SECOND ORDER ODE (Ordinary differential eq^n)

Std. form :

$$y'' = f(x, y, y') \qquad x \in (a,b)$$

x : independent variable , $y = y(x)$: dependent variable

This could be : linear ~~or~~ or non-linear.

Linear 2nd Order ODE :

$$y'' + P(x)\, y' + Q(x)\, y = R(x) \qquad \forall \quad x \in (a,b) \qquad \text{①}$$

* $f(x)$ is said to be linear if
$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \forall \quad x, y \in$
$\alpha, \beta \in R$

$$\Rightarrow \left(\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q\right) y = R(x)$$

$$\Rightarrow \quad D(y) = R(x)$$

Ex. Verify that eq^n ① is linear.
(Prove : $D(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 D(y_1) + \alpha_2 D(y_2)$
$\forall \; y_1, y_2$

sol^n: $D(\alpha_1 y_1 + \alpha_2 y_2) = \left(\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q\right)(\alpha_1 y_1 + \alpha_2 y_2)$

$$\Rightarrow \alpha_1\left(\frac{d^2y_1}{dx^2} + P(x)\frac{dy_1}{dx} + Q(x)y_1\right) + \alpha_2\left(\frac{d^2y_2}{dx^2} + \cdots\right)$$

$$= \alpha_1 D(y_1) + \alpha_2 D(y_2)$$

→ If $R(x) = 0 \Rightarrow$ Eq^n ① is Homogenous

$R(x) \neq 0 \Rightarrow$ Non-Homogenous

$R(x)$: some forced func^n / I/P func^n / load func^n.

Examples.

1.) $y'' + 4y = e^{-x}\sin x \rightarrow$ Non-hom, linear

2.) $yy'' + (y')^3 + xy = 0 \rightarrow$ Hom., Non-linear $(yy'' \;\&\; (y')^3)$

Solution of equation ① is :

A func^n if defined on some interval (open) (a,b) is called a sol^n of eq^n ① if

i) if it is twicely differentiable.

ii) if it satisfies eq ①.

eg. $y'' - y = 0$ $\quad e^x, e^{-x}$

Eg. $y'' + y = 0$ $\quad \sin x, \cos x$

Superposition Principle : (Only applicable on linear & homogenous eq^n)

If $y_1(x)$ and $y_2(x)$ are two sol^{n}s of eq^n : — ②

$y'' + P(x)y' + Q(x)y = 0 \quad \forall x \in (a,b)$

then linear combination of y_1 & y_2, $(c_1y_1 + c_2y_2)$, is again a sol^n of eq^n ②.

Proof:

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

$c_1y_1 + c_2y_2$: $(c_1y_1 + c_2y_2)'' + P(x)[c_1y_1 + c_2y_2]' + Q(x)[c_1y_1 + c_2y_2]$

$$= c_1\left(\frac{d^2y_1}{dx^2} + P(x)y_1' + Q(x)y_1\right) + c_2\left(y_2'' + P(x)y_2'' + Q(x)y_2\right)$$

$$= 0$$

Non-hom

Eg. $y'' + y = 1$: Non-hom. $\rightarrow$ y_1, y_2 : two sol^n s

$y_1 = 1 + \sin x$ $\quad y_2 = 1 + \cos x$

$a(1+\sin x) + b(1+\cos x)$ $\quad \forall\ a, b$

$a + b + a\sin x + b\cos x$

let $a = 1$ $\quad b = 2$ $\quad$ (To disprove)

$3 + \sin x + 2\cos x$

$y'' + y = -\sin x - 2\cos x + 3 + \sin x + 2\cos x = 3 \neq 1$

$\Rightarrow$ Superposition principle is not applicable here.

Non-linear

$yy'' - xy' = 0$ $\qquad x^2(2) - x(2x) = 0$ ✓

$y_1 = x^2$ $\quad y_2 = 1$ $\qquad y\frac{d^2y}{dx^2} = x\frac{dy}{dx}$

Let $a = -1$ $\quad b = 0$

$\Rightarrow$ $-1(x^2) + 0 = -x^2$

$yy'' - xy' = (-x^2)(-2) - x(-2x) \neq 0$

$\Rightarrow$ superposition principle is not applicable here.

The set of sol^n of a linear & hom. differential eq^n is called sol^n space.

* sol^n set : vector space

* First of all, we have to show that the space is not empty.

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- We can take $y=0$: trivial solⁿ of solⁿ space (eqⁿ ② type)
- From superposition principle, vector add^n & property is satisfied
  scalar multpⁿ
- We can prove solⁿ set to be a vector space using this principle.

- $y' = f(x, y)$: General solⁿ, Particular solⁿ, Singular solⁿ
  (Particular solⁿ: use condⁿ $(y(x_0) = y_0)$)

* For linear problems, only general solⁿ & particular solⁿ exist.

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Eg. $(y')^2 - xy' + y = 0$ → Non-linear & 1st order

$y = cx + c^2$ → general solⁿ

$y = \frac{x^2}{4}$ (also satisfy the given eqⁿ) ⇒ singular solⁿ : which can't be obtain from general solⁿ.

* Linear eqⁿ doesn't have singular solⁿ.

$\frac{dy}{dx} + P(x)y = Q(x)$ ⇒ ~~So~~ $y = cy_1$

→ $y'' + P(x)y' + Q(x)y = R(x)$ —① (Non-hom)

$y'' + P(x)y' + Q(x)y = 0$ —② (Hom)

The general solⁿ is of the form :

$c_1y_1 + c_2y_2$ (y'' comes in the eqⁿ) so; 2 arbitrary const.

Assume $y_1 = ky_2$

⇒ $c_1y_1 + c_2y_2 = Cy_2$ (Only 1 arbitary const.)

⇒ can't be G.S. of 2nd order linear D.E.

Hence, y_1 and y_2 can't be written in terms of each other to form G.S.

$\Rightarrow$ If $y_1 = ky_2 \Rightarrow y_1$ & y_2 are linearly dependent.

(we don't look for $y_1, y_2 \neq 0$ otherwise it will become dependent)

Eg. $y'' + y' = 0$

$y_1 = 1$, $y_2 = e^{-x}$

Here, y_1 & y_2 : L.I.

(If $\frac{y_1}{y_2}$ = const. $\Rightarrow y_1$ and y_2 are L.D.)

General sol^n : $\boxed{y(x) = c_1 + c_2 e^{-x}}$

Def^n : For 2nd Order homogenous eq^n (2), a general sol^n will be of the form $y(x) = c_1 y_1(x) + c_2 y_2(x)$

where $y_1(x)$ and $y_2(x)$ are linearly Independent.

Initial value Problem :

$y'' + P(x) y' + Q(x) y = 0$

$y(x_0) = y_0$, $y'(x_0) = y_1$

eg. $y'' - y = 0$ $\quad y(0) = 4$, $y'(0) = -2$

$y_1 = e^x \quad y_2 = e^{-x}$ (By inspection)

$\frac{y_1}{y_2} = \frac{e^x}{e^{-x}} = e^{2x} \neq$ const.

$\Rightarrow e^x$ & e^{-x} : L.I.

$y(x) = c_1 e^x + c_2 e^{-x}$ — general sol^n

$y(0) = c_1 + c_2 = 4$ — (1)

$y'(x) = c_1 e^x - c_2 e^{-x}$

$y'(0) = c_1 - c_2 = -2$

$c_1 = 1 \quad c_2 = 3$

$\boxed{y(x) = e^x + 3e^{-x}}$ $\rightarrow$ Particular sol^n OR sol^n of IVP

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Basis Basis :

$\xrightarrow{}$ sol^n For eq^n ②

If y_1 and y_2 are L.I., then y_1 & y_2 form the general sol^n of ② & it also spans the whole vector space & $\{y_1, y_2\}$ form a basis for the sol^n space of eq^n ②

* More than one basis can exist.
* The basis of sol^n space of eq^n ② is also known as the Fundamental system (F.S.)
* Dimension of sol^n space for eq^n ② : 2

Observation :

$\{y_1, y_2\}$ be F.S. for eq^n ②

$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$ (Non-Singular Matrix)

2×2 $\quad 2\times 1$

$$Ay = \begin{bmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{bmatrix}$$

The rows of Ay is again a F.S. (or basis)

- if $ad = bc$ $\quad \frac{a}{b} = \frac{\cancel{c}}{d} = k \quad a = kb \quad \overset{c}{\cancel{d}} = k\cancel{d} \quad \frac{ky_1 + y_2}{ky_1 + \cancel{b}y_2} = 1$

elements of Ay will become L.D.

→ Using one known sol^n find another

y_1 is known for eq^n — ②

Find y_2 s.t. y_1 and y_2 are L.I.

Variation of parametric factor

$cy_1(x)$: again sol^n of eq^n ②

Let $y_2(x) = \overrightarrow{v(x)\, y_1(x)}$ unknown $\quad \frac{y_2(x)}{y_1(x)} = v(x) \neq$ const. ⇒ L.I.

Since y_2 is sol^n of eq^n ②

$y_2' = v'y_1 + vy_1'$

$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1''$

$$+ P(x)\left[v'y_1 + vy_1'\right] + Q[y_1 v] = 0$$

eq^n ② ⇒ $v''y_1 + 2v'y_1' + vy_1'' + P(x)\left[v'y_1 + vy_1'\right] + Q[y_1 v] = 0$

$$\underbrace{v(y_1'' + Py_1' + Qy_1)}_{0} + v'(2y_1' + P(x)y_1) + v''y_1 = 0$$

$$\frac{v''}{v'} = -\frac{(2y_1' + Py_1)}{y_1} = -\frac{2y_1'}{y_1} - P$$

$$\Rightarrow \log v' = -2\log y_1 - \int P\,dx$$

$$v' = \frac{1}{y_1^2} e^{-\int P dx}$$

$$\Rightarrow \boxed{v = \int \frac{1}{y_1^2} e^{-\int P dx}\, dx}$$

$y_2 = v y_1$

→ Only when $v \neq$ const, $y_2 = v y_1$

here $y_1 \neq 0$, $e^{\int -P dx} \neq 0 \Rightarrow v \neq 0$

Hence, y_1 & y_2 : L.I.

G.S. : $\boxed{y(x) = c_1 y_1 + c_2 y_1 \int \frac{1}{y_1^2} e^{-\int P dx}\, dx}$

Eg. Find a basis sol^n of the following problem

$$x^2 y'' - xy' + y = 0 \quad ; \quad x > 0$$

sol^n $y_1 = x$ is a sol^n $\qquad P(x) = -1/x$

$y_2(x) = vx$

$$v = \int \frac{1}{y_1^2} e^{-\int P dx} \qquad -\int P\,dx = \int \frac{1}{x}\,dx = \log x$$

$$v = \int \frac{1}{x^2}\, x\, dx = \log x$$

$$y_2 = y_1 \log x = x \log x$$

Basis : $\{x, x\log x\}$

General sol^n : $c_1 x + c_2 x \log x$

$F(x, y, y', y'') = 0$ → Implicit form

$y'' = f(x, y, y')$ → Explicit form

If ~~DE~~ in Implicit form
(i) x is missing
(ii) y is missing

02-1x

The general 2^{nd} order ODE is

$$F(x, y, y', y'') = 0 \quad \forall \quad x \in (a, b)$$

↓

when dependent variable is missing

I dependent variable is missing (y)

$$f(x, y', y'') = 0$$

take $y' = p$, $y'' = \frac{dp}{dx}$

$$\boxed{g\left(x, p, \frac{dp}{dx}\right) = 0}$$ 1^{st} order eq^n in p

solve for p, then solve for y'.

eg $xy'' - y' = 3x^2$ $\qquad y' = p \Rightarrow y'' = p'$

$xp' - p = 3x^2$

$\Rightarrow p' - \frac{p}{x} = 3x$

$$I.F. = e^{\int -\frac{1}{x} dx} = 1/x$$

$$\frac{p}{x} = 3x + c_1$$

$$p = 3x^2 + c_1 x$$

$$y' = 3x^2 + c_1 x$$

$$\Rightarrow y = x^3 + \frac{c_1 x^2}{2} + c_2$$

II. Independent variable is missing (x)

$f(y, y', y'') = 0$

$y' = p \qquad y'' = \frac{dp}{dx} = \frac{dp}{dy} \times \frac{dy}{dx} = p\frac{dp}{dy}$

$g\left(y, p, p\frac{dp}{dy}\right) = 0$

Eg. $y'' + k^2 y = 0$

$y' = p \qquad y'' = p\frac{dp}{dy}$

$\Rightarrow p\frac{dp}{dy} + k^2 y = 0 \Rightarrow \frac{dp}{dy} + \frac{k^2}{p} y = 0$

$\Rightarrow \int p\,dp = -k^2 \int y\,dy$???

$\Rightarrow p^2 = -k^2y^2 + c^2$

$\Rightarrow p^2 + k^2y^2 = a^2k^2$??

$p = \pm\sqrt{a^2k^2 - k^2y^2}$

$p = \frac{dy}{dx}$

$\Rightarrow \frac{1}{k}\int \frac{dy}{\sqrt{a^2-y^2}} = \pm\int dx + c_1$

$\Rightarrow \frac{1}{k}\sin^{-1}\left(\frac{y}{a}\right) = \pm x + c_1$

$\Rightarrow y = a\sin(kx + b)$

$\Rightarrow y = c_1 \sin kx + c_2 \cos kx$

<u>2nd order Homogenous eq^n</u>:

$y'' + P(x)y' + Q(x)y = 0 \quad \forall \quad x \in (a,b),\ x \neq 0$

* Special case when $P(x)$, $Q(x)$ are constants

Eg : $y'' + ay' + by = 0$

* std. way of solving 2nd order homogenous eqⁿ with const. coefficient funcⁿ

Method for solving :

$y'' + ay' + by = 0$ [we choose $e^{\lambda x}$ as solⁿ as we again get same expⁿ with const. coeff]

choose $y = e^{\lambda x}$ as solⁿ of above eqⁿ. ($e^{\lambda x}$ can't help to find solⁿ with variable coff)

$y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

substituting the values, we get

$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$

$\Rightarrow \lambda^2 + a\lambda + b = 0$ → characteristic eqⁿ of arbitary eqⁿ

$\Rightarrow \lambda = \dfrac{-a \pm \sqrt{a^2-4b}}{2}$

$y = e^{\frac{-a+\sqrt{a^2-4b}}{2}x}, \quad e^{\frac{-a-\sqrt{a^2-4b}}{2}x}$

CASE I : λ has distinct and real roots when $a^2 - 4b > 0$.

λ_1, λ_2 → corresponding real and distinct roots. Thus, we have 2 solⁿs :

$y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$

$\dfrac{y_1}{y_2} = e^{(\lambda_1 - \lambda_2)x}$ (Non-constant funcⁿ)

$\Rightarrow y_1$ & y_2 are L.I. $\quad y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

CASE II : Complex roots λ_1 and λ_2 :

$\lambda_1 = \alpha + i\beta \qquad \lambda_2 = \alpha - i\beta$

The two solⁿs are :

① → $e^{\lambda_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$

② → $e^{\lambda_2 x} = e^{(\alpha - i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$

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if we write

$$y(x) = \underbrace{c_1 e^{\lambda_1 x}}_{y_1} + \underbrace{c_2 e^{\lambda_2 x}}_{y_2}$$

then it is wrong because y will be a complex func^n

adding y_1 and y_2 and dividing it by 2, we get

$$\frac{y_1 + y_2}{2} = e^{\alpha x} \cos \beta x$$

also, $\frac{y_1 - y_2}{2i} = e^{\alpha x} \sin \beta x$

$$y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

[By superposition Principle]

$$\Rightarrow y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

CASE-III : λ has equal roots $(a^2 - 4b = 0)$

$\lambda = -a/2 \Rightarrow y(x) = e^{-a/2 x}$

so $y_1(x) = e^{-a/2 x}$, $y_2(x) = e^{-a/2 x}$

so, $y_2(x)$ is not acceptable (otherwise y_1 & y_2 are L.D.)

~~let~~

$\therefore y_2 = v y_1 \Rightarrow v = \int \frac{1}{y_1^2} e^{-\int P dx} dx$

$P(x) = a$ (coeff. of y')

$\Rightarrow v = \int \frac{1}{e^{-ax}} \cdot e^{-\int a \cdot dx} dx = x \Rightarrow v = x$

$\Rightarrow y_1(x) = e^{-a/2 x}$, $y_2 = x e^{-a/2 x}$

$$y(x) = c_1 e^{-a/2 x} + c_2 e^{-a/2 x} \cdot x$$

Eg. $y'' + y' - 2y = 0$ $\quad y(0) = 4 \quad y'(0) = -5$

$a = 1 \quad b = -2$

$y = e^{\lambda x} \quad \lambda^2 + \lambda - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$

$\Rightarrow \lambda = -2, 1$

$$y = c_1 e^{-2x} + c_2 e^{x}$$

$y(0) = c_1 + c_2 = 4$
$y'(0) = -2c_1 + c_2 = -5$ $\Big\}$ $\Rightarrow$ $c_1 = +1$, $c_2 = 3$

$\Rightarrow y(x) = 3e^{-2x} + e^{x}$

Eg $y'' + 2y' + 2y = 0$

$a = 2 \quad b = 2$

$\lambda^2 + 2\lambda + 2 = 0 \qquad D < 0$

$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$\alpha_1 = -1 + i \qquad \alpha_2 = -1 - i$

$\alpha = -1 \qquad \beta = 1$

$y(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$

Eg. $y'' - 4y' + 4y = 0 \qquad y(0) = 3 \qquad y'(0) = 1$

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2, 2$

$y_1 = e^{+4/2x} = e^{+2x}$

$y_2 = v y_1 \qquad v = \int \frac{1}{e^{4x}} e^{-\int -4dx} = \int dx = x$

$= x e^{2x}$

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with ~~cons~~ variable coffecients (Homogenous)

$\Rightarrow x^2 y'' + axy' + by = 0 \quad , x > 0 \quad \Rightarrow$ Euclid - Cauchy eqn

Let $y = x^m \Rightarrow y' = mx^{m-1} \quad , y'' = m(m-1)x^{m-2}$

$\Rightarrow m(m-1)x^m + amx^m + bx^m = 0$

$\Rightarrow x^m [m^2 - m + am + b] = 0 \qquad x > 0$

$\Rightarrow \boxed{m^2 + (a-1)m + b = 0}$ Characteristic Auxilliary eqn

$m = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$

Case-1 : real & distinct roots m_1 & m_2.

$y_1 = x^{m_1} \qquad y_2 = x^{m_2}$

here, $\frac{y_1}{y_2} = x^{(m_1 - m_2)} \neq$ constant (L.I.)

hence, y_1 and y_2 form a basis

$$\boxed{y(x) = c_1 x^{m_1} + c_2 x^{m_2}}$$ General sol^n

Case II : Complex roots

$m_1 = \mu + i\nu \qquad m_2 = \bar{m}_1 = \mu - i\nu$

$y_1(x) = x^{\mu + i\nu} = (e^{\ln x})^{\mu + i\nu}$

~~$y_2(x)$~~

$(e^{\ln x})^{i\nu} = e^{i\nu \ln x}$ [of form $e^{i\theta}$]

$= \cos(\nu \ln x) + i \sin(\nu \ln x)$

$y_1(x) = \overset{x^\mu}{\cancel{e^{\mu \ln x}}}[\cos(\nu \ln x) + i \sin(\nu \ln x)]$

$y_2(x) = \overset{x^\mu}{\cancel{e^{\mu \ln x}}}[\cos(\nu \ln x) - i \sin(\nu \ln x)]$

- adding and dividing by 2 :

$x^\mu \cos(\nu \ln x) = \tilde{y}_1(x)$

- subtracting & dividing by $2i$:

$x^\mu \sin(\nu \ln x) = \tilde{y}_2(x)$

form a basis

$$\rightarrow \boxed{y(x) = x^\mu [c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)]}$$ General sol^n

Case III : Double Root :

$m = \frac{1-a}{2} \qquad y_1(x) = x^{(1-a)/2}$

$y_2 = v y_1$

$v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}$ $\quad (P(x) \rightarrow a/x, \quad -\int a \ln x\, dx)$

$= \int \frac{1}{\cancel{y_1^2}} \frac{x^{-a}}{x^{(1-a)}} dx = \ln x$

$$\boxed{y_2 = x^{(1-a)/2} \ln x}$$

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$$\boxed{y = x^{(1-a)/2}\left[c_1 + c_2 \ln x\right]}$$ General sol^n

2^{nd} Order linear and non-homogenous eq^n :

$$y'' + P(x)y' + Q(x)y = R(x), \quad x \in (a,b) \quad \text{--- ①}$$

Corresponding homogenous eq^n : $y'' + P(x)y' + Q(x)y = 0$ --- ②

$\downarrow$

$y_h = c_1 y_1 + c_2 y_2$ [General sol^n of eq^n ②]

y_p – particular sol^n of eq^n ①
(which doesn't involve any parameter)

Now, let $y(x)$ be a sol^n of eq^n ①

$y(x) - y_p(x)$: sol^n of eq^n ②

$$(y(x) - y_p(x))'' + P(x)(y(x) - y_p(x))' + Q(x)(y(x) - y_p(x))$$
$$= R(x) - R(x) = 0$$

Theorem: If y_h is the general sol^n of the homogenous eq^n ② & y_p is any particular sol^n of non-homogenous eq^n ①, then $\boxed{y(x) = y_h + y_p}$ is general sol^n of eq^n ①.

Theorem: Let y_1 and y_2 are two sol^n of eq^n ① on an open interval I. Then,

(i) $y = y_1 - y_2$ is a sol^n of ②.

proof:

$\xrightarrow{\text{linear}}$

$L(y) = y'' + P(x)y' + Q(x)y$

$\Rightarrow L(y) = R(x)$

$L(y_1 - y_2) = L(y_1) - L(y_2)$ [$\because$ linear]

$= R(x) - R(x) = 0$

$\Rightarrow (y_1 - y_2)$ is sol^n of eq^n ②

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(ii) let z be sol^n of ① on I and z_1 be any sol^n of eq^n ②. Then,

$y = z + z_1$ is a sol^n of eq^n ① on I

proof:

$L(z) = R(x)$, $L(x_1) = 0$

$L(z + z_1) = L(z) + L(z_1) = R(x) + 0$

$\Rightarrow z + z_1$ is sol^n of eq^n ①

Ex. $y'' - y = -2\sin x$

$\hookrightarrow y'' - y = 0$ $b = -1$

$\lambda^2 - 1 = 0$

$\Rightarrow \lambda = \pm 1$

$y = e^x, e^{-x}$

$y_h = c_1 e^x + c_2 e^{-x}$

$\Rightarrow$ If we take $y_p = 0$, then every y_n will be the sol^n of eq^n ① ???

$\Downarrow$

$y_p = 0$ is not sol^n of non-hom eq^n. It is only sol^n where $R(x) = 0$

$\Rightarrow y_p = \sin x$

$$\boxed{y(x) = c_1 e^x + c_2 e^{-x} + \sin x}$$

Eg. let $R_1(x)$ and $R_2(x)$ are two cont. $func^n$. let y_i's are particular sol^n of

$y'' + P(x)y' + Q(x)y = R_i(x)$, $i = 1, 2$

Show that $(y_1 + y_2)$ is a particular sol^n of

$y'' + P(x)y' + Q(x)y = R_1(x) + R_2(x)$

(i) Method of undetermine coeffecients :

Limitations : a) It works for 2^{nd} order eq^n with constant coeffecients

b) It only works with some particular form of $R(x)$

(ii) Methods of variations of Parameters : General sol^n of corresponding eq^n (2)

$$y_h(x) = C_1 y_1 + C_2 y_2$$

$$y(x) = y_h(x) + y_p(x)$$

$$y_p(x) = \underbrace{v_1(x) y_1 + v_2(x) y_2}_{\text{unknown}}$$

$$y_p'(x) = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' = (v_1' y_1 + v_2' y_2) + (v_1 y_1' + v_2 y_2')$$

Let $v_1' y_1 + v_2' y_2 = 0$ (enforcing this)

$$y_p'(x) = v_1 y_1' + v_2 y_2'$$

$$y_p''(x) = v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''$$

substituting $y_p(x)$, $y_p'(x)$ and $y_p''(x)$ in eq^n (1), we get

$$v_1(\underbrace{y_1'' + P y_1' + Q y_1}_{0}) + v_2(\underbrace{y_2'' + P y_2' + Q y_2}_{0})$$

$$+ v_1' y_1' + v_2' y_2' = R(x)$$

$$\boxed{v_1' y_1' + v_2' y_2' = R(x)}$$

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = R(x)$$

solving for v_1' and v_2', we get

$$v_1' = \frac{-y_2 R(x)}{y_1 y_2' - y_1' y_2} = \frac{-y_2 R(x)}{W(y_1, y_2)}$$

$$v_2' = \frac{y_1 R(x)}{y_1 y_2' - y_1' y_2} = \frac{y_1 R(x)}{W(y_1, y_2)}$$

Assume $W(y_1, y_2) = 0$

$$y_1 y_2' - y_1' y_2 = 0$$

Dividing whole eq^n by y_1^2, we get

$$\frac{y_1 y_2' - y_1' y_2}{(y_1)^2} = 0$$

$$\Rightarrow \quad \frac{d}{\,}\left(\frac{y_2}{y_1}\right)' = 0$$

$$\Rightarrow \quad y_2 = k y_1$$

$$V_1(x) = \int \frac{-y_2 \, (R(x))}{W(y_1, y_2)} \, dx$$

$$V_2(x) = \int \frac{y_1 \, R(x)}{W(y_1, y_2)} \, dx$$

$$y_p(x) = y_1 \int \frac{-y_2 \, R(x)}{W(y_1, y_2)} \, dx + y_2 \int \frac{y_1 \, R(x)}{W(y_1, y_2)} \, dx$$

Ex. Find particular sol^n of $y'' + y = \operatorname{cosec} x$.

Sol^n: Take $y'' + y = 0$

$\{\sin x, \cos x\} \to$ basis

$y_1(x) = \sin x \qquad y_2(x) = \cos x$

$y_1'(x) = \cos x \qquad y_2'(x) = -\sin x$

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0$$

$$V_1(x) = \int \cos x \operatorname{cosec} x \, dx = \ln|\sin x| + \underbrace{C}_{}$$

not taken as we are already finding out constants

$$V_2(x) = \int \frac{\sin x \operatorname{cosec} x \, dx}{-1} = -x$$

$$y_p = \sin x \ln(\sin x) - x \cos x$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

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$$\boxed{y(x) = c_1 \sin x + c_2 \cos x + \sin x \ln(\sin x) - x \cos x}$$ Ans.

Higher Order linear Differential Eq^{n}s :

(Coupled Harmonic Oscillator)

$$y^n + a_1 y^{n-1} + \ldots + a_{n-1} y' + a_n y = f(x)$$

$$y^n + a_1 y^{n-1} + \ldots + a_{n-1} y' + a_n y = 0$$

$$y(x) = e^{\lambda x}$$

$$\boxed{\lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0} \Rightarrow \text{Auxilliary eq}^n$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n) = 0$$

Case-I : λ_i's are distinct real roots

$y_2(x) = e^{\lambda_2 x}$ $\quad \{y_1(x), y_2(x) \ldots, y_n(x)\} \rightarrow$ basis

$W(y_1, y_2, \ldots, y_n) \neq 0$

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \ldots + c_n e^{\lambda_n x}$$

Case-II : $\lambda_1 = \lambda_2 = \ldots = \lambda_k$ are real roots with multiplicity k.

$y_1 = e^{\lambda_1 x}$, $y_2 = x e^{\lambda_1 x}$, $y_3 = x^2 e^{\lambda_1 x}$ ...

$y_k = x^{k-1} e^{\lambda_1 x}$

$$(c_1 + c_2 x + c_3 x^2 \ldots + c_k x^{k-1}) e^{\lambda_1 x}$$

Case III : Complex Roots :

$\lambda_1 = a + ib \qquad \lambda_2 = a - ib$

$$e^{ax}(A\cos bx + B\sin bx)$$

If $a+ib$ and $a-ib$ are roots with multiplicity k.

$$y(x) = e^{ax}\left[(A_1 + A_2 x + \dots + A_k x^{k-1})\cos bx + (B_1 + B_2 x + \dots + B_k x^{k-1})\sin bx\right]$$

Ques. $y'''' - 2y''' + 2y'' - 2y' + y = 0$

$y(x) = e^{\lambda x}$

$\lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1 = 0$

$(\lambda-1)(\lambda^3 - \lambda^2 + \lambda - 1) = 0$

$(\lambda-1)(\lambda-1)(\lambda^2+1) = 0$

$\lambda = 1, 1, i, -i$

1	1	-2	2	-2	1
	0	1	-1	1	-1
1	1	-1	1	-1	0
	0	1	0	1	
	1	0	1	0	

Existence and Uniqueness of IVP (2nd order)

$$y'' + P(x)y' + Q(x)y = 0 \quad \forall\ x \in [a,b] \quad \text{—— ②}$$

Initial Condn (I.C.) : $y(x_0) = y_0$, $y'(x_0) = y_1$ —— ③

Theorem : Let $P(x)$, $Q(x)$ are continuous and real valued funcn on $[a,b]$. Then, the IVP ② – ③ has one and only one solution in $[a,b]$.

→ Geometrical Interpretation :

IVP ② – ③ has an unique soln on $[a,b]$ that passes through specific point (x_0, y_0) with specific slope y_1.

In other words, a soln of IVP over all $[a,b]$ is completely determined by its value and the value of its derivative at a single point.

Eg $y'' + y = 0$ $\quad y(0) = 0 \quad y'(0) = 1$

$y_1(x) = \sin x \qquad y_2(x) = \cos x \qquad y = c_1 \sin x + c_2 \cos x$

Only y_1 satisfies both the condns

⇒ It has only one soln : $y = \sin x$

•) if $y(0) = 1$, $y'(0) = 0$

Only $y_2(x) = \cos x$ will satisfy the I.C.

* The soln completely changes on varying y_0, x_0

Theorem :
$$\left.\begin{aligned} &\overbrace{y'' + P(x)y' + Q(x)y}^{L(y)} = R(x)\ \forall\ x \in [a,b] \\ &y(x_0) = a , \quad y'(x_0) = b \end{aligned}\right\} \text{—— Ⓐ}$$

Assume $P(x)$, $Q(x)$, $R(x)$ are real valued continuous funcn on $[a,b]$

Prove that IVP (A) has unique sol^n.

Proof: Assume y_1 and y_2 are two distinct sol^{n}s of IVP (A)

$z = y_1 - y_2$

$= \cancel{y_1} \quad L(y_1) = R(x) \quad ; \quad L(y_2) = R(x)$

$\rightarrow \quad L(z) = \underset{\text{(linear)}}{L(y_1 - y_2)} = L(y_1) - L(y_2) = 0 \quad \forall \; x \in [a,b]$

$\rightarrow \quad (y_1 - y_2)(x_0) = a - a = 0$
$y_1(x_0) - y_2(x_0)$

$\rightarrow \quad z'(x_0) = y_1'(x_0) - y_2'(x_0) = 0$

$$\begin{cases} L(z) = 0 & \text{—— (B)} \\ z(x_0) = 0 & z'(x_0) = 0 \end{cases}$$

From earlier theorem, IVP (B) has one and only one sol^n $\Rightarrow$ ~~$y_1 - y_2 = 0$~~

$\Rightarrow$ ~~$\boxed{y_1 = y_2}$~~

$\Rightarrow$ ~~IVP (A) has a unique sol^n.~~

$z = 0$ is a sol^n for IVP (B) (satisfies IVP (B))

Since sol^n is unique $\Rightarrow z = 0$ is only sol^n

$\Rightarrow \quad y_1 - y_2 = 0 \quad \Rightarrow \quad \boxed{y_1 = y_2}$

$\Rightarrow$ IVP (A) has a unique sol^n.

???

Theorem: Let $y_1(x)$ and $y_2(x)$ be L.I. sol^{n}s of homogenous eq^n :

$$y'' + P(x)y' + Q(x)y = 0 \quad \forall \; x \in [a,b] \quad \text{—— (2)}$$

Then $c_1 y_1 + c_2 y_2$

is the general sol^n of (2) on $[a,b]$ in the sense that

lemma
Propositions } using these, we prove theorems

corollary → consequence of theorem
↓
can be proved using theorem

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every sol^n of ② on $[a,b]$ can be obtained from by suitable choices of c_1 and c_2.

Proof. Let $y(x)$ be any sol^n of eq^n ②.

Claim : $y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \forall \quad x \in [a,b]$

* If we can find c_1 & c_2 such that $y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) \Rightarrow$ we can prove above theorem.

differentiating once,

$$y'(x) = c_1 y_1'(x) + c_2 y_2'(x)$$

$$\Rightarrow \quad y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0)$$
$$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0)$$

$$\left[\begin{array}{cc|c} y_1(x_0) & y_2(x_0) & y(x_0) \\ y_1'(x_0) & y_2'(x_0) & y'(x_0) \end{array}\right] \sim \left[\begin{array}{cc|c} y_1 & y_2 & y \\ y_1' & y_2' & y' \end{array}\right]$$

for this to have unique sol^n, A should be invertible

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(y_1, y_2)\Big|_{x_0 = x_0} = (y_1 y_2' - y_1' y_2)\Big|_{x = x_0}$$

$\Rightarrow$ unique

Lemma

If $y_1(x)$ and $y_2(x)$ are any two sol^n of eq^n ② on $[a,b]$, then the Wronskian $W(y_1, y_2)$ is either identically zero or never zero on $[a,b]$ (Can't be zero at some point and non-zero at some other point)

Pf : $W(y_1, y_2)(x) : y_1 y_2' - y_1' y_2$

$W'(y_1, y_2)(x) : y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_1'' y_2$

$$\boxed{W' = y_1 y_2'' - y_1'' y_2}$$

$y_1'' + P(x) y_1' + Q(x) y_1 = 0 \quad \times y_2 \longrightarrow (a)$

$y_2'' + P(x) y_2' + Q(x) y_2 = 0 \quad \times y_1 \longrightarrow (b)$

(a) - (b) : $\underbrace{y_1'' y_2 - y_2'' y_1}_{-W'} + P(x) \underbrace{[y_1' y_2 - y_2' y_1]}_{-W} + Q(x) \underbrace{[\cancel{y_1 y_2} - \cancel{y_1 y_2}]}_{0} = 0$

$- W(x) - P(x) W(x) = 0$

$\Rightarrow W'(x) + P(x) W(x) = 0 \rightarrow$ 1st order linear

$\frac{W'(x)}{W(x)} = -P(x)$

$W(x) = c e^{-\int P dx} \rightarrow$ never zero $e^{\sim}$

$\Rightarrow W(x) \equiv 0 \iff c = 0$
$W(x) \neq 0 \iff c \neq 0$ $\quad | \; x \in [a, b]$

Lemma: If y_1 and y_2 are two solⁿs of eqⁿ (2) in $[a, b]$, then they are L.D. on $[a, b]$ if and only if their $W(y_1, y_2)$ is identically zero $\forall \; x \in [a, b]$

y_1, y_2 are LD $\iff W(y_1, y_2) \equiv 0$
$W(y_1, y_2) \neq 0 \iff y_1 \;\&\; y_2$ are L.I. $\quad | \; \forall \; x \in [a, b]$

Pf : Assume $y_1 \;\&\; y_2 \rightarrow$ L.D.

(if any of them $0 \Rightarrow W = 0 \quad (y_1 y_2'' - y_2 y_1')$)

so assume, both are non-zero

$y_2 = c y_1 \quad$ (L.D.)

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$= y_1 c y_1' - c y_1 y_1' = 0$ ✓

Converse: If $W(y_1, y_2) = 0$

$y_1 y_2' - y_1' y_2 = 0$

Divide by y_1^2

$\Rightarrow \dfrac{y_1 y_2' - y_1' y_2}{y_1^2} \Rightarrow \dfrac{d}{}\left(\dfrac{y_2}{y_1}\right) = 0 \Rightarrow y_2 = k y_1$

$\Rightarrow y_1, y_2$: L.D.

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y_1 and y_2 are two L.I. sol^n of

① — $y'' + P(x) y' + Q(x) = 0 \quad \forall x \in [a,b]$

Let $y(x)$ be any sol^n of ①

$\exists !$ arbitary const. c_1 & c_2 s.t.

② — $y(x) = c_1 y_1 + c_2 y_2 \quad \forall x \in [a,b]$

I.C. $y(x_0) = y_0 \quad y'(x_0) = y_1$

Enough to show $\exists !$ c_1 & c_2 s.t.

$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0)$

$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0)$

$$W(y_1, y_2)\Big|_{x=x_0} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

Lemma: y_1 & y_2 are sol^n of eq^n ①

Then $W(y_1, y_2)$ is either identically zero on $[a,b]$ or never zero

Lemma: ~~$W(y_1, y$~~ y_1 & y_2 are L.D. $\Leftrightarrow W(y_1, y_2) = 0 \quad \forall x \in [a,b]$

$W(y_1, y_2) \neq 0 \Leftrightarrow y_1$ & y_2 are L.I.

To find L.I. or L.D.

1. $y_1/y_2 = k \Rightarrow$ L.D

2. $W(y_1, y_2) = 0 \Rightarrow$ L.D.

Ex. $y'' + y = 0$

$y_1(x) = \sin x$ $\quad y_2(x) = \cos x$

$y_1'(x) = \cos x$ $\quad y_2'(x) = -\sin x$

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \Rightarrow \text{L.I.}$$

Ex. Prove that $y = x$ & $y = \sin x$ can't be a basis sol^n of any 2nd order differential eq^n:

$$y'' + p(x)y' + q(x)y = 0 \qquad x \in I \qquad \text{— (1)}$$

$y_1' = 1$ $\quad y_2' = \cos x$

$W(y_1, y_2) \neq 0 \Rightarrow$ can't be applied here.

Proof: since x & $\sin x$ are L.I.

$\therefore W(x, \sin x) \neq 0 \quad \forall x \in I$

$\Rightarrow x\cos x - \sin x$

at $x = 0$, $W(x, \sin x) = 0$

This is 0 at only one point.

By lemma, it should be 0 over all $x \in I$ ~~but~~ (should be L.D.) but here, both are L.I.

Ex. y_1 and y_2 are two sol^{n}s of (1) (above) with a common zero at any point in I. Show that y_1 & y_2 are L.D.

Common zero : $y_1(x_0) = y_2(x_0) = 0$

$$W(y_1, y_2)\Big|_{x=x_0} = y_1 y_2' - y_1' y_2\Big|_{x=x_0} = 0$$

By lemma, $W(y_1, y_2) = 0 \quad \forall x \in I$

$\Rightarrow y_1$ and y_2 are L.D

$y'' + P(x)y' + Q(x)y = R(x) \quad x \in I$

$y'' + P(x)y' + Q(x)y = 0$

* If coefficients are variable, there is no std. method to solve

Power Series Method :

$x_0 \in R$

$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 - - -$

is called a power series in x around x_0

x_0 - centre

-) $x_0 = 0$, $\sum_{n=0}^{\infty} a_n x^n \rightarrow$ check convergence

Def^n: The series $\sum_{n=0}^{\infty} a_n x^n$ converges at a point x if the limit $\lim_{m \to \infty} \sum_{n=0}^{m} a_n x^n$ exists. The limit is the sum func^n of the series.

Ex. (i) $\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!\,x^2 + \ldots$: converges only for $x=0$, (divergent)

(ii) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$: converges to e^x

(iii) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \rightarrow R = 1$

↳ Converges for $|x| < 1$
Diverges for $|x| \geq 1$

→ $R = 1$ - radius of convergence

converges for $|x| < R$
diverges for $|x| > R$

For (i), $R = 0$
For (ii), $R = \infty$ (converges for all x)

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$0 < R < \infty$

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Def^n : suppose $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$ with $R > 0$ and denote its sum func^n as $f(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$\rightarrow$ If it is convergent, we can differentiate it.

$$f'(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

if convergent,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

We can derive that

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{(compare from Taylor's series)}$$

Def^n : let $f : I \to R$ be a func^n, $x_0 \in I$. Then, f is called analytic around x_0 iff $\exists$ a $\delta > 0$ s.t.

$$③ \; — \; f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \forall \, x \text{ with } |x - x_0| < \delta$$

$f(x)$ has a power series representation in neigbourhood of x_0.

In this case $a_n = \frac{f^n(x_0)}{n!}$ & ③ is the Taylor series expression of f at x_0.

Eg. $\cos x$, $\sin x$, f^n's which are infinitely times differentiable.

* Power series method can be used if we have variable coefficients (which can't be solved by any previous methods)

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* Always make coeff. of $y'' = 1$.

Ex $y'' + y = 0$ Here, $P(x) = 0$ $Q(x) = 1$

$Q(x) = 1 = \sum_{n=0}^{\infty} a_n x^n$ where $a_1, a_2 \dots a_n \dots = 0$

(analytic func^n)

$P(x)$: also analytic

$\Rightarrow$ $P(x)$ and $Q(x)$ are analytic at $x_0 = 0$

Let sol^n of D.E. be : $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1) a_{n+1} x^n$

$y''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 \dots + (n+1)(n+2) a_{n+2} x^n$

$\Rightarrow$ $\cancel{(2a_2 + 6a_3 x + 12a_4 x^2 \dots) + (a_0 + a_1 x + a_2 x^2 \dots) = 0}$

$\Rightarrow$ $\cancel{(a_2 + a_0) + (a_3 + a_1)x + (a_4 + a_2)x^2 \dots}$

$\Rightarrow$ $y(x)$ satisfies above D.E. $\Rightarrow$

$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0 = 0 + 0 \cdot x + 0 \cdot x^2 + \dots$

$= (2a_2 + 6a_3 x + \dots + (n+1)(n+2) a_{n+2} x^n \dots) + (a_0 + a_1 x \dots + a_n x^n + \dots) = 0$

$= (2a_2 + a_0) + (6a_3 + a_1)x + \dots + ((n+1)(n+2) a_{n+2} + a_n) x^n + \dots = 0$

$\Rightarrow$ $2a_2 + a_0 = 0$ $\quad$ $6a_3 + a_1 = 0$ $\quad$ $\boxed{(n+1)(n+2) a_{n+2} + a_n = 0}$ $\quad n \geq 0$

$\Rightarrow$ $a_2 = \frac{-1}{2} a_0$ $\quad$ $a_3 = \frac{-1}{6} a_1$

$\cancel{(n+1)(n+2)}\, a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$ $\Rightarrow$ $a_4 = -\frac{a_2}{3 \times 4} = \frac{a_0}{2 \cdot 3 \cdot 4} = \frac{a_0}{4!}$

$a_5 = -\frac{a_3}{4 \cdot 5} = \frac{a_1}{5!}$, $\quad a_6 = -\frac{a_4}{5 \cdot 6} = -\frac{a_0}{6!}$

$\Rightarrow$ $y(x) = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 \dots$

$\Rightarrow$ $y(x) = \left[1 - \frac{1}{2}x^2 + \frac{x^4}{4!} - \frac{x^6}{6!} \dots\right] a_0 + \left[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{x^5}{7!} \dots\right] a_1 x$

$\Rightarrow$ $\boxed{y(x) = a_0 \cos x + a_1 \sin x}$

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Existence of a Power Series sol^n :-

$$y'' + P(x)y' + Q(x)y = R(x) \quad x \in I \quad \text{——} \quad ①$$

Theorem: Let $P(x), Q(x), R(x)$ admits a power ~~sol^n~~ series representation around a point $x = x_0 \in I$, with non-zero radius of convergence R_1, R_2 & R_3 respectively. $R = \min\{R_1, R_2, R_3, R_4\}$
Then eq^n ① has a sol^n $y(x)$ with power series representation around x_0 and Radius of $y(x)$ is R.

Def^n: A point x_0 is called ordinary point of ① if $P(x), R(x), Q(x)$ admit power series rep^n (with non-zero radius of convergence) around $x = x_0$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Eg. $xy'' + y = 0$

$$y'' + \frac{y}{x} = 0$$

$P(x) = \frac{1}{x}$ = not defined at $x = 0$

$\Rightarrow x_0 = 0$ is not an ordinary point

It can't have $y(x) = \sum_{n=0}^{\infty} a_n (x)^n$

but it may have sol^n $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$

Eg. $(x-1)y'' + (\sin x)y = 0 \qquad x_0 = 0$ or 1

$$y'' + \left(\frac{\sin x}{x-1}\right) y = 0$$

$x_0 = 0$ ✓

$x_0 = 1$ ✗ Not ordinary point $\Rightarrow$ singular point

Def^n: x_0 is called a singular point for eq^n ① if x_0 is not an ordinary point

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Legendre Eq^n :-

$(1-x^2)y'' - 2xy' + p(p+1)y = 0 \qquad -1 < x < 1$

p is real no. , $p \in \mathbb{R}$

→ Legendre Eq^n of order p.

sol^n of above eq^n is known as Legendre $func^n$
↳ spl $func^n$

⇒ $P(x) = \dfrac{-2x}{(1-x^2)} \qquad Q(x) = \dfrac{p(p+1)}{1-x^2}$

choose $x_0 = 0$ as ordinary point in this case

Ex. Find power series representation of $P(x)$ and $Q(x)$ around $x_0 = 0$

$P(x) = \sum_{n=0}^{\infty} b_n x^n \quad , \quad b_n = \dfrac{P^n(0)}{n!}$

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$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substituting in the eq^n,

$(1-x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$

⇒ $\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\text{to make it } x^n..} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$

$n-2 = m$ ⇒ term becomes $\sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} x^m = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

⇒ $\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=\cancel{2}\,0}^{\infty} n(n-1) a_n x^n - 2\sum_{n=\cancel{1}\,0} n a_n x^n + p(p+1)\sum_{n=0} a_n x^n = 0$

for $n=0,1$ term $= 0$, replace 2 by 0 — for $n=0$, term $= 0$, replace 1 by 0

⇒ $\sum_{n=0}\left((n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n\right)x^n = 0$

Equating coeffecient of x^n as zero,

$(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0 \quad , \quad n \geq 0$

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$$(n+1)(n+2)a_{n+2} + (p-n)(p+n+1)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{(p-n)(p+n+1)a_n}{(n+1)(n+2)}, \quad n \geq 0} \quad \text{--- (2)}$$

$n=0$ $\quad a_2 = -\frac{p(p+1)a_0}{2}$, $\quad n=1 \quad a_3 = -\frac{(p-1)(p+2)a_1}{2\cdot 3}$

$n=2$ $\quad a_4 = -\frac{(p-2)(p+3)a_2}{3\cdot 4} = \frac{(p-2)p(p+1)(p+3)a_0}{4!}$

$n=3$ $\quad a_5 = -\frac{(p-3)(p+4)}{4\cdot 5} = \frac{(p-3)(p-1)(p+2)(p+4)a_1}{5!}$

$$y = a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{(p-2)(p)(p+1)(p+3)}{4!}x^4 - \dots\right]$$

$$+ a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 \dots\right]$$

→ legendre func

$y = a_0 y_1(x) + a_1 y_2(x)$: General solⁿ of (1)

where both $y_1(x)$ and $y_2(x)$ are series funcⁿ.

contain even power of x → contain odd power of x

To check :

1.) y_1 & y_2 are L.I or not

2.) y_1 & y_2 are convergent.

→ $y_1/y_2 \neq$ constant funcⁿ ⇒ y_1 & y_2 are L.I.

Convergence

→ Case - I : p is not an integer

$n = 2k$ (even) ⇒ talking only about y_1 series, $n+1 = 2k+2$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a_{2k+2}}{a_{2k}}\right| = \left|\frac{a_{2k+2}\,x^{2k+2}}{a_{2k}\,x^{2k}}\right|$$

$$\lim_{n\to\infty} \left|\frac{-(p-2k)(p+2k+1)}{(2k+1)(2k+2)}\frac{a_{2k}\,x^{2k+2}}{x^2}\right| \to |x|^2$$

for $k \to \infty$

y_1 is convergent for $|x| < 1$

$\boxed{R = 1}$

* If $p = 2$ all terms after x^2 in y_1 will be 0, * y_1 : polynomial, y_2 : series
$p = 3$ all terms after x^5 in y_2 will be 0.

y_2 : polynomial
y_1 : series

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→ Case-II : p is non - -ive integer

If p is even, sum of y_1 terminates to a polynomial while y_2 is infinite series

If p is odd, sum of y_2 terminates to a polynomial while y_1 is infinite series.

??

$$\boxed{a_n = -\frac{(n+1)(n+2)}{(p-n)(p+n+1)} a_{n+2} \quad \text{for } n \le p-2} \quad \text{— (3)}$$

↳ can't be less than p

Def^n: A polynomial sol^n $P_n(x)$ of the Legendre ~~the~~ eq^n (1) is called legendre polynomial whenever $P_n(1) = 1 \ \forall n$

$n=0$: $a_0 = 1$ **

(*) — $$\boxed{a_n = \frac{(2n)!}{2^n (n!)^2}} = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n!}$$ $\frac{2^n (2n)!}{2^n (n!)^2}$

In eq^n (3), for $n = p-2$

$$a_n = -\frac{(p-1)(p) \cdot a_p}{2(2p-1)}$$

$$a_{n-2} = -\frac{(n-1)n}{2(2n-1)} a_n \quad \text{— (4)}$$

$$a_{n-2} = -\frac{(n)(n-1)\, a_n}{2(2n-1)} = -\frac{n(n-1)}{2(2n-1)} \frac{(2n)!}{2^n (n!)^2}$$

$$= -\frac{n(n-1)\, a_n}{2(2n-1)} \frac{2n(2n-1)(2n-2)!}{2^n\, n(n-1)!\, n(n-1)(n-2)!}$$

$$\boxed{a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!(n-2)!}} \quad \text{— (5)}$$

$\boxed{n = n-2}$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$= \frac{(2n-4)!}{2^n\, 2!\,(n-2)!\,(n-4)!} \qquad \{\text{From } ⊛\}$$

In general for $n-2m \geq 0$

$$\boxed{a_{n-2m} = \frac{(-1)^m\,(2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!}}$$

substituting these coefficients

(2) $$\boxed{P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m\,(2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!}\, x^{n-2m}}$$ legendre polynomial of ~~order~~ degree n

$M = \left[\frac{n}{2}\right]$ {greatest integer $\leq \frac{n}{2}$} $= \frac{n}{2}$ or $\left(\frac{n-1}{2}\right)$

Verify:-

$$= \frac{(2n)!\, x^n}{2^n\,(n!)^2} - \frac{(2n-2)!\, x^{n-2}}{2^n\,(n-1)!\,(n-2)!} + \ldots$$

$n=0$: $P_0(x) = 1$

$n=1$ $P_1(x) = \frac{2\cdot 1}{2\cdot 1}$ ~~$\frac{1\cdot n}{2}$~~ $= x$

$n=2$ $\boxed{M=1}$ $P_2(x)$: $\frac{4!\cdot x^2}{4\cdot 2!\cdot 2!} - \frac{2!\; x^0}{2^2\cdot 1!\cdot 1!} = \frac{3}{2}x^2 - \frac{1}{2}$

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$$\frac{d^n}{dx^n} x^{2n-2m} = \frac{(2n-2m)!}{(n-2m)!} \qquad (3)$$

For eg., $n=1$

$$\frac{d}{dx} x^{2-2m} = (2-2m)\, x^{2-2m-1}$$

$$= \frac{(2-2m)(1-2m)!}{(1-2m)!}\, x^{1-2m}$$

$$= \frac{(2-2m)!}{(1-2m)!}\, x^{1-2n}$$

From eq^n (2) and (3),

$$P_n(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m}{2^n\, m!\,(n-m)!} \frac{d^n}{dx^n} x^{2n-2m}$$

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* The sequence of polynomials are orthogonal to each other

$[n/2]$ से आगे की ~~terms ke liye~~ it will be 0.

we can write n here as $\frac{d^n}{dx^n} \sum$ -- remains same

Since the series is convergent

$$P_{(n)}(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-m)!} (x^2)^{n-m} (-1)^m$$

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n}$$ Rodrigue's formula

$n=0 \quad P_0(x) = 1$

$n=1 \quad P_1(x) = \frac{1}{2} 2(x) = x$

Sequence of legendre Polynomials

$$\{P_n(x)\}_{n=0}^{\infty}$$

Orthogonal Properties of legendre Polynomial

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & n = m \end{cases}$$

Proof :- $f(x) \in C^n[-1,1]$: $f(x), f'(x), \ldots, f^n(x)$: cont over $[-1,1]$

$$\int_{-1}^{1} f(x) P(x)\, dx = \int_{-1}^{1} f(x) \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2-1)^n]\, dx$$

$$= \int \frac{1}{2^n n!} \left\{ \left[f(x) \frac{d^{n-1}}{dx^{n-1}} [\underbrace{(x^2-1)^n}_{y^n}] \right]_{-1}^{1} - \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n\, dx \right\}$$

from 1 to -1, $x^2-1=0$ → This term will become 0

if we diff., we will get of form $y \cdot \underbrace{n!}_{const.}$

$$= \frac{-1}{2^n \cdot n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n\, dx$$: In each case, 1st term will be zero

$$I = \frac{(-1)^n}{2^n \cdot n!} \int_{-1}^{1} f^n(x) (x^2-1)^n\, dx$$

Case-I $m > n$ or $n > m$: let $m < n$

Let $f(x) = P_m(x) \in C^n[-1,1]$ {we know that}

$$I = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} P_m^n(x) (x^2-1)^n\, dx = 0 \quad \left[\frac{d^n}{dx^n} P_m(x) \right]$$

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$$I = \int_{-1}^{1} P_m(x) P_n(x) = 0$$

$\to$ $m > n$: change the role of m and n.

$f \in C^m$

$$I = \int_{-1}^{1} f(x) P_m(x)\, dx \quad \text{Replace } f(x) = P_n(x)$$

Case-II $\Rightarrow$ $n = m$

$$f(x) = P_n(x)$$

$$f^n(x) = \frac{d^n}{dx^n}\left(\frac{1}{2^n . n!} \frac{d^n}{dx^n}(x^2-1)^n\right) = \frac{1}{2^n n!} \frac{d^{2n}}{dx^{2n}}(x^2-1)^n$$

only non-zero term will be leading coefficient : x^{2n}
diff $2n$ times $\Rightarrow (2n)!$

$$= \frac{1}{2^n . n!} (2n)!$$

$$I = \frac{(-1)^n}{2^n . n!} \frac{(2n)!}{2^n . n!} \int_{-1}^{1} (x^2-1)^n\, dx$$

$$= \frac{2.(2n)!}{2^{2n}(n!)^2} \underbrace{\int_0^1 (1-x^2)^n\, dx}_{I_1} \qquad x = \sin\theta, \quad dx = \cos\theta\, d\theta$$

$$I_1 = \int_0^{\pi/2} \underset{①}{\cos^{2n}\theta}\ \underset{②}{\cos\theta}\, d\theta = \int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta$$

By parts

$$I_1 = \cancel{\cos^{2n}\theta \sin\theta \Big|_0^{\pi/2}} + 2n \int \sin\theta \cos^{2n-1}\theta \sin\theta\, d\theta .$$

Using ①

$$\int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta = \frac{2n}{2n+1} \int_0^{\pi/2} \cos^{2n-1}\theta\, d\theta \quad — (4)$$

$$I_1 = \frac{2n-2}{2n-1} \int_0^{\pi/2} \cos^{2n-3}\theta\, d\theta$$

In (4)

$$RHS = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \int_0^{\pi/2} \cos^{2n-3}\theta\, d\theta$$

$$= \frac{2n}{2n+1} \cdots \frac{2}{3} \int_0^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{2^n . n!}{1 \cdot 3 \cdots (2n+1)(2n-1)} \times \frac{2^n}{2^n} = \frac{2^{2n}(n!)^2}{(2n)!(2n+1)} \quad — (5)$$

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From eq^n ⑥,

$$I = \frac{2\,(2n)!}{2^{2n}\,(n!)^2} \times \frac{2^{2n}\cdot n!}{(2n)!\,(2n+1)} = \frac{2}{2n+1}$$

$$\boxed{I = \frac{2}{2n+1}}$$

•) $1 = P_0(x)$

$x = P_1(x)$

$x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$

$x^n = \sum_{n=0}^{\infty} a_n P_n(x)$ $\Rightarrow$ $\{P_n(x)\}_{n=0}^{\infty}$ is basis for set of Polynomial

→ given $f(x)$, find a_n in terms of $f(x)$ and $P_n(x)$, use orthogonal property to determine this coeffecient

$f_n(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ Multiply by $P_m(x)$, Integrate $\int_{-1}^{1}$
Then use orthogonal principal.

27-03-17

① Use orthogonality of $P(x)$ $\{P_n(x)\}_{n=0}^{\infty}$

$$\int f(x)\cdot P_m(x)\,dx = \int_{-1}^{1}\sum_{n=0}^{\infty} a_n P_n(x)\,P_m(x)\,dx$$

* $a_n P_n(x)$ converges to $f(x)$ ⇒ integration can be taken inside Σ

$$= \sum_{n=0}^{\infty}\int_{-1}^{1} a_n P_n(x)\,P_m(x)\,dx \quad \Rightarrow \text{ will be 0 excep when } m=n$$

$$= a_n \int_{-1}^{1} P_n^2(x)\,dx = a_n\cdot\frac{2}{2n+1}$$

$$\boxed{a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)\,P_m(x)\,dx}$$

Theorem : Generating function:

$h(t) = \frac{1}{\sqrt{1-2xt+t^2}}$: Coeffecient of power series will legendre polynomial.

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n, \quad t \neq 1 \quad (t:\text{ small})$$

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legendre polynomial : $(1-x^2)y'' - 2xy' + p(p+1)y = 0 \qquad x \in R$

$$P(x) = \frac{-2x}{1-x^2} \qquad Q(x) = \frac{p(p+1)}{1-x^2}$$

$x = \pm 1$ are singular points of above eq^n (not ordinary points)

Def^n: A singular point x_0 of ~~above~~ eq^n $y'' + P(x)y + Q(y) = 0$ is said to be regular if the func^{n}s $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are analytic at $x = x_0$, otherwise, it is called irregular singular point.

we are allowing singularity in $P(x)$ at most upto $\frac{1}{x-x_0}$ (power can't be more than 1) & in $Q(x)$ at most upto $\frac{1}{(x-x_0)^2}$ (power can't be more than 2)

For legendre polynomial

Case-I $\quad x_0 = 1$

$$(x-x_0)P(x) = (1-x)\left[+\frac{2x}{1-x^2}\right] = \frac{2x}{1+x} \text{ ; analytic at } x=1$$

$$(x-x_0)^2 Q(x) = (x-1)^2 \frac{p(p+1)}{1-x^2} = \frac{p(p+1)(1-x)}{1+x} = 0 \text{ at } x=1 \Rightarrow \text{analytic}$$

Hence, $x_0 = 1$ is regular singular point.

Case-II $\quad x_0 = -1$

$$-(x+1)\left[\frac{2x}{1-x^2}\right] = -\frac{2x}{1-x} \text{ : analytic}$$

$$(x+1)^2\left[\frac{p(p+1)}{1-x^2}\right] \text{ : analytic}$$

Bessel's Equation

③ — $x^2y'' + xy' + (x^2-p^2)y = 0$, $\quad p$: non-ive real no.

$$\Rightarrow y'' + \frac{y'}{x} + \left(1 - \frac{p^2}{x^2}\right)y = 0$$

$x = 0$: not an ordinary point

Regular point → $(x-0)\left(\frac{1}{x}\right) = 1$: analytic

$$(x-0)^2\left[1-\frac{p^2}{x^2}\right] = x^2\left[1-\frac{p^2}{x^2}\right] = x^2 - p^2 = -p^2 \text{ at } x=0$$
$\Rightarrow$ analytic

* $x=0$: Regular Point

$y = \sum a_n x^n$: can't be solⁿ of eqⁿ (3)

Theorem: Any differential eqⁿ of the form $y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0$ — (4)

where the function $p(x)$ and $q(x)$ are analytic at $x=0$, has atleast one solⁿ of the form

Frobenious series solⁿ $\Rightarrow$ $\boxed{y = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+m}}$ [using Euler-Cauchy eqⁿ]

solⁿ of Bessel's eqⁿ:

$p(x) = \sum_{n=0}^{\infty} p_n x^n$ $\quad q(x) = \sum_{n=0}^{\infty} q_n x^n$ [p_n and q_n : analytic]

substitute in eqⁿ (4),

$$y'' + \left(\frac{p_0 + p_1 x + p_2 x^2 + \cdots}{x}\right) y' + \frac{(q_0 + q_1 x \ldots)}{x^2} y = 0$$

$$y' = \sum_{n=0}^{\infty} a_n (n+m) x^{n+m-1} \qquad y'' = \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-2}$$

$a_0 \neq 0$ (~~we can choose~~ m such that $a_0 x, a_0 x^2 \ldots$ comes)

$\Rightarrow \sum_{n=0} a_n (n+m)(m+n-1) x^{n+m-2}$

Coeff. of a_0 (All constant terms in resulting series)

(5) — $\boxed{m(m-1) + m p_0 + q_0 = 0}$ Indicial Eqⁿ

$a_1(\quad) = 0 \Rightarrow m = m_1$ & $m = m_2$ get 2 reccurrence relⁿs.

we will get 2 solⁿs m_1 and m_2. Corresponding 2 recurrence relations can be obtained.

Case-I : Roots not differing by an integer (distinct roots)

m_1 and m_2 solⁿs of eqⁿ (5)

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n \; ; \; y_2 = x^{m_2} \sum_{n=0}^{\infty} b_n x^n \; : \text{L.I. roots.}$$

Case-II : Equal ~~Double~~ roots $m_1 = m_2$

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = y_1(x) \ln x + x^{m_1} \sum_{n=0}^{\infty} b_n x^n$$

Case-III : distinct roots differing by integer

$$y_1(x) = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = k y_1 \ln x + x^{m_2} \sum_{n=0}^{\infty} b_n x^n$$

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consider : (all 2nd homogenous O.D.E. can be written in this form)

$$① \text{ — } \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[\lambda q(x) + r(x)\right] y = 0 \qquad x \in [a,b]$$

Eg. $(1-x^2) y'' - 2xy' + n(n+1) y = 0$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0 \quad , \quad \lambda = n(n+1)$$

so, $p(x) = 1-x^2 \quad q(x) = 1 \quad r(x) = 0$

I.C. — $y(x_0) = y_0 \quad y'(x_0) = y_1$

By Existence & Uniqueness theorem, $\exists$ unique sol^n in $[a,b]$

$y(a) = y_0$, $y(b) = y_1$

Boundary value problem

Eg. Vibration ~~of~~ on string (elastic) : Model can be represented by

$$y'' + \lambda y = 0$$

B.C. $y(0) = 0 \qquad y(\pi) = 0$

$p(x) = 1 \quad q(x) = 1 \quad r(x) = 0 \qquad a = 0 \, , \, b = \pi$

We are looking for non-trivial sol^n

Case-I λ is negative

Let $\lambda = -n^2$

$y'' - n^2 y = 0$

Auxiliary eq^n: $(y = e^{mx})$

$m^2 - n^2 = 0 \Rightarrow m = \pm n$

$y(x) = c_1 e^{nx} + c_2 e^{-nx}$

$y(0) = c_1 + c_2 = 0$

$y(\pi) = c_1 e^{n\pi} + c_2 e^{-n\pi} = 0$ | $\Rightarrow c_1 = c_2 = 0$

$y(x) \equiv 0$ — trivial sol^n.

Case-II: $\lambda = 0$

$y'' = 0$

$y(x) = c_1 + c_2 x$

$y(0) = c_1 = 0$

$y(\pi) = c_1 + c_2 \pi = 0$ | $c_1 = c_2 = 0$

$y(x) \equiv 0$

Case-III: λ is +ive $\Rightarrow \lambda = n^2$

$m^2 + n^2 = 0$

$m = \pm in$

$y(x) = A\cos nx + B\sin nx$

$y(0) = A = 0$

$y(\pi) = B\sin n\pi = 0$

$B = 0$ leads to trivial sol^n again.

Assume $B \neq 0 \Rightarrow \sin n\pi = 0 \rightarrow n = 0, \pm 1, \pm 2$

if $n = 0$: Case-II $\Rightarrow n \neq 0$

$\Rightarrow n = \pm 1, \pm 2, \ldots$

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We ~~got~~ take only +ive values of eigen value

Corresponding to each value, we have one sol^n (let)

$y_n(x) = B_n \sin nx$, for $n = 1, 2, \ldots$

$\lambda = n^2 = 1, 4, 9, \ldots$ (eigen values)

The sol^n corresponding to each eigen values are called as <u>Eigen function</u>.

$\{\lambda_n\}_{n=1}^{\infty} \rightsquigarrow \{y_n\}_{n=1}^{\infty}$

eigen value — eigen func^n

<u>Few Observation</u> :

(i) Eigen values are uniquely determined by the problem

(ii.) Eigen func^n are determined upto a non-zero constant factor (B_n is always variable)

(iii.) $\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$

(iv) $\lambda_1 < \lambda_2 < \lambda_3 \ldots < \lambda_n < \lambda_{n+1} < \ldots$

(Eigen values of the BVP are +ive real no.s which can be arranged in an increasing sequence)

(v.) n^{th} eigen func^n vanishes at the end points and exactly $(n-1)$ zeroes inside the interval

→ $\sin x$ $[0, \pi]$ no zero inside interval

$\sin 2x$ $[0, \pi]$ 1 zero " "

$\sin nx$ $[0, \pi]$ $(n-1)$ zeroes

⇒ Consider more general problem

$$[p(x)y']' + [\lambda q(x) + r(x)]y = 0 \quad \forall \quad x \in [a,b] \quad \text{— (1)}$$

homogenous B.C. $\left.\begin{aligned} c_1 y(a) + c_2 y'(a) = 0 \\ d_1 y(b) + d_2 y'(b) = 0 \end{aligned}\right\}$ — (2)

where c_1 or $c_2 \neq 0$ and d_1 or $d_2 \neq 0$ —— 3

Assume $p(x)$ and $q(x) > 0 \quad \forall x \in [a,b]$

The BVP 1 - 2 is called as Strum - Liouville Problem

Properties of Eigen func's

$$\{\lambda_n\}_{n=1}^{\infty} \rightsquigarrow \{y_n(x)\}_{n=1}^{\infty}$$

λ_m and λ_n are two distinct eigen values

$y_m(x)$ and $y_n(x)$ are two corresponding eigen func of eq 1

$\Rightarrow (p(x) y_m')' + (\lambda_m q(x) + r(x)) y_m = 0$ —— $* y_n$

and, $(p(x) y_n')' + (\lambda_n q(x) + r(x)) y_n = 0$ —— $* y_m$

$\Rightarrow (p(x) y_m')' y_n - (p(x) y_n')' y_m + (\lambda_m - \lambda_n) q(x) y_m y_n = 0$

$\Rightarrow \int_a^b (\lambda_m - \lambda_n) q(x) y_m y_n dx = \int_a^b (p(x) y_n')' y_m \, dx - \int_a^b (p(x) y_m')' y_n \, dx$

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$= \int_a^b$

$= y_m p(x) y_n' \Big|_a^b - \int y_m' p(x) \cancel{y_n'} \, dx - y_n p(x) y_m' \Big|_a^b + \int y_n' \cancel{p(x) y_m'} \, dx$

$= \cancel{y_m y_n' (p(b) - p(a)) - y_n y_m' (p(b) - p(a))}$

$= y_m(b) p(b) y_n'(b) - y_m(a) p(a) y_n'(a) - y_n(b) p(b) y_m'(b) + y_n(a) p(a) y_m'(a)$

$= p(b) [y_m(b) y_n'(b) - y_n(b) y_m'(b)] - p(a) [y_m(a) y_n'(a) - y_n(a) y_m(a)]$

$= p(b) W(b) - p(a) W(a)$

y_m and y_n are eigen func^n corresponding to λ_m & λ_n

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$\rightarrow$ $y(a) = 0$ & $y(b) = 0$, $y'(a) = 0$ & $y'(b) = 0$

Putting $a = 0$ in eq^n 3

$$\int_a^b q y_m y_n \, dx = 0 \rightarrow \text{for } m \neq n$$

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$$[p(x) y']' + [q(x)\lambda + r(x)] y = 0 \quad x \in (a,b) \text{ --- } (1)$$

B.C.
$$\begin{cases} c_1 y(a) + c_2 y'(a) = 0 \\ d_1 y(b) + d_2 y'(b) = 0 \end{cases} \longrightarrow (2) \quad p(x), q(x) > 0 \quad x \in [a,b]$$

where c_1 or $c_2 \neq 0$ and d_1 or $d_2 \neq 0$ — (3)

<u>Orthogonality</u> of <u>Eigen func^{n}s w.r.t. weight func^n</u> $(q(x))$

$$(\lambda_m - \lambda_n) \int_a^b q(x) y_m(x) y_n(x) = p(b) W(b) - p(a) W(a)$$

Special case: $y(a) = 0$ & $y(b) = 0$ — (i)

OR, $y'(a) = 0$ & $y'(b) = 0$ — (ii)

$$W(y_m, y_n)\Big|_{x=b} = y_m(b) y_n'(b) - y_m'(b) y_n(b) = 0$$

$$W(y_n, y_m)\Big|_{x=a} = y_n(a) y_m'(a) - y_n'(a) y_m(a) = 0$$

$\Rightarrow$ $\cancel{(\lambda_m - \lambda_n)} \int_a^b q(x) y_m y_n = 0$ for B.C. (i) & (ii)

y_m and y_n are eigen func^n satisfying eq^n 2 and 3

$c_1 y_m(a) + c_2 y_m'(a) = 0$ $\rightarrow$ $AX = b$

$c_1 y_n(b) + c_2 y_n'(b) = 0$

Earlier we had 4 variables

$$A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This system has uniqe sol^n of c_1 & c_2 if determinant of coeffecient matrix:

It has unique sol^n if rank A = rank $[A|b]$

$X = A^{-1} b$

$AX = 0$

$\rightarrow$ unique sol^n: $|A| = 0$

$$\begin{vmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{vmatrix} = 0$$

we also know c_1 or $c_2 \neq 0$

$\Rightarrow W(y_m, y_n)\Big|_{x=a} = 0 \Rightarrow W(a) = 0$

Similarly, $W(b) = 0$

Hence for B.C. 2,

$$\boxed{\int_a^b q(x)\, y_m y_n \, dx = 0 \quad , \quad m \neq n}$$

Ques. Find coeffecients a_n s.t. for any given func^n $f(x)$, ↳ continuous func^n

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

multiplying $q(x) y_m$ both sides

$$\int_a^b q\, y_m f(x) = \sum_{n=1}^{\infty} \int_a^b q\, a_n y_m y_n(x)\, dx$$

$$\int_a^b q(x) y_m f(x) = a_m \int_a^b q(x)\, y_m^2(x)\, dx$$

$$a_m = \frac{\int_a^b q(x) y_m(x) f(x)\, dx}{\int_a^b q(x)\, y_m^2(x)\, dx} \Rightarrow \text{non-zero}$$

Def^n: 1. BVP 1 – 3 with $p(x), q(x) > 0$ on $[a,b]$ and continuous on $[a,b]$ are called regular problem.

2. If one of $p(x), q(x)$ vanish or become infinite at the end points, or the interval itself is infinite then, we call the problem as singular.

Eg. $[(1-x^2)y']' + \lambda y = 0 \quad -1 \le x \le 1, \quad \lambda = n(n+1)$

$P(x) = 1 - x^2$: vanish at $x = -1, 1$

$\Rightarrow$ singular problem (sturm liouville problem on $[-1,1]$)

(Legendre Polynomial)

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Ques. Show that every eigen func^n is unique except for a constant factor.

PROOF: Let u and v be two eigen func^s corresponding to an integer value λ.

claim: u is constant multiple of v, $W(u,v) = 0$

Proof:
$$(pu')' + (q\lambda + r)u = 0 \quad \times v$$
$$(pv')' + (q\lambda + r)v = 0 \quad \times u$$
λ is same for both

$$\Rightarrow (pu')'v - (pv')'u = 0$$
$$\Rightarrow (p'u' + pu'')v = (p'v' + pv'')u$$
$$\Rightarrow p(u''v - v''u) + p'(u'v - v'u) = 0$$
$$\Rightarrow p\,W'(v,u) + p'W(v,u) = 0$$
$$\Rightarrow [p\,W(u,v)]' = 0$$
$$\Rightarrow p\,W(u,v) = C$$

From B.C. ②, we get $C = 0$
$$\Rightarrow \boxed{W(u,v) = 0}$$
Hence Proved

For legendre polynomial:
$((1-x^2)y')' + \lambda y = 0$ $\quad q(x) = 1 \quad -1 \le x \le 1 \quad \lambda = n(n+1)$

		eigen func^n
for $n = 0$	$\lambda = 0$	P_0
$n = 1$	$\lambda = 2$	P_1
$n = 2$	$\lambda = 6$	P_2
⋮	⋮	⋮

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For any n, eigen value is $n(n+1)$.

eigen func^n is $P_n(x)$

Put $q(x) = 1$ $a = -1$ $b = 1$

$$\int_{-1}^{1} y_m(x)\, y_n(x)\, dx = 0 \quad \text{for } m \neq n$$