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Eigen Values and eigen vectors

A real no. λ is an eigen value of an n-square matrix iff $\exists$ a non-zero vector x s.t.

$$AX = \lambda X$$

$$\Rightarrow (A - \lambda I_n) X = 0$$

$x \rightarrow$ eigen vector of A corresponding to eigen value λ.

$\because$ X is non-zero vector,

$$\Rightarrow (A - \lambda I_n) = 0$$

$$\Rightarrow |A - \lambda I_n| = 0 \quad \rightarrow \text{characteristic eq}^n$$

The set $E_\lambda = \{X : AX = \lambda X\} \rightarrow$ eigen space of λ.

$\rightarrow$ $X = 0$: a trivial sol^n of $AX = \lambda X$.

5/11/19

2 classes missing

Vector Space

Real vector space

Add^n

I. Closure Property of Add^n: Let $a, b \in R \Rightarrow a + b \in \mathbb{R}$.
Or, we can say that R is closed w.r.t addition or real no. satisfy closure property w.r.t add^n

II. Commutative Property of Add^n:

$$a + b = b + a \quad , \quad a, b \in \mathbb{R}$$

III. Associative Property of Add^n

$$(a + b) + c = a + (b + c) \quad , \quad \forall a, b, c \in R$$

IV. Additive Identity

$$a + 0 = a = 0 + a \quad \forall a \in R$$

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V Additive Inverse : $a \in R$

$$a + (-a) = 0 = (-a) + a$$

additive inverse

Multiplication :

I $a, b \in R \Rightarrow a * b \in R$

II $a \cdot b = b \cdot a \quad \forall\ a, b \in R$

III $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall\ a, b, c \in R$

IV $a \cdot 1 = a = 1 \cdot a \quad \forall\ a \in R$

V $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a \quad \forall\ a \in R$

multiplicative inverse

* Multiplicative is distributive ~~of~~ over addition.

$a, b, c \in R$

$\Rightarrow a \cdot (b+c) = a \cdot b + a \cdot c \rightarrow$ Left distribution law

$(b+c) \cdot a = b \cdot a + c \cdot a \rightarrow$ Right " "

* Division only has Right distributive property :

~~Ex.~~ $7/(4+5) \neq 7/4 + 7/5$

$(4+5)/7 = 4/7 + 5/7$

Real vector space

A non-empty set V is said to be a real vector space if there are two operations : vector addition $(\oplus)$ and scalar multiplication $(\odot)$ s.t. $\forall$ vectors $u, v, w \in V$ and $a, b \in R$, the following properties are satisfied :

1. $u \oplus v \in V$ (Closure Property)
2. $u \oplus v = v \oplus u$ (Commutative)
3. $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ (Ass.)
4.

Elements of vector space are vectors
Real " " scalars

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4. $\exists$ some element $0 \in V$ s.t

$u \oplus 0 = u = 0 \oplus u$

5. $\exists$ $-u \in V$ s.t.

$u \oplus (-u) = 0 = (-u) \oplus u$

6. $a \odot u \in V$

7. $a \odot (u \oplus v) = a \odot u \oplus a \odot v$

8. $(a * b) \odot u = a \odot (b \odot u)$

$\downarrow$ sum of Real no.

9. $1 \odot u = u$

NOTE :

i) Any scalar multiplied with a zero vector gives a zero vector

$a \odot 0 = 0$ $\oplus 0$

$a \odot 0 \oplus 0 = a \odot 0 + (-a \odot 0)$

$= a \odot (0 \oplus (0) \oplus (-a \odot 0) = a \odot 0 + (-a \odot 0) = 0$

For clarity, use + & .

$\Rightarrow a \cdot 0 + 0 = a \cdot 0 = a \cdot 0 + a \cdot 0 + (-a \cdot 0)$

$= a \cdot (0 + a) + (-a \cdot 0) = a \cdot 0 + (-a \cdot 0) = 0$

iii) If scalar 0 is multiplied with any vector, it gives zero vector

ie, $0 \odot u = 0$

(iii) $(-1) \odot u$ gives additive inverse of u $[-u]$.

Ex. The set $\mathbb{R}$ of all real no. is a vector space wrt the following operations :

(i) $u \oplus v = u + v$ (vector add^n)

(ii) $a \odot u = a u$ (scalar multp^n) $\forall$ $a, u, v \in R$.

Sol^n $V = \mathbb{R}$. $\rightarrow$ Have to verify all properties of R.

Study R_1, R_2 $\quad R_n \to n$ - dimensional

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Q. Prove / Disprove.

Sub-space: subset of space satisfying all properties of that space

→ If V is a vector space and W is a subset of V s.t. W is also a vector space under the same operations as in V, then W is called a subspace of V.

Ex. The set $W = \{[x_1, 0] : x_1 \in R\}$

$W \in R^2$, $V \in R^2$

Th^m: A subset W of a vector space V is a subspace of V iff W is closed wrt vector addition and scalar multiplication. (and vice versa)

NOTE: It is also easy to show that a subset W of a vector space V is a subspace of V iff

$a \odot u \oplus b \odot v \in W \quad \forall \quad a, b \in R$ and $u, v \in V$

Ex. Show that the set $W = \{[x_1, 0], x_1 \in R\}$ is subspace of R^2.

Sol^n Let $u = [x_1, 0]$, $v = [y_1, 0]$ $\quad u, v \in W$ and $a \in R$

Then

$u \oplus v = [x_1 + y_1, 0] \in W$ (by closer property) → vector add^n

$a \cdot u = a \cdot [x_1, 0] = [ax_1, 0] \in W$ → scalar multiplic

⇒ W is closed under 2 op^{n}s vector add^n & scalar multip^n.

Hence, W is a subspace in R^2.

Ex. Verify whether $W = \{[x, y], x - y = 0, x, y \in R\}$ is a subspace in R^2.

Sol^n Let $u = [x_1, x_2]$, $v = [y_1, y_2] \in W$ and $a \in R$

⇒ $x_1 - x_2 = 0$ and $y_1 - y_2 = 0$

$u \oplus v = [x_1 + y_1, x_2 + y_2] \in W$ $\quad [\because (x_1 + y_1) - (x_2 + y_2) = 0]$

$a \odot u = [ax_1, ax_2] \in W$ $\quad [\because ax_1 - ax_2 = a(x_1 - x_2) = 0]$

⇒ W is a subspace in R^2

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Ex. $W = \left\{ \begin{vmatrix} p & q \\ r & s \end{vmatrix} : ps \cdot qr \neq 0, \; p, q, r, s \in R \right\}$ is subspace of $M_{22} \to$ Matrix of dimension 2×2.

Let $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $v = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in W$

$u \oplus v = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$

$\Rightarrow$ W is not a subspace of M_{22}

Ex. Verify whether

$W = \{ [x, y] : y = x^2, \; x, y \in R \}$ is a subspace in R^2.

~~Let $u = [x_1, x_2]$, $v = [y_1, y_2] \in W$~~

$\Rightarrow$

Let $u = [1, 1]$ & $v = [2, 4]$

(i.) $u \oplus v = [1,1] \oplus [2,4] = [3,5] \notin W$

$\Rightarrow$ W is not a subspace in R^2.

→ Let W_1 and W_2 be 2 sub-spaces of vector space V.

Then,

(i) $W_1 \cap W_2$ is a subspace of V.

(ii) $W_1 \cup W_2$ need not be a subspace of V.

(iii) $W_1 \cup W_2$ is a subspace of V iff either $W_1 \subset W_2$ or $W_2 \subset W_1$

[Prove yourself]

Span of a Set

Let S be any subset of vectorspace V. Then, the set of all the linear combinations of finite number of members of S is called span of S. It is denoted by span(S) or L(S)

$\therefore \; L(S) = span(S) = \left\{ a_1 v_1 + a_2 v_2 + \dots + a_n v_n \;\; s.t. \; a_i \in R, \; v_i \in S, \; i = 1, 2, \dots, n \right\}$

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Ex. $S = \{(1,0), (0,1)\}$

then $L(S) = \{a(1,0) + b(0,1) : a, b \in R\}$
$= \{(a,b) : a,b \in R\} = \mathbb{R}^2$

Ex. Show that span of the set $S = \{[2,3,4], [1,5,7], [3,11,13]\}$ is $\mathbb{R}^3$.

By definition,

$L(S) = \{a[2,3,4] + b[1,5,7] + c[3,11,13] : a,b,c \in \mathbb{R}\}$

$= \{[2a+b+3c,\ 3a+5b+11c,\ 4a+7b+13c] : a,b,c \in \mathbb{R}\}$

for simplified span

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore L(S) = a(1,0,0) + b(0,1,0) + c(0,0,1)$
$= [(a,b,c) : a,b,c \in \mathbb{R}] \equiv \mathbb{R}^3$

Ex. $\text{Span}(\phi) = ?$

$\text{span}(\phi) = \{0\}$

Thm: Let S be subset of a vector space V. Then,

(i) $L(S)$ is a subset of V

(ii) $L(S)$ is a subspace of V

(iii) $L(S)$ is the minimal subspace of V containing S.

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Dimension :

The number of elements in the basis of a vector space is called its dimensions

→ If a vector space has a finite dimension, then we call it as finite dimensional vector space.

Ex. Let $B = [(1,0), (0,1)]$ of R^2

so, it has 2 vectors $\Rightarrow \dim(R^2) = 2$

Ex. $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

↓

4 vectors of M_{22} $\Rightarrow$ $\dim(M_{22}) = 4$

Thm: The maximal LI subset of a spanning set of a vector space forms a basis of the vector space.

We'll use LI method to find Maximal LI subset :

Method :

- Write the given vectors as the columns in a matrix & find REF.
- The columns carrying the leading entries correspond to the LI vectors

Ex. Find a maximal LI subset of the set

$S = \{(1,0), (2,1), (1,5)\}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 5 \end{bmatrix}$$

maximal LI $\{(1,0), (2,1)\}$

Ex. $S = \{ [1,0], [2,1], [1,5] \}$ spans R^2.
What is the basis in R^2 from this set?

Sol^n:

$$\cancel{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix}}$$

Maximal LI forms a basis
so, $\{ [1,0], [2,1] \}$ forms a basis

Linear Transformation

Let V and W be 2 vector spaces. Then, a function $f: V \to W$ is said to be linear transformation (LT) if and only if $\forall\, v_1, v_2 \in V$ and $c \in R$, we have

$$f(v_1 + v_2) = f(v_1) + f(v_2) \text{ and}$$
$$f(cv_1) = c\, f(v_1)$$

Ex. $f: M_{mn} \longrightarrow M_{nm}$ given by:
$f(A) = A^T$ is LT.

Sol^n: Let $A, B \in M_{mn}$ and $c \in R$

$$f(A+B) = (A+B)^T$$
$$= A^T + B^T$$
$$= f(A) + f(B)$$

$$f(cA) = (cA)^T = cA^T = c\, f(A)$$
$\Rightarrow$ f is LT.

Ex. $f: R^3 \to R^3$ given by:
$f(x_1, x_2, x_3) = \{ x_1, x_2, -x_3 \}$ is a LT.

Sol^n Let $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$

$$f(X+Y) = \cancel{(X+Y, x} (x_1 + y_1, x_2 + y_2, -x_3 - y_3)$$
$$= (x_1, x_2, -x_3) + (\ldots$$

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$= f(X) + f(Y)$ ✓

$f(cX) = \{cx_1, cx_2, -cx_3\}$

$= c\,\{x_1, x_2, -x_3\} = c\,f(X)$ ✓

$\Rightarrow f$ is LT.

Ex. Let A be a $m \times n$ matrix. ~~and~~ Then $f : R^n \to R^m$ given by:

$f(X) = AX$ is a LT.

Sol^n Let $X_1, X_2 \in R^n$ and $a \in R$. Then,

- $f(X_1 + X_2) = A(X_1 + X_2) = AX_1 + AX_2 = f(X_1) + f(X_2)$
- $f(aX_1) = A(aX_1) = a(AX_1) = af(X_1)$

$\Rightarrow f$ is a LT.

Note : A LT $f : V \to V$ is called a linear operator on V.

Theorems :

I. If $L : V \to W$ is a LT ~~and~~, then $L(0) = 0$

and $L(a_1v_1 + a_2v_2) = a_1 L(v_1) + a_2 L(v_2)$ $\forall\, a_1, a_2 \in R$ & $v_1, v_2 \in V$

Proof : Let $u \in V$ then

- $L(0) = L\{(u) + (-u)\} = L(u) + L(-u)$

$= L(u) + (-1)L(u) = 0$.

- $L(a_1v_1 + a_2v_2) = L(a_1v_1) + L(a_2v_2)$

$= a_1 L(v_1) + a_2 L(v_2)$

II. The composition of two LT is also a LT. ie,

if $L_1 : V_1 \to V_2$ & $L_2 : V_2 \to V_3$ are LT, then

$L_2 \circ L_1 : V_1 \to V_3$ is a LT

Proof : Let $u, v \in V_1$ and $a \in \mathbb{R}$

Given : $L_1 : V_1 \to V_2$ & $L_2 : V_2 \to V_3$ are LT

We have $(L_2 \circ L_1)(u+v) = L_2\left[L_1(u+v)\right]$

$= L_2\left[L_1(u) + L_1(v)\right] = L_2[L_1(u)] + L_2[L_1(v)]$

$= L_2 \circ L_1(u) + L_2 \circ L_1(v)$ ✓

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$(L_2 o L_1)(au) = L_2[L_1(au)] = L_2[aL_1(u)] = a L_2[L_1(u)]$

$= a L_2 o L_1(u)$

$\Rightarrow$ $L_2 o L_1$ is a LT.

III. If $L : V \rightarrow W$ is a LT and V_1 & W_1 are subspace of V and W resp., then

$L(V_1) = \{L(u) : u \in V_1\}$ is a subspace of W

and $L^{-1}(W_1) = \{v : L(v) \in W_1\}$ is a subspace of V.

Proof:

To prove:

$L(V_1) = \{L(v) : v \in V_1\}$ is a subspace of W

Let $L(u), L(v) \in L(V_1)$ and $a \in \mathbb{R}$.

Then, $u, v \in V_1$ and we have

$L(u) + L(v) = L(u+v) \in L(V_1)$

since $u, v \in V_1$

Also, $aL(u) = L(au) \in L(V_1)$

$\because au \in V_1$ $\Rightarrow$ $L(V_1)$ is a subspace of W.