$y'' + P(x)\,y' + Q(x)\,y = R(x) \qquad x \in I$

$y'' + P(x)\,y' + Q(x)\,y = 0$

* If coefficients are variable, there is no std method to solve

Power Series Method :

$x_0 \in R$

$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 \ldots$

is called a power series in x around x_0

x_0 - centre

·) $x_0 = 0$, $\sum_{n=0} a_n x^n$ → check convergence

Def^n : The series $\sum_{n=0} a_n x^n$ converges at a point x if

the limit $\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ exists. The limit is the sum func^n of the series.

Ex. (i) $\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\,x^2 + \ldots$: converges only for $x = 0$, (divergent)

(ii) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$: converges to e^x

(iii) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$ → $R = 1$

↳ Converges for $|x| < 1$

Diverges for $|x| \geq 1$

→ $R = 1$ - radius of convergence

converges for $|x| < R$

diverges for $|x| > R$

For (i), $R = 0$

For (ii), $R = \infty$ (converges for all x)

Date
Page

$0 < R < \infty$

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Def^n: suppose $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$ with $R > 0$ and denote its sum func^n as $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

→ If it is convergent, we can differentiate it.

$$f'(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

if convergent,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

We can derive that

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{(Compare from Taylor's series)}$$

Def^n: Let $f : I \to R$ be a func^n, $x_0 \in I$. Then, f is called analytic around x_0 iff $\exists$ a $\delta > 0$ s.t.

③ — $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \forall\, x$ with $|x - x_0| < \delta$

$f(x)$ has a power series representation in neighbourhood of x_0. In this case $a_n = \frac{f^n(x_0)}{n!}$ & ③ is the Taylor series expression of f at x_0.

Eg. $\cos x$, $\sin x$, f^ns which are infinitely times differentiable.

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* Power series method can be used if we have variable coefficients (which can't be solved by any previous methods)

Gunjan
Date 20-3-17
Page

* Always make coeff. of y'' = 1.

Ex $y''+y=0$ Here, $P(x)=0$ $Q(x)=1$

$Q(x) = 1 = \sum_{n=0}^{\infty} a_n x^n$ where $a_1, a_2, \dots a_n \dots = 0$

(analytic func^n)

$P(x)$: also analytic

$\Rightarrow$ $P(x)$ and $Q(x)$ are analytic at $x_0=0$.

Let sol^n of D.E. be : $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1)a_{n+1} x^n$

$y''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 \dots \dots + (n+1)(n+2) a_{n+2} x^n$

$\Rightarrow$ ~~$(2a_2+6a_3x+12a_4x^2\dots)+(a_0+a_1x+a_2x^2\dots)=0$~~

$\Rightarrow$ ~~$(2a_2+a_0)+(6a_3+a_1)x+(a_4+a_2)x^2\dots$~~

$\Rightarrow$ $y(x)$ satisfies above D.E. $\Rightarrow$

$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0 = 0 + 0\cdot x + 0\cdot x^2 + \dots$

$= (2a_2 + 6a_3 x + \dots + (n+1)(n+2)a_{n+2} x^n \dots) + (a_0 + a_1 x \dots + a_n x^n + \dots) = 0$

$= (2a_2 + a_0) + (6a_3 + a_1)x + \dots + ((n+1)(n+2)a_{n+2} + a_n)x^n + \dots = 0$

$\Rightarrow$ $2a_2 + a_0 = 0$ $\quad$ $6a_3 + a_1 = 0$ $\quad$ $\boxed{(n+1)(n+2)a_{n+2} + a_n = 0}$ $\quad n \geq 0$

$\Rightarrow$ $a_2 = -\frac{1}{2}a_0$ $\quad$ $a_3 = -\frac{1}{6}a_1$

~~$(n+1)(n+2)$~~ $a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$ $\Rightarrow$ $a_4 = -\frac{a_2}{3\times4} = \frac{a_0}{2\cdot3\cdot4} = \frac{a_0}{4!}$

$a_5 = -\frac{a_3}{4\cdot5} = \frac{a_1}{5!}$, $a_6 = -\frac{a_4}{5\cdot6} = -\frac{a_0}{6!}$

$\Rightarrow$ $y(x) = a_0 + a_1 x - \frac{1}{2}a_0 x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 \dots$

$\Rightarrow$ $y(x) = \left[1 - \frac{1}{2}x^2 + \frac{x^4}{4!} - \frac{x^6}{6!} \dots\right] a_0 + \left[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{x^5}{7!} \dots\right] a_1 x$

$\boxed{y(x) = a_0 \cos x + a_1 \sin x}$

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Page

Existence of a Power Series sol^n :-

$$y'' + P(x)y' + Q(x)y = R(x) \quad x \in I \quad \text{--- (1)}$$

Theorem: Let $P(x), Q(x), R(x)$ admits a power sol^n series representation around a point $x = x_0 \in I$, with non-zero radius of convergence R_1, R_2 & R_3 respectively. $R = \min\{R_1, R_2, R_3, R_4\}$. Then eq^n (1) has a sol^n $y(x)$ with power series representation around x_0 and Radius of $y(x)$ is R.

Def^n: A point x_0 is called ordinary point of (1) if $P(x), R(x), Q(x)$ admit power series rep^n (with non-zero radius of convergence) around $x = x_0$.

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Eg. $xy'' + y = 0$

$$y'' + \frac{y}{x} = 0$$

$P(x) = \frac{1}{x}$: not defined at $x = 0$

" $x_0 = 0$ is not an ordinary point

It can't have $y(x) = \sum_{n=0}^{\infty} a_n (x)^n$

but it may have sol^n $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$

Eg. $(x-1)y'' + (\sin x)y = 0 \quad x_0 = 0$ or 1

$$y'' + \left(\frac{\sin x}{x-1}\right) y = 0$$

$x_0 = 0$ ✓

$x_0 = 1$ ✗ Not ordinary point ⇒ singular point

Def^n: x_0 is called a singular point for eq^n (1) if x_0 is not an ordinary point

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Date
Page

Legendre Eq^n :-

$(1-x^2)y'' - 2xy' + p(p+1)y = 0$ $\quad -1 < x < 1$

p is real no. , $p \in \mathbb{R}$

→ Legendre eq^n of order p

sol^n of above eq^n is known as Legendre $func^n$ ↳ spl $func^n$

→ $P(x) = \frac{-2x}{(1-x^2)}$ $\quad Q(x) = \frac{p(p+1)}{1-x^2}$

choose $x_0 = 0$ as ordinary point in this case

Ex Find power series representation of $P(x)$ and $Q(x)$ around $x_0 = 0$

$P(x) = \sum_{n=0} b_n x^n$, $b_n = \frac{P^n(0)}{n!}$

2-3-17

$y(x) = \sum_{n=0}^{\infty} a_n x^n$, $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substituting in the eq^n,

$(1-x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x\{\sum_{n=1}^{\infty} n a_n x^{n-1} + p(p+1)\sum_{n=0} a_n x^n = 0$

→ $\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\text{to make it } x^n} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$

$n-2 = m$ → term becomes $\sum_{m=0}^{\infty}(m+1)(m+2)a_{m+2}x^m = \sum_{n=0}(n+1)(n+2)a_{n+2}x^n$

→ $\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=\cancel{2}0}^{\infty} n(n-1)a_n x^n - 2\sum_{n=\cancel{1}0} n a_n x^n + p(p+1)\sum_{n=0} a_n x^n = 0$

for n=0,1 term = 0, replace 2 by 0 $\quad$ for n=0, term = 0, replace 1 by 0

→ $\sum_{n=0}((n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n)x^n = 0$

Equating coeffecient of x^n as zero,

$(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0$, $n \geq 0$

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Date
Page

$(n+1)(n+2)a_{n+2} + (p-n)(p+n+1)a_n = 0$

$$\boxed{a_{n+2} = -\frac{(p-n)(p+n+1)a_n}{(n+1)(n+2)}, \quad n \geq 0} \quad \text{--- (2)}$$

$n=0 \quad a_2 = -\frac{p(p+1)}{2}a_0 \quad , \quad n=1 \quad a_3 = -\frac{(p-1)(p+2)}{2\cdot 3}a_1$

$n=2 \quad a_4 = -\frac{(p-2)(p+3)}{3\cdot 4}a_2 = \frac{(p-2)p(p+1)(p+3)}{4!}a_0$

$n=3 \quad a_5 = -\frac{(p-3)(p+4)}{4\cdot 5} = \frac{(p-3)(p-1)(p+2)(p+4)}{5!}a_1$

$$y = a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{(p-2)(p)(p+1)(p+3)}{4!}x^4 - \cdots\right]$$

$$+ a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 \cdots\right]$$

→ legendre func^n

$y = a_0 y_1(x) + a_1 y_2(x)$: General sol^n of (1)

where both $y_1(x)$ and $y_2(x)$ are series func^n.

contain even power of x ↳ contain odd power of x.

To check :

1.) y_1 & y_2 are L.I. or not

2) y_1 & y_2 are convergent.

→ $y_1/y_2 \neq$ constant func^n ⇒ y_1 & y_2 are L.I.

convergence

→ Case - I : p is not an integer

$n = 2k$ (even) ⇒ talking only about y_1 series, $n+1 = 2k+2$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a_{2k+2}}{a_{2k}}\right| = \left|\frac{a_{2k+2}\,x^{2k+2}}{a_{2k}\,x^{2k}}\right|$$

$$\lim_{n\to\infty} \left|\frac{-(p-2k)(p+2k+1)}{(2k+1)(2k+2)}\frac{a_{2k}\,x^{2k+2}}{x^2}\right| \to |x|^2$$

for $k \to \infty$

y_1 is convergent for $|x| < 1$

$\boxed{R = 1}$

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* If $p = 2$ all terms after x^2 in y_1 will be 0, * y_1 : polynomial
$p = 3$ all terms after x^5 in y_2 will be 0. y_2 : series

y_2 : polynomial
y_1 : series

→ Case-II : p is non-ive integer

If p is even, sum of y_1 terminates to a polynomial while y_2 is infinite series

If p is odd, sum of y_2 terminates to a polynomial while y_1 is infinite series.

??

$$\boxed{a_n = -\frac{(n+1)(n+2)}{(p-n)(p+n+1)} \underbrace{a_{n+2}}_{\text{can't be less than } p} \text{ for } n \le p-2} \quad — ③$$

(discontinuous at $x = p-1, p$)

Def^n: A polynomial sol^n $P_n(x)$ of the Legendre ~~func~~ eq^n ① is called Legendre polynomial whenever $P_n(1) = 1 \;\forall\, n$

$n = 0$: $a_0 = 1$

**

$$\boxed{a_n = \frac{(2n)!}{2^n (n!)^2}} = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n!} \qquad \frac{{}^{2n}C_n}{2^n (n!)^2}$$ — ✱

In eq^n ③, for $n = p-2$

$$a_n = -\frac{(p-1)(p)}{2(2p-1)} \cdot a_p$$

$$a_{n-2} = -\frac{(n-1)n}{2(2n-1)} a_n \quad — ④$$

$$a_{n-2} = -\frac{(n)(n-1)\, a_n}{2(2n-1)} = -\frac{n(n-1)}{2(2n-1)} \frac{(2n)!}{2^n (n!)^2}$$

$$= \frac{-n(n-1)\, a_n}{2(2n-1)} \; \frac{2n(2n-1)(2n-2)!}{2^n\, n(n-1)!\; n(n-1)(n-2)!}$$

$$\boxed{a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!(n-2)!}} \quad — ⑤$$

$\boxed{n = n-2}$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n+3)} a_{n-2}$$

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Date
Page

$$= \frac{(2n-4)!}{2^n\, 2!\,(n-2)!\,(n-4)!} \quad \{\text{From } ⊛\}$$

In general for $n-2m \geq 0$

$$\boxed{a_{n-2m} = \frac{(-1)^m (2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!}}$$

substituting these coeffecients

(2) $$\boxed{P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!} x^{n-2m}}$$ — legendre polynomial of ~~order~~ degree n

$M = \left[\frac{n}{2}\right]$ {greatest integer $\leq \frac{n}{2}$} $= \frac{n}{2}$ or $\left(\frac{n-1}{2}\right)$

Verify :-

$$= \frac{(2n)!\, x^n}{2^n (n!)^2} - \frac{(2n-2)!\, x^{n-2}}{2^n (n-1)!\,(n-2)!} + \ldots$$

$n=0$: $P_0(x) = 1$

$n=1$ $P_1(x) = \frac{2\cdot 1}{2\cdot 1} \cancel{\frac{1\cdot n}{2}} = x$

$n=2$ $P_2(x)$: $\frac{4!\cdot x^2}{4\cdot 2!\cdot 2!} - \frac{2!\; x^0}{2^2\cdot 1!\cdot 1!} = \frac{3}{2}x^2 - \frac{1}{2}$

$\boxed{M=1}$

24-3-17 $$\frac{d^n}{dx^n} x^{2n-2m} = \frac{(2n-2m)!}{(n-2m)!} \quad \text{— (3)}$$

for eg., $n=1$

$$\frac{d}{dx} x^{2-2m} = (2-2m)\, x^{1-2m-1}$$

$$= \frac{(2-2m)(1-2m)!}{(1-2m)!} x^{1-2m}$$

$$= \frac{(2-2m)!}{(1-2m)!} x^{1-2n}$$

From eq^n (2) and (3),

$$P_n(x) = \sum_{m=0}^{[M/2]} \frac{(-1)^m}{2^n\, m!\,(n-m)!} \frac{d^n}{dx^n} x^{2n-2m}$$

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* The sequence of polynomials are orthogonal to each other

$[n/2]$ से आगे की ~~Terms ke liye~~ it will be 0.

~~we can write n here as~~

Since the series is convergent → $\frac{d^n}{dx^n}\Sigma$ ---- remains same

$$P_{(n)}(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\sum_{m=0}^{[n/2]}\frac{n!}{m!(n-m)!}(x^2)^{n-m}(-1)^m$$

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n}$$ ← Rodrigue's formula

$n=0$ $\quad P_0(x) = 1$

$n=1$ $\quad P_1(x) = \frac{1}{2}2(x) = x$

Sequence of legendre Polynomials

$\{P_n(x)\}_{n=0}^{\infty}$

Orthogonal Properties of legendre Polynomial

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & m\neq n \\ \frac{2}{2n+1} & n=m \end{cases}$$

Proof:- $f(x)\in C^n[-1,1]$: $f(x), f'(x), \ldots f^n(x)$: cont over $[-1,1]$

$$\int_{-1}^{1} f(x)P(x)\,dx = \int_{-1}^{1} f(x)\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]\,dx$$

$$= \frac{1}{2^n n!}\left\{\left[f(x)\frac{d^{n-1}}{dx^{n-1}}\underset{y^n}{\underbrace{[(x^2-1)^n]}}\right]_{-1}^{1} - \int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right\}$$

from 1 to -1, $x^2-1=0$ → this term will become 0

⇐ if we diff., we will get of form $y.\underset{\to const.}{n!}$

$$= \frac{-1}{2^n.n!}\int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

: In each case, 1st term will be zero

$$I = \frac{(-1)^n}{2^n.n!}\int_{-1}^{1} f^n(x)(x^2-1)^n\,dx$$

Case-I $\quad m>n$ or $n>m$: let $m<n$

let $f(x) = P_m(x)\in C^n[-1,1]$ $\quad$ {we know that}

$$I = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} P_m^{\,n}(x)(x^2-1)^n\,dx = 0 \quad \left[\frac{d^n}{dx^n}P_m(x)\right]$$

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$$I = \int_{-1}^{1} P_m(x)\, P_n(x) = 0$$

→ $m > n$: change the role of m and n.

$f \in C^m$

$$I = \int_{-1}^{1} f(x)\, P_m(x)\, dx$$ Replace $f(x) = P_n(x)$

Case-II ⇒ $n = m$

$$f(x) = P_n(x)$$

$$f^n(x) = \frac{d^n}{dx^n}\left(\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\right) = \frac{1}{2^n n!}\frac{d^{2n}}{dx^{2n}}(x^2-1)^n$$

↓ only non-zero term will be leading coefficient : x^{2n}

diff $2n$ times ⇒ $(2n)!$

$$= \frac{1}{2^n n!}(2n)!$$

$$I = \frac{(-1)^n}{2^n n!}\frac{(2n)!}{2^n n!}\int_{-1}^{1}(x^2-1)^n\, dx$$

$$= \frac{2(2n)!}{2^{2n}(n!)^2}\underbrace{\int_0^1 (1-x^2)^n\, dx}_{I_1}$$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$I_1 = \int_0^{\pi/2} \cos^{2n}\theta \cos\theta\, d\theta = \int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta$$

① ②

By parts

$$I_1 = \cos^{2n}\theta \sin\theta \Big|_0^{\pi/2} + 2n\int \cancel{\sin^{2n}}\theta \cos^{2n-1}\theta \sin\theta\, d\theta$$

Using ① $$\int_0^{\pi/2}\cos^{2n+1}\theta\, d\theta = \frac{2n}{2n+1}\int_0^{\pi/2}\cos^{2n-1}\theta\, d\theta \quad \text{—— (4)}$$

$$I_1 = \frac{2n-2}{2n-1}\int_0^{\pi/2}\cos^{2n-3}\theta\, d\theta$$

In (4)

$$\text{RHS} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\int_0^{\pi/2}\cos^{2n-3}\theta\, d\theta$$

$$= \frac{2n}{2n+1}\cdots\frac{2}{3}\int_0^{\pi/2}\cos\theta\, d\theta$$

$$= \frac{2^n \cdot n!}{1\cdot 3\cdots(2n+1)(2n-1)} \times \frac{2^n}{2^n} = \frac{2^{2n}(n!)^2}{(2n)!(2n+1)}$$

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from eq^n ⊛,

$$I = \frac{2\,(2n)!}{2^{2n}\,(n!)^2} \times \frac{2^{2n}\,n!}{(2n)!\,(2n+1)} = \frac{2}{2n+1}$$

$$\boxed{I = \frac{2}{2n+1}}$$

•) $1 = P_0(x)$

$x = P_1(x)$

$x^2 = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)$

$$x^n = \sum_{n=0}^{\infty} a_n P_n(x) \Rightarrow \{P_n(x)\}_{n=0}^{\infty} \text{ is basis for set of Polynomials}$$

→ given $f(x)$ find a_n in terms of $f(x)$ and $P_n(x)$. Use orthogonal property to determine this coefficient

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

Multiply by $P_m(x)$. Integrate $\int_{-1}^{1}$, then use orthogonal principal

27-03-17

① Use orthogonality of $f(x)$ $\{P_n(x)\}_{n=0}^{\infty}$

$$\int f(x)\, P_m(x)\, dx = \int_{-1}^{1} \sum_{n=0}^{\infty} a_n P_n(x)\, P_m(x)\, dx$$

∵ $a_n P_n(x)$ converges to $f(x)$ → integration can be taken inside Σ

$$= \sum_{n=0}^{\infty} \int_{-1}^{1} a_n P_n(x)\, P_m(x)\, dx \Rightarrow \text{will be 0 except when } m = n$$

$$= a_n \int_{-1}^{1} P_n^2(x)\,dx = a_n \cdot \frac{2}{2n+1}$$

$$\boxed{a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x)\, P_m(x)\, dx}$$

Theorem: Generating function:

$$h(t) = \frac{1}{\sqrt{1-2xt+t^2}}$$

: Coefficient of power series will be Legendre polynomial.

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \quad t \neq 1 \ (t: \text{small})$$

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Date
Page

legendre polynomial : $(1-x^2)y'' - 2xy' + p(p+1)y = 0 \qquad x \in R$

$P(x) = \frac{-2x}{1-x^2} \qquad Q(x) = \frac{p(p+1)}{1-x^2}$

$x = \pm 1$ are singular points of above eq^n (not ordinary points)

Def^n: A singular point x_0 of ~~above~~ $y'' + P(x)y' + Q(x)y = 0$ eq^n is said to be regular if the func^{n}s $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at $x = x_0$, otherwise, it is called irregular singular point.

we are allowing singularity in $P(x)$ at most upto $\frac{1}{x-x_0}$ & in $Q(x)$ at most upto $\frac{1}{(x-x_0)^2}$

power can't be more than 1

power can't be more than 2

For legendre polynomial

Case-I $\quad x_0 = 1$

$(1-x_0)P(x) = (1-x)\left[+\frac{2x}{1-x^2}\right] = \frac{2x}{1+x}$: analytic at $x = 1$

$(x-x_0)Q(x) = (x-1)^2 \frac{p(p+1)}{1-x^2} = \frac{p(p+1)(1-x)}{1+x} = 0$ at $x = 1$

$\Rightarrow$ analytic

Hence, $x_0 = 1$ is regular singular point.

Case-II $\quad x_0 = -1$

$-(x+1)\left[\frac{2x}{1-x^2}\right] = \frac{-2x}{1-x}$: analytic

$(x+1)^2\left[\frac{p(p+1)}{1-x^2}\right]$: analytic

Bessel's Equation

(3) $\quad x^2y'' + xy' + (x^2-p^2)y = 0 \qquad p$: non-ive real no.

$\Rightarrow y'' + \frac{y'}{x} + \left(1 - \frac{p^2}{x^2}\right)y = 0$

$x = 0$: not an ordinary point

Regular point $\quad (x-0)\left(\frac{1}{x}\right) = 1$: analytic

$(x-0)^2\left[1-\frac{p^2}{x^2}\right] = x^2\left[1-\frac{p^2}{x^2}\right] = x^2-p^2 = -p^2$ at $x=0$
$\Rightarrow$ analytic

$\Rightarrow x=0$: Regular Point

$y=\sum a_n x^n$: can't be solⁿ of eqⁿ (3)

Theorem: Any differential eqⁿ of the form $y''+\frac{p(x)}{x}y'+\frac{q(x)}{x^2}y=0$ — (4)

where the function $p(x)$ and $q(x)$ are analytic at $x=0$, has atleast one solⁿ of the form

Frobenius series solⁿ $\Rightarrow$ $\boxed{y=x^m\sum_{n=0}^{\infty}a_n x^n = \sum_{n=0}^{\infty}a_n x^{n+m}}$ [using Euler-Cauchy Eqⁿ]

solⁿ of Bessel's eqⁿ :

$p(x)=\sum_{n=0}^{\infty}p_n x^n$ $\quad q(x)=\sum_{n=0}^{\infty}q_n x^n$ [p_n and q_n : analytic]

substitute in eqⁿ (4),

$$y''+\left(\frac{p_0+p_1x+p_2x^2+\dots}{x}\right)y'+\frac{(q_0+q_1x\dots)}{x^2}y=0$$

$$y'=\sum_{n=0}a_n(n+m)x^{n+m-1} \qquad y''=\sum_{n=0}^{\infty}a_n(n+m)(n+m-1)x^{n+m-2}$$

$a_0\neq 0$
(~~we can choose~~ m such that a_0x, a_0x^2... comes)

$\Rightarrow \sum_{n=0}a_n(n+m)(m+n-1)x^{n+m-2}$

Coeff. of a_0 (All constant terms in resulting series)

(5) — $\boxed{m(m-1)+mp_0+q_0=0}$ Indicial Eqⁿ

$a_1(\quad)=0 \Rightarrow m=m_1$ & $m=m_2$ get 2 recurrence relⁿs.

We will get 2 solⁿs m_1 and m_2. Corresponding 2 recurrence relations can be obtained.

Case-1 : Roots not differing by an integer (distinct roots)

m_1 and m_2 solⁿs of eqⁿ (5)

$y_1=x^{m_1}\sum_{n=0}^{\infty}a_nx^n$; $y_2=x^{m_2}\sum_{n=0}^{\infty}b_nx^n$: L.I. roots.

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Date
Page

Case-II Equal Double roots $m_1 = m_2$

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = y_1(x) \ln x + x^{m_1} \sum_{n=0}^{\infty} b_n x^n$$

Case-III : distinct roots differing by integer

$$y_1(x) = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = k y_1 \ln x + x^{m_2} \sum_{n=0}^{\infty} b_n x^n$$

29-03-17

Consider : (all homogeneous 2nd O.D.E. can be written in this form)

(1) — $$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[\lambda q(x) + r(x)\right] y = 0 \qquad x \in [a,b]$$

Eg. $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0, \quad \lambda = n(n+1)$$

so, $p(x) = 1-x^2 \quad q(x) = 1 \quad r(x) = 0$

I.C — $y(x_0) = y_0 \quad y'(x_0) = y_1$

By Existence & Uniqueness theorem, ∃ unique solⁿ in $[a,b]$

$y(a) = y_0, \quad y(b) = y_1$

Boundary value problem

Eg. Vibration on string (elastic) : Model can be represented by

$$y'' + \lambda y = 0$$

B.C $y(0) = 0 \quad y(\pi) = 0$

$p(x) = 1 \quad q(x) = 1 \quad r(x) = 0 \quad a = 0, \ b = \pi$

We are looking for non-trivial solⁿ

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Gunjan

Date

Page

Case-I: λ is negative

let $\lambda = -n^2$

$$y'' - n^2 y = 0$$

$\frac{d^2y}{}$ Auxiliary eq^n: $(y = e^{mx})$

$$m^2 - n^2 = 0 \Rightarrow m = \pm n$$

$$y(x) = c_1 e^{nx} + c_2 e^{-nx}$$

$$y(0) = c_1 + c_2 = 0$$
$$y(\pi) = c_1 e^{n\pi} + c_2 e^{-n\pi} = 0$$
$\Rightarrow c_1 = c_2 = 0$

$y(x) \equiv 0$ — trivial sol^n.

Case-II: $\lambda = 0$

$$y'' = 0$$
$$y(x) = c_1 + c_2 x$$
$$y(0) = c_1 = 0$$
$$y(\pi) = c_1 + c_2\pi = 0$$
$c_1 = c_2 = 0$

$$y(x) \equiv 0$$

Case-III: λ is +ve $\Rightarrow \lambda = n^2$

$$m^2 + n^2 = 0$$
$$m = \pm in$$

$$y(x) = A\cos nx + B\sin nx$$
$$y(0) = A = 0$$
$$y(\pi) = \cancel{A\cdot 0}\ B\sin n\pi = 0$$

$B = 0$ leads to trivial sol^n again

Assume $B \neq 0 \Rightarrow \sin n\pi = 0 \Rightarrow n = 0, \pm 1, \pm 2$

if $n = 0$: Case-II $\Rightarrow n \neq 0$

$\Rightarrow n = \pm 1, \pm 2, \ldots$

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We ~~got~~ take only +ive values of eigen values

Corresponding to each value, we have one solⁿ. (let)

$y_n(x) = B_n \sin nx$, for $n = 1, 2, \ldots$

$\lambda = n^2 = 1, 4, 9, \ldots$ (eigen values)

The solⁿ corresponding to each eigen values are called as Eigen function

$\{\lambda_n\}_{n=1}^{\infty} \longrightarrow \{y_n\}_{n=1}^{\infty}$

eigen value → eigen func^n

Few Observation

(i) Eigen values are uniquely determined by the problem

(ii) Eigen funcⁿ are determined upto a non-zero constant factor (B_n is always variable)

(iii) $\lambda_n \to \infty$ as $n \to \infty$

(iv) $\lambda_1 < \lambda_2 < \lambda_3 \ldots < \lambda_n < \lambda_{n+1} < \ldots$

(Eigen values of the BVP are +ive real no.s which can be arranged in an increasing sequence)

(v) n^{th} eigen funcⁿ vanishes at the end points and exactly $(n-1)$ zeros inside the interval

→ $\sin x$ $[0, \pi]$ no zero inside interval

$\sin 2x$ $[0, \pi]$ 1 zero " "

$\sin nx$ $[0, \pi]$ $(n-1)$ zeroes

⇒ Consider more general problem

$$[p(x) y']' + [\lambda q(x) + r(x)] y = 0 \quad \forall \quad x \in [a, b] \quad \text{--- (1)}$$

homogenous B.C.

$$\left.\begin{aligned} c_1 y(a) + c_2 y'(a) &= 0 \\ d_1 y(b) + d_2 y'(b) &= 0 \end{aligned}\right\} \text{--- (2)}$$

Date
Page

where c_1 or $c_2 \neq 0$ and d_1 or $d_2 \neq 0$ — ③

Assume $p(x)$ and $q(x) > 0 \quad \forall \; x \in [a,b]$

The BVP ① - ② is called as Strum - Liouville Problem

Properties of Eigen func^n's

$$\{\lambda_n\}_{n=1}^{\infty} \rightsquigarrow \{y_n(x)\}_{n=1}^{\infty}$$

λ_m and λ_n are two distinct eigen values.

$y_m(x)$ and $y_n(x)$ are two corresponding eigen func^n of eq^n ①.

$$\Rightarrow (p(x)y_m')' + (\lambda_m q(x) + r(x))y_m = 0 \quad — \quad * y_n$$

$$\text{and, } (p(x)y_n')' + (\lambda_n q(x) + r(x))y_n = 0 \quad — \quad * y_m$$

$$\Rightarrow (p(x)y_m')'y_n - (p(x)y_n')'y_m + (\lambda_m - \lambda_n)q(x)y_m y_n = 0$$

$$\Rightarrow \int_a^b (\lambda_m - \lambda_n)q(x)y_m y_n \, dx = \int_a^b (p(x)y_n')'y_m \, dx - \int_a^b (p(x)y_m')'y_n \, dx$$

③

$$= \int_a^b$$

$$= y_m p(x) y_n' \Big|_a^b - \int \cancel{y_m' p(x) y_n'} \, dx - y_n p(x) y_m' \Big|_a^b + \int \cancel{y_n' p(x) y_m'} \, dx \quad 0$$

$$= \cancel{y_m y_n'(p(b) - p(a)) - y_n y_m'(p(b) - p(a))}$$

$$= y_m(b)p(b)y_n'(b) - y_m(a)p(a)y_n'(a) - y_n(b)p(b)y_m'(b) + y_n(a)p(a)y_m'(a)$$

$$= p(b)\left[y_m(b)y_n'(b) - y_n(b)y_m'(b)\right] - p(a)\left[y_m(a)y_n'(a) - y_n(a)y_m'(a)\right]$$

$$= p(b)W(b) - p(a)W(a)$$

y_m and y_n are eigen func^n corresponding to λ_m & λ_n

Date
Page

→ $y(a)=0$ & $y(b)=0$, $y'(a)=0$ & $y'(b)=0$

Putting $a=0$ in eq^n 3

$$\int_a^b q\, y_m y_n\, dx = 0 \rightarrow \text{for } m \neq n$$

31-03-17

$$[p(x)\,y']' + [q(x)\lambda + r(x)]\,y = 0 \qquad x \in (a,b) \quad \text{——(1)}$$

B.C.

$$\begin{cases} c_1 y(a) + c_2 y'(a) = 0 \\ d_1 y(b) + d_2 y'(b) = 0 \end{cases} \longrightarrow (2) \qquad p(x), q(x) > 0 \quad x \in [a,b]$$

where c_1 or $c_2 \neq 0$ and d_1 or $d_2 \neq 0$ —— (3)

Orthogonality of Eigen func^{n}s w.r.t. weight func^n ($q(x)$)

$$(\lambda_m - \lambda_n)\int_a^b q(x)\, y_m(x)\, y_n(x) = p(b)\,W(b) - p(a)\,W(a)$$

Special case: $y(a)=0$ & $y(b)=0$ —— (i)

or, $y'(a)=0$ & $y'(b)=0$ —— (ii)

$$W(y_m, y_n)\Big|_{x=b} = y_m(b)\,y_n'(b) - y_m'(b)\,y_n(b) = 0$$

$$W(y_n, y_m)\Big|_{x=a} = y_n(a)\,y_m'(a) - y_n'(a)\,y_m(a) = 0$$

→ $\cancel{(\lambda_m-\lambda_n)}\int_a^b q(x)\, y_m y_n = 0$ for B.C. (i) & (ii)

y_m and y_n are eigen func^n satisfying eq^n 2 and 3

$c_1 y_m(a) + c_2 y_m'(a) = 0$ → $AX = b$

$c_1 y_n(b) + c_2 y_n'(b) = 0$

Earlier, we had 4 variables

$$A\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This system has unique sol^n of c_1 & c_2 if determinant of coefficient matrix:

It has unique sol^n if rank A = rank $[A|b]$

$X = A^{-1} b$

$AX = 0$

→ unique sol^n : $|A| \neq 0$

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$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Date
Page

$$\begin{vmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{vmatrix} = 0$$

we also know c_1 or $c_2 \neq 0$

$\Rightarrow$ $W(y_m, y_n)\Big|_{x=a} = 0$ $\Rightarrow$ $W(a) = 0$

similarly, $W(b) = 0$

Hence for B.C. 2,

$$\boxed{\int_a^b q(x)\, y_m y_n \, dx = 0 \quad , \quad m \neq n}$$

Ques. Find coeffecients a_n s.t. for any given func^n $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

↳ continuous func^n

Multiplying $q(x) y_m$ both sides

$$\int_a^b q\, y_m f(x) = \sum_{n=1}^{\infty} \int_a^b q\, a_n y_m y_n(x)\, dx$$

$$\int_a^b q(x) y_m f(x) = a_m \int_a^b q(x)\, y_m^2(x)\, dx$$

$$a_m = \frac{\int_a^b q(x)\, y_m(x) f(x)\, dx}{\int_a^b q(x)\, y_m^2(x)\, dx} \quad \Rightarrow \text{non-zero}$$

Def^n: 1. BVP 1 – 3 with $p(x), q(x) > 0$ on (a,b) and continuous on $[a,b]$ are called <u>regular</u> problem.

2. If one of $p(x), q(x)$ vanish or become infinite at the end points, or the interval itself is infinite. Then, we call the problem as <u>singular</u>.

Eg. $[(1-x^2)y']' + \lambda y = 0$ $\quad -1 \le x \le 1$, $\lambda = n(n+1)$

$P(x) = 1 - x^2$: vanish at $x = -1, 1$

$\Rightarrow$ <u>singular problem</u> (sturm liouville problem on $[-1,1]$)

(legendre Polynomial)

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Date
Page

Que. Show that every eigen func^n is unique except for a constant factor.

PROOF: Let u and v be two eigen func^{n}s corresponding to an integer value λ.

claim: u is constant multiple of v, $W(u,v) = 0$

Proof: $(pu')' + (q\lambda + r)u = 0 \times v$ } λ is same for both
$(pv')' + (q\lambda + r)v = 0 \times u$

$\Rightarrow (pu')'v - (pv')'u = 0$

$\Rightarrow (p'u' + pu'')v = (p'v' + pv'')u$

$\Rightarrow p(u''v - v''u) + p'(u'v - v'u) = 0$

$\Rightarrow pW'(v,u) + p'W(v,u) = 0$

$\Rightarrow [pW(u,v)]' = 0$

$\Rightarrow pW(u,v) = c$

From B.C. ②, we get $c = 0$

$\Rightarrow \boxed{W(u,v) = 0}$

Hence Proved

For legendre polynomial:
$((1-x^2)y')' + \lambda y = 0$ $\quad q(x) = 1$ $\quad -1 \le x \le 1$ $\quad \lambda = n(n+1)$

		eigen func^n
for $n = 0$	$\lambda = 0$	P_0
$n = 1$	$\lambda = 2$	P_1
$n = 2$	$\lambda = 6$	P_2
⋮	⋮	⋮

Gunjan
Date
Page

For any n, eigen value is $n(n+1)$.
eigen func^n is $P_n(x)$

Put $q(x) = 1$ $a = -1$ $b = 1$

$$\int_{-1}^{1} y_m(x)\, y_n(x)\, dx = 0 \quad \text{for } m \neq n$$

3-4-17 Laplace Transformation

$y''(x) + P(x)y' + Q(x)y = R(x) \leftarrow$ Input

$y(x_0) = y_0$, $y'(x_0) = y_1$

$R(x)$ can be discontinuous

$$f(t) \longrightarrow \boxed{L} \xrightarrow{F(s)} \boxed{L^{-1}} \longrightarrow \text{get the sol}^n$$

* This again comes from power series

(analog) $\sum_{n=0}^{\infty} a_n x^n = A(x)$

$\sum_{n=0}^{\infty} a(n) x^n = A(x)$

$\rightarrow$ $a(n) = 1 \ \forall n$ $\quad A(x) = \frac{1}{1-x}$, $|x| < 1$

$\rightarrow$ $a(n) = \frac{1}{n!}$ $\quad A(x) = e^x$

The power series can further be transformed as:

(continuous) $\int_0^{\infty} f(t)\, x^t\, dt$ $\quad x = e^{\ln x} \Rightarrow x^t = e^{t \ln x}$

$0 < x < 1$ $\quad \ln x < 0$

let $\ln x = -s$

$$\int_0^{\infty} f(t)\, e^{-st}\, dt = L[f(t)] = F(s)$$

Integral transformation of $f(t)$

e^{-st} $\downarrow$ Kernel of this transformation

Laplace Transformation

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Ex. 1 $f(t) = 1$

$$\int_0^\infty e^{-st}\, dt$$

$$= \lim_{r \to \infty} \int_0^r e^{-st}\, dt$$

$$= \lim_{r \to \infty} \left[\frac{e^{-st}}{-s}\right]_0^r = \frac{1}{s}, \quad s > 0$$

$$L\{1\} = \frac{1}{s}, \quad s > 0$$

Ex. 2 $f(t) = e^{at}$

$$\int_0^\infty e^{(a-s)t}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \frac{1}{s-a}, \quad s > a$$

$$L\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

* Exponential shifting formula
* s-shifting formula

$L\{f(t)\} = F(s)$ [Given] $= \int_0^\infty e^{-st} f(t)\, dt$, $s>0$ — (1)

$e^{at} f(t) \xrightarrow{L}$? (in terms of $F(s)$)

$$L\{e^{at} f(t)\} = \int_0^\infty f(t)\, e^{-(s-a)t}\, dt \quad \text{— (2)}$$

$= F(s-a)$, $s>a$ [compare with (1)]

$$\boxed{e^{at} f(t) \longrightarrow F(s-a), \quad s > a}$$

→ $L\{1\} = \frac{1}{s} = F(s)$, $s > 0$

$e^{at} \cdot 1 \xrightarrow{L} F(s-a) = \frac{1}{s-a}$, $s - a > 0$ or $s > a$

→ Laplace transformation is linear $\quad$ ($\because$ Integration is linear)

$$L\{af(t) + b\,g(t)\} = a\,L\{f(t)\} + b\,L\{g(t)\}$$

Date
Page

Ex. $L\{\cos at\} = \int_0^\infty \cos at\, e^{-st}\,dt = (1)$

$$\cos at = \frac{1}{2}\left[e^{iat} + e^{-iat}\right]$$

$$L\{\cos at\} = \frac{1}{2}\left\{L\{e^{iat}\} + L\{e^{-iat}\}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{s}{s^2+a^2}$$

Ex. $L\{\sin at\} - ?$

Ex. $L\{e^{at}\cos\omega t\} = \int_0^\infty \cos\omega t\, e^{-(s-a)}\,dt = \frac{s-a}{(s-a)^2+\omega^2}$

<u>Laplace Transformation of Polynomial</u>

$$L\{t^n\} = \int_0^\infty t^n e^{-st}\,dt$$

$$= \underbrace{t^n \frac{e^{-st}}{-s}\Bigg|_0^\infty}_{(1)} - \int_0^\infty n\,t^{n-1}\frac{e^{-st}}{-s}\,dt$$

(1) → $\lim_{t\to\infty} \frac{t^n}{e^{st}} \Rightarrow 0$ (By L'Hospital's Rule)

$$L\{t^n\} = \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt = \frac{n}{s}L\{t^{n-1}\} = \frac{n(n-1)}{s^2}L\{t^{n-2}\}$$

$$= \frac{n(n-1)\ldots 1}{s^n}L\{t^0\} = \frac{n!}{s^{n+1}}$$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}} \quad : \text{when } n \in \mathbb{Z}$$

• a is real no., $a > 0$

$$L\{t^a\} = \int_0^\infty t^a e^{-st}\,dt \qquad st = x, \quad s\,dt = dx$$

$$= \int_0^\infty \left(\frac{x}{s}\right)^a e^{-x}\frac{dx}{s} = \frac{1}{s^{a+1}}\underbrace{\int_0^\infty x^a e^{-x}\,dx}_{\text{gamma func}^n}$$

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Date
Page

$$\boxed{L\{t^a\} = \frac{1}{s^{a+1}} \gamma(a+1)}$$ $\gamma(n+1) = n!$

→ $a = n$ (+ive integer)

$$L\{t^n\} = \frac{1}{s^{n+1}} \gamma(n+1) = \frac{n!}{s^{n+1}}$$

Inverse Laplace Transformation :

Ques $\frac{1}{s(s+3)} \xrightarrow{L^{-1}} ??$

$$\frac{1}{s(s+3)} = \frac{1/3}{s} + \frac{(-1/3)}{s+3} = \frac{1}{3} - \frac{1}{3}e^{-3t}$$

⇒ $\int_0^\infty e^{-st} f(t)\, dt$ — Improper Integral.

for which value of t, this becomes definite integral ?

Assumptions :-

① f(t) should not grow rapidly

② It should be almost exponential growth like e^t (not like e^{t^2}) {so that it may cancel e^{-st} ...}

$|f(t)| \le Me^{kt}$, $t \geq 0$, $k \geq 0$, $M > 0$

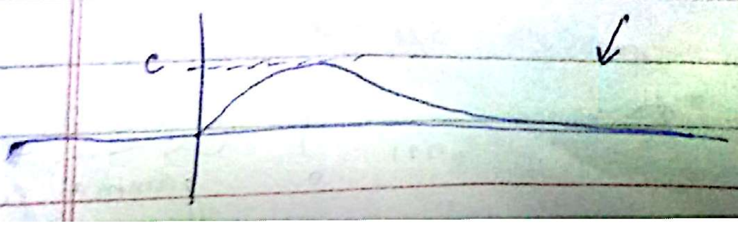
⇒ f(t) of exponential type.

Ques $f(t) = \sin t$: exponential type

$|\sin t| \le Me^{kt}$

we know, $|\sin t| \le 1 = 1 \cdot e^{0t}$ ⇒ $M=1$, $k=0$, $t \geq 0$

Ques $f(t) = t^n$ $c = t^n/e^t$ $|t^n| \le ce^t$

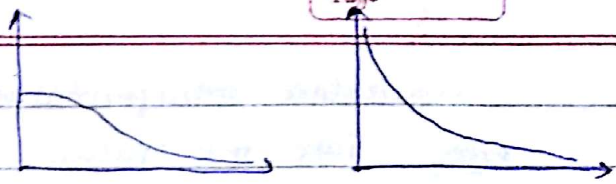
c


$\boxed{c = n!}$

⇒ exponential type

Date
Page

Ques. $f(t) = \frac{1}{t}$



$\int_0^\infty \frac{1}{t} e^{-st}\, dt \rightarrow$ Improper $\quad e^{-st}$

↳ at $t \to \infty$, but is unbounded

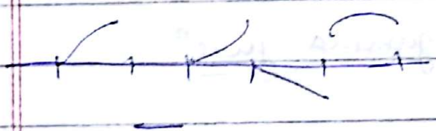
$\frac{1}{t}$ ~~fast~~ goes faster than e^{-st} → not exponential type

Ques. e^{t^2} : is not bounded as $|e^{t^2}| \geq e^{kT}$, no matter how big k is

5-4-17

Def^n: $f(t)$ is said to be piecewise continuous in $[a,b]$ if $f(t)$ is continuous in finitely many subintervals of $[a,b]$ and has finite limit at either end points of each sub-interval

Eg.



Existence of Laplace Transformations

Let $f(t)$ be piecewise continuous on every finite interval for $t \geq 0$ and satisfies

$$|f(t)| \leq Me^{kt}, \quad \forall\ t \geq 0,\ k \geq 0,\ M > 0$$

or $f(t)$ is of exponential type

Then $L\{f(t)\}$ exists $\forall\ s > k$

Proof :- $|L\{f(t)\}| = \left| \int_0^\infty f(t) e^{-st}\, dt \right| \leq \int_0^\infty |f(t) e^{-st}\, dt|$

$$\leq M \int_0^\infty e^{kt} e^{-st}\, dt = M \left. \frac{e^{-(s-k)t}}{-(s-k)} \right|_0^\infty$$

$$= \frac{M}{s-k} \quad \text{for } s > k$$

$$|L\{f(t)\}| \leq \frac{M}{s-k} \quad \text{for } s > k$$

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→ Laplace transformation of $f(t)$ is unique if it exists

proof : Take two values $F(s_1)$ & $F(s_2)$. Take difference

$$F(s_1) - F(s_2) = 0$$

* The condition

$$|f(t)| \leq Me^{kt}$$

is sufficient but not necessary.

Ex. Take $f(t) = 1/\sqrt{t}$: $\frac{1}{\sqrt{t}} \to \infty$ as $t \to 0$ ⇒ Not exponential type

$$L\{f(t)\} = \int_0^\infty \frac{1}{\sqrt{t}} e^{-st} dt \qquad st = x, \quad s\,dt = dx$$

$$= \int_0^\infty \frac{\sqrt{s}}{\sqrt{x}} e^{-x} \frac{dx}{s} = \int_0^\infty \frac{1}{\sqrt{sx}} e^{-x} dx$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty (x)^{-1/2} e^{-x} dx \to \text{gamma func}^n$$

$$= \frac{1}{\sqrt{s}} \gamma\left(-\frac{1}{2}+1\right)$$

$$= \frac{\gamma(1/2)}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

Ex. $\cosh t$, $\sinh t$

$\downarrow$ $\frac{e^{at}+e^{-at}}{2}$, $\downarrow$ $\frac{e^{at}-e^{-at}}{2}$

$$L\{\cosh t\} = \int_0^\infty \frac{e^{at}+e^{-at}}{2} \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{s}{s^2-a^2}, \quad s > |a|$$

Gunjan
Date
Page

solving ODE by using L.T. :

$y'' + ay' + by = h(t)$ → need not be continuous

$y(0) = y_0$, $y'(0) = y_0'$

Take L.T. of ODE → get Algebraic eq^n in $\Psi(s)$ $\xrightarrow{\text{Solve for } \Psi(s)}$ $\Psi(s) = \frac{P(s)}{Q(s)}$ $\xrightarrow{L^{-1}\{\Psi(s)\}}$ $y(t)$

Laplace Transformation of derivatives of $f(t)$
↳ exponential type

$$L\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt$$
② ①

$$= e^{-st} f(t) \Big|_0^\infty + s \int_0^\infty e^{-st} f(t)\, dt$$

$$= 0 - f(0) + sL\{f(t)\}$$

$$\Rightarrow \boxed{L\{f'(t)\} = sL\{f(t)\} - f(0)} \quad \text{(1)}$$

$$\rightarrow L\{f''(t)\} = \int_0^\infty f''(t)\, e^{-st}\, dt$$
② ①

$$= e^{-st} f'(t) \Big|_0^\infty$$

or use eq^n — (1)

$$L\{f''(t)\} = sL\{f'(t)\} - f'(0)$$
$(f'(t))' = g'$
$$= s[sL\{f(t)\} - f(0)] - f'(0)$$

$$\boxed{L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)}$$
$$= s^2 F(s) - sf(0) - f'(0)$$

* $f'(t)$ need not be of exponential type

Theorem : If $f(t), f'(t), f''(t), \ldots f^{n-1}(t)$ is continuous $\forall\ t \geq 0$
$|f(t)| \leq Me^{kt}$ for $k \geq 0$, $t \geq 0$, $M > 0$
$f^n(t)$: piecewise continuous

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Then $\boxed{L\{f^n(t)\} = s^n F(s) - s^{(n-1)} f(0) - s^{(n-2)} f'(0) \ldots - f^{(n-1)}(0) \quad \text{for } s > k}$

Corollary: Let $f'(t)$ be piecewise continuous for $t \geq 0$

Also, let $f(0) = 0$

Then $L\{f'(t)\} = sF(s) - \cancel{f(0)}^{0}$

$L\{f'(t)\} = sF(s)$

$\boxed{L^{-1}\{sF(s)\} = f'(t)}$

$F(s) \xrightarrow{L^{-1}} f(t)$ (known)

$sF(s) \xrightarrow{L^{-1}} f'(t)$ [if $f(0)=0$]

Ex. Find $L\{\sin^2 t\}$

$f(t) = \sin^2 t \qquad f(0) = 0$

$f'(t) = 2\sin t\cos t = \sin 2t$

$L\{\sin 2t\} = L\{f'(t)\} = s\, L\{f(t)\}$

$\cancel{= \dfrac{s \cdot 2}{s^2+4}} = \dfrac{2s}{s^2+4}$

$\dfrac{2}{s^2+4} = s\, F(s)$

$F(s) = \dfrac{2}{s(s^2+4)} \xrightarrow{L^{-1}} \sin^2 t$

Ex. $f(t) = t\sin at \qquad$ {use f''}

6-4-17. $y'' + 2y' + \cancel{8}y = e^{-t}$

$y(0) = -1, \quad y'(0) = 1$

Assume $L\{y(t)\} = Y(s)$

$L\{y'(t)\} = sY(s) + 1$

$L\{y''(t)\} = s^2 Y - sy(0) - y'(0)$

$= s^2 Y + s - 1$

$L\{y'' + 2y' + y\} = L\{e^{-t}\}$

$\Rightarrow s^2Y + s - 1 + 2sY + 2 + Y = \frac{1}{s+1}$

$\Rightarrow (s+1)^2 Y + (s+1) = \frac{1}{s+1}$

$Y(s) =$

$\Rightarrow (s+1)^2 Y = \frac{1}{s+1} - (s+1) \Rightarrow \frac{1 - (s+1)^2}{(s+1)^3} = \frac{1}{(s+1)^3} - \frac{1}{(s+1)^4}$

polynomial

<u>Inverse Laplace Transformation</u>

$y(t) = e^{-t}\frac{t^2}{2} - e^{-t}$

$\frac{1}{s^3} \xrightarrow{L^{-1}} \frac{t^2}{2}$

$\Rightarrow \boxed{y(t) = \frac{1}{2}e^{-t}(t^2 - 2)}$

$F(s) \xrightarrow{L^{-1}} f(t)$

$F(s-a) \xrightarrow{L^{-1}} e^{at} f(t)$

* No need to find general solⁿ of ~~...~~ corresponding homogeneous eqⁿ.

* No need of finding particular solⁿ

<u>Laplace Transformation of Integration of f(t)</u>

$L\{\int_0^t f(\tau)d\tau\} = ?$ [In terms of $L\{f(t)\} = F(s)$]

→ 1st you have to show existence

$\left\{ \begin{array}{l} f(\tau) \text{ : piecewise continuous} \\ |f(t)| \le Me^{kt} \quad \text{[exponential type]} \end{array} \right.$

<u>Proof</u>: $g(t) = \int_0^t f(\tau)\,d\tau$, $g(t)$ is also piecewise continuous

$|g(t)| = \left|\int_0^t f(\tau)d\tau\right| \le \int_0^t |f(\tau)d\tau| \le M\int_0^t e^{k\tau}d\tau = \frac{M}{k}[e^{kt} - 1]$

$< \frac{M}{k}e^{kt}$

$\Rightarrow g(t)$: exponential type.

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$g'(t) = f(t)$ except at point of discontinuity of $f(t)$

$L\{g'(t)\} = L\{f(t)\}$

$s \cdot L\{g(t)\} - \cancel{g(0)}^{0} = F(s)$

$L\{g(t)\} = \frac{F(s)}{s}$

$$\boxed{L\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}, \quad s > K}$$

* $F(s) \xrightarrow{L^{-1}} f(t)$

$\frac{F(s)}{s} \xrightarrow{L^{-1}} \int_0^t f(\tau)\,d\tau$

Eg. Find inverse of ~~$L\{f(t)\}$~~ $= \frac{1}{s(s^2+\omega^2)}$

$F(s)$ $\frac{1}{s^2+\omega^2} \xrightarrow{L^{-1}} \frac{\sin \omega t}{\omega} = f(t)$

$\frac{F(s)}{s} = L\left\{\int_0^t f(\tau)\,d\tau\right\}$

$= \frac{1}{\omega}\int_0^t \sin \omega \tau \, d\tau$

$= \frac{1}{\omega^2}(1 - \cos \omega t)$

Eg. Find Inverse L.T. of $\frac{1}{s^2(s^2+\omega^2)}$

Integrate above expression

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Date
Page

Differentiation & Integration of Transform

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\frac{d}{ds} F(s) = F'(s) = \frac{d}{ds}\left(\int_0^\infty f(t)\, e^{-st}\, dt \right)$$

$$= \int_0^\infty (-t)\, f(t)\, e^{-st}\, dt = -L\{t f(t)\}$$

$$\boxed{F'(s) = -L\{t f(t)\}}$$

Eg. $\sin \omega t \xrightarrow{L} \dfrac{\omega}{s^2+\omega^2}$

$$t \sin \omega t \xrightarrow{L} -\frac{d}{ds}\left\{ \frac{\omega}{s^2+\omega^2} \right\} = +\frac{\omega(2s)}{(\omega^2+s^2)^2}$$

Integration :-

$f(t)$ → exponential type

→ piecewise continuous for $t \geq 0$

$\lim_{t \to 0^+} \dfrac{f(t)}{t}$ exists

$$\boxed{L^{-1}\left\{ \int_s^\infty F(u)\, du \right\} = \frac{f(t)}{t}}$$

Proof $\displaystyle\int_s^\infty F(u)\, du = \int_s^\infty \int_0^\infty f(t)\, e^{-ut}\, dt\, du = \int_0^\infty \int_s^\infty f(t) e^{-ut}\, dt\, du$

$$= \int_0^\infty f(t)\left(\int_s^\infty e^{-ut}\, du \right) dt = \int_0^\infty f(t) \left(\frac{e^{-ut}}{-t} \right)\Bigg|_s^\infty dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-us}}{us} \right] dt = \cancel{\frac{e^{-st}}{st} \int_0^\infty f(t)\, dt} = \int_0^\infty \frac{e^{-st}}{t} f(t)\, dt$$

$$= L\left\{ \frac{f(t)}{t} \right\} \qquad \left\{ \because \lim_{t \to 0^+} \frac{f(t)}{t} \text{ exists} \right\}$$

Eg. L.T. of $\frac{e^t \sin at}{t}$

1st find L.T of $e^t \sin at$ $\longrightarrow \frac{a}{s^2+a^2} = F(s)$

$\therefore e^t \sin at \xrightarrow{L} F(s-1) = \frac{a}{(s-1)^2+a^2}$.

$$\frac{e^t \sin at}{t} = \int_s^\infty \frac{a}{(u-1)^2+a^2}\,du$$

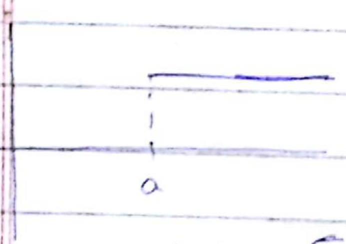
or $$\frac{\sin at}{t} = \int_s^\infty \frac{a}{u^2+a^2}\,du = \tan^{-1}\left(\frac{s}{a}\right)$$

$$e^t \frac{\sin at}{t} \longrightarrow \tan^{-1}\left(\frac{s-1}{a}\right)$$

10-4-17

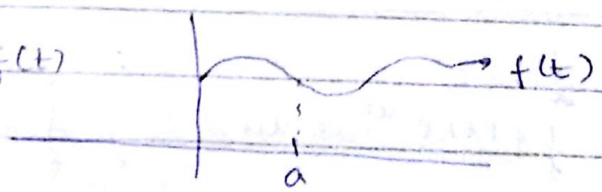
Unit step functions

$$u(t-a) \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t \geq a \end{cases}$$

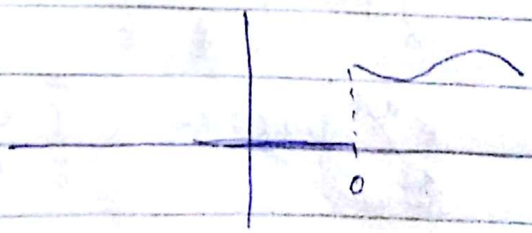


$$u(t) \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$$

let $f(t)$

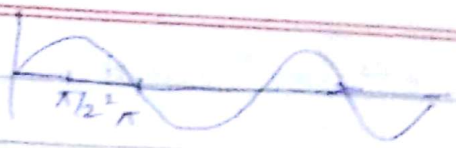


$$u(t-a)\,f(t) = \begin{cases} 0 & t<a \\ f(t) & t>a \end{cases}$$



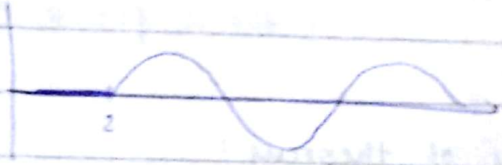
$\int f(t)\ F(s) = \int_0^\infty f(t) e^{-st}\, dt$

Eg. $f(t) = \sin t$

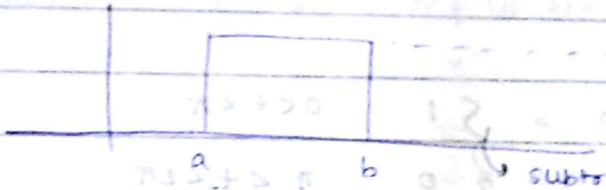


$u(t-2)\sin t$

→ $u(t-2)\sin(t-2)$

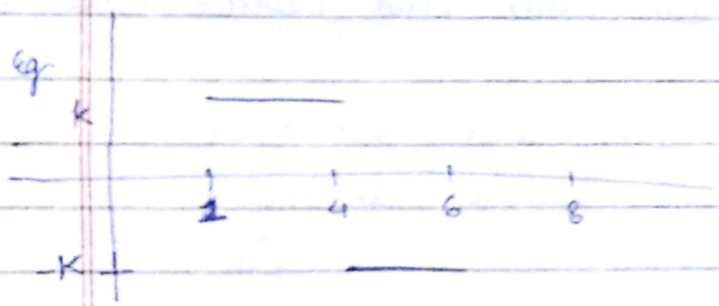


→ To get func^n like :



$f(t) = u(t-a) - u(t-b)$

Eg.



$\Rightarrow k[u(t-1) - u(t-4)]$

$- k[u(t-4) - u(t-6)]$

$\Rightarrow k[u(t-1) - 2u(t-4) + u(t-6)]$

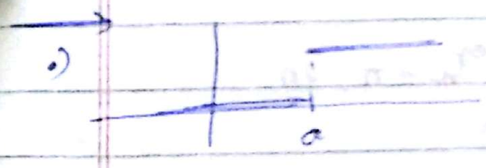
Theorem: 2nd shifting th^m or t-shifting th^m :

f is of exponential type, piecewise continuous

$L\{u(t-a) f(t-a)\} = e^{-as} L\{f(t)\} = e^{-as} F(s)$

$e^{-as} F(s) \xrightarrow{L^{-1}} u(t-a) f(t-a)$

→ •)



∵ piecewise continuous

$L\{u(t-a)\} = \int_0^\infty u(t-a) e^{-st}\, dt$

$= \int_a^\infty e^{-st}\, dt = \left. \frac{e^{-st}}{-s} \right|_a^\infty = \frac{e^{-as}}{s}$

$\boxed{L\{u(t)\} = \frac{1}{s}}$

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Proof :-

$$e^{-as} F(s) = e^{-as}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty f(t)\, e^{-s(a+t)}\,dt$$ let $t+a=\tau$

$$= \int_a^\infty f(\tau-a)\, e^{-\tau s}\,d\tau = \int_0^\infty u(\tau-a) f(\tau-a)\, e^{-\tau s}\,d\tau$$

$$= L\{u(\tau-a) f(\tau-a)\}$$

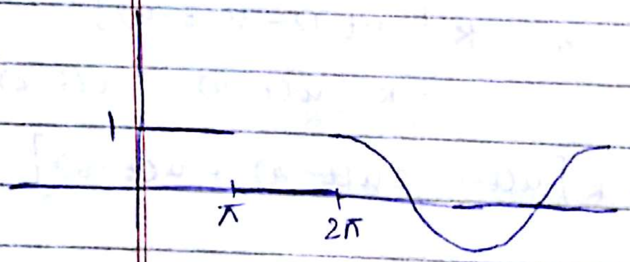
$$= L\{u(t-a) f(t-a)\}$$: Hence Proved.

- **Consequences of theorem :**

$L\{u(t-a) f(t)\} = e^{-as} L\{f(t+a)\}$ shifting of t in $f(t)$.

Eg. $f(t) = \begin{cases} 1 & 0<t<\pi \\ 0 & \pi<t<2\pi \\ \cos t & t \geq 2\pi \end{cases}$ Find $L\{f(t)\}$

In terms of unit step funcⁿ, then find Laplace transfmⁿ



$$f(t) = u(t) - u(t-\pi) + 0 + (u(t-2\pi)) \underbrace{\cos t}_{f(t)}$$

$u(t-a) \xrightarrow{L} \dfrac{e^{-as}}{s}$ $L\{\cos t\}$?

$$F(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-2\pi s} L\{\cos(t+2\pi)\}$$

$$= \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-2\pi s} \cdot \frac{s}{s^2+1}$$

there is a point of discontinuity at $x = \pi, 2\pi$

→ discont. at π

Ques $\dfrac{s e^{-\pi s}}{s^2+4} \Rightarrow \dfrac{s}{s^2+4}(e^{-\pi s}) = \cancel{u(t-\pi)\cos 2t}$

$= e^{-\pi s} L\{\cos 2t\}$

$= u(t-\pi)\cos 2t$

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Convolution

$$f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = g(t) * f(t) \quad \text{[Commutative]}$$

$L\{f(t)\} = F(s)$ $\quad L\{g(t)\} = G(s)$ $\quad$ [Known]

$$\boxed{F(s)\, G(s) \xrightarrow{L^{-1}} f * g(t)}$$

Proof :- $F(s)\, G(s) = \left(\int_0^\infty f(\tau) e^{-s\tau} d\tau\right) \cdot G(s)$ $\quad$ func^n of s ; taken inside

$$= \int_0^\infty f(\tau) e^{-s\tau} G(s)\, d\tau$$

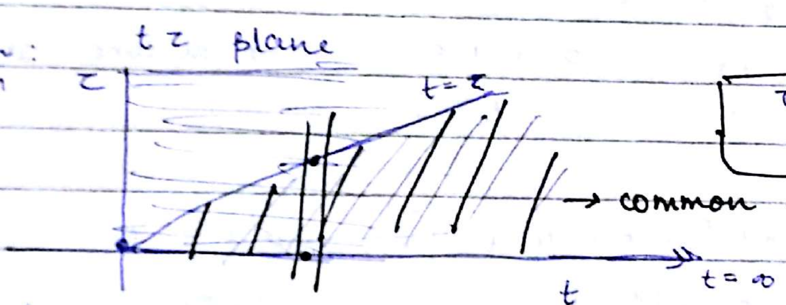
(using t-shifting th^m)
$e^{-as} F(s) \xrightarrow{L^{-1}} u(t-a) f(t-a)$

$$= \int_0^\infty f(\tau) L\{u(t-\tau)\, g(t-\tau)\}\, d\tau$$

$$= \int_0^\infty f(\tau) \left(\int_0^\infty u(t-\tau) g(t-\tau) e^{-st} dt \right) d\tau$$

$$= \int_0^\infty f(\tau) \left(\int_\tau^\infty e^{-st} g(t-\tau)\, dt \right) d\tau$$

Changing order of integration :



$\tau \le t < \infty$
$0 \le \tau < \infty$

$$= \int_{t=0}^{\infty} \int_{\tau=0}^{t} f(\tau) e^{-st} g(t-\tau)\, dt\, d\tau = \int_{t=0}^\infty e^{-st} \left(\int_{\tau=0}^{t} f(\tau) g(t-\tau)\, d\tau \right) dt$$

$$= \int_0^\infty e^{-st} f*g(t)\, dt = L\{f*g(t)\}$$

Eg. $\dfrac{1}{s^2(s-a)}$ $\quad \underbrace{\dfrac{1}{s^2}}_{F(s)} * \underbrace{\dfrac{1}{s-a}}_{G(s)} \xrightarrow{L^{-1}} f * g(t)$

$\dfrac{1}{s^2} \to t$ $\qquad \dfrac{1}{s-a} \to e^{at}$

more app : $u(t-a) e^{at}$

$$F(s) \cdot G(s) \xrightarrow{L^{-1}} \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t \tau e^{a(t-\tau)} d\tau = e^{at} \cancel{f(t)} \int_0^t \tau e^{-a\tau} d\tau$$

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12-04-17

Integral Equation :-

$$y(t) = t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$$

$$y(t) = t + y * \sin(t) \qquad L\{y(t)\} = Y(s)$$

Take Laplace transformation

$$Y(s) = \frac{1}{s^2} + Y(s)\cdot\frac{1}{s^2+1}$$

$$Y(s) = \frac{s^2+1}{s^4} - \frac{1}{s^2} + \frac{1}{s^4}$$

$$y(t) = t + \frac{t^3}{6}$$

Differential Eqns with discont. coefficients :-

Ex. $y'' + 3y' + 2y = f(t)$

$y(0) = y'(0) = 0$

$$f(t) = \begin{cases} 4t & 0 < t < 1 \\ 8 & t \geq 1 \end{cases} \rightarrow \text{dis cont. at } t = 1$$

$$f(t) = 4t\,[u(t) - u(t-1)] + 8\,u(t-1)$$

$$f(t) = 4t\,u(t) + 4u(t-1) - (t-1)\,4u(t-1)$$

$$L\{y'\} = sY(s) - y(0) = sY(s)$$

$$L\{y''\} = s^2Y(s) - s(y(0)) - y'(0) = s^2Y(s)$$

$$L\{f(t)\} = 4 \times \frac{1}{s^2} + \frac{4e^{-s}}{s} - \frac{e^{-s}}{s^2}$$

$$y'' + 3y' + 2y = f(t)$$

$$3sY(s) + s^2Y(s) + 2Y(s) = \frac{4}{s^2} + \frac{4e^{-s}}{s} - \frac{e^{-s}}{s^2}$$

$$Y(s+1)(s+2) = \frac{4 - e^{-s} + 4se^{-s}}{s^2}$$

$$Y(s) = \cancel{\frac{4 - e^{-s} + 4se^{-s}}{}} \; \frac{4}{s^2(s+1)(s+2)} + \frac{4e^{-s}}{s(s+1)(s+2)} - \frac{e^{-s}}{s^2(s+1)(s+2)}$$

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using partial fraction

$y(t) = -3 - e^{-2t} + 4e^{-t} + 2t + u(t-1)\left[5 + 3e^{-2(t-1)} - 6e^{-(t-1)} - 2(t-1)\right]$

L.T. for 2nd O.D.E. with variable coeffecients

$\begin{cases} t y''(t) + y'(t) + t\, y(t) = 0 \\ y(0) = 1 \quad y'(0) = 0 \end{cases}$

$y''(t) + \frac{1}{t} y'(t) + y(t) = 0$

$t=0$: ~~reg~~ singular point (regular) $\rightarrow P(t) = t^m \sum_{n=0}^{\infty} a_n t^n$

$y(t) \xrightarrow{L} Y(s)$

→ Applying L.T.

$t\, y(t) \xrightarrow{L} -\frac{d}{ds} Y(s)$

$L\{t\, y''(t)\}$

$t y'' \rightarrow -\frac{d}{ds}\{s^2 Y(s) - s\, y(0) - \underset{0}{y'(0)}\}$

$t y'' \xrightarrow{L} -\frac{d}{ds}\left[s^2 Y(s) - s\right]$

$-\left[2s Y(s) - 1 + s^2 Y'\right]$

$t\, y(t) \xrightarrow{L} -Y'(s)$

substituting,

$-\left[s^2 Y' + 2s Y(s) - 1\right] + (sY(s) - 1) - Y'(s) = 0$

$-(s^2+1) Y' + (-2s + s) Y = 0$

$(s^2+1) Y' + sY = 0$

$\Rightarrow \int \frac{dY}{Y} = -\int \frac{s}{s^2+1} ds$

$\log Y = -\frac{1}{2} \log|s^2+1| + \log c$

$Y = \frac{c}{(s^2+1)^{1/2}}$

$\boxed{\begin{array}{l} t\, y(t) \xrightarrow{L} -Y'(s) \\ t^2 y(t) \xrightarrow{L} Y''(s) \end{array}}$

$Y = c\,(s^2+1)^{-1/2}$

$= \frac{c}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2}$

$= \frac{c}{s}\left(1 - \frac{1}{2s^2} + \frac{(-1/2)(-3/2)}{2} \times \frac{1}{s^4} \cdots\right)$

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$(1\cdot 3\cdot 5 \ldots (2n-1))\ (1\cdot 2\cdot 4\cdot 6 \ldots 2n)$

$(n!)^2 = 2^n n!$

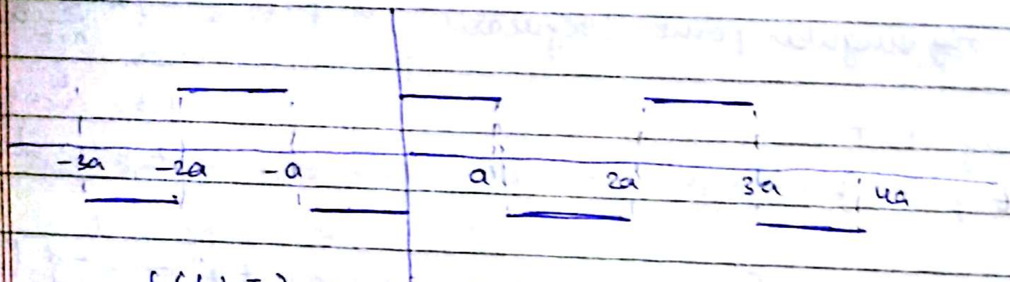
$(2n!) = 2^n \cdot n!\ (1\cdot 2 \cdots$

$$Y(s) = \frac{c}{s}\left(1 - \frac{1}{2s^2} + \frac{3}{2^3 s^4} \ldots + \frac{(-1)^n 1\cdot 3\cdot 5 \ldots (2n-1)}{2^n n!\ s^{2n}} + \ldots\right)$$

$$= c\sum_{n=0}^{\infty} \frac{(-1)^n \cancel{1\cdot 3\cdot 5 \ldots (2n-1)}\,(2n)!}{2^{2n}(n!)^2\ s^{2n+1}}$$

$$= c\left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} \ldots\right] \rightarrow \text{Bessel's func}^n \text{ of zeroth order } I_0(t)$$

➡ **Periodic function** :-



$f(t+T) = f(t)\ \forall\ t$

f is periodic func^n of period T

$f(t+nT) = f(t)\ \forall\ t \geq 0,\ n \in \mathbb{Z}$

Theorem : $f(t)$ is a periodic func^n of period T

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$$

Proof : $L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$

$$= \left\{\int_0^T + \int_T^{2T} + \ldots + \int_{nT}^{n(n+1)T} f(t)\, e^{-st}\, dt\right\}$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{n(n+1)T} f(t)\, e^{-st}\, dt \qquad t = y + nT$$

$$= \sum_{n=0}^{\infty} \int_0^T \overset{f(y)}{f(y+nT)}\, e^{-s(y+nT)}\, dy$$

$$= \sum_{n=0}^{\infty} e^{-snT} \int_0^T f(y)\, e^{-sy}\, dy$$

$$= \left\{\int_0^T f(y)\, e^{-sy}\, dy\right\}\left[1 + e^{-sT} \ldots \ldots\right]$$

$$= \frac{1}{1-e^{-sT}} \int_0^T f(y)\, e^{-sy}\, dy$$ Hence Proved

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$(2n!) = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \cdot (n+1) \cdots\cdots (2n-1) \cdot (2n)$

$= 2^n \, (1 \cdot 3 \cdot$

For graph drawn earlier,

$$f(t) = \begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases}$$

$$\left[\frac{e^{-st}}{-s} \right]_0^a$$

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a e^{-st}\, dt \mp \int_a^{2a} e^{-st}\, dt \right\}$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1}{s} - \frac{e^{-as}}{s} - \frac{e^{-as}}{s} + \frac{e^{-2as}}{s} \right]$$

$$= \frac{1}{s} \tanh\left(\frac{as}{2}\right)$$

Ques. $F(s)$: given Find $f(t)$ s.t. $L\{f(t)\} = F(s)$??
can we find $f(t)$ always ??

Always, power of s is more in denominator than in numerator. So, we can find $f(t)$ only if

$$\lim_{s \to \infty} F(s) = 0$$

If $F(s) = \left(1 + \underbrace{\frac{s}{s+1} + \frac{s^2}{s+1} + \frac{s^3}{s+1} \cdots}_{} \right)$

can't find L^{-1} of these terms, so it will be better to make them 0 and solve the question further.

17-4-17

Fourier series

→ can be represented in terms of sine or cosine func^n's (Periodic func^n's)

→ It was Originated to solve Heat eq^n by Joseph Fourier.

→ Mathematical advantage :

{ drawback of Taylor series : analytic at some point, only valid / accurate in neighbourhood of that point }

This can be defined in terms of cosine & sine func^n everywhere

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Periodic func^n:

$f(x+p) = f(x) \quad \forall x$

f is periodic func^n with period p

$f(x+np) = f(x) \quad : \forall x ; n \in \mathbb{Z}$

→ $f(x) = \text{const} \quad \forall x$: Periodic

→ f & g : periodic func^n of period p

$\alpha f + \beta g$: again periodic with period p.

* If a periodic func^n $f(x)$ has a smallest period $p > 0$, then it is called fundamental period of $f(x)$

eg. F.P. of $\sin x$ & $\cos x$: 2π

const. func^n doesn't have any F.P.

→ Aim is to decompose a periodic func^n in terms of sin & cos func^n in the form of trigonometric series :

$$\boxed{a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx} \quad : \text{approximation of } f(x)$$

here, a_0, a_n, b_n : fourier coeffecients, determined from given $f(x)$

• **Euler formula for Fourier Coeffecients.**

Assume $f(x)$ is periodic func^n of period 2π. $f(x)$ is integrable over a period 2π.

let $$f(x) = a_0 + \underbrace{\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}_{\text{convergent (assume now) uniformly}} \quad \text{—— (1)}$$

1) Determination of a_0

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right) dx$$

$$= \int_{-\pi}^{\pi} a_0\,dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx)\,dx$$

→ $$\int_{-\pi}^{\pi} a_n \cos nx\,dx = 2a_n \int_0^{\pi} \cos nx\,dx = 2a_n \left. \frac{\sin nx}{n} \right|_0^{\pi} = \boxed{0}$$

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$\int_{-\pi}^{\pi}$ even func^n $= 2\int_0^{\pi} \cdots dx$

$\int_{-\pi}^{\pi}$ odd func^n $= 0$

$b_n \int_{-\pi}^{\pi} \sin nx \, dx = 0$

$\Rightarrow \int_{-\pi}^{\pi} f(x)\, dx = 2\pi a_0$

$$\Rightarrow \boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx}$$

2) Determination of a_n :

Multiply $\cos mx$ in eq^n ①. Integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} a_0 \cos mx \, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos mx \cos nx \, dx + \int_{-\pi}^{\pi} b_n \sin nx \cos mx \, dx$$

$$\rightarrow a_0 \int_{-\pi}^{\pi} \cos mx \, dx = 2a_0 \int_0^{\pi} \cos mx \, dx = \boxed{0}$$

$$\rightarrow \int_{-\pi}^{\pi} a_n \cos nx \cos mx \, dx = \frac{a_n}{2} \int_{-\pi}^{\pi} \cos(m+n)x + \cos(m-n)x$$

Use: orthogonality of $\cos nx$ & $\sin nx$ from $-\pi$ to π :

(Prove this)

$$\int_{-\pi}^{\pi} \cos nx \cos mx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases} \qquad \{\cos nx\}_{n=1}^{\infty}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases} \qquad \{\sin nx\}_{n=1}^{\infty}$$

$$\Rightarrow a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0 & n \neq m \\ \pi a_n & n = m \end{cases}$$

$$\rightarrow \int_{-\pi}^{\pi} b_n \sin nx \cos mx \, dx = \frac{b_n}{2} \int_{-\pi}^{\pi} \sin(m+n)x + \cos(n-m)x \, dx$$

$$= \frac{b_n}{2} [0] = \boxed{0}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \cos mx \, dx = \pi a_m$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx}$$

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Assignment - 01

1. Given : $A = (a_{ij})_{m \times n}$ $B = (b_{ij})_{m \times n}$

$A^T = (a_{ji})_{n \times m}$ $B^T = (b_{ji})_{n \times m}$

$B^T A^T$ ~~$= (b_{ji})_{n \times m}$~~

i-j th element of $(AB)^T$ = j-i th element of (A B)

= sum of product of corresponding elements of j th row of A and i th coloumn of B

= " " j th row of B^T & j th column of A^T

= i-j th element of $B^T A^T$

$\Rightarrow (AB)^T = B^T A^T$

Hence Proved

3) Determination of b_n :

Multiply sin mx into eqn (1) & integrate from $-\pi$ to π

$\int_{-\pi}^{\pi} a_n \sin mx \, dx = 0$ $\int b_n \sin nx \sin mx \, dx = \pi$, m = n
other term = 0

$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = b_m \pi$

$$\Rightarrow \boxed{b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx}$$

The fourier series of $f(x)$: $a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

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Eg. $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$ $f(x)$: periodic with period 2π

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\cdot dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx\right] = \frac{1}{\pi}\left[-\frac{\sin nx}{n}\Big|_{-\pi}^{0} + \frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx\,dx + \int_{0}^{\pi}\sin nx\,dx\right] = \frac{1}{\pi}\left[\frac{\cos nx}{n}\Big|_{-\pi}^{0} - \frac{\cos nx}{n}\Big|_{0}^{\pi}\right]$$

$\boxed{b_n = \frac{4}{n\pi}}$ $$= \frac{1}{\pi}\left[\frac{1}{n} - \left(\frac{-1}{n}\right)^n - \left(\left(\frac{-1}{n}\right)^n - \frac{1}{n}\right)\right]$$

$$b_n = \frac{2}{n\pi}\left[1 - \cos n\pi\right] = \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

$$b_n = \begin{cases} 0 & n \in \text{even} \\ \frac{4}{n\pi} & n \in \text{odd} \end{cases}$$

$$f(x) \simeq \sum_{n=1}^{\infty} b_n \sin nx = b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x \ldots$$

$$= \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots\right]$$

→ To find whether convergent or not, use ratio test. we will find that it is convergent

Points of discontinuity

- at $x = 0$, $f(x) = 0$ (we are getting) but $f(x) \neq 0$ at $x = 0$
  ⇒ we are getting avg. of L.H. value & R.H. value

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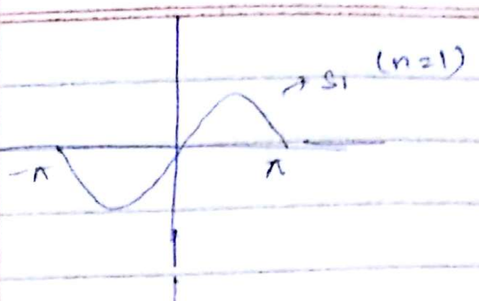
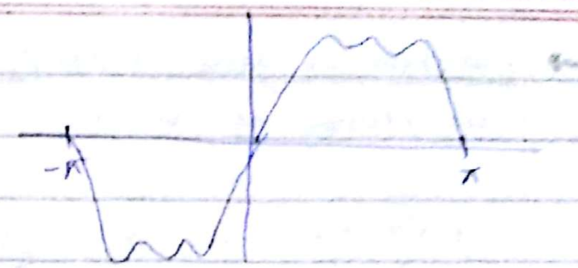
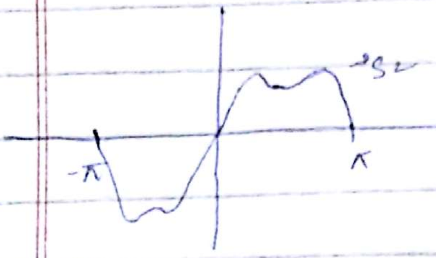
→







eventually, it converges to given $f(x)$ as $n \to \infty$.

→ At $x = \frac{\pi}{2}$ [continuous]

$$1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\cdots\right]$$

$$\cancel{\frac{\pi}{4}} \; \cancel{= 1 + \frac{1}{3}} \quad \boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots = \frac{\pi}{4}} \quad \to \text{appxm}^n \text{ of } \frac{\pi}{4}$$

19-4-17

Theorem : (i) $f(x)$ is periodic func^n with period 2π.

(ii) $f(x)$ is piecewise continuous in $[-\pi, \pi]$ and has left hand & right hand limit at each point of the interval. (f is integrable)

Then the fourier series of $f(x)$, ie,

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

exists and converges to the sum func^n $f(x)$, ~~expe~~ except at a point x_0 at which $f(x)$ is discontinuous.

The sum of the series is the avg. of the left & right hand limits of $f(x)$ at x_0

→ In prev. eg, points of discontinuity : ~~$-\pi$~~, 0, ~~π~~

∴ ~~S_n~~ series converges to $\frac{4}{\pi}(\cdots\cdots)$ except

at $x = 0$

↓

$$\text{sum of series} = \frac{(1) + (-1)}{2} = 0$$

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→ **Function of any period $p = 2L$**

use change of scale

set $v = \frac{\pi x}{L}$ ⇒ $x = \frac{Lv}{\pi}$

$p = 2L$

$f(x) \rightsquigarrow g(v)$ → periodic func^n with $p = 2\pi$

$x \to -L$ ⇒ $v \to -\pi$

$x \to L$ ⇒ $v \to \pi$

→ $g(v)$ is a periodic func^n of period 2π

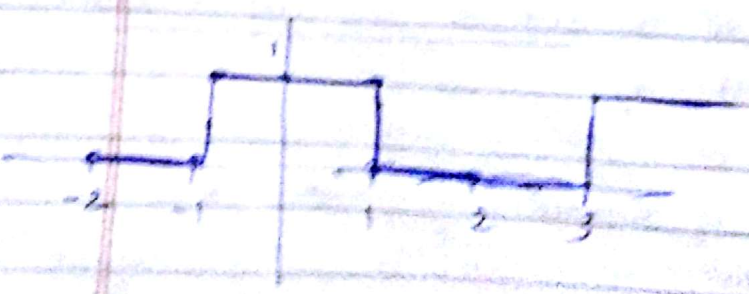
$$g(v) = a_0 + \sum_{n=1}^{\infty} (\tilde{a}_n \cos nv + \tilde{b}_n \sin nv)$$

fourier series

$$\boxed{\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \\ a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}}$$

→ $\tilde{a}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\,dv$

Eg. $f(x) = \begin{cases} 0 & -2 < x < -1 \\ 1 & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$



$\boxed{T = 4}$

$\boxed{L = 2}$

Date
Page

$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} dx = \frac{1}{2}$

$a_n = \frac{1}{2}\int_{-1}^{1} \cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\left[\sin\frac{n\pi x}{2}\right]_{-1}^{1} \times \frac{2}{n\pi} =$

$= \frac{2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0 & n = \text{even} \\ 2/n\pi & n = 1, 5, 9 \ldots \\ -2/n\pi & n = 3, 7, 11, \ldots \end{cases}$

$b_n = \frac{1}{2}\int_{-1}^{1} \sin\frac{n\pi x}{2}\,dx = \frac{-1}{2}\left[\cos\frac{n\pi x}{2}\right]_{-1}^{1} = 0 \quad \forall n$

$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x \ldots\right]$

$\Rightarrow$ For points of discontinuity, it converges to $\frac{1}{2}$, not to $f(x)$.

$(x=1)$: $\left(\lim_{h\to 1^-} f(x) + \lim_{h\to 1^+} f(x)\right)\frac{1}{2} = \frac{1}{2}(1+0) = \frac{1}{2}$

(1) $f(x)$ is even function :-

$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}}\ \underbrace{\sin\frac{n\pi x}{L}}_{\text{odd}}\,dx = 0 \quad \Rightarrow \text{odd}$

$a_0 = \frac{1}{2L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}}\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx$

$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}}\ \underbrace{\cos\frac{n\pi x}{L}}_{\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$ even $\times$ even = even

$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}$: Fourier cosine series

(2) $f(x)$ is odd func^n :-

$a_0 = \frac{1}{2L}\int_{-L}^{L} \underbrace{f(x)}_{\text{odd}}\,dx = 0$

$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{odd}}\ \underbrace{\cos\frac{n\pi x}{L}}_{\text{even}}\,dx = 0$: odd

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$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{odd}}\ \underbrace{\sin\frac{n\pi}{L}x}_{\text{odd}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

odd · odd = even

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x}$$

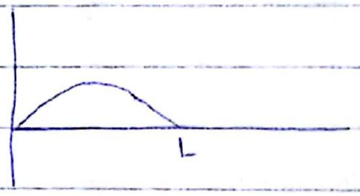
~~Foti~~ Fourier sine series

→ For period = 2π $\{L = \pi\}$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

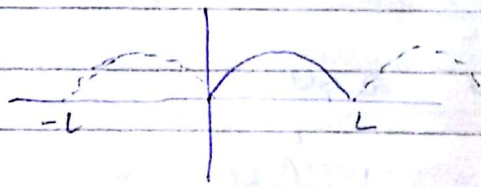
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

→ **Function with fixed boundary**



To get fourier series in only terms of only cosine ~~term~~ / sine func^n

Case-I: Half-Range Expansion.



$$g(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \end{cases}$$

↓

Even extension of $f(x)$

$g(x)$: periodic func^n with period $2L$

$g(x+2L) = g(x)\ \forall\ x$

$g(x)$: even func^n (graph)

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$$a_0 = \frac{1}{L}\int_0^L g(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L g(x)\cos\frac{n\pi}{L}x\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

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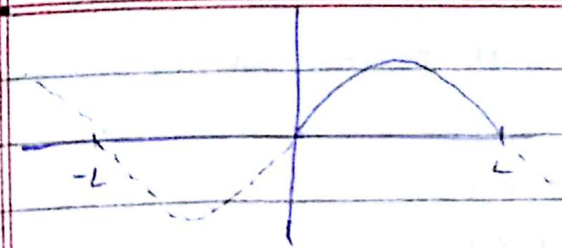
Date
Page

Case-II



$$g(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \end{cases}$$

$\downarrow$

odd extension of $f(x)$

$g(x)$: odd func^n, periodic with period $2L$

$$g(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

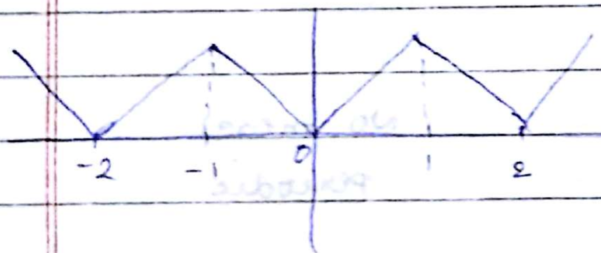
$$b_n = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi}{L} x \, dx = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Que. Find cosine series of $f(x)$ defined on $0 < x < L$.

You need to make an even extension of $f(x)$

Eg. Find Fourier cosine series of $f(x) = x$ $(0 < x < 1)$

$\boxed{L = 1}$



$g(x+2) = g(x) \ \forall \ x$

$$g(x) = \begin{cases} x & 0 < x < 1 \\ -x & -1 < x < 0 \end{cases}$$

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{1}\int_0^1 g(x)\, dx = \frac{1}{1}\int_0^1 x \, dx = \frac{1}{2}$$

$$a_n = \frac{2}{1}\int_0^1 x \cdot \cos n\pi x \, dx = 2\left[x \frac{\sin n\pi x}{n\pi} - \int \frac{\sin n\pi x \, dx}{n\pi} \right]$$

$$= 2\left[\frac{x \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2} \right]\Bigg|_0^1$$

$$= 2\left[\frac{\sin n\pi}{n\pi} + \frac{\cos n\pi}{(n\pi)^2} - \frac{1}{(n\pi)^2} \right]$$

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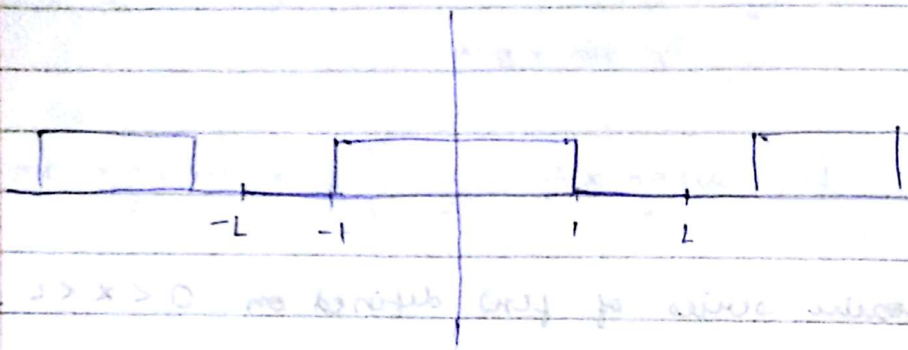
Used in complex func

Date 21/4/17

Page

→ $f_L(x)$ – is periodic func^n with period $2L$

$$f_L(x) = \begin{cases} 0 & -L < x -1 \\ 1 & -1 < x < 1 \\ 0 & 0 < x < L \end{cases}$$



$$\lim_{L \to \infty} f_L(x) = \begin{cases} 0 & -\infty < x < -1 \\ 1 & -1 < x < 1 \\ 0 & 1 < x < \infty \end{cases}$$

or

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{elsewere} \end{cases}$$, No more periodic

∴ Fourier series of $f_L(x)$:

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

let $\omega_n = \frac{n\pi}{L}$

$$= \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left\{\cos\omega_n x \int_{-L}^{L}\left(f_L(v)\cos\omega_n v\,dv\right) + \sin\omega_n x \int_{-L}^{L}\left(f_L(v)\sin\omega_n v\,dv\right)\right\}$$

Let $\Delta\omega = \omega_{n+1} - \omega_n$

$$= \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

$$\Rightarrow \frac{1}{L} = \frac{\Delta\omega}{\pi}$$

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$\lim_{L \to \infty} f_L(x) = f(x) \to$ non-periodic periodic

$\frac{1}{L} \to 0$ as $L \to \infty$

Δw is very small when $L \to \infty$

$\downarrow$

dw

$\omega_n \to \omega$ as $L \to \infty$

(Fourier series) $f(x) = \frac{dw}{\pi} \int_0^{\infty} \left\{ \cos wx \int_{-\infty}^{\infty} f(v) \cos wv \, dv + \sin wx \int_{-\infty}^{\infty} f(v) \sin wv \, dv \right\}$

$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv$

$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv$

$f(x) = \int_0^{\infty} \left(A(w) \cos wx + B(w) \sin wx \right) dw$

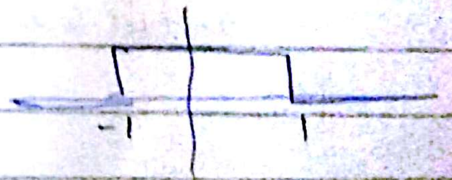
Theorem: (i) $f(x)$ is piecewise continuous in each finite interval and has left and right hand derivatives at every point.
(ii) $f(x)$ is integrable absolutely on whole real line.

Then,

$f(x)$ can be represented by a Fourier Integral.

At a point x_0 where $f(x)$ is discontinuous, the values of the Fourier integral equals the avg. of left and right hand limits of $f(x)$ at that point.

Eg. $f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$



Fourier Integral of this $f(x)$:

$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv$

Date
Page

$$= \frac{1}{\pi}\int_{-1}^{1} \cos \omega v \, dv = \frac{2}{\pi \omega} \sin \omega$$

$$B(\omega) = \frac{1}{\pi}\int_{-1}^{1} \sin \omega v \, dv = 0$$

$$f(x) = \int_{0}^{\infty} \frac{2}{\pi\omega} \sin\omega \cos\omega x \, d\omega$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin\omega \cos\omega x}{\omega} \, d\omega$$

Ex

$$= \frac{2}{\pi}$$

$x = 1$ $\quad f(1^{+}) = 0 \qquad f(1^{-}) = 1$

$$\therefore f(1) = \frac{1+0}{2} = \frac{1}{2}$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\omega \cos\omega x}{\omega} \, d\omega = \begin{cases} 0 & |x| > 1 \\ 1/2 & x = -1, 1 \\ 1 & -1 < x < 1 \end{cases}$$

$$\int_{0}^{\infty} \frac{\sin\omega \cos\omega x}{\omega} \, d\omega = \begin{cases} 0 & |x| > 1 \\ \pi/4 & x = 1 \\ \pi/2 & |x| < 1 \end{cases}$$

$x = 0$ is point of continuity

$$\boxed{\int_{0}^{\infty} \frac{\sin\omega}{\omega} \, d\omega = \frac{\pi}{2}}$$

$$Si(u) = \int_{0}^{u} \frac{\sin\omega}{\omega} \, d\omega$$

$\longrightarrow$ sine integral

$$\boxed{\lim_{u \to \infty} Si(u) = \frac{\pi}{2}}$$

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1.) $f(x)$: even func^n $\sin\omega x$ $f(x)$

$B(\omega) = 0$ (odd × even = odd)

$f(x) = \int_{-\infty}^{\infty} A(\omega)\cos\omega x \, d\omega$, $A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos\omega v \, dv$

cosine integral

2.) $f(x)$: odd func^n : $f(x)$ $\cos\omega x$

$A(\omega) = 0$ (odd × even = odd)

$f(x) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin\omega v \, dv$ ← sine integrals

Q. Find fourier cosine and sine integral of

$f(x) = e^{-kx}$, $k>0$, $x>0$

1.) cosine Integrals :

$A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos\omega v \, dv = \frac{2}{\pi}\int_0^{\infty} e^{-kv}\cos\omega v \, dv$

① ② $= I$

$I = \frac{-k}{k^2+\omega^2} e^{-kv}\left(-\frac{\omega}{k}\sin\omega v + \cos\omega v\right)$

$v \to 0$ $v \to \infty$

$I = \frac{-k}{k^2+\omega^2}$ $I = 0$

$A(\omega) = \frac{2k}{\pi(\omega^2+k^2)}$

$f(x) = \int_0^{\infty} A(\omega)\cos\omega x \, d\omega$

$= \frac{2k}{\pi}\int_0^{\infty} \frac{\cos\omega x}{k^2+\omega^2} d\omega = e^{-kx}$

$\Rightarrow \boxed{\int_0^{\infty} \frac{\cos\omega x}{k^2+\omega^2} d\omega = \frac{\pi}{2k} e^{-kx}}$ → valid for $k>0$, $x>0$

Laplace Integral

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Ass 11, Q7(b) with $k=1$

Date
Page

29 Sine Integral :

$$B(\omega) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin\omega v\,dv = \frac{2\omega}{\pi(k^2+\omega^2)}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\omega}{k^2+\omega^2}\sin\omega x\,d\omega = e^{-kx}$$

$$\boxed{\int_0^\infty \frac{\omega}{k^2+\omega^2}\sin\omega x\,d\omega = \frac{\pi}{2}e^{-kx}}$$

Laplace Integral

Ass 11

7.(a)
$$\int_0^\infty \frac{\cos x\omega + \omega\sin x\omega}{1+\omega^2}\,d\omega = \begin{cases} \frac{\pi}{2} & x=0 \\ 0 & x<0 \\ \pi e^{-x} & x>0 \end{cases}$$

0 is point of discontinuity $\Rightarrow (\pi+0/2) = \pi/2$

$$f(x) = \begin{cases} 0 & x<0 \\ \pi e^{-x} & x>0 \end{cases}$$

$$A(\omega) = \frac{1}{\pi}\int_0^\infty \pi e^{-v}\cos v\omega\,dv = \frac{1}{\pi(1+\omega^2)}$$
$(x<0,\ f(x)=0)$

$$B(\omega) = \frac{1}{\pi}\int_0^\infty \pi e^{-v}\sin\omega v\,dv = \frac{\omega}{1+\omega^2}$$

$$f(x) = \int_0^\infty\left(\frac{1}{1+\omega^2}\cos\omega x + \frac{\omega}{1+\omega^2}\sin\omega x\right)d\omega$$

8. Find $A(\omega)$ s.t. $\frac{1}{\pi}\int_0^\infty \cos\omega x\,d\omega = \frac{1}{1+x^2}$

even func^n

$$f(x) = \frac{1}{1+x^2}$$

(Cosine Integral)

$$A(\omega) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+v^2}\cos v\omega\,dv = (\quad)e^{-\omega}$$

Use 7(b) to find $A(\omega)$ with $k=1$

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