Sketching of Vector Function

$$\vec{V} = x\hat{i}$$

$$\vec{V} = x^2\hat{i}$$

$$\vec{V} = -y\hat{i} + x\hat{j}$$

Gradient Operator (∇)

$T(x,y) = -(\cos^2 x + \cos^2 y)^2$

T=T(x,y,z)

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz.$$

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}), \end{aligned}$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Geometrical Interpretation of the Gradient

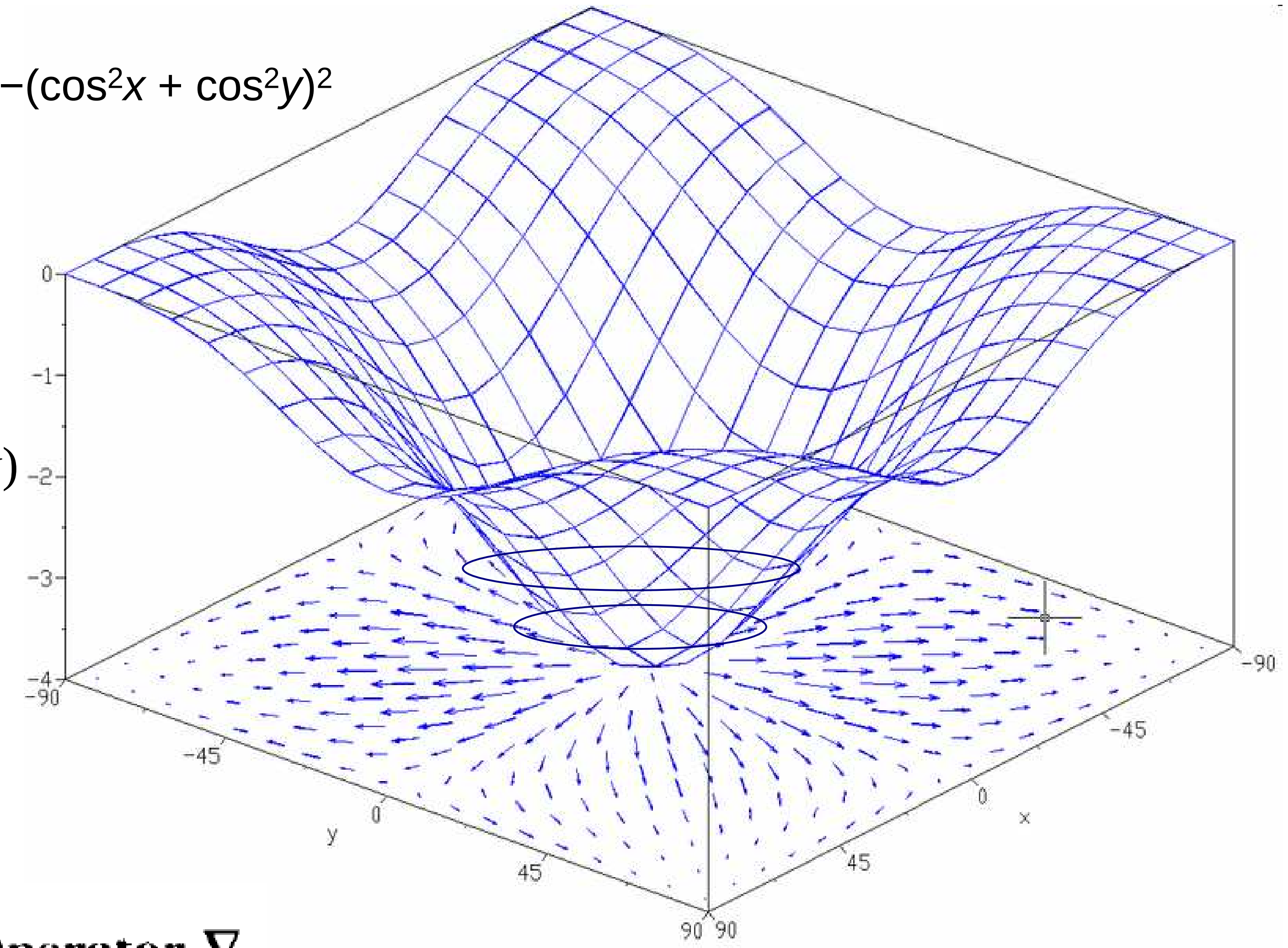
$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T||d\mathbf{l}| \cos\theta$$

when $\theta = 0$ (for then $\cos\theta = 1$)

The gradient ∇T points in the direction of maximum increase of the function T.

The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.

$T(x,y) = -(\cos^2 x + \cos^2 y)^2$



The Operator ∇

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.}$$

http://en.wikipedia.org/wiki/Gradient

$$\nabla T = \underbrace{\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) T}_{\text{Vector Field}}$$

Scalar Field

The Divergence

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

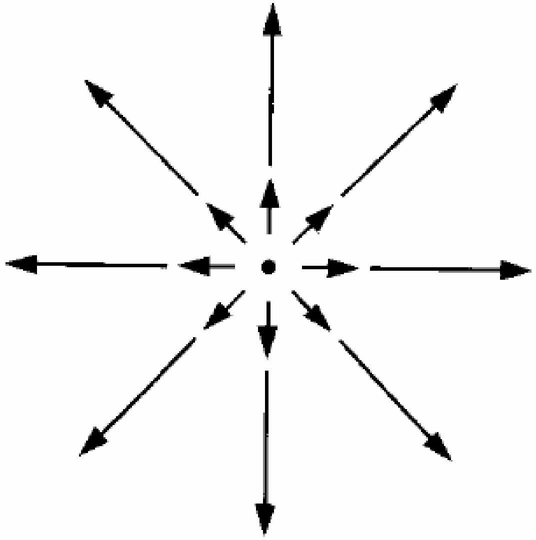
$$\nabla \cdot \vec{V} \neq \vec{V} \cdot \nabla$$

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

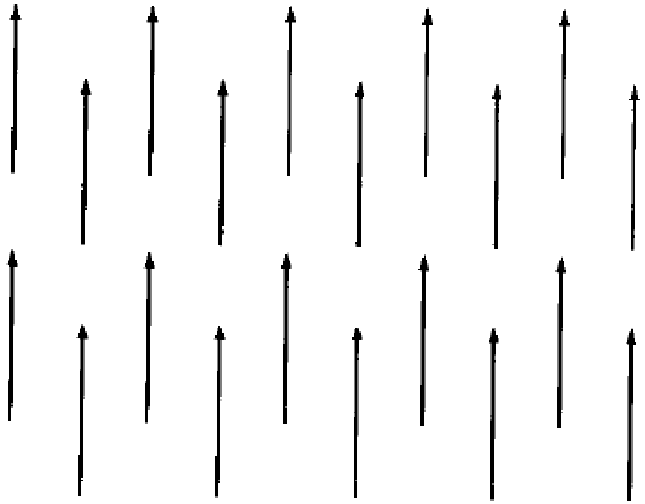
(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

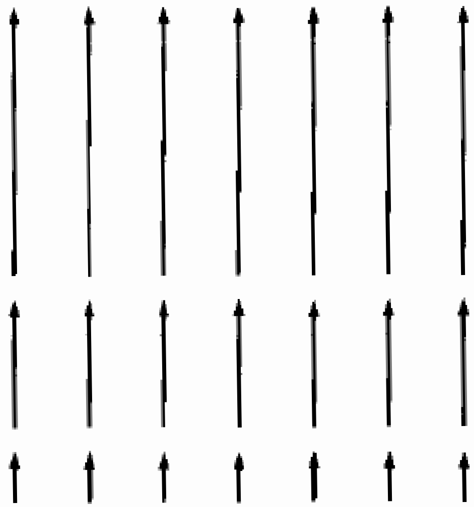
$$\mathbf{v}_a = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$



$$\mathbf{v}_b = \hat{\mathbf{z}}$$



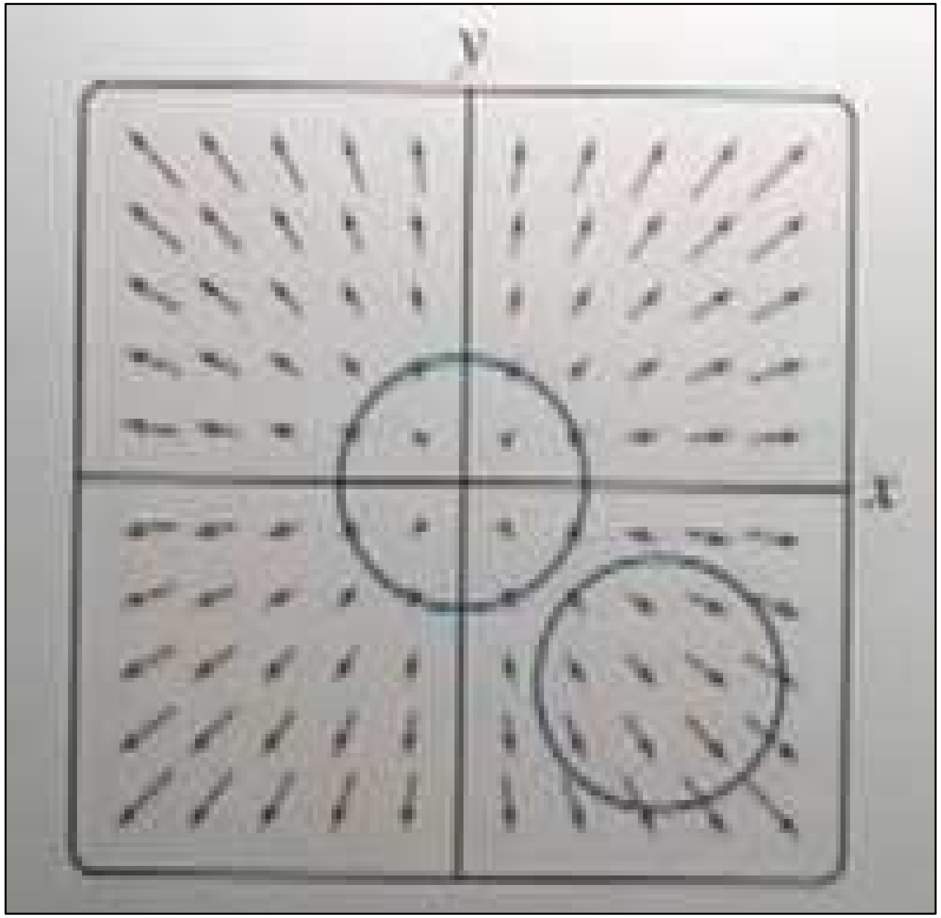
$$\mathbf{v}_c = z\,\hat{\mathbf{z}}$$



Introduction to Electrodynamics by David. J. Griffiths

In many cases, the divergence of a vector function at point P may be predicted by considering a closed surface surrounding P and analyzing the flow over the boundary, keeping in mind that at P:

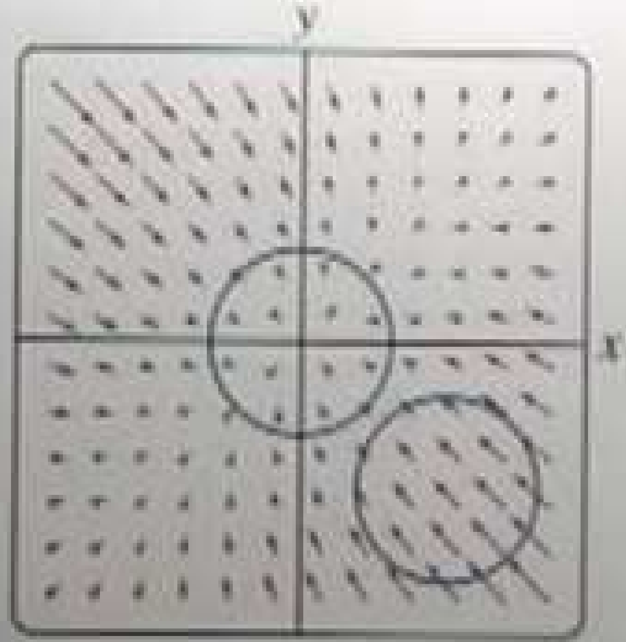
$$\nabla \cdot \vec{F} = \text{outflow} - \text{inflow}$$



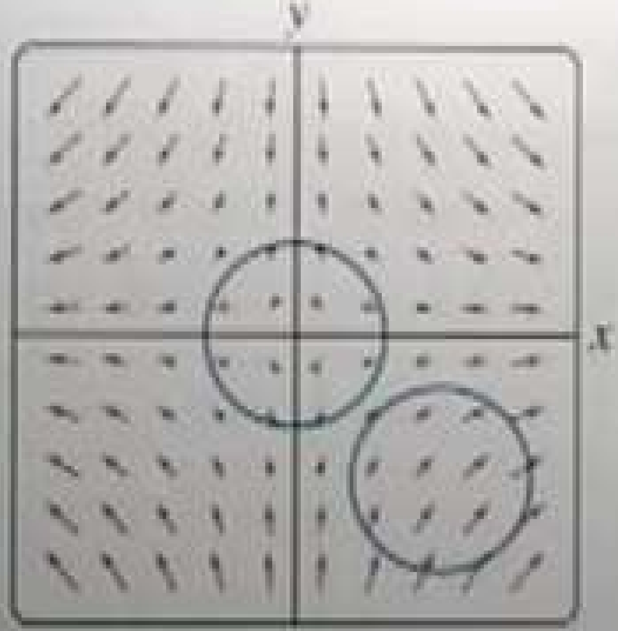
$$\vec{V} = x\hat{i} + y\hat{j}$$

$$\vec{\nabla} \cdot \vec{V} = 2$$

In 3-D, divergence is a measure of Change of flux per unit volume



(B) The force field
$\mathbf{F} = \langle y - 2x, x - 2y \rangle$
with $\operatorname{div}(\mathbf{F}) = -4$.
There is a net inflow
into every circle.

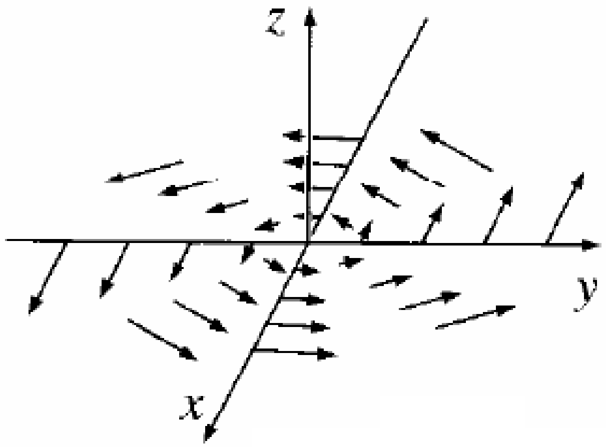


(C) The force field $\mathbf{F} = \langle x, -y \rangle$ with $\operatorname{div}(\mathbf{F}) = 0$. The flux through every circle is zero.

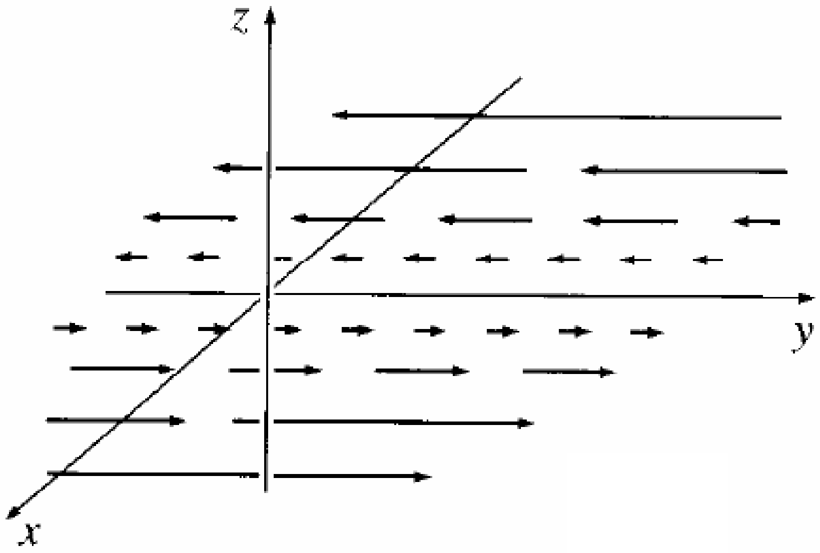
The Curl

$$
\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}
$$

$$
= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)
$$

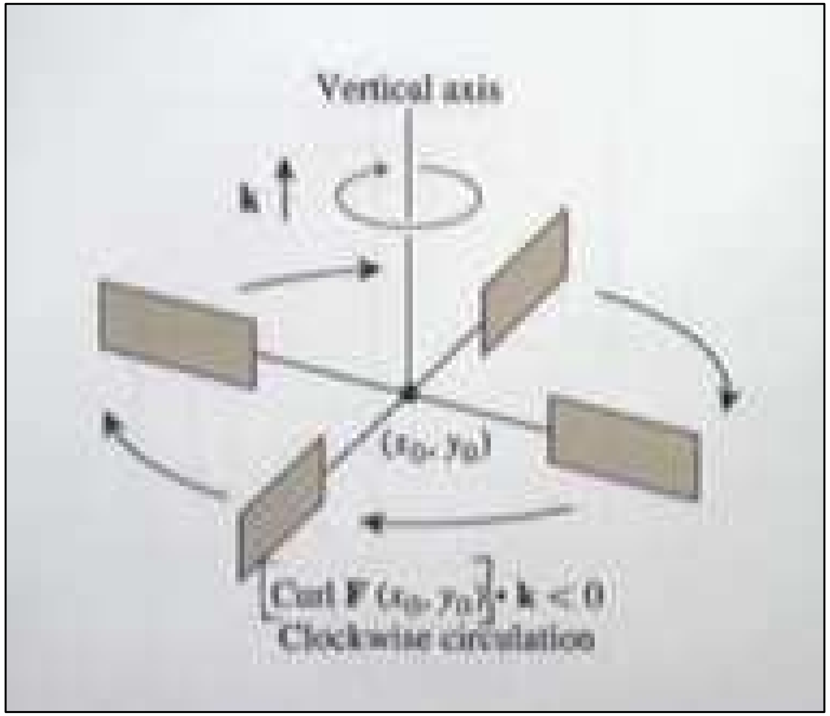


$\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$



$\mathbf{v}_b = x\hat{\mathbf{y}}$

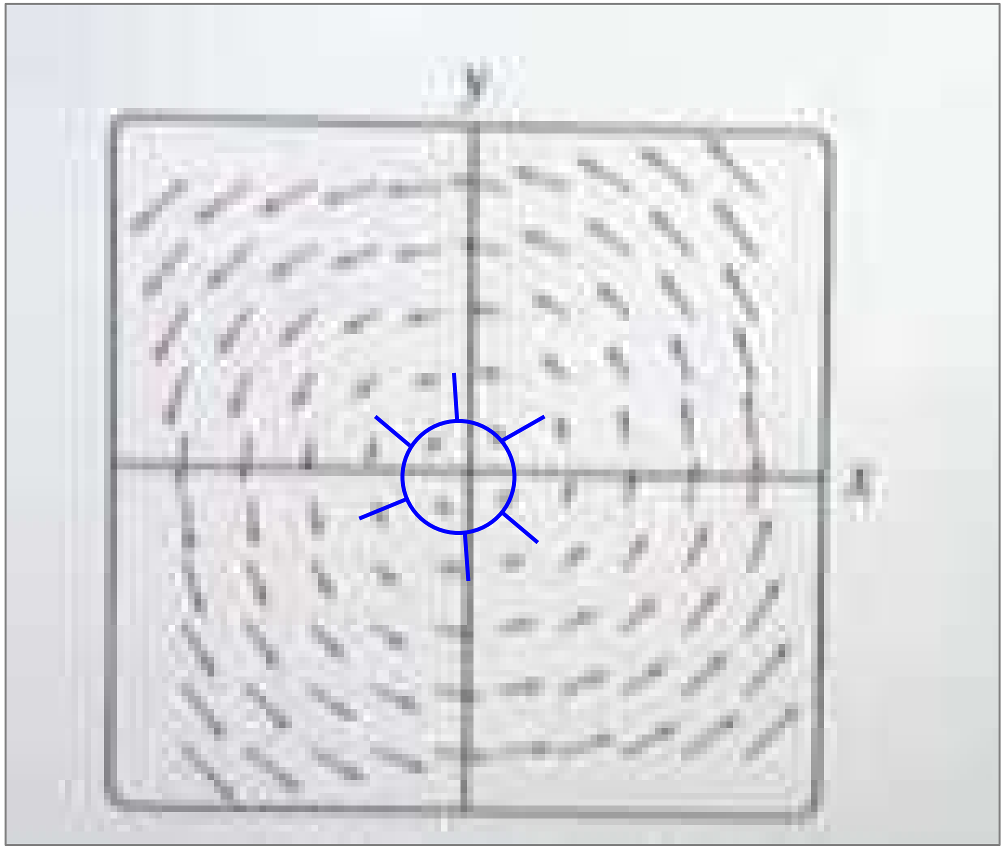
Paddle wheel analysis



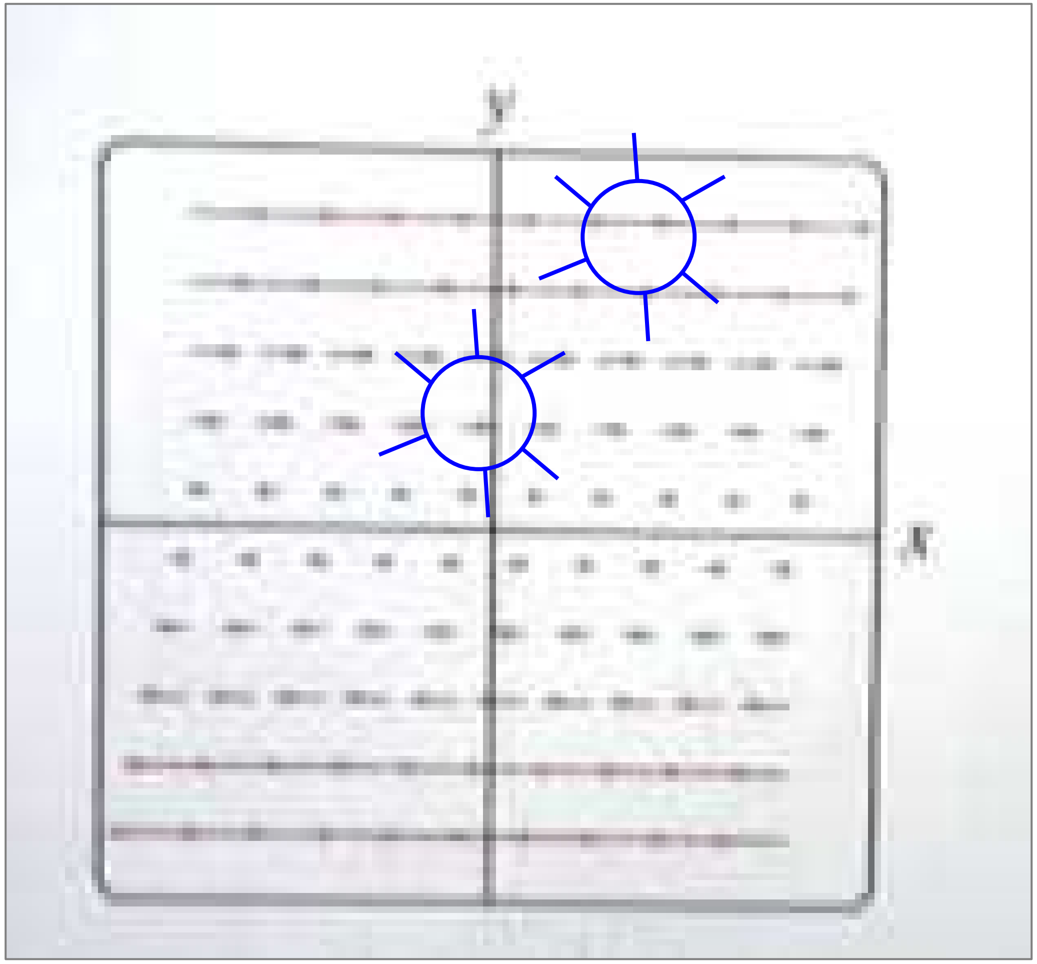
$$[\nabla \times \vec{\boldsymbol{F}}(\boldsymbol{x}_0, \boldsymbol{y}_0)] \cdot \hat{\boldsymbol{k}} < 0$$

$$\vec{F}(x, y) = -y\hat{i} + x\hat{j}$$

$$(\vec{\nabla} \times \vec{F}) \cdot \hat{k} = 2$$



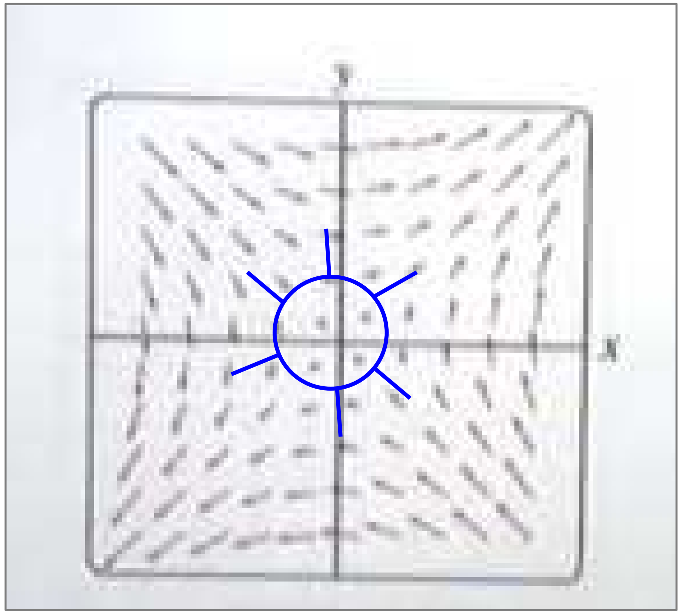
$$\vec{F}(x,y) = y\hat{i}$$

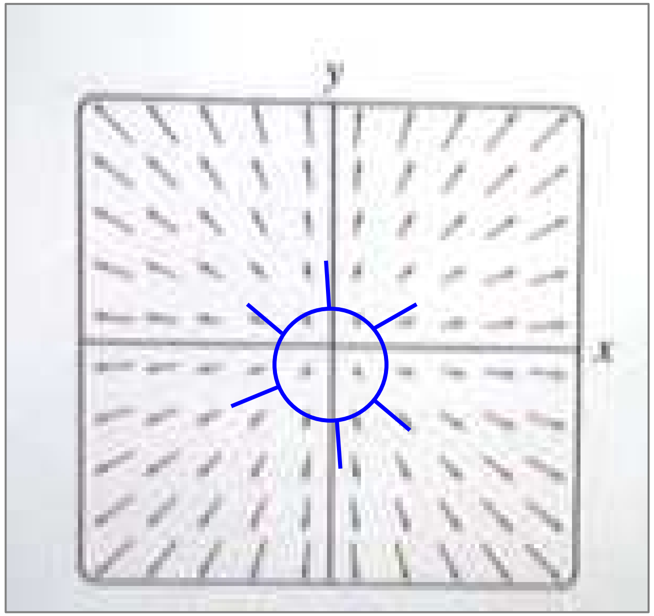


$$(\nabla \times \vec{F}) \cdot \hat{k} = -1$$

$$\vec{\boldsymbol{F}}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{y}\hat{\boldsymbol{i}} + \boldsymbol{x}\hat{\boldsymbol{j}}$$

$$\vec{F}(x,y) = x\hat{i} + y\hat{j}$$





$$\nabla \times \vec{\boldsymbol{F}} = 0$$

Summary

Gradient of a Scalar Field $\longrightarrow$ **Vector Field**

$$\underset{\text{Scalar Field}}{\nabla T} = \underbrace{\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) T}_{\text{Vector Field}}$$

Divergence of a Vector Field $\longrightarrow$ **Measure of Change of flux per unit volume**

Curl of a Vector Field $\longrightarrow$ **Measure of degree of rotation of the field**