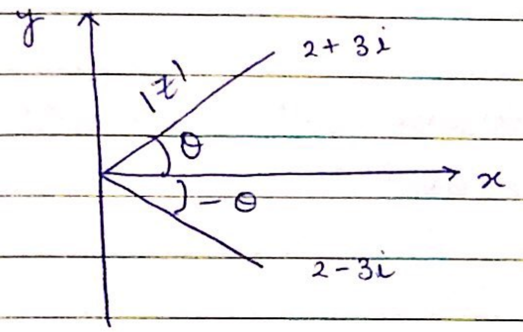
INTRODUCTION TO COMPLEX NO.

→ i is representation for $\sqrt{-1}$, not a simplification of $\sqrt{-1}$

Complex No. : $a + bi$

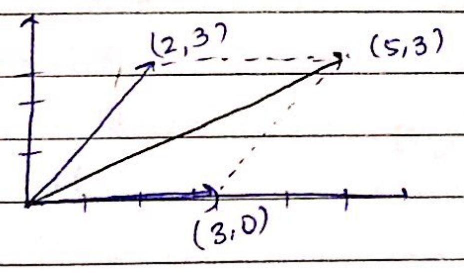
On graph : $x + iy \equiv (x, y)$

Conjugate of $a + ib$: $a - ib$
↳ dir^n of angle is changed.

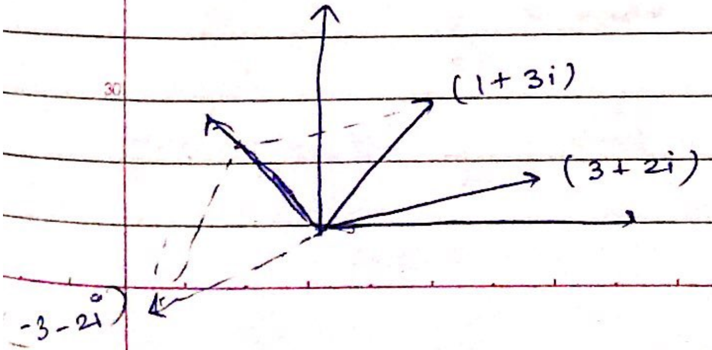


* we can't compare 2 complex no. because they don't come in a line (like 1, 2, 3, ...), but they form a plane.

$(2 + 3i) + (3 + 0i)$



Subtraction : Additive inverse. $1 + 3i - (3 + 2i)$



Multiplication :-

$z_1 = a + ib \qquad z_2 = c + id$

$z_1 z_2 = (a+ib)(c+id) = ac + i(ad+bc) - bd$

→ The complex no. system is a natural extension of the real no. system.

→ $i^2 = (0,1)(0,1) = (-1,0)$ or $i^2 = -1$

→ $z^2 = zz \,, \; z^3 = z^2 z$

→ Inverse of z : z^{-1} s.t.

$$zz^{-1} = 1$$

→ If we square a no. the distance will be ~~doubled~~ squared from ~~a~~ origin & angle will be doubled.

Polar form :

1.) $z = 1 + i$

$r = \sqrt{2}$

$\theta = \pi/4$

Polar form : $\sqrt{2}\, e^{i\pi/4}$

3.) $z = -1 + i$

$r = \sqrt{2}$

$\theta = 3\pi/4$

$\sqrt{2}\, e^{i 3\pi/4}$

2.) $z = -1 - i$

$r = \sqrt{2}$

$\theta = -3\pi/4$

Polar form : $\sqrt{2}\, e^{(-3\pi/4)i}$

4.) $z = 1 - i$

$r = \sqrt{2}$

$\theta = -\pi/4$

$\sqrt{2}\, e^{-i\pi/4}$

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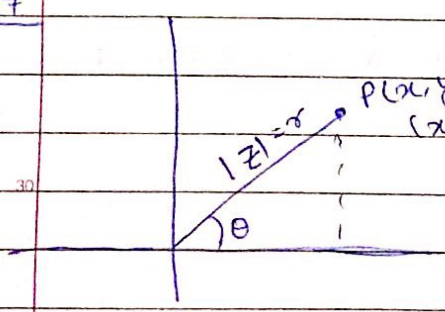

$|z| = \sqrt{x^2+y^2} = r$

$\tan\theta = \dfrac{y}{x}$

$\sin\theta = \dfrac{y}{r} \Rightarrow y = r\sin\theta$

$x = r\cos\theta$

$z = x + iy$
$= r\cos\theta + i\, r\sin\theta$
$z = r(\cos\theta + i\sin\theta) \Rightarrow \boxed{z = re^{i\theta}}$

Euler's Formula :

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

→ $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

To calculate z^7, it will be hard. So, in such case, we should use polar form.

$z_1 = r_1 e^{i\theta_1}$ $\quad z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$z_1 / z_2 = r_1 / r_2 \, e^{i(\theta_1 - \theta_2)}$

* $z = 0$ can't be written in polar coordinate system.
(can't define θ)

- In case of Polar form :

Arguement :

$$\text{Arg } z = \begin{cases} \tan^{-1}(y/x) & \text{if } x \geq 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \ \& \ y > 0 \\ \tan^{-1}(y/x) - \pi & \text{if } x < 0 \ \& \ y < 0 \end{cases}$$

arg z - General arguement

$z = \sqrt{2}\, e^{i\pi/4}$

* If we add $2n\pi$ to the θ, it doesn't change z

$\arg z = \operatorname{Arg} z + 2n\pi \quad , \quad n = 0, \pm 1$

→ **Principal Arguement :-**

$-\pi \leq \operatorname{Arg} z \leq \pi$

→ **General Arguement :-**

$\text{arguement } z = \arg z = \operatorname{Arg} z + 2n\pi$

We can also write as:

$z = r e^{i(\underbrace{\theta}_{\operatorname{Arg} z} + 2n\pi)} = r e^{i\theta}$

Eg. $(\sqrt{3} + i)^{61} = z$

$\operatorname{Arg} \theta = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$z = 2 e^{i\pi/6}$

$z^{61} = 2^{61} e^{i(61\pi/6)} = 2^{61} e^{i[10\pi + \pi/6]} = 2^{61} e^{i\pi/6}$

$= 2^{61} \left[\sqrt{3}/2 + i\, 1/2 \right]$

$$\boxed{z^{61} = 2^{60} (\sqrt{3} + i)}$$

$$\boxed{\arg(z_1 z_2) = \arg z_1 + \arg z_2}$$

But in general

$\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg} z_1 + \operatorname{Arg} z_2$: It will be true when real part is positive

$z_1 = -1 \quad z_2 = i$

$z_1 z_2 = -i$

$\operatorname{Arg}(z_1) = \pi \quad \operatorname{Arg}(z_2) = \pi/2$

$\operatorname{Arg}(z_1 z_2) = -\pi/2$

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Eg. $z = \sqrt{3}+i$

$z = \sqrt{3}-i$

$z = -\sqrt{3}+i$

$z = \sqrt{3}-i$

ω is n^{th} root of z if $\boxed{\omega^n = z}$

De - moiver's theorem :-

$$(e^{i\theta})^n = e^{in\theta}$$

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

→ if $n=2$

$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2$

$= \cos^2\theta - \sin^2\theta + 2i(\cos\theta\sin\theta)$

$\cancel{\cos 2\theta} +$

→ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\qquad \sin 2\theta = 2\sin\theta\cos\theta$

→ $n=3$

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta - i\sin^3\theta + 3i\cos\theta\sin\theta(\cos\theta + i\sin\theta)$

$= \cos^3\theta - 3\cos\theta\sin^2\theta +$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$

→ $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

PMI :-

let say it is true for $n=k$

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos\theta\cos k\theta - \sin k\theta\sin\theta \; + i\,[\sin k\theta\cos\theta + \sin\theta\cos k\theta]$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

Hence proved

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Roots

ω is n^{th} root of a no. z if $\omega^n = z$ — (1)

$$\left.\begin{aligned} z &= re^{i\theta} \\ \omega &= Re^{i\phi} \end{aligned}\right\} \text{— (2)}$$

Using (1) & (2), [z is given, have to find ω]

$\omega = z^{1/n}$

$Re^{i\phi} = (re^{i\theta})^{1/n} = \left[re^{i(\theta+2k\pi)}\right]^{1/n}$, $k = 0, \pm1, \pm2$

Eg. $z = -8i$, find cubic root

$= r^{1/n} e^{i(\theta/n + 2k\pi/n)}$

$= r^{1/n} e^{i\theta/n} e^{i2k\pi/n}$

eg. $z = -8i$, find cubic root

$= (-8i)^{1/3} = 2(-i)^{1/3} = 2(e^{-\pi/2\, i})^{1/3}$

$= 2e^{-\pi/6\, i}$

$= 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

$= 2\left[\frac{\sqrt{3}}{2} - \frac{i}{2}\right] = \sqrt{3} - i$

Here, we only get 1 sol^n but we should get 3 sol^{n}s

Eg. $z = (-1)^{1/2}$

$z = (e^{i\pi})^{1/2}$

$= e^{\pi/2\, i} = i \Rightarrow$ we get only 1 root

→ If in previous eq.

$(-8i)^{1/3} = 2(-i)^{1/3} = 2\left(e^{-\pi/2 i + 2k\pi i}\right)^{1/3}$

$= 2e^{i\pi/6} e^{\frac{2k\pi i}{3}}$

take

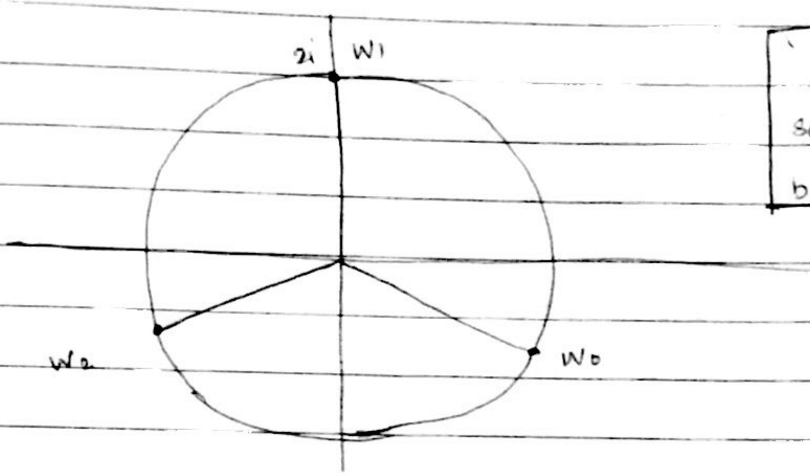
Radius in all cases is 2

$k=0 \quad w_0 = 2e^{-i\pi/6} = \sqrt{3} - i$

$k=1 \quad w_1 = 2e^{-i\pi/6} \cdot e^{2\pi/3 i} = 2e^{i\pi/2} = 2i$

$k=2 \quad w_2 = 2e^{-\pi/6} \cdot e^{4\pi/3 i} = 2e^{i7\pi/6} = -(\sqrt{3}+i)$

• $k=3 \quad w_3 = 2e^{-i\pi/6} \cdot e^{2\pi i} = 2e^{-i\pi/6} = w_0$



All roots lie at same radius & diff angles of b/w all are same

Eg. Find the n^{th} root of

(a) 1 (b) −1 (c) i (d) $-i$

$z = -i$

$\omega = z^{1/n} = (-i)^{1/n} = \left(e^{-\pi/2 i}\right)^{1/n} = \left(e^{-\pi/2 i + 2k\pi i}\right)^{1/n}$

$= e^{-\pi/2n\, i} \cdot e^{\frac{2k\pi}{n} i}$

* Take values of k continuously. ~~so that you get w in ...~~

→ $-(2i)(\sqrt{3}-i)(\sqrt{3}+i) = -2i(3+1) = -8i$

We know that $w_n = (r)^{1/n} e^{i\theta/n} \cdot e^{2k\pi/n\, i}$

Take $d = e^{2\pi/n\, i}$

$d^k = e^{2k\pi/n\, i}$

$$\omega_0 = \sqrt[n]{r}\ e^{i\theta/n}$$

$$w_1 = \sqrt[n]{r}\ e^{i\theta/n} \cdot e^{i2\pi/n}$$

$$= \omega_0 d$$

$$w_2 = \sqrt[n]{r} \cdot e^{i\theta/n} \cdot e^{i2\pi/n} \cdot e^{i2\pi/n}$$

$$= w_1 d = w_0 d^2$$

$$w_3 = w_0 d^3$$

$$\vdots$$

$$\boxed{w_{n-1} = w_0\, d^{n-1}}$$

* Sum of roots $= 0$

$$w_0 + w_1 + w_2 + \cdots + w_{n-1}$$

$$= w_0 + w_0 d + w_0 d^2 \cdots + w_0 d^{n-1}$$

$$= w_0 \left(1 + d + d^2 \cdots + d^{n-1}\right)$$

$$= w_0 \frac{(1-d^n)}{1-d}, \quad d \neq 1$$

$$d^n = \left(e^{2\pi/n\, i}\right)^n = 1$$

$$\Rightarrow \text{sum} = w_0 \frac{(1-1)}{1-d} = 0$$

* Product of roots

$$w_0 w_1 w_2 \cdots w_n$$

$$= w_0\, w_0 d\, w_0 d^2 \cdots w_0 d^{n-1}$$

$$= (w_0)^n\, d^{(n-1)n/2}$$

$$= w_0^n\, e^{\frac{2\pi}{n} i \frac{(n)(n-1)}{2}}$$

$$= w_0^n\, e^{(n-1)\pi i}$$

$$= z\,(-1)^{n-1}$$

if $n =$ odd : $+$ve $(+1)$

$n =$ even : $-$ve (-1)

Eg. $(-1)^{1/4}$

Product of roots : $(-1)(-1)^{4-1} = 1$

eg. 1) $(-1+i)^{1/3} = \left[\sqrt{2}\, e^{(3\pi/4 + 2k\pi)i}\right]^{1/3}$

2) $(-2\sqrt{3} - 2i)^{1/4} = 4^{1/4} \left\{ e^{(7\pi/6 + 2k\pi)i} \right\}^{1/4}$

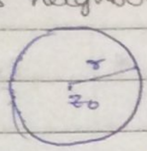
Eg. $z^4 = -2\sqrt{3} - 2i$ (same as above example)

$w = (-2\sqrt{3} - 2i)^{1/4}$

→ Complex no. exist in plane so neighbourhood will be considered wrt circle.

Eq^n of circle : $|z - z_0| = r$

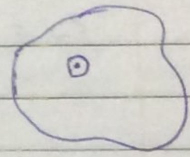
Inside circle : $\{z : |z - z_0| \le r\}$ (considered as neighbourhood.



Neighbourhood : (Nbd)

The point that satisfies the inequality $|z - z_0| < r$ is called the neighbourhood of z_0.

9/8/17 ~~Interior Point~~ : OR



A delta neighbourhood of a point z_0 is the set of all points z s.t. $|z - z_0| < \delta$, for any positive no. δ. ie

$Nbd_\delta(z_0) = \{z : |z - z_0| < \delta\}$ (open circle)

Interior Point :

A point z_0 is called interior point of a set S if we can find a δ nbd of z_0 (atleast one) all of whose points belong to S.

Boundary Point :

If every δ nbd of z_0 contains points belonging to S and also points not belonging to S, then it is boundary point.

Exterior Point :

A point which is neither interior nor boundary point.

Open Set :

An open set is a set which consist only of interior points

Connected Set :

A set is said to be connected if any two points of the set can be joined by a path consisting of straight line segments all points of which are in S.

Domain :

An open and connected set.

* $z_0 = \{z : |z - 0| < \frac{1}{100}\}$

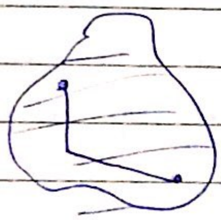
If we have a point $1+i$ ⇒ ~~$z_0 = \{z$~~ Nbd : $\{z : |z - (1+i)| < \delta\}$

$C_r = \{z : |z - z_0| < r\}$ $\quad$ $C_r' = \{z : |z - z_0| \leq r\}$

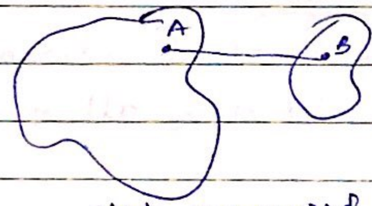
⇓

includes boundary points also.

• $S = \{z : |z - 3| \leq 7\}$: Not an open set (also includes boundary points)

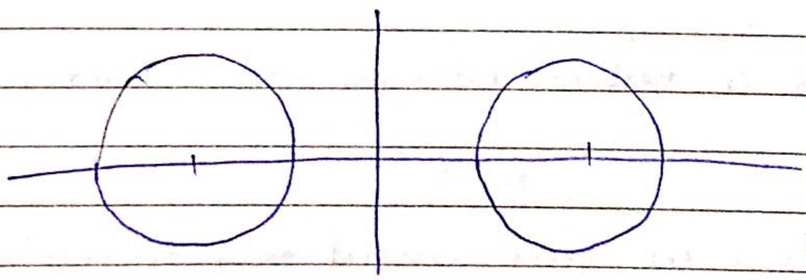


connected



Not connected (Line segment consist of path points which don't belong to S)

$S = \{z : |z - 4| < 1 \cup |z + 4| < 1\}$



Not connected, but open

⇓

Not a domain.

Function :

A func^n from A to B is a rule of correspondance that assigns to each element in A to some element in B.

we have not written unique value bcoz in case of func^n's like $f(z) = z^{1/2}, z^{1/3} \dots$ we will get more than 1 outputs (func^n are multivalued)

→ $f(z) = z^{1/2}$
$= \sqrt{r}\ e^{i(\theta/2 + k\pi)}$

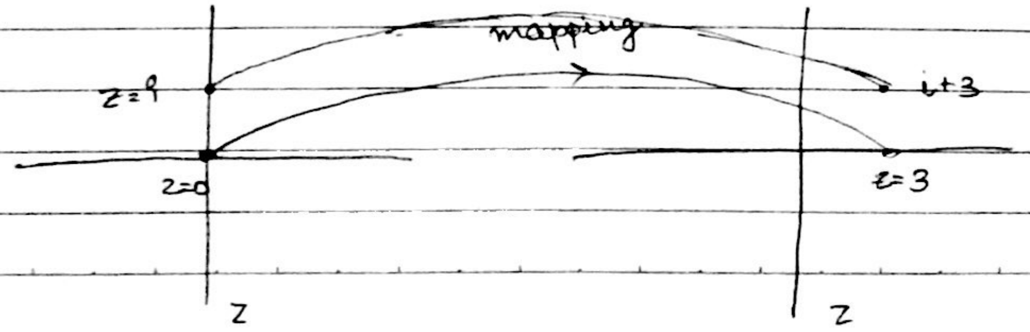
$$f(z) = \begin{cases} \sqrt{r}\, e^{i\theta/2} & k=0 \\ \sqrt{r}\, e^{i(\theta/2+\pi)} & k=1 \end{cases}$$

→ Polynomial func^n :
$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n$$

→ Rational func^n :
$$\frac{p(x)}{q(x)} = \frac{b_0 z^n + b_1 z^{n-1} + \dots + b_n}{c_0 z^n + c_1 z^{n-1} + \dots + c_n} \quad ; \quad q(x) \neq 0$$

→ Exponential func^n :
$f : A \longrightarrow B \qquad A, B \subseteq \mathbb{C}$ → complex set

Let say $f(z) = z + 3$
here, i/p : 2-D o/p : 2-D
hence, we can't map it on x-y plane → can be used when i/p is 1-D & o/p is also 1-D



- $f(z) = e^z = re^{i\phi}$

$\Rightarrow e^{x+iy} = re^{i\phi}$

$\Rightarrow e^x \cdot e^{iy} = r\, e^{i(\phi + 2k\pi)}$

Comparing the powers,

$e^x = r$ & $y = \phi + 2k\pi$

$\rightarrow \boxed{x = \ln r, \quad y = \phi + 2k\pi}$

$\rightarrow z = (\ln r) + i\,(\phi + 2k\pi)$

Ex. $e^z = 1 + i$. Find z

$e^z = \sqrt{2}\, e^{i(\pi/4 + 2k\pi)}$

$e^{x+iy} = \sqrt{2}\, e^{i(\pi/4 + 2k\pi)}$

$e^x = \sqrt{2}$ $\quad y = \frac{\pi}{4} + 2k\pi$

$\rightarrow z = \frac{\ln 2}{2} + i\left(\frac{\pi}{4} + 2k\pi\right)$ —— Ans.

$\rightarrow e^{z_1} e^{z_2} = e^{z_1 + z_2}$

$= e^{(x_1 + x_2)}\, e^{i(y_1 + y_2)}$

$\frac{e^{z_1}}{e^{z_2}} = e^{(x_1 - x_2) + i(y_1 - y_2)}$

$\rightarrow$ Logarithmic func^n :

$-1 = e^{i(\pi + 2k\pi)}$

$\log(-1) = \log(e^{(\pi + 2k\pi)i}) = (\pi + 2k\pi)\,i$

for $k = 0$, $\boxed{\log(-1) = \pi i}$

- generalised case :

$e^w = z \Rightarrow \boxed{\ln z = w}$

$\rightarrow e^w = re^{i(\theta + 2k\pi)}$

$\rightarrow e^w = e^{\ln r + i(\theta + 2k\pi)}$

$\Rightarrow$ $\boxed{w = \ln r + i\,(\theta + 2k\pi)}$ $\Rightarrow$ $\boxed{w = \ln|z| + i \arg z}$

$\Rightarrow$ $\ln z = \ln r + i\,(\theta + 2k\pi)$

$\ln(-1) = \overset{0}{\ln(|-1|)} + i\,(\pi + 2k\pi)$

$= \pi i\,(2k+1)$

Principal Logarithm : $\operatorname{Log} z$

General/ Simple Logarithm : $\log z$

$\log(-1) = \pi i\,(2k+1)$

$\operatorname{Log}(-1) = \pi i$

$\boxed{\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z}$

Here also,

$\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$

$z_1 = z_2 = -1$

$\operatorname{Log}((\cancel{-1})(\cancel{-1}))^{1} = \cancel{\operatorname{Log}(\pi^2)}\ 0$

$\operatorname{Log} z_1 + \operatorname{Log} z_2 = 2\pi i$

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* If we take only principal value, we will only get single valued function.

→ Complex Exponents :

when $z \neq 0$ and the exponent c is any complex no, the function z^c is defined by means of the eqn

$$z^c = e^{c(\log z)}$$

$$\left\{ \begin{aligned} e^z = e^{x+iy} &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + i e^x \sin y \end{aligned} \right\}$$

$$z^c = e^{\log z^c} = e^{c \log z}$$

Ques Calculate i^{-2i} $\Rightarrow$ $z = i$, $c = -2i$

$$\log z = \ln|z| + i \arg z$$

$$= \ln|i| + i\left(\frac{\pi}{2} + 2k\pi\right)$$

$$= \left(2k + \frac{1}{2}\right)\pi i$$

$$i^{-2i} = e^{-2i \log i}$$

$$= e^{-2i(2k+1/2)\pi i}$$

$$= e^{2(2k + 1/2)\pi}$$

② i^{1+i} ④ $(-1)^{1/\pi}$ ⑥ $(1-i)^{5i}$

③ 2^{1+i} ⑤ $(1+i)^{i}$

~~②~~ $\cancel{i^{1+i}}$

take $k=0$

② $e^{(1+i)\log i} = e^{(1+i)(\frac{\pi}{2} + \cancel{2k\pi})i} = e^{-\pi/2 + \pi/2 i}$

$= ie^{-\pi/2}$ (1 value)

~~③~~ $\cancel{2^{1+i} = e^{(1+i)\log 2}}$

④ $e^{1/\pi \log(-1)} = e^{1/\pi(\pi i + 2k\pi i)} = e^{i(2k+1)}$

$$e^{i(2k+1)} = \cos(2k+1) + i\sin(2k+1)$$

③ $2^{1+i} = e^{(1+i)\log 2}$

NOTE :

→ $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$ {It is equal when real part is positive but not when real part is negative}

→ Similarly,

$\log(z_1 z_2) = \log z_1 + \log z_2$

$\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$ {same reason as above}

→ Similarly,

$(z_1 z_2)^c = z_1^c z_2^c$

but for principal value of

$(z_1 z_2)^c \neq z_1^c z_2^c$

Ex. $z_1 = 1+i \quad z_2 = 1-i \quad z_3 = -1-i$

$(z_1 z_2)^i = ?$, $(z_2 z_3)^i = ?$ {Principal values}

{Principal value of (P.V.) $2^{1+i} = e^{(1+i)\log 2} = e^{(1+i)(\ln 2 + 0)} = e^{\ln 2} e^{i \ln 2}$
$= 2\,[\cos(\ln 2) + i \sin(\ln 2)]$}

$z_1 z_2 = (1+i)(1-i) = 1+1 = 2$

$(z_1 z_2)^i = 2^i = e^{i \ln 2} = \cos(\ln 2) + i \sin(\ln 2)$

$(z_1)^i \cdot (z_2)^i = (1+i)^i \cdot (1-i)^i$

$= e^{i \log(1+i)} \cdot e^{i \log(1-i)}$

$= e^{i(\ln\sqrt{2} + \pi/4)} \; e^{i(\ln\sqrt{2} - \pi/4)}$

$= e^{i(\ln 2)}$

LHS = RHS ✓

$\Rightarrow (z_1 z_2)^i = z_1^i z_2^i$ ✓

$z_2 z_3 = (1-i)(-1-i) = -1 - i + i - 1 = -2$

$z_2 z_3 = -2^i$

$(z_2 z_3)^i = (-2)^i = e^{i \log(-2)} = e^{i(\ln 2 + i\pi)}$

$z_2^i \cdot z_3^i = (1-i)^i \cdot (-1-i)^i$

$= e^{i(\ln\sqrt{2} - i\pi/4)} \, e^{i(\ln\sqrt{2} - 3\pi/4)} = e^{i(\ln 2 - i\pi)}$

(Real part is -ive)

$$(z_2 z_3)^i \neq z_2^i \, z_3^i$$

we are getting difference of 2π in the exponent part

→ <u>Trigonometric functions</u>:

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{—— (1)}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \quad \text{—— (2)}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Rightarrow \cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{—— (3)}$$

<u>Or</u> $\sin z = \sin(x + iy)$

$$= \sin x \cos iy + \cos x \sin iy$$

→ $\cos iy = \dfrac{e^{-y} + e^{y}}{2}$ $\qquad \sin iy = \dfrac{e^{-y} - e^{y}}{2i} = \dfrac{i^2 (e^{y} - e^{-y})}{2i}$

⇓

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

↳ hyperbolic

→ $\boxed{\begin{aligned} \cos iy &= \cosh y \\ \sin iy &= i\sinh y \end{aligned}}$

$$\Rightarrow \sin z = \sin x \cosh y + i \cos x \sinh y$$

unbounded ⟵ unbounded unbounded

hence, $\sin z$ is always unbounded.

Now, using (3), we get

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{ix - y} - e^{-ix + y}}{2i}$$

(1) Here we are getting a difference of 2π.
we can't use (3) if we are calculating principal values

Also, $\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$
$= \cos x \cosh y - i \sin x \sinh y$

absolute value is $x^2+y^2 \Rightarrow$ entire value is always ↑ng,
hence unbounded even where ∃ a '−'

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Eg $\log x^y = x \log y$] —— (3)
$\log(i^3) = 3 \log i = 3\left(\ln|i| + i\pi/2\right) = \boxed{\frac{3\pi}{2} i}$
On contrary,
$\log(i^3) = \log(-i) = \ln|-i| + i(-\pi/2) = \boxed{-\frac{\pi}{2} i}$ } —— (1)
This is correct answer

* We are using $\log z = \ln|z| + i\arg z$
$= \ln|z| + i\,\mathrm{Arg}\,z + 2k\pi i$

Hence, at different values of k, we will get different values of $\log z$

→ $\log(i^2) = 2\log i = 2 \times \frac{\pi}{2} i = \pi i$

$\log(i^2) = \log(-1) = \pi i$

} Here, we are getting same values

Now,
$\log(i^3) = \log(i^2 \cdot i) = \log(\underset{z_1}{-1} \times \underset{z_2}{i}) \Rightarrow$ Here real part of z_1 is −ive, thus we were getting diff of 2π

whereas
$\log(i^2) = \log(\underset{z_1}{i} \cdot \underset{z_2}{i})$ Here, real parts are +ive thus, we are getting the same results.

Ques. Find all the roots of the eq^n:
$\log z = i\frac{\pi}{2}$

$z = e^{i(\pi/2 + 2k\pi)}$

- Function
  |
  Limit
  |
  Continuity
  |
  Differentiability
  ↓
  Analytic

The reverse flow is automatic, i.e., if a func^n is analytic, then it will be differentiable, and so on ...

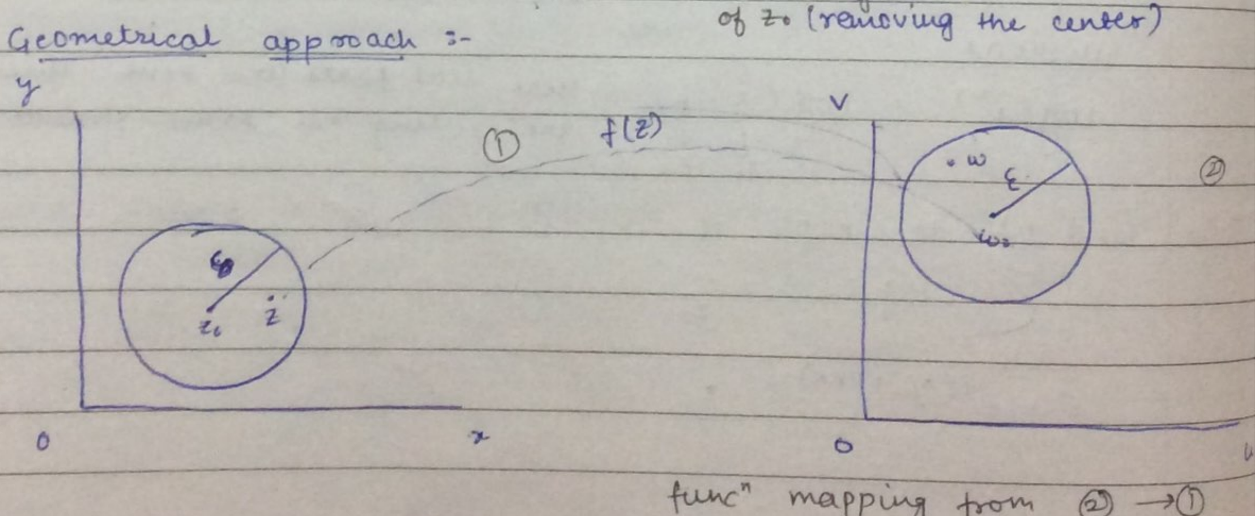
Limit :

Let a func^n f be defined at all points z in some deleted neighbourhood of z_0. The statement that the limit of $f(z)$ as z approaches z_0 is a no. w_0 or $\lim_{z \to z_0} f(z) = w_0$ — ①

Statement ① means that for each positive no. ϵ, $\exists$ a positive no. $\underset{w_0}{\delta}$ s.t.

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad \underbrace{0 < |z - z_0| < \delta}_{\text{deleted neighbourhood of } z_0 \text{ (removing the center)}} \quad — (*)$$

Geometrical approach :-



func^n mapping from ② → ①

(*) :→ all points in ① (circle) should have images in ② (circle)

They are not crossing the range / boundary

This means that the points $w = f(z)$ can be made arbitrary close to w_0 if we choose the point z close ~~to~~ ~~zero~~ enough to z_0, but distinct from it.

Examples :

(1) $\lim_{z \to 0} \frac{z}{|z|} =$

(2) $\lim_{z \to 0} \frac{(Re(z) - Im(z))^2}{|z|^2}$

(3) $\lim_{z \to 0} \frac{Re\, z}{|z|}$

(4) $\lim_{z \to 0} \left[\frac{1}{1 - e^{1/x}} + i y^2 \right]$

(5) $\lim_{z \to 0} \frac{Im\, z}{|z|}$

(6) $\lim_{z \to 0} \frac{Re(z^2)}{|z|^2}$

(1) $\lim_{z \to 0} \frac{x + iy}{\sqrt{x^2 + y^2}} = \lim_{x,y \to 0} \frac{x + iy}{\sqrt{x^2 + y^2}}$

Iterative limit

$\lim_{\Delta x \to 0, \Delta y \to 0} \frac{x + iy}{\sqrt{x^2 + y^2}} = \lim_{\Delta y \to 0} i \frac{y}{\sqrt{y^2}} = i$

$\lim_{\Delta y \to 0, \Delta x \to 0} \frac{x + iy}{\sqrt{x^2 + y^2}} = \lim_{\Delta x \to 0} \frac{x}{\sqrt{x^2}} = 1$

Limit does not exist

(2) $\lim_{z \to 0} \frac{(x - y)^2}{x^2 + y^2} = \lim_{x \to 0, y \to 0} \frac{x^2 + y^2 - 2xy}{x^2 + y^2} = \lim_{x \to 0, y \to 0} 1 - \frac{2xy}{x^2 + y^2}$

~~$\lim_{y \to 0}$~~ $= 1$ In both cases, we get same values (1) above

Take $y = mx$

$\lim_{x \to 0} \frac{(x - mx)^2}{x^2 + m^2 x^2} = \lim_{x \to 0} \frac{(1 - m)^2}{1 + m^2}$

Depends on m. Hence, Limit D.N.E.

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* Iterative limit $\not\Longrightarrow$ Original limit
(which is asked in the question)

(3) $$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$$

Take $y = mx$

$$\lim_{x\to 0} \frac{1}{\sqrt{1+m^2}}$$: depends on m $\Rightarrow$ limit D.N.E.

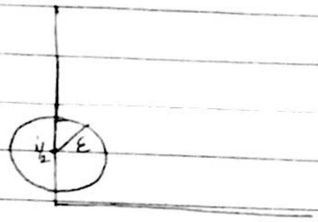
(4.)

Ques. Show that if $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

$z_0 = 1$

(we have to check value at $z = 1$)



Let $|z-1| < \delta$

then $\left|f(z) - i/2\right|$

$$= \left|\frac{i\bar{z}}{2} - \frac{i}{2}\right|$$

$$= \left|\frac{i}{2}(\bar{z}-1)\right| = \left|\frac{i}{2}\right| \left|\bar{z}-1\right|$$

$$= \frac{|z-1|}{2} < \epsilon \quad \text{whenever} \quad 2\epsilon = \delta$$

Hence Proved

Eg. $\lim_{z \to i} z^2 = -1$

$z_0 = i$

$|z - i| < \delta$

$$|f(z) - f(z_0)| = |z^2+1| = |z^2 - i^2| = |z-i|\,|z+i|$$

$$= |z-i|\,|z-i+2i|$$

$$\leq |z-i|\{|z-i| + |2i|\} = \delta(\delta+2) \quad \{|a+b| \leq |a| + |b|\}$$

$$\leq \epsilon$$

whenever $\delta^2 + 2\delta = \epsilon$

$\delta(\delta+2) = \epsilon$

$\delta^2 + 2\delta = \epsilon$

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Continuity: A function f is said to be continuous at a point z_0 if the following 3 conditions hold:

1.) $\lim_{z \to z_0} f(z)$ exist

2.) $f(z_0)$ exist

3.) $\lim_{z \to z_0} f(z) = f(z_0)$

For each positive no. ϵ, $\exists$ a positive no. δ s.t.

$|f(z) - f(z_0)| < \epsilon$, whenever $|z - z_0| < \delta$

{Here, we also check value at the point while we don't do so in case of finding the limit}.

Q. $f(z) = \text{Log } z$ (Principal)

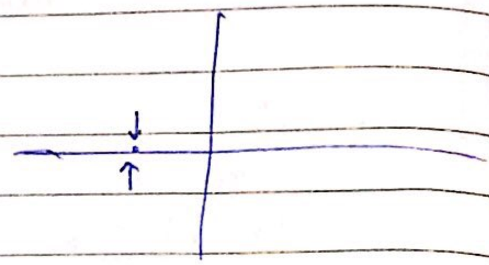
Is it continuous on the negative real axis ?

$$\underset{\downarrow}{\overset{\text{discont}}{\text{Log } z}} = \underset{\downarrow}{\overset{\text{cont}}{\ln |z|}} + i \underset{\downarrow}{\overset{\text{discont}}{\text{Arg } z}}$$

gets disturbed when real part gets negative (not continuous itself)

Case-I : y : +ive

Case-II : y : -ive



$$\begin{cases} +\pi & x < 0, y > 0 \\ -\pi & x < 0, y < 0 \end{cases}$$

Hence, limit from both sides are not equal

$\Rightarrow$ It is not continuous on negative real axis.

(On any other point, it will be continuous).

Differentiability :

Let $f(z)$ be a single valued func^n defined in a domain D.

The func^n $f(z)$ is said to be differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

The limit is called derivative of $f(z)$ at the point $z = z_0$ & it is denoted by $f'(z_0)$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Substitute $z - z_0 = \Delta z$

$$\Rightarrow f'(z) = \lim_{z \to z_0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

OR

For each $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t.

$$\cancel{\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right| < \varepsilon \text{ whenever } |z_0| < \delta}$$

$$\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - f'(z_0) \right| < \epsilon \text{ whenever } |z - z_0| < \delta$$

Q. $f(z) = \bar{z}$

(1) is continuous at $z = 0$ ✓

(2) is differentiable at $z = 0$ ✗

$f(z) = \bar{z}$

$= x - iy$: continuous

$\delta - \epsilon$ def^n :

$$f(z) = \begin{cases} x - iy & z \neq 0 \\ 0 & z = 0 \end{cases}$$

$\cancel{|f(z) - f(0)| < \varepsilon}$

$\cancel{|\bar{z}| < \epsilon \Rightarrow |z| < \epsilon}$

$|z - z_0| < \delta \Rightarrow |z| < \delta$

$|f(z) - f(z_0)| = |\bar{z}| = |z| < \epsilon$ whenever $\varepsilon = \delta$

if $\delta = \epsilon \Rightarrow$

$|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$

$\Rightarrow$ It is continuous at $z_0 = 0$

(2) $\dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$f'(0) = \lim_{\Delta z \to 0} \frac{(\overline{z_0 + \Delta z}) - 0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i \Delta y}$$

take $y = mx$

$$\Rightarrow \lim_{x \to 0} \frac{1 - im}{1 + im} \Rightarrow \text{limit D.N.E. (depends on } m)$$

Ques. Prove that if $f(z) = z^2$, then $f'(z) = 2z$.

$$\lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - (z_0)^2}{\Delta z} = f'(z_0)$$

$$= \lim_{\Delta z \to 0} \frac{2\Delta z z_0 + \Delta z_0^2}{\Delta z} = f'(z_0)$$

$$\Rightarrow 2z_0 = f'(z_0)$$

$$\Rightarrow \boxed{f'(z) = 2z}$$

Hence Proved.

Ques. Check the differentiability for

1.) $|z|^2$ 2.) $\dfrac{(Re\, z)^2}{|z|}$ 3.) $Im\ z$ 4.) $Re\ z$

It will be deficult to solve above problem by def^n.

Cauchy - Riemann Eq^n :

Let $f(z) = u(x,y) + i\,v(x,y)$ is defined & continuous at a point $z = x+iy$ and differentiable at z itself. Then, at that point, first order ~~different~~ partial derivative exists and satisfies the C-R eq^n :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or $u_x = v_y$ & $u_y = -v_x$

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Solⁿ (1) $|z|^2 = x^2 + y^2 + i\,0$

$f(z) = u + iv$

$\Rightarrow u = x^2 + y^2$ & $v = 0$

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 2y$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ at $x = 0$

Hence, funcⁿ could be differentiable at origin

* not satisfy condⁿ (C-R eqⁿ) $\longrightarrow$ not differentiable

(3) $f(z) = y$

$v = y \quad u = 0$

$\frac{\partial v}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial y} = 1 \neq \frac{\partial u}{\partial x} = 0$: Not differentiable

proof

C-R eqⁿ :-

$\Delta z = \Delta x + i\,\Delta y, \quad z = x + iy$

$z + \Delta z = (x + \Delta x) + i\,(y + \Delta y) \qquad f(z) = u + iv$

$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{f((x+\Delta x) + i(y+\Delta y)) - f(x+iy)}{\Delta x + i\Delta y}$

$= \frac{u(x+\Delta x + i(y+\Delta y)) + iv(x+\Delta x + i(y+\Delta y)) - u(x+iy) - iv(x+iy)}{\Delta x + i\Delta y}$

$= \cancel{u(\Delta x + i\Delta y) + iv(\Delta x + i\Delta y)}$

$= \lim_{\Delta z \to 0} \frac{u(x+\Delta x + i(y+\Delta y)) - u(x+iy)}{\Delta x + \Delta y} + i\left(\frac{v(x+\Delta x + i(y+\Delta y)) - v(x+iy)}{\Delta x + \Delta y}\right)$

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Iterative Limit:

i) $\Delta x \to 0$

$$\lim_{\Delta y \to 0} \frac{u(x+i(y+\Delta y)) - u(x+iy)}{i\,\Delta y} + i\left(\frac{v(x+i(y+\Delta y)) - v(x+iy)}{i\,\Delta y}\right)$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{—— (1)}$$

ii) $\Delta y \to 0$

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x+iy) - u(x+iy)}{\Delta x} + i\left(\frac{v(x+\Delta x+iy) - v(x+iy)}{\Delta x}\right)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (2)$$

$$\frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$ to be differentiable

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

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* Satisfaction of CR eq^{n}s at (x_0, y_0) $\Rightarrow$ $f(z)$ may or may not be differentiable at (x_0, y_0)
* dissatisfaction of CR eq^n at (x_0, y_0) $\Rightarrow$ $f(z)$ is non-differentiable at (x_0, y_0)

**

Theorem: Let a func^n $f(z) = u + iv$ is defined in some ϵ nbd of (x_0, y_0) and first order partial derivative u_x, u_y, v_x & v_y exist in that nbd. If partial derivatives satisfy C-R eq^{n}s at (x_0, y_0) and are continuous at (x_0, y_0), then $f'(z)$ exist at (x_0, y_0) and $f'(z)$ is given by:

$$f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0)$$

$$f(x_0+\Delta x,\ y_0+\Delta y) - f(x_0,y_0) = f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

if $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

then $f(x,y)$ is continuous at (x_0, y_0) —— (1)

PROOF :

$$f(z_0) = u(x_0,y_0) + i\,v(x_0,y_0)$$

$$\Delta z = \Delta x + i\,\Delta y$$

$$f(z_0+\Delta z) = u(x_0+\Delta x,\ y_0+\Delta y) + i\,v(x_0+\Delta x,\ y_0+\Delta y)$$

$f(z_0+\Delta z) - f(z_0) =$ (Using continuity of 2 variables) —— (3)

$$\Delta u + i\,\Delta v$$

where $\Delta u = u(x_0+\Delta x,\ y_0+\Delta y) - u(x_0,y_0)$

$\Delta v = v(x_0+\Delta x,\ y_0+\Delta y) - v(x_0,y_0)$

acc to (1),

$$\Delta u = u_x(x_0,y_0)\,\Delta x + u_y(x_0,y_0)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ { since partial derivative are continuous }

In the same manner.

$$\Delta v = v_x(x_0,y_0)\,\Delta x + v_y(x_0,y_0)\,\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$$

where $\epsilon_3, \epsilon_4 \to 0$ as $\Delta x, \Delta y \to 0$

$$= -u_y(x_0,y_0)\,\Delta x + u_x(x_0,y_0)\,\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$$

Eq^n (3) becomes :

~~$u_x(x_0,y_0)\,[\Delta x + i\Delta y] + u_y(x_0,y_0)\,[\Delta y + i\Delta x] + \Delta x\,[\epsilon_1$~~

$$\Delta f(z_0) = (u_x + i v_x)(x_0,y_0)\,\Delta x + (u_y + i v_y)(x_0,y_0)\,\Delta y + \epsilon\,\Delta x + \eta\,\Delta y$$

where $\epsilon = \epsilon_1 + \epsilon_3$

$\eta = \epsilon_2 + \epsilon_4$

By C-R eq^n s, we have

$$\Delta f(z_0) = (u_x + i v_x)\Delta x + (-v_x + i u_x)\Delta y + \epsilon \Delta x + \eta \Delta y$$

$$= (u_x + i v_x)(\Delta x + i \Delta y) + \epsilon \Delta x + \eta \Delta y$$

where $\epsilon, \eta \to 0$ as $\Delta x, \Delta y \to 0$

$$\lim_{\Delta z \to 0} \frac{\Delta f(z_0)}{\Delta z} = (u_x + i v_x)(x_0, y_0) = f'(z_0)$$

Hence Proved

Polar form of C-R eq^n :

$z = z(x,y) = re^{i\theta}$, $z \neq 0$

$f(z) = u(x,y) + i v(x,y)$

$= u(r,\theta) + i v(r,\theta)$

$f(re^{i\theta}) = u(r,\theta) + i v(r,\theta)$ —— ①

differentiating wrt r, we get :

$f'(re^{i\theta})\, e^{i\theta} = u_r + i v_r$ —— ②

differentiating wrt θ, we get :

$f'(re^{i\theta})\, ir e^{i\theta} = u_\theta + i v_\theta$ —— ③

using eq's ② & ③

$ir(u_r + i v_r) = u_\theta + i v_\theta$

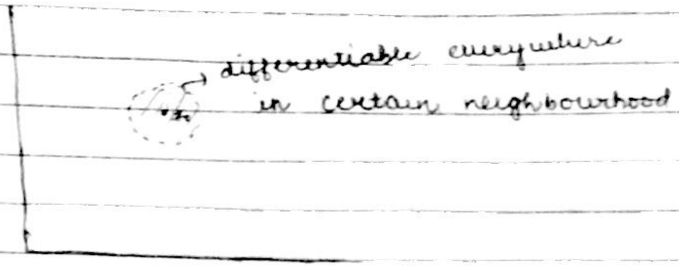
$\Rightarrow$ $\boxed{-r v_r = u_\theta}$, $\boxed{r u_r = v_\theta}$

$\Rightarrow$ $\boxed{u_r = \frac{1}{r} v_\theta \ , \ v_r = \frac{-1}{r} u_\theta}$ $(r \neq 0)$

Analytic Function :

A func^n $f(z)$ is said to be analytic at a point z_0 if $f(z)$ is not differentiable only at z_0 but also in some nbd of the point z_0.

Geometrically :



eg. $f(z) = |z|^2$: differentiable only at the point $z = 0$

- There does not exist any point in nbd where $f(z)$ is differentiable $\Rightarrow$ $f(z)$ is not analytic at $z = 0$

To show $|z|^2$ is diff at $z = 0$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \frac{(z+\Delta z)(\bar{z}+\Delta\bar{z}) - z\bar{z}}{\Delta z}$$

$$= \frac{z\Delta\bar{z} + \bar{z}\Delta z + \Delta z\Delta\bar{z}}{\Delta z}$$

for any non-zero point

$$= \bar{z} + \Delta\bar{z} + z\frac{\Delta\bar{z}}{\Delta z} =$$

If we approach $z+\Delta z \to \Delta z$ along x-axis ($\Delta y \to 0$)

$\lim_{x \to 0}$

$= \bar{z} + z + \Delta x$

along y axis ($\Delta x \to 0$)

$= -z + \bar{z} + \Delta y$

} getting 2 different limits

$\Rightarrow$ Not differentiable at any non-zero point if $z = 0$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{|\Delta z|^2}{\Delta z} = \frac{\Delta z \, \Delta \bar{z}}{\Delta z} = \Delta \bar{z}$$

$\lim_{\Delta z \to 0} \Delta \bar{z} = 0$: unique limit $\Rightarrow$ diff at 0

Q. $f(z) = \frac{1}{z}$: non-differentiable at 0
diff. at other points

$\Rightarrow$ We can find nbd of any point $(z \neq 0)$ where it is differentiable.

$\Rightarrow$ $f(z)$ is analytic everywhere except $z = 0$

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Q. $f(z) = \left(\frac{Re(z) - Im(z)}{|z|^2}\right)^2$: limit d.n.e at $z = 0$
$\Rightarrow$ non-differentiable
$\Rightarrow$ non-analytic at $z = 0$

Q $f(z) = |z|^2 = x^2 + y^2$

$u = x^2 + y^2 \qquad v = 0$

$f(z) = u(x,y) + i\, v(x,y)$ $\{$ $u, v \in$ Real valued func^n of (x, y)

$u_x = \frac{\partial u}{\partial x} = 2x \qquad u_y = 2y$

$v_x = 0 \qquad v_y = 0$

$u_x \neq v_y$ & $u_y = -v_x$ only when $x, y = 0$
$\Rightarrow$ func^n may be differentiable at $z = 0$ (It satisfies C-R eq^n)

- set of all points where it is going may be diff : 1 point

Q.
$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

$$\frac{x^3 - y^3}{x^2+y^2} = u \qquad v = \frac{x^3+y^3}{x^2+y^2}$$

Check at $(0,0)$

$$\lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = u_x(0,0)$$

$$\boxed{\begin{aligned} u_x(a,b) &= \lim_{h\to 0} \frac{u(a+h,b) - u(a,b)}{h} \\ u_y(a,b) &= \lim_{k \to 0} \frac{u(a,b+k) - u(a,b)}{k} \end{aligned}}$$ std. definition of partial derivatives

$$\Rightarrow u_x(0,0) = \lim_{h\to 0} \frac{1}{h}\left(\frac{h^3 - 0}{h^2+0} - 0\right) = 1$$

$$v_y(0,0) = \lim_{k\to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k\to 0} \frac{k^3}{k^2 \times k} = 1$$

$$u_y(0,0) = \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \to 0} \frac{-k^3}{k^2 \times k} = -1$$

$$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h} = \lim_{h\to 0} \frac{h^3}{h^2 \times h} = 1$$

$\Rightarrow$ C-R conditions are satisfied here

to check diff. at $z = 0$ $(z_0 = 0)$

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}$$

Let $\Delta z = z$

$$\lim_{z\to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$$

In below eq. partial derivatives are not continuous, hence, it is non-differentiable.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

$$\cancel{\lim_{z\to 0} \frac{(x^3-y^3)+i(x^3+y^3)}{x^3+ix^2y+xy^2+iy^3}} = \cancel{\lim_{z\to 0}}$$

$\cancel{\lim_{z\to 0}}$

Now, we have to check limit's existance at $z=0$

take $y = mx$

$$\lim_{x\to 0} \frac{(x^3-m^3x^3)+i(x^3+m^3x^3)}{(x^2+m^2x^2)(x+imx)}$$

$$\lim_{x\to 0} \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}$$

$$= \lim_{x\to 0} \frac{(1-m^3)+i(1+m^3)}{1+im^3+im+m^2}$$: depends on m $\Rightarrow$ limit D.N.E.

$\Rightarrow$ $f(z)$ is non-differentiable at $z=0$ but it satisfies C-R eq^n.

Ques. $f(z) = z^2 = x^2-y^2+2ixy$

$u(x,y) = x^2-y^2 \qquad v(x,y) = 2xy$

$u_x = 2x \qquad v_y = 2x$

$u_y = -2y \qquad v_x = 2y$

$\Rightarrow$ It satisfies C-R eq^n. everywhere.

also, partial derivatives are continuous $\Rightarrow$ diff. also

$\Rightarrow$ It is analytic func^n.

Ques. $f(z) = \bar{z} = x-iy$

$u = x \qquad v = -y$

$u_x = 1 \qquad u_y = 0$

$v_y = -1 \qquad v_x = 0$

This doesn't satisfy C-R eq^n.

- Can we use C-R eq^n to check diff. of real func^{n}s? Why / Why not?

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* It is not necessary that for every func^n, ∃ atleast 1 point at which C-R eq^n is satisfied.

Eg. $f(z) = (\text{Re } z)^2$ $\quad f(z) = (\text{Im } z)^2$

Here, also, C-R eq^n d.n.e. at any point.

Ques. If $f(z) = e^{\bar{z}}$, then is $f(z)$ analytic at $z = 0$?

exponential func^n are always continuous. But we have to check for differentiability.

$$e^{\bar{z}} = e^{x-iy}$$
$$= e^x \cdot e^{-iy}$$
$$= (e^x)[\cos(y) - i(\sin y)]$$

$u = e^x \cos y$ $\qquad v = -e^x \sin y$

$u_x = e^x \cos y$ $\qquad v_y = -e^x \cos y$

$u_y = -e^x \sin y$ $\qquad v_x = -e^x \sin y$

It doesn't satisfy C-R eq^n ⇒ it is non-differentiable

Ques. $f(z) = \text{Log } z$

(i) Is it analytic on negative real axis?

→ Not cont. ⇒ non-differentiable on negative real axis

(ii.) Is it analytic at any other point.

$$\text{Log } z = \ln|z| + i \text{ Arg } z$$
$$= \frac{1}{2} \ln|x^2+y^2| + i\left(\tan^{-1}(y/x)\right)$$

take simplest case.

$u = \dfrac{\ln(x^2+y^2)}{2}$ $\qquad v = \tan^{-1}(y/x)$

(Ex. 12.4 Edition - 8) : Erwin Kreyszig

Q. $f(z) = \text{Arg } z$

Summary

→ C-R eq^n : $u_x = v_y$ & $u_y = -v_x$ (necessary cond^n)

Sufficient condition for $f(z)$ to be differentiable :

1) u_x, v_y, u_y, v_x exist
2) satisfies C-R eq^n
3) u_x, v_y, u_y, v_x are continuous

Entire Function :

If $f(z)$ is analytic for every z_0 in z plane, it is called as an entire func^n

Eg: $f(z) = z^2$: entire func^n

$f(z) = \log z$: not entire func^n (not analytic on $-$ve x-axis)

Harmonic Function :

A real valued func^n H of two variables x & y is said to be harmonic in a given domain of x-y plane if throughout the domain, it has continuous partial derivatives of the first and second order and satisfies the equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$$

Eg: $f(z) = z^2$: Complex func^n

$= (x^2 - y^2) + i\,(2xy)$

$u(x,y) = x^2 - y^2$; $v(x,y) = 2xy$

Here, both u & v are real valued func^n

To check harmonic or not (both u & v)

1) Partial derivative should exist

$u_x = 2x$, $u_y = -2y$ ✓

$u_{xx} = 2$, $u_{yy} = -2$

2) $u_{xx} + u_{yy} = 0$ ✓ ⇒ u is harmonic

Eg. $e^z = 0$, $z = ??$ (Find out)

$e^{x+iy} =$ ~~[scribbled out]~~ ⇒ we can't calculate z here.

$e^z = e^{\log 0}$ use $z=0$, hence we can't define arg z, therefore, log can't be defined

similarly, v is also harmonic.

* ~~v is conjugate harmonic of u.~~

* ~~If we have~~

Theorem: If a func^n $f(z) = u(x,y) + i v(x,y)$ is analytic in a domain D, then its components functions u & v are harmonic in D.

If not analytic : we can't say.

Eg. $f(z) = |z|^2 = x^2+y^2+0i$

$u = x^2+y^2$	$v = 0$
$u_x = 2x$ $\quad u_y = 2y$	$v_x = 0$ $\quad v_y = 0$
$u_{xx} = 2$ $\quad u_{yy} = 2$	$v_{xx} = 0$ $\quad v_{yy} = 0$
$u_{xx} + u_{yy} \neq 0$	$v_{xx} + v_{yy} = 0$

* → Not harmonic ⇒ Not analytic

Eg. $f(z) = \bar{z} = x - iy$

$u = x$ $\qquad\qquad v = -y$

$u_x = 1$ $\quad u_y = 0$ $\qquad v_y = -1$ $\quad v_x = 0$

$u_{xx} = 0$ $\quad u_{yy} = 0$ $\qquad v_{yy} = 0$ $\quad v_{xx} = 0$

⇒ u : harmonic $\qquad v$: harmonic

But we know $f(z)$ is non-analytic (non-differentiable)

* ⇒ Harmonic $\not\Rightarrow$ Analytic

Conclusion :

If u & v are harmonic, then $f(z)$ may or may not be analytic.

PROOF (of above theorem) Using C-R eq^n where

$u_x = v_y$, $u_y = -v_x$

⇒ $u_{xx} = v_{yx}$ — ①, $u_{yy} = -v_{xy}$ — ②

Performing ① + ②, we get

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} \quad — ③$$

Using the concept that if u_x & u_y are continuous, then

$u_{xy} = u_{yx}$,

③ becomes :

$u_{xx} + u_{yy} = 0$ { since $f(z)$ is analytic (given) ∴ it is also continuous }

Hence proved

Similarly,

$u_{xy} = v_{yy}$ — ④ , $u_{yx} = -v_{xx}$ — ⑤

Adding ④ & ⑤, we get

$v_{yy} + v_{xx} = 0$

Eg. $u = x^2 - y^2$, if u is harmonic, find conjugate harmonic of u. (This means we have to construct an analytic func^n)

① Here, we have to find v. $u + iv \Rightarrow$ analytic func^n.

② ~~Given~~ As $u = x^2 - y^2$

$u_x = 2x$

Using C-R eq^n,

$v_y = 2x$

$v = 2xy + \phi(x)$ — ① { v_y should be cont. then only we can integrate it }

$u_y = -2y$

Using C-R eq^n,

$-v_x = -2y$

$v_x = 2y$ — ②

$\cancel{v = 2xy + \phi(y)}$ — ②

Equating ~~① with ②~~

Diff. ① wrt x,

$v_x = 2y + \phi'(x)$ — ③

If $\frac{\partial f}{\partial x}$ exist $\Rightarrow$ f is cont

In harmonic defn, $\frac{\partial f}{\partial x}$ & $\frac{\partial^2 f}{\partial x^2}$ exist $\Rightarrow$ f & f' should be continous

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From ② & ③,

$$2y = 2y + \phi'(x)$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \cancel{\boxed{v = 2xy}}\ \text{Ans}$$

$$\Rightarrow \phi(x) = c$$

Now, $v = 2xy + c$

$$\Rightarrow f(z) = u + iv$$

$$= (x^2 - y^2) + i(2xy + c)$$

$$\boxed{f(z) = z^2 + ic}$$

Here, v is conjugate harmonic to u. / of u

Also, we can see that f(z) is analytic here.

Eg. $u = x^2 + y^2$: Not harmonic

Can we calculate v here?

Eg. Re f(z) = constant = u

f(z) : analytic, is f(z) also constant?

Let $u = c$

Since f(z) : analytic $\Rightarrow$ $\nexists$ u, v : harmonic

* If u is harmonic & v is conjugate harmonic of u, then u is a conjugate harmonic of −v.

Eg. $u = \ln|z|$, $u \neq 0$

* Eg. $u = e^{ax}\cos 5y$. For what values of a, u will be analytic?

$u_x = ae^{ax}\cos 5y$, $u_{xx} = a^2 e^{ax}\cos y$

$u_y = -5e^{ax}\sin 5y$, $u_{yy} = -25e^{ax}\cos 5y$

For u to be harmonic,

$$a^2 - 25 = 0$$

$$\Rightarrow \boxed{a = \pm 5}\ \text{Ans.}$$

→ piecewise cont curve : break at finite points, otherwise cont

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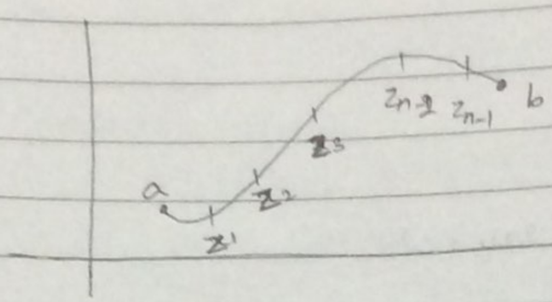
COMPLEX LINE INTEGRAL

Let $f(z)$ be continuous at all points of a curve C, which we shall assume has a finite length.



Subdivide C into n parts by means of the points $z_1, z_2, \ldots z_{n-1}$ choosen arbitrarily and call $z_0 = a$, $z_n = b$. On each arc joining z_{k-1} to z_k, choose a point d_k. From the sum

$$S_n = f(d_1)(z_1 - a) + f(d_2)(z_2 - z_1) \ldots + f(d_n)(b - z_{n-1})$$

on writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S_n = \sum_{k=1}^{n} f(d_k)\, \Delta z_k$$

Let the no of subintervals n ↑es in such a way that the largest of the chord length $|\Delta z_k|$ approaches 0. Then,

$$\int_a^b f(z)\,dz \quad \text{or} \quad \int_C f(z)\,dz$$

Some Definitions :

1. Arc :

A set of points $z = (x, y)$ in the complex plane is said to be an arc if

$$x = x(t), \quad y = y(t), \quad (a \le t \le b)$$

where $x(t)$ & $y(t)$ are continuous func^n s of the real parameter t

* learn definitions also (will come in exam)

2. Simple arc / Jordan arc :

If it does not cross (intersect) itself.

Mathematically,

$$z(t_1) \neq z(t_2) \quad \text{if} \quad t_1 \neq t_2$$

<u>Integral of function $w(t)$</u> :

$$w(t) = u(t) + i\,v(t) \qquad , \qquad a \le t \le b$$

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Eg. $$\int_0^1 (1+it)^2\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt$$

$$= t - \frac{t^3}{3}\Big|_0^1 + i\;t^2\Big|_0^1$$

$$= \frac{2}{3} + i$$

<u>Line integral</u> :-

→ Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$ along the

(a) parabola $x = 2t$, $y = t^2+3$

(b) straight line from $(0,3)$ to $(2,3)$ then from $(2,3)$ to $(2,4)$

(c) a straight line from $(0,3)$ to $(2,4)$.

(a) $dx = 2\,dt \qquad dy = 2t\,dt$

$$I = \int_0^1 \left(2(t^2+3) + 4t^2\right)2\,dt + \left(6t - (t^2+3)\right)2t\,dt$$

$$= \int_0^1 12t^2 + 12 + 12t^2 - 2t^3 - 6t \quad dt$$

$$= \int_0^1 (-2t^3 + 24t^2 - 6t + 12)\,dt = -\frac{2}{4} + \frac{24}{3} - \frac{6}{2} + 12$$

$$= 16.5$$

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(b) $I = \int_{(0,3)}^{(2,3)} I + \int_{(2,3)}^{(2,4)} I$

$\Downarrow$ $y=3$, $dy=0$ $\qquad \Downarrow$ $x=2$, $dx=0$

$= \int_0^2 (6+x^2)\,dx + \int_3^4 (6-y)\,dy$

$= 6x + \frac{x^3}{3}\Big|_0^2 + 6y - \frac{y^2}{2}\Big|_3^4$

$= 12 + \frac{8}{3} + \left[24 - 8 - \left(18 - \frac{9}{2}\right)\right]$

$= 12 + \frac{8}{3} - 2 + \frac{9}{2} \qquad = 10 + \frac{8}{3} + \frac{9}{2}$

$= \frac{60 + 16 + 27}{6} = \frac{103}{6}$

(c) $(0,3) \rightarrow (2,4)$

$(y-3) = \frac{4-3}{2-0}(x)$

$x = 2(y-3)$

$x = 2y - 6 \qquad dx = 2dy$

$I = \int_3^4 (2y + 4(y-3)^2)\,2\,dy + (6(y-3) - y)\,dy$

$= \int_3^4 (2y + 4(y^2 - 6y + 9))2\,dy + (5y - 18)\,dy$

-22

$= \int_3^4 8y^2 - 44y + 72 + 5y - 18 \quad dy$

$= \int_3^4 8y^2 - 39y + 54\,dy$

$y^3/3$

$= 8\left[\frac{64}{3} - \frac{27}{3}\right] - 39\left[\frac{16}{2} - \frac{9}{2}\right] + 54(1)$

$= 8\left[\frac{37}{3}\right] - 39\left[\frac{7}{2}\right] + 54$

$\overset{2}{54} \times 6$

$$= \frac{296}{3} - \frac{273}{2} + \frac{324}{6}$$

$$= \frac{592 - 819 + 324}{6} = \frac{97}{6}$$

* Along different paths, we are getting different solutions.

→ Evaluate $\int_C f(z)\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve

ⓐ $z = t^2 + it$

ⓑ The line from $z = 0$ to $z = 2i$ and then, the line from $z = 2i$ to $z = 4 + 2i$

$f(z) = \bar{z}$ ↓ no where diff. , $f(z) = z$ ↓ everywhere diff.

1) $f(z) = \bar{z}$

$$I = \int_C \bar{z}\,dz = \int_C (x - iy)(dx + idy)$$

$$= \int_C (x\,dx + y\,dy) + i\int_C (x\,dy - y\,dx)$$

2) $f(z) = z$

$$I = \int_C (x+iy)(dx + idy) = \int_C \cancel{x} y\,dy + i\int_C x\,dy + y\,dx$$

1) $$I_a = \int_0^2 t^3 \cdot 2dt + t \cdot dt + i\int_0^2 t^2 \cdot dt - 2t^2 dt$$

$z = t^2 + it$, $x = t^2$, $y = t$

$dx = 2t$ $dy = dt$

$t^2(2t\,dt)$

$$= \frac{2 \times \overset{4}{\cancel{16}}}{\cancel{4}} + \frac{2^2}{2} + i\left[\frac{8}{3} - \frac{2 \cdot 2^3}{3}\right]$$

$$= \overset{8}{\cancel{\frac{16}{3}}} + 2 + i\left[\frac{8}{3} - \frac{16}{3}\right]$$

$$= \boxed{\overset{10}{\cancel{\frac{22}{3}}} + i\left[-\frac{8}{3}\right]} \quad \text{——} \quad ①$$

$$I_b = \int_{z=0\ (0,0)}^{2i\ (0,2)} f(z)\,dz + \int_{z=2i\ (0,2)}^{z=4+2i\ (4,2)}$$

$x = 0 \Rightarrow dx = 0$ $\qquad$ $y = 2 \Rightarrow dy = 0$

$$I_b = \left(\int_0^2 y\,dy + i\int_0^2 x\,dy - y\,dx\right) + \left(\int_0^4 x\,dx + i\int_0^4 0 - 2\,dx\right)$$

$$= \frac{2^2}{2} + i(0) + \frac{4^2}{2} + i(-2\times 4)$$

$$= \boxed{10 - 8i} \quad \text{—— (2)}$$

2) $f(z) = z$ $\qquad$ $I = \int_C x\,dx - y\,dy + i\int_C x\,dy + y\,dx$

I_a: $z = t^2 + it$

$x = t^2 \qquad y = t$

$dx = 2t\,dt \qquad dy = dt$

$$I = \int_0^2 t^2(2t\,dt) - t\,dt + i\int_0^2 t^2\,dt + 2t^2\,dt$$

$$= \int_0^2 2t^3\,dt - t\,dt + i\int_0^2 3t^2\,dt$$

$$= \frac{2^4}{2} - \frac{2^2}{2} + i(2)^3 = \boxed{6 + 8i} \quad \text{—— (3)}$$

$$I_b = \left(\int_0^2 -y\,dy\right) + \left(\int_0^4 x\,dx + i\int_0^4 2\,dx\right)$$

$$= -\frac{2^2}{2} + \frac{4^2}{2} + i\,2\times 4$$

$$= \boxed{6 + 8i} \quad \text{—— (4)}$$

we can see ① ≠ ② but ③ = ④

(we can notice that $\bar{z}$ is not-analytic , z is analytic)

Integral only depends on end points

In case of circle analytic func^n will have integral = 0.
(any other closed loop)

→ When we are changing the parameter, the curve should be also smooth (Here, t^2+it)

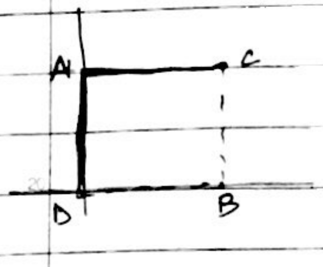
Smooth Curve :

A curve $\gamma : [a,b] \to \mathbb{C}$ is said to be smooth if $\gamma(t)$, $a \le t \le b$ is continuously differentiable

Eg. $t^2 + i(2t)$

$d(t^2 + i2t) = 2t + i2$: continuous ⇒ it is cont. diff.

Eg. Path ⑤ in earlier eg.



It is not differentiable at A (or B)

⇒ ~~not~~ cont. diff. from D to A & then A to C

⇒ peacewise smooth curve / peacewise cont. differentiable

Peacewise Smooth Curve :-

If a curve can be defined on a finite no. of sub-intervals s.t. on each interval, curve is smooth, then it is called as peacewise smooth curve.

Theorem. ML - Inequality :-

let C be a peacewise smooth curve C s.t.

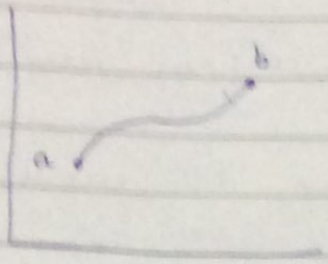
$$z(t) = x(t) + iy(t), \quad a \le t \le b$$

~~and~~ then $\left| \int_C f(z)\, dz \right| \le ML$

Smooth curve $\Rightarrow$ Piecewise smooth curve, but not vice versa

where L is the length of the curve C and $|f(z)| \le M$ everywhere on C.

Proof.



$$|S_n| = \left| \sum_{k=1}^{n} f(\alpha_k)\, \Delta z_k \right|$$

$$\le \sum_{k=1}^{n} |f(\alpha_k)|\, |\Delta z_k| \qquad \{ |a_1 a_2| = |a_1||a_2| \}$$

$$\left| \int f(z)\,dz \right| \le M \underbrace{\sum_{k=1}^{n} |\Delta z_k|}_{\text{length of curve } (\lim_{n\to\infty})} = ML$$

{ Σ length of chord of curve $\le$ length of curve (of $L \to 0$)
But we divide it in small sub-intervals, Thus
Σ of length of chords = length of curve }

Formally:
From the definition of ~~continuous~~ contour integral, we get

$$S_n = \sum_{k=1}^{n} f(\alpha_k)\, \Delta z_k$$

Given $$|S_n| = \left| \sum_{k=1}^{n} f(\alpha_k)\, \Delta z_k \right|$$

$$\le \sum_{k=1}^{n} |f(\alpha_k)|\, |\Delta z_k|$$

$$\le M \sum_{k=1}^{n} |\Delta z_k|$$

It is clear that $L^* = \sum_{k=1}^{n} |z_k - z_{k-1}|$ represents the sum of the length of the chords whose end points are $z_0, z_1, z_2 \dots z_n$

Since a st. line path is the smallest distance b/w any two points, $|z_k - z_{k-1}|$ does not exceed the length of arc joining the point z_{k-1} to z_k. Thus, $n \to \infty$ s.t. the max $|\Delta z_k| \to 0$, we have $L^* \longrightarrow L$ (the length of curve C)

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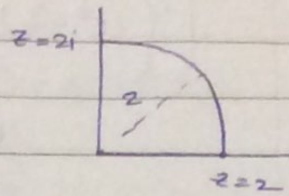
eg. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the 1^{st} quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3-1}\, dz \right| \le \frac{6\pi}{7}$$



$$L = \frac{2\pi r}{4} = \frac{2\pi(2)}{4} = \pi$$

* M : Max. value of func^n on that curve *

take min value of denom

$$f(z) = \frac{z+4}{z^3-1}$$

$$f_1(z) = |z + 4| \le |\overset{2}{z}| + 4 \le 6$$

$$f_2(z) = \frac{1}{|z^3-1|} \le \boxed{\frac{1}{|z|^3-1}} = \frac{1}{7}$$

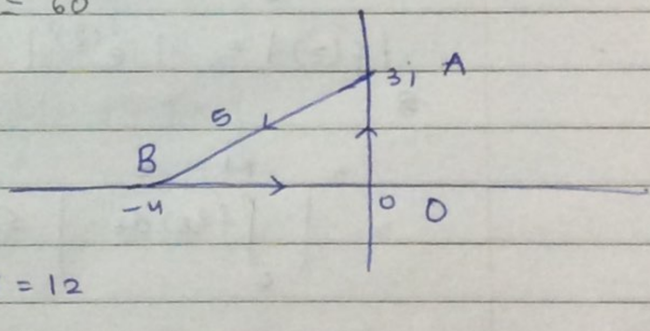
$$\left\{ \cancel{\not\le} \frac{1}{|z|^3+1} \right\}$$

$$\Rightarrow M = \frac{6}{7}$$

$$\Rightarrow \left| \int_C f(z)\, dz \right| \le ML = \frac{6\pi}{7} \quad \text{Hence Proved.}$$

30-8-17

Ex. (Churchill & Brown) Show that if C is the boundary of the triangle with vertices at the points 0, $3i$, and -4 oriented in the counterclockwise direction, then $\left| \int_C (e^z - \bar{z})\, dz \right| \le 60$



dir^n also has a significance
Here, integral also depends on C.

$$I = \int_C (e^z - \bar{z})\, dz \qquad L = 3 + 4 + 5 = 12$$

length of curve $C = 12$

$|f(z)| = |e^z - \bar{z}|$

Using Δ inequality

can be max : 4

$$|e^z - \bar{z}| \le |e^z| + |\bar{z}| = |e^z| + |z|$$

$$|e^z| = |e^x||e^{iy}| = |e^x| \doteq 1 \quad (\text{when } x = 0)$$

$|e^{iy}|$ always $= 1$

$\Rightarrow \quad M = 1+4 = 5$

Using M-L Inequality,

$$\left| \int_C (e^z - \bar{z})\, dz \right| \leq 5 \times 12 = 60$$

Hence Proved

Ques Find the upper bound for the absolute value of the integral

$$I = \int_C e^{(\bar{z})^2}\, dz \quad ;$$

$C : |z| = 1$ traversed in anti-clockwise direction. (Positive dir)

~~Using~~

We can also use:

length of the curve C : $\boxed{L = \int_a^b |z'(t)|\, dt = \int_a^b \left(\sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2}\right) dt \quad a \leq t \leq b}$

Using parametric representations

$x = \cos\theta \qquad y = \sin\theta$

$$L = \int_0^{2\pi} \left(\sqrt{\sin^2\theta + \cos^2\theta}\right) d\theta = 2\pi$$

$$|f(z)| = |e^{(\bar{z})^2}| = |e^{(x-iy)^2}| = |e^{x^2+y^2}| \cdot |e^{-i2xy}|$$

$$= |e^{x^2+y^2}| = e$$

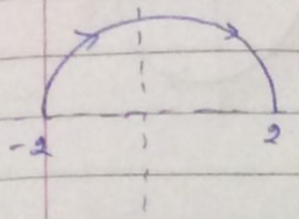
$\Rightarrow \quad M = e$

$$\Rightarrow \left| \int_C f(z)\, dz \right| \leq 2\pi e$$

Ques. Evaluate the following integral

$$I = \int_C \frac{2z+3}{z}\, dz \quad , \text{ where } C \text{ is}$$

ⓐ upper half of the circle $|z| = 2$ in the clockwise dir^n.



$x = 2\cos\theta \qquad y = 2\sin\theta \qquad \pi \le \theta \le 0$

$z(\theta) = 2e^{i\theta}$

$z'(\theta) = 2ie^{i\theta}$

$$\cancel{I = \int_\pi^0 |2ie^{i\theta}|\, d\theta = 2\pi}$$

$$\cancel{I = \int z\, dz = \int 2e^{i\theta}\, d(2e^{i\theta}) = \int 2e^{i\theta}}$$

$$I = \int \frac{2(2e^{i\theta})+3}{2e^{i\theta}}\, d\left(2e^{i\theta}\right) = \int \frac{4e^{i\theta}+3}{2e^{i\theta}} \cdot i\, 2e^{i\theta}\, d\theta$$

$$= i \int (4e^{i\theta} + 3)\, d\theta$$

or $\quad I = \int_C f(z)\, dz = \int_a^b f[z(t)]\, z'(t)\, dt$

$$\Rightarrow I = i \int_\pi^0 (4e^{i\theta}+3)\, d\theta = i \left[\frac{4e^{i\theta}}{i} + 3\theta \right]_\pi^0$$

$$= i \left[\frac{4}{i} - \frac{4(-1)}{i} - 3\pi \right]$$

$$= 8 - 3\pi i$$

ⓑ upper half of the circle $|z| = 2$ in anti-clockwise dir^n.

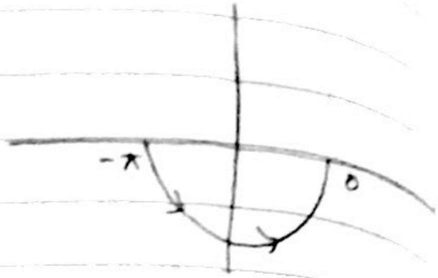
Only limit will change

$$I = i \left[\frac{4e^{i\theta}}{i} + 3\theta \right]_0^\pi = i \left[-\frac{4}{i} - \frac{4}{i} + 3\pi \right]$$

$$= -8 + 3\pi i$$

ⓒ Lower half of circle $|z|=2$ in anti-clockwise dirⁿ
ⓓ " " " " " " clockwise dirⁿ

ⓒ $$I = i\int_{-\pi}^{0}(4e^{i\theta}+3)\,d\theta$$



$$= i\left[\frac{4e^{i\theta}}{i}+3\theta\right]_{-\pi}^{0}$$

$$= i\left[\frac{+4}{i}+\frac{4}{i}+3\pi\right]$$

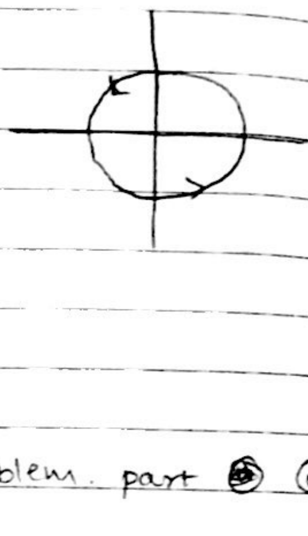
$$= +8+3\pi i$$

ⓓ $$I = i\left[\frac{4e^{i\theta}}{i}+3\theta\right]_{0}^{-\pi}$$

$$= -8-3\pi i$$

ⓔ In the circle $|z|=2$ in the anticlockwise dirⁿ

$$I = i\int_{0}^{2\pi}(4e^{i\theta}+3)\,d\theta$$



$$= i\left[\frac{4e^{i\theta}}{i}+3\theta\right]_{0}^{2\pi}$$

$$= i\left[\frac{4}{i}-\frac{4}{i}+6\pi\right] = 6\pi i$$

Ques. $I = \int z^2\,dz$, repeat the above problem part ⓔ

$$z(t) = 2e^{it}$$

$$z'(t) = 2ie^{it}$$

$$I = \int 4e^{i2t}\,d(2e^{it}) = \int 4e^{i2t}\cdot 2i\,e^{it}\,dt$$

$$= 8i\int_{0}^{2\pi}e^{i3t}\,dt$$

$$= \frac{8i}{3i}\left[e^{i3t}\right]_{0}^{2\pi} = 0$$

* Even if we take $f(z) = z^3$, we will get $I = 0$.
$(z^4, z^5 \ldots z^n)$

→ if $f(z) = z$
$\Rightarrow e^{i2t}$ will come $\Rightarrow I = 0$ again

$\Rightarrow$ $f(z) = z, z^2, z^3, \ldots z^n$: $I = 0$ in closed loop

•) Polynomial, Rational func^{n}s are always analytic $(Q \neq 0)$

→ If $f(z) = 1/z$

$$I = \int \frac{1}{2} e^{-it} \cdot 2ie^{it} dt = i\int_0^{2\pi} dt = 2\pi i$$
↳ constant func^n.

→ $f(z) = z^{-2}$

$$I = \int \frac{1}{\cancel{4}\,2} e^{-it} \cdot 2i\, dt = \frac{i}{2}\int_0^{2\pi} e^{-it} dt = \frac{i}{2}\left(\frac{e^{-it}}{-i}\right)_0^{2\pi}$$
$= 0$

→ $f(z) = z^{-3}$, then again $I = 0$.

⟶

$$I = \oint_C z^n dz = \begin{cases} 2\pi i & \text{when } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

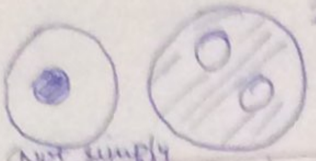
$C : |z| = r$ is traversed in the counter clockwise dir^n.

Proof:
$z(t) = re^{it}$
$z'(t) = ire^{it}$

$$I = \int z^n dz = \int (re^{it})^n \cdot d(re^{it})$$

$$= \int r^n e^{int} \cdot ri\, e^{it} dt = i\int_0^{2\pi} r^{n+1} e^{i(n+1)t} dt$$

$$= i r^{n+1} \left[\frac{e^{i(n+1)t}}{i(n+1)}\right]_0^{2\pi}$$



when $n \neq -1$, $I = 0$

when $n = -1$: $I = i r^0 \int_0^{2\pi} d\theta = 2\pi i$

Hence proved

4/9/17 Basic Properties of Complex Integrals :

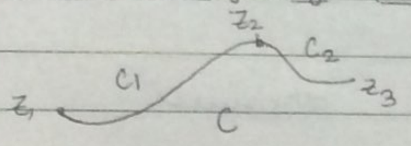
1.) Linearity : If the integrals of f_1 and f_2 over path C exist, so does the integral of $kf_1 + f_2$ over path C.

$$\int_C (kf_1 + f_2)(z)\, dz = k \int_C f_1(z)\, dz + \int_C f_2(z)\, dz$$

2.) Sense Reversal : Integrals over the same path from z_0 to z_1 and from z_1 to z_0 introduces a '$-$' sign as :

$$\int_{z_0}^{z_1} f(z)\, dz = -\int_{z_1}^{z_0} f(z)\, dz$$

3.) Partitioning of Path



$C = C_1 + C_2$

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

(Observe the last second ~~eg~~ example)

Q. $$I = \oint_C (z - z_0)^n\, dz$$

C : $|z - z_0| = r$ traversed in anti-clockwise dir^n.

(Notice that there is only shift in centre) domain : open & connected region

→ simply domain (can draw a circle anywhere which lies in the domain)

$1<|z|<2$

↳ Doubly connected (can't draw circle)

$$z - z_0 = r e^{i\theta}$$

$$z(\theta) = z_0 + r e^{i\theta} \qquad z'(\theta) = r i e^{i\theta}$$

$$I = \int_0^{2\pi} (r e^{i\theta})^n \cdot r i e^{i\theta} = r^{n+1} i \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi}$$

⇒ By shifting centre, the value of integral does not change.

$$I = \begin{cases} 2\pi i & n = -1 \\ 0 & \text{Otherwise} \end{cases}$$

Simply Connected Domain :

In a complex plane, a domain (open and connected set) is said to be simply connected if every simple closed curve in D encloses only points in D . Eg : $|z| \le 3$

↳ not intersect itself

A domain that is not simply connected is called Multiply connected

Eg. $3 < |z| < 5$ (annulus / annular region)

Green's Theorem :

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let D be the region bounded by C.

If L and M(x,y) are func^ns of (x,y) defined on the open ~~regular~~ region containing D and ~~the~~ have continuous partial derivative, then

Learn.

$$\boxed{\oint_C L\,dx + M\,dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\,dy}$$

$$\begin{cases} z(t) & \text{diff} \\ z'(t) & \text{cont.} \\ z'(t) \neq 0 & a < t \le b \end{cases}$$

{ because integration of analytic func^n only depends upon end points, its ~~value~~ integration over a closed loop is always zero }

Cauchy's Theorem

Let $f(z)$ be analytic in a simply connected domain D and $f'(z)$ is continuous in D. Then, for every simple closed curve C in D

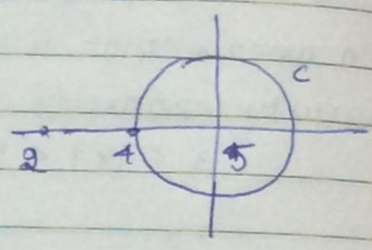
$$\oint_C f(z)\,dz = 0$$

Ex (a) $\oint_{|z|=1} z^2\,dz$ $\qquad\qquad$ $\oint_{|z|=2} e^z\,dz$ ↳ also zero

$\downarrow$
did it previously
(both are analytic func^{n}s)

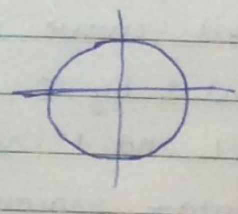
(b) $\oint \frac{z^2+1}{z-2}$, $\quad (|z-5| = 1$, anti-clockwise)

non-analytic on $z = 2$
~~But we have to check only on the curves.~~
$|z-5| = 1$ doesn't include $z = 2$
$\Rightarrow$ Given func^n is analytic within given close curve.
$\Rightarrow \boxed{I = 0}$



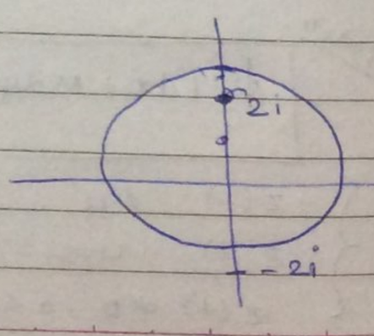
(c) $I = \oint \frac{\cos z}{(z^2+4)}$, $\quad$ 1.) $C : |z| = 1$
2.) $C : |z-i| = 2$

1.) $|z| = 1$



$z = \pm 2i$: not analytic
not included in curve
$I = 0$

2.) $|z-i| = 2$
At one point, it is not defined (singularity)
We can't comment on I.



Proof: (cauchy's theorem)

$$\int_C f(z)\,dz = \int (u+iv)(dx+idy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

① ②

If $f(z)$ is analytic $\Rightarrow$ $f(z)$ is also diff. $\Rightarrow$ ① & ② diff
$\Rightarrow$ u & v diff
$\Rightarrow$ partial derivative of u & v exist.

Using Green's theorem,

$$\iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$= 0$ (Using C-R eq^n)

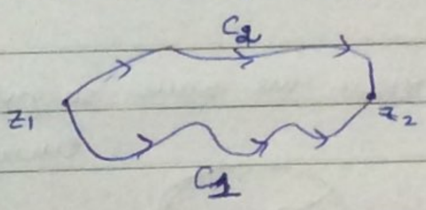
Hence Proved

4-9-17

Independent of Path

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Proof:- Let z_1 and z_2 be any 2 points in domain D. Consider 2 paths C_1 & C_2 in D from z_1 to z_2 without having any common points denoted by C_1 & C_2



Denote $C_2^* = -C_2$
(the path C_2 with orientation reversed)

Integrating then from z_1 over C_1 to z_2 & z_2 over C_2^* to z_1.
This will form a simple closed curve $C = C_1 + C_2^*$
Then applying cauchy's theorem on C for analytic func^n $f(z)$, we get

$$\int_C f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Hence Proved

(This result is also used in cauchy proof as we get same sol^n in diff paths) for now analytic $\equiv$ holomorphic

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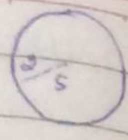
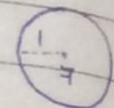
Ex (i) $\int_C \frac{z^2+2}{z^3}\, dz$ $\qquad$ $C : |z-5| = 2$



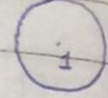
$f(z)$: analytic in given region

$\Rightarrow I = 0$

(ii) $\int_C \frac{\sin z}{(z-1)}\, dz = 0$ $\qquad$ $C : |z-7| = 1$



(iii) $\int_C \tan z \, dz = 0$ $\qquad$ $C : |z| = 1$



$\tan z = \frac{\sin z}{\cos z}$: Not defined at $\frac{\pi}{2}, \frac{3\pi}{2}$ ...... $\rightarrow$ point of singularity

Singularities / Point of singularity

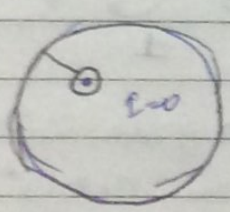
Generally, points where func^n is not analytic (not defined)

① $f(z) = \frac{z^2+2}{z^3}$: $z = 0$: singularity

② $\frac{\sin z}{z-1} = f(z)$ $\qquad$ $z = 1$: "

③ $\tan z = f(z)$ $\qquad$ $z = \frac{(2n+1)\pi}{2}$: "

→ When we have singular point, we covered the curve s.t. everywhere outside smaller circle $I = 0$. So, we



Q. Find the singular point :

i) $f_0(z) = \frac{1}{z}$:

ii.) $f_1(z) = \bar{z}$

(iii) $f_2(z) = \mathrm{Re}\,\bar{z}$

(iv.) $f_3(z) = |z|^2$