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Linear dependencies

Ex:- let $S = \{[1,0],[0,1]\}$ is LI in R^2

Sol:- $v_1 = [1,0]$

$v_2 = [0,1]$

To check LI / LD that

$a_1[1,0] + a_2[0,1] = [0,0] \Rightarrow [a_1, a_2] = [0,0]$

$a_1 = 0$ $\Rightarrow$ S is LI in R^2

$a_2 = 0$

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Ex:- let $S = \{[1,2], [2,4]\}$

Given $v_1 = [1,2]$

$v_2 = [2,4]$

to check LI / LD

$a_1 v_1 + a_2 v_2 = 0$

$a_1 [1,2] + a_2 [2,4] = [0,0]$

$a_1 + 2a_2 = 0$

$2a_1 + 4a_2 = 0$ $\Rightarrow$ $a_1 + 2a_2 = 0$

$\Rightarrow a_1 = -2a_2$

one of the non trivial solut^n

Suppose $a_1 = 2$ then $a_2 = -1$

are not zero.

$\Rightarrow$ S is linearly dependendent

Theorem: Two vectors in a vector space are linearly dependent iff one vector is scalar multiple of the other

Proof:- let v_1 and v_2 are two vectors in V and v_1, v_2 are LD then there exist two real #s a_1 and a_2 out of which a_1, a_2 atleast one is non zero (say $a_1 \neq 0$)

$a_1 v_1 + a_2 v_2 = 0$

$\because a_1 \neq 0$

$a_1 v_1 = -a_2 v_2$

$v_1 = \underbrace{\left(-\frac{a_2}{a_1}\right)}_{\text{scalar}} v_2$

$\Rightarrow$ v_1 is a scalar multiple of v_2

Conversely

Let v_1 be a scalar multiple of v_2

i.e $v_1 = a v_2$

$\Rightarrow a v_2 - v_1 = 0$

or $(-1)v_1 + a v_2 = 0$

linear combinat^n of v_1 & v_2 is zero but scalar $(-1) \neq 0$ $\Rightarrow v_1$ & v_2 are LD

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Ex:- Verify whether the

$S = \{[3,1,-1], [-5,-2,2], [2,2,-1]\}$ is LI in R^3

Sol:- Given $v_1 = [3,1,-1]$

$v_2 = [-5,-2,2]$

$v_3 = [2,2,-1]$ to check LI/LD

$a_1v_1 + a_2v_2 + a_3v_3 = 0$

$\Rightarrow a_1[3,1,-1] + a_2[-5,-2,2] + a_3[2,2,-1] = [0,0,0]$

$\Rightarrow 3a_1 - 5a_2 + 2a_3 = 0$

$a_1 - 2a_2 + 2a_3 = 0$

$-a_1 + 2a_2 - a_3 = 0$

Augumented matrix

$$\left[\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} a_3 = 0 \\ a_2 = 0 \\ a_1 = 0 \end{array}$$

$\Rightarrow v_1, v_2, v_3$ are LI

$S = \{[3,1,-1], [-5,-2,2], [2,2,-1]\}$ is linearly Independent.

Let $S = \{[3,1,-1], [-5,-2,2], [2,2,-1]\}$

$$A = \begin{bmatrix} 3 & 1 & -1 \\ -5 & -2 & 2 \\ 2 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ rank of $A = 3$

$\Rightarrow$ given set is LI

Ex:- Any set containing zero vector is always linearly dependent

while a singleton set containing a non zero vector is LI

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Sol:- let $S = \{0, v_1, v_2, \ldots, v_n\}$ be a set containing a zero vector then we can say

$1 \cdot \vec{0} + 0 \cdot \vec{v_1} + 0 \cdot \vec{v_2} + \ldots + 0 \cdot \vec{v_n} = 0$

$\Rightarrow S$ is LD $\because 1 \neq 0$

let $S = \{v_1\}$

$a_1 v_1 = 0$ but $v_1 \neq 0$

$\Rightarrow a_1 = 0$

$\Rightarrow S$ is LI

Theorem: A finite set S containing atleast two vectors is ~~to~~ LD iff some vectors of S can be expressed as a linear combinatn of other vectors in S.

Th:- A non empty finite subset S of a vectorspace V is LI iff every vector $v \in L(S)$ can be expressed uniquely as a linear combinatn of the members of S.

Proof:- let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of V

Suppose S is LI and v is any member of $L(S)$

$\Rightarrow$ v can be expressed as a linear combinatn of members of S

$\Rightarrow a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = v$

$b_1 v_1 + b_2 v_2 + \ldots + b_n v_n = v$

$(a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \ldots + (a_n - b_n) v_n = 0$

then LI of $S \Rightarrow a_1 - b_1 = 0 \Rightarrow a_1 = b_1$

$a_2 - b_2 = 0 \Rightarrow a_2 = b_2 \ldots, a_n = b_n$

$\Rightarrow$ uniqueness

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Conversely

Assume that every vector v ~~which~~ $\in L(S)$ can be expressed uniquely as a linear combination of members of S.

$\Rightarrow$ To prove $S = \{v_1, v_2, \ldots, v_n\}$ is LI

Let $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$

~~If~~ Also $0 \cdot \vec{v_1} + 0 \cdot \vec{v_2} + \ldots + 0 \cdot \vec{v_n} = 0$

$\Rightarrow 0 \in L(S)$

By uniqueness $a_1 = a_2 = a_3 = \ldots = a_n = 0$

$\Rightarrow S$ is LI

Note :- An ∞ subset S of a vector space V is LI iff every finite subset of S is LI

for ex :-

$S = \{1, x, x^2, \ldots\}$ is an linearly independent set in P (the vector space P is the vector space of all polynomials)

Basis

A subset B of vector space V is a basis of V iff B is LI and $L(B) = V$

$\Rightarrow$ The basis set B is LI and generates / spans the vector space V.

Ex:- The set $B = \{[1,0], [0,1]\}$ is a basis in R^2, called the standard basis of R^2

Sol :- For $B = \{[1,0], [0,1]\}$ to be LI

$a(1,0) + b(0,1) = (0,0)$

$a = 0$ $\Rightarrow$ B is LI

$b = 0$

Also the $L(B) \in R^2$

Since any $(x_1, x_2) \in R^2$ can be written as
$x_1(1,0) + x_2(0,1) = [x_1, x_2]$
A linear combinatⁿ of $B = \{[1,0], [0,1]\}$

Similarly
$\{[1,0,0], [0,1,0], [0,0,1]\}$ is std basis of R^3

The set $B = \{[1,2,1], [2,3,1], [-1,2,-3]\}$ is a basis in R^3

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ -1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ rank of $B = 0$ $\Rightarrow$ B is LI

$L(B) = a[1,0,0] + b[0,1,0] + c[0,0,1]$
$= [a,b,c] \in a,b,c \in R^3$
$\Rightarrow$ B is basis in R^3

Ex:- The set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
is the std basis in M_{22}

The set $B = \{1, x, x^2, ---, x^n\}$ is a std basis of P_n

Ex:- Empty set $\{ \quad \}$ is basis for $V = \{0\}$

Th:- If B_1 is a finite basis of a vector space V and B_2 is any other basis of V then B_2 has same # of vectors as in B_1