]} \qquad [X,Y \text{ are Independent}]$$

$$\rightarrow \frac{\dfrac{\cancel{e^{-\lambda_1}}\lambda_1^k}{(k!)} \cdot \dfrac{\cancel{e^{-\lambda_2}}(\lambda_2)^{n-k}}{(n-k)!}}{\dfrac{\cancel{e^{-(\lambda_1+\lambda_2)}}(\lambda_1+\lambda_2)^n}{(n)!}}$$

$$\rightarrow \frac{(n)!}{(k)!\,(n-k)!}\left[\frac{\lambda_1^k\ .\ \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}\right]$$

$$\rightarrow {}^nC_k\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

Ques X, Y Independent, Identically distributed Geometric Random variable. Find the conditional PMF

$$P\left[\frac{Y=y}{X+Y=n}\right]$$

$$\rightarrow \frac{P[Y=y,\quad X+Y=n]}{P[X+Y=n]} = \frac{P[Y=y,\quad X=n-y]}{P[X+Y=n]}$$

$$\rightarrow \frac{P[Y=y]\ P[X=n-y]}{P[X+Y=n]} \rightarrow \sum_{k=1}^{n-1} P[X=k,\ Y=n-k]$$

$$\rightarrow \frac{(1-p)^{y-1}p\quad(1-p)^{n-y-1}p}{\cancel{(1-p)}\sum_{k=1}^{n-1}P[X=k].\ P[Y=n-k]}$$

$$\rightarrow \frac{(1-p)^{n-2}\ p^2}{\sum_{k=1}^{n-1} p(1-p)^{k-1}\ p(1-p)^{n-k-1}}$$

ASHOKA
Date:
Page No.

$$\rightarrow \frac{p^2(1-p)^{n-2}}{\sum_{R=1}^{n} p^2(1-p)^{n-2}} = \frac{1}{n-1}$$

→ continous Random variable :-

Suppose $f(x,y)$ is a joint pdf of two RV X and Y conditional prob of X given that $Y=y$.

$$f_{X/Y}\left(\frac{x}{y}\right) = \frac{f(x,y)}{f(y)}$$

$$F_{X/Y}\left(\frac{a}{y}\right) = \int_{-\infty}^{a} f_{X/Y}\left(\frac{x}{y}\right) dx$$

$$P[X \leq a \,|\, Y = y]$$

Ques1 Joint density of X and Y :

$$f(x,y) = \begin{cases} \frac{12}{5} x(2-x-y) & ; 0<x<1, \quad 0<y<1 \\ 0 & ; \text{otherwise} \end{cases}$$

$$f_{X/Y}\left(\frac{x}{Y=y}\right) = \frac{f(x,y)}{f_Y(y)} \qquad 0<y<1$$

$$\rightarrow \cancel{\int_{-\infty}^{\infty}} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, \cancel{dy}\, dx$$

$$= \int_0^1 \frac{12}{5}(2x - x^2 - xy)\, \cancel{dy}\, dx$$

$$= \frac{12}{5}\left[\cancel{2xy} - \cancel{x^2y} - \cancel{\frac{xy^2}{2}}\right]_0^1$$

$$\Rightarrow \cancel{\frac{12}{5}\left[2-1-\frac{1}{2}\right]} \Rightarrow \cancel{\frac{12}{5} \times \frac{1}{2} = \frac{6}{5}}$$

$$\rightarrow \frac{12}{5}\left[x^2 - \frac{x^3}{3} - \frac{x^2}{2}y\right]_0^1$$

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ASHOKA
Date:
Page No.

$$\rightarrow \frac{12}{5}\left[1-\frac{1}{3}-\frac{1}{2}\right] = \frac{12}{5}\left[1-\frac{1}{6}\right]$$

$$\Rightarrow \underline{\underline{2}}$$

$$\frac{f(x,y)}{f_Y(y)} = \frac{6x(2-x-y)}{4-3y}$$

$$f(x,y) = \begin{cases} \dfrac{e^{-x/y}\, e^{-y}}{y} & 0<x<\infty \\ & 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

$$P[X>1 \mid Y=y]$$

or $$1 - P[X\le 1 \mid Y=y] = 1 - \int_{-\infty}^{1} f_{X/Y}\left(\frac{x}{y}\right) dx$$

0

conditional density function of X

$$1 - \int_0^1 \left(\frac{1}{y} e^{-x/y}\right) dx$$

$$1 - \left[\frac{1}{\cancel{y}} e^{-x/y} \times \cancel{y}\right]_0^1$$

$$1 + \left[e^{-1/y} - 1\right] \Rightarrow \underline{e^{-1/y}}$$

Q. Why we are taking y as constant ?
Because Random variable is taking values on a point

20|3|18
→

<u>covariance</u> :- The covariance of two Random variables X and Y is denoted by Cov (X, Y) and defined by

$$\text{Cov}(X,Y) = E\left[\underbrace{(X-E(X))}_{\text{deviation of X about its Mean}}(Y-E(Y))\right]$$

Date:
Page No.

covariance measures the Behaviour of two Random variable ; How they Behave together

→ If cov(X, Y) is positive ⇒ Both X and Y are simultaneously increasing or decreasing

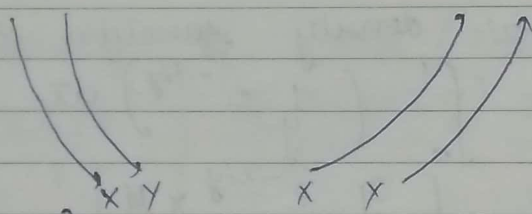
→ If cov(X, Y) < 0 ⇒ one RV increase and another one decrease

→ cov(X, Y) = 0 ⇒ X and Y are uncorelated

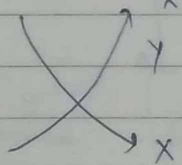
$$\text{cov}(X,Y) = E\Big[XY - Y\underbrace{E(X)}_{\text{constant}} - XE(Y) + \underbrace{E(X)E(Y)}_{\text{constant}}\Big]$$

$$\Rightarrow E(XY) - E(Y)E(X) - E(X)\cancel{E(Y)} + \cancel{E(X)}E(Y)$$

→ $\boxed{\text{cov}(X,Y) \Rightarrow E(XY) - E(X)E(Y)}$



→ cov(x, y) > 0



→ cov(x, y) < 0

ex

→ If there is no corelation b/w 2 ~~Independent~~ Random variable then it is not necessary that they will be Independent

Result

→ If two RV X and Y are Independent than that implies than X and Y are uncorrelated i.e. cov(X, Y) = 0

→ To Prove when X & Y → Independent cov(X, Y) = 0

$$\text{cov}(X,Y) = -E(X)E(Y) + E(XY)$$

$$\rightarrow -E(X)\cancel{E(Y)} + E(X)\cancel{E(Y)}$$

$$\rightarrow 0$$

?

ASHOKA
Date:
Page No.

To prove the last Result :

To prove this let $f_{XY}(x,y)$ is joint pdf of X & Y. Then

$$E(XY) = \iint xy\, f_{XY}(x,y)\, dx\, dy$$

• $\left\{ E[g(X,Y)] = \iint g(X,Y)\, f_{XY}(x,y)\, dx\, dy \right\}$

$$E(XY) = \iint xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$E(XY) = \Big[\underbrace{\int x\, f_X(x)\, dx}_{E(X)}\Big]\Big[\underbrace{\int y\, f_Y(y)\, dy}_{E(Y)}\Big]$$

(because of Independency of X & Y)

Range of X does not depend on Range of Y.

$E(X,Y) \rightarrow E(X)\,E(Y)$

$\rightarrow cov(X,Y) = 0$

→ X & Y are uncorrelated.

• converse of this is not true.

ex Let X (RV) −1 0 1

$P(X=x)$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

define $Y = \begin{cases} 0 & \text{if } X = 0 \\ 1 & \text{if } X = 0 \end{cases}$

uncorrelatedness does not define the Independence of two Random Variables.

$XY = 0 \Rightarrow E(XY) = 0$
constant

$E(X) = -\frac{1}{3} + 0 + \frac{1}{3} = 0$ $\Sigma\, x_i f(x_i)$

$cov(X,Y) = 0$ → X and Y are uncorrelated Variables

• definition of Y
depends on definition of X

Date:
Page No.

Ex → consider RV X & Y where the joint pmf is given by

Y \ X	-1	0	1
-1	0	1/4	0
0	1/4	0	1/4
1	0	1/4	0

→ X

↓ Y

$P(X=-1, Y=-1) = 0$ → RV X and Y are uncorrelated but dependent.

$P(X=-1) = \frac{+1}{4}$

$P(X=0) = \frac{1}{2}$ $\quad P(X=1) = 1/4$

$P(Y=-1) = \frac{1}{4}$

$P(Y=0) = \frac{1}{2}$ $\quad P(Y=1) = 1/4$

Both X and Y are having same distribution because their marginal probabilities are same

$E(X) = \Sigma x_i \, p(x_i)$

$= -1\times\frac{1}{4} + 0\times\frac{1}{2} + 1\times\frac{1}{4} = 0$

$E(Y) = 0$

$E(XY) =$

XY :	-1	0	+1
$P(XY = xy)$	$\underline{0+0}$	$\frac{1}{4}+0+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$	$0+0$

$= \frac{1}{2} + \frac{1}{2} = 1$

$cov(X, Y) \Rightarrow 0 \Rightarrow$ X & Y are uncorrelated

Ques

line

21/3/18

ASHOKA

Date:

Page No.

$P(X=1, Y=-1) = 0 \neq P(X=-1)\ P(Y=-1)$

$\Rightarrow$ X and Y are dependent

iven

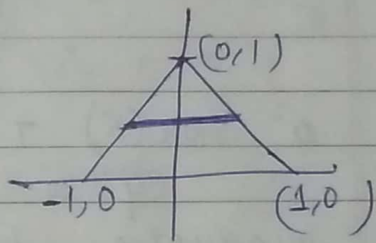
$\rightsquigarrow E(X,Y) = \iint xy\ f_{xy}(x,y)\,dx\,dy$

ques consider two RV X and Y jointly uniformly distributed over triangle with vertices $(-1,0)$ $(1,0)$ $(0,1)$ with joint pdf

$$f_{xy}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in T \\ 0 & \text{otherwise} \end{cases}$$

ore-

show RV X and Y are uncorrelated but dependent.



line eqn $\rightarrow$ $y = mx + c$

$m = 1$

$1 = c$

$\boxed{y = x + 1}$

me

$$E(X) = \int_0^1 \int_{y-1}^{1-y} x\,dx\,dy = 0$$

$$E(XY) = \int_0^1 \int_{y-1}^{1-y} xy\,dx\,dy = 0$$

$\operatorname{cov}(X,Y) = 0$

$\rightarrow$ X & Y are dependent

2/3/18

<u>Correlation coefficient</u>: for two X and Y correlation coefficient is denoted by $\rho(X,Y)$ and defined as $\rightarrow$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y}$$

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ASHOKA

Date:

Page No.

Property of correlation function :-

1* $P(X,Y) \leq 1$

consider

$$E\left\{\left[a\left(X-E(X)\right)+\left(Y-E(Y)\right)\right]^2\right\} \geq 0$$

→ $$E\left[a^2\left(X-E(X)\right)^2+\left(Y-E(Y)\right)^2+2a\left(X-E(X)\right)\left(Y-E(Y)\right)\right] \geq 0$$

→ $$a^2 E\left[X-E(X)\right]^2+E\left[Y-E(Y)\right]^2+2a\,E\left[\left(X-E(X)\right)\left(Y-E(Y)\right)\right] \geq 0$$

→ $$a^2\,Var(X)+Var(Y)+2a\,cov(X,Y) \geq 0$$

eq^n is quadratic in a

quadratic is always greater than zero.

so $D<0$

$$b^2-4ac<0$$

$$\left(2\,cov(X,Y)\right)^2-4\,Var(X)\,Var(Y)<0$$

$$\left(cov(X,Y)\right)^2-Var(X)\,Var(Y)<0$$

$$\frac{\left(cov(X,Y)\right)^2}{Var(X)\,Var(Y)}<1$$

Taking square Root

$$\boxed{\left|\frac{cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}\right|<1}$$ Bounded by 1

↓

This value is in between -1 to 1.

LHS

#

ASHOKA Date: Page No.

Properties →

* $\text{cov}(X,Y) = \text{cov}(Y,X)$
* $\text{cov}(X,X) = \text{Var}(X)$
* $\text{cov}(aX,Y) = a\,\text{cov}(X,Y)$
* $\text{cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \text{cov}(X_i, Y_j)$

Prove this

LHS $\quad \text{cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right)$

$\rightarrow E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} E(X_i)\right)\left(\sum_{j=1}^{n} Y_i - \sum_{j=1}^{n} E(Y_i)\right)\right]$

$\rightarrow E\left[\sum_{i=1}^{n}(X_i - E(X_i))\right]\left(\sum_{j=1}^{n}(Y_j - E(Y_j))\right]$

$\rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[(X_i - E(X_i)\right]\left(Y_j - E(Y_j)\right)$

$\rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} \text{cov}(X_i, Y_j)$ RHS

Let (S, F, P) be a prob model and $A \in F$, $B \in F$

Let $X = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } \bar{A} \text{ occurs} \end{cases}$

$Y = \begin{cases} 1 & \text{if } B \text{ occurs} \\ 0 & \text{if } \bar{B} \text{ occurs} \end{cases}$

$\text{cov}(X,Y) > 0$ Then what about A & B?

$\text{cov}(X,Y)$, $E(XY)$, $E(X)$, $E(Y)$

- $\boxed{E(X) = P(A)}$ $\qquad \boxed{E(Y) = P(B)}$
- $E(XY) = \underline{P(AB)}$

$$XY = \begin{cases} 1 & \text{if A and B both occur} \\ 0 & \text{if A or B doesn't occur} \end{cases}$$

$$\boxed{E(XY) = P(AB)}$$

- $\rightarrow$ $cov(X, Y) > 0$
- $\Rightarrow$ $P(AB) > P(A)\,P(B)$
- $\Rightarrow$ $\dfrac{P(AB)}{P(B)} > P(A)$
- $\Rightarrow$ $P(A/B) > P(A)$

$\rightarrow$ If $cov(X,Y)$ is positive then conditional prob of A and B is greater than Prob of A.

If $cov(X, Y) < 0$

$P(A/B) < P(A)$

If $cov(X, Y) = 0$

$\rightarrow$ $\boxed{\text{unconditional Prob} = \text{Conditional Prob}}$

23/3/18

<u>conditional Distribution</u> :-

(continous Random V)

$$P(A/B) = \frac{P(AB)}{P(B)} \quad \text{provided} \quad P(B) > 0$$

In the similar manner, conditional distribution $F_X(x/M)$ of a random variable X, assuming event M has occurred, is defined as

$$\rightarrow \boxed{F_{X/M}(x/M) = \frac{P(X \le x, M)}{P(M)}}$$

$F_X(x) = P(X \le x)$

when X is continous

ASHOKA
Date:
Page No.

→ conditional distribution : Discrete Random V :-

If $M = \{ Y = y \}$

If X and Y are two Random Var. and
$M = \{ Y = y \}$ then

$$\boxed{F_{X/Y}(x/y) = \frac{P(X \leq x, Y = y)}{P(Y = y)}}$$

If X and Y are discrete Random V. associated to some experiment and $M = \{ Y = y \}$ Then conditional PMF of X given than $\{ Y = y \}$ is

$$\boxed{p_{X/Y}(x/y) = P\left(\frac{X = x, Y = y}{P(Y = y)}\right)}$$

$$\boxed{p_{X/Y}(x/y) = \frac{p(x,y)}{p(y)}}$$ → joint PMF → Marginal PMF

23/3/18

$$P(E/M) = \frac{P(EM)}{P(M)} \qquad P(M) > 0$$

If X is a random variable then conditional ~~probability~~ distribution of X given that M has occured can be thought as the conditional Probability as →

$$F_{X/M}(x/M) = P\left(\frac{X \leq x, M}{P(M)}\right)$$

provided $P(M) > 0$

Conditional Distribution ; discrete case :
X and Y two discrete RV with joint PMF $p_{XY}(x,y)$ Then conditional pmf is given by

$$\boxed{p_{X/Y}(x/y) = P(X = x / Y = y)}$$

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$$p_{X/Y}(x/y) = \frac{P(X=x,\ Y=y)}{P(Y=y)} = \frac{p_{XY}(x,y)}{p_Y(y)}$$

$$\boxed{p_{X/Y}(x/y) = \frac{p_{XY}(x,y)}{p_Y(y)}} \quad p_Y(y) > 0$$

$$\sum p_X(x) = 1$$

$$\sum_x p_{X/Y}(x/y) = \sum_x \frac{p_{XY}(x,y)}{p_Y(y)}$$

$$= \frac{1}{p_Y(y)} \sum_x p_{XY}(x,y)$$

$$\boxed{\sum_x p_{X/Y}(x/y) = \frac{1}{p_Y(y)} \cdot p_Y(y) = 1}$$

* If X and Y are independent :-

$$\boxed{p_{X/Y}(x/y) = p_X(x)}$$

<u>Ques</u> Let the joint pmf of X and Y is given by

X \ Y	-1	0	1
-1	0	1/4	0
0	1/4	0	1/4
1	0	1/4	0

compute pmf of X given that Y = 0, Also compute conditional distribution f^n of X $F_{X/Y}(x/y)$ given $y=0$?

$p_{X/Y}(x/y)$ $y=0$?

Ex ample

Sol

ASHOKA
Date:
Page No.

$p(Y=0) = 1/2$

$$p_{X/Y}(x/0) = \frac{p(x,0)}{p_Y(0)} = \frac{P(X=x,\ Y=0)}{P(Y=0)}$$

$$p_{X/Y}(x/0) = \begin{cases} \frac{1/4}{1/2} = 1/2 & x=-1 \\ 0 & x=0 \\ \frac{1/4}{1/2} = \frac{1}{2} & x=1 \end{cases}$$

$y=0$ always

→ $F_{X/Y}(x/y) = P(X \le x / Y=y)$

$$\boxed{F_{X/Y}(x/y) = \sum_{a \le x} P_{X/Y}(a/y)}$$

→ In last example:

$$F_{X/Y}(x/0) = \begin{cases} 0 & x<-1 \\ 1/2 & -1 \le x < 0 \\ \frac{1}{2}. & 0 \le x < 1 \\ 1 & x > 1 \end{cases}$$

example Let X and Y be two Independent Poission Random v. with parameters λ_1 & λ_2 respectively. Then find out the conditional distribution of X given that $X+Y=n$.

Sol

$$P(X=k/X+Y=n) = \frac{P(X=k,\ X+Y=n)}{P(X+Y=n)}$$

$$\frac{P(X=k,\ Y=n-k)}{P(X+Y=n)} = \frac{P(X=k)\ P(Y=n-k)}{P(X+Y=n)}$$

Since X and Y are Independent

Date:
Page No.

$$\frac{\cancel{e^{-\lambda_1}}\,\lambda_1^{k}}{k!}\;\frac{\cancel{e^{-\lambda_2}}\,\lambda_2^{(n-k)}}{(n-k)!}\left[\frac{n!}{\cancel{e^{-(\lambda_1+\lambda_2)}}\,(\lambda_1+\lambda_2)^{n}}\right] \rightarrow$$

$$\rightarrow \frac{n!}{k!\,(n-k)!}\,\frac{\lambda_1^{k}\,\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^{n}} = {}^{n}C_{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$\rightarrow \quad p_X(x) = \sum_y p_{XY}(x,y)$$

$$= \sum_y p_{XY}(x/y)\, p_Y(y)$$

Conditional distribution : Continuous case : 26/3/18

Let X and Y be two jointly continuous RV's with joint pmf $f_{XY}(x,y)$ then conditional pdf of X given that $\{X=y\}$ is defined as →

$$\boxed{f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)}} \quad \text{provided } f_Y(y)>0$$

Ques

example

$$\int_{-\infty}^{\infty} f_{X/Y}(x/y)\,dx = 1$$

$$\int_{-\infty}^{\infty} \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{f_Y(y)}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \frac{f_X(y)}{f_Y(y)} = 1$$

Hence proved

→ If X and Y are jointly continuous then for any $x \in A$ (A is any set on the Real line)

$$\boxed{P(x\in A \mid Y=y) = \int_{x\in A} f_{X/Y}(x/y)\,dx}$$

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→ we are just defining this f^n in a new real dimension by limiting the value of Y.

$A = (-\infty, a)$

$$F_{X/Y}(a/y) = P(X \le a / Y = y)$$

$$= \int_{-\infty}^{a} f_{X/Y}(x/y)\,dx$$

- It helps us to find out prob of those events when one Random Variable is already assigned with some value.

26/3/18

Ques Let X & Y be two R.V's having the joint pdf

$$f_{XY}(x,y) = \begin{cases} 2 & \text{when } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$P(X \le \frac{2}{3} \mid Y = \frac{3}{4}) = ?$

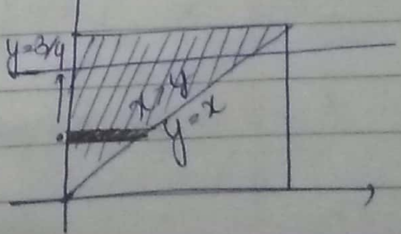
$$P(x \in A \mid Y = y) = \int_{x \in A} \underbrace{f_{X/Y}(x/y)}_{\text{conditional density } f^n}\,dx$$

we need to find out $f_{X/Y}(x/\frac{3}{4})$

→ $$f_{X/Y}(x/3/4) = \frac{f_{XY}(x, 3/4)}{f_Y(3/4)}$$

$$f_Y\left(\frac{3}{4}\right) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_{0}^{3/4} 2\,dx$$



→ x Ranges from 0 to $\frac{3}{4}$

$\boxed{\text{Ans} = \frac{3}{2}}$

from given function

conditional density f^n

$$f_{X/Y}(x/\tfrac{3}{4}) = \begin{cases} \cancel{2}/\tfrac{3}{2} & 0 < x < \cancel{2/3}\ 3/4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 4/3 & 0 < x < 3/4 \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ $P(X \le 2/3 / Y = \tfrac{3}{4}) = \int_0^{2/3} \frac{4}{3} dx$

$\Rightarrow \frac{4}{3} \times \frac{2}{3} \Rightarrow \frac{8}{9}$

$\rightarrow$ when $P(X \le 1 \mid Y = \tfrac{3}{4}) = \int_0^{3/4} f_{X/Y}(x/y)\,dx + \int_{3/4}^{1} f_{X/Y}\left(\frac{x}{y}\right) dx$

we need to add because upto $y = \frac{3}{4}$ $f_{X/Y}(x/\tfrac{3}{4})$ is different & after that it is different

$\rightarrow \int_0^{3/4} \frac{4}{3} dx + \int_{3/4}^{1} 0\,dx$

$\rightarrow \frac{4}{3} \times \frac{3}{4} = 1$

Ex Let X & Y be Independent continuous RV with pdf f_X and f_Y Respectively. Let $Z = X + Y$ Determine conditional ~~Prob~~ density of Z given $X = x$

$F_{Z/X}(z/x) = P(Z \le z \mid X = x) = ?$

$F(z/x)$

$$P(Z \le z \mid X = x) = \int_{-\infty}^{z} f_{Z/X}(t/x)\,dt \qquad \text{—①}$$

$$P(Z \le z \mid X = x) = P(X + Y \le z \mid X = x)$$

$$= P(Y \le z - x \mid X = x)$$

As X and Y are Independent RV.

$$= P(Y \le z - x)$$

Date:

Page No.

$$= \int_{-\infty}^{z-x} f_Y(y)\, dy$$

$$= \int_{-\infty}^{z} f_Y(t-x)\, dt \quad \text{--- (2)} \qquad \text{say } y = t-x$$

comparing (1) and (2)

$$F_{Z/X}(z/x) = \int_{-\infty}^{z} f_{Z/X}(t/x)\, dt$$

$$= \int_{-\infty}^{z} f_Y(t-x)\, dt$$

$$\boxed{f_{Z/X}(z/x) = f_Y(z-x)}$$

ex Suppose X and Y are Independent Identically distributed, geometric RV's with parameter p find the conditional ~~Prob~~ pmf of Y given $X+Y=n$. where $n \geq 2$.

- X and Y can takes values from natural Numbers

$R_X = 1, 2, 3 \ldots$ Because they are geometric Random V.

$X+Y = 2, 3, 4 \ldots$

$$P(Y=y / X+Y=n) = ?$$

→ Y take. $n, n+1, n+2 \ldots$

$$P(Y=y / X+Y=n) = 0$$

for $y = 1, 2, 3 \ldots n-1$

$$P(Y=y / X+Y=n) = \frac{P(Y=y, X+Y=n)}{P(X+Y=n)}$$

$$\rightarrow \frac{P(Y=y, X=n-y)}{P(X+Y=n)} = \frac{P(Y=y)\, P(X=n-y)}{P(X+Y=n)} \quad \text{--- (1)}$$

$$P(X+Y=n) = \sum_{k=1}^{n-1} P(X=k,\ Y=n-k)$$

$$= \sum_{k=1}^{n-1} P(X=k)\ P(Y=n-k)$$

$$= \sum_{k=1}^{n-1} p\cdot(1-p)^{k-1}\quad p(1-p)^{n-k-1}$$

$$\Rightarrow \sum_{k=1}^{n-1} (1-p)^{n-2}\,(p)^2$$

Thus from eq^n ①

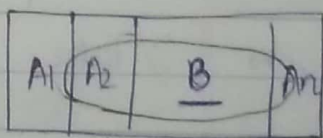
$$P\left(Y=y/_{X+Y=n}\right) = \frac{p^2(1-p)^{n-2}}{\sum_{k=1}^{n-1}(p)^2(1-p)^{n-2}}$$

$$= \frac{1}{n-1}$$

$$P\left(Y=y \mid X+Y=n\right) = \begin{cases} 0 ; & y=n, n+1, n+2 \ldots \\ \frac{1}{n-1} ; & y=1,2,3 \ldots n-1 \end{cases}$$

27/3/18 If set of events $\{A_1, A_2, A_3 \ldots A_n\}$ with $P(A_i)>0$ partitioned the sample space S and if B is an arbitrary event then

$$\boxed{P(B) = \sum_{i=1}^{n} P(B/A_i)\ P(A_i)}$$



→ Total Probability law (Discrete case) :-

Theorem: Let X and Y be two discrete random variables Then for any event B

$$\boxed{P(B) = \sum_{y \in R_y} P(B/Y=y)\ p_Y(y)}$$

Proof : If Y is a discrete RV with pmf $p_Y(y) = P(Y=y)$, then the set $\{[Y=y]\}_{y \in R_Y}$ partitioned the sample space.

Then by total Probability Theorem

$$P(B) = \sum_{y \in R_Y} P(B \mid Y=y)\ P(Y=y)$$

$$\boxed{P(B) = \sum_{y \in R_Y} P(B \mid Y=y)\ p_Y(y)}$$

This Result is just simple extension of the ~~p~~ total Probability Theorem.

<u>Total Probability Law</u> : <u>Continous case</u> :-

Let X be a continous RV. Then for any event B we have :-

$$\boxed{P(B) = \int_{-\infty}^{\infty} P(B \mid X=x)\ f_X(x)\,dx}$$

<u>Example</u> : Let X and Y be Two Independent RV on $(0,1)$. Then find $P(X^3 + Y > 1)$?

<u>Solution</u>

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$f_X(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{XY}(xy) = \begin{cases} 1 & 0 < x < 1 \quad 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

It is multiplication of $f_X(x)$ & $f_X(y)$ because X&Y are Independent ~~R~~ Random Vas.

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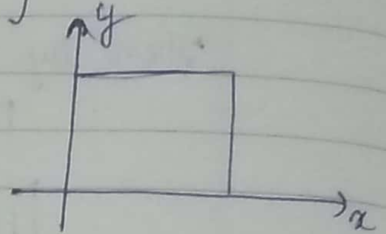
Date:
Page No.

Method 1 :-

$$P(X^3+Y>1) = \iint_A f_{XY}(x,y)\, dx\, dy$$

$$A = \{ (x,y) \in \mathbb{R}^2 \mid x^3+y>1 \}$$

To find out set A in the given region we need to draw a curve $f(x) = 1-x^3$



$\lim_{x\to\infty} f(x) = -\infty$ & $\lim_{x\to-\infty} f(x) = +\infty$

It implies Range of this f^n lies on whole Real line. Range $(-\infty, \infty)$

$f'(x) = -3x^2 < 0 \quad \forall x \in (-\infty, \infty)$

$f(x)$ is always decreasing

$f'(x) = 0 \Rightarrow x=0$ critical point

It can be point of local Maximal or local Minima. $x=0$ is neither a point of local Maxima or Minima as in the nbd of $x=0$ $f(x)$ is always decreasing.

→ $f''(x) = -6x < 0 \quad \forall x \in (0, \infty)$

so on the positive axis function is concave down type and on negative axis function is concave up i.e convex

at $x=0$ $f''(x)=0 \Rightarrow x$ is a point of Inflection (change its concavity)

That Region in ~~the~~ square for which $x^3+y>1$ should get satisfied.

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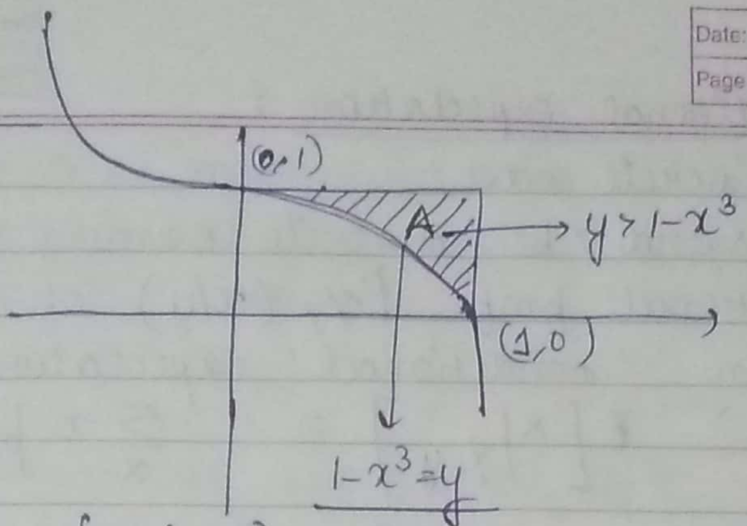
Date:
Page No.

concave up



$\rightarrow \int_{x=0}^{1}\int_{y=1-x^3}^{1} f_{XY}(x,y)\, dx\, dy$ $\qquad y \in (1-x^3, 1)$

$x \in (0,1)$

$\rightarrow \int_0^1 \left[y \right]_{1-x^3}^{1} dx$

$\Rightarrow \int_0^1 \left[\cancel{1} - \cancel{1} + x^3 \right] \Rightarrow \left[\frac{x^4}{4} \right]_0^1 = 1/4$ Ans

Method 2 :-

$P(X^3 + Y > 1) = \int_{-\infty}^{\infty} P\left(X^3 + Y > 1 / (X = x) \right) f_X(x) dx$

$= \int_0^1 P\left(X^3 + Y > 1 / X = x \right) dx$

$= \int_0^1 P\left(Y > 1 - X^3 / X = x \right) dx$

$= \int_0^1 P\left(Y > 1 - x^3 \right) dx$

$= \int_0^1 \left[\int_{1-x^3}^{1} \underline{f_Y(y)} \, dy \right]$

$\Rightarrow \int_0^1 \cancel{1} - \cancel{1} + x^3 \, dy$

$\rightarrow \left[\frac{x^4}{4} \right]_0^1 \Rightarrow \underline{1/4 \text{ Ans}}$

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Date:
Page No.

Conditional Expectation :

Discrete case -

Let X and Y be discrete Random Variable with conditional pmf $p_{X/Y}(x/y)$ of X given Y=y

Then conditional expectation

$$E\left[X|_{Y=y}\right] = \sum_x x\, p_{X/Y}(x/y)$$

$$X,\ p(x). \qquad E(X) = \sum_x x\, p(x)$$

Continous case : Let X and Y be continous RV with conditional pdf $f_{X/Y}(x/y)$ of X given Y=y.

$$E[X|Y=y] = \sum_x x\, p_{X/Y}(x/y) \rightarrow \text{discrete case}$$

$$\boxed{E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X/Y}(x/y)\, dx} \quad \text{continous case.}$$

Ex Let X and Y be Independent RV with geometric distribution of parameter p ; $p \in (0,1)$ Calculate

$$E[Y/X+Y=n] \text{ where } n \geq 2$$

$$\sum_y y\, p_{X/Y}(x/y)$$ conditional pmf we need to find out

$$p_{X/Y+X}\left(\frac{y}{n}\right) = \begin{cases} 0 ; & y \geq n \\ \frac{1}{n-1} ; & y = 1, 2, \ldots n-1 \end{cases}$$

$$\sum_y p_{Y/X+Y}(y/n) = \sum_{y=1}^{n-1} y \cdot \frac{1}{n-1} = \frac{1}{n-1} \sum_{y=1}^{n-1} (n-1)$$

$$= \frac{1}{(n-1)} \frac{(n-1)(n)}{2}$$

$$= \frac{n}{2}$$

Theorem

Proof

→

Ex

ASHOKA
Date:
Page No.

Theorem: Let X, Y be two discrete RV's with joint pmf $p_{XY}(x,y)$ Marginal pmf $p_X(x)$, $p_Y(y)$ respectively then

$$\rightarrow E(y) = \sum_x E\left(Y/_{X=x}\right) p_X(x)$$

unconditional expectation of Y.

Proof: Take R.H.S.

$$\sum_x E\left[Y/_{X=x}\right] p_X(x) = \sum_x \sum_y y\, p_{Y/X}(y/x)\, p_X(x)$$

$$p_{Y/X}(y/x) = \frac{p_{XY}(x,y)}{p_X(x)}$$

$$\cdot \sum_x \sum_y y\, p_{XY}(x,y) = \sum_y y \sum_x p_{XY}(xy)$$

$$= \sum_y y\, p_Y(y)$$

$$= \underline{E(y)}$$

→ for X & Y continous RV; then with joint pdf $f_{XY}(xy)$ Marginal pdf $f_X(x)$ $f_Y(y)$ respectively then

$$E(Y) = \int_{-\infty}^{\infty} E\left[Y/_{X=x}\right] f_X(x)\,dx$$

Total expectation theorem.

Ex: Let X and Y be continous RV with joint pdf

$$f_{XY}(x,y) = \begin{cases} 6(y-x) & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E\left[Y/_{X=x}\right]$ and then $E[Y]$?

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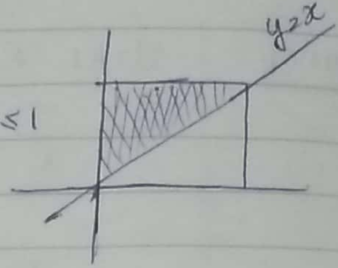
Sol $E[Y|X=x] = \int_{-\infty}^{\infty} y \, f_{Y/X}(y/x)\, dy$

$f_{Y/X}(x/y) = \dfrac{f_{XY}(xy)}{f_X(x)}$

Marginal density f^n of $X \rightarrow$

$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

$= \begin{cases} \int_x^1 6(y-x)\, dy & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$



$= \begin{cases} 3(x-1)^2 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$

$f_{Y/X}(y/x) = \dfrac{f_{XY}(xy)}{f_X(x)}$

$= \begin{cases} \dfrac{2(y-x)}{(x-1)^2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow$ $E[Y|X=x] = \int_x^1 \dfrac{2y(y-x)}{(x-1)^2}\, dy \quad 0 \le x \le 1$

$= \dfrac{2}{(x-1)^2} \int_x^1 y(y-x)\, dy = \boxed{\dfrac{x^2+x+2}{3(x-1)} \quad 0 \le x \le 1}$

$\rightarrow$ expectation of Y

$E(y) = \int E[Y|X=x]\, f_X(x)\, dx$

$= \int_{-\infty}^{\infty} \dfrac{x^2+x+2}{\cancel{3}(\cancel{x-1})}\, \cancel{3}(x-1)^2\, dx$

$$\int_0^1 (x^2+x+2)(x-1)\,dx = \underline{\frac{3}{4}}$$

→ **Complex Valued Random Variables :-**

$X : S \to R$

$Z : X + iY$ where X and Y Both are Real

$Z : S \to C$ (complex plane)

$E(Z) = E(X+iY) = \underline{E(X) + iE(Y)}$

$E(aZ_1 + bZ_2) = aE(Z_1) + b(E(Z_2))$

→ **Characteristic Function** : Let X be a RV then characteristic f^n of X defined by

→ $$\boxed{\phi_X(t) = E[e^{itx}] \quad ; t \in R}$$

characteristic f^n exist for all Random Var.

$\phi_X(t) = E[\cos(tX) + i\sin(tX)]$

→ $\cos(tX)$ & $\sin(tX)$ both are f^n of Random variable and bounded by 1 so their Expectation will also be finite charac. f^n exist for all R.V..

→ $\phi_X(t)$ exists for all RV. X and $t \in R$

$$e^{itX} = \underbrace{\cos(tX)}_{g_1(X)} + i\,\underbrace{\sin(tX)}_{g_2(X)}$$

$|g_1(X)| \leq 1$ & $|g_2(X)| \leq 1$

$\underline{E[e^{itX}]} = E[\cos tX + i\sin tX]$

ensure the existence of RV ch f^n of for all R.V.

It is finite ; exists for all RV. $\underline{t \in R}$

skewness, contosis?
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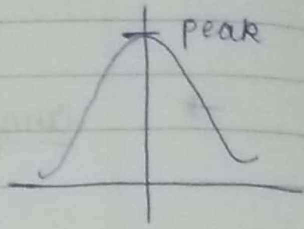
ASHOKA
Date:
Page No.

ex X ~ Bernoulli RV with parameter p;
$\phi_X(t) = ?$



→ Random variable is skewed
characteristic function
$(p), (1-p)$

$$E[g(X)] = \sum_x g(x)\, p_X(x)$$

$$E[e^{itX}] = \sum_x e^{itx}\, p_X(0)$$

$$= e^{it0}\, p(X=0) + e^{it1}\, p(X=1)$$

$$= (1-p) + e^{it}(p)$$

$$\boxed{\phi_X(t) = (1-p) + e^{it}(p)}$$

ex $\phi_X(t)$ for $X \sim N(0,1)$ standard Normal Random v.
$\phi_X(t) = e^{-t^2/2}$ Prove it

2/4/17 If for a RV, X we have distribution function and density function then we have its characteristic function also

Properties of characteristic function :-

① Ex Let X be a RV and a and b are real countable then :

$$\boxed{\phi_{a+bX}(t) = e^{iat}\,\phi_X(bt)}$$

Proof : consider LHS :

$$\phi_{a+bX}(t) = E\left[e^{it(a+bX)}\right]$$

$$= E\left[e^{iat} \cdot e^{ibtX}\right] = e^{iat} \cdot E\left[e^{ibtX}\right]$$

$$= e^{iat}\,\phi_X(bt)$$

ex

②

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ASHOKA
Date:
Page No.

ex
$X \sim N(\mu, \sigma^2)$

$Y = \frac{X-\mu}{\sigma} \sim N(0,1)$ Mean = 0
Variance = 1

$\phi_Y(t) = e^{-t^2/2}$

what is $\phi_X(t)$?

$X = \mu + \sigma(Y)$

$\phi_X(t) = \phi_{\mu+\sigma Y}(t) = e^{i\mu t}\phi_Y(\sigma t)$

$= e^{i\mu t} e^{-\sigma^2 t^2/2}$

② Let X and Y be two Independent Random Var.
then $\boxed{\phi_{X+Y}(t) = \phi_X(t)\,\phi_Y(t)}$

Proof LHS $\phi_{X+Y}(t) = E\left[e^{it(X+Y)}\right]$

$\rightarrow E\left[e^{iXt} \cdot e^{iYt}\right]$

$\rightarrow E\left[e^{itX}\right] \cdot E\left[e^{itY}\right]$
Because X and Y are Independent RV

$\rightarrow \boxed{\phi_X(t)\,\phi_Y(t)}$

corollarly :- Let X_1 ~~and~~, X_2, X_3 ... X_n be n Independent Random variables

$\phi_{X_1+X_2+X_3 \cdots X_n}(t) = \phi_{X_1}(t)\,\phi_{X_2}(t) \cdots \phi_{X_n}(t)$

ex → find out the ch. function of a Random variable $X \sim B(n, p)$.
If X is a Binomial Random variable then it can be written as sum of n Independent ~~Binomial~~ Bernoulli Trials

$X = Y_1 + Y_2 + Y_3 - - Y_n$

Bernoulli Trials

Date:

Page No.

$X = X_1 + X_2 + X_3 \text{ ---- } X_n$

where X_i are n Independent Bernoulli RV with parameter p.

$$\phi_X(t) = \phi_{X_1+X_2+X_3 - X_n}(t)$$

$$= \phi_{X_1}(t) \cdot \phi_{X_2}(t) \cdot \phi_{X_3}(t) \text{ -- } \phi_{X_n}(t)$$

$$\rightarrow \left[p e^{it} + (1-p)\right]^n$$

<u>Theorem</u> :- <u>Uniqueness Theorem</u> :- Let X and Y be Random variables s.t $\phi_X = \phi_Y$. Then $X = Y$ i.e. X and Y will have same distribution.

<u>Proof</u> :- $\phi_X(t) = \phi_X(t)$ $\quad t \in R$

$$E[e^{itX}] = E[e^{itY}]$$

$$\rightarrow e^{itX} = e^{itY}$$

$$\rightarrow X = Y$$

Hence proved.

<u>ex</u> : Let $X \sim B(n_1, p)$ & $Y \sim B(n_2, p)$ with X and Y are Independent.

$$X + Y \sim B(n_1 + n_2, p)$$

Binomially distributed

$$\phi_{X+Y}(t) = \phi_X(t)\, \phi_Y(t)$$

$$\rightarrow \left[p e^{it} + (1-p)\right]^{n_1} \left[p e^{it} + (1-p)\right]^{n_2}$$

$$\rightarrow \left[p e^{it} + (1-p)\right]^{n_1+n_2}$$

RHS is the ch function of a Binomial RV say Z with parameter $(n_1 + n_2, p)$ while LHS is the ch function of $X + Y$.

Date:
Page No.

By uniqueness theorem we can say

$X+Y=Z$

$$\boxed{X+Y \sim B(n_1+n_2, p)}$$

ex $X \sim N(\mu_1, \sigma_1^2)$ & $Y \sim N(\mu_2, \sigma_2^2)$
Both are independent then show that
$X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.
$\rightarrow$

<u>Inequalities</u> :-

1) <u>Jensen's Inequality</u> :- Let X be a RV and $f: I \rightarrow R$ be a <u>convex</u> function where $I \subseteq R$ in an Interval. Then

$$\boxed{E[f(x)] \geq f[E(x)]}$$

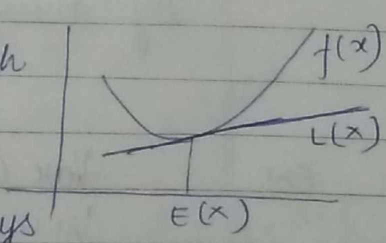
provided expectations exist.

If f is <u>concave</u> then :-

$$\boxed{E[f(x)] \leq f[E(x)]}$$

<u>Proof of this Inequality</u> $\rightarrow$
<u>Assumption</u> :- function f is differentiable
Proof : Let $f: I \rightarrow R$ be differentiable

Let $E(x) = a+bx$ be a line which is tangent to $f(x)$ at $E(x)$.



Due to differentiability $L(x)$ always lie below the curve

• $E[f(x)] \geq E[L(x)]$

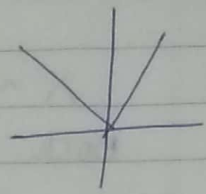
because this line always lie below the curve

$$E[a+bx] = a+bE(x)$$
$$= L(E(x)) = f(E(x))$$

$$\boxed{E(f(x)) \geq f(E(x))}$$

Ex: $f(x) = |x|$

function is defined on $I \to R$

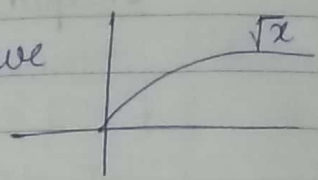


→ By the graph it is clear that graph is con~~cave~~vex.

$$\boxed{E[|X|] \geq |E[X]|}$$ for all Real Values.

Ex: $f(x) = \sqrt{x}$; $x > 0$

concave



$$\boxed{E[\sqrt{x}] \leq \sqrt{E[x]}}$$

Ex: $|E(X)| \leq E[|X|] \leq \sqrt{E(X^2)}$

X^2 is always positive, from previous example we have $E[\sqrt{x}] \leq \sqrt{E[X]}$

→ $E[\sqrt{X^2}] \leq \sqrt{E(X^2)}$ True for all $X \in R$

$X^2 > 0 \quad \forall x \in R$ ↓

→ $E[|X|] \leq \sqrt{E(X^2)}$

by above example

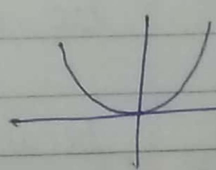
$$|E(X)| \leq E[|X|] \leq \sqrt{E(X^2)}$$

Ex: $Var(X) \geq 0$

$Var(X) = E(X^2) - [E(X)]^2$

→ consider $f(x) = x^2 \quad \forall x \in R$

convex



→ $E(X^2) \geq (E(X))^2$

→ $E(X^2) - [E(X)]^2 \geq 0$

$$\boxed{Var(X) \geq 0}$$

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→ **Moments of a Random Variable :-**

Let r be a positive real number and X be a RV. Then r-th moment of X about a point a is defined by : $E[(X-a)^r]$

- If we set $a = 0$; then $E[X^r]$ is the r^{th} moment of RV X about the origin.
- If we set $a = \mu$ (expectation of X) Then

$E[(X - E(X))^r]$ gives us r^{th} moment of X about its mean.

→ Skewness is obtained with the help of moment

→ variance (spread), skewness, cortosis.

→ we know that $E[X^r]$ exists and is finite if $E[|X^r|]$ exists and is less than ∞

$$\boxed{E[X^r] = \sum_x x^r p_X(x)}$$

$$\sum_x x^r p_X(x) \leq \left| \sum_x x^r p_X(x) \right|$$

$$\leq \sum_x |x^r p_X(x)|$$

PMF always take positive value

$$\leftrightarrow \quad = \sum_x |x^r| \, |p(x)| = E[|X|^r]$$

$$\sum_x x^r p_X(x) \leq \sum_x |x^r| \, |p_X(x)|$$

Date:
Page No.

Ex If the Moment (about origin) of order $q > 0$ exists for a random variable X, then show that Moments of order p exists where $0 < p < q$

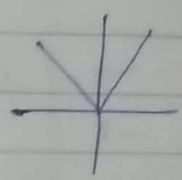
Sol:- $f : (0, \infty) \to \mathbb{R}$ $\qquad f(x) = x^r$ where $r > 1$

$f''(x) = r(r-1)\, x^{r-2}$
convex $\forall\, x \in (0, \infty)$

$\Rightarrow \quad E[X^r] \geq [E(X)]^r$

$[E(X)]^r \leq E[X^r] \leq E[\,|X|^r]$

$\Rightarrow \quad [E(X)] \leq \left(E[\,|X|^r]\right)^{1/r} \quad — (1)$



$0 < p < q$; then $q/p > 1$

Replace r by q/p in (1) we get

$$[E(X)] \leq \left(E(|X|^{q/p}]\right)^{p/q} \quad — (2)$$

set X by X^p we get

$$E[X^p] \leq \left(E[X]^q\right)^{p/q}$$

p^{th} Moment exist if q^{th} Moment exist

If high moment exist low Moment also exists

Ex: Let X be a RV with $E(X) = 10$ show that
$$E[\ln \sqrt{x}] \leq \frac{1}{2} \ln 10$$

Sol $f(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$

$f'(x) = \frac{1}{2x} \qquad f''(x) = -\frac{1}{2x^2} < 0$ concave

$E[\ln \sqrt{x}] \leq \ln \sqrt{10} = \frac{1}{2} \ln 10$