OPTIMIZATION TECHNIQUES

Formulation → Objective funcⁿ → Linear (LP Problem)

Non-linear (NLP Problem)

Optimize Z : Either maximize or minimize

eg. $Z = a_1x_1 + a_2x_2 + \dots + a_nx_n \rightarrow$ LP Problem

$Z = a_1x_1^2 + a_2x_2^2 + \dots + a_nx_n \rightarrow$ NLP Problem

Ques A company sells 2 different products A and B making profits of ₹40 and ₹30 per unit respectively. They are both produced with the help of common production process & sold in 2 different markets. The production process has a total capacity of 30,000 man-hours. It takes 3 hrs to produce a unit of A and 1 hour to produce a unit of B. The market has been surveyed & a company officials feel that max^m no. of unit A & B that can be sold are 8000 and 12000 respectively. Subject to these limitations, products can be sold in any combination. Formulate the problem as an LP model to maximize profit.

Sol^n: Let x_1, x_2 = no. of units of products A and B

$$Z = 40x_1 + 30x_2$$

$3x_1 + x_2 \le 30000$ (man hours)

$\left.\begin{array}{l} x_1 \le 8000 \\ x_2 \le 12000 \end{array}\right\}$ market

$x_1, x_2 \ge 0$ (non-negative constraint)

Some Important Definitions :-

Solutions : The value of decision variable x_j ($j = 1, 2, \dots n$) that satisfy the constraints of LP Problem is said to be constitute the sol^n to that LP Problem.

1. decision variables $(x_1, x_2, \dots)$
2. surplus variables
3. slack variable

2. Feasible solution : The set of decision variable x_j $(j = 1, 2, \dots, n)$ that satisfy all the constraints & non-negativity conditions of an LP Problem simultaneously is said to constitute the feasible solⁿ to that LP Problem.

3. Basic Solution : For a set of m simultaneous eqⁿs in n variables $(n > m)$, a solⁿ obtained by setting $(n-m)$ variables equal to 0 and solving the remaing m eqⁿs in m variables is called the basic solⁿ

Convert to Standard form :-

optimize (Max / Min) $z = c_1x_1 + c_2x_2 - - - + c_nx_n$

subject to constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$$

→ If we want Min (z), we can do Max (−z) = Max (z*) [−z = z*]

$$\text{Max}(z) = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0s_1 + 0s_2 + \dots + 0s_n$$

subject to
$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + s_2 = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m = b_m$$

$(b_1, b_2, \dots, b_m \ge 0)$

Eg. Min (z) = $2x_1 + 3x_2$

subject to $x_1 + x_2 \le 1$

$2x_1 - 3x_2 \le 2$

$\Rightarrow$ Max $(z^*) = 2x_1 + 3x_2 + 0s_1 + 0s_2$

$x_1 + x_2 + s_1 = 1$

$2x_1 - 3x_2 + s_2 = 2$

$\Rightarrow$ We can also write as :

$$Z = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{m} 0\, s_j$$

sub. to

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i \quad x_j, s_i \geq 0$$

Max $Z = cX + 0s$

$AX + s = b$, $x, s \geq 0$

Simplex Method :-

Step I :- Formulate & convert to std. form

Step II :- (Setup the initial solutions) In order to get basic feasible sol^n $[x_B = B^{-1} b]$ (represent in tabular form)

→ $\boxed{\text{coeff of basic variable : } C_B}$

→ Variable in basis : B

→ value of Basic variable b $(= x_B)$

→ $c_j \rightarrow c_1 \quad c_2 \ldots c_n \quad 0 \quad 0 \ldots 0$

Coeff of basic var C_B	Variable in basis B	value of Basic variable $b (= x_B)$	x_1	x_2	...	x_n	s_1	s_2	...	s_m
C_{B_1}	s_1	$x_{B_1} = b_1$	a_{11}	a_{12}		a_{1n}	1	0		0
C_{B_2}	s_2	$x_{B_2} = b_2$	a_{21}	a_{22}		a_{2n}	0	1		0
⋮	⋮	⋮								
C_{Bm}	s_n	$x_{Bm} = b_m$	a_{m1}	a_{m2}		a_{mn}	0	0		1

Always, $b_1, b_2 \dots b_m$ → should be positive no.

Camlin Page
Date / /

$z = \sum c_{B_i} x_i$ $\quad z_j = C_{B_i} x_j$

(Basic variable coeff.)
× values of Basic variable (BV)

= $\sum$ BV coefficient
× j^{th} column of data matrix

Step III : (Test for optimality)

9/8/17

Optimize $z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} 0 s_i$

subject to
$\sum_{j=1}^{n} a_j x_j + s_i = b_i \quad s_i, x_j \geq 0$ for all i, j

- Calculate $(c_j - z_j)$ value for all non-basic variables. To obtain the value of z_j, multiply each element under variable columns with the corresponding element with C_B

(i) If all $(c_j - z_j) \leq 0$ Then the basic feasible sol^n is optimal.

(ii) If atleast 1 column of the coeffecient matrix (a_k) for wh. $(c_k - z_j) > 0$ and other elements are non-negative (ie, a_{ik} ...) then ∄ unbound sol^{n}s to the given problem.

(iii) If atleast 1 $(c_j - z_j) > 0$ and each these column have atleast one positive element a_{ij} for some row, then this indicates that an element improved in the value of the objective func^n z is possible.

Step - IV : Select the variable to enter the basis if case (iii) of step-III holds, then select a variable that has the largest $(c_j - z_j)$ value to enter the new sol^n, ie,

$$c_k - z_k = \text{Max} \{ (c_j - z_j),\ c_j - z_j > 0 \}$$

The column is called key / pivot column

Step V : (Test for feasibility) (variable to leave the basis)

$$\frac{x_{Br}}{a_{rj}} = \text{Min.} \left\{ \frac{x_{ri}}{a_{rj}}, \; a_{rj} > 0 \right\}$$

→ The variable that is to leave the basis is determined by dividing the value in x_B column by the corresponding element in the key column. (Step - IV)

- Make pivot element 1.

Again check Step - I

Ques. Maximize $Z = 3x_1 + 5x_2 + 4x_3$ → linear

sub. to

$$\left.\begin{aligned} 2x_1 + 3x_2 &\leq 8 \\ 2x_2 + 5x_3 &\leq 10 \\ 3x_1 + 2x_2 + 4x_3 &\leq 15 \end{aligned}\right\} \text{RHS is +ive}$$

3 variables (x_1, x_2, x_3)

① Include 3 slack variables (3 constraints)

$$Z = 3x_1 + 5x_2 + 4x_3 + \underline{0}\, s_1 + \underline{0}\, s_2 + \underline{0}\, s_3$$

↳ C_B (check)

sub. to

$$2x_1 + 3x_2 + s_1 = 8$$

$$2x_2 + 5x_3 + s_2 = 10$$

$$3x_1 + 2x_2 + 4x_3 + s_3 = 15$$

$$x_1, x_2, x_3 \geq 0 \qquad s_1, s_2, s_3 \geq 0$$

② Take decision variables = 0, get value of slack variables for initial sol^n.

$x_1 = x_2 = x_3 = 0$ → C_B (check)

$s_1 = 8 \qquad s_2 = 10 \qquad s_3 = 15$

Max $(Z) = 0$

③ (To see whether the current sol^n is optimum or not.)

Compute $(C_j - Z_j) = c_j - C_B B^{-1} a_j = c_j - C_B y_j$

(matrix form)

C_B : coeffecients in initial sol

Tabular form :-

Profit/unit C_B	Variable in Basis	sol^n value	$C_j \to$ 3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3
				pivot element				
0	s_1	8	2	$\boxed{3}$	0	1	0	0
0	s_2	10	0	2	5	0	1	0
0	s_3	15	3	2	4	0	0	1
$z = 0$		z_j	0	0	0	0	0	0
		$c_j - z_j$	3	5	4	0	0	0
				↑ pivot column				

C_B x_1 column

$z_1 = 0(2) + 0(0) + 0(3) = 0$

$z_2 = 0(3) + 0(2) + 0(2) = 0$ (x_2 columns)

$z_3 = 0(0) + 0(5) + 0(4) = 0$ (x_3 column)

$c_1 - z_1 = 3 - 0 = 3$

$c_2 - z_2 = 5 - 0 = 5$

$c_3 - z_3 = 4 - 0 = 4$

$c_j - z_j > 0$, take max. as pivot column (x_2)

Take min $\left(\frac{x_B}{x_2}\right) \Rightarrow \min\left(\frac{8}{3}, \frac{10}{2}, \frac{15}{2}\right) = \frac{8}{3}$

row with element 3 : pivot row

- since this exchange ratio 8/3 is minimum in row 1, the basis variable s_1 is chosen to leave the solution.

R_1 (new) $\to$ R_1 (old) / 3

R_2 (new) $\to$ R_2 (old) $-$ 2R_1 (new)

R_3 (new) $\to$ R_3 (old) $-$ 2R_1 (new)

} make the pivot element 1

$s_1, s_2, s_3, \ldots$

→ coeff. of x_2 is 5 in objective funcⁿ (given) (Z)

Date 1 0 / 1 / 0

C_B	Variable in Basis B	x_B	$c_j \to$ 3: x_1	5: x_2	4: x_3	0: s_1	0: s_2	0: s_3	min ratio x_B/x_3
5	x_2 {pivot column}	8/3	2/3	1	0	1/3	0	0	—
0	s_2	14/3	-4/3	0	key element $\boxed{5}$	-2/3	1	0	14/15 key row
0	s_3	29/3	5/3	0	4	-2/3	0	1	29/12
		z_j	10/3	5	0	5/3	0	0	
Z =	40/3	$c_j - z_j$	-1/3	0	$\boxed{4}$	-5/3	0	0	

↑ key column

$Z = \sum C_B \cdot x_B$

$= 5 \times \frac{8}{3} = \frac{40}{3}$.

here, $x_2 = 8/3$ $s_2 = 14/3$ $s_3 = 29/3$ $x_1 = x_3 = s_1 = 0$

{ feasible solⁿ }

Now, min. ratio : $\frac{14}{15}$, $\frac{29}{12}$ { for pivot row }

(bcoz $c_j - z_j > 0$)

16/8/17

here, to calculate z_j, $(C_{Bj} \times x_j)$

$z_1 = (5)(2/3) + 0(-4/3) + 0(5/3) = 10/3$

$z_2 = (5)(1) + 0(0) + 0(0) = 5$

$z_3 = (5)(0) + 0(5) + 0(4) = 0$ like this

x_3 will enter & s_2 will ~~enter~~ leave

Now, we have to make the pivot element (5) 1.

R_2 (new) $\longrightarrow$ R_2 (old) / 5

R_3 (new) $\longrightarrow$ R_3 (old) $- 4\, R_2$ (new)

Scanned by CamScanner

$\frac{5\times 40}{5\times 3} + \frac{56}{15}$ $\qquad$ $5\times 46/3 + \frac{16}{15}$

Profit/unit C_B	Variable in basis (B)	Sol^n value $b\ (=x_B)$	c_j: 3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3
5	x_2	8/3	2/3	1	0	1/3	0	0
4	x_3	14/15	-4/15	0	1	-2/15	0	0
0	s_3	89/15	$\boxed{41/15}$ pivot element	0	0	2/15	-4/5	1
		z_j	34/15	5	4	17/15	4/5	0
$Z = \frac{256}{15}$		$c_j - z_j$	11/3 ↑	0	0	-17/15	-4/5	0

min ratio : $(8/3)/(2/3) = 4$

$(14/15)/(-4/15) = -$ (always +ive ratio is considered)

$(89/15)/(41/15) = 2.\ldots$

Now, to make the pivot element 1,

$R_3(\text{new}) \longrightarrow R_3(\text{old}) \times 15/41$

$R_2(\text{new}) \longrightarrow R_2(\text{old}) + 15/4\ R_3(\text{new})$

$R_1(\text{new}) \longrightarrow R_1(\text{old}) - 3/2\ R_3(\text{new})$

x_1 will enter and s_3 will leave the table

C_B	B	$b\ (=x_B)$	c_j: 3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3
5	x_2	50/41	0	1	0	15/41	8/41	-10/41
4	x_3	62/41	0	0	1	-6/41	5/41	4/41
3	x_1	89/41	1	0	0	-2/41	-12/41	15/41
		z_j	3	5	4	45/41	24/41	11/41
		$c_j - z_j$	0	0	0	-45/41	-24/41	-11/41

All $c_j - z_j \le 0 \Rightarrow$ we get optimal solution

$$Z = 3x_1 + 5x_2 + 4x_3$$

$$= \frac{3 \times 89}{41} + 5 \times \frac{50}{41} + 4 \times \frac{62}{41}$$

$$= \frac{267 + 250 + 248}{41} = \frac{765}{41}$$

19/8/17

* If one of the inequalities is '$>$' instead of '$\le$' or $b_1, b_2, \ldots, b_n$ is $-$ive, then we can't apply the above steps.

NOTE: (2-PHASE METHOD / BIG-M METHOD)

(i) When the constraints are of '$\le$' type, ie, $\sum_{j=1}^{n} a_{ij} x_j \le b_i$, $x_j \ge 0$ but some of right hand side of constraint are negative, ie, $b_i < 0$

In this case, after adding the non-ive slack variable, ie, s_i $(i = 1, \ldots, n)$, the initial sol^n will be $s_i = -b_i$ $(x_1 = x_2 = x_3 \ldots = x_n)$ It doesn't provide optimal sol^n as it violates the ideal cond^{n}s ($b_i < 0$ here)

(ii) $\sum_{j=1}^{N} a_{ij} x_j \ge b_i$; $x_j > 0$

Here, $\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i$; $x_j \ge 0$, $s_i \ge 0$

here, s_i is negative slack variable
surplus variable

* $\le$ slack variable

$\ge$ surplus variable (negative slack)

$=$ artificial variable

2 methods to

1.) Two-phase method

2.) Big M method

The case (ii) becomes :

$$\sum_{j=1}^{n} a_{ij} x_j - s_i + \underset{\text{artificial variable}}{A_i} = b_i$$

Total no. of variables = $n + m + m = (n+2m)$

An initial basic feasible sol^n of the new system can be obtained by equating $(n+2m-m) = (n+m)$ variables equal to 0. So new sol^n is :

$$A_i = b_i, \quad i = 1, 2, 3, \ldots m \quad \left(\begin{matrix} x_i = 0 \\ s_i = 0 \end{matrix} \right)$$

This is not the original sol^n

Ques. Minimize $Z = x_1 + x_2$

subject to $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7, \quad x_1, x_2 \geq 0$

$s_1, s_2 \quad A_1, A_2 \quad x_1, x_2$: total 6 variables

2-phase method

Max $Z^* = -A_1 - A_2$

sub. to. $2x_1 + x_2 - s_1 + A_1 = 4$

$x_1 + 7x_2 - s_2 + A_2 = 7$

		$c_j \rightarrow$	0	0	0	0	-1	-1	
C_B	variable in basis B	sol^n value	x_1	x_2	s_1	s_2	A_1	A_2	
-1	A_1	4	2	1	-1	0	1	0	4/1
-1	A_2	7	1	$\boxed{7}$	0	-1	0	1	→ 7/7
		Z_j	-3	-8	1	1	-1	-1	
		$C_j - Z_j$	3	8	-1	-1	0	0	
				↑ key col					

* Replace x_1, x_2 everytime by A_1, A_2

Camlin	Page
Date	/ /

→ here, the artificial variable will become the decision variable.

R_2 (new) ⟶ R_2(old) / 7

R_1 (new) ⟶ R_1 (old) − R_2 (new)

$2 - 1/7$

already replaced by x_2, so put x_1 *

C_B	variable in basis B	sol^n value (c_j →)	x_1 (0)	x_2 (0)	s_1 (0)	s_2 (0)	A_1 (−1)	A_2* (−1)
−1	A_1	3	$\boxed{13/7}$	0	−1	1/7	1	−1/7 →
0	x_2	1	1/7	1	0	−1/7	0	1/7
$z^* = -3$		z_j	−13/7	0	+1	−1/7	−1	+1/7
		$c_j - z_j$	13/7 ↑	0	−1	+1/7	0	−8/7

R_1 (new) ⟶ R_1 (old) × 7/13

R_2 (new) ⟶ R_2(old) − $\frac{1}{7}$ R_1 (new)

C_B	variable in basis	sol^n value (c_j →)	x_1 (0)	x_2 (0)	s_1 (0)	s_2 (0)	A_1 (−1)	A_2 (−1)
0	x_1	21/13	1	0	−7/13	1/13	7/13	−1/13
0	x_2	10/13	0	1	1/13	−12/91	−1/13	12/91

$1 - 7/13$

Iteration I : To remove A_1 from sol^n, enter s_2 in the basis by applying the row op^n

* If we take 13/7 as pivot element, we ~~won't get~~ will get unbounded value, so we won't make the above table

We will rather make the following table
(We can make both tables, sol^n will be same)

C_B	variable in basis B	sol^n value $b (= x_B)$	x_1	x_2	s_1	s_2	A_1^*	A_2^*
0	s_2	21	13	0	-1	1	7	-1
0	x_2	4	2	1	1	0	1	8
		z_j	0	0	0	0	0	0
$z^* = 0$		$c_j - z_j$	0	0	0	0	-1	-1

All $c_j - z_j \leq 0 \rightarrow$ Phase-1 completed

$x_1 = 0 \quad x_2 = 4 \quad s_2 = 21 \quad s_1 = 0 \quad A_1 = 0 \quad A_2 = 0$

* Always x_1 & x_2 should enter the 2nd column

Phase 2 : Max $z^* = -x_1 - x_2$

C_B	variable in basis	sol^n value $b (= x_B)$	c_j: x_1 (-1)	x_2 (-1)	s_1 (0)	s_2 (0)
0	s_2	21	$\boxed{13}$	0	-7	1
-1	x_2	4	2	1	-1	0
		z_j	-2	-1	-1	0
		c_j	1	0	-1	0

(A_1 & A_2 have been replaced, so, won't be there in table)

21/13

C_B	variable in basis B	sol^n value b $(= x_B)$	x_1 (-1)	x_2 (-1)	s_1	s_2
-1	x_1	21/13	1	0	-7/13	1/13
-1	x_2	4/13	0	1	1/13	-2/13
		z_j	-1	-1	6/13	1/13
		$c_j - z_j$	0	0	-6/13	-1/13

$-1 + \frac{14}{13}$

SELECT * FROM ___ →

$x_1 = +21/13 \qquad x_2 = +4/13$

$Z^* = \frac{25}{13}$

28-8-17 **BIG - M METHOD :-**

Procedure :

* Assign a large undesirable coeffecient to the artificial variable from the p.o.v. of objective func^n.
* If this objective func^n Z is to be minimized, then a large positive price called penalty is assigned to each artificial var.
* Similarly, if Z is to be maximized, then a large negative price is assigned to each of these variables.
* The penalty is supposed to be assigned by −M for maximization problem and +M for minimization problem, where M > 0.

Step-I Express this L-P Problem in std. form by adding slacks variable, surplus & artificial variables. Assign a 0 coeffecient to both slack and surplus variables. Then, add a very large positive coeffecient ∓M (Maximization case), +M (Minimization case) to the artificial variable in the objective func^n.

Step-II Initial basic feasible sol^n is obtained by assigning zero to original variables.

Step-III Calculate the value of $c_j - z_j$ in the last row of the simplex Table & examine the value.

(i) if all $(c_j - z_j) \geq 0$, then the current basic feasible sol^n is optimal.

(ii) if for a column k, $(c_k - z_k)$ is most negative (smallest value) & all ~~entire~~ entries of ~~these~~ columns are −ve, then the problem has unbounded optimal sol^n.

(iii) If one or more $(c_j - z_j) < 0$ (Minimization) then select the variable to enter into the basis with largest negative $c_j - z_j$ value.

→ We have to remove artificials variables first

→ constraint : equality : add A_n

$\geq \longrightarrow -s_n \xrightarrow{=} -s_n + A_n$

$$c_k - z_k = \text{Min}\{c_j - z_j : c_j - z_j < 0\}$$

Step-IV: Find key row & key element

Step-V: Make the key element 1 by performing row op^n's & make other entries of key col. zero.

Repeat Step-III after this.

Ques. (Using Big-M Method) Minimize $z = 5x_1 + 3x_2$

subject to
$$2x_1 + 4x_2 \leq 12$$
$$2x_1 + 2x_2 = 10$$
$$5x_1 + 2x_2 \geq 10 \;;\; x_1, x_2 \geq 0$$

$$\text{Min } z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$$

sub. to
$$2x_1 + 4x_2 + s_1 = 12$$
$$2x_1 + 2x_2 + A_1 = 10$$
$$5x_1 + 2x_2 - s_2 + A_2 = 10$$

30/8/17

		c_j :	5	3	0	0	M	M	Min ratio
Cost / unit	Variable in basis B	sol^n value $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	
0	s_1	12	2	4	1	0	0	0	6
M	A_1	10	2	2	0	0	1	0	5
M	A_2	10	$\boxed{5}$	2	0	-1	0	1	→ 2
		z_j	7M	4M	0	-M	M	M	
		$c_j - z_j$	5-7M	3-4M	0	M	0	0	

↑ Check step III (Case III)

$R_3(\text{new}) \longrightarrow R_3(\text{old})/5$

$R_2(\text{new}) \longrightarrow R_2(\text{old}) - 2R_3(\text{new})$

R_1 " $\quad R_1$ " "

$2 - 4/5$

$4 - 4/5$ (we can write A_2^*)

Cost/unit	Variable in basis B	Sol^n value $b\ (=x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	Min ratio
C_j:			5	3	0	0	M	M	
0	s_1	8	0	$\boxed{16/5}$	1	$2/5$	0	~~$-2/5$~~	$5/2 \rightarrow$
M	A_1	6	0	$6/5$	0	$2/5$	1	~~$-2/5$~~	5
5	x_1	2	1	$2/5$	0	$-1/5$	0	~~$1/5$~~	5
	Z_j		5	$6M/5 + 2$	0	$2M/5 - 1$	M		
	$C_j - Z_j$		0	$1 - 6M/5$	0	$1 - 2M/5$	0		

$\uparrow$ (under x_2)

No role of A_2 so we can remove/write A_2^*.

Now, we have to remove A_1

$\frac{6}{5} \times \frac{1}{8}$ $\quad 6 - 6/5 \times 5/2 \quad 2/5 \times 5/16$

Cost/unit	Variable in basis B	Sol^n value $b\ (=x_B)$	x_1	x_2	s_1	s_2	A_1	Min ratio
3	x_2	$5/2$	0	1	$5/16$	$1/8$	0	20
M	A_1	3	0	0	$-3/8$	$\boxed{1/4}$	1	$12 \rightarrow$
5	x_1	1	1	0	$-1/8$	$-1/4$	0	—
	Z_j		5	3	$\frac{5}{16} - \frac{3M}{8}$	$\frac{M}{4} - \frac{7}{8}$	M	
	$C_j - Z_j$		0	0	$\frac{3M}{8} - \frac{5}{16}$	$\frac{7}{8} - \frac{M}{4}$	0	

$\uparrow$ (under s_2)

$Z = \frac{25}{2} + 3M$

Cost/unit	Variable in basis B	Sol^n value $b\ (=x_B)$	x_1	x_2	s_1	s_2	~~A_1~~
3	x_2	1	0	1	$1/2$	0	
0	s_2	12	0	0	$-3/2$	1	
5	x_1	4	1	0	$-1/2$	0	
	Z_j		5	3	-1	0	
	$C_j - Z_j$		0	0	1	0	

This is the optimal sol^n

The sol^n is : $x_2 = 1 \quad x_1 = 4 \quad s_1 = 0 \quad s_2 = 12$

Ans. $= 5x_1 + 3x_2$

$= \boxed{23}$

Date 01 / 08 / 17

Duality

(alternative for 2-Phase / Big-M Method)

Duality theorem :-
If an optimal solution exists to either the primal symmetric dual program, then the other program also has an optimal sol^n and the two objective $func^n$ have the same optimal value.

1. A dual variable is defined for each constraint in the primal L-P Problem and vice versa.
Given a primal L-P with m constraints & n variables there exists a dual LP with m-variables with n- constraints & vise- versa

2. The right hand side constraints $b_1, b_2, \ldots b_m$ of the prime LP becomes with coeffecient of dual variables $y_1, y_2, \ldots y_m$ in the objective $func^n$ Z_y.
Also, the coeffecients $c_1, c_2, \ldots c_n$ of the primal variable $x_1, x_2 \ldots x_n$ in the objective $func^n$ becomes the RHS constraint in the dual LP Problem

<u>Primal LP</u>		<u>Dual LP</u>
i.) $b_1, b_2, \ldots b_m$	$\longrightarrow$	coeff. of $y_1, y_2 \ldots y_m$ in Z_y
ii.) coeff. of $x_1, x_2 \ldots x_n$	$\longrightarrow$	right hand side const. in dual LP Problem.
iii.) Maximization primal LP with const. $\le$	$\longrightarrow$	Minimization dual LP with constraints $\ge$

Eg. Max $z = x_1 - x_2 + 3x_3$: Primal (LP Program

sub. to $x_1 + x_2 + x_3 \le 10$

$2x_1 - x_2 - x_3 \le 2$

$2x_1 - 2x_2 - 3x_3 \le 6$

Dual LP problem : $\min\ z_y = 10y_1 + 2y_2 + 6y_3$

sub. to $y_1 + 2y_2 + 2y_3 \ge 1$

$y_1 - 2y_2 - 2y_3 \ge -1$

$y_1 - 2y_2 - 3y_3 \ge 3$

$y_1, y_2, y_3 \ge 0$

Take transpos of x_i matrix

→ Optimal solⁿ in both the cases will be same.

→ We can get ans. in less no. of iterations in case of dual LP Problem.

4/9/17 **Dual Simplex Method**

$\min\ z = c_1x_1 + c_2x_2 \ldots + c_nx_n$

sub to $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$

$\vdots$

$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \ge b_i$ $\quad -b_i$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$

Algorithm :-

Step-I : (Determine the initial solⁿ)
Convert to S.F. by adding slack, surplus and artificial variables.

Step-II : (Test the optimality of the solⁿ)
If all solⁿ value are positive ($x_{b_i} \ge 0$), then no need to apply this method, because improved solⁿ can be obtained by simplex method itself. Otherwise, go to Step-III.

Step-III : (Test the feasibility of the solⁿ)
If there exists a row, say r, for which the solⁿ value is -ive ($x_{br} < 0$) and all elements in row r and column j

are positive (ie $y_{rj} \geq 0, \forall j$), then the current ~~basic~~ sol^n is infeasible, otherwise, go to step-IV.

Step-IV (Obtain the improved sol^n)

(i) Select a basic variable associated with the row (called key row) that has largest negative value, ie,

$$x_{B_r} = \min \{ x_{B_i} : x_{B_i} < 0 \}$$

(ii) Determine the min ratio only for those column that have a -ve element in row r. Then, select a non-basic variable for entering into the basis associated with the column for which

$$\min \left\{ \frac{c_j - z_j}{y_{rj}} \quad y_{rj} < 0 \quad \forall j \right\}$$

Ques Max $z = -3x_1 - 2x_2$

sub. to $x_1 + x_2 \geq 1$ ⋮ $-x_1 - x_2 \leq -1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \geq 10$; $x_2 \leq 3$, $x_1, x_2 \geq 0$

$-x_1 - 2x_2 \leq -10$

Max $z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

$-x_1 - x_2 + s_1 = -1$

$x_1 + x_2 + s_2 = 7$

$-x_1 - 2x_2 + s_3 = -10$

$x_2 + s_4 = 3$

C_B	Variable in basis B	sol^n value	x_1	x_2	s_1	s_2	s_3	s_4
		c_j	-3	-2	0	0	0	0
0	s_1	-1	-1	-1	1	0	0	0
0	s_2	7	1	1	0	1	0	0
0	s_3	$\boxed{-10}$ →	-1	$\boxed{-2}$	0	0	1	0
0	s_4	3	0	1	0	0	0	1
		z_j	0	0	0	0	0	0
		$c_j - z_j$	-3	-2	0	0	0	0

Min Ratio : $-3/-1$

Scanned by CamScanner

Camlin Page

Date 06 / 09 / 17

$$x_{B_r} = \min \{ x_{B_i} : x_{B_i} < 0 \}$$

08-09-17

Iteration - I :

	c_j :		-3	-2	0	0	0	0
C_B	Variable in Basis B	$b\,(=x_B)$	x_1	x_2	s_1	s_2	s_3^*	s_4
0	s_1	4	-1/2	0	1	0	-1/2	0
0	s_2	2	1/2	0	0	1	1/2	0
-2	x_2	5	1/2	1	0	0	-1/2	0
0	s_4	-2 →	$\boxed{-1/2}$	0	0	0	1/2	1
		z_j	-1	-2	0	0	+1	0
	B_r	$c_j - z_j$	-2	0	0	0	-1	0

atleast 1 is negative & all $c_j - z_j \leq 0$

~~$R_4 \to R$~~

*won't consider this col eventhough min value exists here as s_3 has left the basis already

		c_j	-3	-2	0	0		0
C_B	B	$b\,(=x_B)$	x_1	x_2	s_1	s_2	s_3^*	s_4^*
0	s_1	6	0	0	1	0	-1	-1
0	s_2	0	0	0	0	1	1	1
-2	x_2	3	0	1	0	0	0	1
-3	x_1	4	1	0	0	0	-1	-2
		z_j	-3	-2	0	0	3	4
		$c_j - z_j$	0	0	0	0	-3	-4

all $B_r > 0$ ⇒ This is optimal sol^n

$x_1 = 4$, $x_2 = 3$ $s_1 = 6$

-5 -1/2 $s_2 = 0$ $s_3 = 0$ $s_4 = 0$

3

$$z = -6 - 12 = \underline{-18}$$

Ques. The NCAA is making plane for distributing tickets to upcoming regional basketball championships. Upto 10,000 available seats will be divided b/w the media, the competing universities & the general public. Media people are admitted free, but the NCAA recieves \$ 45 per ticket from universities & \$.100 per ticket from general public & Atleast 500 tickets must be reserved for the media, & atleast half as many tickets should go to the competing universities as to the general public. Within these restrictions the NCAA wishes to find the allocation that raises the most money.

let no. of tickets given to media : x_1

universities : x_2

general public : x_3

$$\text{Max } z = 45x_2 + 100x_3$$

sub. to

$$x_1 + x_2 + x_3 \leq 10{,}000$$

$$x_1 \geq 500$$

$$2x_2 \geq x_3 \quad , \quad x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

CONVEX ANALYSIS

Hyperplanes and Half spaces :

A hyperplane in E^n generalised the definition of straight line :

$$c_1x_1 + c_2x_2 + c_3x_3 = z \quad \text{in} \quad E^3$$

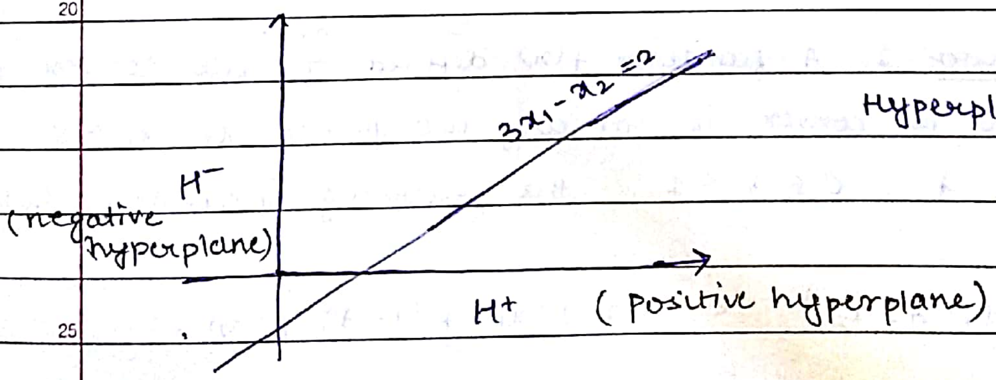
Eg. $2x_1 + 3x_2 - 4x_3 = 5$

A hyperplane in E^n is the set of vectors (points) X of form $\underset{\text{locus}}{H} = \{ X : cX = z \}$

where c is a non-zero vector in E^n and is called normal to the hyperplane & z is scalar.

Equivalently, a hyperplane may be defined as a set of all points $X = (x_1, x_2, \dots, x_n)$ satisfying the eq^n :

$$c_1x_1 + c_2x_2 + \dots + c_nx_n = z \quad (\text{Not all } c_j = 0)$$



Hyperplane in E^2.

$H_+ = \{ x : cx \geq z \}$: set of points below st. line

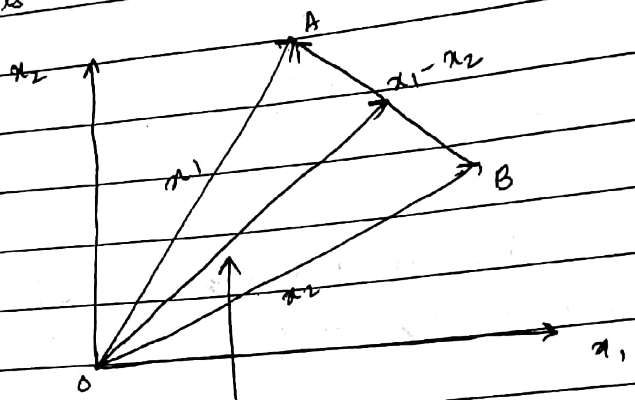
$H_- = \{ x : cx \leq z \}$: set of points above st. line

$H_+ : 3x_1 + 2x_2 - x_3 \geq 5$

$H_- : 3x_1 + 2x_2 - x_3 \leq 5$

Date 13 / 09 / 17

Convex Set : A subset S of E^n is said to be convex if for all pair of points $x_1, x_2 \in S$ any convex combination $\lambda x_1 + (1-\lambda) x_2$ for $0 \le \lambda \le 1$ (The line segment) is also contained in S



$\lambda x_1 + (1-\lambda) x_2$, $0 \le \lambda \le 1$

→ $x_1 \in S$, $x_2 \in S$, then $\lambda x_1 + (1-\lambda) x_2 \in S$. also.

Eg. $S = \{x_1, x_2, x_3, x_4\}$

$\underline{\underline{x_1, x_2}}$: $\lambda x_1 + (1-\lambda) x_2 = x_4$ (let)

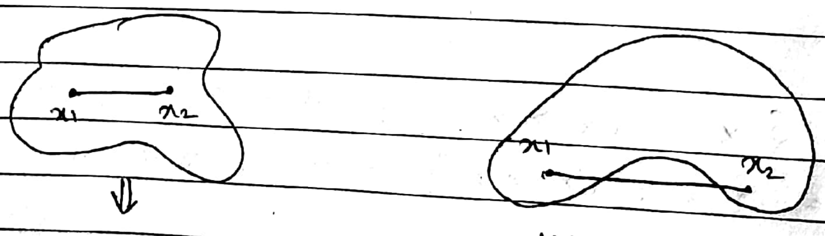
$\Downarrow$

~~[illegible]~~ convex set

Convex Function : A function $f(x)$ defined ~~on~~ over the convex set S is said to be convex if for any two points $x_1, x_2 \in S$ and all scalar λ , $0 \le \lambda \le 1$, the following inequality hold

$$f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

→



$\Downarrow$

Convex set

~~we can't take~~ $x_1, x_2 \in S$

(line segment $\notin S$)

$\Downarrow$

not Convex set

Camlin Page
Date / /

Convex Linear Combinations

Let $x_1, x_2, \ldots, x_n$ be set of vectors in E^n and λ_j $(j = 1, 2, \ldots, n)$ be non-negative real no. s.t. $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$ then the vector x is given by

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = \sum_{j=0}^{n} \lambda_j x_j$$

and is called Convex Linear Combination of vectors in E^n.