The LNM Institute of Information Technology
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MATH-II ■ Assignment #2

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1. Which of the following are vector space:
   (a) R(Q) (b) C(Q) (c) R(C) (d) R(R)
   Where, Q: set of all rational numbers; R: set of all real numbers; C: set of all complex numbers.

a, b, d will satisfy all the properties of vector space, however © does not satisfy scalar multiplication. i.e; to a vector space $a \in F, \alpha \in V \Rightarrow a\alpha \in V$, but in ©
$2i \in C$ & $3 \in R$ but $(2i)*(3) = 6i \notin R$, Here $R = V$ & $F = C$.

2. Prove that the set $C[a,b]$ of all real valued continuous functions defined on the closed interval $[a,b]$ forms a real vector space if (i) addition is defined by $(f+g)(x) = f(x) + g(x)$, $f, g \in C[a,b]$, (ii) Multiplication by a real number r is defined by $(rf)(x) = rf(x)$, $f \in C[a,b]$. Prove that the subset $D[a,b]$ of all real valued differentiable functions defined on $[a,b]$ is a subspace of $C[a,b]$.

Can easily verified

Yes, $C[a,b]$ is a vector space and $D[a,b] \subseteq C[a,b]$. Additionally, it will satisfy $af + g \in D[a,b]$ $\forall a \in F$ & $f, g \in D[a,b]$

3. Which of the following are the subspaces of $\mathbb{R}^3$:
   (a) $\{(x,y,z) | x \geq 0\}$; (b) $\{(x,y,z) | x + y = z\}$; (c) $\{(x,y,z) | x = y^2\}$; (d) $\{(x,y,z) | xy = 0\}$.

NO — (a): $a = -2$, $\alpha = (1,1,1)$ but $a\alpha \notin$

(b): yes

(c): NO — $\alpha = (1,1,1)$, $a = 2$, but $a\alpha$

(d): NO — $\alpha = (1,0,0)$ & $\beta = (0,1,0) \in$ but $\alpha + \beta$ does not belongs to the collection

4. Express the polynomial $v = x^2 + 4x - 3$ in $P(x)$ as a linear combination of the polynomials $p_1 = x^2 - 2x + 5$, $p_2 = 2x^2 - 3x$, $p_3 = x - 1$. Apply $v = aP_1 + bP_2 + cP_3$ and solve, we get
$a = 5/9,\ b = 2/9,\ c = 25/9$

5. Determine whether the following sets of vectors are linearly independent or not
   (a) $S = \{(1,0,2,1), (1,3,2,1), (4,1,2,2)\}$ of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & -6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -6 & -2 \end{bmatrix}$$ in echelon form all rows are non-zero so L.I.

   (b) $S = \{(1,2,6), (-1,3,4), (-1,-4,-2)\}$ of $\mathbb{R}^3$.

Similarly $$\begin{bmatrix} 1 & 2 & 6 \\ -1 & 3 & 4 \\ -1 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & 5 & 10 \\ 0 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & 5 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$ L.I

   (c) $S = \{u+v, v+w, w+u\}$ in a vector space V given that $\{u, v, w\}$ is linearly independent. Apply the condition $a\alpha + b\beta + c\gamma = 0$ here $\alpha = u+v$, $\beta = v+w$, $\gamma = w+u$
so $a(u+v) + b(v+w) + c(w+u) = 0 \Rightarrow (a+c)u + (a+b)v + (b+c)w = 0$
As u, v, w are L.I. so $a+c = 0$, $a+b = 0$, $b+c = 0$, it gives $a = b = c = 0$

   (d) $S = \{(1,2,0), (3,-1,1), (4,1,1)\}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 1 \\ 0 & -7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ as one row is zero in echelon form so Linearly dependent

6. Let V be the vector space of functions from R into R. Show that the functions $f(x) = \sin(x)$, $g(x) = e^x$, $h(x) = x^2$ are linearly independent.

Check using the condition of LI & LD, i.e;

$a\,f(x) + b\,g(x) + c\,h(x) = 0$

$a \sin x + b e^x + c x^2 = 0$

it gives $a = b = c = 0$ so linearly independent

I Am) If $\{\alpha_1, \alpha_2 \cdots \alpha_n\}$ are L.D. then for some non-zero a_i

$a_1\alpha_1 + \cdots + a_k\alpha_k + \cdots + a_n\alpha_n = 0$

if $a_k \neq 0$ $\quad \alpha_k = -\frac{1}{a_k}(a_1\alpha_1 + \cdots + a_{k-1}\alpha_{k-1} + a_{k+1}\alpha_{k+1} \cdots + a_n\alpha_n)$

it shows that α_k is a linear combination of $\alpha_1, \cdots \alpha_n$

7. If the set of the vectors $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ in a vector space V over a field F be linearly dependent, then at least one of them is a linear combination of the remaining others.

no 8 → it is echelon form with no zero row so L.I.

8. Check whether the following four vectors in R^4 form a basis of R^4: (1,1,1,1), (0,1,1,1), (0,0,1,1), (0,0,0,1). Now to show span solve

$(x, y, z, w) = a(1,1,1,1) + b(0,1,1,1) + c(0,0,1,1) + d(0,0,0,1)$

Sol If we solve it ~~we get~~ the solution always exist for all values of x, y, z & w so span

9. Check which of the following polynomials in $P_2(x)$ form a basis of $P_2(x)$:

(a) $\{1, x, x^2\}$ — It is LI & span $P_2(x)$ so basis of $P_2(x)$

(b) $\{1 + x + x^2\}$ — It does not span $P_2(x)$

(c) $\{1, x, x^2, 1 + x + x^2\}$ — It is linearly dependent (LD)

(d) $\{1, 1 + x, 1 + x + x^2\}$. — It is LI & span $P_2(x)$ so basis.

Span + Linearly Independent = Basis.

10. Find the dimension of the following vector spaces

(a) $\{A : A$ is 2×3 real matrices$\}$. As it basis has six element so dimension is six

(b) $\{A : A$ is 3×3 real upper - triangular matrices$\}$.

Basis is $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$

As basis has six element so its dimension is six

(c) $\{A : A$ is 3×3 real symmetric matrices$\}$

Similarly dim = 6

(d) $\{A : A$ is 2×2 real skey-symmetric matrices$\}$ dim = 1 as basis $\left\{\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right\}$

11. Find the dimension and basis of the subspace W of $M_{2,3}$ spanned by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 3 \\ 7 & 5 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 2 & 4 & 3 & 7 & 5 & 6 \\ 1 & 2 & 3 & 5 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dim W = 2, basis $\left\{\begin{bmatrix}1&2&1\\3&1&2\end{bmatrix}, \begin{bmatrix}0&0&1\\1&3&2\end{bmatrix}\right\}$.

12. For what value of k, the matrix A has rank 2 if

$$\text{(I)} \quad A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 2 & 2 & 0 & 1 \\ 5 & 4 & 3 & k \end{bmatrix} \qquad \text{(II)} \; B = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 0 & 9 & 3 \\ 2 & 2 & 0 & 1 \\ 4 & 2 & 6 & k \end{bmatrix}.$$

(I) ns

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 2 & 2 & 0 & 1 \\ 5 & 4 & 3 & k \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -6 & -1 \\ 0 & 4 & -12 & k-5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -6 & -1 \\ 0 & 0 & 0 & k-3 \end{bmatrix}$$

if $k-3=0$ rank become 2

i.e, $k=3$

Apply row operation :-

(II)
$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 0 & 9 & 3 \\ 2 & 2 & 0 & 1 \\ 4 & 2 & 6 & k \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -6 & -1 \\ 0 & 2 & -6 & k-4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -6 & -1 \\ 0 & 2 & -6 & k-4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -6 & -1 \\ 0 & 0 & 0 & k-3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 - R_2$$

this entry should be zero, for the rank of matrix 2

thus $k-3=0 \Rightarrow k=3$