Solutions for Math 311 Assignment #7

(1) Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{\overline{z}^2 - 1}\right| \le \frac{\pi}{3}$$

where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ lying in the first quadrant.

Proof. For $|z| = 2$, $|\overline{z}^2 - 1| \ge |\overline{z}^2| - |1| = 3$. Therefore,

$$\left|\frac{1}{\overline{z}^2 - 1}\right| \le \frac{1}{3}$$

for $|z| = 2$. Consequently,

$$\left|\int_C \frac{dz}{\overline{z}^2 - 1}\right| \le \frac{1}{3}\int_C |dz| = \frac{\pi}{3}.$$

□

(2) Show that if C is the boundary of the triangle with vertices at the points 0, $3i$ and -4 oriented counterclockwise, then

$$\left|\int_C (e^z - \overline{z})dz\right| \le 60.$$

Proof. For $z \in C$, $\mathrm{Re}(z) \le 0$. Therefore, $|e^z| \le 1$ for $z \in C$. Also it is clear that $|z| \le 4$ for $z \in C$. Therefore,

$$\left|\int_C (e^z - \overline{z})dz\right| \le \int_C (|e^z| + |z|)|dz| \le 5\int_C |dz| = 60.$$

□

(3) Let C_R be the circle $|z| = R$ $(R > 1)$ oriented counterclockwise. Show that

$$\left|\int_{C_R} \frac{\operatorname{Log} z}{z^2} dz\right| < 2\pi\left(\frac{\pi + \ln R}{R}\right)$$

and then

$$\lim_{R\to\infty} \int_{C_R} \frac{\operatorname{Log} z}{z^2} dz = 0.$$

Proof. Since $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$,

$$|\operatorname{Log}(z)| \le \big|\ln|z|\big| + |\operatorname{Arg}(z)| = \ln R + |\operatorname{Arg}(z)|.$$

for $|z| = R$. The equality holds only if $\operatorname{Arg}(z) = 0$. And since $-\pi < \operatorname{Arg}(z) \le \pi$,

$$|\operatorname{Log}(z)| < \ln R + \pi$$

for $|z| = R$. Therefore,

$$\left|\int_{C_R} \frac{\operatorname{Log} z}{z^2} dz\right| < \frac{\pi + \ln R}{R^2} \int_{C_R} |dz| = 2\pi \left(\frac{\pi + \ln R}{R}\right).$$

By L'Hospital's rule,

$$\lim_{R\to\infty} \frac{\pi + \ln R}{R} = \lim_{R\to\infty} \frac{(\pi + \ln R)'}{(R)'} = \lim_{R\to\infty} \frac{1}{R} = 0$$

and hence

$$\lim_{R\to\infty} \int_{C_R} \frac{\operatorname{Log} z}{z^2} dz = 0.$$

□

(4) Compute

$$\int_{-1}^{1} z^i dz$$

where the integrand denote the principal branch

$$z^i = \exp(i \operatorname{Log} z)$$

of z^i and where the path of integration is any continuous curve from $z = -1$ to $z = 1$ that, except for its starting and ending points, lies above the real axis.

Solution. Let C be a continuous curve from $z = -1$ to $z = 1$ which lies above the real axis except for -1 and 1. That is, C is given by $z = w(t)$ for $a \le t \le b$, where $w(a) = -1$, $w(b) = 1$ and $\operatorname{Im}(w(t)) > 0$ for $a < t < b$.

We have

$$\left(\frac{z^{1+i}}{1+i}\right)' = z^i$$

for $z \in \mathbb{C}\backslash(-\infty, 0]$. Therefore,

$$\begin{aligned}
\int_C z^i dz &= \lim_{s\to a^+} \int_s^b (w(t))^i dw(t) \\
&= \lim_{s\to a^+} \left.\frac{(w(t))^{1+i}}{1+i}\right|_s^b \\
&= \frac{1}{1+i} \lim_{s\to a^+} (\exp((1+i)\operatorname{Log} w(b)) - \exp((1+i)\operatorname{Log} w(s))) \\
&= \frac{1}{1+i}(1 - \lim_{s\to a^+} \exp((1+i)\operatorname{Log} w(s))) \\
&= \frac{1}{1+i}(1 - \exp(-\pi + \pi i)) \\
&= \frac{1}{1+i}(1 + e^{-\pi}) = \frac{1-i}{2}(1 + e^{-\pi}).
\end{aligned}$$

(5) Apply Cauchy Integral Theorem to show that

$$\int_C f(z)dz = 0$$

when C is the unit circle $|z| = 1$, in either direction, and when

(a) $f(z) = \dfrac{z^2}{z-3}$;
(b) $f(z) = \tan z$;
(c) $f(z) = \operatorname{Log}(z+2)$.

Proof. (a) Since $f(z)$ is analytic in $\{z \neq 3\}$, $f(z)$ is analytic everywhere in $\{|z| \leq 1\}$. So $\int_C f(z)dz = 0$ by CIT.

(b) Since $f(z)$ is analytic in $\{z \neq k\pi + \pi/2 : k \text{ integer}\}$, $f(z)$ is analytic everywhere in $\{|z| \leq 1\}$ since $|k\pi + \pi/2| > 1$ for all integers k. So $\int_C f(z)dz = 0$ by CIT.

(c) Since $\operatorname{Log}(z)$ is analytic in $\mathbb{C}\backslash(-\infty, 0]$, $f(z)$ is analytic in $\mathbb{C}\backslash(-\infty, -2]$. Clearly, $(-\infty, -2] \cap \{|z| \leq 1\} = \emptyset$. Therefore, $f(z)$ is analytic everywhere in $\{|z| \leq 1\}$. So $\int_C f(z)dz = 0$ by CIT. $\square$

(6) Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$ and $y = \pm 1$ and C_2 be the positively oriented circle $|z| = 4$. Apply Cauchy Integral Theorem to show that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

when

(a) $f(z) = \dfrac{1}{3z^2+1}$;

(b) $f(z) = \dfrac{z+2}{\sin(z/2)}$;

(c) $f(z) = \dfrac{z}{1-e^z}$.

Proof. It suffices to show that $f(z)$ is analytic in the region

$$D = \{|z| \le 4\} \backslash \{|x| < 1, |y| < 1\}$$

between the two curves C_1 and C_2.

(a) Since $f(z)$ is analytic in $\{z \neq \pm\sqrt{3}i/3\}$ and $\pm\sqrt{3}i/3 \notin D$, $f(z)$ is analytic everywhere in D. Therefore, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ by CIT.

(b) Note that $\sin(z/2) = 0$ if and only if $z/2 = k\pi$ for some integer k. It follows that $f(z)$ is analytic in $\{z \neq 2k\pi : k \text{ integer}\}$. Since $2k\pi \in \{|x| < 1, |y| < 1\}$ for $k = 0$ and $2k\pi \in \{|z| > 4\}$ for integers $k \neq 0$, $f(z)$ is analytic everywhere in D. Therefore, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ by CIT.

(c) Note that $1 - e^z = 0$ if and only if $z = 2k\pi i$ for some integer k. It follows that $f(z)$ is analytic in $\{z \neq 2k\pi i : k \text{ integer}\}$. Since $2k\pi i \in \{|x| < 1, |y| < 1\}$ for $k = 0$ and $2k\pi i \in \{|z| > 4\}$ for integers $k \neq 0$, $f(z)$ is analytic everywhere in D. Therefore, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ by CIT. $\square$