Morera's theor

Let $f(z)$ is continuous in a domain D

& $\oint_C f(z)\,dz = 0$.

for every simple closed path in D. Then $f(z)$ is analytic in D.

Sequence and Series of complex numbers

$S_n : \mathbb{N} \longrightarrow \mathbb{C}$

$S_n = \{ i^n / n \}$, $n \in \mathbb{N}$

$= \{ i, \frac{-1}{2}, \frac{-i}{3}, \ldots\ldots \}$

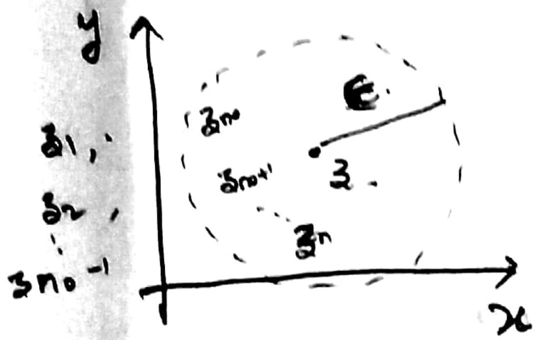
$$\boxed{S_n \longrightarrow S \quad , \quad \lim_{n\to\infty} S_n = S}$$

$$\lim_{n\to\infty} \frac{i^n}{n} = 0.$$

Convergence of Sequence :-

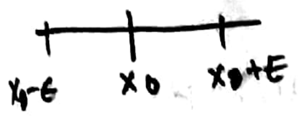
A sequence $\{z_n\} \to z$ iff for every $\epsilon > 0$ $\exists$ a $n_0 \in \mathbb{N}$ st $|z_n - z| < \epsilon$ whenever $n > n_0$.

$$\lim_{n\to\infty} z_n = z.$$



$z_1, z_2 \cdots z_{n_0-1}$ lie outside the circle.

$z_{n_0}, z_{n_0+1} - - -$ lie inside the circle.



Exr $\{i^n\}$, does not converge, which means diverges.

$\{1/n^2 + i\}$, converges to i.

$|z_n - z| < \epsilon \Rightarrow \left|\frac{1}{n_0^2} + i - i\right| < \epsilon$.

$|1/n_0^2| < \epsilon \Rightarrow 1/\epsilon < n_0^2$

$$\boxed{n_0 > 1/\sqrt{\epsilon}.}$$

Exr $z_n = \{i^n/n\} \to 0.$

① $|z_n - z| = \left|\frac{i^n}{n} - 0\right| = 1/n < \epsilon$.

$n_0 \geq 1/\epsilon$

$|z_n - z| < \epsilon$ whenever $n \geq n_0$

② $|z_n - z| = \left|\frac{1}{n^2} + i - i\right| = 1/n^2 < \epsilon$.

$$n_0 > \frac{1}{\sqrt{\epsilon}}.$$

$|z_n - z| < \epsilon$ whenever $n \geq n_0$.

if we cannot find n_0, we can say sequence diverges.

<u>Theorem</u>:- Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, 3 \ldots)$ and $z = x + iy$, then $\lim_{n\to\infty} z_n = z$, iff ①

$\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. ②

<u>Proof</u>:- if ② holds, for a given $\epsilon > 0$, $\exists$ $n_1, n_2 \in \mathbb{N}$ s.t $|x_n - x| < \epsilon$, whenever $n > n_1$ & $|y_n - y| < \epsilon$ whenever $n > n_2$

Let $n_0 \geq \max\{n_1, n_2\}$

then it is obvious that $|x_n - x| < \epsilon$ when $n > n_0$

and $|y_n - y| < \epsilon$ when $n > n_0$.

$$|(x_n + i y_n) - (x + iy)|$$

$$= |(x_n - x) + i(y_n - y)| \leq |x_n - x| + |y_n - y|$$

$$< \epsilon + \epsilon.$$

$$< 2\epsilon \quad \text{whenever } n \geq n_0.$$

$\Rightarrow z_n \to z$.

Conversly assume that

(i) holds

i.e for given $\epsilon > 0$, $\exists$ a $n_0 \in \mathbb{N}$ s.t $|z_n - z| < \epsilon$

whenever $n > n_0$.

$$\left.\begin{aligned} |x_n - x| &\leq |(x_n - x) + i(y_n - y)| < \epsilon \\ |y_n - y| &\leq |(x_n - x) + i(y_n - y)| < \epsilon. \end{aligned}\right\} \; n \geq n_0$$

Ex:- $z_n = -2 + i \dfrac{(-1)^n}{n^2}$

$\lim\limits_{n\to\infty} x_n = -2$, $\quad \lim\limits_{n\to\infty} y_n = 0$

$$|z_n - z| = |(x_n - x) + i(y_n - y)|$$

$$\leq |x_n - x| + |y_n - y|$$

$$\leq 2\epsilon.$$

$|z_n - z| < 2\epsilon$, $n \geq n_0$

$|x_n - x| < \epsilon$, $n > n_0$.

$|-2 + \cancel{2}| < \epsilon$, $n > n_1$

$0 < \epsilon$.

$$|y_n - y| < \epsilon = \left| \frac{i(-1)^n}{n^2} - 0 \right| < \epsilon.$$

$$= \frac{(-1)^n}{n^2} < \epsilon.$$

$z_n \longrightarrow -2$

$\{x_n\}$. $\sum_{n=0}^{\infty} x_n = x_0 + x_1 + x_2 + \cdots \rightarrow S$

iff Sequence of partial sums of $S_n = \sum_{i=1}^{n} S_i \rightarrow S$.

as $n \to \infty$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} x_i = \sum_{i=1}^{\infty} x_i = S.$$

Convergence of series of complex number:- An infinite series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots$$

of complex numbers converges to the sum S iff the sequence $S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_n$ of partial sum converges to S.

We then write

$$\sum_{n=1}^{\infty} z_n = S \quad \text{i.e}$$

$$\lim_{N\to\infty} S_n = \lim_{N\to\infty} \sum_{n=1}^{N} z_n = \sum_{n=1}^{\infty} z_n = S.$$

S_N = remainder term

$= S_{N+1} + S_{N+2} + \cdots$

$|S_N - S| = |S_N|$

In the case of convergence of an infinite series, sequence by remainder are $\{P_n\} \to 0$

Exp $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ whenever $|z|<1$

$1+z+z^2+ \cdots + z^n = \frac{1-z^{n+1}}{1-z}, \quad (z \neq 1)$

$S_N = 1+z+z^2+ \cdots z^{n-1} = \frac{1-z^n}{1-z}$.

$|S_N - S| = \left| \frac{1-z^n}{1-z} - \frac{1}{1-z} \right| = \left| \frac{-z^n}{1-z} \right|$.

$= |P_N|$

$|P_N| \to 0$ whenever $|z|<1$ (converges)

$\not\to 0$ whenever $|z|>1$ (diverges).

Theorem: Suppose $z_n = x_n + i y_n$ $(n = 1, 2, 3, \ldots)$ and $S = X + iY$, then $\sum_{n=1}^{\infty} z_n = S$, iff $\sum_{n=1}^{\infty} x_n = X$ & $\sum_{n=1}^{\infty} y_n = Y$.

Theorem :- If a series $\sum_{n=1}^{\infty} z_n$ converges then

$\lim_{n\to\infty} z_n = 0.$ [Necessary $cond^n$]

proof :- $S_n = z_1 + z_2 + \cdots + z_n.$

$z_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}.$$

$$= S - S = 0.$$

Absolute Convergence :-

An infinite Series $\sum_{n=0}^{\infty} z_n$ is called absolute convergent is $\sum_{n=0}^{\infty} |z_n|$ converges.

if $\sum z_n$ converges but $\sum |z_n|$ doesn't converge then the series is called conditionally convergent.

Absolute convergence $\Rightarrow$ conditionally convergent.

$\not\Leftarrow$

$\sum_{n=0}^{\infty} (-1)^n \, 1/n$ converges. but doesn't converges absolutely

i.e it is conditionally convergent.

Comparision test :- A Series $\sum z_n$ converges absolutely if $\exists$ a convergent series of the terms $\sum b_n$ s.t

$$|z_n| \leq b_n \quad \forall \; n \in \mathbb{N}$$

Ratio test :- If a series $\sum_{n=1}^{\infty} z_n$ is s.t $\lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = l$, then .

1. Series converges absolutely if $l < 1$
2. Series diverges if $l > 1$
3. test fails if $l = 1$.

Ex:- $\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!}$

Ratio test $\lim_{n \to \infty} \frac{(100+\cancel{75i})^{n+1}}{\cancel{(n+1)!}} \times \frac{\cancel{n!}}{(100+\cancel{75i})^n}$

$$\Rightarrow \lim_{n \to \infty} \frac{100+75i}{n+1}$$

$\cdot \; n \to \infty$, $\lim_{n \to \infty} \frac{100+75i}{n+1} \longrightarrow 0$

$$l = 0.$$

$l < 1$, hence it converges.

Root test :-

If a series $\sum_{n=1}^{\infty} z_n$ is s.t $\lim_{n\to\infty} |z_n|^{1/n} = l$,

then

① Series converges absolutely if $l<1$

② Series diverges if $l>1$.

③ test fails if $l=1$.

Exr $\sum_{n=0}^{\infty} \frac{(3i)^n}{n^n}$, $\lim_{n\to\infty} \left[\left(\frac{3i}{n}\right)^n\right]^{1/n}$

$\lim_{n\to\infty} 3i/n \longrightarrow 0$, $n\to\infty$, $l=0$

$l<1$, hence it converges absolutely.

Power series :-

A power series is an infinite series of the form.

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\cdots$$

where z_0 and a_n are complex constraints and z is a point in the stated region.

$\sum a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$

Ex:- $\sum_{n=0}^{\infty} z^n$.

$|z| < 1$ converges.

$|z| \geq 1$ diverges.

$a_n = 1$, $z_0 = 0$.

Ex: $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $a_n = 1/n!$, $z_0 = 0$

Ratio test.

$$\lim_{n\to\infty} \frac{\cancel{z^{n+1}}}{\cancel{(n+1)!}} \times \frac{\cancel{n!}}{\cancel{z^n}} = \lim_{n\to\infty} \frac{z}{n+1} \longrightarrow 0$$

$n \longrightarrow \infty$

$l = 0 < 1$, hence it converges absolutely.

Ex: $\sum n!\, z^n$.

converges only when $z = 0$ or diverges.

Radius of convergence:-

The smallest circle with centre z_0 that includes all points at ~~a given~~ which a given power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, converges, is called circle of convergence. Let R denote the radius of that circle, then R is called radius of convergence.

$$|z-z_0| = R.$$

How to determine radius of convergence of $\sum a_n (z-z_0)^n$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |z-z_0| = l.$$

Given power series converges if $l < 1$.

i.e $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |z-z_0| < 1.$ or.

$$|z-z_0| < \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

Set $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, then the given power series converges every where in the circle

$$|z-z_0| < R.$$

where.

$$\boxed{R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|}$$

$$\boxed{R = \lim_{n\to\infty} 1/|a_n|^{1/n}}$$

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Exr $\sum_{n=0}^{\infty} \frac{n!}{n^n} \cdot (z+\pi)^n$,

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}}{n^n} \times \frac{\cancel{(n+1)^{n+1}}\,(n+1)^n}{\cancel{(n+1)!}\,\cancel{(n+1)}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}$$

$$= e^{\cancel{n}(\cancel{x}+1)\cancel{n}-\cancel{x})}$$

= e ~~an~~ ~~[illegible]~~,

Taylor Series :- Suppose that a function $f(z)$ is analytic throughout a disk $|z-z_0|<R_0$. Then $f(z)$ has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

where $a_n = \frac{f^{(n)}(z_0)}{n!}$, $n=0,1,2,\dots,n$.

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$$

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Proof: Let z be any point inside c.

Let c_1 be a circle centred at z_0.

and enclosing the point z.

Then by cauchy integral formula.

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-z)}\, dw. \quad \text{—} (1).$$

$$\frac{1}{w-z} = \frac{1}{(w-z_0) - (z-z_0)}$$

$$= \frac{1}{(w-z_0)} \left[\frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)} \right]$$

$$\cancel{\sum_{n=0}^{\infty} z^n = \ldots} \quad \frac{1}{1-z} = \sum_{n=0}^{N} z^n + \frac{z^n}{1-z}.$$

$$\frac{1}{w-z} = \left[\frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)} \right] \frac{1}{(w-z_0)}$$

$$= \frac{1}{w-z_0} \left[1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{w-z_0}\right)^{n+1} + \frac{\left(\frac{z-z_0}{w-z_0}\right)^n}{1 - \frac{z-z_0}{w-z_0}} \right]$$

$$= \frac{1}{(w-z_0)} + \frac{z-z_0}{(w-z_0)^2} + \frac{(z-z_0)^2}{(w-z_0)^3} + \cdots + \frac{(z-z_0)^{n-1}}{(w-z_0)^n} + \frac{\left(\frac{z-z_0}{w-z_0}\right)^n}{(w-z)}$$

from (1)

$$f(z) = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{(w-z_0)}dw + \frac{(z-z_0)}{2\pi i}\int_{c_1} \frac{f(w)\,dw}{(w-z_0)^2} + \frac{(z-z_0)^2}{2\pi i}\int_{c_1}\frac{f(w)}{(w-z_0)^3}dw$$

$$+ \cdots + \frac{(z-z_0)^{n-1}}{2\pi i}\int_{c_1}\frac{f(w)}{(w-z_0)^n}dw + \frac{1}{2\pi i}\int_{c_1}\left(\frac{z-z_0}{w-z_0}\right)^n \frac{f(w)}{(w-z)}dw$$

By using cauchy integral formula.

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots$$

$$\cdots + \frac{(z-z_0)^{n-1}}{(n-1)!}f^{(n-1)}(z_0) \longrightarrow X_n.$$

$$\boxed{\lim_{n\to\infty} X_n = 0}$$

Since w lie on c_1, thus $\left|\frac{z-z_0}{w-z_0}\right| = \gamma < 1$ &

$f(w)$ is bounded by M on c_1

$$w - z = (w - z_0)(z - z_0)$$

$$= r_1 - (z - z_0)$$

where r_1 is the radius C_1

$$|\rho_n| = \left| \frac{1}{2\pi i} \int_{C_1} \left(\frac{z-z_0}{w-z_0}\right)^n \frac{f(w)}{(w-z)} \, dw. \right|$$

$$\leq \frac{1}{2\cancel{\pi i}} \int_{C_1} \frac{r^n M}{r - (z-z_0)} \, 2\cancel{\pi} r_1$$

$$\longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

$$\Rightarrow f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots\cdots$$

Exr $f(z) = \cancel{A} \cdot e^z$.

$$f(z) = f(0) + f'(0) + \frac{z^2}{2!} f''(0) + \cdots \qquad \Sigma a_n z^n z_0$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$\underbrace{\hspace{10cm}}$$

[Taylor series about origin] Maclauren series Expansion

$f^{(n)}(z) = e^z \Rightarrow f^{(n)}(0) = 1$

$$a_n = \frac{f^{(n)}(0)}{n!} = 1/n!$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

$$= |z| < \infty.$$

Exr $\sinh z$

$\cancel{f^{(n)}(z) = \sinh z - i \sin}$

$\cancel{\sinh z = \frac{e^z - e^{-z}}{2z}}$ $\quad \sinh z = -i \sin iz.$

$$= (-\cancel{i}) \frac{e^{-z} - e^{z}}{2\cancel{i}} = \frac{e^z - e^{-z}}{2}.$$

$$= 1/2 \left[\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \right]$$

$$= 1/2 \sum_{n=0}^{\infty} [1-(-1)^n] \frac{z^n}{n!}$$, if n is even then $1-(-1)^n = 0$.

$$\sinh z = \frac{1}{2} \sum_{n=0}^{\infty} \left[1-(-1)^{2n+1}\right] \frac{z^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{\cancel{2}} \sum_{n=0}^{\infty} \cancel{2} \frac{z^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad |z| < \infty$$

Exr $f(z) = 1/1-z$, as a power series expansion, $|z|<1$

$$= \sum_{n=0}^{\infty} z^n.$$

$$f(z) = \frac{1}{1-z}, \quad f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}, \quad (n = 0, 1, 2 \ldots)$$

$$\Rightarrow f^{(n)}(0) = n!.$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

$$a_n = 1.$$

$$\Rightarrow f(z) = 1 + z + z^2 + \cdots \qquad |z| < 1.$$

$$\frac{1}{1+z} = \cancel{\frac{1}{1-(-z)}} = \sum_{n=0}^{\infty} (-1)^n (z^n), \quad |z| < 1.$$

$$1/z = \frac{1}{1+(z-1)} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad |z-1| < 1$$

Ex:- $f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1+2z^2}{z^3(1+z^2)}$

$$= \frac{2(1+z^2)-1}{z^3(1+z^2)} = \frac{1}{z^3}\cdot\left[2 - \frac{1}{1+z^2}\right]$$

$$= \frac{1}{z^3}\left[2 - \sum_{n=0}^{\infty}(-1)^n z^{2n}\right] \quad |z|<1$$

$$= \frac{1}{z^3}\cdot\left[1 + z^2 - z^4 + z^6 - z^8 + \cdots\right]$$

$$= \underbrace{\frac{1}{z^3} + \frac{1}{z}} - z + z^3 - z^5 \ldots \ldots \quad |z|<1$$

Ex:- $\int_C \frac{1+2z^2}{z^3+z^2}\,dz$

Laurent series:-

Suppose that a z^n $f(z)$ is analytic throughout the annular region.

$R_1 < |z-z_0| < R_2$ centred at z_0 & let C denote any contour around z_0 and lying in that domain. Then at each point in the domain

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

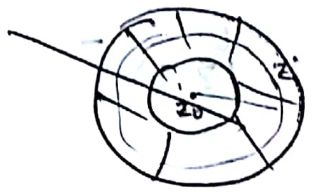
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad , \quad n = 0, 1, 2, \ldots$$

&

$$b_n = \frac{1}{2\pi i} \int \frac{f(z)}{(z-z_0)^{-n+1}} dz. \quad , \quad n = 1, 2, \ldots$$

Laurent's series:-

$$R_1 < |z - z_0| < R_2$$



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

$$+ \sum_{n=0}^{\infty} b_n (z.$$

$$\Rightarrow \boxed{\int_C f(z) dz = 2\pi i b_1}$$

Ex:- $f(z) = e^z.$

Replace z by $1/z$

$0 < |z| < \infty.$

$$f(z) = e^{1/z}$$

$$= \sum_{n=0}^{\infty} (1/z)^n \, 1/n! = 1 + 1/1!z + 1/2!z^2 + \ldots$$

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$a_0 + a_1(z-z_0)^2 + \dots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$

$$\int_C e^{1/z} dz = 1/2\pi i .$$

Ex $f(z) = -1/(z-1)(z-2)$

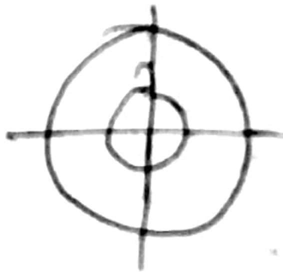
$f(z)$ is not analytic at $z = 1, 2$

$f(z) = 1/(z-1) - 1/(z-2)$

D_1: $|z| < 1$

D_2: $1 < |z| < 2$

D_3: $2 < |z| < \infty$



for D_1; $|z| < 1$, $|z/2| < 1$

$$f(z) = \frac{-1}{1-z} + \frac{1}{2} \cdot \frac{1}{1-z/2}$$

$$= -\sum_{n=0}^{\infty} z^n + 1/2 \sum_{n=0}^{\infty} (z/2)^n$$

$$= \sum_{n=0}^{\infty} z^n/2^{n+1} - \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

for D2 :- $1 < |z| < 2$.

$1/|z| < 1$ & $|z|/2 < 1$.

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} - 1/2 \cdot \frac{1}{1-z/2}$$

$$= 1/z \sum_{n=0}^{\infty} (1/z)^n - 1/2 \sum_{n=0}^{\infty} (z/2)^n.$$

$$= \sum_{n=0}^{\infty} 1/z^{n+1} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{z^n}.$$

for D3 :- $\oint_c \frac{1}{(z-1)(z-2)} dz = 2\pi i$.

$2 < |z| < \infty$.

$\frac{2}{|z|} < 1$, then $1/|z| < 1$.

$$f(z) = 1/z \frac{1}{1-1/z} - \frac{1}{z} \frac{1}{1-2/z}.$$

$$= 1/z \sum_{n=0}^{\infty} (1/z)^n - 1/z \sum_{n=0}^{\infty} (2/z)^n$$

$$= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}.$$

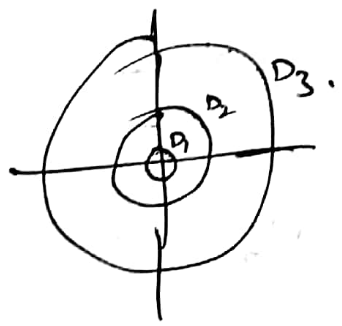
Exr $f(z) = \dfrac{1}{z(z^2-3z+2)} = \dfrac{1}{z(z-1)(z-2)}$

$$= \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$D_1 : 0 < |z| < 1$

$D_2 : 1 < |z| < 2$

$D_3 : 2 < |z| < \infty$



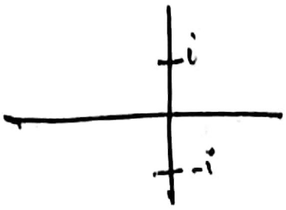
Isolated Singular point :-

A point $z = z_0$ is called singular pt for a func $f(z)$ if $f(z)$ ~~and~~ fails to be analytic at $z = z_0$, but is analytic at some point in every nbd of z_0.

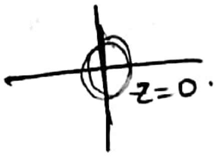
A singular point $z = z_0$ is called isolated singular point if there exist a deleted nbd of z_0 st $f(z)$ is analytic everywhere in the nbd.

$$0 < |z - z_0| < \epsilon.$$

Ex $f(z) = \dfrac{z+1}{z^3(z^2+1)}$, $z = 0, \pm i$.



Ex $\text{Log}\, z = \ln r + i\theta$. $r > 0$, $(-\pi < \theta < \pi)$.



Ex $f(z) = \dfrac{1}{\sin(\pi/z)}$

(i) f^n is not analytic at $z=0$ and $z = 1/n$, $n = \pm 1, \pm 2 \ldots$

Residue :- if $z = z_0$ is an isolated singular point then $\exists$ a region $0 < |z - z_0| < R$ where $f(z)$ is analytic through out then.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

Here coefficient b_1 i.e coefficient of $\dfrac{1}{z-z_0}$ in laurent series expansion is called Residue of $f(z)$ at $z = z_0$.

$$b_1 = \operatorname*{Res}_{z=z_0} f(z) \qquad b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$$

or $\int_C f(z)\,dz = 2\pi i\, b_1$

$$= 2\pi i \operatorname*{Res}_{z=z_0} f(z).$$

where c is +vely oriented simple closed contour lie in $0 < |z-z_0| < R_2$

Classification of Isolated Singular point :- Part of laurent series expansion consisting of negative power of $(z-z_0)$ i.e $\sum_{n=1}^{\infty} b_n (z-z_0)^n$ is principal point of $f(z)$

Poles :- If principal part of $f(z)$ contains only finite number of terms such that $b_m \neq 0$, but $b_{m+1} = 0 = b_{m+2}$ ----

then $z = z_0$ is called pole of order m.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n}.$$

$(0 < |z-z_0| < R_2)$. , with $b_m \neq 0$.

Alternate definition :- If there exist a $\lim_{z \to z_0} (z-z_0)^m f(z) = l$ (finite & non zero) then $z = z_0$ is called pole of order m.

Exp $f(z) = z^2 - 2z + 3 / z - 2$

$$f(z) = \frac{z(z-2)+2}{z-2} = z + \frac{3}{z-2} = 2 + (z-2) + \frac{3}{z-2}$$

$z=2$ simple pole.

$$\lim_{z \to 2} f(z-2)f(z) = 3 \quad . \quad \text{, pole of order 1}$$

$$\oint f(z)\,dz = 2\pi i \times 3 = 6\pi i .$$ → coefficient

Ex $f(z) = \sinh z / z^4$,

$z=0$ is pole of order 3. as

$$f(z) = 1/z^4 \{ z + z^3/3! + z^5/5! + \cdots \}$$

$$= 1/z^3 + 1/z\,3! + z/5! + \cdots$$

Highest power in negative $= 3$. → order of pole

$$\lim_{z \to 0} \frac{z^3 \sinh z}{z^4} = 1.$$

Removable singular point: If principal point of $f(z)$ does not contains any term then $z = z_0$ is called removable singularity.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (0 < |z-z_0| < R_2)$$

In such case singularity can be removed by defining func $f(z)$ at $z=z_0$ in such a way that it becomes analysis at $z=z_0$.

$$f(z_0) = a_0,$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad |z-z_0| < R_2$$

Alternate definition

If $\lim_{z \to z_0} f(z)$ exists & finite then $z=z_0$ is called removable singularity.

We can remove singularity by defining $f(z_0) = 1$.

Ex2 $f(z) = \dfrac{1-\cos z}{z^2}$

$$= 1/z^2 \left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right]$$

$$= 1/z^2 \left[\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right]$$

$$= 1/2! - \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots$$

$f(0) = 1/2!$.

$$f(z) = \begin{cases} 1/2 & z = 0 \\ \frac{1-\cos z}{z^2} & z \neq 0. \end{cases}$$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{1-\cos z}{z^2}$$

$$= \lim_{z \to 0} \frac{\sin z}{2z} = 1/2.$$

Essential singular point :-

If principal point of $f(z)$ contains finite no of term then $z - z_0$ is called Essential singularity.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} b_n (z-z_0)^{-n}$$

$$0 < |z-z_0|$$

Alternate definition :-

If there does not exist a fine integer m such that

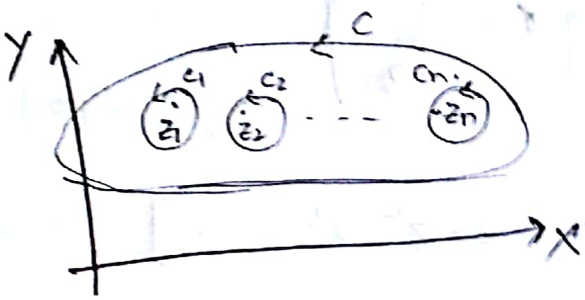
$$\lim_{z \to z_0} (z-z_0)^m f(z) = l \quad (\text{finite } \& \text{ non zero})$$

then $z = z_0$ is called Essential singularity.

Cauchy residue theorem:-

Let c be a positively oriented simple closed contour if a function $f(z)$ is analytic inside and on c except for a finite no. of points say $z_1, z_2, \ldots z_n$ inside c, then

$$\int_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}_{z=z_i} f(z)$$



Cauchy's theorem for multi-connected region.

$$\int_C f(z)\,dz - \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = 0. \quad \Rightarrow \int_C f(z)\,dz = \sum_{i=1}^{n} f(z)\,dz.$$

$$\Rightarrow \int_C f(z)\,dz = 2\pi i\,(B_1 + B_2 + \cdots B_n).$$

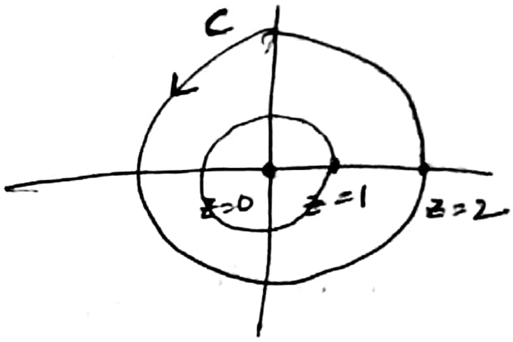
$$0 < |z_1 - z_2| < r_2$$

$$\int_C f(z)\,dz = 2\pi i B_2, \quad B_2 = \operatorname{Res}_{z=z_2} f(z)$$

where $B_i = \operatorname{Res}_{z=z_i} f(z)$, $\quad i = 1(1)n$

Exr $\int \frac{5z-2}{z(z-1)} dz$

$C: |z|=2$



$$f(z) = \frac{5z-2}{z(z-1)}$$

$$0 < |z| < 1$$

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5z-2}{z}\left[\cancel{1} - \frac{1}{1-z}\right]$$

$$= 5 - \frac{2}{z}\left[-1 - z - z^2 - z^3 \dots\right]$$

$$\operatorname*{Res}_{z=0} f(z) = 2 = b_1$$

$$0 < |z-1| < 1$$

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z(z-1)}$$

$$= \frac{5}{z} + \frac{3}{z(z-1)}$$

$$= \left(5 + \frac{z}{z-1}\right)\left(\frac{1}{(z-1)+1}\right)$$

$$= \left(5 + \frac{z}{z-1}\right)\left(1 - (z-1) + (z-1)^2 + \dots\right)$$

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$$f(z) = \frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{z-1}$$

Residue at pole:-

Let $f(z)$ has a pole of order m at $z = z_0$. The

$$\underset{z=z_0}{\text{Res}}\, f(z) = \frac{1}{(n-1)!} \lim_{z \to z_n} \left[\frac{d^{n-1}}{dz^{n-1}} \left\{ (z-z_0)^m f(z) \right\} \right]$$

Proof:-

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_{m-1}}{(z-z_0)^m}$$

$$+ \frac{b_n}{(z-z_0)^n}, \quad b_n \neq 0, \quad 0 < |z-z_0| < R_2$$

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2}$$

$$+ b_{m-1}(z-z_0) + b_m.$$

where $b_m \neq 0$, $|z-z_0| < R_2$

$\Rightarrow f^n (z-z_0)^m \cdot f(z)$ is analytic ~~series~~ in. $|z-z_0| < R_2$ as RHS is a power series in expansion there.

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) = (n+m)(n+m-1) + \cdots$$

$$(n+2) \sum_{n=0}^{\infty} a_n (z-z_0)^{n+1} + (n-1)!\, b_1$$

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Take $\lim_{z \to z_0}$ both side $\cdot \lim_{z \to z_0} \left[\cancel{\quad} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) \right] = (m-1)! b_1$

Ex $f(z) = \frac{z^3 + 2z}{(z-i)^3}$.

If $f(z)$ has a pole of order m at $z = z_0$ then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left\{ (z-z_0)^m f(z) \right\} \right]$$

Zeroes of Analytic functions :-

Let a function f is analytic at $z = z_0$. It has zero of order m iff there is a function g which is analytic and non zero at z_0

s.t $f(z) = (z-z_0)^m g(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad |z-z_0| < R_2$$

$$a_0 = a_1 = \text{------} = a_{m-1} = 0.$$

$$f(z) = a_m (z-z_0)^m + a_{m+1} (z-z_0)^{m+1} + \text{----}$$

$$= (z-z_0)^m \left[a_m + a_{m+1} (z-z_0) + a_{m+2} (z-z_0)^2 + \text{---} \right]$$

$= (z-z_0)^m g(z)$

$g(z) = b_0 + b_1 (z-z_0) + b_2 (z-z_0)^2 + \cdots$

$b_0 \neq 0 \qquad |z-z_0| < r_2$

$\Rightarrow f(z) = b_0 (z-z_0)^m + b_1 (z-z_0)^{m+1} + b_2 (z-z_0)^{m+2} + \cdots$

$|z-z_0| < r_2$

$f^{(m)}(z_0) = m!\, b_0 \neq 0.$

Zeros and poles :-

Suppose that

two functions p & q are analytic at a point z_0

(b) $p(z_0) \neq 0$ and q has a zero of order m at $z = z_0$

then quotient function $p(z)/q(z)$ has a pole of order m at $z = z_0$.

$p(z)/q(z)$

$z = z_0$ is an isolated singular pt for $z = z_0$.

$q(z) = (z-z_0)^m g(z)$, where $g(z_0) \neq 0$.

$$\frac{p(z)}{(z-z_0)^n g(z)}$$

$$\frac{1}{(z-z_0)^n}\left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right], \quad |z-z_0| < R_2$$

$a_0 \neq 0$.

Exr $\frac{1}{z(i^z-1)}$ pole of order 2 at $z_0 = 0$.

$p(z) = 1$, $q(z) = z(i^z - 1)$

$p(0) \neq 0$.

Corollary:- if $q(z)$ has a zero of order m at $z = z_0$ then $\frac{1}{q(z)}$ has a pole of order m at $z = z_0$.

Evaluation of Improper Integrals:- In calculas, improper integral of continuos f^n $f(x)$ over $[0,\infty)$ is defined by the integral

$$\int_0^\infty f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx$$

when the limit on R.H.S. exists, the $\int_0^\infty f(x)\,dx$, is said to converge to that limit.

Similarly, In calculus, improper integral of continuous fn $f(x)$ over $(-\infty, \infty)$ is defined by integral

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\,dx$$

when the limit on R.H.S exists, then $\int_0^\infty f(x)\,dx$, is said to converge to that limit.

Cauchy principle value (PV) is defined by the number

$$\boxed{P.V \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.}$$

if Improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ converges

$\Rightarrow$ Its PV exists.

Exr $\cancel{\int_{-\infty}^{\infty} x\,dx}$

$$\text{P.V} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

$$= \lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx + \lim_{R\to\infty} \int_{0}^{R} f(x)\,dx$$

Exr $\int_{-\infty}^{\infty} x\,dx$.

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R\to 0} \left[\int_{-R}^{R} x\,dx \right]$$

$$= \lim_{R\to 0} \left[x^2/2 \right]_{-R}^{R}$$

$$= \lim_{R\to 0} \left[\cancel{\frac{R^2}{2}} - \cancel{\frac{R^2}{2}} \right]$$

$$= 0.$$

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} x\,dx + \lim_{R_2\to 0} \int_{0}^{R_2} x\,dx$$

$$= \lim_{R_1\to\infty} \frac{x^2}{2}\Big]_{-R_1}^{0} + \lim_{R_2\to 0} \frac{x^2}{2}\Big]_{0}^{R_2}.$$

$$= \lim_{R_1 \to \infty} \frac{-R_1^2}{2} + \lim_{R_2 \to \infty} \frac{R_2^2}{2}$$

Convergence of Improper Integral.

$\Rightarrow$ Existence of cauchy P.V.

$\nLeftarrow$

If $f(x)$ is an even function over $(-\infty, \infty)$, then

Existence of cauchy PV $\Rightarrow$ convergence of improper integral. $\int_{-\infty}^{\infty} f(x)\,dx$.

Suppose P.V $\int_{-\infty}^{\infty} f(x)\,dx$ exists where $f(x)$ is an even function. P.V. $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx$ exists.

$f(-x) = f(x) \quad \forall \quad -\infty < x < \infty$.

$$\int_{-R}^{R} f(x)\,dx = 2\int_{0}^{R} f(x)\,dx \quad \text{since } f(x) \text{ is even.}$$

$$\int_{-R_1}^{0} f(x)\,dx + \int_{0}^{R_2} f(x)\,dx = \frac{1}{2}\int_{-R_1}^{R_1} f(x)\,dx + \frac{1}{2}\int_{-R_2}^{R_2} f(x)\,dx.$$

Take R_1 & $R_2 \longrightarrow \infty$.

$$\frac{1}{2} \lim_{R_1 \to \infty} \int_{-R_1}^{R_1} f(x)\,dx + \frac{1}{2} \lim_{R_2 \to \infty} \int_{-R_2}^{R_2} f(x)\,dx = PV \int_{-\infty}^{\infty} f(x)\,dx.$$

$$\Rightarrow \lim_{R_1 \to \infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} f(x)\,dx$$

$$= P.V. \int_{-\infty}^{\infty} f(x)\,dx.$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) \text{ converges.}$$

If $f(x)$ is a continuous fn over $-\infty < x < \infty$ then.

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1 \to \infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} f(x)\,dx.$$

$$I_1 \quad + \quad I_2$$

Improper Integral of rational function:-

$$\int_{-\infty}^{\infty} f(x)\,dx.$$

$f(x) = p(x)/q(x)$, where both p & q

are polynomial with real coefficients and do not have a factor in common. Also $q(x)$ does not have zeros on the real axis and hence at least one zero above x-axis

Let $q(x)$ have finite no. of zeroes say $z_1, z_2, \dots z_n$, which lie above x-axis.

If one contour the function

$$f(z) = p(z) / q(z)$$

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}_{z=z_i} f(z)$$

If $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$, then.

$$\text{P.V} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \cdot \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)$$

If $f(x)$ is even then.

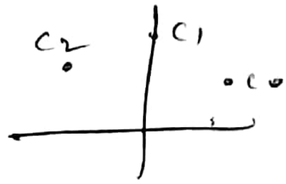
$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \cdot \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)$$

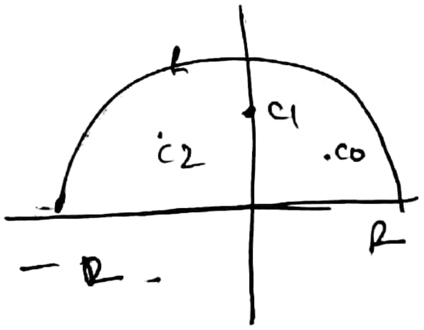
$$\int_0^{\infty} \frac{x^2}{1+x^6}\,dx$$

Consider $f(z) = z^2 / 1+z^6$. Zeroes of $1+z^6$ are

$$c_k = \exp\left[i\cdot\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right)\right], \quad k = 0,1,2 \ldots$$

c_0, c_1, c_2 .. lies above x-axis





$R > 1$

$$\int_{-R}^{R} \frac{x^2}{1+x^6}\,dx + \int_{C_R} \frac{z^2}{1+z^6}\,dz = 2\pi i\,(B_0 + B_1 + B_2)$$

c_0, c_1, c_2 are simple pole of $f(z)$

$$B_k = \lim_{z\to c_k} (z-c_k)\,f(z) = \lim_{z\to c_k} \frac{(z-c_k)\,z^2}{1+z^6}$$

$$= \lim_{z\to c_k} \frac{2z(z-c_k) + z^2}{6z^5} = \frac{c_k^2}{6c_k^5}$$

$$= 1/6c_k^3, \quad k = 0,1,2$$

$B_0 = 1/6i \quad , B_1 = \frac{-1}{6i} \quad , B_2 = 1/6i.$

$$\int_{-R}^{R} \frac{x^2}{1+x^6}\, dx + \int \frac{z^2}{1+z^6}\, dz = 2\pi i \left(\cancel{\frac{1}{6i}} \cancel{- \frac{1}{6i}} + \frac{1}{6i} \right)$$

$$\Downarrow_0 \qquad = \pi/3.$$

$$|f(z)| = \frac{|z^2|}{|1+z^6|} \leq \frac{|z^2|}{||z|^6 - 1|} = \frac{R^2}{R^6-1}$$

$$\left| \int_{C_R} f(z)\, dz \right| \leq \frac{R^2}{R^6-1} \cdot \pi R = \frac{\pi R^3}{R^6-1} \rightarrow 0, \text{ as } R \rightarrow \infty.$$

$$PV \int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\, dx = \pi/3.$$

$$\int_0^{\infty} x^2/1+x^6\, dx = \pi/6.$$

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Improper Integral in Fourier's Analysis:-

$$\int_0^\infty f(x)\cos ax\,dx \qquad \int_0^\infty f(x)\sin ax\,dx$$

$a > 0$ constraint

$f(x) = p(x)/q(x)$, where p & q are polynomials having no factors in ~~conn~~ common.

$f(x)$ does not have any zero on real axis has atleast one zero above real axis.

$f(z)\,e^{iaz}$.

$$|\sin az|^2 = \sin^2 ax + \sinh^2 ay.$$

$$|\cos az|^2 = \cos^2 ax + \sinh^2 ay.$$

$$\sinh ay = \frac{e^{ay} - e^{-ay}}{2}.$$

$$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax}\cdot e^{-ay}|$$

$$= e^{-ay} \to 0 \text{ as } y \to 0.$$

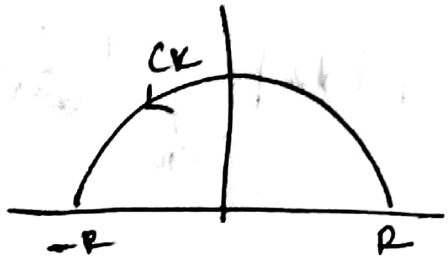
$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}\, dx = \frac{2\pi}{e^3}.$$

$$f(z)\, e^{i3z} = 1/(z^2+1)^2 \; e^{i3z}.$$

Roots of $q(z)$ i.e $(z^2+1)^2$ does not lie on the x-axis and have only imaginary zeroes which are $z = \pm i$ and both are of order 2.

$f(z)\, e^{i3z}$ analytic every except $z = \pm i$, $z = \pm i$ are the poles of order 2 of $f(z)\, e^{i3z}$.

Among them $z = i$, lie in the upper half plane.



$$\int_{-R}^{R} f(x)\, e^{i3x}\, dx + \int_{C_R} f(z)\, e^{i3z} = 2\pi i\; B_1$$

$$B_1 = \operatorname{Res}_{z=i} f(z)\, e^{i3z}$$

$$B_1 = 1/1!\; \lim_{z\to i} \left[\frac{d}{dz} \left\{ \frac{e^{i3z}}{(z+i)^2} \right\} \right]$$

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$f(z)e^{iaz}$.

$z=z_0$ is a pole of order m of $f(z)$ then

$$\underset{z=i}{\text{Res}}\ f(z)\,e^{i3z} = \frac{1}{1!}\left[\frac{d}{dz}\left\{(z-i)^2 f(z)\,e^{i3z}\right\}\right]$$

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{\sin 3x}{(x^2+1)^2} = 0.$$

$$\int_{-\infty}^{\infty}\frac{\cos 3x}{(x^2+1)^2}\,dx = 2\pi/e^3 \qquad |z_1+z_2| \geq \big||z_1|-|z_2|\big|$$

$$\left|\text{Re}\int_{C_R} f(z)\,e^{i3z}\,dz\right| \leq \left|\int_{C_R} f(z)\,e^{i3z}\,dz\right|$$

$$\leq \int_{C_R} |f(z)|\,|e^{i3z}|\,|dz| = \int_{C_R}\left|\frac{1}{(z^2+1)^2}\right||e^{i3z}|\,dz$$

$$\leq \int_{C_R}\frac{e^{-3ay}}{(R^2-1)^2}\,|dz| = \frac{e^{-3ay}}{(R^2-1)^2}\,\pi R = \frac{\pi/R^3\ e^{-3ay}}{(1-1/R^2)^2} \to 0 \text{ as } R\to\infty$$

$$|z_1+z_2| \geq \big||z_1|-|z_2|\big|$$

$$|z^2+1| \geq R^2-1$$

$$\Rightarrow \quad \frac{1}{|z^2+1|^2} \leq \frac{1}{(R^2-1)^2}$$

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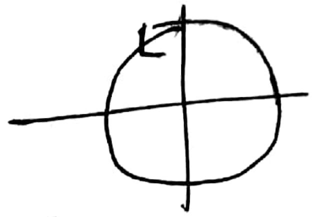
Indefinite Integral :-

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta.$$



$\theta \in [0, 2\pi)$

$0 \le \theta \le 2\pi.$

$z = e^{i\theta}$

$|\ dz = ie^{i\theta} d\theta \Rightarrow d\theta = dz/iz.$

$\cos\theta = \dfrac{z+z^{-1}}{2}, \quad \sin\theta = \dfrac{z-z^{-1}}{2i}.$

$$\int_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) dz/iz.$$

Ex: $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$. $(-1<a<1)$

$$\int_C \frac{2/a}{z^2+\left(\frac{2i}{a}\right)z-1}\, dz. \quad C\ |z|=1.$$

$z_1 = \left(\dfrac{-1+\sqrt{1-a^2}}{a}\right)i, \quad z_2 = \left(\dfrac{-1-\sqrt{1-a^2}}{a}\right)i.$

$$\int_C^{\infty} \frac{2/a}{(z-z_1)(z-z_2)}\, dz.$$

here $|z_2| > 1$, because of $-1 < a < 1$

$$|z_1 z_2| = 1 \Rightarrow |z_1| < 1$$

$z = z_1$ is a simple pole.

$$B_1 = \underset{z=z_1}{\text{Res}}\, f(z) = \lim_{z \to z_1} \frac{2/a}{(z-z_2)} = \frac{2/a}{z_1 - z_2}$$

$$= \frac{\cancel{2a}\; 2/a}{(\cancel{2}\sqrt{1-a^2}/\cancel{a})i}$$

$$B_1 = 1/(\sqrt{1-a^2})i$$

$$\int_c^\infty \frac{2/a}{(z-z_1)(z-z_2)}\, dz = 2\pi \cancel{i}\; \cancel{1} / (\sqrt{1-a^2})\cancel{i} = \frac{2\pi}{\sqrt{1-a^2}}$$

Partial Differential Equations

Black - Scholes Equations :-

$$f_t + rs f_s + r^2 s^2 \frac{\partial^2 f}{\partial s^2} = rf. \quad , \quad f = f(s).$$

to product the value of a particular stock

r = risk free ~~return~~ rate of return.

r = validity constant.

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Navier stokes equation :-

$$U_t + (U \cdot \nabla) U = -\nabla p + \gamma \Delta U.$$

fluid mechanics-

Definition of PDE:- A p.d.e is an Equation $F(x, y, \ldots z, z_x, z_y \ldots z_{xx}, z_{yy} \ldots) = 0$ invoking two (or more) independent variables $x, y \ldots$ etc and a dependent variable $z(x, y \ldots)$ in some domain D and partial derivatives such as $z_x, z_y \ldots, z_{xx}, z_{xy}, z_{yy} \ldots$ etc.

Order of PDE:- is defined by the order of the highest ordered derivatives appearing in the PDE.

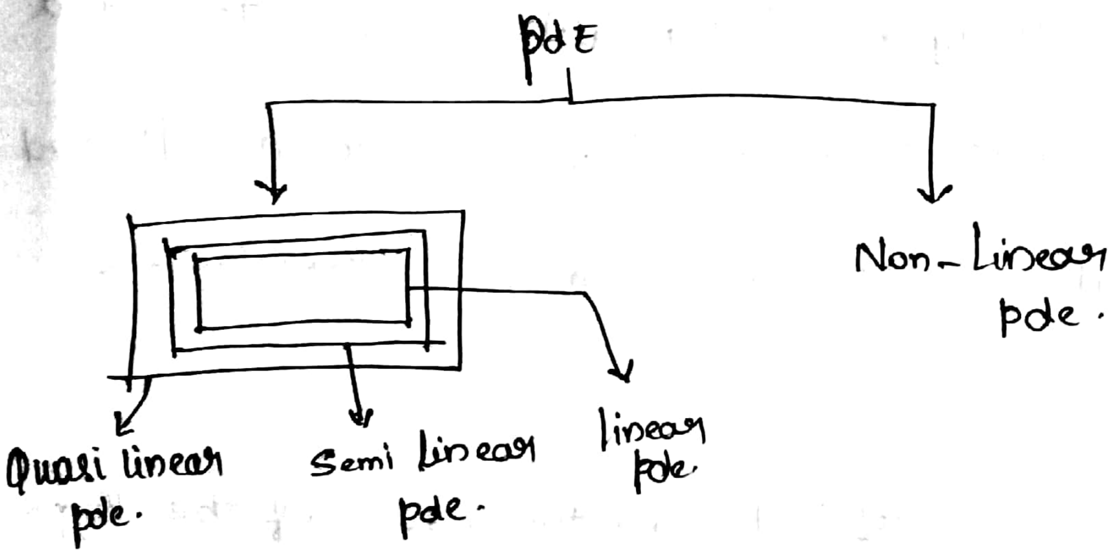
Classification of PDE:-

Quasi linear PDE:- A pde is said to be quasi linear if it is linear in the highest order derivative.

Semi linear PDE:- A quasi linear pde is said to be semi linear if the co-efficients of the highest order derivatives donot contain either the dependent variable or its derivative

Linear PDE :- A semi linear pde is said to be linear if it is linear in the dependent variable and its derivatives.

Non-linear pde :- A pde which is not quasilinear, is said to be non linear.



Ex: $UU_{xx} + U_t = xt$ — 2^{nd} order. quasi linear.

$\cancel{U_{xx} + UU_t}$

$U_{xx} + UU_t = \frac{xt}{U}$ — Semi linear Pde.

Ex Surface of revolution

$z = F(r)$, $r = (x^2+y^2)^{1/2}$.

where z-axis is the axis of revolution

$$\partial z/\partial x = F'(r)\, \partial r/\partial x = F'(r) \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$$

$$= \frac{F'(r)x}{r} \implies F'(r) = \frac{r\, \partial z/\partial x}{x}.$$

$$\partial z/\partial y = \frac{r\, \partial z/\partial x\, y}{xr} \Rightarrow \boxed{x\, \partial z/\partial y - y\, \partial z/\partial x = 0.}$$

$$F(x, y, z, p, q) = 0.$$

$$\boxed{xq - yp = 0}$$

$f(x, y, z, p, q) = 0$, $p = z_x, q = z_y$.

$z = z(x, y)$

Ex $F(U, V) = 0$ —— (1)

$U = U(x, y, z)$, $V = V(x, y, z)$

F is arbitrary f^n of U and V having first-order partial derivative w.r.t U & V.

Diff ① wrt x & y respectively

$$\frac{\partial F}{\partial U}\left(\frac{\partial U}{\partial x}+\frac{\partial U}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial U}{\partial V}\left(\frac{\partial V}{\partial x}+\frac{\partial V}{\partial z}p\right)=0. \quad — ①$$

$$\frac{\partial F}{\partial U}\left(\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}q\right)+\frac{\partial U}{\partial V}\left(\frac{\partial V}{\partial y}+\frac{\partial U}{\partial z}q\right)=0. \quad — ②.$$

on Eliminating $\frac{\partial F}{\partial U}$ & $\frac{\partial F}{\partial V}$ from ① & ②

we get.

→ first order pde.

$$\frac{\partial(U,V)}{\partial(y,z)}p+\frac{\partial(U,V)}{\partial(z,x)}q=\frac{\partial(U,V)}{\partial(x,y)} \quad — (A).$$

where $\frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} = U_xV_y - U_yV_x$ is the Jacobian.

Ex: $(x-a)^2+(y-b)^2+z^2=1$

$f(x,y,z,a,b)=0, \quad z=z(x,y).$

$2(x-a)+2zp=0 \Rightarrow \quad x-a=-zp$

$2(y-b)+2zq=0 \Rightarrow \quad y-b=-zq$

$p^2 z^2 + q^2 z^2 + z^2 = 1$

$$\boxed{z^2 (p^2 + q^2 + 1) = 1}$$

$F(x, y, z, a, b) = 0$. — (1).

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p = 0 \quad \text{— (2)}$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q = 0 \quad \text{— (3)}.$$

Eliminating a & b from (1), (2), & (3) we can get

$f(x, y, z, p, q) = 0$.

Classification of first order PDE:-

$f(x, y, z, p, q) = 0$ — (1).

(1) Quasi linear PDE:- (1) has the form

$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$.

ex:- $xz^2 p + x^2 y z q = x^2 y^2 z^2$

(2) Semi linear:- (1) has the form.

$P(x, y)p + Q(x, y)q = R(x, y, z)$

$$x^2y^2p + x^3y^3q = xyz^2$$

Linear PDE:-

① has the form

$$P(x,y)\,p + Q(x,y)\,q = R(x,y)\,z + S(x,y).$$

$$xy^2p + x^3y^3q = xyz.$$

Non linear PDE:- Those PDE which do not belong to above three category

$$p^2+q^2=1.$$

Solution of 1st order PDE:-

Consider a 1st order PDE

$$f(x,y,z,p,q) = 0 \quad \text{—— ①}.$$

A continuously differentiable function $z = z(x,y)$ in $(x,y) \subseteq \mathbb{R}\times\mathbb{R}$ is called the solⁿ of ①, if z and its partial derivatives w.r.t x & y (i.e p & q) satisfy ①.

in D -

Note that a solution $z = z(x, y)$ can be intepretated as a surface in 3-D shaped. The solⁿ $z = z(x, y)$ is also called integral surface of the pde ①.

There are different types of integral surface.

i) **Complete Integral** :- A two parametric family of solutions $z = F(x, y, a, b)$ is called a complete integral of ①.

General Integral :-
Any relation of the form $F(u, v) = 0$ where $u = u(x, y, z)$, $v = v(x, y, z)$ and F is an arbitrary smooth function is called general integral of ① If z, p, q determined by $F(u, v) = 0$ satisfy ①.

$z = f(x, y, a, b)$ · Complete Integral.

$f(u, v) = 0$ general solⁿ.

Singular Integral :-
Another ~~situation~~ solution of pde can be obtained by finding an envelope of complete integral

(i.e two parametric family of solution surface $z = f(x, y, a, b)$). This envelope is obtained by eliminating a & b from

$$z = f(x, y, a, b), \quad f_a = 0, \quad f_b = 0$$

Exr consider the pde.

$$f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0.$$

complete integral or family of surfaces is given by.

$$z = F(x, y, a, b) = ax + by + a^2 + b^2 \quad \text{——— (1)}.$$

$$F_a = x + 2a = 0 \quad \text{——— (2)}$$

$$F_b = y + 2b = 0 \quad \text{——— (3)}.$$

Substituting $a = -x/2$ & $b = -y/2$ in (1) we get

$$\boxed{4z = -(x^2 + y^2)} \quad \text{singular sol}^n.$$

which is a paraboloid of revolution.

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Cauchy problem :-

$$f(x,y,z,p,q)=0. \quad -①.$$

To find the integral surface of ① which contains an initial curve.

$$\Gamma : x = x_0(\lambda), \; y = y_0(\lambda), \; z = z_0(\lambda), \; \lambda \in I.$$

IVPr

Lagrange's Method. to solve quasi linear first order pde:

Let P, Q, R, be continuously differentiable functions (of x, y & z). Then general soln of

$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$$

is $F(U,V)=0$, where, $U = u(x,y,z) = c_1$, & $V = V(x,y,z) = c_2$ are two linear independent solutions of the quasi linear equations.

$$dx/P = dy/Q = dz/R. \; \&$$

F is an arbitrary smooth function.

Proof:- Let $U = c_1$ & $V = c_1$ be the soln of AE, then

$$dU = 0.$$

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$$U_x dx + U_y dy + U_z dz = 0$$

But $dx/P = dy/Q = dz/R . = c$

It gives

$$PU_x + QU_y + RU_z = 0 \quad \text{---} \quad (*).$$

$$PU_x + QU_y + RU_z = 0 \quad \text{---} \quad (**)$$

$$\frac{P}{\frac{\partial(U,V)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(U,V)}{\partial(z,x)}} = \frac{R}{\frac{\partial(U,V)}{\partial(x,y)}} = c_3 .$$

Also recall that $F(U,V)=0$ leads to pde

$$\frac{\partial(U,V)}{\partial(y,z)} p + \frac{\partial(U,V)}{\partial(z,x)} q = \frac{\partial(U,V)}{\partial(x,y)},$$

$$P = \frac{\partial(U,V)}{\partial(y,z)} c_3 , \quad Q = \quad , \quad R = .$$

$Pp + Qq = R$ leads to.

$$\frac{\partial(U,V)}{\partial(y,z)} p + \frac{\partial(U,V)}{\partial(z,x)} q = \frac{\partial(U,V)}{\partial(x,y)}.$$

(ii) $\dfrac{dx - dy}{x^2 - y^2} = \dfrac{dz}{(x+y)z}$

$$\frac{dx - dy}{x - y} = dz/z.$$

$$x - y = c_2 z.$$

$\Rightarrow$ $\Gamma\left(\frac{1}{x} - \frac{1}{y}, \frac{z}{x-y}\right)$ ...

Exr Cauchy problem. $xp + yq = z$.

B.Γ: $x_0 = \lambda^2$, $y_0 = \lambda + 1$, $z_0 = \lambda$

$$\frac{dx}{x} = \frac{dy}{y} = dz/z \qquad y/x = c_1$$

$$y/z = c_2.$$

Now if Γ is the curve on integral surface then

$$\frac{\lambda+1}{\lambda} = c_2 \text{ \& } \frac{\lambda+1}{\lambda^2} = c_1$$

$(c_1 - 1)c_1 = c_2$ or $(U-1)U = V$.

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Exr $yz\,p + xz\,q = xy$.

$$dx/yz = dy/xz = dz/xy.$$

(i) $x\,dx - y\,dy = 0 \Rightarrow x^2 - y^2 = c_1$

(ii) $y\,dy - z\,dz = 0 \Rightarrow y^2 - z^2 = c_1$

$$F(x^2-y^2,\ y^2-z^2) = 0 \quad \text{General soln.}$$

Exr $(z^2 - 2yz - y^2)p + x(y+z)q = x(y-z)$

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$ $\Rightarrow$ here x can easily removed.

(i) $\dfrac{dy}{y+z} = \dfrac{dz}{y-z} \Rightarrow y\,dy - z\,dz - z\,dy - y\,dz = 0$

$$y\,dy - z\,dz - (z\,dy + y\,dz) = 0.$$

$$d(y^2/2) - d(z^2/2) - d(yz) = 0$$

on integration, we get.

$$y^2 - z^2 - 2yz = c_1$$

use x, y, z as a Langrange's multiplier we get

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{x(y+z)} = dz/x(y-z)$$

$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0$

$$x^2 + y^2 + z^2 = c_2$$

$$F\left(y^2 - z^2 - 2yz,\ x^2 + y^2 + z^2\right) = 0.$$

Exr $x^3 p + y(3x^2 + y) q = z(2x^2 + y)$

$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} = \frac{dz}{z(2x^2+y)}$$

here use $-\frac{1}{x}, \frac{1}{y}, \frac{-1}{z}$ as multiplier.

$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} = \frac{dz}{z(2x^2+y)} = \frac{-\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{0}.$$

$$-1/x\,dx + 1/y\,dy - 1/z\,dz = 0.$$

this will occur when we differentiate this eqn.

$$\log(-x) + \log(y) + \log(-z) = \log(c_1)$$

$$\log(y/xz) = \log c_1.$$

$$y/xz = c_1.$$

$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} \Rightarrow \frac{(3x^2+y)\,dx}{x^3} = \frac{dy}{y} = \frac{(3x^2+y)\,dx + dy}{x^3+y}.$$

$$\Rightarrow \frac{(3x^2+y)\,dx + dy + x\,dy.}{x^3 + y + xy} = dy/y.$$

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$d(\log(x^3+y+xy)) = d(\log y)$

$$x^3+y+xy / y = C_2$$

Choose the path do not . any other variable.

Second order PDE :-

Let $D \subseteq R^2$ be a smooth Domain;

let R, S and T $(D \to R)$ be smooth function then a PDE is said to be 2nd order semi linear pde, if it can be written in this form.

$$R(x,y)\,U_{xx} + S(x,y)\,U_{xy} + T(x,y)\,U_{yy} + g(x,y,u,u_x,u_y) = 0$$

$R^2+S^2+T^2 \neq 0$ It has to satisfy this also inorder to ~~be~~ be a 2nd order Pde.

$U(x,y)$ - having ~~constant~~ continuous derivative.

classification of pde :-

① Hyperbolic :- ① is hyperbolic in D if $S^2-4RT > 0$ in D

② Parabolic :- $S^2-4RT = 0$.

③ Elliptic :- $S^2-4RT < 0$.

Exr $U_{tt} = c^2 U_{xx}$ $c > 0$

1D wave eqn.

$U \to$ dependent
$x, t \to$ independent.

here $U(t, x)$

$U_{tt} - c^2 U_{xx} = 0$.

($\because$ U_{tx} coeff $= 0$).

$R = 1$, $S = 0$, $T = -c^2$

$S^2 - 4RT = 4c^2 > 0 \Rightarrow$ hyperbolic.

Exr $U_t = \sigma U_{xx}$ — Heat eqn.

$R = 0$, $S = 0$, $T = -\sigma$.

$S^2 - 4RT = 0 - 4(0 \times) = 0 \Rightarrow$ parabolic.

Exr $U_{xx} + U_{yy} = 0$ Laplace eqn.

$R = 1$, $S = 0$, $T = 1$.

$0 - 4(1) = -4 < 0$, Elliptical.

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