ODE → 1 independent variable

PDE → 2 ≥ 1 " "

PARTIAL DIFFERENTIAL EQUATIONS

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- App^ns of PDE :

1. Black - Scholes Eq^n : used in mathematical finance to predict value of given stock.

$$f_t + r S f_s + \sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} = r f$$

f : $f(S)$

r : ~~rate~~ risk free rate of return

σ = Volatity const.

2. Navier - stoke's Eq^n : used in Fluid mechanics

3 D

$$u_t + (u_0 \nabla) u = -\nabla_p + \nu \nabla u$$

$$\nabla u = 0 \quad \text{in} \quad \Omega$$

$$u = 0 \quad \text{on} \quad d\Omega$$

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PDE :-

1^st order PDE is of the form :

(1) — $f(x, y, z, z_x, z_y) = 0$

independent variable

$z = z(x, y)$ ← dependent variable

$$z_x = \frac{\partial z}{\partial x}, \quad z_y = \frac{\partial z}{\partial y}$$

2^{nd} Order PDE :

$$f(x, y, z(x,y), z_{xx}, z_{xy}, z_{yy}, z_x, z_y) = 0$$

- PDE with more than 2 independent variables :

$$f(x, y, t, \ldots, z, z_{xx}, z_{yy}, z_{tt}, \ldots) = 0$$

$$z = z(x, y, t, \ldots)$$

Order of PDE : Highest order partial derivatives which appears in the PDE.

Eg. $z_{xx} + 2x z_x^3 + z_y = 0$ Order : 2
$\longrightarrow$ Quasilinear & semi linear

Classification of PDEs

1.) Quasilinear PDE

PDE is said to be Quasilinear if the highest order derivatives are linear

2) Semi-linear PDE

A quasilinear PDE is semi-linear if the coefficients of highest order derivatives do not contain dependent variable or its derivatives.

* All ~~sem~~ semi-linear are ~~qu~~ quasilinear

3.) Linear Eq^n

A semilinear PDE is said to be linear, if it is linear in the ~~linear~~ dependent variable & its derivatives

4) Non-Linear

If a PDE is not quasilinear, then it is non-linear.

All linear are semilinear
All semi linear are quasi linear

highest order

eg. $u u_{xx} + u_t = xt$ ⇒ 2nd Order quasilinear PDE
$u = u(x,t)$: Quasi linear ✓
Not semilinear
Not linear

eg. $x u_{xx} + u u_t = xt$
not linear (product of dependent variables)
2nd Order semi linear PDE

eg. $x u_{xx} + u_t = xt$
2nd Order Linear PDE.

eg. $(u_{xx})^2 + u_t = xt$: Not quasilinear ⇒ Non-linear
2nd order PDE

eg. $u_{xx} + (u_t)^2 = xt$: 2nd order semi linear PDE.
↳ Not linear

Classification of 1st order PDE :

1.) Quasilinear PDE → p & q have to be linear

$f(x, y, z, p, q)$

$p = z_x \quad q = z_y \quad z = z(x,y)$

$$z^2 x p + e^z y q = x^2 \sin y$$

$$P(x,y,z)\, p + Q(x,y,z)\, q = R(x,y,z) \Rightarrow$$

2.) semi linear PDE : not func^n of z

$$P(x,y)\, p + Q(x,y)\, q = R(x,y,z) \Rightarrow$$

eg. $x y^2 p + e^x \sin y\, q = z^3 \sin y$

diff. b/w 2.) & 3.) ⇒ $R(x,y,z)$ is allowed to be non-linear

3.) Linear PDE :

$$P(x,y)\, p + Q(x,y)\, q = R(x,y) + z\, S(x,y)$$

won't contain $z^2, z^3 \ldots$

4.) Non-linear PDE

$$f(x, y, z, p, q) = 0$$

Eg. $pq = 0$.

Originating 1st Order PDE

Consider an eq^n : $\quad x, y$: independent

$$x^2 + y^2 + (z-c)^2 = a^2 \quad \because \text{eq}^n \text{ of sphere}$$

diff. wrt x :

$$2x + \cancel{2y\, y_x}^{\,0} + 2(z-c)z_x = 0 \quad \text{—— (1)}$$

Similarly,

$$\cancel{2x\, x_y}^{\,0} + 2y + 2(z-c)z_y = 0 \quad \text{—— (2)}$$

$$2x + 2(z-c)p = 0$$

$$2y + 2(z-c)q = 0$$

$$\Rightarrow \quad qx + (z-c)pq = 0$$

$$py + (z-c)pq = 0$$

$$\boxed{qx - py = 0}$$ — 1st order linear PDE characterizes the eq^n of spheres with center at z-axis.

Ex. → Consider an eq^n of the form :

(1) —— $F(x, y, z, a, b) = 0 \quad a, b \to$ const. $\quad x, y \to$ independent variable

diff. wrt. x :

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=2} + \frac{\partial F}{\partial z}\underbrace{\frac{\partial z}{\partial x}}_{p} = 0$$

$$\Rightarrow \quad \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \quad \text{—— (2)}$$

diff. wrt. y :

$$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \quad \text{—— (3)}$$

F, a & b need to be eliminated from these 3 eq^{n}s.
we get a PDE : (Use above eq here)

$$f(x, y, z, p, q) = 0 \quad \text{(non-linear PDE)}$$

Surface of Revolution

All the surfaces with z - axis as the axis of revolution are of the form :

(4) $z = F(r)$ $\qquad r = \sqrt{x^2+y^2}$

diff wrt x,

$$p = z_x = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} = F'(r) \cdot \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x$$

$$p = \left(\frac{x}{r}\right) F'(r) \quad \text{—— (5)}$$

diff wrt y,

$$q = z_y = \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial y} = F'(r) \cdot \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y$$

$$q = \left(\frac{y}{r}\right) F'(r) \quad \text{—— (6)}$$

$\Rightarrow$ $\boxed{py - xq = 0}$ again, linear 1st order PDE

20/10/17 In general, consider the surface of the form

$$F(u, v) = 0 \quad \text{—— (1)}$$

where $u = u(x, y, z)$ and $v = v(x, y, z)$ are two known func^n of x, y & z.

$x, y \rightarrow$ ~~independent variable~~

Diff. eq^n (1) wrt x

$$\frac{\partial F}{\partial u} \cdot \left[\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial u}{\partial y} \cdot \cancel{\frac{\partial y}{\partial x}}^{\,0} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}\right] +$$

$$\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial v}{\partial y} \cdot \cancel{\frac{\partial y}{\partial x}}^{\,0} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x}\right] = 0$$

→ In ODE
1st order ⇒ 1-D soln
2nd order ⇒ 2-D soln

→ But in PDE, it is not so

$$\Rightarrow \frac{\partial F}{\partial u}\left[\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\right] = 0 \quad \text{--- (2)}$$

Similarly, diff. wrt y, we get:

$$\frac{\partial F}{\partial v}\left[\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\right] = 0 \quad \text{--- (3)}$$

Eliminate $\frac{\partial F}{\partial u}$ & $\frac{\partial F}{\partial v}$ from eqn (2) & (3).

$$\boxed{\frac{\partial(u,v)}{\partial(y,z)}\, p + \frac{\partial(u,v)}{\partial(z,x)}\, q = \frac{\partial(u,v)}{\partial(x,y)}}$$

— ~~Not~~ ~~Non~~ Linear P.D.E. (could be Quasilinear / ~~Semi-linear / linear~~)

where $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

(Jacobian of u, v w.r.t. x & y)

Ex. 1) consider: $(x-a)^2 + (y-b)^2 + z^2 = 1$ [1st type] Sphere

leads to non-linear PDE.

Verify it.

2.) $z = x^n f(y/x)$ [2nd type] ↳ Rotation

leads to a 1st order PDE.

verify it.

Find the PDE which characterizes the above surfaces?

System of Surfaces

$$f(x, y, z, c) = 0 \quad \text{—— (4)}$$

$c \to$ parameter

$$f(x, y, z, a, b) = 0 \quad : \quad 2 \text{ parameter} \quad \text{—— (5)}$$

Def^n: Envelope of one-parameter system :

The surface determined by eliminating the parameter 'C' between the eq^n :

$$f(x, y, z, c) = 0 \quad \& \quad \frac{\partial}{\partial c} f(x, y, z, c) = 0$$

is called the envelope of one-parameter system (4)

Eg. $x^2 + y^2 + (z-c)^2 = 1$

diff. w.r.t. c

$$F(x, y, z, c) = x^2 + y^2 + (z-c)^2 - 1 = 0$$

$$\frac{\partial}{\partial c} F(x, y, z, c) = -2(z-c) = 0 \Rightarrow \boxed{z = c}$$

substituting $z = c$, envelope of above eq^n is :

$$x^2 + y^2 = 1$$

(unit circle on x-y plane)

$\to$ Cross-section of system of surfaces (here, spheres)

Def^n: Envelope of two-parameter system :

Consider system of surfaces. (5)

The surface obtained by eliminating a, and b from eq^{n}s

$$f(x, y, z, a, b) = 0, \quad \frac{\partial f}{\partial a} = 0 \text{ and } \frac{\partial f}{\partial b} = 0$$

is called the envelope of two-parameter system (5).

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eg. $(x-c)^2 + (y-d)^2 + z^2 = 1$

$f(x,y,z,c,d) = (x-c)^2 + (y-d)^2 + z^2 - 1 = 0$

$\frac{\partial f}{\partial c} = -2(x-c) = 0 \Rightarrow \boxed{x = c}$

$\frac{\partial f}{\partial d} = -2(y-d) = 0 \Rightarrow \boxed{y = d}$

$\Rightarrow$ The envelope is : $z^2 = 1$ or $z = \pm 1$ [2 parallel planes]

Solution of 1st Order PDE

Consider

$$f(x, y, z, p, q) = 0 \quad \text{where } p = z_x,\ q = z_y \quad (6)$$

(i) A func^n $z = z(x, y)$ should satisfy eq^n (6)

(ii) The func^n z is continuously differentiable (because p & q must exist) on $(x, y) \in D \subset R \times R$

A sol^n $z = z(x, y)$ exists in 3D space (ie, $(x, y, z(x,y)) \in R^3$) can be interpreted as surface & hence, is called integral surface of PDE (6)

Classification based on solutions :-

1. Complete Integral or Complete sol^n :

$f(x, y, z, a, b) = 0$ lead to PDE of 1st order. Any such relation which contain 2 arbitary const. a & b, and is a sol^n of a 1st order PDE is said to be complete integral

Ex. $x^2 + y^2 + (z-c)^2 = a^2$

Sol^n : $qx - py = 0$: 1st order

2. General Integral or General sol^n :

$f(u, v) = 0$ provides a sol^n of 1st order PDE, known as General Integral

Ex. ~~$f(u,v) = 0$~~ ~~sol^n~~ $\frac{\partial(u,v)}{\partial(z,x)} + p\frac{\partial(u,v)}{\partial(y,z)} = \frac{\partial(u,v)}{\partial(x,y)}$

will give rise to $f(u,v) = 0$

Singular Integral :

The sol^n obtained from the envelope of the two-parameter family is known as singular point. This is obtained by eliminating a & b from

$$z = F(x, y, a, b), \quad \frac{\partial F}{\partial a} = 0, \quad \frac{\partial F}{\partial b} = 0$$

Ex $\quad z - px - qy - p^2 - q^2 = 0$

$$z = p^2 + q^2 + px + qy$$

Check, this $\quad ax + by + a^2 + b^2$ is a complete integral.

$$z = F(x, y, a, b)$$

Find singular $sol^n s$:

$$\frac{\partial F}{\partial a} = x + 2a = 0, \qquad \frac{\partial F}{\partial b} = y + 2b = 0$$

$$\Rightarrow x = -2a \qquad\qquad y = -2b$$

$$\Rightarrow 4z = -(x^2 + y^2) \quad \text{— singular } sol^n.$$

M - III

PDE

Cauchy Value Problem (or Initial Value Problem)

Objective : To find an integral surface of the given PDE

$$f(x, y, z, p, q) = 0 \quad \text{--- (1)}$$

which contain an initial curve

$$C : x = x_0(s), \; y = y_0(s), \; z = z_0(s), \; s \in I$$

The cauchy value problem is to find a solⁿ $z(x, y)$ of the PDE (1) s.t.

$$z_0(s) = z(x_0(s), y_0(s)) \quad \forall \; s \in I$$

If we can solve quasilinear $\Rightarrow$ we can solve for linear & semi-linear also.

Langrage Method (for solving Quasilinear Problems)

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z) \quad \text{--- (2)}$$

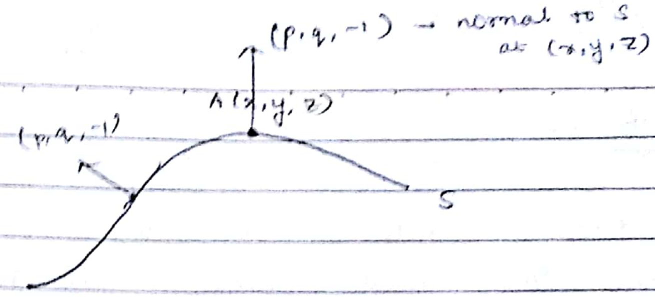
P, Q, R are smooth funcⁿ in $D \subset \mathbb{R}^3$

$P, Q, R \in C^1(D) \Rightarrow$ (derivatives of P, Q, R are continuously differentiable)

$P, Q, R : D \rightarrow \mathbb{R}$ don't vanish simultaneously

$z = z(x, y)$ is an integral surface in xyz - space

$$S : \{ z = z(x, y) : (x, y) \in D' \subset \mathbb{R} \times \mathbb{R} \}$$



Eqⁿ (1) becomes:

$$Pp + Qq - R = 0$$

We can say that : (P, Q, R) are orthogonal to $(p, q, -1)$

eqⁿ (2) is equivalent to say that vector $(p, q, -1)$ and (P, Q, R) are orthogonal at each point $A \in S$

$P\hat{i} + Q\hat{j} + R\hat{k}$ lies on the tangent plane at A.
(then only, they'll be orthogonal)

For a curve $C: x = x(t),\ y = y(t),\ z = z(t)$ on S

we have $(P\hat{i} + Q\hat{j} + R\hat{k}) \parallel (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$

diff. wrt t

equivalently,

$$\boxed{\frac{\dot{x}}{P} = \frac{\dot{y}}{Q} = \frac{\dot{z}}{R}}$$ – characteristic eqⁿ of PDE – (2) ——— (3)

or $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Thᵐ: The general solⁿ of the PDE (2) is

Implicit form ← $F(u, v) = 0$ (or $u = G(v)$ or $v = H(u)$) → explicit form

where F is an arbitary smooth funcⁿ of u and v.

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solⁿ of characteristic eqⁿ (3)

Proof: $\underbrace{u(x, y, z) = c_1 \ \&\ v(x, y, z) = c_2}$ → called characteristic curves (here) are two solⁿs of characteristic eqⁿ (3)

$\Rightarrow \quad du = 0 \quad$ and $\quad dv = 0$

$\Rightarrow \quad u_x\,dx + u_y\,dy + u_z\,dz = 0 \quad$ and $\quad v_x\,dx + v_y\,dy + v_z\,dz = 0$

But $\quad \dfrac{\overset{dx}{\dot{x}}}{P} = \dfrac{\overset{dy}{\dot{y}}}{Q} = \dfrac{\overset{dz}{\dot{z}}}{R} = k$ (let)

$$\left.\begin{aligned} &\Rightarrow \quad P u_x + Q u_y + R u_z = 0 \\ &\text{In the same way,} \\ &\quad P v_x + Q v_y + R v_z = 0 \end{aligned}\right\}$$

Solving for P, Q and R, we get.

$\cancel{\dfrac{P}{u_y v_z}}$

$$k = \frac{P}{\dfrac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\dfrac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\dfrac{\partial(u,v)}{\partial(x,y)}} \quad \text{—— (4)}$$

$(u_y v_z - v_y u_z)$

Since $F(u,v) = 0$ leads to PDE of form

$$\frac{\partial(u,v)}{\partial(y,z)}\,p + \frac{\partial(u,v)}{\partial(z,x)}\,q = \frac{\partial(u,v)}{\partial(x,y)} \quad \text{[seen in Revolution of surface Part]} \quad \text{—— (5)}$$

Comparing eq^n (4) & (5), we get

$$\boxed{P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)}$$

Eg. $\quad x^2 p + y^2 q - (x+y)z = 0$

Sol^n: The characteristic is given by:

take on RHS first

$$\underset{\text{coeff. of } p}{\frac{dx}{x^2}} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$\Rightarrow \quad \dfrac{dx}{x^2} = \dfrac{dy}{y^2} \Rightarrow -\dfrac{1}{x} = -\dfrac{1}{y} + c$

$$\Rightarrow \frac{1}{x} - \frac{1}{y} = c_1 = u(x,y) \quad \text{—— (1)}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{dx - dy}{x^2 - y^2}$$

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\Rightarrow \frac{dx - dy}{x - y} = \frac{dz}{z}$$

$$\Rightarrow \frac{d(x-y)}{x-y} = \frac{dz}{z}$$

$$\Rightarrow \log|x-y| = \log|z| + \log k$$

$$\Rightarrow (x-y) = kz = c_2 z$$

$$\Rightarrow \frac{x-y}{z} = c_2 = v(x,y) \quad \text{—— (2)}$$

Now, we have a sol^{n}s $u(x,y)$ & $v(x,y)$

$$F(u(x,y), v(x,y)) = F(c_1, c_2)$$

$$= \boxed{F\left(\frac{1}{x} - \frac{1}{y}, \frac{x-y}{z}\right) = 0}$$

This will give general sol^n for given PDE

Only need is it should be ~~sol~~ diff.

To get exact sol^n, we must eliminate F using Cauchy Value Problem.

$$F(\quad) \equiv \phi\left(\frac{1}{x}, \frac{-1}{y}\right) = G\left(\frac{x-y}{z}\right)$$

$$\text{or} \quad \frac{x-y}{z} = H\left(\frac{1}{x}, \frac{-1}{y}\right)$$

Ex: $xp + yq = z$ containing the curve

$C : x_0 = s^2, \; y_0 = s+1, \; z_0 = 1$

Sol^n: Characteristic eq^n:

$$\frac{dx}{x} + \frac{dy}{y} = \frac{dz}{z}$$

$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log|y| = \log|z| + \log c_1$

$\Rightarrow \frac{y}{z} = c_1 = u(x, y, z)$

$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{y}{x} \cancel{\frac{x}{y}} = c_2 = v(x, y, z)$

Two independent sol^n for this characteristic eq^n.

General sol^n : $F(u, v) = 0$

$\Rightarrow F\left(\frac{y}{z}, \frac{y}{x}\right) = 0$

Option 1

$\frac{y}{z} = c_1 \Rightarrow \frac{s+1}{s} = c_1$] ~~Relation b/w~~ c_1 & c_2 (find)

$\frac{y}{x} = c_2 \Rightarrow \frac{s+1}{s^2} = c_2$

$\Rightarrow$ we get : $(c_1 - 1)\, c_1 = c_2$

$\Rightarrow \left(\frac{y}{z} - 1\right)\frac{y}{z} = \frac{y}{x} \Rightarrow \frac{(y-z)y}{z^2} = \frac{y}{x}$

$\Rightarrow \boxed{(y-z)x = z^2}$

Option 2.

$\frac{y}{x} = G\left(\frac{y}{z}\right) \rightarrow$ we have to determine G.

$\Rightarrow \frac{s+1}{s^2} = G\left(\frac{s+1}{s}\right)$ $G(t)$: in terms of t

$\frac{s+1}{s} = t \Rightarrow 1 + \frac{1}{s} = t \Rightarrow \frac{1}{s} = t - 1 \Rightarrow s = \frac{1}{t-1}$

$\Rightarrow \dfrac{\frac{1}{t-1} + 1}{\frac{1}{(t-1)^2}} = G(t)$

put $t = y/z$

$\Rightarrow t(t-1) = G(t)$

$\Rightarrow G(y/z) = (y/z)(y/z - 1) = \frac{y}{x}$

$\Rightarrow \boxed{(y-z)\,x = z^2}$

→ $u(x,y,z) = c_1$
$v(x,y,z) = c_2$ } → characteristic curve

Independent solⁿ :

$\nabla u \times \nabla v \neq 0$

The general solⁿ : $F(u,v) = 0$

$(p,q,-1)$ is normal ?? ∵

$F(x,y,z)$: $z = z(x,y)$ — Integral surface (Explicit form)

$F(x,y,z) = z(x,y) - z = 0$ — Implicit solⁿ

$\nabla F = (z_x, z_y, -1) = (p,q,-1)$: normal derivative to S at (x,y,z)

⇒ $P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$
is equivalent to
$(P,Q,R) \cdot \nabla F = 0$

$(P,Q,R) \parallel (\dot{x}, \dot{y}, \dot{z})$

Eg. $yzp + xzq = xy$

Solⁿ Characteristic eqⁿ :

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

$$\frac{dx}{yz} = \frac{dy}{xz} \Rightarrow x\,dx - y\,dy = 0 \Rightarrow x^2 - y^2 = c_1 = u(x,y,z)$$

$$\frac{dy}{xz} = \frac{dz}{xy} \Rightarrow y\,dy - z\,dz = 0$$
$$\Rightarrow y^2 - z^2 = c_2 = v(x,y,z)$$

$F(x^2-y^2,\ y^2-z^2) = 0$ → Implicit form

⇒ $x^2 - y^2 = G(y^2 - z^2)$ or $y^2 - z^2 = H(x^2 - y^2)$ → Explicit form

→ Not every implicit form can be converted into explicit form

Eg.. $(z^2 - 2yz - y^2)\,p + x(y+z)\,q = x(y-z)$

Sol^n: Characteristic eq^n:

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} \qquad \text{—— (1)}$$

$$\frac{dy}{y+z} = \frac{dz}{y-z} \Rightarrow y\,dy - z\,dy = y\,dz + z\,dz$$

$$\Rightarrow \cancel{\frac{y^2}{2} - yz} = \cancel{yz + \frac{z^2}{2}} + c$$

$$\Rightarrow y\,dy - z\,dz - \underbrace{z\,dy - y\,dz}_{C_1} \qquad d(yz)'$$

$$\Rightarrow \frac{y^2}{2} - \frac{z^2}{2} - yz = C_1$$

or $\quad y^2 - z^2 - 2yz = C_1 = u(x,y,z)$

Modifying (1) : (Componendo and Dividendo)

$$= \frac{x\,dx + y\,dy + z\,dz}{xz^2 - \cancel{2xyz} - \cancel{xy^2} + \cancel{xy^2} + \cancel{xyz} + \cancel{xzy} - \cancel{xz^2}} = \frac{}{0} = K$$

0 also makes sense here.

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$ { ~~we are not dividing here,~~ it's like some notation.}

$\Rightarrow x^2 + y^2 + z^2 = C_2 = v(x,y,z)$

$$F(y^2 - z^2 - 2yz,\; x^2 + y^2 + z^2) = 0$$

Eg. $2p + 3q + 8z = 0$

Find the integral curve containing the following curves:

i) σ : $z = 1 - 3x$ on line $y = 0$

ii) σ : $z = x^2$ on line $2y = 1 + 3x$

iii) σ : $z = e^{-4x}$ on line $2y = 3x$

1st, find general sol^n, for this, find characteristic eq^n,

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{-8z}$$

$2\,dy - 3\,dx = 0$

$\Rightarrow 2y - 3x = C_1 = u(x,y,z)$

$$\frac{dx}{2} = \frac{dz}{-8z} \Rightarrow -4x = \ln z + \ln C_2$$

$$\Rightarrow e^{-4x} = z C_2$$

$$\Rightarrow \frac{e^{-4x}}{z} = C_2 = v(x,y,z)$$

or $\quad z\,e^{4x} = C_2 = v(x,y,z)$

general sol^n :

$$F(2y-3x,\ z e^{4x}) = 0$$

or $\quad z e^{4x} = G(2y-3x)$ — ①

eg i) σ : $z = 1-3x$ on $y=0$

$$(1-3x)e^{4x} = G(-3x)$$

put $-3x = s \Rightarrow x = -s/3$

$\Rightarrow$ ~~(1+s) e~~

$$G(s) = (1+s)\,e^{-4s/3}$$

From ①

~~$(1+3x)e^{4x}$~~ $\Rightarrow$ ~~$z e^{4x} = G(-3x)$~~

From ①,

$$z e^{4x} = (1+2y-3x)\,e^{-4(2y-3x)/3}$$

$+4x$

$$\Rightarrow z = (1+2y-3x)\,e^{-8/3 y}$$

ii) σ : $z = x^2$ on line $2y = 1+3x$

$$(x^2)e^{4x} = G(1+3x-3x) = G(1)$$

or $\quad x^2 = G(1)\,e^{-4x} \Rightarrow$ no possible value of x

polynomial $\quad$ const $\quad$ exponential

$\Rightarrow$ NO SOL^n

iii) $z = e^{-4x}$ on $2y = 3x$

$$e^{-4x}\cdot e^{4x} = G(0)$$

$\Rightarrow G(0) = 1 \Rightarrow$ func^n takes value 1 at $x=0$

~~G(t):~~

$\Rightarrow$ can be: e^t, $\cos t$, $1+t$, $1+t^5$ ...

$\Downarrow$

$\boxed{\text{Infinite many sol}^n\text{s}}$

→ cauchy problem may have:
unique solⁿ
no solⁿ
infinitely many solⁿ

eg. $(2xy-1)p + (z-2x^2)q = 2(x-yz)$

Find Integral surface containing the curve

$\gamma : x_0(s) = 1, \quad y_0(s) = 0, \quad z_0(s) = s$

$$\frac{z\,dx}{z(2xy-1)} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)}\left(\frac{x}{x}\right)$$

try to make denominator = 0 & exact diff eqⁿ in numerator

① $\Rightarrow \dfrac{z\,dx + dy + x\,dz}{\cancel{2xyz} - \cancel{z} + \cancel{z} - \cancel{2x^2} + \cancel{2x^2} - \cancel{2xyz}} = k$

$\Rightarrow z\,dx + x\,dz + dy = 0$

$\Rightarrow xz + y = c_1 = u(x,y,z)$

② $\dfrac{2x\,dx + 2y\,dy + dz}{\cancel{4x^2y} - \cancel{2x} + \cancel{2yz} - \cancel{4x^2y} + \cancel{2x} - \cancel{2yz}}$

$\Rightarrow x^2 + y^2 + z = c_2 = v(x,y,z)$

~~For~~ verify c_1 & c_2 are independent solⁿs.

$F(u, v) = 0$

$\Rightarrow xz + y = G(x^2+y^2+z)$
or $x^2+y^2+z = H(xz+y)$

} both may not help to get exact ⇒ need to take care while choosing.

$xz + y = G(x^2+y^2+z)$
$s = G(1+s)$
$1+s = t$
$G(t) = t-1$

$\Rightarrow xz + y = (x^2+y^2+z) - 1$

$1 + s = H(s)$

$\Rightarrow x^2+y^2+z = 1 + (xz+y)$

Ex. $x^3 p + y(3x^2+y) q = z(2x^2+y)$

$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} = \frac{dz}{2zx^2+zy}$$

(1)
$$\frac{-x^{-1}dx + y^{-1}dy - z^{-1}dz}{-x^2+3x^2+y-2x^2-y}$$

$$\Rightarrow -\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0$$

(2)
$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} \Rightarrow \left(\frac{3x^2+y}{x^3}\right) dx = \frac{dy}{y}$$

$$\Rightarrow \frac{(3x^2+y)\,dx + dy}{x^3+y} = \frac{(3x^2+y)\,dx + dy + x\,dy}{x^3+y+xy} = \frac{dy}{y}$$

Exact

$$\Rightarrow \frac{d(x^3+y+xy)}{x^3+y+xy} = \frac{dy}{y}$$

$$\Rightarrow \frac{x^3+y+xy}{y} = c_2$$

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Second Order PDE

The std. second ~~ordrd~~ order PDE in 2 independent variable

$$f(x, y, z, z_{xx}, z_{yy}, z_{xy}, z_x, z_y) = 0$$

(Semi-linear PDE)

$\Downarrow$

$u \rightarrow$ dependent variable
$u = u(x, y)$

can be represented as:

(1)
$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

A, B, C, D, E, F, G : func^n of independent variables x & y
(D, E, F, G may also be func^n of u)

same eq^n may be classified in diff. form depending on domain

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$A^2+B^2+C^2 \neq 0$, A, B, C are continuous as possess continuous partial derivative of as high order as necessary

Def^n: A func^n $u(x,y)$ is said to be regular sol^n of

(2) —— $Au_{xx} + Bu_{xy} + Cu_{yy} + g(x,y,u,u_x,u_y) = 0$

in $D \subset R \times R$ if $u \in C^2(D)$ (upto 2nd order derivative are continuous)

and the func^n u & it's derivatives satisfies (2) for all $x, y \in D$

Genesis of 2nd Order PDE

$f \in C^2(D)$ & $u = f(x+at)$

$u_x = f'(x+at)$ $\quad u_t = af'(x+at)$

$u_{xx} = f''(x+at)$ $\quad u_{tt} = a^2 f''(x+at)$

$\Rightarrow \boxed{u_{tt} = a^2 u_{xx}}$: leads to 2nd Order PDE

Classification of 2nd Order PDE (Parabola, Ellipse, Hyperbola)

(3) —— $\underbrace{ax^2 + bxy + cy^2}$ + dx + ey + f = 0

principal part (~~classi~~ classifications depends on these variables only)

Normal form

→ $b^2 - 4ac > 0$: Hyperbola $\left(x^2/a^2 - y^2/b^2 = 1\right)$

→ $b^2 - 4ac = 0$: Parabola $(x^2 = y)$

→ $b^2 - 4ac < 0$: Ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$

For PDE :

Principal part of (2) :

$Lu = A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy}$

i) $B^2(x,y) - 4A(x,y)C(x,y) > 0 \Rightarrow (x,y)$ — hyperbolic PDE

ii) $= 0 \Rightarrow$ — parabolic PDE

iii) — elliptic PDE

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eg. $u_{xx} - x^2 u_{yy} = 0$

$A(x,y) = 1$ $\qquad$ $C(x,y) = -x^2$

$B(x,y) = 0$ (1)

$B^2 - 4AC = 0 - 4(1)(-x^2) = 4x^2 \geq 0$

$$\Rightarrow u_{xx} - x^2 u_{yy} = 0 \rightarrow \begin{cases} \rightarrow \text{hyperbolic} & x \neq 0 \\ \rightarrow \text{Parabolic} & x = 0 \end{cases}$$

eg. $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} = \underbrace{\frac{y^2}{x} u_x + \frac{x^2}{y} u_y}$

not included in classification but should be defined $(x \neq 0)$

$A = y^2$ $\qquad$ $B = -2xy$ $\qquad$ $C = x^2$

$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0 \Rightarrow$ Parabolic PDE

(But we need to take care of u_x & u_y, & they should be defined, so $x \neq 0$)

eg. $u_{xx} + x^2 u_{yy} = 0$

$B^2 - 4AC = 0 - 4(1)(x^2) = -4x^2$

$\Rightarrow$ ellipse for $x \neq 0$

parabolic for $x = 0$

eg. $u_{xx} + x u_{yy} = 0$

$\Rightarrow -4x$

$$\begin{cases} \text{Parabolic} & : x = 0 \\ \text{Elliptic} & : x > 0 \\ \text{hyperbolic} & : x < 0 \end{cases}$$

Here, we will study : 2^{nd} order Semi-Linear PDE with 2 independent variables

<u>Cannonical (Normal) form of 2^{nd} Order PDE</u> :

good for hyperbolic & ~~ellip~~ parabolic (may not help in case of elliptic)

(1) —— $A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0$

$A^2 + B^2 + C^2 \neq 0$ (all A, B, C can't be 0 at same time)

; otherwise, it won't be 2^{nd} order !

Our aim : $(x, y) \rightsquigarrow (\xi, \eta)$, $\xi = \xi(x, y)$

$u(x, y) \rightsquigarrow u(\xi, \eta)$ $\quad \eta = \eta(x, y)$

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \quad \text{(Assume)}$$

Thus, transformation is invertible

$u_\xi = u_\xi(\xi, \eta)$

$u_x = u_\xi \xi_x + u_\eta \eta_x$ (chain Rule)

$\cancel{u_{xx} \rightarrow u_\xi (\xi_x)^2 \pm u_\xi \xi_{xx} + u_\eta (\eta_x)^2}$

$$u_{xx} = (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) \xi_x + u_\xi \xi_{xx}$$
$$+ (u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x) \eta_x + u_\eta \eta_{xx}$$

$$u_{xx} = u_{\xi\xi} \xi_x^2 + 2 u_{\xi\eta} \eta_x \xi_x + u_\xi \xi_{xx} + u_{\eta\eta} \eta_x^2 + u_\eta \eta_{xx}$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_{xy} = (u_{\xi\xi} \xi_y + u_{\xi\eta} \eta_y) \xi_x + u_\xi \xi_{xy}$$
$$+ (u_{\eta\xi} \xi_y + u_{\eta\eta} \eta_y) \eta_x + u_\eta \eta_{xy}$$
$$= u_{\xi\xi} \xi_y \xi_x + u_\xi \xi_{xy} + u_{\xi\eta} \eta_y \xi_x + u_{\eta\xi} \xi_y \eta_x$$
$$+ u_{\eta\eta} \eta_y \eta_x + u_\eta \eta_{xy}$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{yy} = \left(u_{\xi\xi} \xi_y + u_{\xi\eta} \eta_y \right) \xi_y + u_\xi \xi_{yy}$$
$$+ \left(u_{\eta\xi} \xi_y + u_{\eta\eta} \eta_y \right) \eta_y + u_\eta \eta_{yy}$$
$$= u_{\xi\xi} \xi_y^2 + 2 u_{\xi\eta} \xi_y \eta_y + u_{\xi} \xi_{yy} + u_{\eta\eta} \eta_y^2 + u_\eta \eta_{yy}$$

Principal part :

$$A u_{xx} + B u_{xy} + C u_{yy} = u_{\xi\xi} \left(A \xi_x^2 + B \xi_x \xi_y + C \xi_y^2 \right)$$
$$+ u_{\xi\eta} \left(2A \xi_x \xi_y + B \left(\xi_x \eta_y + \eta_x \xi_y \right) + 2 \xi_x \eta_y \right)$$
$$+ u_{\eta\eta} \left(A \eta_x^2 + B \eta_x \eta_y + C \eta_y^2 \right) + h(\xi, \eta, u, u_\xi, \ldots$$

We choose ξ and η such that Jacobian $\neq 0$.

$\Rightarrow$ Eqn (1) becomes :

(2) — $\bar{A}(\xi_x ; \xi_y) u_{\xi\xi} + 2\bar{B}(\xi_x, \xi_y ; \eta_x, \eta_y) u_{\xi\eta} + \bar{A}(\eta_x ; \eta_y) u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$ ~~— (3)~~ (2)

coeff of $u_{\xi\xi}$ (func^n of something)

$$\bar{A}(u ; v) = A u^2 + B u v + C v^2$$
$$\bar{B}(u_1, v_1 ; u_2, v_2) = A u_1 u_2 + \frac{1}{2} B (u_1 v_2 + u_2 v_1) + C v_1 v_2$$

Eg. $\bar{B}^2(\xi_x, \xi_y ; \eta_x, \eta_y) - 4\bar{A}(\xi_x ; \xi_y) \bar{A}(\eta_x ; \eta_y) = 0$

$$= (B^2 - 4AC)\,(\xi_x \eta_y - \xi_y \eta_x)^2$$

> 0 (hyperbola) ; Jacobian > 0 $(\neq 0)$

(*)

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1) <u>Hyperbolic PDE</u> : $B^2 - 4AC > 0$ (Wave eq^n)

consider the quadratic eq^n :

$$A\alpha^2 + B\alpha + C = 0$$

We've 2 real and distinct roots : $\lambda_1(x,y)$, $\lambda_2(x,y)$ (say)

We choose $\xi(x,y)$ & $\eta(x,y)$ s.t.

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y} \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y} \qquad — (3)$$

$$\Rightarrow \xi_x = \lambda_1 \xi_y \quad \text{and} \quad \eta_x = \lambda_2 \eta_y$$

$$\begin{aligned}\bar{A}(\xi_x ; \xi_y) &= A\xi_x^2 + B\xi_x \xi_y + C\xi_y^2 \\ &= A\lambda_1^2 \xi_y^2 + B\lambda_1 \xi_y^2 + C\xi_y^2 \\ &= (A\lambda_1^2 + B\lambda_1 + C)\xi_y^2 \qquad (\lambda_1 : \text{root of eq}^n) \\ &= 0\end{aligned}$$

Similarly, $\bar{A}(\eta_x, \eta_y) = 0$

~~$2\bar{B}(\xi_1, \xi_2 ; u$~~

$$2\bar{B}(\xi_x, \xi_y ; \eta_x, \eta_y)\, u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$$

Since $\bar{B} > 0$

$$\boxed{u_{\xi\eta} = Q(\xi, \eta, u, u_\xi, u_\eta)}$$ canonical form for Hyperbolic case.

Eg. $u_{\xi\eta} = k$

$$\Rightarrow u_\xi = f(\xi)$$

$$u = \int f(\xi)\, d\xi + g(\eta)$$

$$= F(\xi) + g(\eta)$$

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ξ & η are sol^{n}s of eq^{n}s ③

$\rightarrow$ $\xi_x - \lambda_1 \xi_y = 0$

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0}$$

$d\xi = 0$
$\Rightarrow \xi = C_2$

$$\frac{dy}{dx} + \lambda_1 = 0$$

Assume $f_1(x,y) = c_1$ satisfies above eq^n

General sol^n

$F(c_1, c_2) = 0 \Rightarrow$

$F(f_1(x,y), \xi) = 0$

$\Rightarrow \xi = G_1(f_1(x,y))$

In particular, the simplest one is

$$\boxed{\xi = f_1(x,y)}$$

$\eta_x - \lambda_2 \eta_y = 0$

The sol^n of this PDE is :

$$\boxed{\eta = f_2(x,y)}$$

where $f_2(x,y)$ is sol^n of $\frac{dy}{dx} + \lambda_2 = 0$

Now, η & ξ are known, so we can find the canonical form.

Ex. $u_{xx} = x^2 u_{yy}$ (Find canonical form)

$A = 1$ $B = 0$ $C = -x^2$

$B^2 - 4AC = 0 + 4x^2 > 0 \quad \forall \quad x \neq 0$

$\Rightarrow$ Hyperbolic Type

Consider eq^n :

$$1\alpha^2 + 0 + (-x^2) = 0$$

$$\alpha^2 - x^2 = 0$$

$$\Rightarrow \cancel{x = \pm 1} \qquad \lambda_1(x,y) = +x$$

$$\Rightarrow \cancel{\lambda_1, \lambda_2 = \pm 1} \qquad \lambda_2(x,y) = -x$$

Choose ξ & η s.t.

$$\xi_x = \lambda_1 \xi_y \quad \& \quad \cancel{\eta_x = \lambda_2 \eta_y}$$

$$\Rightarrow \xi_x - x\,\xi_y = 0$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{-x} = \frac{d\xi}{0}$$

$+\lambda_2$

$$\Rightarrow \frac{dy}{dx} + x = 0 \quad \& \quad \frac{dy}{dx} - x = 0$$

$$\Rightarrow \underbrace{y + \frac{x^2}{2}}_{f_1} = C_1 \quad \& \quad \underbrace{y - \frac{x^2}{2}}_{f_2} = C_2$$

$$\xi(x,y) = y + \frac{x^2}{2} \qquad \eta(x,y) = y - \frac{x^2}{2}$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$= u_\xi (x) + u_\eta (-x) = x(u_\xi - u_\eta)$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\eta_x\xi_x + u_\xi \xi_{xx} + u_{\eta\eta}\eta_x^2 + u_\eta \eta_{xx}$$

$$= u_{\xi\xi}(x^2) + 2u_{\xi\eta}(-x^2) + u_\xi(1) + u_{\eta\eta}(x^2) + u_\eta(-1)$$

$$u_{xx} = x^2\left[u_{\xi\xi} + u_{\eta\eta} - 2u_{\xi\eta}\right] + \left[u_\xi - u_\eta\right]$$

Similarly,

$$\boxed{u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}}$$

Put in eq^n ①

$$A \quad u_{xx} = x^2 u_{yy}$$

$$\Rightarrow x^2 \left[\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}} \right] + \left[u_\xi - u_\eta \right] = x^2 \left[\cancel{u_{\xi\xi}} + 2u_{\eta\xi} + \cancel{u_{\eta\eta}} \right]$$

Checkpoint : coeff. of $u_{\xi\xi}$ & $u_{\eta\eta}$ should be 0 in case of hyperbolic PDE.

$$\Rightarrow \quad 4u_{\xi\eta} x^2 = u_\xi - u_\eta$$

$$\Rightarrow \quad u_{\xi\eta} = \frac{u_\xi - u_\eta}{4x^2} = \boxed{\frac{u_\xi - u_\eta}{4(\xi - \eta)}}$$

canonical form.

$$= Q(\xi, \eta, u, u_\xi, u_\eta)$$

Q) Parabolic PDE : $B^2 - 4AC = 0$ (Heat Eq^n)

$A\alpha^2 + B\alpha + C = 0$: Repeated Real roots $= \lambda(x, y)$

Choose $\xi(x, y)$ s.t.

$$\frac{\partial \xi}{\partial x} = \lambda \frac{\partial \xi}{\partial y} \quad \left[\text{makes } \bar{A} = 0 \right]$$

This choice of ξ makes the coeff. of $u_{\xi\xi}$ as 0.

$$\bar{A}(\xi_x ; \xi_y) = \xi_y^2 \underbrace{(A\lambda^2 + B\lambda + C)}_{=0}$$

Choose $\eta(x, y)$ s.t. (ξ & η should be independent func^{n}s)

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0 \quad \text{OR} \quad \nabla\xi \times \nabla\eta \neq 0$$

here $\bar{A}(\eta_x, \eta_y)$ may not be equal to 0

From eq^n ⊛,

$$(\bar{B}^2 - 4\bar{A}\bar{C}) = 0$$

$$\bar{B}^2(\quad) - 4\bar{A}(\quad) \ldots = 0 \Rightarrow \bar{B}^2 = 0$$

$\Rightarrow \quad \bar{B} = 0$

Using ②, the canonical form is reduced to
$\bar{A}\underbrace{(\eta_x, \eta_y)}_{\text{non-0}} u_{\eta\eta} = G(\xi, \eta, \dots)$

$$\Rightarrow \quad \boxed{u_{\eta\eta} = Q(\xi, \eta, u, u_\xi, u_\eta)}$$

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Ques. Find canonical form of (solve it iff possible)

$u_{xx} + 2u_{xy} + u_{yy} = 0 \qquad A = 1 \quad B = 2 \quad C = 1$

$B^2 - 4AC = 4 - 4 = 0 \Rightarrow$ parabolic for all x and y

$\alpha^2 + 2\alpha + 1 = 0$

$\Rightarrow (\alpha + 1)^2 = 0$

$\Rightarrow \alpha = -1$

Choose $\xi(x, y)$ s.t.

$\Rightarrow \dfrac{\partial \xi}{\partial x} = (-1)\dfrac{\partial \xi}{\partial y}$

$\Rightarrow \xi_x + \xi_y = 0$: Linear 1st order PDE

$\dfrac{dy}{dx} + \lambda = 0$

$\Rightarrow \dfrac{dy}{dx} - 1 = 0$

$\Rightarrow \boxed{y - x = c_1}$

$\Rightarrow \xi(x, y) = y - x$

$\dfrac{dx}{1} = \dfrac{dy}{1}$

$x - y = c$.

$\eta(x, y) = ??$

$\eta_x = \lambda \eta_y - X$

Choose $\eta(x, y)$ s.t.
$\dfrac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$

Let $\eta(x, y) = x + y$ (can choose any value)

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$\xi = y - x$

$\xi_x = -1 \qquad \xi_{xx} = 0 \qquad \xi_{xy} = 0$

$\xi_y = 1 \qquad \xi_{yy} = 0$

$\eta(x,y) = x + y$

$\eta_x = 1$ all other $= 0$

$\eta_y = 1$

$$u_{xx} = u_{\xi\xi}\,\xi_x^2 + 2u_{\xi\eta}\,\eta_x \xi_x + u_{\eta\eta}\,\eta_x^2 + \cancel{u_\eta \eta_{xx}} + \cancel{u_\xi \xi_{xx}}$$

$$= u_{\xi\xi}(1) + 2u_{\xi\eta}(-1) + u_{\eta\eta}$$

$$u_{xy} = u_{\xi\xi}(-1) + 0 + u_{\xi\eta}(-1) + u_{\eta\xi}(1) + u_{\eta\eta}$$

$$= -u_{\xi\xi} - \cancel{u_{\xi\eta}} + \cancel{u_{\eta\xi}} + u_{\eta\eta}$$

$$= u_{\eta\eta} - u_{\xi\xi}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

substitute in given problem :

$$\Rightarrow u_{\xi\xi} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}} + 2\left[\cancel{u_{\eta\eta}} - u_{\xi\xi}\right] + u_{\xi\xi} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}} = 0$$

$$\Rightarrow 4u_{\eta\eta} + \underbrace{0\,(u_{\xi\xi}) + 0\,(u_{\eta\xi})}_{} = 0$$

In case of parabola, these should be 0. (checkpoint)

$\eta = x+y$

$\xi = y-x$

or $\boxed{u_{\eta\eta} = 0} \rightarrow$ Canonical form

$\Rightarrow u_\eta = f(\xi)$

$\Rightarrow u(\xi,\eta) = \int f(\xi)\,d\eta + g(\xi)$

$u(\xi,\eta) = f(\xi)\,\eta + g(\xi)$

$$\boxed{u(x,y) = f(y-x)(x+y) + g(y-x)}$$

Here, we are able to get explicit form.

If we check from here,

$u_x =$

$u_{yy} =$

⋮

this will satisfy parabolic eqn.

For same eg, take $\eta(x,y) = x$

$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0 \Rightarrow$ Find corresponding soln.

3) Elliptic PDE : $B^2 - 4AC < 0$

Consider : $A\alpha^2 + B\alpha + C = 0$

Suppose $\lambda_1(x,y)$ and $\lambda_2(x,y)$: ~~two~~ distinct imaginary roots

$\Rightarrow \lambda_1 = \bar{\lambda}_2$

Choose $\xi(x,y)$ and $\eta(x,y)$ s.t.

$\xi_x = \lambda_1 \xi_y$ and $\eta_x = \lambda_2 \eta_y$

$\Rightarrow \bar{A}(\xi_x, \xi_y) = 0 = \bar{A}(\eta_x, \eta_y)$

$\Rightarrow$ Canonical form is :

$$u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{— (1)}$$

complex canonical form

Since $\xi(x,y)$ and $\eta(x,y)$ are complex characteristic curve and we want a real ~~form~~ curve.

(We can add and subtract to get real curve using the superposition principle)

$\xi(x,y) = \alpha + i\beta$ and $\eta(x,y) = \alpha - i\beta$

[ξ & η are conjugate bcoz λ_1 & λ_2 are conjugate]

$\alpha(\xi,\eta) = \alpha = \frac{1}{2}(\xi + \eta) \qquad \beta = \frac{1}{2i}(\xi - \eta) = \beta(\xi,\eta)$

α & β are two real characteristic curves

We've to get real canonical form in (α, β) form

$$u(x,y) \longrightarrow \underbrace{u(\xi,\eta) \longrightarrow u(\alpha,\beta)}_{\text{Using this part:}}$$

$$u_\xi = u_\alpha \alpha_\xi + u_\eta \beta_\eta = \frac{1}{2} u_\alpha + \frac{1}{2i} u_\beta$$

$$u_\eta = u_\alpha \alpha_\eta + u_\beta \beta_\eta = \frac{1}{2} u_\alpha - \frac{1}{2i} u_\beta$$

$$u_{\xi\eta} = \frac{1}{2}\left(u_{\alpha\alpha}\alpha_\eta + u_{\alpha\beta}\beta_\eta\right) + \frac{1}{2i}\left(u_{\beta\alpha}\alpha_\eta + u_{\beta\beta}\beta_\eta\right)$$

$$= \frac{1}{2}\left(u_{\alpha\alpha}\cdot\frac{1}{2} + u_{\alpha\beta}\left(\frac{-1}{2i}\right)\right) + \frac{1}{2i}\left(u_{\beta\alpha}\cdot\frac{1}{2} + u_{\beta\beta}\left(\frac{-1}{2i}\right)\right)$$

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$= \frac{1}{4} u_{\alpha\alpha} - \frac{1}{4i} \cancel{u_{\alpha\beta}} + \frac{1}{4i} \cancel{u_{\beta\alpha}} + \frac{1}{4} u_{\beta\beta}$

$\Rightarrow \boxed{u_{\xi\eta} = \frac{1}{4} (u_{\alpha\alpha} + u_{\beta\beta})}$

The real canonical form is :

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \psi (\alpha, \beta, u, u_\alpha, u_\beta)}$$

Ex. $u_{xx} + x^2 u_{yy} = 0$

$A = 1 \quad B = 0 \quad C = x^2$

$B^2 - 4AC = -4x^2 < 0 \quad \forall \; x \neq 0 \Rightarrow$ Elliptic for all $x \neq 0$

$\lambda^2 + x^2 = 0$

$\lambda = \pm ix \Rightarrow \lambda_1 = ix \qquad \lambda_2 = -ix$

Choose $\xi(x, y)$ and $\eta(x,y)$ s.t.
~~$\xi_x = ix \xi_y$~~

The 2 associated ODE are :

$\frac{dy}{dx} + ix = 0 \quad \& \quad \frac{dy}{dx} - ix = 0$

$\Rightarrow y + i\frac{x^2}{2} = C_1 \quad \& \quad y - i\frac{x^2}{2} = C_2$

$\xi(x,y) = y + i\frac{x^2}{2} \quad \& \quad \eta(x,y) = y - i\frac{x^2}{2}$

$\alpha(\xi, \eta) = \frac{1}{2}(2y) = y$ (real part) $\qquad \beta(\xi, \eta) = \frac{x^2}{2}$ (Imaginary part)

directly get $u(x,y) \longrightarrow u(\alpha, \beta)$

$u_x = u_\alpha \cancel{\alpha_x} + u_\beta \beta_x = u_\beta x$

$u_{xx} = 1 \cdot u_\beta + x(\cancel{u_{\beta\alpha} \alpha_x} + u_{\beta\beta} \beta_x)$

$= u_\beta + x^2 u_{\beta\beta}$

$u_y = u_\alpha \alpha_y + \cancel{u_\beta \beta_y} = u_\alpha$

$u_{yy} = u_{\alpha\alpha} \alpha_y + \cancel{u_{\alpha\beta} \beta_y} = u_{\alpha\alpha}$

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substitute :

$\Rightarrow \quad u_\beta + x^2 u_{\beta\beta} + x^2 u_{\alpha\alpha} = 0$

$\Rightarrow \quad \boxed{x^2 (u_{\beta\beta} + u_{\alpha\alpha}) + u_\beta = 0}$ ~~Canonical form.~~

$x^2 = 2\beta$ (write all x and y in terms of α & β)

$\Rightarrow \quad 2\beta (u_{\alpha\alpha} + u_{\beta\beta}) + u_\beta = 0$

$\Rightarrow \quad \boxed{u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\beta}{2\beta}}$ Canonical

* If $u_{xx} + 4u_{yy} = 0$ is given, it is already in canonical form. so, no need to proceed further.

Same has to be applied in case of hyperbolic & parabolic.

Canonical form.

* In elliptic, we can't get soln (can't integrate $u_{\alpha\alpha}$ & $u_{\beta\beta}$) form So, they are of not help.

eg. $u_{xx} - 2(\sin x)u_{xy} - (\cos^2 x) u_{yy} - u_y \cos x = 0$

$B = -2\sin x \qquad A = 1 \qquad C = -\cos x$

$B^2 - 4AC = 4\sin^2 x + 4\cos^2 x = 4 > 0 \Rightarrow$ Hyperbolic

$\alpha^2 - (2\sin x)\alpha + (-\cos^2 x) = 0$

$\alpha = \sin x \pm 1$

$\lambda_1 = \sin x + 1 \qquad \lambda_2 = \sin x - 1$

$\frac{dy}{dx} + (\sin x + 1) = 0$ & $\frac{dy}{dx} + (\sin x - 1) = 0$

$\Rightarrow \quad y - \cos x + x = c_1$ & $y - \cos x - x = c_2$

$\xi = y - \cos x + x \qquad \eta = y - \cos x - x$

$u(x,y) \to u(\xi, \eta)$

Find $u_{xx}, u_{yy}, u_{xy}, u_y$, will get $\boxed{u_{\eta\xi} = 0}$

$u_y = u_\xi \xi_y + u_\eta \eta_y$

$= u_\xi (1) + u_\eta (1) = u_\xi + u_\eta$

$u_\eta = f(\eta)$

$u = \eta f(\xi) + g(\eta)$

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$$= \frac{1}{4} u_{\alpha\alpha} - \frac{1}{4i} \cancel{u_{\alpha\beta}} + \frac{1}{4i} \cancel{u_{\beta\alpha}} + \frac{1}{4} u_{\beta\beta}$$

$$\Rightarrow \boxed{u_{\xi\eta} = \frac{1}{4} (u_{\alpha\alpha} + u_{\beta\beta})}$$

The real canonical form is:

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \psi (\alpha, \beta, u, u_\alpha, u_\beta)}$$

Ex. $u_{xx} + x^2 u_{yy} = 0$

$A = 1 \quad B = 0 \quad C = x^2$

$B^2 - 4AC = -4x^2 < 0 \quad \forall \; x \neq 0 \Rightarrow$ Elliptic for all $x \neq 0$

$\lambda^2 + x^2 = 0$

$\lambda = \pm ix \Rightarrow \lambda_1 = ix \qquad \lambda_2 = -ix$

~~Choose $\xi(x,y)$ and $\eta(x,y)$ s.t. $\xi_x = ix\,\xi_y$~~

The 2 associated ODE are:

$$\frac{dy}{dx} + ix = 0 \quad \& \quad \frac{dy}{dx} - ix = 0$$

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$$= u_\beta + x^2 u_{\beta\beta}$$

$$u_y = u_\alpha \alpha_y^{1} + \cancel{u_\beta \beta_y}^{0} = u_\alpha$$

$$u_{yy} = u_{\alpha\alpha} \alpha_y + \cancel{u_{\alpha\beta}\beta_y}^{0} = u_{\alpha\alpha}$$

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substitute:

$\Rightarrow u_\beta + x^2 u_{\beta\beta} + x^2 u_{\alpha\alpha} = 0$

$\Rightarrow \boxed{x^2 (u_{\beta\beta} + u_{\alpha\alpha}) + u_\beta = 0}$ ~~Canonical~~ form.

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anonical form.

So, they are of not help.

Grad

eg. $u_{xx} - 2(\sin x)u_{xy} - (\cos^2 x) u_{yy} - u_y \cos x = 0$

$B = -2\sin x \qquad A = 1 \qquad C = -\cos^2 x$

$B^2 - 4AC = 4\sin^2 x + 4\cos^2 x = 4 > 0 \Rightarrow$ Hyperbolic

$\alpha^2 - (2\sin x)\alpha + (-\cos^2 x) = 0 \qquad \alpha = \dfrac{2\sin x \pm \sqrt{4\sin^2 x + 4\cos^2 x}}{2}$

$\alpha = \sin x \pm 1$

$A_1 = \sin x + 1 \qquad A_2 = \sin x - 1$

$\dfrac{dy}{dx} + (\sin x + 1) = 0$ & $\dfrac{dy}{dx} + (\sin x - 1) = 0$

$\Rightarrow y - \cos x + x = C_1$ & $y - \cos x - x = C_2$

$\xi = y - \cos x + x \qquad \eta = y - \cos x - x$

$u(x,y) \to u(\xi, \eta)$

Find $u_{xx}, u_{yy}, u_{xy}, u_y$, will get $\boxed{u_{\eta\xi} = 0}$

$u_\eta = f(\eta)$

$u_\xi = \eta f(\xi) + g(\eta)$

$u_y = u_\xi \xi_y + u_\eta \eta_y$

$= u_\xi (1) + u_\eta (1) = u_\xi + u_\eta$

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$u_y = u_\xi + u_\eta$

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y$

$= \underline{u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}}$

$= u_\xi \sin x + u_\eta \sin x$

$u_{xx} = u_\xi \xi_x + u_\eta \eta_x = u_\xi(\sin x + 1) + u_\eta(\sin x - 1)$

$u_{xx} = \cos x\, u_\xi + u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + \cos x\, u_\eta + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x$

$= \cos x(u_\xi + u_\eta) + u_{\xi\xi}(\sin x + 1) + u_{\eta\eta}(\sin x - 1) + u_{\xi\eta}[$

$\cancel{u_{yx} = u_{\xi x}}$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$= u_\xi(\sin x + 1) + u_\eta(\sin x - 1)$

$= (u_\xi + u_\eta)\sin x + (u_\xi - u_\eta)$

$u_{xx} = \cos x(u_\xi + u_\eta) + \sin x(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x) + (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x - u_{\eta\xi}\xi_x - u_{\eta\eta}\eta_x)$

$= \cos x(u_\xi + u_\eta) + \sin x[u_{\xi\xi}(\sin x + 1) + u_{\xi\eta}(\sin x - 1) + u_{\xi\eta}(\sin x + 1) + u_{\eta\eta}(\sin x - 1)] + [u_{\xi\xi}(\sin x + 1) + u_{\xi\eta}(\sin x - 1) - u_{\xi\eta}(\sin x + 1) - u_{\eta\eta}(\sin x - 1)]$

$= \cos x(u_\xi + u_\eta) + \sin x[u_{\xi\xi}(\sin x + 1) + u_{\eta\eta}(\sin x - 1) + 2u_{\xi\eta}\sin x] + (u_{\xi\xi}(\sin x + 1) + u_{\eta\eta}(\sin x - 1) - 2u_{\eta\xi})$

$u_{xy} = \cancel{(\sin x + 1)}[u$

$(\sin x + 1)[u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y \cancel{+ u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y}]$

$+ (\sin x - 1)[u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y]$

Eq^n becoms :

$u_{xx} - 2\sin x\,(u_{xy}) - (\cos^2 x)u_{yy} - u_y \cos x = 0$

13/11/17 **One-dimensional wave eqⁿ**

Vibration on a infinite string :

eqⁿ : $u_{tt} = c^2 u_{xx}$, $-\infty < x < \infty$, $t > 0$ $(c > 0)$

IC : ① — $u(x,0) = f(x)$

② — $u_t(x,0) = g(x)$ $\Big\}$ $-\infty < x < \infty$

$f(x)$ is the initial position of the string

$g(x)$ is the initial velocity of the string at x.

$A = 1$, $B = 0$, $C = -c^2$

$B^2 - 4AC = +4c^2 > 0 \ \forall\ c \neq 0 \Rightarrow$ Hyperbolic PDE

$A\alpha^2 + B\alpha + C = 0$

$\Rightarrow \alpha^2 - c^2 = 0 \Rightarrow \alpha = \pm c$

$\lambda_1 = c$ $\quad \lambda_2 = -c$ $\quad \boxed{\lambda(t,x)}$

choose ξ and η s.t. :

$\frac{dx}{dt}\ \cancel{\frac{dy}{dx}} + c = 0$ & $\frac{dy}{dx} - c = 0$

$\Rightarrow \boxed{x\ \cancel{y}\ \cancel{t} - ct = \xi}$ $\quad \boxed{x\ \cancel{y} + ct = \eta}$ (let us choose in this way)

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$u(x,y) \longrightarrow u(\xi,\eta)$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x$

$u_t = u_\xi \xi_t + u_\eta \eta_t = c u_\eta - c u_\xi$

$u_{tt} = c\left[u_{\eta\xi}\xi_t + u_{\eta\eta}\eta_t - u_{\xi\xi}\xi_t - u_{\xi\eta}\eta_t\right]$

$$\boxed{\text{The canonical form is: } u_{\eta\xi} = 0}$$

$-2c^2 u_{\eta\xi} + c^2 u_{\eta\eta} + c^2 u_{\xi\xi} = c^2 u_{\xi\xi} + c^2 u_{\eta\eta} + 2u_{\eta\xi} c^2$

$u_{\xi\eta} = 0$

$\Rightarrow u_\xi = f(\xi)$

$\Rightarrow u = F(\xi) + G(\eta)$

$u(x,t) = F(x-ct) + G(x+ct)$

F & G are arbitrary, smooth funcⁿ

Using initial condⁿs,

$u(x,0) = f(x) \Rightarrow$

$u(x,0) = F(x) + G(x) = f(x)$ —— (3)

$u_t(x,t) = -cF'(x-ct) + cG'(x+ct)$

$u_t(x,0) = -cF'(x) + cG'(x) = g(x)$ —— (4)

Integrating (4) wrt x for x_0 to x, we get

$-cF(x) + cG(x) = \int_{x_0}^{x} g(s)\,ds$ —— (5)

Solving for $F(x)$ and $G(x)$, we get

$F(x) = \frac{1}{2c}\left[cf(x) - \int_{x_0}^{x} g(s)\,ds\right]$

$G(x) = \frac{1}{2c}\left[cf(x) + \int_{x_0}^{x} g(s)\,ds\right]$

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Therefore,

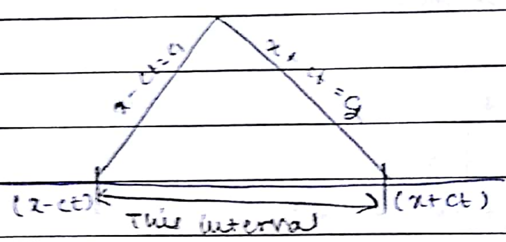
$$u(x,t) = \frac{1}{2c}\Big[\underbrace{c\,f(x-ct) - \int_{x_0}^{x-ct} g(s)\,ds}_{F(x-ct)} + \underbrace{c\,f(x+ct) + \int_{x_0}^{x+ct} g(s)\,ds}_{G(x+ct)}\Big]$$

$$\boxed{u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

De-Alembert solⁿ for 1-D wave Eqⁿ

* $f \in C^2$ (twice differentiable), $g \in C^1$ s.t. $u \in C^2$ will be the regular solⁿ of wave eqⁿ.

$\Longrightarrow$ $x - ct = c_1$ and $x + ct = c_2$



$[x-ct, x+ct]$: domain of dependence for (x,t)

PROPERTIES :-

1.) If $f(x)$ and $g(x)$ are odd funcⁿ, then solⁿ $u(x,t)$ is also an odd funcⁿ (Prove $u(-x,t) = -u(x,t)$)

2.) If $f(x)$ and $g(x)$ are even funcⁿ, then $u(x,t)$ is also even ie $u(-x,t) = u(x,t)$

3.) Periodic initial data yield periodic solⁿ

$f(x+2L) = f(x) \quad \forall x$

$g(x+2L) = g(x) \quad \forall x$

then,

$u(x+2L, t) = u(x,t)$

Infact, $u\left(x, t + \frac{2L}{c}\right) = u(x,t)$ [In this case]

SPECIAL CASE

(i) Initial position of string is at rest

$f(x) = 0$

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds$$

(ii) Initial velocity at x is 0

$g(x) = 0$

then $u(x,t) = \frac{1}{2} \left[f(x-ct) + f(x+ct) \right]$

Eg. $u_{tt} = c^2 u_{xx}$, $x \in R$, $t > 0$

(i) IC : $u(x,0) = \sin x$ $x \in \mathbb{R}$

$u_t(x,0) = 0$

$$u(x,t) = \frac{1}{2} \left[\sin(x-ct) + \sin(x+ct) \right]$$

$$= \frac{1}{2} \left[2 \sin x \cos ct \right]$$

(ii.) ~~Eg.~~ $u(x,0) = 0$ $u_t(x,0) = \sin x$

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \cancel{g(s)} \sin x \, dx$$

$$= \frac{1}{2c} \left[\cos(x-ct) + \cos(x+ct) \right]$$

Vibration on semi infinite string

$u_{tt} = c^2 u_{xx}$ $0 < x < \infty$, $t > 0$, $c > 0$

IC : $u(x,0) = f(x)$

$u_t(x,0) = g(x)$

Zero Condn (boundary condn) $u(0,t) = 0$ $t > 0$

* If we use De-Alembert sol^n for $x - ct < 0 \Rightarrow t > c/x$, then $f(x-ct)$ will become $-ve$, but we have $f(x)$ $\forall$ $x \geq 0$

" It is meaningless for $t > x/c$

Re-write IC's as below

$$\left.\begin{aligned} u(x,0) &= F(x) \\ u_t(x,0) &= G(x) \end{aligned}\right\} \quad -\infty < x < \infty$$

where

$$F(x) = \begin{cases} f(x) & 0 < x < \infty \\ -f(-x) & -\infty < x < 0 \end{cases}$$

$$G(x) = \begin{cases} g(x) & 0 < x < \infty \\ -g(-x) & -\infty < x < 0 \end{cases}$$

$F(x)$ and $G(x)$ are odd ~~exeng~~ extensions of $f(x)$ & $g(x)$

Now, De-Alembert sol^n becomes

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

We've to verify the above sol^n satisfies ICs. (original)

i) At $t = 0$

~~let~~ $u(x,0) = \frac{1}{2}\left[F(x) + F(x)\right] + \frac{1}{2c}\cancel{\int_x^x G(s)\,ds}^{\,0}$

$= F(x) = f(x)$ $[x > 0]$

ii) diff. wrt t, we get

$$u_t(x,t) = \frac{1}{2}\left[-c\,F'(x-ct) + c\,F'(x+ct)\right]$$

$$+ \frac{1}{2c}\left[c\,G(x+ct) + c\,G(x-ct)\right]$$

Leibnitz Rule

{Using : $\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)\,dx = b'(t)\,f(b(t)) - a'(t)\,f(a(t))$ }

$$u_t(x,0) = \frac{1}{2}\left[-cF'(x) + cF'(x)\right] + G(x)$$
$$= G(x) = g(x) \quad (x>0)$$

iii.) $u(0,t) = \frac{1}{2}\left[F(-ct) + F(ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} G(s)\,ds$

$$= \frac{1}{2}\left[-F(ct) + F(ct)\right] + 0 = 0 \quad \left[F \,\&\, G : \text{odd func}^n\text{s}\right]$$

SPECIAL CASE :

i.) initial velocity $g(x) = 0$

$\Rightarrow G(x) = 0$

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] \quad \forall\ x \in \mathbb{R}$$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] & x > ct \\ \frac{1}{2}\left[-f(-x-ct) - f(x+ct)\right] & x < ct \end{cases}$$

$\Downarrow$

$\frac{1}{2}\left[f(x+ct) - f(ct-x)\right]$

↳ defined for $x < ct$

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Vibration on finite string

(1) → $u_{tt} = c^2 u_{xx} \quad 0 < x < l, \quad t > 0$

$u(x,0) = f(x)$; & $u_t(x,0) = g(x)$, $0 < x < l$

$u(0,t) = u_t(0,t) = 0$ } $t > 0$

$u(l,t) = u_t(l,t) = 0$

Consider the ICs :

$u(x,0) = F(x)$, $u_t(x,0) = G(x)$, $-l < x < l$

$$F(x) = \begin{cases} f(x) & 0<x<l \\ -f(-x) & -l<x<0 \end{cases} \qquad G(x) = \begin{cases} g(x) & 0<x<l \\ +g(-x) & -l<x<0 \end{cases}$$

F & G are odd extension of $f(x)$ & $g(x)$

$$\left.\begin{aligned} F(x+2l) &= F(x) \quad \forall x \\ G(x+2l) &= G(x) \quad \forall x \end{aligned}\right\} \text{// Make F and G periodic}$$

F & G are periodic func^n ⇒ can be represented as ~~FS~~ a Fourier series.

Therefore, $F(x)$ and $G(x)$ Fourier sine expansions :
(odd func^n)

~~$F(x) = \frac{2}{l}\int_0^l F(x)$~~

$$F(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{l}x\right) \; ; \; G(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$$

$$a_n = \frac{2}{l}\int_0^l \underbrace{\cancel{F(x)}}_{f(x)} \sin\left(\frac{n\pi}{l}x\right)dx \; ; \; b_n = \frac{2}{l}\int_0^l \underbrace{\cancel{G(x)}}_{g(x)} \sin\left(\frac{n\pi}{l}x\right)dx$$

Since
$F(x) = f(x)$. $0<x<l$
$G(x) = g(x)$ $0<x<l$

The De-Alembert sol^n, now, will be :

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} a_n\left(\sin\left(\frac{n\pi}{l}(x-ct)\right) + \sin\left(\frac{n\pi}{l}(x+ct)\right)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi s}{l}\right)ds$$

$F(x)$ & $G(x)$ should be periodic and $F, G \in C^2$ for :

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + \frac{1}{2c}\sum_{n=1}^{\infty} b_n\left[\left(-\cos\frac{n\pi s}{l}\right)*\frac{l}{n\pi}\right]_{x-ct}^{x+ct}$$

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$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + \frac{l}{\pi c}\sum_{n=1}^{\infty} \frac{b_n}{n} \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)}$$

Condⁿs required for fourier ~~sees~~ series existence:

$\{ f \in C^2$, f''' is piecewise continuous
$g(x) \in C^1$ & g'' " "

then $u \in C^2$ is regular solⁿ of wave eqⁿ

Separation of variables method (Alternative method)
For above problem:

$u(x,t) = X(x)\, T(t)$: solⁿ of PDE (1)
$\Rightarrow$ It satisfies PDE eqⁿ (1)

$u_t(x,t) = X(x)\cdot \dot{T}$, $u_{tt} = X\ddot{T}$
$u_x = X'T$ $u_{xx} = X''T$

Satisfying in PDE:

$$X\ddot{T} = c^2 X'' T$$

$$\underbrace{\frac{X''}{X}}_{\text{funcⁿ of } X} = \underbrace{\frac{\ddot{T}}{c^2 T}}_{\text{funcⁿ of } T} = \lambda$$

$\Rightarrow X'' - \lambda X = 0$, $\ddot{T} - \lambda c^2 T = 0$ (non-trivial solⁿs are required)

To find above values, we should have 2 boundary condⁿs

$u(0,t) = 0$
$\Rightarrow X(0)\,T(t) = 0$ But $T \neq 0$ (If $T=0$, it'll lead to trivial solⁿ)
$\Rightarrow X(0) = 0$ —— (1)
$u(l,t) = 0$
$\Rightarrow X(l)\,T(t) = 0$
$\Rightarrow X(l) = 0$ —— (2)

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Case-I : $\lambda > 0$, $\lambda = +n^2$

$\cancel{m^2}$ $\cancel{n^3}$

$m^2 - n^2 = 0 \Rightarrow m = \pm n$

$\cancel{\lambda^2 + \lambda + \alpha^2}$

$X(x) = c_1 e^{nx} + c_2 e^{-nx}$ — general solⁿ

$X(0) \Rightarrow c_1 + c_2 = 0$
$X(l) \Rightarrow c_1 e^{nl} + c_2 e^{-nl} = 0$ $\Big\}$ $c_1 = c_2 = 0$

This gives trivial solⁿ, we don't need this

Case-II : $\lambda = 0$

$X'' = 0$

$X = A + Bx$

$X(0) = \cancel{c_2} = 0$
$X(l) \Rightarrow c_1 \cancel{l} \cancel{+ c_2} = 0$ $\Big\}$ $c_1 = c_2 = 0$

Again, we get a trivial solⁿ

Case-III : $\lambda < 0$, $\lambda = -n^2$

$m^2 + n^2 = 0$

$m = \pm in$

$X = A \cos nx + B \sin nx$

$X(0) = A = 0$

$X(l) = \cancel{A \cos} B \sin nl = 0$ $B \neq 0$
(if $B = 0 \Rightarrow$ trivial solⁿ)

$\Rightarrow \sin nl = 0$

$nl = K\pi$, $K \in \mathbb{Z}$

$\boxed{n = \dfrac{K\pi}{l}}$, $K \in \mathbb{Z}$

$\lambda = -n^2 = -\dfrac{K^2\pi^2}{l^2} = \cancel{-\dfrac{n^2\pi^2}{l^2}}$, $\cancel{n = 1, 2, 3, \ldots}$

eigen value : $\lambda_K = -\dfrac{K^2\pi^2}{l^2}$, $K = 1, 2, 3 \ldots$
(For eigen value, we take only +ve K)

Eigen funcⁿ : $X_K(x) = B_K \sin\left(\dfrac{K\pi x}{l}\right)$

Putting in $\ddot{T} - \lambda c^2 T = 0$

$$\ddot{T}_n + \frac{n^2\pi^2 c^2}{l^2} T_n = 0$$

$$m^2 + \left(\frac{n^2\pi^2 c^2}{l^2}\right) = 0$$

$$m = \pm i \left(\frac{n\pi c}{l}\right)$$

$$T_n(t) = C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right) \quad \text{- general sol}^n$$

$$u_n(x,t) = X_n(x) \cdot T_n(x) \quad n = 1, 2, \ldots$$

$$= \left[a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right)$$

Superposition principle implies :

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi}{l} ct\right)\right] \sin\left(\frac{n\pi x}{l}\right)$$

IC : (to eliminate const)

$$u(x,0) = \underbrace{\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)}_{\Downarrow} = f(x)$$

Looks like ~~fourier sine series of f(x)~~

so, we should choose a_n s.t. f(x) becomes fourier seri

Choose $u(x,0)$ s.t. a_n become Fourier sine ~~coeff. of f(x)~~

$$\boxed{a_n = \frac{2}{l} \int f(x) \sin\left(\frac{n\pi x}{l}\right) dx}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-a_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi ct}{l}\right) + b_n \left(\frac{n\pi c}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \underbrace{b_n \left(\frac{n\pi c}{l}\right)}_{\text{This should be coeff. of fourier sine series}} \sin\left(\frac{n\pi x}{l}\right) = g(x)$$

$$\Rightarrow b_n\left(\frac{n\pi c}{l}\right) = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

($\Rightarrow$ Choose $u_t(x,0)$ s.t b_n becomes Fourier ~~sine~~ series coeff. of $g(x)$)

$$\Rightarrow b_n = \left(\frac{2}{\cancel{l}}\right)\left(\frac{\cancel{l}}{n\pi c}\right)\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\Rightarrow \boxed{b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx}$$

Ex. $u_{tt} - c^2 u_{xx} = 0$ $\quad 0 < x < 2\pi \quad , \quad t > 0$

$$\left.\begin{aligned} u(x,0) &= \cos x - 1 = f(x) \\ u_t(x,0) &= 0 = g(x) \end{aligned}\right\} \quad 0 \leq x \leq 2\pi$$

$$\left.\begin{aligned} u(2\pi, t) &= 0 \\ u(0, t) &= 0 \end{aligned}\right\} \quad t \geq 0$$

Find sol^n using De-Alembert's sol^n & method of separation variable

$g(x) = 0$

$\Rightarrow b_n = 0$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$u(x,t) = \sum a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + \frac{l}{\pi c}\sum_{n=1}^{\infty}\frac{b_n}{n}\sin\frac{n\pi x}{l}\sin\left(\frac{n\pi ct}{l}\right)$$

Non-Homogenous PDE :

1) with Homogenous ICs :

① — PDE : $u_{tt} + c^2 u_{xx} = F(x,t)$ $\qquad x \in \mathbb{R},\ t > 0$

② — ICs : $u(x,0) = 0$, $u_t(x,0) = 0$, $x \in \mathbb{R}$

We know the sol^n for :

$$\begin{cases} \text{PDE} : u_{tt} + c^2 u_{xx} = 0 \quad ; \quad x \in \mathbb{R},\ t > 0 \\ \text{ICs} : u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

Consider

$v(x,t;s)$ satisfies the following problem

$v_t - c^2 v_{xx} = 0$ —③, $x \in \mathbb{R}$, $s > t > 0$ boundary $cond^n$.

$$\left.\begin{aligned} v(x,s;s) &= 0 \\ v_t(x,s;s) &= F(x,s) \end{aligned}\right\} \text{—④}$$

Initial data is given at $t = s$ (instead of $t = 0$)

Using De-Alembert's sol^n for PDE ③-④ :

$$v(x,t;s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} F(r,s)\, dr$$

(1st term will be cancelled)

We define $u(x,t) = \int_0^t v(x,t;s)\, ds$

claim : $u(x,t)$ is sol^n of PDE ① - ②.

Verification :

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t)\, dx = \int_{a(t)}^{b(t)} f_t(x,t)\, dx + b'(t) f(b(t),t) - a'(t) f(a(t),t)$$

at $t=s$, $v(\ ) = 0$ (IC)

$$u_t(x,t) = 1 \cdot \underbrace{\cancel{v(x,t;t)}}_{0} + \int_0^t v_t(x,t;s)\, ds$$

$a(t) = 0$

$b(t) = t$

$$u_{tt}(x,t) = \underbrace{v_t(x,t;t)}_{F(x,t)} + \int_0^t v_{tt}(x,t;s)\, ds$$

$$u_{tt}(x,t) = F(x,t) + \int_0^t c^2 v_{xx}(x,t;s)\, ds$$

$$u_{xx} = \int_0^t v_{xx}(x, t ; s)\, ds$$

$$\Rightarrow \quad u_{tt}(x,t) = F(x,t) + c^2 u_{xx}$$

$\Rightarrow$ $u(x,t)$ is sol^n of PDE ① – ②

Hence Proved.

→ The sol^n of original problem ① – ② is.

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(r,s)\, dr\, ds$$

$$\boxed{u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(r,s)\, dr\, ds}$$

2.) with non-homogenous ICs :

⑤ — IC : $u(x,0) = f(x)$

$u_t(x,0) = g(x)$

Now, we'll use superposition principle.

Find $u_1(x,t)$ s.t.

$$u_{tt} - c^2 u_{xx} = 0 \quad , \quad x \in R, \quad t > 0$$

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x)$$

Find $u_2(x,t)$ s.t.

$$u_{tt} - c^2 u_{xx} = F(x,t) \quad , \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x,0) = 0 \quad , \quad u_t(x,0) = 0$$

$u(x,t) = u_1(x,t) + u_2(x,t)$ satisfies eq^n ① – ② ⑤

$$u_1(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds$$

$$u_2(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(r,s)\, dr\, ds$$

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(7) — $u_{tt} - c^2 u_{xx} = F(x,t)$, $0 < x < l,\ t > 0$

(8) — IC: $u(x,0) = f(x)$
$u_t(x,0) = g(x)$ } $0 \le x \le l$

(9) — $u(0,t) = 0$ $u(l,t) = 0$ (Boundary - condⁿ) $\Rightarrow u_t(0,t) = u_t(l,t) = 0$

Prove that PDE (7) – (9) has unique solⁿ if it exist.

Proof: Assume u_1 and u_2 are 2 solⁿs of (7) - 9

$v(x,t) = u_1 - u_2$

$v(x,t)$ will satisfy

$v_{tt} - c^2 v_{xx} = 0$ gives $v_x(x,0) = 0$

$v(x,0) = 0$, ~~$v_t(0,t) = 0$~~ $v_t(x,0) = 0$.

BC : $v(0,t) = v(l,t) = 0$

$\Rightarrow v_t(0,t) = v_t(l,t) = 0$

$\Rightarrow$ $f(x) = g(x) = 0 \Rightarrow$ De-Alembert's solⁿ will give $v = 0$. But we don't know if it is unique or not.

$\rightarrow$ consider $E(t) = \frac{1}{2}\int_0^l \left(c^2 v_x^2(x,t) + v_t^2(x,t)\right) dx$
(Energy term)

$E(t) > 0 \quad \forall \quad t > 0$

$$\frac{dE}{dt} = \frac{1}{\cancel{2}}\int_0^l \left(\cancel{2}c^2 v_x v_{xt} + \cancel{2} v_t v_{tt}\right) dx$$
↳ Integrate w.r.t by parts

$$= \int_0^l v_t v_{tt}\, dt + \cancel{c^2 v_x v_t \Big|_0^l}^{\,0} - c^2 \int_0^l v_{xx} v_t\, dx$$

$$= \int_0^l v_t \left(\cancel{v_{tt} - c^2 v_{xx}}^{\,0}\right) dx = 0$$

(6)

$\Rightarrow E(t) = \text{const.} \quad \forall \ t$

$$E(0) = \frac{1}{2}\int_0^l \left(c^2 \cancel{v_x^2(x,0)}^{\,0} + \cancel{v_t^2(x,0)}^{\,0}\right) dx$$

(10)

$\Rightarrow E(0) = 0$

$\Rightarrow E(t) = 0 \ \forall \ t$

$V(x,t) = g(t)$
$V_t(x,t) = g'(t) = 0$
$\Rightarrow g(t) = c$

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$E(t) = 0 \Rightarrow V_x(x,t) = 0$ & $V_t(x,t) = 0$ [both terms are positive, should be individually 0]

- Only possible if
$V_{\cancel{xt}}(x,t) = \text{const.}$ $\forall$ $x \in [0,l)$, $t \geq 0$

we know $V(x,0) = 0$

$\Rightarrow$ $\boxed{V(x,t) = 0}$

$\Rightarrow$ $u_1 = u_2$

Eg $u_{tt} = u_{xx} + x^2 - t$, $x \in \mathbb{R}, t > 0$
$u(x,0) = u_t(x,0) = 0$ $x \in \mathbb{R}$.

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Heat Conduction Problem

→ heat can't come out of this

$u_t = k u_{xx}$, $0 < x < l$, $t > 0$

IC $u(x,0) = f(x)$, $0 < x < l$

BC $u(0,t) = u(l,t) = 0$, $t \geq 0$

$k > 0$

$k = \dfrac{\alpha}{C_p \rho}$ → Thermal Conductivity

ρ → mass density

C_p → heat capacity

We will solve this PDE by method of separation of variables

~~A = 0 B =~~ $A = k$ $B = 0$ $C = 0$

$\Rightarrow B^2 - 4AC = 0 \Rightarrow$ It is parabolic type

The canonical form is :

$u_{\eta\eta} = Q(\xi, \eta, u, u_\xi, u_\eta)$

Can't use D'Alembert's sol^n as RHS ≠ 0

We can reduce this PDE into :

$u(x,t) = X(x)\, T(t)$

$u_t = X \dot{T}$

$u_{xx} = X'' T$

$X \dot{T} = k X'' T$

$\Rightarrow \dfrac{X''}{X} = \dfrac{\dot{T}}{kT} = \lambda$ (const)

func^n of x alone ↳ func^n of t alone

$X'' + \lambda X = 0$, $\dot{T} - \lambda k T = 0$

applying ~~IC~~ BC :

$X(0) = X(l) = 0$

Case-I : $\lambda > 0$, $\lambda = \alpha^2$

$m^2 - \alpha^2 = 0 \Rightarrow m = \pm \alpha$

$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

~~$X(0)$~~ B.C. $\Rightarrow$ $X(x) = 0$

Case - II : $\lambda = 0$

$$X'' = 0$$

$$\Rightarrow X = Ax + B$$

BCs $\Rightarrow$ $X(x) = 0$

Case - III

$$\lambda < 0, \quad \lambda = -\alpha^2$$

$$m^2 + \alpha^2 = 0 \Rightarrow m = \pm i\alpha$$

$$X(x) = A\cos\alpha x + B\sin\alpha x$$

$$X(0) = A = 0$$

$$X(l) = B\sin\alpha l = 0$$

$$\Rightarrow \sin\alpha l = 0 \qquad \text{(B can't be 0)}$$

$$\Rightarrow \alpha l = n\pi \quad \forall\, n \in Z$$

$$\Rightarrow \alpha_n = \frac{n\pi}{l}$$

Eigen values: $\lambda_n = -\alpha^2 = -\frac{n^2\pi^2}{l^2}, \quad n = 1, 2, 3 \ldots$ (Because $n=0$ gives $\lambda = 0$ trivial soln)

$$\boxed{X_n(x) = B_n \sin\left(\frac{n\pi}{l}\right)x}, \quad n = 1, 2, 3, \ldots$$

$$\dot{T} - \lambda k T = 0 \Rightarrow \dot{T} + \left(\frac{n^2\pi^2}{l^2}\right)k T_n = 0$$

$$\Rightarrow \boxed{T_n(t) = C_n \exp\left(-\frac{n^2\pi^2}{l^2} kt\right)}$$

$$u_n(x,t) = X_n(x)\, T_n(t)$$

superposition principle implies

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum X_n T_n$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 t}{l^2}\right) \sin\left(\frac{n\pi x}{l}\right)}$$

To eliminate a_n, use ICs

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$u(x,0) = f(x)$

$\Rightarrow \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$

Choose a_n s.t. $u(x,0)$ become Fourier sine series of $f(x)$

$$\boxed{a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx}$$

Condⁿs needs for this solⁿ to exist:

$f(x) \in C^2$

f''' is piecewise continuous.

Eg.

$u_t = u_{xx} \quad 0 < x < l \quad t > 0$

$u(0,t) = u(l,t) = 0 \quad 0 \le x \le l$

$u(x,0) = \underbrace{x(l-x)}_{f(x)} \quad 0 \le x \le l$

Find heat eqⁿ

$$a_n = \frac{2}{l}\int_0^l x(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

<u>Non-homogenous problem</u>:

(1) — $u_t - k u_{xx} = F(x,t) \quad 0 \le x \le l, \; t > 0$

(2) — $u(x,0) = f(x)$

(3) — $u(0,t) = u(l,t) = 0 \quad t \ge 0$

We can use Duhamel principle to find solⁿ of Non-homogenous problem.

→ This problem has unique solⁿ if it exists.

Proof: Let $u_1(x,t)$ and $u_2(x,t)$ be 2 solⁿs of PDE (1) — (3)

$\Rightarrow v(x,t) = u_1(x,t) - u_2(x,t)$ will satisfy

$v_t = k v_{xx}$

$v(x,0) = 0$

$v(0,t) = v(l,t) = 0$

Claim : To show that v is identically zero.

$$E(t) = \frac{1}{2k}\int_0^l v^2(x,t)\,dx$$

$$E(t) \geq 0 \quad \forall \quad t \geq 0$$

differentiating wrt t

$$\frac{d}{dt}(E(t)) = \frac{1}{2k}\int_0^l 2v(x;t)\, \underbrace{v_t(x,t)}_{k v_{xx}}\, dx$$

$$= \int_0^l \underset{①}{v(x,t)}\ \underset{②}{v_{xx}(x,t)}\, dx$$

Integrating by parts

$$= v v_x \Big|_0^l - \int_0^l v_x v_x \, dx$$

$$= -\int_0^l v_x^2 \, dx$$

$$E'(t) < 0 \quad \forall \quad t > 0$$

~~Funcⁿ is ↓ng~~

$$E(0) = \frac{1}{2k}\int_0^l v^2(x,0)\, dx = 0$$

Funcⁿ is positive & ↓ng and $E(0) = 0$

$\Rightarrow E(t) = 0 \quad \forall \quad t \geq 0$

$\Rightarrow v^2(x,t) = 0$

$\Rightarrow v(x,t) = 0 \quad \forall \quad x \in [0,l], \quad t \geq 0$

$\Rightarrow u_1(x,t) = u_2(x,t)$

Hence Proved

→ With non-homogenous B.C. :

$u_t = k u_{xx}, \quad 0 < x < l, \quad t > 0$ —— (4)

IC : $u(x,0) = f(x) \quad 0 \leq x \leq l$ —— (5)

BC : $u(0,t) = a, \quad u(l,t) = b \quad t \geq 0$ —— (6)

*

i) Find a particular solⁿ of PDE & B.C.

ii.) Find solⁿ of corresponding homogenous B.C.

Then add the solⁿ

if 3 points ⇒ polynomial

(Interpolation to make st. line (from 2 points))

i.) Assume $u_p(x,t) = cx + d$ is particular solⁿ satisfying BC ⑥

$u_p(0,t) = d = a$

$u_p(l,t) = cl + a = b \Rightarrow c = \frac{b-a}{l}$

Particular solⁿ is : $\boxed{u_p(x,t) = \left(\frac{b-a}{l}\right)x + a}$

We can check that it ~~depends~~ satisfies PDE also.

ii.) Consider

$V_t - kV_{xx} = 0 \qquad 0 < x < l \qquad t \geq 0$

BC $\quad V(0,t) = 0 \quad , \quad V(l,t) = 0 \qquad t \geq 0$

IC $\quad V(x,0) = f(x) - u_p(x,0) = F(x) \quad , \quad 0 \leq x \leq l.$

$$\Rightarrow V(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 kt}{l^2}\right) \sin\left(\frac{n\pi x}{l}\right)$$

where $a_n = \frac{2}{l}\int_0^l F(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$$\boxed{u(x,t) = V(x,t) + u_p(x,t)}$$

Eg