All are mutually exclusive

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$$P\left(\bigcup_{i=0}^{3}(x=i)\right) = \sum_{i=0}^{3} P\{x=i\}$$

Q 3 Balls are to be Randomly selected without Replacement from a bag containing 20 Balls no. 1 - 20. If we bet atleast one of the Ball that are drawn has a numbers as large as or greater than 17. What is the Prob. that we win the bet?

Sol :- Largest no. in the Ball is to 17, 18, 19, 20

X = the largest Ball among selected.

$X = 3 - 20$. (Because min no of those 2 Balls can only be 1 or 2)

$$P\{X=i\} = \frac{{}^{i-1}C_2}{{}^{20}C_3} \qquad i = 3, 4, 5 \ldots 20$$

largest selected ball is 17

$$P\{\underline{X=17}\} = \frac{{}^{16}C_2}{{}^{20}C_3}$$

then the other two balls can be of Range 1-16

$$P\{X=18\} = \frac{{}^{17}C_2}{{}^{20}C_3}$$

Total Probability = $P(X=17) + P(X=18) + P(X=19) + P(X=20)$ (To win the bet)

$$P\{X=19\} = \frac{{}^{18}C_2}{{}^{20}C_3}$$

$$P\{X=20\} = \frac{{}^{19}C_2}{{}^{20}C_3}$$

Q 3 Balls are Randomly chosen from a Bag containing 3W, 3 Red, 5 Black Balls. Suppose that we win $1 for each white Ball selected and loose $1 for each Red Ball selected. Find the Probability that we win the Money.

Sol For Black No loss (Nothing)

X - denote the total winning amount

$X = 0, \pm 1, \pm 2, \pm 3$

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$P\{X=1\} + P\{X=2\} + P\{X=3\}$

$P\{X=1\} = \dfrac{3C_1\ 5C_2 + 3C_2\ 3C_1}{11C_3} = P\{X=-1\}$

$P\{X=2\} = \dfrac{3C_2\ 5C_1}{11C_3} = P\{X=-2\}$

$P\{X=3\} = \dfrac{3C_3}{11C_3} = P\{X=-3\}$

Taking summation of all values

Probability $= 1/3$

Properties of Random variable :-

1) If X and Y are random variables (rv) then $X+Y$, $X-Y$, XY, X/Y $(Y \neq 0)$ are also rv. moreover $aX+bY$ is rv, $a, b \in R$

ii) if X is rv, $f: R \to R$ is function then $f(X)$ is also rv

iii) if X and Y are two rv and f is a function of two variables then $f(X, Y)$ are also rv.

- Discrete Random variable :- ~~An~~ Random variable whose range is either finite or countably infinite is DRV. otherwise called continous Random variable. eg measurement of height

- Probability Mass function :- corresponding to only discrete Random function. It is denoted by

PMF $\to f_X(\cdot)$

In probability :-

$f_X(x) = P\{X = x\}$

Definition :- A real valued function (f) defined by $f(x) = P\{X=x\}$ is called Probability Mass function (pmf of d rv)

$f_X(x)$

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ex Tossing of 2 coins :-

$\Omega = \{ (HH), (TT), (HT), (TH) \}$

X = no. of heads.

$$f_X(x) = \begin{cases} 1/4 & \text{if } x = 0 \\ 2/4 = 1/2 & \text{if } x = 1 \\ 1/4 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

your choice want to define it or not ← 0

This the Prob. Mass function of X.

- **Properties of Probability Mass function :-**

→ Let 'f' be a PMF of a Random variable X then

1) $f(x) \geq 0$ for all $x \in R$

2) $S = \{ x \in R : f(x) > 0 \}$ is either finite or countably Infinite

3) $\sum_{x \in X} f(x) = 1$ → x is taking all the values of X

1). $f_X(x)$ is denoting Prob. which is always greater than zero

2). It is talking about Range of the Random variable which is discrete → and it is always finite or countably Infinite

3). All the Probabilities are Mutually exclusive so Total Prob will always be one.

→ If you have a f^n $R \rightarrow R$ which satisfy all the above four properties Then their exist a Random variable X whose PMF is given by $f_X(x)$. So there Must exist a Random variable whose PMF is $f_X(x)$.

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20/1/17

- The pmf of a random variable X given by
$P\{X=x\} = \frac{c\lambda^x}{x!}$; $x = 0,1,2,3\ldots$ $\lambda > 0$;

(a) $P\{X=0\}$; (b) $P\{X>2\}$

$\sum_{x=0} P\{X=x\} = 1 = \sum_{x=0} \frac{c\lambda^x}{(x)!} = 1$

$c\sum_{x=0}^{\infty} \frac{\lambda^x}{(x)!} = 1$

$ce^{\lambda} = 1$ $\boxed{c = e^{-\lambda}}$

$P\{X=x\} = \frac{e^{-\lambda}\lambda^x}{(x)!}$

(a) $P\{X=0\} = \frac{e^{-\lambda}\lambda^0}{(0)!} = e^{-\lambda}$

$P\{X>2\} = 1 - P\{X=0\} - P\{X=1\} - P\{X=2\}$

$= \cancel{1-e^{-\lambda}}$

Discrete uniform Random Variable :-

Suppose random variable X takes finitely many values {say n} with probability of taking each values is same then X is called D.R.V. Suppose X takes $\{x_1, x_2, x_3 \ldots x_n\}$ values →

$1 = P\left[\bigcup_{i=1}^{n} [X = x_i]\right] = \sum_{i=1}^{n} P\{X = x_i\}$

(Because of whole sample space)

$= nP\{X = x_i\}$ $i = 1, 2, \ldots n$

$\boxed{P\{X = x_i\} = \frac{1}{n}}$

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~~Ber~~ Bernoulli Random variable :-

Suppose a trial or an experiment whose outcomes can be classified as success or failure is performed.

Suppose :- $X = 1$ denote outcome is success with Prob. P.

$X = 0$ failure prob. $(1-P)$.

$$f(0) = f_X(0) = \left.\begin{array}{l} P\{X=0\} = 1-p \\ P\{X=1\} = p \end{array}\right] \quad - (1)$$

a random variable whose pmf is given by eqn (1) is called Bernoulli Random Variable.

Binomial Random variable :- Suppose 'n' independent trials, each of which Results in a success with probability p or in failure with prob. $(1-P)$

X = denotes the number of success occurred

$$P\{X=k\} = {}^nC_k \, p^k (1-p)^{n-k} \quad -(2)$$

a rv whose pmf is given by eqn (2) is called Binomial rv with parameter (n,p).

$$\sum_{k=0} {}^nC_k \, p^k (1-p)^{n-k} = [p + (1-p)]^n = 1$$

$(1,p)$

$$P\{X=0\} = {}^1C_0 \, p^0 (1-p)^1 = 1-p$$

$$P\{X=1\} = {}^1C_1 \, p \, (1-p)^{1-1} = p$$

Binomial Random variable is a special case of Bernoulli Random variable with parameter $(1,p)$.

Q 5 fair coins are tossed if the outcomes are assumed Independent Find the pmf of the no. of heads obtained !

$X =$ number of heads occur

Binomial RV with $(5, 1/2)$

PMF :-

$P\{X=0\} = {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = 1/32$

$P\{X=1\} = {}^5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 = 5/32$

$P\{X=2\} = {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = 10/32$

$P\{X=3\} = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = 10/32$

$P\{X=4\} = {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 = 5/32$

$P\{X=5\} = {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 = 1/32$

Q It is known that screw produced by a company will be defective with prob 0.01 independently of each other. The company sells the screws in packets of 10 and offer a Money back guarantee that atmost 1 of the 10 screws is defective what proportion of the packets sold Must the company Replace ?

$X =$ denotes the number of defective screws in the packet.

$P\{X>1\} = 1 - P\{X=0\} - P\{X=1\}$

This is Binomial Random variable with Parameter $(10, 0.01)$ because all screws are Independent.

$P\{X>1\} = 1 - {}^{10}C_0 (0.01)^0 (0.99)^{10} - {}^{10}C_1 (0.01)^1 (0.99)^9$

$= 0.004$

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<u>Poisson Random Variable</u> :- A Random variable whose pmf is given as

$$P\{X=k\} = \frac{e^{-\lambda}\lambda^k}{(k)!}, \quad k=0,1,2 \ldots$$

$\lambda > 0$ is called poisson Random Variable

→ It is used as an approximation for Binomial Random variable :-

<u>Suppose</u> :- X is a B.R.V. with parameter (n,p) also n is large and p is so small s.t. np is of moderate size. $\boxed{\lambda = np}$

$$P\{X=k\} = {}^nC_k \, p^k (1-p)^{n-k} \qquad k=0,1,2\ldots$$

$$= \frac{n!}{(k)!(n-k)!} p^k (1-p)^{n-k}$$

$$= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left[1-\frac{\lambda}{n}\right]^{n-k}$$

$$P\{X=k\} = \frac{n(n-1) \text{---} (n-(k-1))}{n^k} \frac{\lambda^k}{(k!)} \cdot \left(\frac{(1-\lambda/n)^n}{(1-\lambda/n)^k}\right) \text{---} (1)$$

<u>Taking limiting case</u> :-

$$\left[1-\frac{\lambda}{n}\right]^n \quad n \to \infty \quad \cong e^{-\lambda}$$

$$\left[1-\frac{\lambda}{n}\right]^k \quad n \to \infty \quad \cong 1$$

$$\frac{n(n-1) \text{----} [n-(k-1)]}{n^k} \cong 1$$

Putting all these values in eqⁿ (1)

$$\boxed{P\{X=k\} \cong \frac{e^{-\lambda}\lambda^k}{(k)!}}$$

$\lambda = np$ no. of Mistakes

ex ① No. of Misprinting on a page of a Book.
ex ② No. of person in the community who survive to age 100.

$\lambda = np$ (For approximation)

n = no of people who survive to age 100.

Q Suppose that the probability that an item produced by a Machine will be defective is 0.1. Find the probability that a sample of 10 item will contain at most one defective item.

Sol: Binomial Random variable :-

$$\underbrace{{}^{10}C_0 (\cdot 1)^0 (0.9)^{10}}_{P\{X=0\}} + \underbrace{{}^{10}C_1 (\cdot 1)^1 (0.9)^9}_{P\{X=1\}} = 0.7361$$

By Poisson Random Variable :-

$$P\{X=0\} + P\{X=1\}$$

$$e^{-1} + e^{-1}$$

$$\approx 0.7358$$

$\lambda = np$

$\lambda = 1$

31/1/18 Geometric Random Variable :-

Suppose Independent trial is having probability p; $0 < p < 1$ of being a success are performed until a success occur.

X = denotes the no. of trials needed to occur a success.

$$P\{X = n\} = (1-p)^{n-1} \, p \quad \text{— ①}$$

p → Probability

Any Random variable whose PMF is given by eqⁿ① is called geometric Random variable

$(1-p)^{n-1}$: 1st $(n-1)$ trials fails and last one is a success

nth trial is a success

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Q- An urn contains N white Balls and M Black Balls. Balls are randomly selected one at a time until a ~~one~~ Black Ball is obtained. If we assume that each Ball selected is Replaced before the next one is drawn. What is the Prob. that exactly n draws needed (2) Atleast k draws needed.

Solution: X = denotes the no. of draws needed to select a Black Ball.

Geometric RV

$$P\{X=n\} = (1-p)^{n-1}\, p$$

$$p = \frac{M}{N+M}$$

$$P\{X=n\} = \left(\frac{M}{M+N}\right)\left(1-\frac{M}{M+N}\right)^{n-1}$$

$$= \left(\frac{M}{M+N}\right)\left(\frac{N}{M+N}\right)^{n-1}$$

(ii) $P\{X \geq k\}$ discrete random variable

$$= \sum_{n=k} p(1-p)^{n-1}$$

$$\rightarrow P\{X \geq k\} = (1-p)^{n-1}$$

$$= \left(1-\frac{M}{M+N}\right)^{k-1}$$

$$= \left(\frac{N}{M+N}\right)^{k-1}$$

$$= \left(1-\frac{M}{M+N}\right)^{k-1} = (1-p)^{k-1}$$

$$p\sum_{n=k}(1-p)^{n-1}$$

$$\rightarrow \left(\frac{M}{M+N}\right)\sum_{n=k}\left(\frac{N}{M+M}\right)^{n-1}$$

By Geometric Variable

$$\rightarrow \quad \frac{M}{M+N}\left[\frac{\left(N/M+N\right)^{k-1}}{1-\frac{N}{M+N}}\right] - \left(\frac{N}{M+N}\right)^{k}$$

(Absolutely continous)

→ **continous Random variable :-**

(i) life of a transistor

(ii) time that a train arrive at a specific station.

A Random variable is called continous R.V. if ∃ a non-negative function (f or f_X) defined on R i.e. $f_X : R \to R^+ \cup \{0\}$ having the property that for any subset B of R.

$$P[X \in B] = \int_B f_X(x)\,dx$$

$$= \int_B f(x)\,dx$$

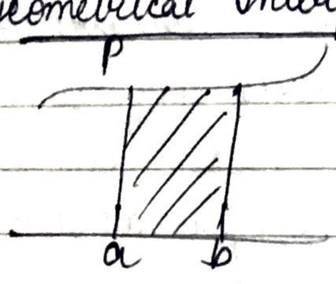
f is called pdf (prob. density f^n) of continous r.v X.

if $B = [a,b]$

$$P[X \in [a,b]] = P[a \leq X \leq b] = \int_a^b f(x)\,dx$$

Area Bounded by $f(x)$ from Interval A to B

Geometrical Interpretation of pdf.



- if $a = b$ (Prob. at a particular point)

$$P[X = a] = \int_a^a f(x)\,dx = 0$$

Including or excluding the end points does not effect on the probability of the function

All are same

\# $P[a < x \le b] = P[a \le x \le b]$

* $P[a \le x < b] = P[a < x < b]$

1. For f to be pdf — It should be non negative; satisfy Axiom 3 Normalisation Property

Question $f_X(x) = \begin{cases} 1/2\sqrt{x} & ; \ 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$

It is Valid pdf

$$\int_0^1 \frac{1}{2\sqrt{x}}\,dx = 2 \times \frac{1}{2}\left[x^{1/2}\right]_0^1 \Rightarrow \frac{1}{2} \times 2$$

$\Rightarrow 1$

Also it is non negative so PDF.

- Pdf is used to find the probability of an event but is not used to Represent probability of an event. Pdf itself is not a probability so it is not Restricted b/w 0 and on. It can have any value

- But In pmf $f_X(k) = P\{X = k\}$ It is the probability of an event.

→ Continous Uniform Random variable :-
A random variable that takes any values in interval $[a,b]$ and any two sub Interval of equal length have same probability. Then X is called CURV ~~denoting~~ denoted by $U[a,b]$ or $U(a,b)$

Pdf of CURV → $f(x) = \begin{cases} 1/b-a & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

Symbolically we denote CURV by $X \sim U[a,b]$

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Ques Buses arrive at a specific stop at 15 min interval starting 7 a.m. i.e. If a passenger arrive at the stop at a time that is uniformly distributed b/w 7 and 7:30 a.m. What is the probability that he waits less than 5 Minutes for a bus
(ii) He waits More than 10 Minutes

Solution :- X = denotes the no. of minutes past 7 that a passenger arrive at the stop
X is the uniform RV over [0, 30]

$$\text{Pdf} = \begin{cases} \frac{1}{30} & ; \quad 0 < x < 30 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$P\{\cancel{X}10 \leq X \leq 15\} + P[25 \leq X \leq 30]$$

$$\int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx = \frac{10}{30} = \frac{1}{3}$$

(ii) $P\{\cancel{5 \leq X \leq 10}\ 0 \leq X \leq 5\} + P\{15 \leq X \leq 20\}$

$$= \frac{1}{3}$$

Q. Why answers ~~are~~ same ?
Because for CURV for equal length sub Intervals probability is same.

5/2/18 ~~Uniform Ro~~ <u>Cummulative</u> <u>Distribution</u> <u>Function</u> :- (CDF):

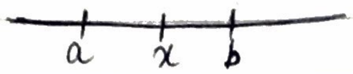
For uniform Random variable we have

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$F_X(x) = \sum_{\text{all } x \in X} p(x) = P(X \leq x)$$

↓
For discrete Random variable

eg



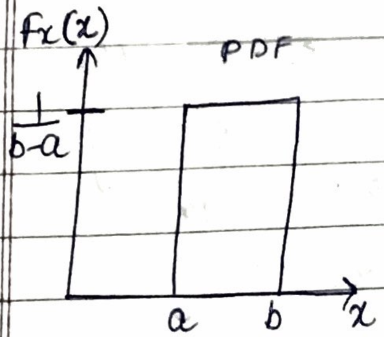
$$P(X \leq x) = \int_{-\infty}^{a} f_X(x)\,dx + \int_{a}^{x} f_X(x)\,dx$$

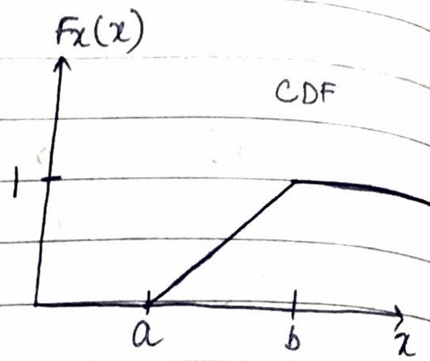
$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < a \\ \dfrac{x - a}{b - a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

→ **Application of cummulative distribution function :** uniform Random variable

- Signal system quantization process of sample signals prior to encoding in any digital communication system.

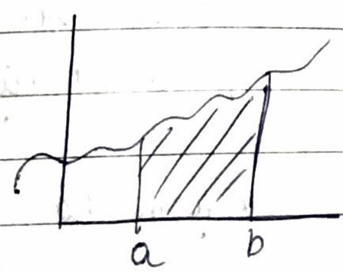
→





* Distribution function is always a non ↓sing function its derivative give us probability density function that is always positive

all same
$$P(a \leq x \leq b) = \int_{a}^{b} f_X(x)\,dx$$
$P(a \leq x < b)$
$P(a < x < b)$
$P(a < x \leq b)$



only for continous uniform Random variable.

→ **Exponential Random variable :-**

A continous Random variable whose pdf is given by :-

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

then X is said to be exponentially distributed with parameter $\lambda > 0$

$X \sim E(\lambda)$ $\qquad\qquad R_X : [0, \infty)$

<u>CDF for exponential random variable</u> :-

$$F_X(x) = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{a} \lambda e^{-\lambda x}\,dx + \int_{a}^{x} \lambda e^{-\lambda x}\,dx$$

$$\left.\frac{\cancel{\lambda} e^{-\lambda x}}{-\cancel{\lambda}}\right]_{-\infty}^{a} + \left.\frac{\lambda e^{-\lambda x}}{-\lambda}\right]_{a}^{x}$$

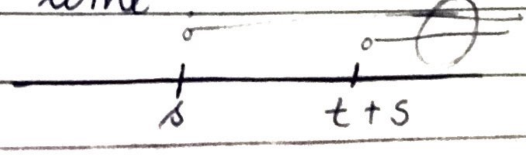
$$e^{-\lambda a} - \frac{1}{\lambda}\left[e^{-x^2} - e^{-\lambda a}\right]$$

$$\begin{array}{c} F_X(x) \\ \Downarrow \\ P(X \leq x) \\ \Downarrow \\ \int_{-\infty}^{\infty} f_X(x)\,dx \end{array} = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

\# Probability that X takes values $\geq t+s$ given that it has taken values only $\geq s$.

$$P\left(X \geq t+s \,/\, X \geq s\right)$$

If X: Life time of an equipment. Then this probability will be what is the probability that X will survive $t+s$ time when it has already survived s time



$$P(X \geq t+s / X \geq s) = \frac{P(X \geq t+s)}{P(X \geq s)}$$

$$\rightarrow \frac{1 - P(X < t+s)}{1 - P(X < s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = \underline{P(X \geq t)}$$

$$\boxed{P(X \geq t+s / X \geq s) = P(X \geq t)}$$

\# Memoryless property

→ distribution of the amount of time (starting from now) unitil some specific event occurs. (All these type of things are

→ defined by exponential Random variable).

Ex suppose the lifetime of an aeroplane is exponentially distributed with parameter $\lambda = \frac{1}{10}$ Someone bought this aeroplane. Find out the probability that aeroplane will not fail in the next 5 years.

Sol Directly apply Memoryless Property.

t_0 $\qquad$ $t_0 + 5$

$$P(X \geq t_0 + 5 / X \geq t_0) = e^{-5\lambda}$$

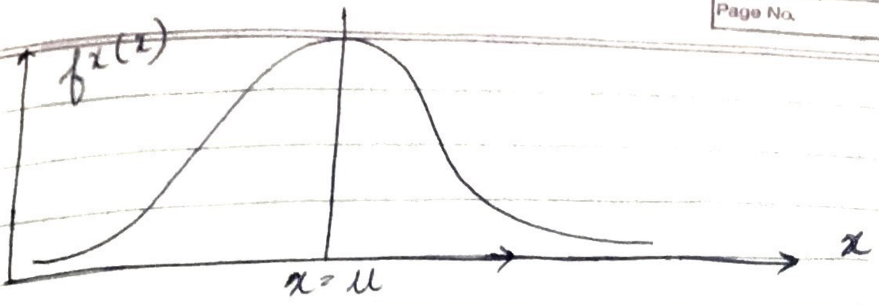
$$= e^{-5/10}$$

$$= e^{-0.5} = \underline{0.368}$$

7/2/18 Normal Random variable :- we say that a R.V. X is normal (Gaussian) r.v if its pdf is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

$$X \sim N(\mu, \sigma^2)$$



$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-u}{\sigma}\right)^2} dy$$

$$= G\left(\frac{x-u}{\sigma}\right)$$

$$F_X(x) = P(X \le x) = +\int_{-\infty}^{x} f_Y(y)\, dy$$

$$G(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$ [Prob. f^n of standard Normal ~~value~~ RV]

- $X \sim N(0,1) \Rightarrow$ X is said to be standard normal RV

Theorem If X is standard Normal RV with parameters u, σ^2 $X \sim N(u, \sigma^2)$ then show that $Y = \alpha X + \beta$ $\sim N(u\alpha + \beta, \alpha^2\sigma^2)$

X is normally distributed with parameters u, σ^2
Then Y is normally distributed with $u\alpha + \beta$ and $\alpha^2\sigma^2$.

Proof
$$F_Y(a) = P(Y \le a) = P(\alpha X + \beta \le a)$$

$$P\left(X \le \frac{a-\beta}{\alpha}\right) = F_X\left(\frac{a-\beta}{\alpha}\right)$$

$$= \int_{-\infty}^{a-\beta/\alpha} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-u}{\sigma}\right)^2} dx$$

$v = \alpha x + \beta \Rightarrow dv = \alpha\, dx$

$$F_Y(y) = \int_{-\infty}^{a} \frac{1}{\alpha\sigma\sqrt{2\pi}} \exp\left\{-\frac{(v-(\alpha\mu+\beta))^2}{2\alpha^2\sigma^2}\right\} dv$$

$\Downarrow$

$$F_Y(y) \sim N(\alpha u + \beta, \alpha^2\sigma^2)$$

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$X \sim N(\mu, \sigma^2)$

$Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

$\downarrow$

It is only a parameter defined as $\dfrac{X-\mu}{\sigma}$

By previous question

$\alpha = \dfrac{1}{\sigma} \quad \beta = -\dfrac{\mu}{\sigma}$

$N\left(\dfrac{1}{\sigma} \times \mu + -\dfrac{\mu}{\sigma}, \dfrac{1}{\sigma^2} \times \sigma^2\right)$

$N(0,1)$

✓ Hence Proved

Example Let $X \sim N(-5,4)$

(a) $P(X<0)$ (b) $P(-7<X<-3)$

Solution $P(X<0) = F_X(0)$ $\quad \mu = -5$, $\sigma^2 = 4$

$G\left(\dfrac{0-(-5)}{2}\right) = G(2.5)$

$P(Z \leq 2.5)$

$= 0.99376$

It is true only for cont. r.v.

$P(-7<X<-3)$

$P(X \leq -3) - P(X \leq -7)$

$F_X(-3) - F_X(-7)$

$G(1) - G(-1) = 2\{G(1)-0.5\}$

Symmetric about the pt $\mu = 0$

$\Rightarrow 2(0.84134 - 0.5)$

$= 0.66\ldots$

9/2/18 **Cummulative Distribution function :-**

$F_X(x) = P(X \leq x)$ defined for every

x from $-\infty$ to ∞.

(i) $F_X(x)$ is non decreasing.

(ii) $\lim\limits_{x \to \infty} F_X(x) = 1$

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(iii) $\lim_{x \to -\infty} F_X(x) = 0$

① **Proof** :- $a < b$

$\overset{\displaystyle a \qquad b}{\longrightarrow}$

Event $\{x \leq a\}$ must contain in $\{x \leq b\}$
Then $P\{X \leq a\}$ must be smaller Then $P\{X \leq b\}$

or $P\{X \leq a\}$ can be Replaced by $F_X(a)$
so $F_X(a)$ is smaller than $F_X(b)$
$\Rightarrow F_X(x)$ is non decreasing.

② **Proof** $\underline{P(X \leq \infty)} = \lim_{x \to \infty} F(x)$
each & every point of Real line is contained so That prob. will be 1.

③ No point is Included so zero.

$\rightarrow \boxed{P(a < X \leq b) = F_X(b) - F_X(a)}$

$P(a < X \leq b) = P(X \leq b) - P(X \leq a)$

Sample space

- countable (finite)
  - → Range of X also countable
  - $F_X(x)$ will be discrete.
- Uncountable
  - → Range of X may or may not be countable
  - $F_X(x)$ can be continous or can be discrete.

for $0 < X \leq 12$ → This is uncountable sample space

$X = \{0, 1, 2, 3 \ldots \}$
upto some finite value
countable sample space

Pointer on wheel of chance

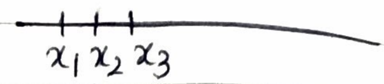
Distribution Function :- of a discrete r.v. is

p.m.f.

$$p(x) = P(X = x)$$

$x : x_1, x_2, x_3 \ldots$

$p(x_i) > 0$



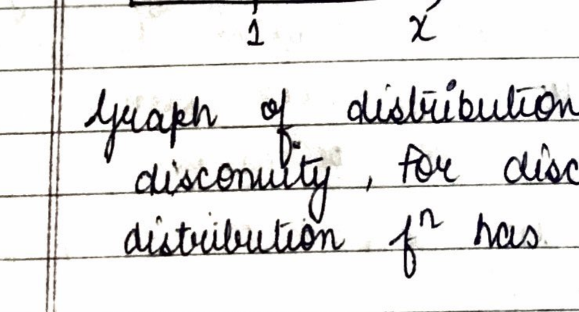
$p(x_i) = 0$ for other x. That function will qualify for pMf $\boxed{\Sigma(p(x_i)) = 1}$

$$F_X(x) = \sum_{\text{all } x_i < x} p(x_i)$$

ex1 Bernoulli Trials $X = \begin{cases} 1 & \text{success} \\ 0 & \text{failure} \end{cases}$

$P(X=1) = p$, $P(X=0) = 1-p$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$



Graph of distribution f^n has finite no. of disconuity, for discrete Random variable, distribution f^n has step function.

→ for a RV if distrubution f^n is given then to find (for previous question)

$$P(X=0) = F_X(0) - F_X(0^-)$$
$$= 1 - p - 0 = 1-p$$

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$P(X=1) = F_X(1) - F_X(1^-)$
$= 1-(1-p) = p$

$\boxed{P(X=a) \Rightarrow F_X(a) - F_X(a^-)}$ for discrete Random variable

Distribution f^n of a continous R.V. is defined by :-

$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$

pdf

* continous RV will have a continous distribution function.

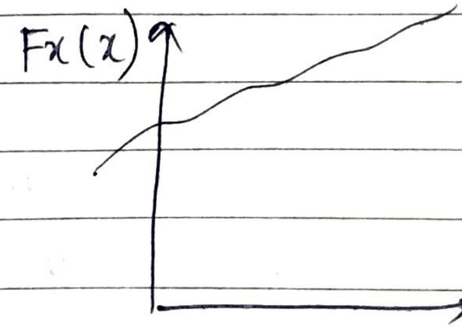
If $F_X(x)$ is differentiable then

$$\boxed{\frac{d}{dx} F_X(x) = f_X(x) = \text{pdf}}$$

$$\frac{d}{dx} F(x) = \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}$$

greater value

→ $f_X(x) \geq 0 \quad \forall x$



$\int_a^b f_X(x)dx = P(a<X<b)$

$P(a<X\leq b) = F_X(b) - F_X(a)$
$= \int_{-\infty}^{b} f_X(x)dx - \int_{-\infty}^{a} f_X(x)dx$
$= \int_a^b f_X(x)dx = P(a\leq X\leq b)$

$P(X=a) = F_X(a) - F_X(a^-) = 0$

Expectations of a Random (Discrete) Variable :-

Let X be a discrete Random variable with pmf $f_X(x)$. Then the expectation of rv X denoted by E(x) and defined as :

$$E(x) = \sum_{x \in X} x f_X(x)$$

provided the RHS series converges absolutely i.e. $\sum_{x \in X} |x| f_X(x) < \infty$

$$E(x) = \frac{\sum_{x \in X} x f_X(x)}{\sum_{x \in X} f_X(x)} = 1$$

It is the summation of all probabilities ←

E(x) is also the weighted average.

CONTINOUS :

→ Let X be a continous rv with pdf $f_X(x)$:-

$$E(x) = \int_{-\infty}^{\infty} x f_X(x)$$

provided the Improper Integral in RHS converges absolutely.

ex Let X be a random variable taking values of rolling a die.

$X = 1, 2, 3, 4, 5, 6$

pmf $f_X(x) = 1/6 \quad ; \quad x = 1, 2, 3, 4, 5, 6$

expectation of this Random variable :-

$$E(x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times 1/6 + 6 \times 1/6$$

$$= 1/6 [21] = 7/2$$

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ex $X \to$ Random Variable taking values 2^k with probability 2^{-k} i.e. $\frac{1}{2^k}$; $k = 1, 2, 3, 4 \ldots$ resp

Find expectation of X.

$$E(X) = \sum_{x \in X} x f_X(x) = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k}$$

$$= \sum_{k=1}^{\infty} 1$$

This series is not convergent as $a_n = 1$

$\lim_{n \to \infty} a_n = 1 \neq 0$.

→ $\sum_{x \in X} |x| f(x)$ does not converge absolutely so in this expectation is not defined properly.

ex $X = 2^k$ and -2^k with probability 2^{-k} for $k = 1, 2, 3 \ldots$

. Firstly check whether the series is convergent or not!

$$\sum_{x \in X} |x| f_X(x) = \sum_{k=1} \left\{ 2^k \cdot \frac{1}{2^k} + 2^k \cdot \frac{1}{2^k} \right\}$$

$$= \sum_{k=1} 2.$$

Expectation of a function of Random Variable

when we want to calculate expectation of X^2 instead of X.

Let X be a discrete Random variable with pmf $f_X(x)$. Let $g: R \to R$

$$\boxed{E[g(x)] = \sum_{x \in X} g(x) f_X(x)}$$

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If X is continous RV with pdf $f_X(x)$

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

ex-1 X → RV takes values -1, 0, 1 with probability 0.3, 0.5, 0.2 respectively. Find the expectation of X^2.

$$E(X^2) \Rightarrow g(x) = X^2$$

$$E(g(x)) = \sum_{x \in X} x^2 f_X(x)$$

$$\rightarrow 1 \times 0.3 + 0 \times 0.5 + 1 \times 0.2$$

$$\rightarrow 0.5$$

$\boxed{E(X) = \mu \text{ (Mean)}}$

- Properties of Mean :

$$E(aX+b) = aE(X) + b$$

$$E(aX+b) = \sum_{x \in X} (ax+b)\, f_X(x)$$

$$= a \sum_{x \in X} x f_X(x) + b \sum_{x \in X} f_X(x)$$

$$\boxed{E(aX+b) = aE(x) + b}$$

expecation is nothing but mean of the function

- Variance : It is the measurement of scattering of R.V. around the mean (expected value).

* [If values are scattered near the mean; variance is small when values are scattered far from the mean; variance is large

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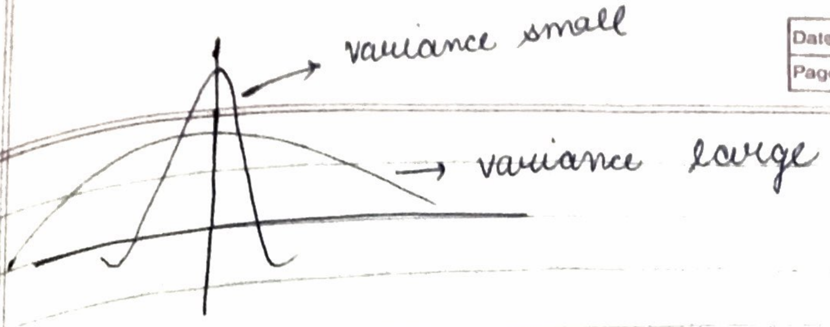

\# Let X be a Random variable then Variance is defined as :-

$$\boxed{Var(x) = E[(x-\mu)^2]} \rightarrow \text{non negative}$$

Square Root of Var (x) is Standard Deviation

only when var is non negative $\boxed{S.D = \sqrt{Var(X)}}$

\# Variance of a constant is always zero as mean of constant k is k $(k-k)=0$ because it measures the scattering of Random Variable.

① → $Var(aX+b) = E[(aX+b) - \underbrace{\mu(aX+b)}_{\text{Mean of } aX+b}]^2$

$$= E[(aX+b) - (a\mu+b)^2]$$

$$= E[(aX-a\mu)^2]$$

$$= E[a^2(X-\mu)^2]$$

$$= a^2 E[(X-\mu)^2]$$

$$\boxed{Var(aX+b) = a^2 Var(X)}$$

② $\boxed{Var(X) = E(X^2) - [E(X)]^2}$

Suppose X is a discrete RV with pmf $f_X(x)$:

$$Var(X) = E[(X-\mu)^2]$$

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$$= \sum_{x \in X} (x-\mu)^2 f_X(x)$$

$$= \sum_{x \in X} (x^2+\mu^2-2x\mu) f_X(x)$$

$$\Rightarrow \sum_{x \in X} x^2 f_X(x) + \sum_{x \in X} \mu^2 f_X(x) - \sum_{x \in X} 2\mu x f_X(x)$$

$$\Rightarrow E(X^2) + \mu^2 \sum_{x \in X} f_X(x) - 2\mu \sum_{x \in X} x f_X(x)$$

$$\Rightarrow E(X^2) + \mu^2 . 1 - 2\mu \underbrace{E(X)}_{\mu} \quad 2\mu^2$$

$$\Rightarrow E(X^2) - \mu^2$$

$$\Rightarrow \boxed{E(X^2) - [E(X)]^2}$$

13/02/18 **Properties of Expectation operator :-**

P-1 if $X \geq 0$, $E(X) \geq 0$

suppose X is a discrete Random variable with pmf $f_X(x)$.

$$E(X) = \sum_{x \in X} x f_X(x)$$

$$= \sum_{x=-\infty}^{\infty} x f_X(x)$$

$$\boxed{\begin{array}{c} x<0 \\ f_X(x)=0 \end{array}} \qquad = 0 + \sum_{x=0}^{\infty} x f_X(x)$$

$x \geq 0$; $f_X(x) \geq 0$

$$\sum_{x=0}^{\infty} x f_X(x) \geq 0$$

$$E(x) \geq 0$$

\# If X is continous R.V. width pdf $f_X(x)$

$$E(X) : \int_{-\infty}^{\infty} x f_X(x)$$

$$= 0 + \int_0^{\infty} x f_X(x) \qquad E(x) \geq 0$$

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P-2 if $X \geq Y$, then $E(X) \geq E(Y)$

$X \geq Y$

$X - Y \geq 0$

by P-1 $E(X-Y) \geq 0$

$E(X) - E(Y) \geq 0$

$$\boxed{E(X) \geq E(Y)}$$

P-3 $|E(X)| \leq E(|X|)$

$X \leq |X|$

By P-2 $E(X) \leq E(|X|)$ — ①

$-X \leq |X|$

By P-2 $E(-X) \leq E(|X|)$ — ②

$\Rightarrow -E(X) \leq E(|X|)$

$\Rightarrow E(X) \geq -E(|X|)$ — ③

From eqn ① and ③

$-E(|X|) \leq E(X) \leq E(|X|)$

$$\boxed{|E(X)| \leq E(|X|)}$$

Expectation and Variance of discrete R.V. :-

(i) **Bernoulli R.V. :-** (probability - p)

X - BRV

$f_X(x) = p$; if $x = 1$

$= 1-p$; if $x = 0$

$P\{X=0\} = 1-p$

$P\{X=1\} = p$

$E(X) = \sum_{x \in X} x f_X(x)$

x taking only two values

$= 0 \cdot (1-p) + 1(p)$

$$\boxed{E(X) = p}$$

$$\boxed{\text{expectation} = p}$$

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Variance :-

$$Var(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \sum_{x \in X} x^2 f_X(x)$$

$$= (0)^2(1-p) + (1)^2(p) = p$$

$$\boxed{Var(x) = p - p^2 = p(1-p)}$$

2. Binomial Random Variable :

→ $f_X(x) = {}^nC_x \, p^x (1-p)^{n-x} \quad ; x = 0, 1, 2$

$$E(x) = \sum_{x \in X} x \, {}^nC_x (p)^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x . \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{x . n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)[n-1-(x-1)]!} p^{x-1} (1-p)^{n-1-(x-1)}$$

$$= np \underbrace{\sum_{x=1}^{n} {}^{n-1}C_{x-1} (p)^{x-1} (1-p)^{n-1-(x-1)}}$$

$$= np [p + (1-p)]^{n-1}$$

$$\boxed{E(x) = np}$$

opening the summation :-

$${}^{n-1}C_0 \, p^0 (1-p)^{n-1} + {}^{n-1}C_1 \, p^1 (1-p)^{n-2}$$

$$\text{———} \quad {}^{n-1}C_{n-1} \, p^{n-1} (1-p)^0$$

→ This summation is binomial expansion of $[p + (1-p)]^{n-1}$

<u>Variance</u> :-

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x \in X} x^2 f_X(x)$$

$$= \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} (x^2 - x + x) \, {}^nC_x \, p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \cdot {}^nC_x \, p^x (1-p)^{n-x} + \underline{\sum_{x=0}^{n} x \, {}^nC_x \, p^x (1-p)^{n-x}}$$

→ $$\sum_{x=2}^{n} x(x-1) \, {}^nC_x \, p^x (1-p)^{n-x} + np$$

→ $$\sum_{x=2}^{n} \frac{x(x-1)(n)!}{(n-x)! \, x!} p^x (1-p)^{n-x} + np$$

→ $$(n)(n-1) p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)! \, [n-2-(x-2)]!} p^{x-2} (1-p)^{[n-2-(x-2)]} + np$$

→ $$(n)(n-1) p^2 [p + (1-p)]^{n-2} + np$$

→ $$n(n-1)p^2 + np = E(X^2)$$

<u>Variance</u> :- $n(n-1)p^2 + np - n^2p^2$

$$\Rightarrow \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\boxed{Var \Rightarrow np(1-p)}$$

$$= \underline{npq}$$

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3. **Poission Random Variable :-**

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{(x)!} \quad ; x = 0, 1, 2 \ldots$$

$$E(X) = \sum_{x=0}^{\infty} \frac{x\, e^{-\lambda} \cdot \lambda^x}{(x)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{x \cdot \lambda^x}{(x)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!}$$

→ By this 1st term is negative ! Negative ! not defined

$$E(X) = e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$\Rightarrow \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$\Rightarrow \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{E(x) = \lambda}$$

$$E(X^2) = \sum_{x=0}^{\infty} \frac{x^2\, e^{-\lambda}\lambda^x}{(x)!}$$

$$\Rightarrow \sum_{x=0}^{\infty} (x^2 - x + x) \frac{e^{-\lambda}\lambda^x}{(x)!}$$

$$\Rightarrow \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{(x)!} + \sum_{x=0}^{\infty} \frac{x\, e^{-\lambda}\lambda^x}{(x)!}$$

$$\Rightarrow \sum_{x=2}^{\infty} \cancel{x}(\cancel{x-1}) \frac{e^{-\lambda}\lambda^x}{(x-2)!} + \lambda$$

$$\Rightarrow e^{-\lambda} \cdot \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$E(X^2) = e^{-\lambda}\lambda^2 e^{\lambda} + \lambda$$

$$\boxed{E(x^2) = \lambda(\lambda+1)} \Rightarrow \lambda^2 + \lambda$$

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$$Var(X) = E(X^2) - [E(X)]^2$$
$$\Rightarrow \lambda^2 + \lambda - \lambda^2$$

$$\boxed{Var(X) = \lambda}$$

14/2/18 Expectation and Variance of Geometric Random Variable :

PMF $f_X(x) = p(1-p)^{x-1}$

$$E(X) = \sum_{x \in X} x f_X(x)$$

$$\Rightarrow \sum_{x=\phi}^{\infty} x\, p(1-p)^{x-1}$$

$$\Rightarrow \sum_{x=1}^{\infty} x p(1-p)^{x-1}$$

$$\rightarrow \sum_{x=1}^{\infty} (\overbrace{x+1-1}) \, p(1-p)^{x-1}$$

$$\rightarrow \sum_{x=1}^{\infty} (x-1)\, p\, (1-p)^{x-1} + \sum_{x=1}^{\infty} p(1-p)^{x-1}$$

$$\rightarrow \sum_{x=1}^{\infty} (x-1)\, p\, (1-p)^{x-1} + \frac{\cancel{p}}{\cancel{1}-(\cancel{1}-p)}$$

Put $x-1=i$ $\quad \sum_{i=0}^{\infty} i\, p(1-p)^{i} + 1$

$$\rightarrow \sum_{i=1}^{\infty} i\, p\, (1-p)^{i} + 1$$

expectation of i or expectation of x

$$\rightarrow (1-p) \underbrace{\sum_{i=1}^{\infty} i\, p\, (1-p)^{i-1}} + 1$$

$$E(X) = (1-p)\, E(X) + 1$$
$$E(X)[\cancel{1} - \cancel{1} + p] = 1$$

$$\boxed{E(X) = 1/p}$$

that question

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Variance :

$$E(X^2) = \sum_{x=1} x^2 p (1-p)^{x-1}$$

$$= \sum_{x=1} [x-1+1]^2 p (1-p)^{x-1}$$

$$= \sum_{x=1} [(x-1)^2 + 1 + 2(x-1)] p (1-p)^{x-1}$$

$$\Rightarrow \sum_{x=1} (x-1)^2 p (1-p)^{x-1} + 2\sum_{x=1} (x-1) p (1-p)^{x-1} + \underbrace{\sum_{x=1} p (1-p)^{x-1}}_{1}$$

$$\Rightarrow \sum_{x=1} (x-1)^2 p (1-p)^{x-1} + 2\sum_{x=1} (x-1) p (1-p)^{x-1} + 1$$

$\underline{x-1 = i}$

$$E(X^2) \rightarrow \underbrace{\sum_{i=0} i^2 p(1-p)^i}_{} + 2\sum_{i=0} i\, p(1-p)^i + 1$$

$$\rightarrow \sum_{i=1} i^2 p(1-p)^i + 2\sum_{i=1} i p(1-p)^i + 1$$

$$\rightarrow (1-p) \underbrace{\sum_{i=1} i^2 p(1-p)^{i-1}}_{E(i^2) = E(X^2)} + 2(1-p) \underbrace{\sum_{i=1} i p(1-p)^{i-1}}_{E(i) = E(X)} + 1$$

$$E(X^2) = (1-p) E(X^2) + 2(1-p) E(X) + 1$$

$$E(X^2)[\cancel{1} - \cancel{1} + p] = \frac{2(1-p)}{p} + 1$$

$$E(X^2) = \frac{2-2p+p}{p^2}$$

$$\boxed{E(X^2) = \frac{2-p}{p^2}}$$

$$\text{Variance } (X) = E(X^2) - [E(X)]^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$\boxed{Var(X) \Rightarrow \frac{1-p}{p^2} = \frac{q}{p^2}}$$

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Continous Random Variable :-

$X \to$ CRV ; pdf = $f_X(x) = \dfrac{1}{\pi(1+x^2)}$ $-\infty < x < \infty$

Find its expectation.

$E(X) = \int_{-\infty}^{\infty} x f_X(x)$ provided $\int_{-\infty}^{\infty} |x| f_X(x) < \infty$

Sol

$\to \int_{-\infty}^{\infty} |x| \dfrac{1}{\pi(1+x^2)}$

$-x = t$
$-dx = dt$

$\Rightarrow \int_{-\infty}^{0} \dfrac{-x}{\pi(1+x^2)} dx + \int_{0}^{\infty} \dfrac{x}{\pi(1+x^2)} dx$

$\Rightarrow \int_{0}^{\infty} \dfrac{x}{\pi(1+x^2)} dx + \int_{0}^{\infty} \dfrac{x}{\pi(1+x^2)} dx$

$\Rightarrow 2 \int_{0}^{\infty} \dfrac{x}{\pi(1+x^2)} dx$

$= \dfrac{2}{\pi} \int_{0}^{\infty} \dfrac{x}{1+x^2} dx$

$= \dfrac{2}{\pi} \lim_{y\to\infty} \int_{0}^{y} \dfrac{x}{1+x^2} dx = \dfrac{1}{\pi} \lim_{y\to\infty} \log(1+y^2)$

$\Rightarrow \underline{\infty}$

This Improper Integral is not convergent.
Example where expectation is not defined.

1) Uniform Continous Random Variable :-

$X \to$ UCRV

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} ; & a \le x \le b \\ 0 ; & \text{otherwise} \end{cases}$$

$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{a}^{b} x \dfrac{1}{b-a} dx$

$= \dfrac{1}{b-a} \int_{a}^{b} x\, dx = \dfrac{1}{(b-a)} \left[\dfrac{x^2}{2}\right]_a^b$

$\Rightarrow \dfrac{b^2-a^2}{2(b-a)} = \dfrac{b+a}{2}$

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Variance $= E(X^2) - [E(X)]^2$

$E(X^2) \Rightarrow \int_a^b x^2 \frac{1}{(b-a)} dx$

$\Rightarrow \frac{1}{(b-a)} \left[\frac{x^3}{3}\right]_a^b$

$\Rightarrow \frac{b^3 - a^3}{3(b-a)} = \frac{a^2+b^2+ab}{3}$

Variance $= \frac{a^2+b^2+ab}{3} - \left[\frac{a+b}{2}\right]^2$

$\Rightarrow \frac{a^2+b^2+ab}{3} - \left[\frac{a^2+b^2+2ab}{4}\right]$

$\Rightarrow \frac{4a^2+4b^2+4ab - 3a^2 - 3b^2 - 6ab}{12}$

$\Rightarrow \frac{a^2+b^2-2ab}{12}$

2) **Normal Random Variable** (continous) :-

$X \to$ NCRV

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$ with parameters μ & σ

$Z = \frac{X-\mu}{\sigma}$ The Z is standard normal continous Random Variable with parameter 0 and 1.

Pdf of This Standard NCRV

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$X = \sigma Z + \mu$

$$\left\{ \begin{array}{l} E(X) = \sigma E(Z) + \mu \\ Var(X) = \sigma^2 Var(Z) + 0 \end{array} \right\}$$

we are not doing this by usual method because Integration becomes so long & calculation is hard

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$$E(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-z^2/2}\, dz$$

taken $-\frac{z^2}{2} = t$ $\qquad -z\,dz = dt$

$$E(z) = \frac{1}{\sqrt{2\pi}} \left[-e^{-z^2/2} \right]_{-\infty}^{\infty}$$

$\boxed{E(z) = 0}$ expectation of Standard Normal CRV

$$E(z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\, dz$$

$$E(z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \cdot \underbrace{z \cdot e^{-z^2/2}\, dz}$$

Using By Parts

$$E(z^2) = 0 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

$z/\sqrt{2} = t$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

$$\int_{0}^{\infty} e^{-x^2}\, dx \qquad x^2 = t$$

$2x\,dx = dt$

$dx = dt / 2x\, \sqrt{t}$

$$\frac{1}{2} \int_0^{\infty} e^{-t \times dt / 2\sqrt{t}} \Rightarrow \cancel{\frac{1}{2} \int_0^{\infty} e^{-\sqrt{t}\,dt}}$$

$$\rightarrow \frac{1}{2} \int_0^{\infty} \cancel{d}\; t^{-1/2} e^{-t}$$

$$\rightarrow \frac{1}{2} \int_0^{\infty} t^{1/2 - 1} e^{-t}\, dt \rightarrow \frac{\sqrt{\pi}}{2}$$

$$\rightarrow \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

$$Var(z) = E(z^2) - [E(z)]^2$$

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$$E(Z^2) = \frac{1}{\sqrt{\pi}} \cdot \pi = 1$$

$$\boxed{Var(Z) = 1}$$

$$\boxed{E(X) = 0 + \mu = \mu}$$

$$\boxed{Var(X) = \sigma^2 \cdot 1 = \sigma^2}$$

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Exponential Random Variable :-

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases} \quad ; \quad \text{for } \lambda > 0$$

Improper Integeral

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x) = \int_0^{\infty} x^n \lambda e^{-\lambda x} \, dx$$

$$\rightarrow \left[-x^n e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} n x^{n-1} e^{-\lambda x} \, dx$$

$$E(X^n) = 0 + n \int_0^{\infty} x^{n-1} e^{-\lambda x} \, dx$$

$$E(X^n) = \frac{n}{\lambda} \int_0^{\infty} \lambda x^{n-1} e^{-\lambda x} \, dx$$

$$\boxed{E(X^n) = \frac{n}{\lambda} E(X^{n-1})}$$

for $n=1$ $\quad E(X) = \frac{1}{\lambda} \rightarrow \boxed{E(1) = \frac{1}{\lambda}}$

for $n=2$ $\quad E(X^2) = \frac{2}{\lambda} \times E(X) = \frac{2}{\lambda^2}$

$$\boxed{Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}}$$

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→ Distribution function of the function of Random Variable:

CDF $F_X(x) = P[X \le x] = \begin{cases} \sum_{x \le X} f_X(t) ; & X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(t)dt , & X \text{ is continous} \end{cases}$

If X is continous rv
→ $f_X(x) = \frac{d}{dx}\left[F_X(x)\right]$

If X is discrete.
$f_X(x) = F(a) - F(a^-)$

$F_X(a^-) = \lim_{n \to \infty} F_X\left(a - \frac{1}{n}\right)$

$P[a \le X \le b] = F_X(b) - F_X(a)$ continous

ex1 X is discrete RV ; $\underbrace{f_X(k) = P[X = k] = \frac{1}{9}}$ (PMF)
$k = -3, -2, -1, 0, 1, 2, 3 \ldots$

$g(x) = Y = |X|$; $g(x) = Y$ is another Random variable
PMF of Y

$Y = 0, 1, 2, 3 \ldots$

- $(Y=0) = (X=0) \Rightarrow P[Y=0] = P[X=0] = \frac{1}{9} = f_Y(0)$
- $(Y=k) = (X=-k) \cup (X=k) \Rightarrow P[Y=k] = P[X=-k] + P[X=k]$
$= 2/9$
$k = 1, 2, 3 \ldots$

$P[Y=y] = \sum_{x : g(x) = y} f_X(x) = \sum_{x : g(x) = y} P[X = x]$

⇒ How to find distribution f^n of RV y! just using the definition

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$$F_Y(t) = P[y \leq t] = \sum_{k \leq t} f_Y(k) = \sum_{k \leq t}\left[\sum_{x : g(x) = k} f_X(x)\right]$$

Distribution f^n of Exponential Random variable :-

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx = \int_{-\infty}^{0} 0 + \int_{0}^{a} f_X(x)\,dx$$

$$= \int_0^a \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_0^a = 1 - e^{-\lambda a}$$

$$\boxed{F_X(a) = \begin{cases} 1 - e^{-\lambda a} & a \geq 0 \\ 0 & a < 0 \end{cases}}$$

ex $X \to$ uniformly continous RV over $(0,1)$; F_X distribution function

$Y = X^n$

find distrubution f^n of Y and PDF of Y.

$$F_Y(y) = P[Y \leq y] = P[X^n \leq y]$$

$$= P[X \leq y^{1/n}] = F_X(y^{1/n})$$

as distrubution f^n of X is given F_X

$U(0,1)$

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} \dfrac{x - \alpha}{\beta - \alpha} & \alpha \leq x \leq \beta \quad 0 \leq x \leq 1 \\ 0 & x < \alpha \\ 1 & \text{if } x > \beta \end{cases}$$

Putting $\alpha = 0$ $\beta = 1$

$$\boxed{F_X(x) = x}$$

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$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$$\boxed{F_X(y^{1/n}) = y^{1/n}}$$ Distribution function

$F_Y(y) = y^{1/n}$ $\quad 0 \le y \le 1$

differentiating it

PDF $\boxed{f_Y(y) = \frac{1}{n} y^{1/n - 1}}$ $\quad 0 \le y \le 1$

\# Now when $X =$ only continously RV

$F_X =$ distribution function

$Y = X^2$

Find DF and PDF

$F_Y(y) = P[Y \le y] = P[X^2 \le y]$

$= P[-\sqrt{y} \le X \le \sqrt{y}]$

$$F_Y(y) = \begin{cases} F_X[+\sqrt{y}] - F_X[-\sqrt{y}] & ; \; y \ge 0 \\ 0 & ; \; y < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) & y \ge 0 \\ 0 & y < 0 \end{cases}$$

\# This can be used as a standard Result.

Let $X =$ exponential Random variable

$Y = X^2$

DF $F_Y(y) = \begin{cases} 1 - e^{-\lambda\sqrt{y}} - 0 & y \ge 0 \\ 0 & y < 0 \end{cases}$

PDF $f_Y(y) = \begin{cases} \frac{\lambda}{2\sqrt{y}} e^{-\lambda\sqrt{y}} & ; \; y \ge 0 \\ 0 & y < 0 \end{cases}$

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$X \rightarrow$ continous RV

$Y = |X|$

$$F_Y(y) = P[Y \leq y]$$

$$= P[|X| \leq y]$$

$$= P[-y \leq X \leq y]$$

$$F_Y(y) = F_X(y) - F_X(-y)$$

$$=$$

$$f_Y(y) = f_X(y) + f_X(-y)$$