CSE237 - Introduction to the Theory of Computation	January 24, 2002
Problem Set 1 - Solutions	

Due January 31, 2002.

1. Prove by induction that for any natural number n, $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

   Proof:
   The proof is in two parts: the basis and the inductive step; we prove each in turn.

   BASIS: If $n = 0, n^3 + (n+1)^3 + (n+2)^3 = 0 + 1 + 8 = 9$, which is divisible by 9. Thus, it is true when n=0.

   INDUCTION: Now, suppose it is true for some $n \geq 0$. We need to show that it is true for n+1. We have:

$$\begin{aligned}(n+1)^3 + (n+2)^3 + (n+3)^3 &= (n+1)^3 + (n+2)^3 + (n+3)(n^2 + 6n + 9) \\ &= (n+1)^3 + (n+2)^3 + n^3 + 9n^2 + 36n + 27 \\ &= n^3 + (n+1)^3 + (n+2)^3 + 9(n^2 + 4n + 3)\end{aligned}$$

   Since $9|n^3 + (n+1)^3 + (n+2)^3$ (by assumption) and $9|9(n^2 + 4n + 3)$, we know that it is true for n+1.

   From the basis and the inductive step, we know that $n^3 + (n+1)^3 + (n+2)^3$ is true for any natural number n.

2. Find all of the solutions to the equation $x^2 + 1 = y^2$ over the integers. Prove that there are no other solutions. (The phrase "over the integers" means that x and y are only allowed to take integer values.)

   Approach1:

$$x^2 + 1 = y^2 \Rightarrow (y+x)(y-x) = 1$$

   Because, x and y are integers, $y + x$ and $y - x$ are also integers. Observe that $y + x$ and $y - x$ cannot be zero. So we have:

$$|y + x| \geq 1$$

$$|y - x| \geq 1$$

   However, both of $|y + x|$ and $|y - x|$ cannot be greater than 1, otherwise, either the product is greater than 1 or one of them is not an integer. Thus, we know:

$$|y + x| = 1$$

$$|y - x| = 1$$

   To ensure $(y + x)(y - x) = 1$, two cases arise:

   Case 1:

$$y + x = 1$$

$$y - x = 1$$

We have the solution $(x, y) = (0, 1)$.

Case 2:

$$y + x = -1$$

$$y - x = -1$$

We have the solution $(x, y) = (0, -1)$.

Based on our deduction, they are the only solutions.

Approach2:

$$x^2 + 1 = y^2 \Rightarrow x^2 < y^2$$

Therefore, $|y|$ must be greater than $|x|$. Let $|y| = |x| + d, d \geq 1$. We have:

$$x^2 + 1 = x^2 + 2|x|d + d^2$$

$\Rightarrow 1 = 2|x|d + d^2 \geq d^2 (|x|d \geq 0) \Rightarrow |d| \leq 1 \Rightarrow d = 0, 1$

We can discuss from two cases:

Case 1:

$d = 0 \Rightarrow 2|x|d + d^2 = 0 \neq 1 \Rightarrow$ No solution

Case 2:

$$d = 1 \Rightarrow 2|x|d + d^2 = 2|x| + 1 = 1 \Rightarrow x = 0 \Rightarrow y = 1, -1$$

So the only solutions are (0, 1) and (0, -1).

3. Let w be a string in $\{0, 1\}^*$. How many ways are there to write w as xy, where x and y are also strings in $\{0, 1\}^*$?

   Let $|w| = n$. To write w as xy, x must be a prefix of w. For each possible x, y can be derived accordingly. Since there are totally $n + 1$ prefixes of w (including ϵ), w can be written as xy in $|w| + 1$ ways.

4. (a) Let L_1 and L_2 be two finite languages over the alphabet $\{0, 1\}$. Define $L_1 \circ L_2 = \{xy | x \in L_1, y \in L_2\}$. Prove an upper bound on the size of $L_1 \circ L_2$ in terms of $|L_1|$ and $|L_2|$ and show that your upper bound cannot be improved.

   Proof:

   From the definition of $L_1 \circ L_2$, we know that for each $x \in L_1$, if we take any string $y \in L_2$, xy will be in $L_1 \circ L_2$. So for each $x \in L_1$, we can obtain $|L_2|$ strings belonging to $L_1 \circ L_2$. There are totally $|L_1|$ possible x's. Therefore, totally, we can obtain $|L_1| \times |L_2|$ strings belonging to $L_1 \circ L_2$. Notice that we have not missed any strings of $L_1 \circ L_2$. As a result, $|L_1 \circ L_2| \leq |L_1 \times |L_2|$.

If there is no duplication among those obtained $|L_1| \times |L_2|$ strings, the size of $L_1 \circ L_2$ will be equal to the upper bound. For example, if $L_1 = L_2 = \{0, 1\}$, $L_1 \circ L_2 = \{00, 01, 11, 11\}$, whose size is 4 ($= |L_1| \times |L_2|$). So the upper bound cannot be improved.

(b) Show, also, that there is a pair of languages A and B so that $|A \circ B|$ is strictly less than the upper bound you have established.

Let $A = \{0, 01\}$, $B = \{10, 0\}$. Then $A \circ B = \{010, 00, 0110\}$. Obviously, $|A \circ B| = 3 < 4 = |A| \times |B|$.