Chapter 7

Integer Linear Programming

"The key is not to prioritize what's on your schedule, but to schedule your priorities."

– R. Covey

PREVIEW

In this chapter, Gomory's cutting plane method, and Branch and Bound method have been discussed for solving an extension of LP model called linear integer LP model. In a linear integer LP model one or more of the variables must be integer due to certain managerial considerations.

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- understand the limitations of simplex method in deriving integer solution to linear programming problems.
- apply cutting plane methods to obtain optimal integer solution value of variables in an LP problem.
- apply Branch and Bound method to solve integer LP problems.
- appreciate application of integer LP problem in several areas of managerial decision-making.

CHAPTER OUTLINE

7.1 INTRODUCTION

In linear programming, each decision variable, slack and/or surplus variable is allowed to take any discrete or fractional value. However, there are certain real-life problems in which the fractional value of the decision variables has no significance. For example, it does not make sense to say that 1.5 men will be working on a project or 1.6 machines will be used in a workshop. The integer solution to a problem can, however, be obtained by rounding off the optimum value of the variables to the nearest integer value. This approach can be easy in terms of economy of effort, time, and the cost that might be required to derive an integer solution. This solution, however, may not satisfy all the given constraints. Secondly, the value of the objective function so obtained may not be the optimal value. All such difficulties can be avoided if the given problem, where an integer solution is required, is solved by integer programming techniques.

Integer LP problems are those in which some or all of the variables are restricted to integer (or discrete) values. An integer LP problem has important applications. Capital budgeting, construction scheduling, plant location and size, routing and shipping schedule, batch size, capacity expansion, fixed charge, etc., are few problems that demonstrate the areas of application of integer programming.

7.2 TYPES OF INTEGER PROGRAMMING PROBLEMS

Linear integer programming problems can be classified into three categories:

(i) *Pure (all) integer programming problems* in which all decision variables are restricted to integer values.

(ii) *Mixed integer programming problems* in which some, but not all, of the decision variables are restricted to integer values.

(iii) *Zero-one integer programming problems* in which all decision variables are restricted to integer values of either 0 or 1.

The broader classification of integer LP problems and their solution methods are summarized in Fig. 7.1 In this chapter, we shall discuss two methods: (i) Gomory's cutting plane method and (ii) Branch and Bound method, for solving integer programming problems.

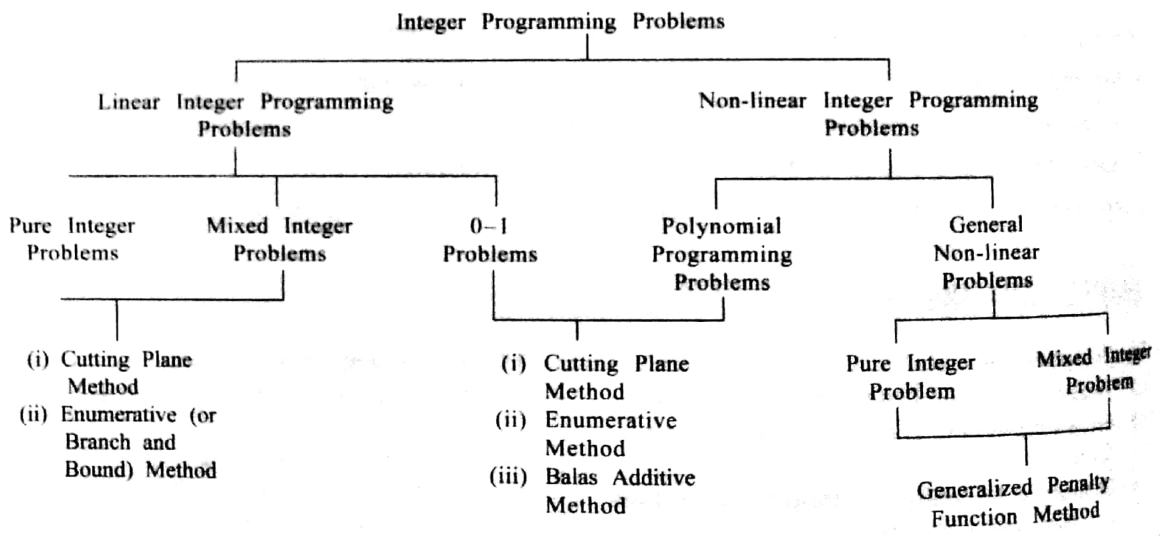

Fig. 7.1 Classification of Integer LP Problems and their Solution Methods

The pure integer linear programming problem in its standard form can be stated as follows:

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the constraints

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m
\end{aligned}
$$

and $x_1, x_2, \ldots, x_n \geq 0$ and are integers.

7.3 ENUMERATION AND CUTTING PLANE SOLUTION CONCEPT

The cutting-plane method to solve integer LP problems was developed by R.E. Gomory in 1956. This method is based on creating a sequence of linear inequalities called *cuts*. Such a *cut* reduces a part of the feasible region of the given LP problem, leaving out a feasible region of the integer LP problem. The hyperplane boundary of a cut is called the *cutting plane.*

A **cut** is the linear constraint added to the given LP problem constraints.

Illustration Consider the following linear integer programming (LIP) problem

Maximize $Z = 14x_1 + 16x_2$

subject to the constraints

(i) $4x_1 + 3x_2 \leq 12$, (ii) $6x_1 + 8x_2 \leq 24$

and $x_1, x_2 \geq 0$ and are integers.

Relaxing the integer requirement, the problem is solved graphically as shown in Fig. 7.2. The optimal solution to this LP problem is: $x_1 = 1.71, x_2 = 1.71$ and Max $Z = 51.42$. This solution does not satisfy the integer requirement of variables x_1 and x_2.

Rounding off this solution to $x_1 = 2, x_2 = 2$ does not satisfy both the constraints and therefore, the solution is infeasible. The dots in Fig. 7.2, also referred to as *lattice points*, represent all of the integer solutions that lie within the feasible solution space of the LP problem. However, it is difficult to evaluate every such point in order to determine the value of the objective function.

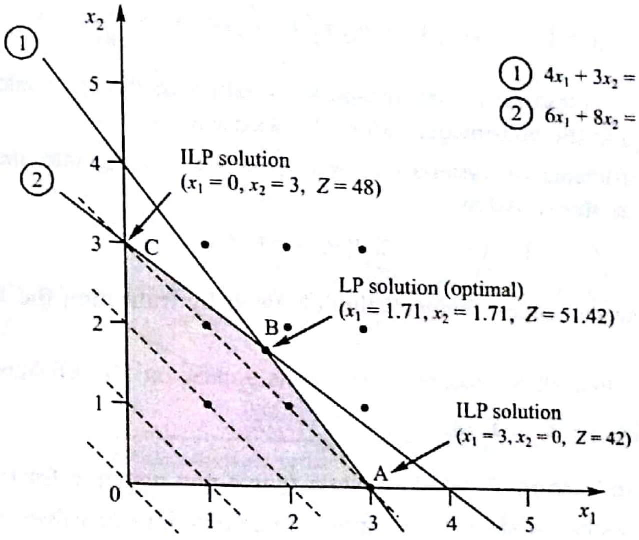

Fig. 7.2 Concept of Cutting Plane

In Fig. 7.2, it may be noted that the optimal lattice point C, lies at the corner of the solution space OABC, obtained by cutting away the small portion above the dotted line. This suggests a solution procedure that successively reduces the feasible solution space until an integer-valued corner is found.

The optimal integer solution is: $x_1 = 0, x_2 = 3$ and Max $Z = 48$. The lattice point, C is not even adjacent to the most desirable LP problem solution corner, B.

Hyperplane boundary of a cut is called the cutting plane.

Remark Reducing the feasible region by adding extra constraints (cut) can never give an improved objective function value. If Z_{IP} represents the maximum value of objective function in an ILP problem and Z_{LP} the maximum value of objective function in an LP problem, then $Z_{IP} \leq Z_{LP}$.

7.4 GOMORY'S ALL INTEGER CUTTING PLANE METHOD

In this section, a procedure called *Gomory's all-integer algorithm* will be discussed for generating 'cuts' (additional linear constraints) so as to ensure an integer solution to the given LP problem in a finite number of steps. Gomory's algorithm has the following properties.

(i) Additional linear constraints never cutoff that portion of the original feasible solution space that contains a feasible integer solution to the original problem.

(ii) Each new additional constraint (or hyperplane) cuts off the current non-integer optimal solution to the linear programming problem.

7.4.1 Method for Constructing Additional Constraint (Cut)

Gomory's method begins by solving an LP problem ignoring the integer value requirement of the decision variables. If the solution so obtained is an integer, i.e. all variables in the '$\mathbf{x_B}$'-*column* (also called basis) of the simplex table assume non-negative integer values, the current solution is the optimal solution to the given ILP problem. However, if some of the basic variables do not have non-negative integer value, an additional linear constraint called the *Gomory constraint* (or cut) is generated. After having generated a linear constraint (or cutting plane), it is added to the bottom of the optimal simplex table. The new problem is then solved by using the dual simplex method. If the optimal solution, so obtained, is again a non-integer, then another cutting plane is generated. The procedure is repeated until all basic variables assume non-negative integer values.

Procedure

In the optimal solution simplex table, select a row called *source row* for which basic variable is non-integer. Then to develop a 'cut', consider only fractional part of the coefficients in source row. Such a cut is also referred to as *fractional cut*.

Suppose the basic variable x_r has the largest fractional value among all basic variables required to assume integer value. Then the rth constraint equation (row) from the simplex table can be rewritten as

$$x_{Br}\ (= b_r) = 1 \cdot x_r + (a_{r1}\, x_1 + a_{r2}\, x_2 + \ldots) = x_r + \sum_{j \neq r} a_{rj}\, x_j \qquad (1)$$

where x_j ($j = 1, 2, 3, \ldots$) represents all the non-basic variables in the rth constraint (row), except the variables x_r and b_r ($= x_{Br}$) is the non-integer value of variable x_r.

Decomposing the coefficients of variables x_j and x_r as well as x_{Br} into integer and non-negative fractional parts in Eq. (1) as shown below:

$$[x_{Br}] + f_r = (1 + 0)\, x_r + \sum_{j \neq r} \{[a_{rj}] + f_{rj}\}\, x_j \qquad (2)$$

where $[x_{Br}]$ and $[a_{rj}]$ denote the largest integer value obtained by truncating the fractional part from x_{Br} and a_{rj} respectively.

Rearranging Eq. (2) so that all the integer coefficients appear on the left-hand side, we get

$$f_r + \{[x_{Br}] - x_r - \sum_{j \neq r} [a_{rj}]\, x_j\} = \sum_{j \neq r} f_{rj}\, x_j \qquad (3)$$

where f_r is strictly a positive fraction ($0 < f_r < 1$) while f_{rj} is a non-negative fraction ($0 \le f_{rj} \le 1$).

Since all the variables (including slacks) are required to assume integer values, the terms in the brackets on the left-hand side as well as on the right-hand side must be non-negative numbers. Since the left-hand side in Eq. (3) is f_r plus a non-negative number, we may write it in the form of the following inequalities (4)

$$f_r \le \sum_{j \neq r} f_{rj}\, x_j \qquad (5)$$

or

$$\sum_{j \neq r} f_{rj}\, x_j = f_r + s_g \quad \text{or} \quad -f_r = s_g - \sum_{j \neq r} f_{rj}\, x_j$$

where s_g is a non-negative slack variable and is also called *Gomory slack variable*.

Equation (5) represents *Gomory's cutting plane constraint*. When this new constraint is added to the bottom of optimal solution simplex table, it would create an additional row in the table, along with a column for the new variable s_g.

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7.4.2 Steps of Gomory's All Integer Programming Algorithm

An iterative procedure for the solution of an all integer programming problem by Gomory's cutting plane method can be summarized in the following steps.

Step 1: Initialization Formulate the standard integer LP problem. If there are any non-integer coefficients in the constraint equations, convert them into integer coefficients. Solve the problem by the simplex method, ignoring the integer value requirement of the variables.

Step 2: Test the optimality

(a) Examine the optimal solution. If all basic variables (i.e. $x_{Bi} = b_i \geq 0$) have integer values, then the integer optimal solution has been obtained and the procedure is terminated.

(b) If one or more basic variables with integer value requirement have non-integer solution values, then go to Step 3.

Step 3: Generate cutting plane Choose a row r corresponding to a variable x_r that has the largest fractional value f_r and follow the procedure to develop a 'cut' (a Gomory constraint) as explained in Eqn. (5):

$$f_r = s_g - \sum_{j \neq r} f_{rj}\, x_j, \quad \text{where} \quad 0 \leq f_{rj} < 1 \quad \text{and} \quad 0 < f_r < 1$$

If there are more than one variables with the same largest fraction, then choose the one that has the smallest profit/unit coefficient in the objective function of maximization LP problem or the largest cost/unit coefficient in the objective function of minimization LP problem.

Step 4: Obtain the new solution Add this additional constraint (cut) generated in Step 3 to the bottom of the optimal simplex table. Find a new optimal solution by using the *dual simplex method*, i.e. choose a variable that is to be entered into the new solution having the smallest ratio: $\{(c_j - z_j)/y_{ij} ;\ y_{ij} < 0\}$ and return to Step 2. The process is repeated until all basic variables with integer value requirement assume non-negative integer values.

The procedure for solving an ILP problem is summarized in a flow chart shown in Fig. 7.3.

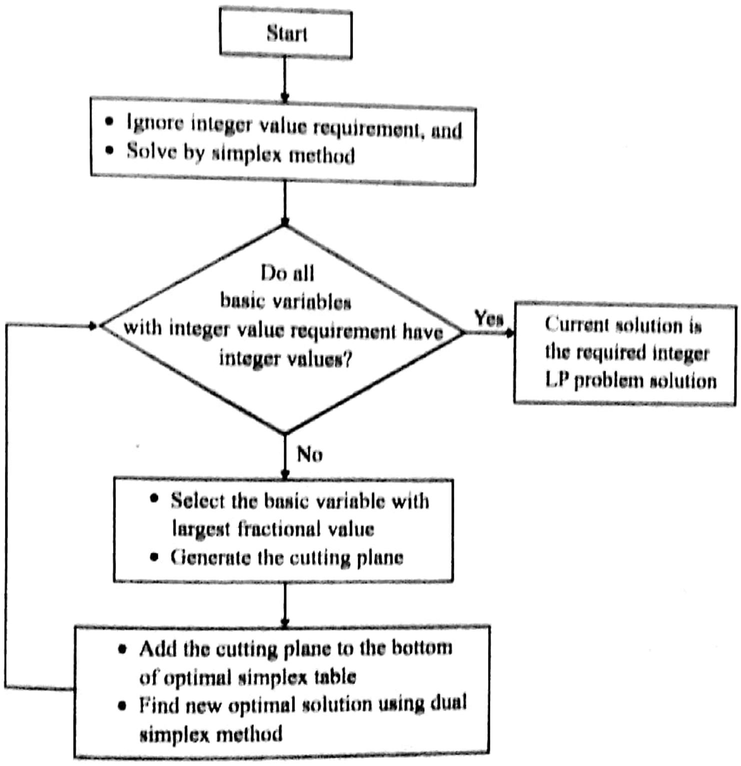

Fig. 7.3 Flow Chart for Solving Integer LP Problem

Example 7.1 Solve the following Integer LP problem using Gomory's cutting plane method.

Maximize $Z = x_1 + x_2$

subject to the constraints

(i) $3x_1 + 2x_2 \leq 5$, (ii) $x_2 \leq 2$

$x_1, x_2 \geq 0$ and are integers.

Solution Step 1: Obtain the optimal solution to the LP problem ignoring the integer value restriction by the simplex method.

	$c_j \rightarrow$		1	1	0	0
1	x_1	1/3	1	0	1/3	
1	x_2	2	0	1	0	
$Z = 7/2$	$c_j - z_j$		0	0	$-1/3$	

Table 7.1 Optimal Non-integer Solution

In Table 7.1, since all $c_j - z_j \leq 0$, the optimal solution of LP problem is: $x_1 = 1/3$, $x_2 = 2$ and Max $Z = 7/2$.

Step 2: In the current optimal solution, shown in Table 7.1 all basic variables in the basis (x_B-column) did not assume integer value. Thus solution is not desirable. To obtain an optimal solution satisfying integer value requirement, go to step 3.

Step 3: Since x_1 is the only basic variable whose value is a non-negative fractional value, therefore consider first row (x_1-row) as *source row* in Table 7.1 to generate Gomory cut as follows:

$$\frac{1}{3} = x_1 + 0.x_2 + \frac{1}{3}s_1 - \frac{2}{3}s_2 \qquad (x_1\text{-source row})$$

The factoring of numbers (integer plus fractional) in the x_1-source row gives

$$\left(0+\frac{1}{3}\right) = (1 + 0)\, x_1 + \left(0+\frac{1}{3}\right) s_1 + \left(-1+\frac{1}{3}\right) s_2$$

Each of the non-integer coefficients is factored into integer and fractional parts in such a manner that the fractional part is strictly positive.

Rearranging all of the integer coefficients on the left-hand side, we get

$$\frac{1}{3} + (s_2 - x_1) = \frac{1}{3}s_1 + \frac{1}{3}s_2$$

Since value of variables x_1 and s_2 is assumed to be non-negative integer, left-hand side must satisfy

$$\frac{1}{3} \leq \frac{1}{3}s_1 + \frac{1}{3}s_2 \qquad \text{(Ref. Eq. 4)}$$

$$\frac{1}{3} + s_{g_1} = \frac{1}{3}s_1 + \frac{1}{3}s_2 \quad \text{or} \quad s_{g_1} - \frac{1}{3}s_1 - \frac{1}{3}s_2 = -\frac{1}{3} \qquad \text{(Cut I)}$$

where s_{g_1} is the new non-negative (integer) slack variable.

Adding this equation (also called Gomory cut) at the bottom of Table 7.1, the new values so obtained is shown in Table 7.2.

	$c_j \rightarrow$		1	1	0	0	0
1	x_1	1/3	1	0	1/3	$-2/3$	0
1	x_2	2	0	1	0	10	0
0	s_{g_1}	$-1/3$	0	0	$-1/3$	$-1/3$	1 →
$Z = 7/2$	$c_j - z_j$		0	0	$-1/3$	$-1/3$	0
	Ratio: Min $(c_j - z_j)/y_{3j}$ (< 0)		–	–	1 ↑	1	–

Table 7.2 Optimal but Infeasible Solution

Step 4: Since the solution shown in Table 7.2 is infeasible, apply the dual simplex method to find a feasible as well as an optimal solution. The key row and key column are marked in Table 7.2. The new solution is obtained by applying the following row operations.

$$R_3(\text{new}) \rightarrow R_3(\text{old}) \times -3; \quad R_1(\text{new}) \rightarrow R_1(\text{old}) - (1/3)\, R_3(\text{new})$$

The new solution is shown in Table 7.3.

		$c_j \rightarrow$	1	1	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value b $(= x_B)$	x_1	x_2	s_1	s_2	s_{g1}
1	x_1	0	1	0	0	–1	1
1	x_2	2	0	1	0	1	0
0	s_{g_1}	1	0	0	1	1	–3
$Z = 2$		$c_j - z_j$	0	0	0	0	–1

Table 7.3 Optimal Integer Solution

Since all $c_j - z_j \leq 0$ and value of basic variables shown in x_B-column of Table 7.3 is integer, the solution: $x_1 = 0$, $x_2 = 2$, $s_{g_1} = 1$ and Max $Z = 2$, is an optimal basic feasible solution of the given ILP problem.

Example 7.2 Solve the following Integer LP problem using the cutting plane method.

Maximize $Z = 2x_1 + 20x_2 - 10x_3$

subject to the constraints

(i) $2x_1 + 20x_2 + 4x_3 \leq 15$, (ii) $6x_1 + 20x_2 + 4x_3 = 20$

and $x_1, x_2, x_3 \geq 0$ and are integers.

Also show that it is not possible to obtain a feasible integer solution by simple rounding off method.

Solution Adding slack variable s_1 in the first constraint and artificial variable in the second constraint, the LP problem is stated in the standard form as:

Maximize $Z = 2x_1 + 20x_2 - 10x_3 + 0s_1 - MA_1$

subject to the constraints

(i) $2x_1 + 20x_2 + 4x_3 + s_1 = 15$, (ii) $6x_1 + 20x_2 + 4x_3 + A_1 = 20$

and $x_1, x_2, x_3, s_1, A_1 \geq 0$ and are integers.

The optimal solution of the LP problem, ignoring the integer value requirement using the simplex method is shown in Table 7.4

		$c_j \rightarrow$	2	20	–10	0
Basic Variables Coefficient c_B	Variables in Basis B	Basic Variables Value b $(= x_B)$	x_1	x_2	x_3	s_1
20	x_2	5/8	0	1	1/5	3/40
2	x_1	5/4	1	0	0	–1/4
$Z = 15$		$c_j - z_j$	0	0	–14	–1

Table 7.4 Optimal Non-integer Solution

The non-integer optimal solution shown in Table 7.4 is: $x_1 = 5/4$, $x_2 = 5/8$, $x_3 = 0$ and Max $Z = 15$.

To obtain an optimal solution satisfying integer value requirement, we proceed to construct Gomory's constraint. In this solution, the value of both basic variables x_1 and x_2 are non-integer. Since the fractional part of the value of basic variable $x_2 = (0 + 5/8)$ is more than that of basic variable x_1 $(= 1 + 1/4)$, the x_2-row is selected for constructing Gomary cut as follows:

$$\frac{5}{8} = 0 \,.\, x_1 + x_2 + \frac{1}{5}\, x_3 + \frac{3}{40}\, s_1 \quad (x_2\text{-source row})$$

The factoring of the x_2-source row yields

$$\left(0 + \frac{5}{8}\right) = (1 + 0)\, x_2 + \left(0 + \frac{1}{5}\right) x_3 + \left(0 + \frac{3}{40}\right) s_1$$

$$\frac{5}{8} - x_2 = \frac{1}{5} x_3 + \frac{3}{40} s_1 \quad \text{or} \quad \frac{5}{8} \leq \frac{1}{5} x_3 + \frac{3}{40} s_1$$

On adding a slack variable s_{g_1}, the Gomory's fractional cut becomes:

$$\frac{5}{8} + s_{g_1} = \frac{1}{5}x_3 + \frac{3}{40}s_1 \quad \text{or} \quad s_{g_1} - \frac{1}{5}x_3 - \frac{3}{40}s_1 = -\frac{5}{8} \quad \text{(Cut I)}$$

Adding this additional constraint at the bottom of optimal simplex Table 7.4, the new values so obtained are shown in Table 7.5.

Iteration 1: Remove the variable s_{g_1} from the basis and enter variable s_1 into the basis by applying the dual simplex method. The new solution is shown in Table 7.6.

Table 7.5 Optimal but Infeasible Solution

		$c_j \rightarrow$	2	20	−10	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_{g_1}
20	x_2	5/8	0	1	1/5	3/40	0
2	x_1	5/4	1	0	0	−1/4	0
0	s_{g_1}	−5/8	0	0	−1/5	−3/40	1 →
$Z = 15$		$c_j - z_j$	0	0	−14	−1	0
		Ratio: Min $(c_j - z_j)/y_{3j}$ (< 0)	–	–	70	40/3 ↑	–

Table 7.6

		$c_j \rightarrow$	2	20	−10	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_{g_1}
20	x_2	0	0	1	0	0	−1
2	x_1	10/3	1	0	2/3	0	−10/3
0	s_1	25/3	0	0	8/3	1	−40/3
$Z = 20/3$		$c_j - z_j$	0	0	−34/3	0	−40/3

The optimal solution shown in Table 7.6 is still non-integer. Therefore, one more fractional cut needs to be generated. Since x_1 is the only basic variable whose value is a non-negative fractional value, consider the x_1-row (because of largest fractional part) for constructing the cut:

$$\frac{10}{3} = x_1 + \frac{2}{3}x_3 - \frac{10}{3}s_{g_1} \quad (x_1\text{-source row})$$

The factoring of the x_1-source row yields

$$\left(3+\frac{1}{3}\right) = (1+0)\,x_1 + \left(0+\frac{2}{3}\right)x_3 + \left(-4+\frac{2}{3}\right)s_{g_1}$$

$$\frac{1}{3} + (3 - x_1 + 4s_{g_1}) = \frac{2}{3}x_3 + \frac{2}{3}s_{g_1} \quad \text{or} \quad \frac{1}{3} \le \frac{2}{3}x_3 + \frac{2}{3}s_{g_1}$$

On adding another Gomory slack variable s_{g_2}, the second Gomory's fractional cut becomes:

$$\frac{1}{3} + s_{g_2} = \frac{2}{3}x_3 + \frac{2}{3}s_{g_1} \quad \text{or} \quad s_{g_2} - \frac{2}{3}x_3 - \frac{2}{3}s_{g_1} = -\frac{1}{3} \quad \text{(Cut II)}$$

Adding this cut to the optimal simplex Table 7.6, the new table so obtained is shown in Table 7.7.

	$c_j \rightarrow$		2	20	−10	0	0	0	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_{g1}	s_{g2}	
20	x_2	0	0	1	0	0	−1	0	
2	x_1	10/3	1	0	2/3	0	−10/3	0	
0	s_1	25/3	0	0	8/3	1	−40/3	0	
0	s_{g2}	−1/3	0	0	−2/3	0	−2/3	1	→
$Z = 20/3$	$c_j - z_j$		0	0	−34/3	0	−40/3	0	
	Ratio: Min $(c_j - z_j)/y_{4j}$ (< 0)		—	—	17 ↑	—	20	—	

Table 7.7 Optimal but Infeasible Solution

Iteration 2: Enter non-basic variable x_3 into the basis to replace basic variable s_{g2} by applying the dual simplex method. The new solution is shown in Table 7.8.

	$c_j \rightarrow$		2	20	−10	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_{g1}	s_{g2}
20	x_2	0	0	1	0	0	1	0
2	x_1	3	1	0	0	0	−4	0
0	s_1	7	0	0	0	1	−16	4
−10	x_3	1/2	0	0	1	0	1	−3/2
$Z = 1$	$c_j - z_j$		0	0	0	0	−2	−17

Table 7.8 Optimal but Non-Integer Solution

The optimal solution shown in Table 7.8 is still non-integer because variable x_3 does not assume integer value. Thus, a third fractional cut needs to be constructed with the help of the x_3-row:

$$\frac{1}{2} = x_3 + s_{g_1} - \frac{3}{2}\, s_{g_2} \quad (x_3\text{–source row})$$

$$\left(0 + \frac{1}{2}\right) = (1 + 0)x_3 + (1 + 0)s_{g_1} + \left(-2 + \frac{1}{2}\right)s_{g_2}$$

$$\frac{1}{2} + \left(2s_{g_2} - x_3 - s_{g_1}\right) = \frac{1}{2}s_{g_2} \quad \text{or} \quad \frac{1}{2} \le \frac{1}{2}s_{g_2}$$

The required Gomory's fractional cut obtained by adding slack variable s_{g_3} is:

$$\frac{1}{2} + s_{g_3} = \frac{1}{2}s_{g2} \quad \text{or} \quad s_{g_3} - \frac{1}{2}s_{g_2} = -\frac{1}{2} \qquad \text{(Cut III)}$$

Adding this cut to the bottom of the optimal simplex Table 7.8, the new table so obtained is shown in Table 7.9.

	$c_j \rightarrow$		2	20	−10	0	0	0	0	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_{g1}	s_{g2}	s_{g3}	
20	x_2	0	0	1	0	0	1	0	0	
2	x_1	3	1	0	0	0	−4	1	0	
0	s_1	7	0	0	0	1	−16	4	0	
−10	x_3	1/2	0	0	1	0	1	−3/2	0	
0	s_{g3}	−1/2	0	0	0	0	0	−1/2	1	→
$Z = 1$	$c_j - z_j$		0	0	0	0	−2	−17	0	
	Ratio: Min $(c_j - z_j)/y_{5j}$ (< 0)		—	—	—	—	—	34 ↑	—	

Table 7.9

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Iteration 3: Remove the variable s_{g3} from the basis and enter variable s_{g2} into the basis by applying the dual simplex method. The new solution is shown in Table 7.10.

		$c_j \rightarrow$	2	20	–10	0	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_{g1}	s_{g2}	s_{g3}
20	x_2	0	0	1	0	0	1	0	0
2	x_1	2	1	0	0	0	–4	0	2
0	s_1	3	0	0	0	1	–16	0	8
–10	x_3	2	0	0	1	0	1	0	–3
0	s_{g2}	1	0	0	0	0	0	1	–2
$Z = -16$		$c_j - z_j$	0	0	0	0	–2	0	–34

Table 7.10
Optimal Solution

In Table 7.10, since value of all basic variables is an integer value and all $c_j - z_j \le 0$, the current solution is an integer optimal solution: $x_1 = 2$, $x_2 = 0$, $x_3 = 2$ and Max $Z = -16$.

Example 7.3 The owner of a readymade garments store sells two types of shirts – Zee-shirts and Button-down shirts. He makes a profit of Rs 3 and Rs 12 per shirt on Zee-shirts and Button-down shirts, respectively. He has two tailors, A and B, at his disposal, for stitching the shirts. Tailors A and B can devote at the most 7 hours and 15 hours per day, respectively. Both these shirts are to be stitched by both the tailors. Tailors A and B spend 2 hours and 5 hours, respectively in stitching one Zee-shirt, and 4 hours and 3 hours, respectively on stitching a Button-down shirt. How many shirts of both types should be stitched in order to maximize the daily profit?

(a) Formulate and solve this problem as an LP problem.
(b) If the optimal solution is not integer-valued, use Gomory technique to derive the optimal integer solution. [*Delhi Univ., MBA, 2008*]

Mathematical formulation Let x_1 and x_2 = number of Zee-shirts and Button-down shirts to be stitched daily, respectively.

Then the mathematical model of the LP problem is stated as:

Maximize $Z = 3x_1 + 12x_2$

subject to the constraints

(i) Time with tailor A : $2x_1 + 4x_2 \le 7$
(ii) Time with tailor B : $5x_1 + 3x_2 \le 15$

and $x_1, x_2 \ge 0$ and are integers.

Solution (a) Adding slack variables s_1 and s_2, the given LP problem is stated into its standard form as:

Maximize $Z = 3x_1 + 12x_2 + 0s_1 + 0s_2$

subject to the constraints

(i) $2x_1 + 4x_2 + s_1 = 7$, (ii) $5x_1 + 3x_2 + s_2 = 15$

and $x_1, x_2, s_1, s_2 \ge 0$ and are integers

The optimal solution of the LP problem, obtained by using the simplex method is given in Table 7.11.

		$c_j \rightarrow$	3	12	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2
12	x_2	7/4	1/2	1	1/4	0
0	s_2	39/4	7/2	0	–3/4	1
$Z = 21$		$c_j - z_j$	–3	0	–3	0

Table 7.11
Optimal Non-integer Solution

The non-integer optimal solution shown in Table 7.11 is: $x_1 = 0$, $x_2 = 7/4$ and Max $Z = 21$.

To obtain the integer-valued solution, we proceed to construct Gomory's fractional cut, with the help of x_2-row (because of largest fraction value) as follows:

$$\frac{7}{4} = \frac{1}{2}x_1 + x_2 + \frac{1}{4}s_1 \qquad (x_2\text{-source row})$$

$$\left(1+\frac{3}{4}\right) = \left(0+\frac{1}{2}\right)x_1 + (1+0)\,x_2 + \left(0+\frac{1}{4}\right)s_1$$

or

$$\frac{3}{4} + (1 - x_2) = \frac{1}{2}x_1 + \frac{1}{4}s_1 \quad \text{or} \quad \frac{3}{4} \le \frac{1}{2}x_1 + \frac{1}{4}s_1$$

On adding Gomory slack variable s_{g_1}, the required Gomory's fractional cut becomes:

$$\frac{3}{4} + s_{g1} = \frac{1}{2}x_1 + \frac{1}{4}s_1$$

or

$$s_{g_1} - \frac{1}{2}x_1 - \frac{1}{4}s_1 = -\frac{3}{4} \qquad \text{(Cut I)}$$

Adding this additional constraint to the bottom of the optimal simplex Table 7.11, the new table so obtained is shown in Table 7.12.

	$c_j \rightarrow$		3	12	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
12	x_2	7/4	1/2	1	1/4	0	0
0	s_2	39/4	7/2	0	–3/4	1	0
0	s_{g_1}	–3/4	–1/2	0	–1/4	0	1 →
$Z = 21$		$c_j - z_j$	–3	0	–3	0	0
	Ratio:	$\text{Min}\,(c_j - z_j)/y_{3j}\ (< 0)$	6 ↑	–	12	–	–

Table 7.12 Optimal but Infeasible Solution

Iteration 1: Remove variable s_{g_1} from the basis and enter variables x_1 into the basis by applying dual simplex method. The new solution is shown in Table 7.13.

	$c_j \rightarrow$		3	12	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
12	x_2	1	0	1	0	0	1
0	s_2	9/2	0	0	–5/2	1	7
3	x_1	3/2	1	0	1/2	0	–2
$Z = 33/2$		$c_j - z_j$	0	0	–3/2	0	–6

Table 7.13 Optimal but Non-integer Solution

The optimal solution shown in Table 7.13 is still non-integer. Therefore, by adding one more fractional cut, with the help of the x_1-row, we get:

$$\frac{3}{2} = x_1 + \frac{1}{2}s_1 - 2\,s_{g_1} \qquad (x_1\text{-source row})$$

$$\left(1+\frac{1}{2}\right) = (1+0)\,x_1 + \left(0+\frac{1}{2}\right)s_1 + (-2+0)\,s_{g_1}$$

$$\frac{1}{2} + (1 - x_1 + 2s_{g_1}) = \frac{1}{2}s_1 \quad \text{or} \quad \frac{1}{2} \le \frac{1}{2}s_1$$

On adding Gomory slack variable s_{g_2}, the required Gomory's fractional cut becomes:

$$\frac{1}{2} + s_{g_2} = \frac{1}{2} s_1 \quad \text{or} \quad s_{g_2} - \frac{1}{2} s_1 = -\frac{1}{2} \quad \text{(Cut II)}$$

Adding this cut to the optimal simplex Table 7.13, the new table so obtained is shown in Table 7.14.

		$c_j \rightarrow$	3	12	0	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}	s_{g_2}
12	x_2	1	0	1	0	0	1	0
0	s_2	9/2	0	0	– 5/2	1	7	0
3	x_1	3/2	1	0	1/2	0	2	0
0	s_{g_2}	– 1/2	0	0	– 1/2	0	0	1 →
$Z = 33/2$		$c_j - z_j$	0	0	– 3/2	0	– 6	0
		Ratio: Min $(c_j - z_j)/y_{4j}$ (< 0)	0	0	3 ↑	0	0	0

Table 7.14
Optimal but Infeasible Solution

Iteration 2: Remove variable s_{g_2} from the basis and enter variable s_1 into the basis by applying the dual simplex method. The new solution is shown in Table 7.15.

		$c_j \rightarrow$	3	12	0	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}	s_{g_2}
12	x_2	1	0	1	0	0	0	0
0	s_2	7	0	0	0	1	7	–5
3	x_1	1	1	0	0	0	2	1
0	s_1	1	0	0	1	0	0	–2
$Z = 15$		$c_j - z_j$	0	0	0	0	–6	–3

Table 7.15
Optimal Solution

In Table 7.15, since all basic variables have assumed integer value and all $c_j - z_j \leq 0$, the current solution is an integer optimal solution. Thus, the company should produce $x_1 = 1$ Zee-shirt, $x_2 = 1$ Button-down shirt in order to yield Max (profit) Z = Rs 15.

SELF PRACTICE PROBLEMS A

1. Solve the following all integer programming problems, using Gomory's cutting plane algorithm:

(i) Max $Z = x_1 + 2x_2$
subject to (a) $2x_2 \leq 7$, (b) $x_1 + x_2 \leq 7$
(c) $2x_1 \leq 11$
and $x_1, x_2 \geq 0$ and integers.

(ii) Max $Z = 2x_1 + 1.7x_2$
subject to (a) $4x_1 + 3x_2 \leq 7$, (b) $x_1 + x_2 \leq 4$
and $x_1, x_2 \geq 0$ and integers.

(iii) Max $Z = 3x_1 + 2x_2 + 5x_3$
subject to (a) $5x_1 + 3x_2 + 7x_3 \leq 28$
(b) $4x_1 + 5x_2 + 5x_3 \leq 30$
and $x_1, x_2, x_3 \geq 0$ and integers.

(iv) Max $Z = 3x_1 + 4x_2$
subject to (a) $3x_1 + 2x_2 \leq 8$, (b) $x_1 + 4x_2 \geq 10$
and $x_1, x_2 \geq 0$ and integers.

(v) Max $Z = 4x_1 + 3x_2$
subject to (a) $x_1 + 2x_2 \leq 4$, (b) $2x_1 + x_2 \leq 6$
and $x_1, x_2 \geq 0$ and integers.

(vi) Max $Z = 7x_1 + 9x_2$
subject to (a) $-x_1 + 3x_2 \leq 6$, (b) $7x_1 + x_2 \leq 35$
and $x_1, x_2 \geq 0$ and integers.

2. An airline owns an ageing fleet of Boeing 737 jet airplanes. It is considering a major purchase of up to 17 new Boeing models 757 and 767 jets. The decision must take into account several cost and capacity factors including the following: (i) the airline can finance up to Rs 4,000 million in purchases; (ii) each Boeing 757 will cost Rs 350 million, while each Boeing 767 will cost Rs 220 million; (iii) at least one-third of the planes purchased should be the longer-ranged 757; (iv) the annual maintenance budget is to be no more than Rs 80 million; (v) the annual maintenance cost per 757 is estimated to be Rs 8,00,000 and it is Rs 5,00,000 for each 767 purchased; (vi) each 757 can carry 1,25,000 passengers per year, while each 767 can fly 81,000 passengers annually. Formulate this problem as an integer programming problem in order in order to maximize the annual passenger carrying capacity and solve the problem by using the cutting plane method.

3. An air conditioning and refrigeration company has been awarded a contract for the air conditioning of a new computer installation. The company has to make a choice between two alternatives: (a) Hire one or more refrigeration technicians for 6 hours a day, or (b) hire one or more part-time refrigeration apprentice technicians for 4 hours a day. The wage rate of refrigeration technicians is Rs 400 per day, while the corresponding rate for apprentice technicians is Rs 160 per day. The company does not want to engage the technicians on work for more than 25 man hours per day. It also wants to limit the charges of technicians to Rs 4,800. The company estimates that the productivity of a refrigeration technician is 8 units and that of part time apprentice technician is 3 units. Formulate and solve this problem as an integer LP problem to enable the company to select the optimal number of technicians and apprentices. [*Delhi Univ., MBA, 2002*]

4. A manufacturer of toys makes two types of toys, A and B. Processing of these two toys is done on two machines X and Y. The toy A requires two hours on machine X and six hours on machine Y. Toy B requires four hours on machine X and five hours on machine Y. There are sixteen hours of time per day available on machine X and thirty hours on machine Y. The profit obtained on both the toys is the same, i.e. Rs 5 per toy Formulate and solve this problem as an integer LP problem to determine the daily production of each of the two toys? [*Delhi Univ., MBA, 2003*]

5. The ABC Electric Appliances Company produces two products: Refrigerators and ranges. The production of these takes place in two separate departments. Refrigerators are produced in department I and ranges in department II. Both these are sold on a weekly basis. Due to the limited facilities in the departments the weekly production cannot exceed 25 refrigerators in department I and 35 ranges in department II. The company regularly employs a total of 50 workers in the departments. The production of one refrigerator requires two man-weeks of labour and of one range one man-week. A refrigerator contributes a profit of Rs 300 and a range of Rs 200. Formulate and solve this problem as an integer LP problem to determine the units of refrigerators and ranges that the company should produce to realize the maximum profit? [*Delhi Univ, MBA, 2005*]

6. XYZ Company produces two types of tape recorders: Reel-to-reel model and cassette model, on two assembly lines. The company must process each tape recorder on each assembly line. It has found that the this whole process requires the following amount of time:

Assembly Line	*Reel-to-reel*	*Cassette*
1	6 hours	2 hours
2	3 hours	2 hours

The production manager says that line 1 will be available for 40 hours per week and line 2 for only 30 hours per week. After the mentioned hours of operation each line must be checked for repairs. The company realizes a profit of Rs 300 on each reel-to-reel tape recorder and Rs 120 on each cassette recorder. Formulate and solve this problem as an integer LP problem in order to determine the number of recorders of each type to be produced each week so as to maximize profit. [*Delhi Univ., MBA, 2001*]

7. A company that manufactures metal products is planning to buy any of the following three types of lathe machines – manual, semi-automatic and fully-automatic. The manual lathe machine costs Rs 1,500, while the semi-automatic and fully-automatic lathe machines cost Rs 4,000 and Rs 6,000 respectively. The company's budget for buying new machines is Rs 1,20,000. The estimated contribution towards profit from the manual, semi-automatic and fully-automatic lathe machines are: Rs 16, Rs 20 and Rs 22, respectively. The available floor space allows for the installation of only 36 new lathe machines. The maintenance required on fully automatic machines is low and the maintenance department can maintain 50 fully-automatic machines in a year. The maintenance of a semi-automatic machine takes 20 per cent more time than that of a fully automatic machine and a manually-operated machine takes 50 per cent more time for maintenance of a fully-automatic machine. Formulate and solve this problem as an integer LP problem in order to determine the optimal number of machines to be bought.

8. A stereo equipment manufacturer can produce two models – A and B – of 40 and 80 watts of total power, each. Each model passes through three manufacturing divisions 5 namely 1, 2 and 3, where model A takes 4, 2.5 and 4.5 hours each and model B takes 2, 1 and 1.5 hours each. The three divisions have a maximum of 1,600, 1,200 and 1,600 hours every month, respectively. Model A gives a profit contribution of Rs 400 each and B of Rs 100 each. Assuming abundant product demand, formulate and solve this problem as an integer LP problem, to determine the optimal product mix and the maximum contribution. [*Delhi Univ., MBA, 2004*]

9. A manufacturing company produces two types of screws – metal and wooden. Each screw has to pass through the slotting and threading machines. The maximum time that each machine can be run is 150 hours per month. A batch of 50 wooden screws requires 2 minutes on the threading machine and 3 minutes on the slotting machine. Metal screws of the same batch size require 8 minutes on the threading machine and 2 minutes on the slotting machine. The profit contribution for each batch of wooden and metal screws is Re 1 and Rs 2.50, respectively. Formulate and solve this problem as an integer LP problem in order to determine the optimal product mix for maximum profit contribution.

10. A dietician for a hospital is considering a new breakfast menu that includes oranges and cereal. This breakfast must meet the minimum requirements for the Vitamins A and B. The number of milligrams of each of these vitamins contained in a purchasing unit, for each of these foods, is as follows:

Vitamin	*Milligrams per Purchasing Unit of Food*		*Minimum Requirement (mg)*
	Oranges (doz)	*Cereal (box)*	
A	1	2	20
B	3	2	50

The cost of the food ingredients is Rs 15 per dozen for oranges and Rs 12.50 per box for cereal. For dietary reasons, at least one unit of each food type must be used in the menu plan. Formulate and solve this problem as an integer programming problem.

11. The dietician at a local hospital is planning the breakfast menu for the maternity ward patients. She is primarily concerned with

Vitamin E and iron requirements, for planning the breakfast. According to the State Medical Association (SMA) new mothers must get at least 12 milligrams of Vitamin E and 24 milligrams of iron from breakfast. The SMA handbook reports that a scoop of scrambled egg contains 2 milligrams of Vitamin E and 8 milligrams of iron. The handbook also recommends that new mothers should eat at least two scoops of cottage cheese for their breakfast. The dietician considers this as one of the model constraints. The hospital's accounting department estimates that one scoop of cottage cheese costs Rs 2 and one scoop of scrambled egg also costs Rs 2. The dietician is attempting to determine the optimum breakfast menu that satisfies all the requirements and minimizes the total cost. The cook insists that he can serve foods by only full scoop, thus necessitating an integer solution. Determine the optimum integer solution to the problem. *[Delhi Univ., MBA (HCA), 2002, 2003]*

12. A building contractor has just won a contract to build a municipal library building. His present labour workforce is inadequate to immediately take up this work as the force is already involved with other jobs on hand. The convertor must therefore immediately decide whether to hire one or more labourers on a full-time basis (eight hours a day each) or to allow overtime to one or more of the existing labour force (five hours a day each). Extra labourers can be hired for Rs 40 per day (for eight hours) while overtime costs Rs 43 per day (for five hours per day). The contractor wants to limit his extra payment to Rs 400 per day and to use no more than twenty labourers (both full-time and overtime) because of limited supervision. He estimates that the new labour employed on a full-time basis will generate Rs 15 a day in profits, while overtime labour Rs 20 a day. Formulate and solve this problem as an integer LP problem to help the building contractor to decide the optimum labour force.

13. The ABC company requires an output of at least 200 units of a particular product per day. To accomplish this target it can buy machines A or B or both. Machine A costs Rs 20,000 and B Rs 15,000. The company has a budget of Rs 2,00,000 for the same. Machines A and B will be able to produce 24 and 20 units, respectively of this product per day. However, machine A will require a floor space of 12 square feet while machine B will require 18 square feet. The company only has a total floor space of 180 square feet. Formulate and solve this problem as an integer LP problem to determine the minimum number of machines that should be purchased. *[Delhi Univ., MBA, 2001]*

14. A company produces two products A and B. Each unit of product A requires one hour of engineering services and five hours of machine time. To produce one unit of product B, two hours of engineering and 8 hours of machine time are needed. A total of 100 hours of engineering and 400 hours of machine time is available. The cost of production is a non-linear function of the quantity produced as given in the following table:

Product A		*Product B*	
Production (units)	*Unit Cost (Rs)*	*Production (units)*	*Unit Cost (Rs)*
0– 50	10	0– 40	7
50–100	8	40–100	3

The unit selling price of product A is Rs 12 and of product B is Rs 14. The company would like a production plan that gives the number of units of A and the number of units of B to be produced that would maximize profit. Formulate and solve this problem as an integer linear programming problem to help the company maximize its total revenue. *[Delhi Univ., MBA, 2003]*

15. XYZ Corporation manufactures an electric device, final assembly of which is accomplished by a small group of trained workers operating simultaneously on different devices. Due to space limitations, the working group may not exceed ten in number. The firm's operating budget allows Rs 5,400 per month as salary for the group. A certain amount of discrimination is evidenced by the fact that the firm pays men in the group Rs 700 per month, while women doing the same work receive Rs 400. However, previous experience has indicated that a man will produce about Rs 1,000 in value added per month, while a woman worker adds Rs 900. If the firm wishes to maximize the value added by the group, how many men and women should be included? (A non-integer solution for this problem will not be accepted).

16. A manufacturer of baby dolls makes two types of dolls. One is sold under the brand name 'Molina' and the other under 'Suzie'. These two dolls are processed on two machines – A and B. The processing time for each 'Molina' is 2 hours and 6 hours on machines A and B, respectively and that for each 'Suzie' is 1 hour and 5 hours on machines A and B, respectively. There is 16 hours of time available per day on machine A and 30 hours on machine B. The profit contribution from a 'Molina' is Rs 6 and that from a 'Suzie' is Rs 18. Formulate and solve this problem as an integer LP problem to determine the optimal weekly production schedule of the two dolls. *[Delhi Univ, MBA (PSM), 2004]*

17. The dietician at the local hospital is planning the breakfast menu for the maternity ward patients. She is planning a special non-fattening diet, and has chosen cottage cheese and scrambled eggs for breakfast. She is primarily concerned with Vitamin E and Iron requirements in planning the breakfast.

According to the State Medical Association (SMA) new mothers must get at least 12 milligrams of Vitamin E and 24 milligrams of iron from breakfast. The SMA handbook reports that a scoop of cottage cheese contains 3 milligrams of Vitamin E and 3 milligrams of iron. An average scoop of scrambled egg contains 2 milligrams of Vitamin E and 8 milligrams of iron. The SMA handbook recommends that new mothers should eat at least two scoops of cottage cheese for their breakfast. The dietician considers this as one of the model constraints.

The hospital accounting department estimates that a scoop of cottage cheese costs Re 1, and a scoop of scrambled egg also costs Re 1. The dietician is attempting to determine the optimum breakfast menu that satisfies all the requirements and minimize total cost. The cook insists that he can serve foods by only full scoop, thus necessitating an integer solution. Formulate and solve this problem as an integer LP problem to determine the optimum-integer solution to the problem.

18. A firm makes two products: X and Y, and has total production capacity of 9 tonnes per day, X and Y requiring the same production capacity. The firm has a permanent contract to supply at least 2 tonnes of X and at least 3 tonnes of Y, per day, to another company. Each tonne of X requires 20 machine-hours production time and each tonne of Y requires 50 machine-hours. The daily maximum possible number of machine-hours is 350. All the firm's output can be sold and the profit made is Rs 80 per tonne of X and Rs 120 per tonne of Y. It is required to determine the production schedule for attaining the maximum profit and to calculate this profit. (A non-integer solution for this problem will not be accepted). *[Delhi Univ., MBA, 1999, 2004]*

19. An Airline corporation is considering the purchase of three types of jet planes. The purchase price would be Rs 45 crore for each A type plane; Rs 40 crore for each B type plane and Rs 25 crore for each C type plane. The corporation has resources worth Rs 500 crore for these purchases. The three types of planes, if purchased, would be utilized essentially at maximum capacity. It is estimated that the net annual profit would be Rs 3 million for A type, Rs 2.25 million for B type and Rs 1.5 million for C type planes. Each plane requires one pilot and it is estimated that 25 trained pilots would be available. If only C type planes

are purchased, the maintenance facilities would be able to handle 30 new planes. However, each B type is equivalent to one C type plane and each A type plane is equivalent to one C type plane, in terms of their use of the maintenance facilities.

The management of the corporation wants to know how many planes of each type (considering the fact that number of planes should be an integer) should be purchased in order to maximize its profits.

20. The owner of a ready-made garments store makes two types of shirts: Arrow and Wings. He makes a profit of Rs 10 and Rs 15 per shirt on Arrow and Wings, respectively. To stitch these shirts he has two tailors A and B at his disposal. Tailors A and B can devote, at the most, 12 hours each day. Both these shirts are to be stitched by both the tailors. Tailor A and Tailor B spend 3 hours and 4 hours, respectively, in stitching an Arrow shirt and 4 hours and 3 hours, respectively in stitching a Wings shirt. How many shirts of both types should be stitched in order to maximize daily profits? (A non-integer solution for this problem will not be accepted.) *[Delhi Univ., MBA, 2003]*

21. Suppose five items are to be loaded on a vessel. The weight (W), volume (V) and price (P) are tabulated below. The maximum cargo weight and cargo volume are W = 112 and V = 109, respectively. Determine the most valuable cargo load in discrete unit of each item.

Item	*W*	*V*	*Price (Rs)*
1	5	1	4
2	8	8	7
3	3	6	6
4	2	5	5
5	7	4	4

Formulate and solve this problem as an integer linear programming model.

22. The cutting division of Photo Films Corporation requires from the stock control department, plastic films of 85 feet of fixed unit length that can be cut according to two different patterns. The first pattern would cut each film length into two 35 feet pieces with the remaining 15 feet to scrap. The second pattern will cut each film length into a 35 feet piece and two 25 feet pieces with nothing to scrap. The present order from a customer is for 8 pieces of 35 feet length and six pieces of 25 feet length. What number of plastic films of 85 feet should be cut according to the patterns (assuming both patterns have to be used) in order to minimize the scrap? (A non-integer solution for this problem will not be accepted).

23. A manufacturing company produces two products, each of which requires stamping, assembly and painting operations. Total productive capacity by operation if it were devoted solely to one product or the other is:

Operation	*Productive Capacity (units per week)*	
	Product A	*Product B*
Stamping	50	75
Assembly	40	80
Painting	90	45

Pro-rata allocation of productive capacity is permissible and so is combinations of production of the two products. Demand for the two products is unlimited and the profit on A and B are Rs 150 and Rs 120, respectively. Determine the optimal product mix. (Non-integer solution for this problem will not be accepted.) *[Delhi Univ, MBA, 2007]*

24. Write constraints to satisfy each of the following conditions in a project selection model. The projects are numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.
    (i) Exactly one project from the set (1, 2, 3) must be selected.
    (ii) Project 2 can be selected only if number 10 is selected. However, 10 can be selected without 2 being selected.
    (iii) No more than one project from the set (1, 3, 5, 7, 9) can be selected.
    (iv) If number 4 is selected, then number 8 cannot be selected.
    (v) Projects 4 and 10 must both be selected or both be rejected. *[Delhi Univ., MBA, 2004]*

25. A production manager faces the problem of job allocation between his two production crews. The production rate of crew one is five units per hour and for crew two is six units. The normal working hours for each crew is 8 hours per day. If the firm requires overtime operations each crew can work up to 11 hours (union contract). The firm has a special contract with a customer to provide a minimum of 120 units of the product the next day.

    The union contract calls for working arrangements, where each crew should work at least 8 hours per day, and any overtime not exceeding 3 hours, should be in terms of increments of an hour. The operating costs per hour are Rs 200 for crew one and Rs 220 for crew two. Formulate and solve this problem as an integer programming model to determine the optimum job allocation.

26. A company makes two products, each of which require time on four different machines. Only integer amounts of each can be made. The company wants to find the output mix that maximizes total profits, given the specifications shown in the following table:

Total Machine Hours Available	*Machine*	*Machine Hours (per unit)*	
		Product I	*Product II*
3,400	A	200	500
1,450	B	100	200
3,000	C	400	300
2,400	D	400	100
Profit/unit (Rs 1,000)		6	6

[Delhi Univ., MBA, 2001, 2005]

HINTS AND ANSWERS

1. (i) $x_1 = 4, x_2 = 3$ and Max $Z = 10$
   (ii) $x_1 = 1, x_2 = 2$, and Max $Z = 3.7$
   (iii) $x_1 = 0, x_2 = 0, x_3 = 4$ and Max $Z = 20$
   (iv) $x_1 = 0, x_2 = 4$ and Max $Z = 16$
   (v) $x_1 = 3, x_2 = 0$ and Max $Z = 12$
   (vi) $x_1 = 4, x_2 = 3$ and Max $Z = 55$

4. x_1 and x_2 = number of units of toy A and toy B, respectively, to be produced

   Max $Z = 5x_1 + 5x_2$

   subject to (i) $2x_1 + 5x_2 \le 16$, (ii) $6x_1 + 5x_2 \le 30$

   and $x_1, x_2 \ge 0$ and integers.

   Ans: $x_1 = 3, x_2 = 2$ and Max Z = Rs 25

5. x_1 and x_2 = number of units of refrigerator and range, respectively, to be produced

Max $Z = 300x_1 + 200x_2$

subject to (i) $2x_1 + x_2 \le 60$, (ii) $x_1 \le 25$

(iii) $x_2 \le 35$

and $x_1, x_2 \ge 0$ and integers.

Ans: $x_1 = 13$, $x_2 = 34$ and Max Z = Rs 10,700

14. x_1 and x_2 = number of units of product A and product B, respectively, to be produced

Max $Z = 12x_1 - \{10x_1 - 2(x_1 - 50)\} + 14x_2 - \{7x_2 - 4(x_2 - 40)\}$

subject to (i) $x_1 + 2x_2 \le 100$; (ii) $5x_1 + 8x_2 \le 400$

(iii) $x_1 + 2x_2 \le 100$; (iv) $5x_1 + 8x_2 \le 100$

and $x_1, x_2 \ge 100$ and integers.

16. x_1, and x_2 = number of units of shirt 'Molina' and 'Suzie' respectively, to be produced

Max $Z = 6x_1 + 18x_2$

subject to (i) $2x_1 + 5x_2 \le 16$ (ii) $6x_1 + 5x_2 \le 30$

and $x_1, x_2 \ge 0$ and integers

Ans: $x_1 = 3$, $x_2 = 2$ and Max $Z = 36$

7.5 GOMORY'S MIXED-INTEGER CUTTING PLANE METHOD

In the previous section, Gomary cutting method was discussed for LP problems where all decision variables, slack variables and surplus variables were assumed to have integer values. This implies that this method is applicable only when all variables in the LP problem assume integer values. For example, consider the following constraint:

$$\frac{1}{2}x_1 + x_2 \le \frac{11}{3} \quad \text{or} \quad \frac{1}{2}x_1 + x_2 + s_1 = \frac{11}{3}$$

$$x_1, x_2, s_1 \ge 0 \text{ and integers.}$$

In this equation decision variables x_1 and x_2 can assume integer values only if s_1 is non-integer. This situation can be avoided in two ways:

1. The non-integer coefficients in the constraint equation can be removed by multiplying both sides of the constraint with a proper constant. For example, the above constraint is multiplied by 6 to obtain $3x_1 + 6x_2 \le 22$. However, this type of conversion is possible only when the magnitude of the integer coefficients is small.
2. Use a special cut called *Gomory's mixed-integer cut* or simply *mixed-cut*, where only a subset of variables may assume integer values and the remaining variables (including slack and surplus variables) remain continuous. The details of developing this cut are presented below.

7.5.1 Method for Constructing Additional Constraint (Cut)

Consider the following mixed integer programming problem:

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

and x_j are integers; $j = 1, 2, \ldots, k$ $(k < n)$

Suppose that the basic variable, x_r, is restricted [illegible] and has a la[illegible] value among all those basic variables that are restricted to take in[illegible] rewrite t[illegible] (row) from the optimal simplex table as follows:

$x_{Br} =$ [illegible]

where x_j represents all the non-basic varia[illegible] on-integer value of variable x_r.

Decompose [illegible]

4. (a) $x_1 = 60$, $x_2 = 20$ and Max $Z = 400$
(b) $x_1 = 1$, $x_2 = 1$ and Min $Z = 1$

subject to (i) $0.5x + 0.25y \leq 35$, (ii) $2x + 3y \leq 80$;
and $x, y \geq 0$

24.4 QUADRATIC PROGRAMMING

Among several non-linear programming methods available for solving NLP problems, we shall discuss in this section, an NLP problem with non-linear objective function and linear constraints. Such an NLP problem is called *quadratic programming problem*. The general mathematical model of quadratic programming problem is as follows:

$$\text{Optimize (Max or Min) } Z = \left\{ \sum_{j=1}^{n} c_j x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} x_j d_{jk} x_k \right\}$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

and

$$x_j \geq 0 \text{ for all } i \text{ and } j$$

In matrix notations, QP problem is written as:

$$\text{Optimize (Max or Min) } Z = \mathbf{cx} + \frac{1}{2}\mathbf{x}^T \mathbf{Dx}$$

subject to the constraints

$$\mathbf{Ax} \le \mathbf{b}$$

and

$$\mathbf{x} \ge 0 \qquad (3)$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$; $\mathbf{c} = (c_1, c_2, \ldots, c_n)$; $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$ (4)

$\mathbf{D} = [d_{jk}]$ is an $n \times n$ symmetric matrix, i.e. $d_{jk} = d_{kj}$; $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix

If the objective function in QP problem is of minimization, then matrix $\mathbf{D}$ is symmetric and positive-definite (i.e. the quadratic term $\mathbf{x}^T \mathbf{Dx}$ in $\mathbf{x}$ is positive for all values of $\mathbf{x}$ except at $\mathbf{x} = 0$) and objective function is strictly convex in $\mathbf{x}$. But, if the objective function is of maximization, then matrix $\mathbf{D}$ is symmetric and negative-definite (i.e. $\mathbf{x}^T \mathbf{Dx} < 0$ for all values of $\mathbf{x}$ except for $\mathbf{x} = 0$) and objective function is strictly concave in $\mathbf{x}$. If matrix, $\mathbf{D}$ is null, then the QP problem reduces to the standard LP problem.

24.4.1 Kuhn-Tucker Conditions

The necessary and sufficient Kuhn-Tucker conditions to get an optimal solution to the maximization QP problem subject to linear constraints can be derived as follows:

Step 1: Introducing slack variables s_i^2 and r_j^2 to constraints, the QP problem becomes:

$$\text{Max } f(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} x_j d_{jk} x_k$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j + s_i^2 = b_i; \qquad i = 1, 2, \ldots, m$$

$$-x_j + r_j^2 = 0; \qquad j = 1, 2, \ldots, n$$

Step 2: Forming the Lagrange function as follows:

$$L(\mathbf{x}, \mathbf{s}, \mathbf{r}, \lambda, \mu) = f(\mathbf{x}) - \sum_{j=1}^{n} \lambda_i \{a_{ij} x_j + s_i^2 - b_i\} - \sum_{j=1}^{n} \mu_j \{-x_j + r_j^2\}$$

Step 3: Differentiate $L(\mathbf{x}, \mathbf{s}, \mathbf{r}, \lambda, \mu)$ partially with respect to the components of $\mathbf{x}$, $\mathbf{s}$, $\mathbf{r}$, λ and μ. Then equate these derivatives with zero in order to get the required Kuhn-Tucker necessary conditions. That is,

(i) $\mathbf{c} - \frac{1}{2}(2\,\mathbf{x}^T \mathbf{D}) - \lambda \mathbf{A} + \mu = 0$, or

$$c_j - \sum_{k=1}^{n} x_k d_{jk} - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0; \qquad j = 1, 2, \ldots, n$$

(ii) $-2\lambda\,\mathbf{s} = 0$ or $\lambda_i s_i^2 = 0$, or

$$\lambda_i \left\{ \sum_{j=1}^{n} a_{ij} x_j - b_i \right\} = 0, \qquad i = 1, 2, \ldots, m$$

(iii) $-2\mu\, r = 0$ or $\mu_j r_j = 0$, $\qquad j = 1, 2, \ldots, n$

$\mu_j x_j = 0$, $j = 1, 2, \ldots, n$

(iv) $\mathbf{Ax} + \mathbf{s}^2 - \mathbf{b} = 0$; i.e. $\mathbf{Ax} \le \mathbf{b}$, or

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \ldots, m$$

(v) $-\mathbf{x} + \mathbf{r}^2 = 0$, i.e. $\mathbf{x} \ge 0$, or

$x_j \ge 0$, $\quad j = 1, 2, \ldots, n$

(vi) $\lambda_i, \mu_j, x_j, s_i, r_j \ge 0$

These conditions, except (ii) and (iii), are linear constraints involving $2(n+m)$ variables. The condition $\mu_j x_j = \lambda_i s_i = 0$ implies that both x_j and μ_j as well as s_i and λ_i cannot be basic variables at a time in a non-degenerate basic feasible solution. The conditions $\mu_j x_j = 0$ and $\lambda_i s_i = 0$ are also called *complementary slackness conditions*.

24.4.2 Wolfe's Modified Simplex Method

The Wolf's method for solving a quadratic programming problem can be summarized in the following steps:

Step 1: Introduce artificial variables A_j $(j = 1, 2, \ldots, n)$ in the Kuhn-Tucker condition (i). Then we have

$$c_j - \sum_{k=1}^{n} x_k d_{jk} - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j + A_j = 0$$

For a starting basic feasible solution we shall have $x_j = 0, \mu_j = 0, A_j = -c_j$ and $s_i^2 = b_i$. However, this solution would be desirable if and only if $A_j = 0$ for all j.

Step 2: Apply Phase I of the simplex method to check the feasibility of the constraints $\mathbf{Ax} \leq \mathbf{b}$. If there is no feasible solution, then terminate the solution procedure, otherwise get an initial basic feasible solution for Phase II. To obtain the desired feasible solution solve the following problem:

$$\text{Minimize } Z = \sum_{j=1}^{n} A_j$$

subject to the constraints

$$\sum_{k=1}^{n} x_k d_{jk} + \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j + A_j = -c_j, \quad j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j + s_i^2 = b_i, \quad i = 1, 2, \ldots, m$$

and $$\lambda_i, x_j, \mu_j, s_i, A_j \geq 0 \text{ for all } i \text{ and } j$$

$$\left.\begin{aligned} \lambda_i s_i &= 0 \\ \mu_j x_j &= 0 \end{aligned}\right\} \text{Complementary slackness conditions}$$

Thus, while deciding for a variable to enter into the basis at each iteration, the complementary slackness conditions must be satisfied.

This problem has $2(m+n)$ variables and $(m+n)$ linear constraints, together with $(m+n)$ complementary slackness conditions.

Step 3: Apply Phase II of the simplex method to get an optimal solution to the problem given in Step 2. The solution, so obtained, will also be an optimal solution of the quadratic programming problem.

Example 24.7 Use Wolfe's method to solve the quadratic programming problem:

$$\text{Maximize } Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

subject to the constraint

$$x_1 + 2x_2 \leq 2 \quad \text{and} \quad x_1, x_2 \geq 0$$

[IAS, 1994; Gauhati Univ., MCA, 2000]

Solution Consider non-negativity conditions $x_1, x_2 \geq 0$ as inequality constraints. Add slack variables to all inequality constraints in order to express them as equations. The standard form of QP problem becomes:

$$\text{Maximize } Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

subject to the constraints

$$\text{(i) } x_1 + 2x_2 + s_1^2 = 2, \quad \text{(ii) } -x_1 + r_1^2 = 0, \quad \text{(iii) } -x_2 + r_2^2 = 0$$

and $$x_1, x_2, s_1, r_1, r_2 \geq 0$$

To obtain the necessary conditions, we construct the Lagrange function as follows:

$$L(x_1, x_2, s_1, \lambda_1, \mu_1, \mu_2, r_1, r_2) = (4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2) - \lambda_1 (x_1 + 2x_2 + s_1^2 - 2)$$
$$- \mu_1 (-x_1 + r_1^2) - \mu_2 (-x_2 + r_2^2)$$

The necessary and sufficient conditions for the maximum of L and hence of Z are:

$$\frac{\partial L}{\partial x_1} = 4 - 4x_1 - 2x_2 - \lambda_1 + \mu_1 = 0, \quad \frac{\partial L}{\partial x_2} = 6 - 2x_1 - 4x_2 - 2\lambda_1 + \mu_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 + 2x_2 + s_1^2 - 2 = 0, \quad \frac{\partial L}{\partial s_1} = 2\lambda_1 s_1 = 0$$

$$\frac{\partial L}{\partial \mu_1} = -x_1 + r_1^2 = 0, \quad \frac{\partial L}{\partial \mu_2} = -x_2 + r_2^2 = 0$$

$$\frac{\partial L}{\partial r_1} = 2\mu_1 r_1 = 0, \quad \frac{\partial L}{\partial r_2} = 2\mu_2 r_2 = 0$$

After simplifying these conditions, we get:

(i) $4x_1 + 2x_2 + \lambda_1 - \mu_1 = 4$, (ii) $2x_1 + 4x_2 + 2\lambda_1 - \mu_2 = 6$, (iii) $x_1 + 2x_2 + s_1^2 = 2$

$$\left.\begin{array}{l} \lambda_1 s_1 = 0 \\ \mu_1 x_1 = \mu_2 x_2 = 0 \end{array}\right\} \text{(Complementary conditions)}$$

and $\quad x_1, x_2, \lambda_1, \mu_1, \mu_2, s_1 \geq 0$

Introducing artificial variables A_1 and A_2 in the first two constraints respectively. Then the modified LP problem becomes:

Minimize $Z^* = A_1 + A_2$

subject to the constraints

$$4x_1 + 2x_2 + \lambda_1 - \mu_1 + A_1 = 4$$
$$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + A_2 = 6$$
$$x_1 + 2x_2 + s_1^2 = 2$$

and $\quad x_1, x_2, \lambda_1, \mu_1, \mu_2, A_1, A_2 \geq 0$

The initial basic feasible solution to this LP problem is shown in Table 24.1.

	$c_j \rightarrow$		0	0	0	0	0	0	1	1
c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	λ_1	μ_1	μ_2	s_1	A_1	A_2
1	A_1	4	4	2	1	−1	0	0	1	0 →
1	A_2	6	2	4	2	0	−1	0	0	1
0	s_1	2	1	2	0	0	0	1	0	0
$Z^* = 10$		$c_j - z_j$	−6 ↑	−6	−3	1	1	0	0	0

24.1
Solution

Iteration 1: In Table 24.1, the largest negative values among $c_j - z_j$ values is − 6 corresponding to x_1 and x_2 columns. This means either of these two variables can be entered into the basis. Since $\mu_1 = 0$ (not in the basis), x_1 is considered to enter into the basis. It will replace A_1 in the basis. The new solution is shown in Table 24.2.

c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	λ_1	μ_1	μ_2	s_1	A_2	
	$c_j \rightarrow$		0	0	0	0	0	0	1	
0	x_1	1	1	1/2	1/4	$-1/4$	0	0	0	
1	A_2	4	0	3	3/2	1/2	-1	0	1	
0	s_1	1	0	3/2	$-1/4$	1/4	0	1	0	$\rightarrow$
$Z^* = 4$		$c_j - z_j$	0	-3	$-3/2$	$-1/2$	1	0	0	
				$\uparrow$						

Table 24.2
First Iteration

Iteration 2: In Table 24.2, $\mu_2 = 0$ (not in the basis), therefore x_2 can be introduced into the basis to replace s_1, in the basis. The new solution is shown in Table 24.3.

c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	λ_1	μ_1	μ_2	s_1	A_2	
	$c_j \rightarrow$		0	0	0	0	0	0	1	
0	x_1	2/3	1	0	1/3	$-1/3$	0	$-1/3$	0	
1	A_2	2	0	0	2	0	-1	-2	1	$\rightarrow$
0	x_2	2/3	0	1	$-1/6$	1/6	0	2/3	0	
$Z^* = 2$		$c_j - z_j$	0	0	-2	0	1	2	0	
					$\uparrow$					

Table 24.3
Second Iteration

Iteration 3: In Table 24.3, $s_1 = 0$ (not in the basis), therefore λ_1 can be entered into the basis to replace A_2. The new solution is shown in Table 24.4.

c_B	Basic Variables B	Solution Values $b\ (= x_B)$	x_1	x_2	λ_1	μ_1	μ_2	s_1
	$c_j \rightarrow$		0	0	0	0	0	0
0	x_1	1/3	1	0	0	$-1/3$	1/6	0
0	λ_1	1	0	0	1	0	$-1/2$	-1
0	x_2	5/6	0	1	0	1/6	$-1/12$	1/2
$Z^* = 0$		$c_j - z_j$	0	0	0	0	0	0

Table 24.4
Third Iteration

In Table 24.4, since all $c_j - z_j = 0$, an optimal solution for Phase I is reached. The optimal solution is:

$$x_1 = 1/3,\ x_2 = 5/6,\ \lambda_1 = 1,\ \lambda_2 = 0,\quad \mu_1 = \mu_2 = 0,\ s_1 = 0$$

This solution also satisfies the complementary conditions: $\lambda_1 s_1 = 0$; $\mu_1 x_1 = \mu_2 x_2 = 0$ and the restriction on the signs of Lagrange multipliers, λ_1, μ_1 and μ_2.

Further, as $Z^* = 0$, this implies that the current solution is also feasible. Thus, the maximum value of the given quadratic programming problem is:

$$\begin{aligned}\text{Max } Z &= 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2\\ &= 4(1/3) + 6(5/6) - 2\,(1/3)^2 - 2(1/3)\,(5/6) - 2(5/6)^2 = 25/6\end{aligned}$$

Example 24.8 Use the Wolfe's method to solve the quadratic programming problem:

Maximize $Z = 2x_1 + x_2 - x_1^2$

subject to the constraints

(i) $2x_1 + 3x_2 \le 6$, (ii) $2x_1 + x_2 \le 4$

and $x_1, x_2 \ge 0$

[*Madras, BE(Civil) 2000; Andhra Univ., BE(Mech & Ind) 2001*]

Solution Considering non-negativity conditions $x_1, x_2 \ge 0$ as inequality constraints and add slack variables to all inequalities to express them as equations. After adding slack variables the QP problem becomes:

Maximize $Z = 2x_1 + x_2 - x_1^2$

subject to the constraints

(i) $2x_1 + 3x_2 + s_1^2 = 6$, (ii) $2x_1 + x_2 + s_2^2 = 4$

(iii) $-x_1 + r_1^2 = 0$, (iv) $-x_2 + r_2^2 = 0$

Forming the Lagrange function as follows:

$$L(\mathbf{x}, \mathbf{s}, \lambda, r, \mu) = (2x_1 + x_2 - x_1^2) - \lambda_1 (2x_1 + 3x_2 + s_1^2 - 6)$$
$$- \lambda_2 (2x_1 + x_2 + s_2^2 - 4) - \mu_1 (-x_1 + r_1^2) - \mu_2 (-x_2 + r_2^2)$$

The necessary and sufficient conditions for maximum of L and hence of Z are:

$$\frac{\partial L}{\partial x_1} = -2 - 2x_1 - 2\lambda_1 - 2\lambda_2 + \mu_1 = 0 \quad , \quad \frac{\partial L}{\partial x_2} = 1 - 3\lambda_1 - \lambda_2 + \mu_2 = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0 \quad , \quad \frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0$$

$$\frac{\partial L}{\partial r_1} = -2\mu_1 r_1 = 0 \quad , \quad \frac{\partial L}{\partial r_2} = -2\mu_2 r_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 2x_1 + 3x_2 + s_1^2 - 6 = 0 \quad , \quad \frac{\partial L}{\partial \lambda_2} = 2x_1 + x_2 + s_2^2 - 4 = 0$$

$$\frac{\partial L}{\partial \mu_1} = -x_1 + r_1^2 = 0 \quad , \quad \frac{\partial L}{\partial \mu_2} = -x_2 + r_2^2 = 0$$

After simplifying these conditions, we get:

(i) $2x_1 + 2\lambda_1 + 2\lambda_2 - \mu_1 = 2$ (ii) $3\lambda_1 + \lambda_2 - \mu_2 = 1$

(iii) $2x_1 + 3x_2 + s_1^2 = 6$ (iv) $2x_1 + x_2 + s_2^2 = 4$

(v) $\lambda_1 s_1 = \lambda_2 s_2 = 0$ (vi) $\mu_1 x_1 = \mu_2 x_2 = 0$

and $x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2, s_1, s_2 \ge 0$

Introducing the artificial variables A_1 and A_2 in the first two constraints respectively. Then modified QP problem becomes:

Minimize $Z^* = A_1 + A_2$

subject to the constraints

(i) $2x_1 + 2\lambda_1 + 2\lambda_2 - \mu_1 + A_1 = 2$, (ii) $3\lambda_1 + \lambda_2 - \mu_2 + A_2 = 1$

(iii) $2x_1 + 3x_2 + s_1^2 = 6$, (iv) $2x_1 + x_2 + s_2^2 = 4$

where $\lambda_1 s_1 = \lambda_2 s_2 = 0$; $\mu_1 x_1 = \mu_2 x_2 = 0$

and $x_1, x_2, s_1, s_2, A_1, A_2, \mu_1, \mu_2 \ge 0$

The initial basic feasible solution to this LP problem is shown in Table 24.5.

Table 24.5
Initial Solution

	$c_j \rightarrow$		0	0	0	0	0	0	0	0	1	1	
c_B	Basic Variables B	Solution Values $b (= x_B)$	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_1	A_2	
1	A_1	2	2	0	2	2	–1	0	0	0	1	0	→
1	A_2	1	0	0	3	1	0	–1	0	0	0	1	
0	s_1	6	2	3	0	0	0	0	1	0	0	0	
0	s_2	4	2	1	0	0	0	0	0	1	0	0	
$Z^* = 3$		$c_j - z_j$	–2	0	–5	–3	1	1	0	0	0	0	
			↑										

Iteration 1: In Table 24.5, the largest negative value among $c_j - z_j$ values is -5, but we cannot enter λ_1 (or λ_2) in the basis because of the complementary conditions $\lambda_1 s_1 = \lambda_2 s_2 = 0$. Since $\mu_1 = 0$, x_1 can be entered into the basis with A_1 as the leaving variable. The new solution is shown in Table 24.6.

		$c_j \rightarrow$	0	0	0	0	0	0	0	0	1
c_B	Basic Variables B	Solution Values $b(=x_B)$	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_2
0	x_1	1	1	0	1	1	1/2	0	0	0	0
1	A_2	1	0	0	3	1	0	−1	0	0	1
0	s_1	4	0	3	−2	−2	1	0	1	0	0
0	s_2	2	0	1	−2	−2	1	0	0	1	0
$Z^* = 1$		$c_j - z_j$	0	0	−3	−1	0	1	0	0	0

Table 24.6
First Iteration

Iteration 2: Again, we cannot enter λ_1, λ_2 and μ_1 in the basis in Table 24.6 because s_1, s_2 and x_1, respectively, are already in the basis. Entering x_2 into the basis with s_1 as the leaving variable because $\mu_2 = 0$. The new solution is shown in Table 24.7.

		$c_j \rightarrow$	0	0	0	0	0	0	0	0	1
c_B	Basic Variables B	Solution Values $b(=x_B)$	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_2
0	x_1	1	1	0	1	1	−1/2	0	0	0	0
1	A_2	1	0	0	3	1	0	−1	0	0	1
0	x_2	4/3	0	1	−2/4	−2/3	1/3	0	1/3	0	0
0	s_2	2/3	0	0	−4/3	−4/3	2/3	0	−1/3	1	0
$Z^* = 1$		$c_j - z_j$	0	0	−3	−1	0	1	0	0	1

Table 24.7
Second Iteration

Iteration 3: Since $s_1 = 0$, λ_1 can be entered into the basis in Table 24.7, with A_2 as the leaving variable. The new solution is shown in Table 24.8.

		$c_j \rightarrow$	0	0	0	0	0	0	0	0
c_B	Basic Variables B	Solution Values $b\ (=x_B)$	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2
0	x_1	2/3	1	0	0	2/3	−1/2	1/3	0	0
0	λ_1	1/3	0	0	1	1/3	0	−1/3	0	0
0	x_2	14/9	0	1	0	−4/9	1/3	−2/9	1/3	0
0	s_2	10/9	0	0	0	−8/9	2/3	−4/9	−1/3	1
$Z^* = 0$		$c_j - z_j$	0	0	0	0	0	0	0	0

Table 24.8
Third Iteration

In Table 24.8, all $c_j - z_j = 0$, therefore an optimal solution for Phase I is reached. The optimal solution is:

$$x_1 = 2/3,\ x_2 = 14/9,\ \lambda_1 = 1/3,\ \lambda_2 = 0;\ \mu_1 = \mu_2 = 0,\ s_1 = 0,\ s_2 = 10/9$$

This solution also satisfies the complementary slackness conditions: $\lambda_1 s_1 = \lambda_2 s_2 = 0$; $\mu_1 x_1 = \mu_2 x_2 = 0$ and the restriction on the signs of Lagrange multipliers: λ_1, λ_2, μ_1 and μ_2. Since, $Z^* = 0$, the current solution is also feasible. The maximum value of the objective function of the given quadratic problem is:

$$\text{Max } Z = 2x_1 + x_2 - x_1^2 = 2\,(2/3) + (14/9) - (2/3)^2 = 22/9$$

CONCEPTUAL QUESTIONS A

1. What is meant by quadratic programming? How does a quadratic programming problem differ from a linear programming problem? Give an example. [*Delhi Univ., MBA, 2005*]
2. Is it correct to say that in a quadratic programming problem the objective function and the constraints both should be quadratic? If not, give your own comments. [*AMIE, 2007*]
3. Derive the Kuhn-Tucker necessary conditions for an optimal solution to a quadratic programming problem.
4. Obtain the Kuhn-Tucker conditions for a solution of the problem:

$$\text{Max } Z = \mathbf{cx} + \frac{1}{2}\,\mathbf{x}^T\mathbf{dx},$$

subject to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$.

5. What is quadratic programming? Explain Wolfe's method of solving it.
6. Briefly mention Wolfe's algorithm for solving a quadratic programming problem:

$$\text{Max } Z = \mathbf{cx} + \frac{1}{2}\,\mathbf{x}^T\mathbf{qx}$$

subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$.

SELF PRACTICE PROBLEMS B

Use Wolfe's method for solving the following quadratic programming problems:

1. Max $Z = 2x_1 + 3x_2 - 2x_1^2$

   subject to (i) $x_1 + 4x_2 \leq 4$, (ii) $x_1 + x_2 \leq 2$

   and $x_1, x_2 \geq 0$

2. Min $Z = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$

   subject to $x_1 + x_2 \leq 2$,

   and $x_1, x_2 \geq 0$

3. Max $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

   subject to $3x_1 + 2x_2 \leq 6$

   and $x_1, x_2 \geq 0$

4. Min $Z = x_1^2 + x_2^2 + x_3^2$

   subject to (i) $x_1 + x_2 + 3x_3 = 2$, (ii) $5x_1 + 2x_2 + x_3 = 5$

   and $x_1, x_2, x_3 \geq 0$

5. Max $Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

   subject to $x_1 + 2x_2 \leq 2$

   and $x_1, x_2 \geq 0$ [*IAS (Main), 1994*]

6. Min $Z = x_1^2 + x_2^2 + x_3^2$
   subject to (i) $2x_1 + x_2 - x_3 \le 0$, (ii) $1 - x_1 \le 0$
   and $x_1, x_2, x_3 \ge 0$

Use Beale's method to solve the following quadratic programming problems:

7. Max $Z = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 - 4x_1x_2$
   subject to (i) $x_1 + 2x_2 \le 10$, (ii) $x_1 + x_2 \le 9$
   and $x_1, x_2 \ge 0$

8. Max $Z = 2x_1 + 3x_2 - x_1^2$
   subject to $x_1 + 2x_2 \le 4$,
   and $x_1, x_2 \ge 0$

9. Max $Z = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$
   subject to $x_1 + x_2 \le 4$
   and $x_1, x_2 \ge 0$

10. Min $Z = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$
    subject to $x_1 + x_2 \le 2$
    and $x_1, x_2 \ge 0$

11. Min $Z = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$
    subject to (i) $2x_1 + x_2 \ge 6$, (ii) $x_1 - 4x_2 \ge 0$
    and $x_1, x_2 \ge 0$

12. Max $Z = 2x_1x_2 - 5x_1 - 13x_2 + 3x_2^2 - 10$
    subject to (i) $x_1 + x_2 \le 1$, (ii) $4x_1 + x_2 \ge 2$
    and $x_1, x_2 \ge 0$

HINTS AND ANSWERS

1. $x_1 = 0$, $x_2 = 1$, $\lambda_1 = 1/3$, $\lambda_2 = 5/3$, and Max $Z = 3$
2. $x_1 = 3/2$, $x_2 = 1/2$, $\lambda_1 = 0$, $\lambda_2 = 1$, and Max $Z = 1/2$
3. $x_1 = 4/13$, $x_2 = 33/13$, and Max $Z = 267/13$
4. $x_1 = 81/100$, $x_2 = 7/20$, $x_3 = 7/20$, and Max $Z = 17/20$
5. $x_1 = 1/3$, $x_2 = 5/6$ and Max $Z = 25/6$.
7. $x_1 = 0$, $x_2 = 5$ and Max $Z = 100$
8. $x_1 = 1/4$, $x_2 = 15/8$ and Max $Z = 97/16$
9. $x_1 = 2$, $x_2 = 1$ and Max $Z = 7$
10. $x_1 = 3/2$, $x_2 = 1/2$ and Min $Z = 1/2$
11. $x_1 = 16/5$, $x_2 = 4/5$ and Min $Z = -32/5$.
12. $x_1 = 1/2$, $x_2 = 0$, and Max $Z = -15$.

24.6 SEPARABLE PROGRAMMING

Separable programming is one of the indirect methods used to solve a non-linear programming prob
Indirect methods solve an NLP problem by dealing with one or more linear problems that are extracted

Chapter 23

Classical Optimization Methods

"Management by objective works – if you know the objectives. Ninety percent of the time you don't."

– Drucker, Peter F.

PREVIEW

The classical optimization methods, that are analytical in nature, make use of differential calculus in order to find optimal value for both unconstrained and constrained objective functions.

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- use differential calculus-based methods (also called Classical Optimization Methods) to obtain an optimal solution of problems that involve continuous and differential functions.
- drive necessary and sufficient conditions for obtaining an optimal solution for both unconstrained and constrained, single and multivariable, optimization problems, with equality and inequality constraints.
- make distinction between local, global and inflection extreme points.

CHAPTER OUTLINE

23.1 INTRODUCTION

The classical optimization methods are used to obtain an optimal solution of certain types of problems that involve continuous and differentiable functions. These methods are analytical in nature and make use of differential calculus to find points of maxima and minima for both unconstrained and constrained continuous objective functions. In this chapter, we shall discuss the necessary and sufficient conditions for obtaining an optimal solution of

(i) Unconstrained single and multiple variable optimization problems, and
(ii) Constrained multivariable optimization problems with equality and inequality constraints.

23.2 UNCONSTRAINED OPTIMIZATION

23.2.1 Optimizing Single-Variable Functions

Figure 23.1 depicts the graph of a continuous function $y = f(x)$ of single independent variable, x in the domain (a, b). The *domain* is the range of values of x. The domain limits (or end points) are generally called *stationary (or critical) points*. There are two categories of stationary points: *inflection points and extreme points*. The extreme points are further classified as either *local (or relative) or global (or absolute)* extrema.

Local extreme points represent the maximum or minimum values of the function in the given range of values of the variable. In Fig. 23.1, points a, x_1, x_2, x_3, x_4, x_5 and b are all extrema of $f(x)$. The classical approach to the theory of maxima and minima does not provide a direct method of obtaining global (or absolute) maximum (or minimum) value of a function. It only provides the method for determining the local (or relative) maximum and minimum values.

Mathematically, a function $y = f(x)$ is said to achieve its maximum value at a point, $x = x_0$, if

$$f(x_0 + h) - f(x_0) < 0 \quad \text{or} \quad f(x_0 + h) < f(x_0)$$

where h is a sufficiently small number in the neighbourhood of the point $x = x_0$. In other words, the point x_0 is a local maximum if the value of $f(x)$ at every point in the neighbourhood of x_0 does not exceed $f(x_0)$.

Similarly, a function $f(x)$ is said to achieve its minimum value at a point, $x = x_0$ if:

$$f(x_0 + h) - f(x_0) > 0 \quad \text{or} \quad f(x_0 + h) > f(x_0)$$

When a function has several local maximum and minimum values, the global minimum (in case of cost minimization) or global maximum (in case of profit maximization) is obtained by comparing the values of the function at various extreme points (including the limits of the domain). The global minimum value of a function is the minimum value among all local minimum values of the function in the domain. Similarly, the global maximum value of a function is the maximum value among all local maximum values of the function in the domain. In Fig. 23.1, the point E, i.e. $f(x_4)$ represents the global maximum, whereas the point F, i.e. $f(x_5)$ represents the global minimum.

Local extreme points represent the maximum or minimum values of any continuous function, $y = f(x)$ in the range of values of x.

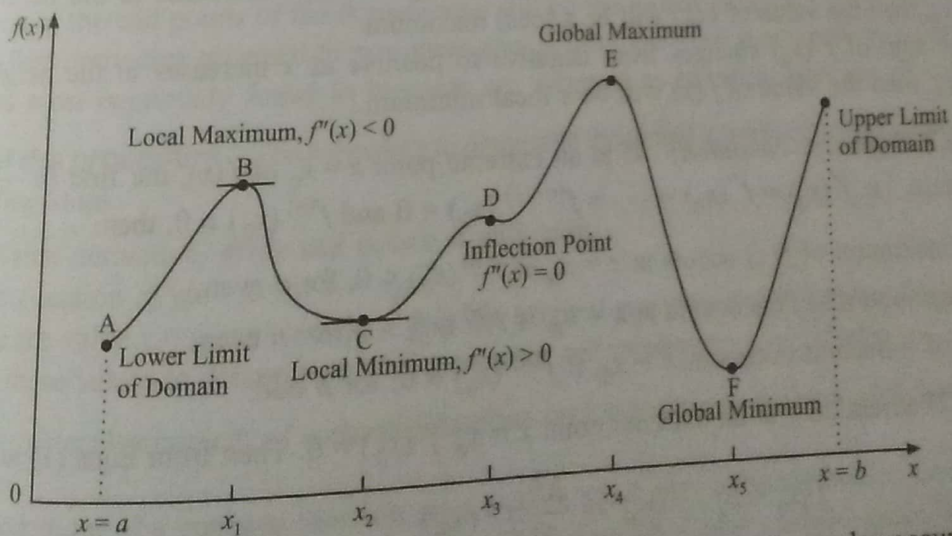

Fig. 23.1 Local and Global Optimum

The global maximum (or minimum) of a function over the larger interval can also occur at an end point of the interval rather than at any local (relative) maximum or minimum point. It is possible for a local maximum value of a function to be less than a local minimum value of the function.

23.2.2 Conditions for Local Minimum and Maximum Value

Theorem 23.1 (*Necessary condition*) A necessary condition for a point x_0 to be the local extrema (local maximum and minimum) of a function $y = f(x)$ defined in the interval $a \le x \le b$ is that the first derivative of $f(x)$ exists as a finite number at $x = x_0$ and $f'(x_0) = 0$.

Proof Let $y = f(x)$ be a given function that can be expanded in the neighbourhood of $x = x_0$ by *Taylor's theorem*. Let at $x = x_0$ the value of $f(x)$ be $f(x_0)$.

Consider two values of x, namely $+h$ and $-h$, in the neighbourhood and either side of $x = x_0$ (h being very small). If maximum is at $x = x_0$, then from definition, $f(x_0) > f(x_0 + h)$ and $f(x_0) > f(x_0 - h)$. That is, $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ are both negative for maximum at $x = x_0$. Further, if minimum is at $x = x_0$, then $f(x_0) < f(x_0 + h)$ and $f(x_0) < f(x_0 - h)$. That is, $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ are both positive for minimum at $x = x_0$. By using Taylor's theorem, we have:

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots + \frac{h^n}{n!} f^n(x_0) + R_n(x_0 + \theta h); \quad 0 < \theta < 1$$

or
$$f(x_0 + h) - f(x_0) = h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots \tag{1}$$

where,
$$R_n(x_0 + \theta h) = \frac{h^{n+1}}{(n+1)!} f^{n+1}(x_0 + \theta h)$$

and is called the remainder.

The expressions $f'(x_0)$ and $f''(x_0)$ represent the first and second derivative of $f(x)$ at $x = x_0$. Similarly,

$$f(x_0 - h) = f(x_0) - h f'(x_0) + \frac{h^2}{2!} f''(x_0) - \ldots$$

$$f(x_0 - h) - f(x_0) = - h f'(x_0) + \frac{h^2}{2!} f''(x_0) - \ldots \tag{2}$$

Global minimum (or maximum) value of a function is the minimum (or maximum) value among all local minimum (or maximum) values of the function.

If h is very small, then neglecting the terms of higher order, we get,

$$f(x_0 + h) - f(x_0) = h f'(x_0) \tag{3}$$

and
$$f(x_0 - h) - f(x_0) = - h f'(x_0) \tag{4}$$

For $x = x_0$ to be a local maximum or minimum value, the sign of $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ must be the same for all $x = x_0 \pm h$. Thus from Eqns (3) and (4) if $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ have the same sign, then $f'(x_0)$ should be zero; otherwise they will have different signs. Hence the *necessary condition for any function f(x) to* have local optimum value at any extreme point $x = x_0$, is that its first derivative $f'(x_0) = 0$.

Remark The distinction between a local minimum and local maximum can also be seen by examining the direction of change of first derivative, $f'(x_0)$ at $x = x_0$.

(i) If the sign of $f'(x_0)$ changes from positive to negative as x increases in the neighbourhood of $x = x_0$, then the value of $f(x)$ will be a local maximum.
(ii) If the sign of $f'(x_0)$ changes from negative to positive as x increases in the neighbourhood of $x = x_0$, then the value of $f(x)$ will be a local minimum.

Theorem 23.2 (*Sufficient condition*) If at an extreme point $x = x_0$ of $f(x)$, the first $(n - 1)$ derivatives of it become zero, i.e. $f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \ne 0$, then:

(i) local maximum of $f(x)$ occurs at $x = x_0$, if $f^{(n)}(x_0) < 0$, for n even,
(ii) local minimum of $f(x)$ occurs at $x = x_0$, if $f^{(n)}(x_0) > 0$, for n even,
(iii) point of inflection occurs at $x = x_0$, if $f^{(n)}(x_0) \ne 0$, for n odd.

Proof From Theorem 23.1 at an extreme point $x = x_0$, $f'(x_0) = 0$. Then from Eqns (1) and (2), we have

$$f(x_0 + h) - f(x_0) = \frac{h^2}{2!} f''(x_0) \tag{5}$$

and
$$f(x_0 - h) - f(x_0) = \frac{h^2}{2!} f''(x_0) \tag{6}$$

neglecting powers of h higher than second. Here, the following three possible cases may arise:

Case 1: If $f''(x_0) > 0$, then both $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ are positive and hence local minimum value of $f(x)$ exists at $x = x_0$.

Case 2: If $f''(x_0) < 0$, then both $f(x_0 + h) - f(x_0)$ and $f(x_0 - h) - f(x_0)$ are negative and hence local maximum value of $f(x)$ exists at $x = x_0$.

Case 3: If $f''(x_0) = 0$, then no information is obtained about the maximum or minimum value of $f(x)$. That is, in this case the function $f(x)$ may have a local maximum, a local minimum, or a point of inflection. Hence, if $f''(x_0) = 0$, then we examine successively higher order derivatives of $f(x)$ at $x = x_0$ until we find a derivative such that $f^{(n)}(x_0) \neq 0,\ n \geq 2$.

If $f^{(n)}(x_0) < 0$, for n even, then $f(x)$ has local maximum value at $x = x_0$. If $f^{(n)}(x_0) > 0$, for n even, then $f(x)$ has local minimum value at $x = x_0$. If n is odd, then $x = x_0$ is the point of inflection (or saddle point).

The *necessary* and *sufficient* conditions for the existence of local maximum and minimum and point of inflection are summarized in Table 23.1. The entire preceding discussion is summarized in Fig. 23.2.

Necessary Condition	Sufficient Condition	Nature of Function	Conclusion
$f'(x_0) = 0$	$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) < 0$, n even	Concave	Local maximum at $x = x_0$
$f'(x_0) = 0$	$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) > 0$, n even	Convex	Local minimum at $x = x_0$
$f'(x_0) = 0$	$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$, n odd	–	Point of inflection at $x = x_0$

Table 23.1 Conditions for Local Maximum, Minimum and Point of Inflection.

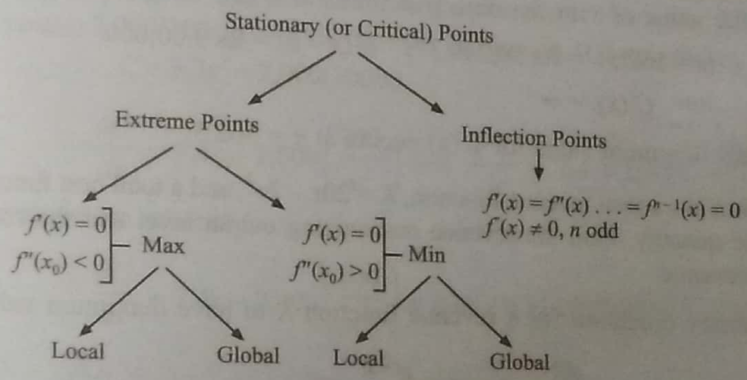

Fig. 23.2 Determination of Critical Point

It becomes easy to find the maximum or minimum values when the function is either convex or concave. If a function is convex, the first derivative set equal to zero must give at least one local minimum. The value of the function at the end points of the domain may still be the global minimum. Similarly, if a function is concave, the first derivative set equal to zero must give at least one local maximum. It is due to this reason that functions most commonly found in business are assumed to be either concave or convex.

Summary of the procedure The procedure to determine the global minimum or maximum is summarized in the following steps:

1. Compute first derivative, dy/dx and equate it with zero.
2. Solve the equation $dy/dx = 0$ for $x = x_0$.
3. Substitute the value $x = x_0$, $x = a$ and $x = b$ in the original equation and determine $f(x_0)$, $f(a)$ and $f(b)$.
4. Compare these values to determine global maximum and minimum, respectively.

Remarks 1. A local minimum of a convex function on a convex set is also a global minimum of that function.

2. A local maximum of a concave function on a convex set is also a global maximum of that function.
3. A local minimum of a strictly convex function on a convex set is also a unique global minimum of that function.
4. A local maximum of a strictly concave function on a convex set is also a unique global maximum of that function.

Example 23.1 A trader receives x units of an item at the beginning of each month. The cost of carrying x units per month is given by:

$$C(x) = \frac{c_1 x^2}{2n} + \frac{c_2 (20n - x)^2}{2n}$$

where c_1 = cost per day of carrying a unit of item in stock (= Rs 10)
c_2 = cost per day of shortage of a unit of item (= Rs 150)
n = number of units of item to be supplied per day (= 30)

Determine the order quantity x that would minimize the cost of inventory.

Solution The necessary condition for a function to have either minimum or maximum value at a point is that its first derivative should be zero. Thus,

$$\frac{dC(x)}{dx} = \frac{c_1 x}{n} - \frac{c_2 (20n - x)}{n} = 0$$

Therefore,
$$x = \frac{20n c_2}{c_1 + c_2} = \frac{(20)(30)(150)}{10 + 150} = 562.5$$

The nature of the extreme point given by x is determined by considering the second derivative.

$$\frac{d^2 C(x)}{dx^2} = \frac{c_1}{n} + \frac{c_2}{n} > 0$$

Since the value of the second derivative is positive, therefore, $x = 562.5$ is a local minimum point.

By substituting the value of x in the objective function $C(x)$, we get

$$C(x = 562.5) = \text{Rs } 56{,}249.37; \quad C(x = 0) = \text{Rs } 9{,}00{,}000$$

$$\lim_{x \to \infty} C(x) = \infty$$

It follows that, a global minimum value for $C(x)$ occurs at $x = 562.5$.

Example 23.2 A firm has a total revenue function, $R = 20x - 2x^2$, and a total cost function, $c = x^2 - 4x + 20$, where x represents the quantity. Find the revenue maximizing output level and the corresponding value of profit, price and total revenue.

Solution The necessary condition for a revenue function R to have maximum value at a point is that:

$$\frac{dR}{dx} = 0 \text{ and } \frac{d^2 R}{dx^2} < 0.$$

Since $R = 20x - 2x^2$, therefore $dR/dx = 0$ gives $20 - 4x = 0$ or $x = 5$. Also $d^2R/dx^2 = -4\ (< 0)$. Since the value of second derivative is negative, the revenue will be maximum at an output level, $x = 5$.

The profit function is given by:

$$\pi = R - C = (20x - 2x^2) - (x^2 - 4x + 20) = 24x - 3x^2 - 20$$

Thus, the total profit at $x = 5$ will be: $P = 24(5) - 3(5)^2 - 20 = 25$.

The price of a product is given by $P = \dfrac{\pi}{x} = 20 - 2x = 10$, at $x = 5$. The maximum revenue at $x = 5$, is $R = 20(5) - 2(5)^2 = 50$.

Example 23.3 The total profit y, in rupees, of a drug company from the manufacturing and sale of x drug bottles is given by, $y = -(x^2/400) + 2x - 80$

(a) How many drug bottles must the company sell in order to achieve the maximum profit?
(b) What is the profit per drug bottle when this maximum is achieved?

Solution Given $y = -(x^2/400) + 2x - 80$. Therefore,

$$\frac{dy}{dx} = -\frac{2x}{400} + 2 = -\frac{x}{200} + 2$$

The first order condition for maximum value of y is $dy/dx = 0$, i.e. $-(x/200) + 2 = 0$ or $x = 400$.

Since $d^2y/dx^2 = -1/200(< 0)$, therefore the company must sell $x = 400$ drug bottles in order to achieve the maximum profit, which is equal to $y = -(400)^2/400 + 2 \times 400 - 80 =$ Rs 320.

Example 23.4 The efficiency E of a small manufacturing concern depends on the workers W and is given by $10E = -(W^3/40) + 30W - 392$. Find the strength of the workers that would give the maximum efficiency.

Solution Given, $10E = -(W^3/40) + 30W - 392$ or $E = -(W^3/400) + 3W - 392$. Therefore,

$$\frac{dE}{dW} = -\frac{3W^2}{400} + 3$$

The first order condition for maximum value of E is $dE/dW = 0$, i.e. $-(3W^2/400) + 3 = 0$ or $W = \pm 20$ (neglecting $W = -20$ because workers cannot be negative in number).

Also $\dfrac{d^2E}{dW^2} = -\dfrac{6W}{400}(< 0)$ at $W = 20$ (a second-order condition for maxima), therefore the efficiency of the workers shall be maximum when they are $W = 20$ in number.

Example 23.5 The cost of fuel for running a train is proportional to the cube of the speed generated in km per hour. When the speed is 12 km per hour, the cost of fuel is Rs 64 per hour. If other charges are fixed, namely Rs 2,000 per hour, find the most economical speed of the train for running a distance of 100 km.

Solution Let x km per hour be the speed of the train. Then, the cost of fuel $= kx^3$, where k is constant of proportionality.

Given that it cost Rs 64 per hour at 12 km/hour. Therefore, $64 = k(12)^3$ or $k = 64/(12)^3 = 0.037$. Hence, cost of fuel $= 0.037x^3$ rupees per hour. The fuel for running a distance of 100 km is: $0.037x^3 \cdot (100/x) = 3.7x^2$

Also, the Fixed cost $= 2{,}000(100/x)$. If C is the cost of running 100 km, then,

$$C = 3.7x^2 + 2{,}000(100/x)$$

$$\frac{dC}{dx} = 7.4x - 2{,}000\left(\frac{100}{x^2}\right) \text{ and } \frac{d^2C}{dx} = 7.4 + 2{,}000\left(\frac{200}{x^3}\right)$$

For maximum or minimum value of C,

$$\frac{dC}{dx} = 7.4x - 2{,}000\left(\frac{100}{x^2}\right) = 0 \text{ or } x = (27{,}027.027)^{1/3} = 30$$

For this value of x, $\dfrac{d^2C}{dx^2} > 0$, i.e. C is minimum. Thus, the most economic speed of the train should be 30 km/hour.

Example 23.6 The production function of a commodity is given by: $Q = 40F + 3F^2 - (F^3/3)$, where Q is the total output and F is the units of inputs.

(a) Find the number of units of input required to give the maximum output.
(b) Find the maximum value of marginal product.
(c) Verify that when the average product is maximum, it is equal to marginal product.

Solution (a) We have, $Q = 40F + 3F^2 - (F^3/3)$; $F \geq 0$.

For maximum or minimum output level,

$$\frac{dQ}{dF} = 40 + 6F = 0 \text{ or } (F + 10)(F - 4) = 0, \text{ i.e. } F = 4 \text{ or } -10. \text{ Also } \frac{d^2Q}{dF^2} = 6 - 2F$$

For $F = 4$, $\dfrac{d^2Q}{dF^2} = 6 - 2(4) = -2\ (< 0)$ and for $F = -10$, $\dfrac{d^2Q}{dF^2} = 6 - 2(-10) = 14(< 0)$.

Thus, the output Q is maximum when $F = 4$ units of input are used.

(c) If signs of determinants do not meet conditions (i) and (ii), then the extreme point may be either a maximum or a minimum or neither. In this case the matrix $\mathbf{H}(\mathbf{x})$ is termed as semi-definite or indefinite.

To illustrate these results, consider $\mathbf{H}(\mathbf{x})$ as:

$$\mathbf{H}(\mathbf{x}) = \begin{bmatrix} \begin{bmatrix} [5] & 3 \\ 3 & 4 \end{bmatrix} & \begin{matrix} 0 \\ -1 \end{matrix} \\ \begin{matrix} 0 & -2 \end{matrix} & 4 \end{bmatrix}$$

The leading principal minors of this matrix are:

$$|5| = 5, \quad \begin{vmatrix} 5 & 3 \\ 3 & 4 \end{vmatrix} = 11 \quad \text{and} \quad \begin{vmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -2 & 3 \end{vmatrix} = 34$$

In general, we need to determine, and evaluate determinates:

$$|a_{11}|; \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}; \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ and so on.}$$

2. Summary of results

Table 23.2

Necessary Condition	*Sufficient Condition*	*Conclusion*
$\nabla f(x_0) = 0$	$\mathbf{H}(x_0)$ is positive definite	Local minimum at $x = x_0$
$\nabla f(x_0) = 0$	$\mathbf{H}(x_0)$ is negative definite	Local maximum at $x = x_0$
$\nabla f(x_0) = 0$	$\mathbf{H}(x_0)$ is indefinite	Point of inflection at $x = x_0$

Example 23.7 Find the second order Taylor's series approximation of the function:

$$f(x_1, x_2) = x_1^2\, x_2 + 5x_1\, e^{x_2}$$

about the point $x_0 = [1, 0]^T$ [AMIE, 2005]

Solution The second order Taylor's series approximation of the function $f(x_1, x_2)$ at $x = x_0$ is:

$$f(x_1, x_2) = f\begin{bmatrix}1\\0\end{bmatrix} + \nabla f\begin{bmatrix}1\\0\end{bmatrix} h + \frac{1}{2!} h^{\mathrm{T}} \mathbf{H}(\mathbf{x}) h, \quad \text{where } x = x_0 + \theta\, h, \text{ and}$$

$$h = \begin{bmatrix}h_1\\h_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x_1 - 1\\x_2\end{bmatrix}$$

$$x_0 + \theta h = \begin{bmatrix}1\\0\end{bmatrix} + \theta\begin{bmatrix}x_1 - 1\\x_2\end{bmatrix} = \begin{bmatrix}1 + \theta x_1 - \theta\\\theta x_2\end{bmatrix}$$

$$\nabla f(x_0) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = \left(2x_1 x_2 + 5e^{x_2},\; x_1^2 + 5x_1 e^{x_2}\right)$$

For $x_0 = [1, 0]^T$, the value of $\nabla f(x_0) = [5,\ 6]$

$$\mathbf{H}(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\, \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2x_2 & 2x_1 + 5e^{x_2} \\ 2x_1 + 5e^{x_2} & 5x_1 e^{x_2} \end{bmatrix}$$

Substituting the values in $f(x_1, x_2)$, we get:

$$f(x_1, x_2) = 5 + [5, 6]\begin{bmatrix}x_1 - 1\\x_2\end{bmatrix} + \frac{1}{2!}\begin{bmatrix}x_1 - 1\\x_2\end{bmatrix}^T \begin{bmatrix} 2x_2 & 2x_1 + 5e^{x_2} \\ 2x_1 + 5e^{x_2} & 5x_1 e^{x_2} \end{bmatrix}\begin{bmatrix}x_1 - 1\\x_2\end{bmatrix}$$

Example 23.8 Consider the function, $f(x) = x_1 + 2x_2 + x_1x_2 - x_1^2 - x_2^2$. Determine the maximum or minimum point (if any) of the function.

Solution The necessary condition for local optimum (maximum or minimum) value is that gradient

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = 0$$

That is, $\partial f/\partial x_1 = 1 + x_2 - 2x_1 = 0$, and $\partial f/\partial x_2 = 2 + x_1 - 2x_2 = 0$. The solution of these simultaneous equations, is: $x_0 = (4/3, 5/3)$.

The sufficient condition can be verified by considering the Hessian matrix as follows:

$$\mathbf{H}(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\, \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2\, \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\det A_1 = \left|\frac{\partial^2 f}{\partial x_1^2}\right| = -2, \text{ and } \det A_2 = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\, \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2\, \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{vmatrix} = 4 - 1 = 3$$

Since the signs of the principal minor determinants of $\mathbf{H}(\mathbf{x})$ are alternating, matrix $\mathbf{H}(\mathbf{x})$ is negative definite and the point $x_0 = (4/3, 5/3)$ is the local maximum of the function $f(x)$.

SELF PRACTICE PROBLEMS A

1. Examine the following functions for extreme points
   (a) $f(x) = 4x^4 - x^2 + 5$
   (b) $f(x) = (3x - 2)^2 (2x - 3)^2$
   (c) $f(x) = x^5/5 - 5x^4/2 + 35x^3/3 - 25x^2 + 24x$
   (d) $f(x) = x^3 - 15x^2 + 10x + 100$
2. Examine the following functions for extreme points:
   (a) $f(x_1, x_2) = 3x_1^2 + x_2^2 - 10$
   (b) $f(x_1, x_2) = 100 (x_1 - x_2^2)^2 + (1 - x_1)^2$
   (c) $f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2$
   (d) $f(x_1, x_2) = 26x_1 - 5x_1^2 + 2x_2 - 10$
3. Determine the local maximum or minimum point (if any) of the following functions:
   (a) $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + x_3^2 + x_1x_2 - 2x_3 - 7x_1 + 12$
   (b) $f(x_1, x_2) = 12x_1x_2 + 5x_2^2$
   (c) $f(x_1, x_2, x_3) = -x_1^2 - 2x_2 - x_3^2 - 2x_1x_2$
   (d) $f(x_1, x_2) = x_1^2 - 4x_1x_2^2 + x_2^2$
   (e) $f(x_1, x_2, x_3) = x_1x_2 + 10x_1 - x_1^2 + x_2^2 - x_3^2$
   (f) $f(x_1, x_2) = x_1^2 + 5x_2^2 + 7x_1x_2^2$
4. Show that a cube curve whose equation is of the form: $y = ax^3 + bx^2 + cx + d$, where $a, b, c, d \neq 0$, has one and only one point of inflection.
5. Show that the demand curves, $p = \dfrac{a}{x+b} - c$ and $p = (a - bx)^2$ are both downward sloping and convex from below.
6. The cost of producing x units of a product is given as: $C(x) = 0.001x^3 - 0.3x^2 + 30x + 42$. Determine whether the cost function is concave up and where it is concave down. Also find the inflection point.
7. A firm's revenue function is given by $R = 80\,D$, where R is gross revenue and D is quantity sold. A production cost function is given by: $C = 1{,}50{,}000 + 60\,(D/900)^2$. Find the total profit function and the number of units to be sold in order to get the maximum profit.
8. A manufacturer can sell 'x' items per week at a price $P = 20 - 0.001x$ rupees each. It costs $Y = 5x + 200$ rupees to produce 'x' items. Determine the number of items the manufacturer has to produce per week for obtaining the maximum profit.
9. An indifference map is defined by the relation: $(x + h)\sqrt{y + k} = a$, where h and k are fixed positive numbers and a is a positive parameter. By expressing y as a function of x and by finding derivatives, show that each indifference curve is downward sloping from below.
10. For a particular process, the average cost is given by $C = 56 - 8x + x^2$, where C is the average cost per unit and x the number of units produced. Find the minimum value of the average cost and the corresponding number of units to be produced.
11. A company has examined its cost structure and revenue structure and has determined that C the total cost, R the total revenue, and x the number of units produced, are related as: $C = 100 + 0.015x^2$ and $R = 3x$. Find the production rate x that will maximize profits of company, and the profit.
12. The demand function for a particular commodity is, $p = 15e^{-x/3}$, where p is the price per unit and x is the number of units

demanded. Determine the price and the quantity for which the revenue (R) is maximum.

13. If the total revenue (R) and total cost (C) function of a firm are given by $R = 30x - x^2$ and $C = 20 + 4x$, where x is the output, find the equilibrium level output of the firm. What is the maximum profit ?

14. Let the total cost (C) function of firm be given by the equation, $C = 300x - 10x^2 + (x^3/3)$, where C stands for cost and x for output. Calculate the output at which:
    (a) marginal cost is minimum,
    (b) average cost is minimum, and
    (c) average cost is equal to marginal cost.

15. A firm produces x units of output per week at a total cost of $(x^3/3) - x^2 + 5x + 3$ rupees. Find the output level at which:
    (i) the marginal cost (MC) and the average variable cost (AC) attain their respective minimum value and
    (ii) $MC = AC$

16. For a product manufactured by a monopolist firm, the unit demand function is: $x = (1/3)(25 - 2p)$, where x is the number of units and p is the price. Let the average cost per unit be Rs 4. Find:
    (a) the revenue function R in terms of price p
    (b) the cost function C in terms of price p
    (c) the profit function p
    (d) the price per unit that maximizes the profit, and
    (e) the maximum profit.

17. A radio manufacturer finds that he can sell x radios per week at Rs p each, where $p = 2\{100 - (x/4)\}$. His cost of production of x radios per week is Rs $\{120x + (x^2/2)\}$. Show that his profit is maximum when the production is 40 radios per week. Also find his maximum profit per week.

18. The price p per unit at which a company can sell all that it produces is given by the function: $p = 300 - 4x$. The cost function is, $C(x) = 500 + 28x$, where x is the number of units produced. Find that value of x which would maximize profit.

19. The demand function faced by a firm is, $p = 500 - 0.2x$ and its cost function is $C = 25x + 1,000$, where p = price, x = output and C = cost. Find the output at which the profit of the firm is maximum. Also find the price it will charge for the maximum profit.

20. There are 60 newly built apartments. All these would be occupied at a rent of Rs 4,500 per month. But one apartment is likely to remain vacant for every Rs 150 increase in rent. An occupied apartment requires Rs 6 month for maintenance. Find the relationship between profit and the number of unoccupied apartments. What is the number of vacant apartments for which the profit is maximum?

HINTS AND ANSWERS

2. (a) $x_0 = (4, -1, 1)$, local minimum
   (b) $x_0 = (0, 0, 0)$, local maximum
   (c) $x_0 = (8, 4, 3)$, local maximum

4. Point of inflection at $x = -b/3a$.

5. $\frac{dp}{dx} = -a/(x+b)^2\ (< 0)$ and $(x+b)^2 > 0$, curve is downwards sloping.

   Again $\frac{d^2p}{dx^2} = \frac{2a}{(x+b)^2}\ (> 0)$, curve is convex from below.

6. $\frac{d^2C}{dx^2} > 0$ for $x > 100$; $\frac{d^2C}{dx^2} < 0$ for $x < 100$ and $\frac{d^2}{dx^2} = 0$ at $x = 100$

7. Profit = Revenue – Cost = $80D - 1,50,000 - 60\,(D/900)^2$; Max. profit at, $D = 5,400$

8. Profit = Revenue – Cost = $(20 - 0.001x)x - (5x + 2,000)$; Max. profit at, $x = 7,500$ units

9. $x + H = \frac{a}{\sqrt{y+k}}\ (< 0)$ or $x = \frac{a}{\sqrt{y+k}} - h$

   $\frac{dy}{dx} = \frac{-a}{2(y+k)^{3/2}}\ (< 0)$; curves are sloping downwards

   $\frac{d^2x}{dx^2} = \frac{3a}{2(y+k)^{5/2}}\ (> 0)$; curves are convex from below

10. $x = 4$, Min $C = 40$.

11. $P = R - C = 3x - 100 - \frac{15x^2}{1,000}$; $x = 100$ units;
    Max. P = Rs. 50 at $x = 100$

12. $R = yx = 15x\,e^{-x/3}$; $x = 3$ or ∞ (absurd); Max. P = Rs 50 at $x = 100$

13. $P = R - C = 26x - x^2 - 20$; $x = 13$, Max P = Rs 149

14. (i) $MC = \frac{dC}{dx} = 300 - 20x + x^2$; $x = 10$;
    (ii) $AC = \frac{C}{x} = 300 - 10x + \frac{x^2}{3}$; $x = 15$
    (iii) $AC = MC$ gives $x = 15$

15. Same as, Q.14

16. (i) $R = x,\ p = (1/3)(25 - 2p)\,p$
    (ii) $C = 40,\ x = 40\ (1/3)(25 - 2p)$
    (iii) $P = x,\ p - C = (1/3)(25p - 2p^2) - (40/3)(25 - 2p)$
    (iv) $p = 105/4$

17. Profit $P = x \cdot p - C$, solve $\frac{dP}{dx} = 0$ for x.

18. Total revenue function, $R = x(300 - 4x)$.
    Profit, $P = R - C = -4x^2 + 272x - 500$. Solve $\frac{dP}{dx} = 0$ for x; $x = 34$.

19. Total profit function, R = Revenue – Cost = $p \cdot x - (25x + 1,000)$
    $= 475x - 1,000 - 0.2x^2$;
    Solve $\frac{dP}{dx} = 0$ for x; $x = 1187.50$; $P = 262.50$.

20. Let x be the vacant apartments;
    Profit = Revenue – Cost = $(4,500 + 150x)(60 - x) - 6x$.

23.3 CONSTRAINED MULTIVARIABLE OPTIMIZATION WITH EQUALITY CONSTRAINTS

In this section, we shall discuss the problem of optimizing a continuous and differentiable function subject to equality constraints. That is:

Optimize (max or min) $Z = f(x_1, x_2, \ldots, x_n)$

subject to the constraints

$$h_i(x_1, x_2, \ldots, x_n) = b_i; \quad i = 1, 2, \ldots, m$$

In matrix notation the above problem can also be written as:

Optimize (max or min) $Z = f(\mathbf{x})$

subject to the constraints

$$g_i(\mathbf{x}) = 0, \quad i = 1, 2, \ldots, m$$

where
$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \tag{10}$$

and
$$g_i(\mathbf{x}) = h_i(\mathbf{x}) - b_i; \quad b_i \text{ is a constant} \tag{11}$$

Here it is assumed that $m < n$ to get the solution.

There are various methods for solving the above defined problem. But in this section, we shall discuss only two methods:

(i) *Direct Substitution Method*, and
(ii) *Lagrange Multipliers Method.*

23.3.1 Direct Substitution Method

Since the constraint set $g_i(\mathbf{x})$ is also continuous and differentiable, any variable in the constraint set can be expressed in terms of the remaining variables. Then it is substituted into the objective function. The new objective function, so obtained, is not subject to any constraints and hence its optimum value can be obtained by the unconstrained optimization method, discussed in the previous section.

Sometimes this method is not convenient, particularly when there are more than two variables in the objective function and are subject to constraints.

Example 23.9 Find the optimum solution of the following constrained multivariable problem.

Minimize $Z = x_1^2 + (x_2 + 1)^2 + (x_3 - 1)^2$

subject to the constraint

$$x_1 + 5x_2 - 3x_3 = 6,$$

and $x_1, x_2, x_3 \geq 0$ [*AMIE, 2005*]

Solution Since the given problem has three variables and one equality constraint, any one of the variables can be removed from Z, with the help of the equality constraint. Let us choose variable x_3 to be eliminated from Z. Then, from the equality constraint, we have:

$$x_3 = \frac{(x_1 + 5x_2 - 6)}{3}$$

Substituting the value of x_3 in the objective function, we get:

$$Z \text{ or } f(x) = x_1^2 + (x_2 + 1)^2 + \frac{1}{9}(x_1 + 5x_2 - 9)^2$$

The necessary condition for minimum of Z is that the gradient

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = 0 \tag{12}$$

That is,

$$\frac{\partial Z}{\partial x_1} = 2x_1 + \frac{2}{9}(x_1 + 5x_2 - 9) = 0 \tag{13}$$

$$\frac{\partial Z}{\partial x_2} = 2(x_2 + 1) + \frac{10}{9}(x_1 + 5x_2 - 9) = 0$$

On solving these equations, we get $x_1 = 2/5$ and $x_2 = 1$

To find whether the solution, so obtained, is minimum or not, we must apply the sufficiency condition by forming a Hessian matrix. The Hessian matrix for the given objective function is

$$\mathbf{H}(\mathbf{x}_1, \mathbf{x}_2) = \begin{bmatrix} \dfrac{\partial^2 Z}{\partial x_1^2} & \dfrac{\partial^2 Z}{\partial x_1\, \partial x_2} \\ \dfrac{\partial^2 Z}{\partial x_2\, \partial x_1} & \dfrac{\partial^2 Z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 20/9 & 10/9 \\ 10/9 & 20/9 \end{bmatrix}$$

Since the matrix is symmetric and principal diagonal elements are positive, $\mathbf{H}(\mathbf{x}_1, \mathbf{x}_2)$ is positive definite and the objective function is convex. Hence, the optimum solution to the given problem is, $x_1 = 2/5$, $x_2 = 1$, $x_3 = -1/5$ and Min $Z = 28/5$.

23.3.2 Lagrange Multipliers Method

In this method an additional variable in each of the given constraints is added. Thus, if the problem has n variables and m equality constraints, then m additional variables are to be added so that the problem would have $n + m$ variables. Before discussing the general method, let us illustrate its salient features through the following simple problem that involves only three variables:

Necessary condition for a problem with $n = 3$ and $m = 1$ Consider the NLP problem

$$\text{Optimize (max or min) } Z = f(x_1, x_2, x_3) \tag{14}$$

subject to the constraint

$$g(x_1, x_2, x_3) = 0 \tag{15}$$

Let an optimum value of Z occur at a point $(x_1, x_2, x_3) = (a, b, c)$ at which at least one of the partial derivatives $\partial g/\partial x_1$, $\partial g/\partial x_2$, $\partial g/\partial x_3$ does not vanish. Thus, we may proceed as follows:

(i) Choose one variable say x_3 in constraint (15) and express it in terms of the remaining two variables such that $x_3 = h(x_1, x_2)$

(ii) Substitute the value of x_3 into the objective function (14). We then get:

$$Z = f\{(x_1, x_2), h(x_1, x_2)\}$$

From unconstrained optimization methods, we know that the necessary condition for local optimum is that all first derivatives with respect to x_1 and x_2 must be zero; that is:

$$\frac{\partial Z}{\partial x_j} = 0; \quad j = 1, 2 \tag{16}$$

Applying the chain rule for differentiation on (16) we get:

$$\frac{\partial Z}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_3} \cdot \frac{\partial h}{\partial x_j}; \quad j = 1, 2$$

But from Eq. (15), we have:

$$\frac{\partial g}{\partial x_j} + \frac{\partial g}{\partial x_3} \cdot \frac{\partial h}{\partial x_j} = 0; \quad j = 1, 2$$

$$\frac{\partial h}{\partial x_j} = -\frac{(\partial g/\partial x_j)}{(\partial g/\partial x_3)}; \quad \frac{\partial g}{\partial x_3} \neq 0, \quad j = 1, 2$$

at the point $(x_1, x_2, x_3) = (a, b, c)$.

Since optimum occurs at the point (a, b, c) we have:

$$\frac{\partial Z}{\partial x_j} = \frac{\partial f}{\partial x_j} - \left[\frac{\partial f}{\partial x_3} \cdot \left\{ \frac{\partial g/\partial x_j}{\partial g/\partial x_3} \right\} \right] = 0, \text{ at } (x_1, x_2, x_3) = (a, b, c). \tag{17}$$

As $\partial g/\partial x_3 \neq 0$, we define a quantity λ, called *Lagrange multiplier* as given below. The value of λ represents the amount of change in the objective function due to the per unit change in the constraint limit, i.e.

$$\frac{\partial f}{\partial x_3} - \lambda \frac{\partial g}{\partial x_3} = 0 \text{ , at } (x_1, x_2, x_3) = (a, b, c)$$

or
$$\lambda = \frac{(\partial f/\partial x_3)}{(\partial g/\partial x_3)}$$

Equation (17) can now be written as:

$$\frac{\partial Z}{\partial x_j} = \left(\frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j}\right) = 0, \quad j = 1, 2 \tag{18}$$

at $(x_1, x_2, x_3) = (a, b, c)$ and the constraint equation

$$g(x_1, x_2, x_3) = 0 \tag{19}$$

is also satisfied at the extreme (or critical) points, $x_1 = a$, $x_2 = b$ and $x_3 = c$. The conditions (18) and (19) are called necessary conditions for a local optimum, provided not all $(\partial g/\partial x_j)$, $j = 1, 2$ become zero at the extreme points.

The necessary conditions given by Eqs (18) and (19) can be obtained very easily by forming a function L, called the *Lagrange function*, as:

$$L(x_j, \lambda) = f(x_j) - \lambda g(x_j), \quad j = 1, 2, 3 \tag{20}$$

We must, then, partially differentiate $L(x_j, \lambda)$ with respect to x_j ($j = 1, 2, 3$) and λ and equate them with zero. The following equations provide the necessary conditions for local optimum:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j} = 0, \quad j = 1, 2, 3 \tag{21}$$

$$\frac{\partial L}{\partial \lambda} = g(x_j) = 0, \quad j = 1, 2, 3$$

These equations can be solved for the unknown x_j ($j = 1, 2, 3$) and λ.

Remark The necessary conditions, so obtained, become sufficient conditions for a maximum (or minimum) if $f(x)$ is concave (or convex), with equality constraints.

Example 23.10 Obtain the necessary conditions for the optimum solution of the following problem:

Minimize $f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5}$

subject to the constraint

$$g(x_1, x_2) = x_1 + x_2 - 7 = 0$$

and $x_1, x_2 \geq 0$ *[Kerala Univ., MSc (Maths), 2001]*

Solution Forming the Lagrangian function, we obtain

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$$

The necessary conditions for the minimum of $f(x_1, x_2)$ are given by:

$$\frac{\partial L}{\partial x_1} = 6e^{2x_1+1} - \lambda = 0 \quad \text{or} \quad \lambda = 6e^{2x_1+1}$$

$$\frac{\partial L}{\partial x_2} = 2e^{x_2+5} - \lambda = 0, \quad \text{or} \quad \lambda = 2e^{x_2+5}$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 7) = 0$$

On solving these three equations in three unknowns, we obtain:

$$x_1 = (1/3)(11 - \log 3), \quad \text{and} \quad x_2 = 7 - (1/3)(11 - \log 3).$$

Necessary conditions for a general problem Consider the non-linear programming problem:

Optimize $Z = f(\mathbf{x})$

subject to the constraint

$$h_i(\mathbf{x}) = b_i$$

or

$$g_i(\mathbf{x}) = h_i(\mathbf{x}) - b_i = 0 \quad i = 1, 2, \ldots, m \quad \text{and} \quad m \leq n; \ \mathbf{x} \in E^n$$

The necessary conditions (21) for a function to have a local optimum at the given points can be extended to the case of a general problem with n variables and m equality constraints.

Multiply each constraint with an unknown variable λ_i ($i = 1, 2, \ldots, m$) and subtract each from the objective function, $f(x)$ to be optimized. The new objective function now becomes:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \, g_i(\mathbf{x}); \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^T$$

where $m < n$. The function $L(\mathbf{x}, \lambda)$ is called the *Lagrange function.*

The necessary conditions for an unconstrained optimum of $L(\mathbf{x}, \lambda)$, i.e. the first derivatives, with respect to $\mathbf{x}$ and λ of $L(\mathbf{x}, \lambda)$ must be zero, are also necessary conditions for the given constrained optimum of $f(\mathbf{x})$, provided that the matrix of partial derivatives $\partial g_i / \partial x_j$ has rank m at the point of optimum.

The necessary conditions for an optimum (max or min) of $L(\mathbf{x}, \lambda)$ or $f(\mathbf{x})$ are the $m + n$ equations to be solved for $m + n$ unknown $(x_1, x_2, \ldots, x_n; \lambda_1, \lambda_2, \ldots, \lambda_m)$.

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = 0; \quad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda_i} = -g_i; \quad i = 1, 2, \ldots, m$$

These $(m + n)$ necessary conditions also become sufficient conditions for a maximum (or minimum) of the objective function $f(\mathbf{x})$, in case it is concave (or convex) and the constraints are equalities, respectively.

Sufficient conditions for a general problem Let the Lagrangian function for a general NLP problem, involving n variables and m ($< n$) constraints, be

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \, g_i(\mathbf{x})$$

Further, the necessary conditions

$$\frac{\partial L}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda_i} = 0; \quad \text{for all } i \text{ and } j$$

for an extreme point to be local optimum of $f(\mathbf{x})$ is also true for optimum of $L(\mathbf{x}, \lambda)$.

Let there exist points x and λ that satisfy the equations

$$\nabla L(\mathbf{x}, \lambda) = \nabla f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \, g_i(\mathbf{x}) = 0$$

and

$$g_i(\mathbf{x}) = 0, \quad i = 1, 2, \ldots, m$$

Then the sufficient condition for an extreme point $\mathbf{x}$ to be a local minimum (or local maximum) of $f(\mathbf{x})$ subject to the constraints $g_i(\mathbf{x}) = 0$ ($i = 1, 2, \ldots, m$) is that the determinant of the matrix (also called *Bordered Hessian matrix*)

$$\mathbf{D} = \begin{bmatrix} \mathbf{Q} & \mathbf{H} \\ \mathbf{H}^{\mathbf{T}} & \mathbf{0} \end{bmatrix}_{(m+n)\times(m+n)}$$

is positive (or negative), where

$$\mathbf{Q} = \left[\frac{\partial^2 L(x, \lambda)}{\partial x_i \, \partial x_j}\right]_{n \text{Ÿ} n} ; \quad \mathbf{H} = \left[\frac{\partial g_i(x)}{\partial x_j}\right]_{m \times n}$$

Conditions for maxima and minima The sufficient condition for the maxima and minima is determined by the signs of the last $(n - m)$ principal minors of matrix D. That is,

1. If starting with principal minor of order $(m + 1)$, the extreme point gives the maximum value of the objective function when signs of last $(n - m)$ principal minors alternate, starting with $(-1)^{m+n}$ sign.
2. If starting with principal minor of order $(2m + 1)$, the extreme point gives the maximum value of the objective function when all signs of last $(n - m)$ principal minors are the same and are of $(-1)^m$ type.

Example 23.11 Solve the following problem by using the method of Lagrangian multipliers.

Minimize $Z = x_1^2 + x_2^2 + x_3^2$

subject to the constraints

(i) $x_1 + x_2 + 3x_3 = 2,$ (ii) $5x_1 + 2x_2 + x_3 = 5$

and $x_1, x_2 \geq 0$

Solution The Lagrangian function is

$$L(\mathbf{x}, \lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda_1 (x_1 + x_2 + 3x_3 - 2) - \lambda_2 (5x_1 + 2x_2 + x_3 - 5)$$

The necessary conditions for the minimum of Z give us the following:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0; \qquad \frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2 = 0; \qquad \frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0$$

The solution of these simultaneous equations gives:

$$\mathbf{x} = (x_1, x_2, x_3) = (37/46, 16/46, 13/46);\ \lambda = (\lambda_1, \lambda_2) = (2/23, 7/23) \text{ and } Z = 193/250$$

To see that this solution corresponds to the minimum of Z, apply the sufficient condition with the help of a matrix:

$$\mathbf{D} = \left[\begin{array}{ccc:cc} 2 & 0 & 0 & 1 & 5 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ \hdashline 1 & 1 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 \end{array}\right]$$

Since $m = 2$, $n = 3$, so $n - m = 1$ and $2m + 1 = 5$, only one minor of D of order 5 needs to be evaluated and it must have a positive sign; $(-1)^m = (-1)^2 = 1$. Since $|D| = 460 > 0$, the extreme point, $\mathbf{x} = (x_1, x_2, x_3)$ corresponds to the minimum of Z.

Necessary and sufficient conditions when concavity (convexity) of objective function is not known, with single equality constraint

Let us consider the non-linear programming problem that involves n decision variables and a single constraint:

Optimize $Z = g(\mathbf{x})$

subject to the constraint

$$g(\mathbf{x}) = h(\mathbf{x}) - \mathbf{b} = 0; \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \geq 0$$

Multiply each constraint by Lagrange multiplier λ and subtract it from the objective function. The new unconstrained objective function (Lagrange function) becomes:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda\, g(\mathbf{x})$$

The necessary conditions for an extreme point to be an optimum (max or min) point are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j} = 0; \quad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda} = -g(\mathbf{x}) = 0$$

From the first condition we obtain the value of λ as:

$$\lambda = \frac{(\partial f/\partial x_j)}{(\partial g/\partial x_j)}; \quad j = 1, 2, \ldots, n$$

The sufficient conditions for determining whether the optimal solution, so obtained, is either maximum or minimum, need computation of the value of $(n-1)$ principal minors, of the determinant, for each extreme point, as follows:

$$\Delta_{n+1} = \begin{vmatrix} 0 & \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} & \cdots & \dfrac{\partial g}{\partial x_n} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial^2 f}{\partial x_1^2} - \lambda \dfrac{\partial^2 g}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \dfrac{\partial^2 g}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \dfrac{\partial^2 g}{\partial x_1 \partial x_n} \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \dfrac{\partial^2 g}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} - \lambda \dfrac{\partial^2 g}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \dfrac{\partial^2 g}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{\partial g}{\partial x_n} & \dfrac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \dfrac{\partial^2 g}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \dfrac{\partial^2 g}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} - \lambda \dfrac{\partial^2 g}{\partial x_n^2} \end{vmatrix}$$

If the signs of minors Δ_3, Δ_4, Δ_5 are alternatively positive and negative, then the extreme point is a local maximum. But if sign of all minors Δ_3, Δ_4, Δ_5 are negative, then the extreme point is a local minimum.

Example 23.12 Use the method of Lagrangian multipliers to solve the following NLP problem. Does the solution maximize or minimize the objective function ?

Optimize $Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

subject to the constraint

$$g(x) = x_1 + x_2 + x_3 = 20$$

and $\quad x_1, x_2, x_3 \geq 0$

Solution Lagrangian function can be formulated as:

$$L(\mathbf{x}, \lambda) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 - \lambda(x_1 + x_2 + x_3 - 20)$$

The necessary conditions for maximum or minimum are:

$$\frac{\partial L}{\partial x_1} = 4x_1 + 10 - \lambda = 0; \qquad \frac{\partial L}{\partial x_2} = 2x_2 + 8 - \lambda = 0$$

$$\frac{\partial L}{\partial x_3} = 6x_3 + 6 - \lambda = 0; \qquad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 20) = 0$$

Putting the value of x_1, x_2 and x_3 in the last equation $\partial L/\partial \lambda = 0$ and solving for λ, we get $\lambda = 30$. Substituting the value of λ in the other three equations, we get an extreme point: $(x_1, x_2, x_3) = 5, 11, 4$.

To prove the sufficient condition of whether the extreme point solution gives maximum or minimum value of the objective function we evaluate $(n-1)$ principal minors as follows:

$$\Delta_3 = \begin{vmatrix} 0 & \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial^2 f}{\partial x_1^2} - \lambda \dfrac{\partial^2 g}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\, \partial x_2} - \lambda \dfrac{\partial^2 g}{\partial x_1\, \partial x_2} \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial^2 f}{\partial x_2\, \partial x_1} - \lambda \dfrac{\partial^2 g}{\partial x_2\, \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} - \lambda \dfrac{\partial^2 g}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = 48$$

Since the sign of Δ_3 and Δ_4 are alternative, therefore extreme point: $(x_1, x_2, x_3) = (5, 11, 4)$ is a local maximum. At this point the value of objective function is, $Z = 281$.

Interpretation of the Lagrange Multiplier

The value of Lagrange multiplier, which was introduced as an additional variable, can be used to provide valuable information about the sensitivity of an optimal value of the objective function to changes in resource levels (right-hand-side values of the constraints).

Recall that the general NLP problem with two decision variables and one equality constraint can be stated as:

Minimize $Z = f(x_1, x_2)$

subject to the constraint:

$$h(x_1, x_2) = b \quad \text{or} \quad g(x_1, x_2) = b - h(x_1, x_2) = 0$$

The necessary conditions to be satisfied for the solution of the problem are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j} = 0, \quad j = 1, 2 \qquad (22)$$

$$\frac{\partial L}{\partial \lambda} = g(x_1, x_2) = 0$$

where $L(x, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$

If we want to observe the effect of change in the optimal value of the objective function with respect to a change in b, we must differentiate the constraint with respect to b. By doing this we get:

$$db - dh = db - \sum_{j=1}^{2} \frac{\partial h}{\partial x_j} dx_j = 0 \qquad (23)$$

Rewriting Eq. (23) as: $\dfrac{\partial f}{\partial x_j} - \lambda \dfrac{\partial g}{\partial x_j} = \dfrac{\partial f}{\partial x_j} + \lambda \dfrac{\partial h}{\partial x_j} = 0$

or $$\frac{\partial h}{\partial x_j} = -\frac{(\partial f/\partial x_j)}{\lambda}, \quad j = 1, 2$$

Substituting the value of $(\partial h/\partial x_j)$ in Eq. (23), we get:

$$db = -\sum_{j=1}^{2} \frac{1}{\lambda} \frac{\partial f}{\partial x_j} \cdot dx_j = -\frac{df}{\lambda}, \quad \text{since} \quad df = \sum_{j=1}^{2} \frac{\partial f}{\partial x_j} dx_j$$

Hence $\lambda = -df/db$. This relationship indicates that if we increase (or decrease) b, the value of λ would indicate approximately how much the optimal value of objective function would decrease (or increase). Thus depending on the value of λ (positive, negative or zero) it will provide a different estimation of the value of the change in the objective function.

SELF PRACTICE PROBLEMS B

Obtain the solution of the following problems by using the method of Lagrangian multipliers:

1. Min $Z = -2x_1^2 + 5x_1x_2 - 4x_1^2 + 18x_1$
   subject to $x_1 + x_2 = 7$, and $x_1, x_2 \geq 0$
2. Min $Z = 3x_1^2 + x_2^2 + x_3^2$
   subject to $x_1 + x_2 + x_3 = 2$, and $x_1, x_2, x_3 \geq 0$
3. Min $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$
   subject to (i) $x_1 + x_2 + x_3 = 15$, (ii) $2x_1 - x_2 + 2x_3 = 20$
   and $x_1, x_2, x_3 \geq 0$
4. Max $Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$
   subject to $x_1 + 2x_2 = 2$, and $x_1, x_2 \geq 0$
5. Max $Z = 7x_1 - 0.3x_1^2 + 8x_2 - 0.4x_2^2$
   subject to $4x_1 + 5x_2 = 100$, and $x_1, x_2 \geq 0$
6. Min $Z = x_1^2 + x_2^2 + x_3^2$
   subject to $4x_1 + x_2^2 + 2x_3 = 14$, and $x_1, x_2, x_3 \geq 0$
7. Use the method of Lagrange multipliers to solve the following NLP problem. Does the solution maximize or minimize the objective function?

   Optimize $Z = x_1^2 - 10x_1 + x_2^2 - 6x_2 + x_3^2 - 4x_3$

   subject to $x_1 + x_2 + x_3 = 7$ and $x_1, x_2, x_3 \geq 0$
8. Find the dimensions of a rectangular parallelepiped with the largest volume whose sides are parallel to the coordinate planes, to be inscribed in the ellipsoid

$$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

9. A positive quantity b is to be divided into n parts in such a way that the product of n parts is a maximum. Use Lagrangian multiplier method to obtain the optimal subdivision.

HINTS AND ANSWERS

1. $x_1 = 4.95$, $x_2 = 2.045$, and Min $Z = 21.63$
2. $x_1 = 0.81$, $x_2 = 0.35$, $x_3 = 0.28$ and Min $Z = 0.84$
3. $x_1 = 33/9$, $x_2 = 10/3$, $x_3 = 8$, $\lambda_1 = 40/9$, $\lambda_2 = 52/9$ and $Z = 91.1$. Since $|D| = 72$, therefore $x = (x_1, x_2, x_3)$ is a minimum point
4. $x_1 = 1/3$, $x_2 = 5/6$ and Max $Z = 4.166$
5. $x_1 = 12.06$, $x_2 = 10.35$, and Max $Z = 80.73$
6. $x_1 = 81/100$, $x_2 = 7/20$, $x_3 = 7/25$ and Min $Z = 857/1{,}000$.
7. $x_1 = 4$, $x_2 = 2$, $x_3 = 1$ and $\lambda = -2$. Since both $\Delta_3 = -4$ and $\Delta_4 = -12$ are negative, the extreme point $\mathbf{x} = (x_1, x_2, x_3)$ is a local minimum point, and gives, Min $Z = -35$.
8. Formulate, $L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$; where $f(x, y, z) = xyz$, is the volume of a parallelopiped.

   Differentiate partially L with respect to x, y, z and λ and equate them equal to zero. Solve four equations to get first $\lambda = (3/2) \cdot xyz$ and then $x = a/\sqrt{3}$, $y = b/\sqrt{3}$ and $z = c/\sqrt{3}$.
9. Max $Z = x_1, x_2, \ldots, x_n$

   subject to $\quad x_1 + x_2 + \ldots + x_n = b, \; x_j \geq 0 \; (j = 1, 2, \ldots, n)$

   Formulate the Lagrangian function

   $$L(x, \lambda) = (x_1 \cdot x_2 \cdot \ldots x_n) - \lambda (x_1 + x_2 + \ldots + x_n - b)$$

   Differentiate L with respect to $x_1, x_2, \ldots, x_n$ and λ to get necessary condition equations. Solve these equations to first get $\lambda = n (x_1 \cdot x_2 \cdot \ldots \cdot x_n)/b$ and then by substitution, $x_1 = x_2 = \ldots = x_n = b/n$; Min $Z = (b/n)^n$.

23.4 CONSTRAINED MULTIVARIABLE OPTIMIZATION WITH INEQUALITY CONSTRAINTS

In this section the necessary and sufficient conditions for a local optimum of the general non-linear programming problem, with both equality and inequality constraints will be derived. The Kuhn-Tucker conditions (necessary as well as sufficient) will be used to derive optimality conditions. Consider the following general non-linear LP problem:

23.4.1 Kuhn-Tucker Necessary Conditions

Optimize $Z = f(\mathbf{x})$

subject to the constraints

$$g_i(\mathbf{x}) \leq 0, \quad \text{for} \quad i = 1, 2, \ldots, m$$

where $\quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $g_i(\mathbf{x}) = h_i(\mathbf{x}) - b_i$

Add non-negative slack variables s_i $(i = 1, 2, \ldots, m)$ in each of the constraints to convert them to equality constraints. The problem can then be restated as:

Optimize $Z = g(\mathbf{x})$

subject to the constraints

$$g_i(\mathbf{x}) + s_i^2 = 0, \; i = 1, 2, \ldots, m$$

The s_i^2 has only been added to ensure non-negative value (feasibility requirement) of s_i and to avoid adding $s_i \geq 0$ as an additional side constraint.

The new problem is the constrained multivariable optimization problem with equality constraints with $n + m$ variables. Thus, it can be solved by using the Lagrangian multiplier method. For this, let us form the Lagrangian function as:

$$L(\mathbf{x}, \mathbf{s}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \left[g_i(\mathbf{x}) + s_i^2 \right]$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$ is the vector of Lagrange multiplier.

The necessary conditions for an extreme point to be local optimum (max or min) can be obtained by solving the following equations:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x_j} = 0, \quad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda_i} = -[g_i(x) + s_i^2] = 0, \qquad i = 1, 2, \ldots, m$$

$$\frac{\partial L}{\partial s_i} = -2x_i \lambda_i = 0, \qquad i = 1, 2, \ldots, m$$

The equation $\partial L/\partial \lambda_i = 0$ gives us back the original set of constraints: $g_i(\mathbf{x}) + s_i^2 = 0$. If a constraint is satisfied with equality sign, $g_i(\mathbf{x}) = 0$ at the optimum point $\mathbf{x}$, then it is called an *active* (*binding or light*) constraint, otherwise it is known as an *inactive* (*slack*) constraint.

The equation $\partial L/\partial s_i = 0$, provides us the set of rules: $-2\lambda_i s_i = 0$ or $\lambda_i s_i = 0$ for finding the unconstrained optimum. The condition $\lambda_i s_i = 0$ implies that either $\lambda_i = 0$ or $s_i = 0$. If $s_i = 0$ and $\lambda_i > 0$, then equation $\partial L/\partial \lambda_i = 0$ gives $g_i(\mathbf{x}) = 0$. This means either $\lambda_i = 0$ or $g_i(\mathbf{x}) = 0$, and therefore we may also write $\lambda_i g_i(\mathbf{x}) = 0$.

Since s_i^2 has been taken to be a non-negative (≥ 0) slack variable, therefore $g_i(x) \geq 0$. Hence, the equation $\lambda_i g_i(x) = 0$ implies that when $g_i(x) < 0$, $\lambda_i = 0$ and when $g_i(x) = 0$, $\lambda_i > 0$. However λ_i is unrestricted in sign corresponding to $g_i(x) = 0$.

But if $\lambda_i = 0$ and $s_i^2 > 0$, then the ith constraint is inactive (i.e. this constraint will not change the optimum value of Z^* because $\lambda = \partial Z/\partial b_i = 0$) and hence can be discarded.

Thus the Kuhn-Tucker necessary conditions (when active constraints are known) to be satisfied at a local optimum (max or min) point can be stated as follows:

$$\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = 0, \qquad j = 1, 2, \ldots, n$$

$$\lambda_i g_i(x) = 0,$$

$$g_i(x) \leq 0,$$

$$\lambda_i \geq 0, \qquad i = 1, 2, \ldots, m$$

Remark If the given problem is of minimization or if the constraints are of the form $g_i(\mathbf{x}) \geq 0$, then $\lambda_i \leq 0$. On the other hand if the problem is of maximization with constraints of the form $g_i(\mathbf{x}) \leq 0$, then $\lambda_i \geq 0$.

23.4.2 Kuhn-Tucker Sufficient Conditions

Theorem 23.5 (*Sufficiency of Kuhn-Tucker conditions*) The Kuhn-Tucker necessary conditions for the problem

Maximize $Z = f(\mathbf{x})$

subject to the constraints

$$g_i(\mathbf{x}) \leq 0, \qquad i = 1, 2, \ldots, m$$

are also sufficient conditions if $f(\mathbf{x})$ is concave and all $g_i(\mathbf{x})$ are convex functions of $\mathbf{x}$.

Proof The Lagrangian function of the problem

Maximize $Z = f(\mathbf{x})$ subject to the constraints

$$g_i(\mathbf{x}) \leq 0, \qquad i = 1, 2, \ldots, m$$

can be written as: $L(\mathbf{x}, \mathbf{s}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i [g_i(\mathbf{x}) + s_i^2]$

If $\lambda_i \geq 0$, then $\lambda_i g_i(\mathbf{x})$ is convex and $-\lambda_i g_i(\mathbf{x})$ is concave. Further, since $\lambda_i s_i = 0$, we get $g_i(\mathbf{x}) + s_i^2 = 0$. Thus, it follows that $L(\mathbf{x}, \mathbf{s}, \lambda)$ is a concave function. We have derived that a necessary condition for $f(\mathbf{x})$ to be a relative maximum at an extreme point is that $L(\mathbf{x}, \mathbf{s}, \lambda)$ also have the same extreme point. However, if $L(\mathbf{x}, \mathbf{s}, \lambda)$ is concave, its first derivative must be zero only at one point, and obviously this point must be an absolute maximum for $f(\mathbf{x})$.

Example 23.13 Find the optimum value of the objective function when separately subject to the following three sets of constraints:

Maximize $Z = 10x_1 - x_1^2 + 10x_2 - x^2$,

subject to the constraints

(a) $x_1 + x_2 \leq 14$ (b) $x_1 + x_2 \leq 8$ (c) $x_1 + x_2 \leq 9$

$-x_1 + x_2 \leq 6$ $-x_1 + x_2 \leq 5$ $x_1 - x_2 \geq 6$

and $x_1, x_2 \geq 0$ $x_1, x_2 \geq 0$ $x_1, x_2 \geq 0$

Solution (a) Here the constraints are:

$$g_1(\mathbf{x}) = x_1 + x_2 + s_1^2 - 14 = 0$$
$$g_2(\mathbf{x}) = -x_1 + x_2 + s_2^2 - 6 = 0$$

The Lagrangian function is formulated as:

$$L(\mathbf{x}, \mathbf{s}, \boldsymbol{\lambda}) = (10x_1 - x_1^2 + 10x_2 - x_2^2) - \lambda_1(x_1 + x_2 + s_1^2 - 14) - \lambda_2(-x_1 + x_2 + s_2^2 - 6)$$

The Kuhn-Tucker necessary conditions for a maximization problem are:

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0; \qquad \frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1^2 - 14) = 0; \qquad \frac{\partial L}{\partial \lambda_2} = -(-x_1 + x_2 + s_2^2 - 6) = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0; \qquad \frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0$$

The unconstrained solution (i.e. $\lambda_1 = \lambda_2 = 0$) obtained by solving the first four equations is:

$$x_1 = 5,\ x_2 = 5,\ s_1^2 = 4,\ s_2^2 = 6 \quad \text{and} \quad \text{Max } Z = 50$$

Since both s_1^2 and s_2^2 are positive, the solution is feasible. As the solution, so obtained, is unconstrained, therefore in order to find whether or not the solution is maximum we test the Hessian matrix for the given objective function as:

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 Z}{\partial x_1^2} & \dfrac{\partial^2 Z}{\partial x_1\, \partial x_2} \\ \dfrac{\partial^2 Z}{\partial x_2\, \partial x_1} & \dfrac{\partial^2 Z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

and $$\det A_1 = \left| \frac{\partial^2 Z}{\partial x_1^2} \right| = -2; \ \det A_2 = |H| = 4$$

Since signs of the principal minors of H are alternating, matrix H is negative definite and the point $\mathbf{x} = (4, 4)$ gives the local maximum of the objective function Z.

(b) Here the constraints are:

$$g_1(\mathbf{x}) = x_1 + x_2 + s_1^2 - 8 = 0 \quad \text{and} \quad g_2(\mathbf{x}) = -x_1 + x_2 + s_2^2 - 5 = 0$$

The Lagrangian function is formulated as:

$$L(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{s}) = (10x_1 - x_1^2 + 10x_2 - x_2^2) - \lambda_1(x_1 + x_2 + s_1^2 - 8) - \lambda_2(-x_1 + x_2 + s_2^2 - 5)$$

The Kuhn-Tucker necessary conditions for a maximization problem are:

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0; \qquad \frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1^2 - 8) = 0; \qquad \frac{\partial L}{\partial \lambda_2} = -(-x_1 + x_2 + s_2^2 - 5) = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0; \qquad \frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0$$

The unconstrained solution (i.e. $\lambda_1 = \lambda_2 = 0$) obtained by solving the first four equations is:

$$x_1 = 5,\ x_2 = 5,\ s_1^2 = -2,\ s_2^2 = 5 \quad \text{and} \quad \text{Max } Z = 50$$

Since $s_1^2 = -2$, this solution is infeasible. By again solving these equations for $s_1 = \lambda_2 = 0$ (violated first constraint), we get $x_1 = 4$, $x_2 = 4$, $s_2^2 = 5$, $\lambda_1 = 2$ and Max $Z = 48$. This solution satisfies both the constraints and the conditions $\lambda_1 s_1 = \lambda_2 s_2 = 0$ are also satisfied, therefore the point $\mathbf{x} = (4, 4)$ gives the maximum of objective function Z.

(c) Here the constraints are:

$$g_1(\mathbf{x}) = x_1 + x_2 + s_1^2 - 9 = 0 \quad \text{and} \quad g_2(\mathbf{x}) = -x_1 + x_2 + s_1^2 + 6 = 0$$

The Lagrangian function is formulated as:

$$L(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{s}) = (10x_1 - x_1^2 + 10x_2 - x_2^2) - \lambda_1(x_1 + x_2 + s_1^2 - 9) - \lambda_2(-x_1 + x_2 + s_2^2 + 6)$$

The Kuhn-Tucker necessary conditions for a maximization problem are:

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0; \qquad \frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1^2 - 9) = 0; \qquad \frac{\partial L}{\partial \lambda_2} = -(-x_1 + x_2 + s_2^2 + 6) = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0 \;; \qquad \frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0$$

The unconstrained solution (i.e. $\lambda_1 = \lambda_2 = 0$) obtained by solving the first four equations is: $x_1 = 8$, $x_2 = 2$, $s_1^2 = -1$, $s_2^2 = -6$, and Max $Z = 50$. Since both s_1^2 and s_2^2 are negative, the solution is infeasible.

Solving these four equations again for $s_2 = \lambda_1 = 0$ (violating second constraint), we get:

$$x_1 = 2,\ x_2 = 8,\ s_1^2 = -1,\ \lambda_2 = 6, \text{ and Max } Z = 32$$

This solution is also infeasible, as s_1^2 is negative.

Solving these four equations again for $s_1 = s_2 = 0$ (i.e. $\lambda_1 = \lambda_2 \neq 0$) we get:

$$x_1 = 7.5,\ x_2 = 1.5,\ \lambda_1 = 1,\ \lambda_2 = 6, \text{ and Max } Z = 31.50$$

Since this solution does not violate any of the conditions, therefore, the point $x = (7.5, 1.5)$ gives the maximum of the objective function Z.

Example 23.14 Determine x_1 and x_2 so as to

Maximize $Z = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

subject to the constraints

(i) $x_2 \le 8$, (ii) $x_1 + x_2 \le 10$,

and $x_1, x_2 \ge 0$

Solution Here $f(x_1, x_2) = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

$$g_1(x_1, x_2) = x_2 - 8 \le 0$$

$$g_2(x_1, x_2) = x_1 + x_2 - 10 \le 0$$

The Lagrangian function can be formulated as:

$$L(\mathbf{x}, \mathbf{s}, \boldsymbol{\lambda}) = f(x) - \lambda_1\left[g_1(x) + s_1^2\right] - \lambda_2\left[g_2(x) + s_2^2\right]$$

The Kuhn-Tucker necessary conditions can be stated as:

(i) $\dfrac{\partial f}{\partial x_j} - \sum_{i=1}^{2} \lambda_i \dfrac{\partial g_i}{\partial x_j}$, $j = 1, 2$ (ii) $\lambda_i g_i(x) = 0$, $i = 1, 2$

$12 + 2x_2 - 4x_1 - \lambda_2 = 0$ $\lambda_1(x_2 - 8) = 0$

$21 + 2x_1 - 4x_2 - \lambda_1 - \lambda_2 = 0$ $\lambda_2(x_1 + x_2 - 10) = 0$

(iii) $g_i(x) \le 0$ (iv) $\lambda_i \ge 0$, $i = 1, 2$

$x_2 - 8 \le 0$

$x_1 + x_2 - 10 \le 0$

There may arise four cases:

Case 1: If $\lambda_1 = 0$, $\lambda_2 = 0$, then from Condition (i), we have:

$$12 + 2x_2 - 4x_1 = 0 \quad \text{and} \quad 21 + 2x_1 - 4x_2 = 0$$

Solving these equations, we get $x_1 = 15/2$, $x_2 = 9$. However, this solution violates condition (iii) and therefore it should be discarded.

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Case 2: $\lambda_1 \neq 0, \lambda_2 \neq 0$, then from condition (ii) we have:

$$x_2 - 8 = 0 \quad \text{or} \quad x_2 = 8$$
$$x_1 + x_2 - 10 = 0 \quad \text{or} \quad x_1 = 2$$

Substituting these values in condition (i), we get $\lambda_1 = -27$ and $\lambda_2 = 20$. However, this solution violates the condition (iv) and therefore may be discarded.

Case 3: $\lambda_1 \neq 0, \lambda_2 = 0$, then from conditions (ii) and (i) we have:

$$x_1 + x_2 = 10$$
$$2x_2 - 4x_1 = -12$$
$$2x_1 - 4x_2 = -12 + \lambda_1$$

Solving these equations, we get $x_1 = 2, x_2 = 8$ and $\lambda_1 = -16$. However, this solution violates the condition (iv) and therefore may be discarded.

Case 4: $\lambda_1 = 0, \lambda_2 \neq 0$, then from conditions (i) and (ii) we have:

$$2x_2 - 4x_1 = -12 + \lambda_2$$
$$2x_1 - 4x_2 = -21 + \lambda_2$$
$$x_1 + x_2 = 10$$

Solving these equations, we get $x_1 = 17/4$, $x_2 = 23/4$ and $\lambda_2 = 13/4$. This solution does not violate any of the Kuhn-Tucker conditions and therefore must be accepted.

Hence, the optimum solution of the given problem is: $x_1 = 17/4, x_2 = 23/4, \lambda_1 = 0$ and $\lambda_2 = 13/4$ and Max $Z = 1734/16$.

CONCEPTUAL QUESTIONS

1. Sate and prove Kuhn-Tucker necessary and sufficient conditions in non-linear programming.
2. Discuss the economic interpretation of Lagrangian multipliers, the duality theory, and derive the Kuhn-Tucker conditions for the non-linear programming problem:

   Max $Z = f(x)$

   subject to the constraints

   $g_i(x) \le b_i$; $i = 1, 2, \ldots, m$
3. Explain what is meant by Kuhn-Tucker conditions.

SELF PRACTICE PROBLEMS C

Use the Kuhn-Tucker conditions to solve the following non-linear programming problems:

1. Max $Z = 2x_1^2 + 12x_1x_2 - 7x_2^2$

   subject to $2x_1 + 5x_2 \le 98$, and $x_1, x_2 \ge 0$
2. Max $Z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

   subject to (i) $x_1 + x_2 \le 2$, (ii) $2x_1 + 3x_2 \le 12$

   and $x_1, x_2 \ge 0$ [*IAS (Main), 1992*]
3. Max $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

   subject to $3x_1 + 2x_2 \le 6$, and $x_1, x_2 \ge 0$
4. Max $Z = 7x_1^2 - 6x_1 + 5x_2^2$

   subject to (i) $x_1 + 2x_2 \le 10$, (ii) $x_1 - 3x_2 \le 9$

   and $x_1, x_2 \ge 0$
5. Max $Z = 2x_1 - x_1^2 + x_2$

   subject to (i) $2x_1 + 3x_2 \le 6$, (ii) $2x_1 + x_2 \le 4$

   and $x_1, x_2 \ge 0$
6. Min $Z = (x_1 + 1)^2 + (x_2 - 2)^2$

   subject to $0 \le x_1 \le 2$, $0 \le x_2 \le 1$
7. Max $Z = -2x_1^2 + 3x_1 + 4x_2$

   subject to (i) $x_1 + 2x_2 \le 4$, (ii) $x_1 + x_2 \le 2$

   and $x_1, x_2 \ge 0$
8. Min $Z = 2(x_1 - 4)^2 + 4(x_2 - 5)^2$

   subject to (i) $2x_1 + x_2 \le 8$, (ii) $x_1 + 3x_2 \le 20$,

   (iii) $x_1 + x_2 \ge 1$

   and $x_1, x_2 \ge 0$
9. Define a convex programming problem. What is the Lagrangian function associated with it? Solve the non-linear programming problem:

   Min $Z = -\log x_1 - \log x_2$

   subject to $x_1 + x_2 \le 2$, and $x_1, x_2 \ge 0$
10. Write the Kuhn-Tucker conditions for the following problem and solve them.

    (a) Min $Z = x_1^2 + x_2^2 + x_3^2$

    subject to (i) $2x_1 + x_2 - x_3 \le 0$, (ii) $1 - x_1 \le 0$,

    (iii) $2 - x_2 \le 0$, (iv) $x_3 \ge 0$

(b) Min $Z = x_1^2 + x_2^2 + x_3^2$

subject to

(i) $2x_1 + x_2 \le 5$, (ii) $x_1 + x_3 \le 2$
(iii) $1 - x_1 \le 0$, (iv) $2 - x_2 \le 0$,
(v) $x_3 \ge 0$

[IAS (Main), 1993]

11. A manufacturing firm produces two products A and B. It produces them at the per unit cost of Rs 3 and Rs 5 respectively. The cost of production for these two products is given below:

Number of Units Produced	Cost of Production (Rs)
Product A (x_1)	$60 + 1.2x_1 + 0.001\,x_1^2$
Product B (x_2)	$40 + 2x_2 + 0.005\,x_2^2$

Because of the limited available resources, the firm has to bear within the restrictions.

$$2x_1 + 3x_2 \le 2{,}500 \quad \text{and} \quad x_1 + 2x_2 \le 1{,}500$$

Using the Kuhn-Tucker condition methods determine the optimal level of production of products A and B by the firm.

12. A manufacturing firm produces a product A. The firm has the contract to supply 60 units at the end of the first, second and third months. The cost of producing x units of A in any month is given by x^2. The firm can produce more units of A in any month and carry them to a subsequent month. However, a carrying cost of Rs 25 per unit is charged for carrying units of A from one month to the next. Assuming that there is no initial inventory, determine the number of units of A to be produced in each month so as to minimize the total cost.

HINTS AND ANSWERS

1. $x_1 = 44,\ x_2 = 2,\ \lambda = 100$ and Max $Z = 4{,}900$
2. $x_1 = 1/2,\ x_2 = 3/2,\ x_3 = 0,\ \lambda_1 = 3,\ \lambda_2 = 0$ and Max $Z = 17/2$
3. $x_1 = 4/13,\ x_2 = 33/13$, and Max $Z = 21.3$
4. $x_1 = 48/5,\ x_2 = 1/5$, and Max $Z = 587.72$
5. $x_1 = 2/3,\ x_2 = 14/9,\ \lambda_1 = 1/3,\ \lambda_2 = 0$, and Max $Z = 22/9$
6. $x_1 = 2,\ x_2 = 1,\ \lambda_1 = 6,\ \lambda_2 = 2$, and Min $Z = 10$
9. $x_1 = 1,\ x_2 = 1$, and Min $Z = 0$

12. Let x_1, x_2 and x_3 = number of units of product A produced in first, second and third months, respectively.

Min (total cost)

Z = Production cost + Carrying cost

$$= x_1^2 + x_2^2 + x_3^2 + 40x_1 + 25(x_1 - 60) + 25(x_1 + x_2 - 120)$$

subject to (i) $x_1 + x_2 \ge 120$; (ii) $x_1 + x_2 + x_3 \ge 180$
(iii) $x_1 \ge 60$; and $x_1, x_2, x_3 \ge 0$.

CHAPTER SUMMARY

The classical optimization methods are used to obtain an optimal solution of certain types of problems that involve continuous and differentiable function. These methods are analytical in nature and make use of differential calculus in order to find points of maxima and minima for (a) an constrained single and multiple variable continuous function, and (b) constrained multivariable functions with equality and inequality constraints. In this chapter conditions for local as well as global minimum and maximum value of an unconstrained objective function have been derived followed by numerical exercises. Direct substitution method, Langrange multipliers method and Kuhn-Tucker method have also been discussed to find optimal value of an objective function with equality and inequality constraints, respectively.

CHAPTER CONCEPTS QUIZ

1. A function is said to achieve its maximum value at a point, $x = x_0$ if
   (a) $f(x_0) = f(x_0 + h)$ (b) $f(x_0) > f(x_0 + h)$
   (c) $f(x_0) < f(x_0 + h)$ (d) none of these
2. Level maximum of $f(x)$ occurs at $x = x_0$ provided
   (a) $f^n(x_0) = 0$, for n even (b) $f^3(x_0) < 0$, for n even
   (c) $f^n(x_0) > 0$, for n even (d) $f^n(x_0) > 0$ for n odd
3. The point of inflection occurs at $x = x_0$ provided
   (a) $f^n(x_0) = 0$, for n odd (b) $f^n(x_0) \ne 0$, for n odd
   (c) $f^n(x_0) > 0$, for n odd (d) $f^n(x_0) < 0$ for n odd
4. Hessian matrix $\boldsymbol{H}(\boldsymbol{x})$ is positive definite if all its leading principal minors of order
   (a) 1×1 are positive (b) 1×1 are scalars
   (c) 2×2 are positive (d) 2×2 are scalars
5. The necessary condition for minimum of non-linear objective function, Z value is that the gradient
   (a) $\nabla f(x_0) > 0$ (b) $\nabla f(x_0) < 0$
   (c) $\nabla f(x_0) \ne 0$ (d) $\nabla f(x_0) = 0$

Answers to Quiz

1. (b) 2. (b) 3. (b) 4. (a) 5. (d)

Iteration 3: ... the variable x_1 enters the basis ... 4.5 perform the following row operations ...

R_3 (new) $\rightarrow R_3$ (old) $\times$ 15/41 (*key element*)

$\rightarrow$ (89/15 $\times$ 15/41, 41/15 $\times$ 15/41, 0 $\times$ 15/41, 0 $\times$ 15/41, $-$ 2/15 $\times$ 15/41, $-$ 4/5 $\times$ 15/41, 1 $\times$ 15/41)

$\rightarrow$ (89/41, 1, 0, 0, $-$ 2/41, $-$ 12/41, 15/41)

R_1 (new) $\rightarrow R_1$ (old) $-$ (2/3) R_3 (new)
8/3 $-$ 2/3 $\times$ 89/3 = 50/41
2/3 $-$ 2/3 $\times$ 1 = 0
1 $-$ 2/3 $\times$ 0 = 1
0 $-$ 2/3 $\times$ 0 = 0
1/3 $-$ 2/3 $\times$ $-$ 2/41 = 15/41
0 $-$ 2/3 $\times$ $-$ 2/41 = 8/41
0 $-$ 2/3 $\times$ 15/41 = $-$ 10/41

R_2 (new) $\rightarrow R_2$ (old) + (4/15) R_3 (new)
14/15 + 4/15 $\times$ 89/41 = 62/41
$-$ 4/15 + 4/15 $\times$ 1 = 0
0 + 4/15 $\times$ 0 = 0
1 + 4/15 $\times$ 0 = 1
$-$ 2/15 + 4/15 $\times$ $-$ 2/41 = $-$ 6/41
1/5 + 4/15 $\times$ $-$ 12/41 = 5/41
0 + 4/15 $\times$ 15/41 = 4/41

Table 4.6 Optimal Solution

		$c_j \rightarrow$	3	5	4	0	0	0
Basic Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
5	x_2	50/41	0	1	0	15/41	8/41	$-$ 10/41
4	x_3	62/41	0	0	1	$-$ 6/41	5/41	4/41
3	x_1	89/41	1	0	0	$-$ 2/41	$-$ 12/41	15/41
Z = 765/41		z_j	3	5	4	45/41	24/41	11/41
		$c_j - z_j$	0	0	0	$-$ 45/41	$-$ 24/41	$-$ 11/41

In Table 4.6, all $c_j - z_j < 0$ for non-basic variables. Therefore, the optimal solution is reached with, x_1 = 89/41, x_2 = 50/41, x_3 = 62/41 and the optimal value of Z = 765/41.

Example 4.2 A company makes two kinds of leather belts, belt A and belt B. Belt A is a high quality belt and belt *B* is of lower quality. The respective profits are Rs 4 and Rs 3 per belt. The production of each of type *A* requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1,000 belts per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 of these are available per day. There are only 700 buckles a day available for belt B.

What should be the daily production of each type of belt? Formulate this problem as an LP model and solve it using the simplex method.

Solution Let x_1 and x_2 be the number of belts of type A and B, respectively, manufactured each day. Then the LP model would be as follows:

Maximize (total profit) $Z = 4x_1 + 3x_2$

subject to the constraints

(i) $2x_1 + x_2 \leq 1,000$ (Time availability), (ii) $x_1 + x_2 \leq 800$ (Supply of leather)

(iii) $\left.\begin{array}{l} x_1 \leq 400 \\ x_2 \leq 700 \end{array}\right\}$ (Buckles availability)

and $x_1, x_2 \geq 0$

Standard form Introducing slack variables s_1, s_2, s_3 and s_4 to convert given LP model into its standard form as follows.

Maximize $Z = 4x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to the constraints

(i) $2x_1 + x_2 + s_1 = 1{,}000$, (ii) $x_1 + x_2 + s_2 = 800$

(iii) $x_1 + s_3 = 400$, (iv) $x_2 + s_4 = 700$

and $x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

Solution by simplex method An initial feasible solution is obtained by setting $x_1 = x_2 = 0$. Thus, the initial solution is: $s_1 = 1{,}000$, $s_2 = 800$, $s_3 = 400$, $s_4 = 700$ and Max $Z = 0$. This solution can also be read from the initial simplex Table 4.7.

		$c_j \rightarrow$	4	3	0	0	0	0	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4	Min Ratio x_B/x_1
0	s_1	1,000	2	1	1	0	0	0	1,000/2 = 500
0	s_2	800	1	1	0	1	0	0	800/1 = 800
0	s_3	400	1	0	0	0	1	0	400/1 = 400 →
0	s_4	700	0	1	0	0	0	1	not defined
$Z = 0$	z_j		0	0	0	0	0	0	
	$c_j - z_j$		4	3	0	0	0	0	
			↑						

Table 4.7
Initial Solution

In Table 4.7, since $c_1 - z_1 = 4$ is the largest positive number, we apply the following row operations in order to get an improved basic feasible solution by entering variable x_1 into the basis and removing variable s_3 from the basis.

R_3 (new) $\rightarrow R_3$ (old) $\div$ 1 (*key element*) R_1 (new) $\rightarrow R_1$ (old) $- 2R_3$ (new)

R_2 (new) $\rightarrow R_2$ (old) $- R_3$ (new)

The new solution is shown in Table 4.8.

		$c_j \rightarrow$	4	3	0	0	0	0	
Basic Variables Coefficient c_B	Basis Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_3	s_4	Min Ratio x_B/x_2
0	s_1	200	0	1	1	0	−2	0	200/1 = 200 →
0	s_2	400	0	1	0	1	−1	0	400/1 = 400
4	x_1	400	1	0	0	0	1	0	–
0	s_4	700	0	1	0	0	0	1	700/1 = 700
$Z = 1{,}600$	z_j		4	0	0	0	4	0	
	$c_j - z_j$		0	3	0	0	−4	0	
				↑					

Table 4.8
An Improved Solution

The solution shown in Table 4.8 is not optimal because $c_2 - z_2 > 0$ in x_2-column. Thus, again applying the following row operations to get a new solution by entering variable x_2 into the basis and removing variable s_1 from the basis, we get

R_1 (new) $\rightarrow R_1$ (old) $\div$ 1 (*key element*) R_2 (new) $\rightarrow R_2$ (old) $- R_1$ (new)

R_4 (new) $\rightarrow R_4$ (old) $- R_1$ (new)

The improved solution is shown in Table 4.9.

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Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1 (4)	x_2 (3)	s_1 (0)	s_2 (0)	s_3 (0)	s_4 (0)	Min Ratio x_B/s_3
3	x_2	200	0	1	1	0	−2	0	—
0	s_2	200	0	0	−1	1	1	0	200/1 = 200 →
4	x_1	400	1	0	0	0	1	0	400/1 = 400
0	s_4	500	0	0	−1	0	2	1	500/2 = 250
Z = 2,200		z_j	4	3	3	0	−2	0	
		$c_j - z_j$	0	0	−3	0	2 ↑	0	

Table 4.9
Improved Solution

The solution shown in Table 4.9 is not optimal because $c_5 - z_5 > 0$ in s_3-column. Thus, again applying the following row operations to get a new solution by entering variable s_3, into the basis and removing variable s_2 from the basis, we get

R_2 (new) $\rightarrow R_2$ (old) $\div$ 1 (*key element*) $\quad$ R_1 (new) $\rightarrow R_1$ (old) + $2R_2$ (new)

R_3 (new) $\rightarrow R_3$ (old) − R_2 (new); $\quad$ R_4 (new) $\rightarrow R_4$ (old) − $2R_2$ (new)

The new improved solution is shown in Table 4.10.

Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1 (4)	x_2 (3)	s_1 (0)	s_2 (0)	s_3 (0)	s_4 (0)
3	x_2	600	0	1	−1	2	0	0
0	s_3	200	0	0	−1	1	1	0
4	x_1	200	1	0	1	−1	0	0
0	s_4	100	0	0	1	−2	0	1
Z = 2,600		z_j	4	3	1	2	0	0
		$c_j - z_j$	0	0	−1	−2	0	0

Table 4.10
Optimal Solution

Since all $c_j - z_j < 0$ correspond to non-basic variables columns, the current basic feasible solution is also the optimal solution. Thus, the company must manufacture, $x_1 = 200$ belts of type A and $x_2 = 600$ belts of type B in order to obtain the maximum profit of Rs 2,600.

Example 4.3 A pharmaceutical company has 100 kg of A, 180 kg of B and 120 kg of C ingredients available per month. The company can use these materials to make three basic pharmaceutical products namely 5-10-5, 5-5-10 and 20-5-10, where the numbers in each case represent the percentage of weight of A, B and C, respectively, in each of the products. The cost of these raw materials is as follows:

Ingredient	Cost per kg (Rs)
A	80
B	20
C	50
Inert ingredients	20

The selling prices of these products are Rs 40.5, Rs 43 and 45 per kg , respectively. There is a capacity restriction of the company for product 5-10-5, because of which the company cannot produce more than 30 kg per month. Determine how much of each of the products the company should produce in order to maximize its monthly profit.

[Delhi Univ., MBA, 2004, AMIE, 2005]

Solution Let the P_1, P_2 and P_3 be the three products to be manufactured. The data of the problem can then be summarized as follows:

Product	Product Ingredients			Inert
	A	B	C	
P_1	5%	10%	5%	80%
P_2	5%	5%	10%	80%
P_3	20%	5%	10%	65%
Cost per kg (Rs)	80	20	50	20

Cost of P_1 = 5% × 80 + 10% × 20 + 5% × 50 + 80% × 20 = 4 + 2 + 2.50 + 16 = Rs 24.50 per kg
Cost of P_2 = 5% × 80 + 5% × 20 + 10% × 50 + 80% × 20 = 4 + 1 + 5 + 16 = Rs 26 per kg
Cost of P_3 = 20% × 80 + 5% × 20 + 10% × 50 + 65% × 20 = 16 + 1 + 5 + 13 = Rs 35 per kg

Let x_1, x_2 and x_3 be the quantity (in kg) of P_1, P_2 and P_3, respectively to be manufactured. The LP problem can then be formulated as

Maximize (net profit) Z = (Selling price – Cost price) × (Quantity of product)
$= (40.50 - 24.50)\,x_1 + (43 - 26)\,x_2 + (45 - 35)\,x_3 = 16x_1 + 17x_2 + 10x_3$

subject to the constraints

$$\frac{1}{20}x_1 + \frac{1}{20}x_2 + \frac{1}{5}x_3 \le 100 \quad \text{or} \quad x_1 + x_2 + 4x_3 \le 2{,}000$$

$$\frac{1}{10}x_1 + \frac{1}{20}x_2 + \frac{1}{20}x_3 \le 180 \quad \text{or} \quad 2x_1 + x_2 + x_3 \le 3{,}600$$

$$\frac{1}{20}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_3 \le 120 \quad \text{or} \quad x_1 + 2x_2 + 2x_3 \le 2{,}400$$

$$x_1 \le 30$$

and $x_1, x_2, x_3 \ge 0.$

Standard form Introducing slack variables s_1, s_2 and s_3 to convert the given LP model into its standard form as follows:

Maximize $Z = 16x_1 + 17x_2 + 10x_3 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to the constraints

(i) $x_1 + x_2 + 4x_3 + s_1 = 2{,}000$, (ii) $2x_1 + x_2 + x_3 + s_2 = 3{,}600$
(iii) $x_1 + 2x_2 + 2x_3 + s_3 = 2{,}400$, (iv) $x_1 + s_4 = 30$

and $x_1, x_2, x_3, s_1, s_2, s_3, s_4 \ge 0$

Solution by simplex method An initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$. Thus, the initial solution shown in Table 4.11 is: $s_1 = 2{,}000$, $s_2 = 3{,}600$, $s_3 = 2{,}400$, $s_4 = 30$ and Max $Z = 0$.

		$c_j \rightarrow$	16	17	10	0	0	0	0	
Basic Variables Coefficient c_B	Basis Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Min Ratio x_B/x_2
0	s_1	2,000	1	1	4	1	0	0	0	2,000/1 = 2,000
0	s_2	3,600	2	1	1	0	1	0	0	3,600/1 = 3,600
0	s_3	2,400	1	2	2	0	0	1	0	2,400/2 = 1,200 →
0	s_4	30	1	0	0	0	0	0	1	—
$Z = 0$		z_j	0	0	0	0	0	0	0	
		$c_j - z_j$	16	17 ↑	10	0	0	0	0	

Table 4.11 Initial Solution

Since $c_2 - z_2 = 17$ in x_2-column is the largest positive value, we apply the following row operations in order to get a new improved solution by entering variable x_2 into the basis and removing variable s_3 from the basis.

R_3 (new) → R_3 (old) ÷ 2 (*key element*) R_1 (new) → R_1 (old) – R_3 (new)
R_2 (new) → R_2 (old) – R_3 (new)

The new solution is shown in Table 4.12.

		$c_j \rightarrow$	16	17	10	0	0	0	0	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Min Ratio x_B/x_1
0	s_1	800	1/2	0	3	1	0	– 1/2	0	800/(1/2) = 1,600
0	s_2	2,400	3/2	0	0	0	1	– 1/2	0	2,400/(3/2) = 1,600
17	x_2	1,200	1/2	1	1	0	0	1/2	0	1,200/(1/2) = 2,400
0	s_4	30	1	0	0	0	0	0	1	30/1 = 30 →
$Z = 20{,}400$		z_j	17/2	17	17	0	0	17/2	0	
		$c_j - z_j$	15/2	0	– 7	0	0	– 17/2	0	
			↑							

Table 4.12 Improve Solution

The solution shown in Table 4.12 is not optimal because $c_1 - z_1 > 0$ in x_1-column. Thus, applying the following row operations to get a new improved solution by entering variable x_1 into the basis and removing the variable s_4 from the basis, we get

R_4 (new) $\rightarrow R_4$ (old) ÷ 1 (*key element*); $\quad R_1$ (new) $\rightarrow R_1$ (old) – (1/2) R_4 (new)

R_2 (new) $\rightarrow R_2$ (old) – (3/2) R_4 (new); $\quad R_3$ (new) $\rightarrow R_3$ (old) – (1/2) R_4 (new)

The new solution is shown in Table 4.13.

		$c_j \rightarrow$	16	17	10	0	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	s_4
0	s_1	785	0	0	3	1	0	– 1/2	– 1/2
0	s_2	2,355	0	0	0	0	1	– 1/2	– 3/2
17	x_2	1,185	0	1	1	0	0	1/2	– 1/2
16	x_1	30	1	0	0	0	0	0	1
$Z = 20{,}625$		z_j	16	17	17	0	0	17/2	15/2
		$c_j - z_j$	0	0	– 7	0	0	– 17/2	– 15/2

Table 4.13 Optimal Solution

Since all $c_j - z_j < 0$ corresponding to non-basic variables columns, the current solution is an optimal solution. Thus, the company must manufacture, $x_1 = 30$ kg of P_1, $x_2 = 1{,}185$ kg of P_2 and $x_3 = 0$ kg of P_3 in order to obtain the maximum net profit of Rs 20,625.

4.4 SIMPLEX ALGORITHM (MINIMIZATION CASE)

In certain cases, it is difficult to obtain an initial basic feasible solution of the given LP problem. Such cases arise

(i) when the constraints are of the ≤ type,

$$\sum_{j=1}^{n} a_{ij}\, x_j \le b_i\,, \quad x_j \ge 0$$

and value of few right-hand side constants is negative [i.e. $b_i < 0$]. After adding the non-negative slack variable s_i $(i = 1, 2, \ldots, m)$, the initial solution so obtained will be $s_i = - b_i$ for a particular resource, *i*. This solution is not feasible because it does not satisfy non-negativity conditions of slack variables (i.e. $s_i \ge 0$).

(ii) when the constraints are of the ≥ type,

$$\sum_{j=1}^{n} a_{ij}\, x_j \ge b_i\,, \quad x_j \ge 0$$

After adding surplus (negative slack) variable s_i, the initial solution so obtained will be $-s_i = b_i$ or $s_i = -b_i$

$$\sum_{j=1}^{n} a_{ij}\, x_j - s_i = b_i, \quad x_j \geq 0,\ s_i \geq 0$$

This solution is not feasible because it does not satisfy non-negativity conditions of surplus variables (i.e. $s_i \geq 0$). In such a case, artificial variables, $A_i\ (i = 1, 2, \ldots, m)$ are added to get an initial basic feasible solution. The resulting system of equations then becomes:

$$\sum_{j=1}^{n} a_{ij}\, x_j - s_i + A_i = b_i$$

$$x_j,\ s_i,\ A_i \geq 0, \quad i = 1, 2, \ldots, m$$

These are m simultaneous equations with $(n + m + m)$ variables (n decision variables, m artificial variables and m surplus variables). An initial basic feasible solution of LP problem with such constraints can be obtained by equating $(n + 2m - m) = (n + m)$ variables equal to zero. Thus the new solution to the given LP problem is: $A_i = b_i\ (i = 1, 2, \ldots, m)$, which is not the solution to the original system of equations because the two systems of equations are not equivalent. Thus, to get back to the original problem, artificial variables must be removed from the optimal solution. There are two methods for removing artificial variables from the solution.

- *Two-Phase Method*
- *Big-M Method or Method of Penalties*

The simplex method, both for the minimization and the maximization LP problem, may be summarized through a flow chart shown in Fig. 4.1.

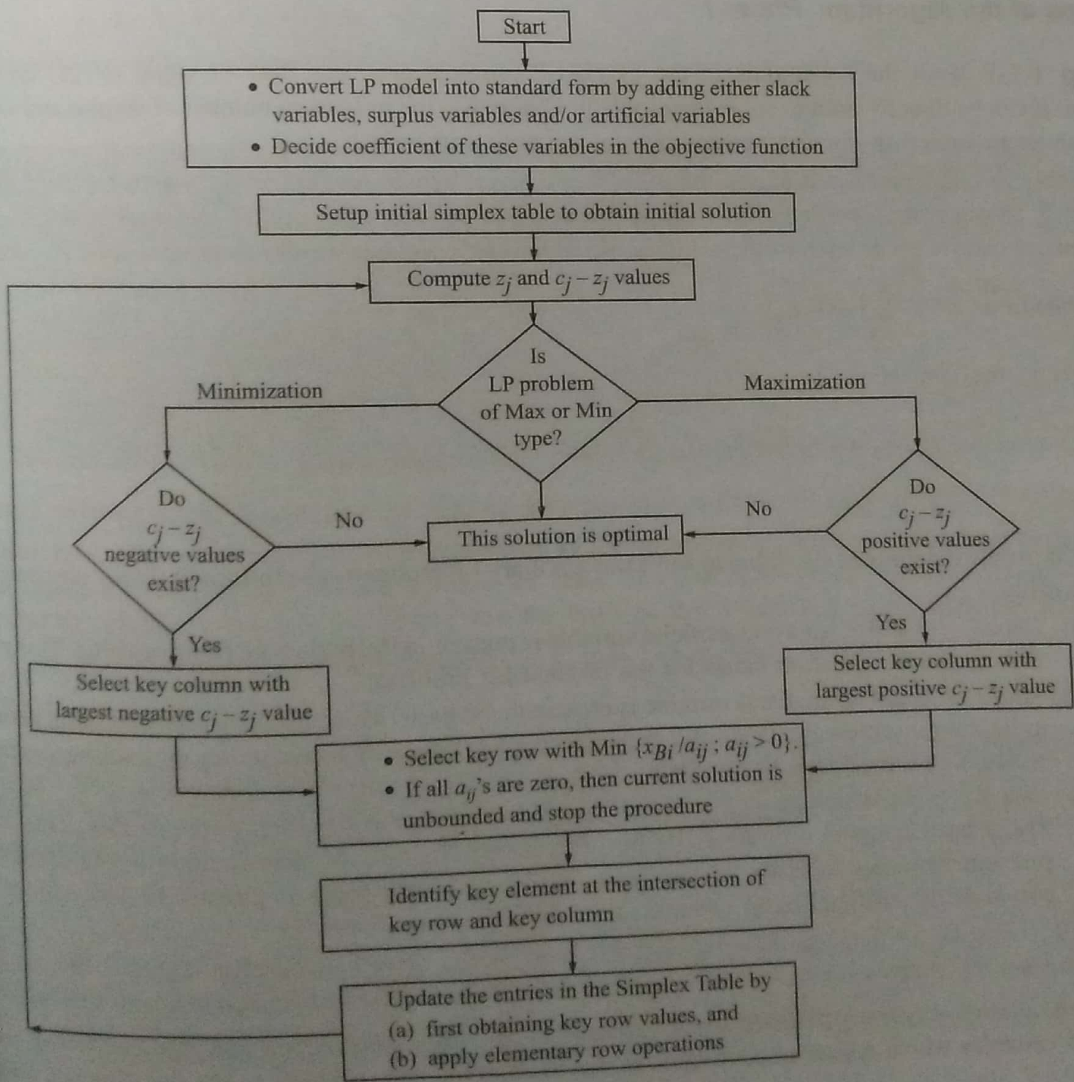

Fig. 4.1 Flow Chart of Simplex Algorithm

Remark Artificial variables have no meaning in a physical sense and are only used as a tool for generating an initial solution to an LP problem. Before the optimal solution is reached, all artificial variables must be dropped out from the solution mix. This is done by assigning appropriate coefficients to these variables in the objective function. These variables are added to those constraints with equality (=) and greater than or equal to (≥) sign.

4.4.1 Two-Phase Method

In the first phase of this method, the sum of the artificial variables is minimized subject to the given constraints in order to get a basic feasible solution of the LP problem. The second phase minimizes the original objective function starting with the basic feasible solution obtained at the end of the first phase. Since the solution of the LP problem is completed in two phases, this method is called the *two-phase method.*

Advantages of the method

1. No assumptions on the original system of constraints are made, i.e. the system may be redundant, inconsistent or not solvable in non-negative numbers.
2. It is easy to obtain an initial basic feasible solution for Phase I.
3. The basic feasible solution (if it exists) obtained at the end of phase I is used as initial solution for Phase II.

Steps of the Algorithm: *Phase I*

An artificial variable is added to the constraints to get an initial solution to an LP problem.

Step 1 (a): If all the constraints in the given LP problem are 'less than or equal to' (≤) type, then Phase II can be directly used to solve the problem. Otherwise, the necessary number of surplus and artificial variables are added to convert constraints into equality constraints.

(b) If the given LP problem is of minimization, then convert it to the maximization type by the usual method.

Step 2: Assign zero coefficient to each of the decision variables (x_j) and to the surplus variables; and assign – 1 coefficient to each of the artificial variables. This yields the following auxiliary LP problem.

$$\text{Maximize } Z^* = \sum_{i=1}^{m} (-1)\, A_i$$

subject to the constraints

$$\sum_{i=1}^{n} a_{ij}\, x_j + A_i = b_i, \qquad i = 1, 2, \ldots, m$$

and $\qquad x_j,\, A_j \geq 0$

Step 3: Apply the simplex algorithm to solve this auxiliary LP problem. The following three cases may arise at optimality.

(a) Max $Z^* = 0$ and at least one artificial variable is present in the basis with positive value. This means that no feasible solution exists for the original LP problem.

(b) Max $Z^* = 0$ and no artificial variable is present in the basis. This means that only decision variables (x_j's) are present in the basis and hence proceed to Phase II to obtain an optimal basic feasible solution on the original LP problem.

(c) Max $Z^* = 0$ and at least one artificial variable is present in the basis at zero value. This means that a feasible solution to the auxiliary LP problem is also a feasible solution to the original LP problem. In order to arrive at the basic feasible solution, proceed directly to Phase II or else eliminate the artificial basic variable and then proceed to Phase II.

Remark Once an artificial variable has left the basis, it has served its purpose and can, therefore, be removed from the simplex table. An artificial variable is never considered for re-entry into the basis.

Phase II: Assign actual coefficients to the variables in the objective function and zero coefficient to the artificial variables which appear at zero value in the basis at the end of Phase I. The last simplex table of

Phase I can be used as the initial simplex table for Phase II. Then apply the usual simplex algorithm to the modified simplex table in order to get the optimal solution to the original problem. Artificial variables that do not appear in the basis may be removed.

Example 4.4 Use two-phase simplex method to solve the following LP problem:

Minimize $Z = x_1 + x_2$

subject to the constraints

(i) $2x_1 + x_2 \geq 4$, (ii) $x_1 + 7x_2 \geq 7$

and $x_1, x_2 \geq 0$

Solution Converting the given LP problem objective function into the maximization form and then adding surplus variables s_1 and s_2 and artificial variables A_1 and A_2 in the constraints, the problem becomes:

Maximize $Z^* = -x_1 - x_2$

subject to the constraints

(i) $2x_1 + x_2 - s_1 + A_1 = 4$, (ii) $x_1 + 7x_2 - s_2 + A_2 = 7$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

where $Z^* = -Z$

Phase I: This phase starts by considering the following auxiliary LP problem:

Maximize $Z^* = -A_1 - A_2$

subject to the constraints

(i) $2x_1 + x_2 - s_1 + A_1 = 4$, (ii) $x_1 + 7x_2 - s_2 + A_2 = 7$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

The initial solution is presented in Table 4.14.

		$c_j \rightarrow$	0	0	0	0	−1	−1	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	Ratio
−1	A_1	4	2	1	−1	0	1	0	4
−1	A_2	7	1	7	0	−1	0	1 →	1 →
$Z^* = -11$		z_j	−3	−8	1	1	−1	−1	
		$c_j - z_j$	3	8 ↑	−1	−1	0	0	

Table 4.14 Initial Solution

Artificial variables A_1 and A_2 are now removed, one after the other, maintaining the feasibility of the solution.

Iteration 1: Applying the following row operations to get an improved solution by entering variable x_2 in the basis and first removing variable A_2 from the basis. The improved solution is shown in Table 4.15. Note that the variable x_1 cannot be entered into the basis as this would lead to an infeasible solution.

R_2 (new) $\rightarrow R_2$ (old) ÷ 7 (*key element*); R_1 (new) $\rightarrow R_1$ (old) − R_2 (new)

		$c_j \rightarrow$	0	0	0	0	−1	−1	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2^*	Ratio
−1	A_1	3	13/7	0	−1	1/7	1	−1/7 →	21 →
0	x_2	1	1/7	1	0	−1/7	0	1/7	−7
$Z^* = -3$		z_j	−13/7	0	1	−1/7	−1	1/7	
		$c_j - z_j$	13/7	0	−1	1/7 ↑	0	−8/7	

Table 4.15 Improved Solution

* This column can permanently be removed at this stage.

Iteration 2: To remove A_1 from the solution shown in Table 4.15, we enter variable s_2 in the basis by applying the following row operations. The new solution is shown in Table 4.16. It may be noted that if instead of s_2, variable x_1 is chosen to be entered into the basis, it will lead to an infeasible solution

R_1 (new) → R_1 (old) × 7; R_2 (new) → R_2 (old) + (1/7) R_1 (new)

Table 4.16 Improved Solution

	c_j →		0	0	0	0	−1	−1
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	s_1	s_2	A_1^*	A_2^*
0	s_2	21	13	0	−7	1	7	−1
0	x_2	4	2	1	−1	0	1	0
$Z^* = 0$		z_j	0	0	0	0	0	0
		$c_j - z_j$	0	0	0	0	−1	−1

* Remove columns A_1 and A_2 from Table 4.16.

Since all $c_j - z_j \le 0$ correspond to non-basic variables, the optimal solution: $x_1 = 0$, $x_2 = 4$, $s_1 = 0$, $s_2 = 21$, $A_1 = 0$, $A_2 = 0$ with $Z^* = 0$ is arrived at. However, this solution may or may not be the basic feasible solution to the original LP problem. Thus, we have to move to Phase II to get an optimal solution to our original LP problem.

Phase II: The modified simplex table obtained from Table 4.16 is represented in Table 4.17.

Table 4.17

	c_j →		−1	−1	0	0	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	s_1	s_2	Min Ratio x_B/x_1
0	s_2	21	13	0	−7	1	21/13 →
−1	x_2	4	2	1	−1	0	4/2
$Z^* = -4$		z_j	−2	−1	1	0	
		$c_j - z_j$	1	0	−1	0	
			↑				

Iteration 1: Introducing variable x_1 into the basis and removing variable s_2 from the basis by applying the following row operations:

R_1 (new) → R_1 (old) ÷ 13 (*key element*); R_2 (new) → R_2 (old) − 2R_1 (new)

The improved basic feasible solution so obtained is given in Table 4.18. Since in Table 4.18, $c_j - z_j \le 0$ for all non-basic variables, the current solution is optimal. Thus, the optimal basic feasible solution to the given LP problem is: $x_1 = 21/13$, $x_2 = 10/13$ and Max $Z^* = -31/13$ or Min $Z = 31/13$.

Table 4.18 Optimal Solution

	c_j →		−1	−1	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	s_1	s_2
−1	x_1	21/13	1	0	−7/13	1/13
−1	x_2	10/13	0	1	1/13	−2/13
$Z^* = -31/13$		z_j	−1	−1	6/13	1/13
		$c_j - z_j$	0	0	−6/13	−1/13

Example 4.5 Solve the following LP problem by using the two-phase simplex method.

Minimize $Z = x_1 - 2x_2 - 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 = 1$

and $x_1, x_2, x_3 \ge 0$.

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Solution After converting the objective function into the maximization form and by adding artificial variables A_1 and A_2 in the constraints, the given LP problem becomes:

Maximize $Z^* = -x_1 + 2x_2 + 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 + A_1 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 + A_2 = 1$

and $x_1, x_2, x_3, A_1, A_2 \geq 0$

where $Z^* = -Z$

Phase I: This phase starts by considering the following auxiliary LP problem:

Maximize $Z^* = -A_1 - A_2$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 + A_1 = 2$ (ii) $2x_1 + 3x_2 + 4x_3 + A_2 = 1$

and $x_1, x_2, x_3, A_1, A_2 \geq 0$

The initial solution is presented in Table 4.19.

		$c_j \rightarrow$	0	0	0	−1	−1
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	x_3	A_1	A_2
−1	A_1	2	−2	1	3	1	0
−1	A_2	1	2	3	4	0	1 →
$Z^* = -3$		z_j	0	−4	−7	−1	−1
		$c_j - z_j$	0	4	7 ↑	0	0

Table 4.19
Initial Solution

To first remove the artificial variable A_2 from the solution shown in Table 4.19, introduce variable x_3 into the basis by applying the following row operations:

R_1 (new) $\rightarrow R_2$ (old) ÷ 4 (*key element*) ; R_1 (new) $\rightarrow R_1$ (old) − $3R_2$ (new)

The improved solution so obtained is given in Table 4.20. Since in Table 4.20, $c_j - z_j \leq 0$ corresponds to non-basic variables, the optimal solution is: $x_1 = 0$, $x_2 = 0$, $x_3 = 1/4$, $A_1 = 5/4$ and $A_2 = 0$ with Max $Z^* = -5/4$. But at the same time, the value of $Z^* < 0$ and the artificial variable A_1 appears in the basis with positive value 5/4. Hence, feasible solution to the given original LP problem does not exist.

		$c_j \rightarrow$	0	0	0	−1
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	x_3	A_1
−1	A_1	5/4	−7/2	−5/4	0	1
0	x_3	1/4	1/2	3/4	1	0
$Z^* = -5/4$		z_j	7/2	5/4	0	−1
		$c_j - z_j$	−7/2	−5/4	0	0

Table 4.20
Optimal but not Feasible Solution

Example 4.6 Use two-phase simplex method to solve following LP problem.

Maximize $Z = 3x_1 + 2x_2 + 2x_3$

subject to the constraints

(i) $5x_1 + 7x_2 + 4x_3 \leq 7$, (ii) $-4x_1 + 7x_2 + 5x_3 \geq -2$, (iii) $3x_1 + 4x_2 - 6x_3 \geq 29/7$

and $x_1, x_2, x_3 \geq 0$. [*Punjab Univ., BE (E & C) 2006 ; BE (IT) 2006*]

Solution Since RHS of constraint 2 is negative, multiplying it by − 1 on both sides and express it as: $4x_1 - 7x_2 - 5x_3 \leq 2$.

Phase I: Introducing slack, surplus and artificial variables in the constraints, the standard form of LP problem becomes:

Minimize $Z^* = A_1$

subject to the constraints

(i) $5x_1 + 7x_2 + 4x_3 + s_1 = 7$, (ii) $4x_1 - 7x_2 - 5x_3 + s_2 = 2$, (iii) $3x_1 + 4x_2 - 6x_3 - s_3 + A_1 = 29/7$

and $x_1, x_2, x_3, s_1, s_2, s_3, A_1 \geq 0$.

Setting variables $x_1 = x_2 = x_3 = s_3 = 0$, the basic feasible solution shown in Table 4.21 to the auxiliary LP problem is: $s_1 = 7, s_2 = 2, A_1 = 29/7$

Table 4.21 Initial Solution

		$c_j \rightarrow$	0	0	0	0	0	0	1	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	A_1	Min Ratio x_B/x_2
0	s_1	7	5	7	4	1	0	0	0	1 →
0	s_2	2	4	−7	−5	0	1	0	0	—
1	A_1	29/7	3	4	−6	0	0	−1	1	29/28
$Z = 29/7$		z_j	3	4	−6	0	0	−1	1	
		$c_j - z_j$	−3	−4	6	0	0	1	0	
				↑						

Since $c_2 - z_2 = -4$ is the largest negative value under x_2-column in Table 4.21, replacing basic variable s_1 with the non-basic variable x_2 into the basis. For this, apply following row operations:

R_1(new) → R_1(old) ÷ 7(*key element*); R_2(new) → R_2(old) + 7R_1(new);

R_3(new) → R_3(old) − 4R_1(new)

to get an improved solution as shown in Table 4.22.

Table 4.22 An Improved Solution

		$c_j \rightarrow$	0	0	0	0	0	0	1	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	A_1	Min Ratio x_B/x_1
0	x_2	1	5/7	1	4/7	1/7	0	0	0	7/5
0	s_2	9	9	0	−1	1	1	0	0	1 } Tie
1	A_1	1/7	1/7	0	−58/7	−4/7	0	−1	1	1 }
$Z = 1/7$		z_j	1/7	0	−58/7	−4/7	0	−1	1	
		$c_j - z_j$	−1/7	0	58/7	4/7	0	1	0	
			↑							

Solution shown in Table 4.22 is not optimal and there is a tie between the s_2 and A_1-rows for the key row. The A_1-row is selected as the key row because preference is given to the artificial variable. Thus (1/7) is the key element. Replace basic variable A_1 in the basis with the non-basic variable x_1. Applying following row operations to get the new solution as shown in Table 4.23.

R_3(new) → R_3(old) ÷ (1/7) key element; R_1(new) → R_1(old) − (5/7) R_3 (new);

R_2(new) → R_2(old) − 9 R_3(new)

Table 4.23 Basic Feasible Solution

		$c_j \rightarrow$	0	0	0	0	0	0
Basic Variables Coefficient c_B	Variables in Basis B	Basic Variables Value $b\,(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
0	x_2	2/7	0	1	42	3	0	5
0	s_2	0	0	0	521	37	1	63
0	x_1	1	1	0	−58	−4	0	−7
$Z = 0$		z_j	0	0	0	0	0	0
		$c_j - z_j$	0	0	0	0	0	0

In Table 4.23, all $c_j - z_j = 0$ under non-basic variable columns. Thus, the current solution is optimal. Also Min $Z^* = 0$ and no artificial variable appears in the basis, and hence this solution is the basic feasible solution to the original LP problem.

Phase II: Using the solution shown in Table 4.23 as the initial solution for Phase II and carrying out computations to get optimal solution as shown in Table 4.24.

Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio x_B/x_3
	$c_j \rightarrow$		3	2	2	0	0	0	
2	x_2	2/7	0	1	42	3	0	5	1/147
0	s_2	0	0	0	521	37	1	63	0 →
3	x_1	1	1	0	−58	−6	0	−7	—
$Z = 25/7$	z_j		3	2	−90	−6	0	−11	
	$c_j - z_j$		0	0	92 ↑	6	0	11	

Table 4.24
Initial Solution

In Table 4.24, $c_3 - z_3 = 92$ is the largest positive value corresponding non-basic variable x_3, replacing basic variable s_2 with non-basic variable x_3 into the basis. For this apply necessary row operations as usual. The new solution is shown in Table 4.25.

Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
	$c_j \rightarrow$		3	2	2	0	0	0
2	x_2	2/7	0	1	0	9/521	−42/521	−41/521
2	x_3	0	0	0	1	37/521	1/521	63/521
3	x_1	1	1	0	0	62/521	58/521	7/521
$Z = 25/7$	z_j		3	2	2	278/521	92/521	65/521
	$c_j - z_j$		0	0	0	−278/521	−92/521	−65/521

Table 4.25
Optimal Solution

Since all $c_j - z_j \le 0$ in Table 4.25, the current solution is the optimal basic feasible solution: $x_1 = 1, x_2 = 2/7$, $x_3 = 0$ and Max $Z = 25/7$.

4.4.2 Big-M Method

Big-M method is another method of removing artificial variables from the basis. In this method, large undesirable (unacceptable penalty) coefficients to artificial variables are assigned from the point of view of the objective function. If the objective function Z is to be minimized, then a very large positive price (called *penalty*) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called *penalty*) is assigned to each of these variables. The penalty is supposed to be designated by $-M$, for a maximization problem, and $+M$, for a minimization problem, where $M > 0$. The Big-M method for solving an LP problem can be summarized in the following steps:

Step 1: Express the LP problem in the standard form by adding slack variables, surplus variables and/or artificial variables. Assign a zero coefficient to both slack and surplus variables. Then assign a very large coefficient $+M$ (minimization case) and $-M$ (maximization case) to artificial variable in the objective function.

Step 2: The initial basic feasible solution is obtained by assigning zero value to decision variables, x_1, x_2, ..., etc.

Step 3: Calculate the values of $c_j - z_j$ in last row of the simplex table and examine these values.

(i) If all $c_j - z_j \ge 0$, then the current basic feasible solution is optimal.

(ii) If for a column, k, $c_k - z_k$ is most negative and all entries in this column are negative, then the problem has an unbounded optimal solution.

(iii) If one or more $c_j - z_j < 0$ (minimization case), then select the variable to enter into the basis (solution mix) with the largest negative $c_j - z_j$ value (largest per unit reduction in the objective function value). This value also represents the opportunity cost of not having one unit of the variable in the solution. That is,

$$c_k - z_k = \text{Min } \{c_j - z_j : c_j - z_j < 0\}$$

Step 4: Determine the key row and key element in the same manner as discussed in the simplex algorithm for the maximization case.

Step 5: Continue with the procedure to update solution at each iteration till optimal solution is obtained.

Remarks At any iteration of the simplex algorithm one of the following cases may arise:

1. If at least one artificial variable is a basic variable (i.e., variable that is present in the basis) with zero value and the coefficient it M in each $c_j - z_j$ $(j = 1, 2, \ldots, n)$ values is non-negative, then the given LP problem has no solution. That is, the current basic feasible solution is degenerate.
2. If at least one artificial variable is present in the basis with a positive value and the coefficients M in each $c_j - z_j$ $(j = 1, 2, \ldots, n)$ values is non-negative, then the given LP problem has no optimum basic feasible solution. In this case, the given LP problem has a *pseudo optimum* basic feasible solution.

Example 4.7 Use penalty (Big-M) method to solve the following LP problem.

Minimize $Z = 5x_1 + 3x_2$

subject to the constraints

(i) $2x_1 + 4x_2 \leq 12$, (ii) $2x_1 + 2x_2 = 10$, (iii) $5x_1 + 2x_2 \geq 10$

and $x_1, x_2 \geq 0$.

Solution Adding slack variable, s_1; surplus variable, s_2 and artificial variables, A_1 and A_2 in the constraints of the given LP problem, the standard form of the LP problem becomes.

Minimize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

subject to the constraints

(i) $2x_1 + 4x_2 + s_1 = 12$, (ii) $2x_1 + 2x_2 + A_1 = 10$ (iii) $5x_1 + 2x_2 - s_2 + A_2 = 10$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

An initial basic feasible solution: $s_1 = 12$, $A_1 = 10$, $A_2 = 10$ and Min $Z = 10M + 10M = 20M$ is obtained by putting $x_1 = x_2 = s_2 = 0$. It may be noted that the columns that correspond to the current basic variables and form the basis (identity matrix) are s_1 (slack variable), A_1 and A_2 (both artificial variables). The initial basic feasible solution is given in Table 4.26.

Since the value $c_1 - z_1 = 5 - 7M$ is the smallest value, therefore variable x_1 is chosen to enter into the basis (solution mix). To decide a current basic variable to leave the basis, calculate minimum ratio as shown in Table 4.26.

		$c_j \rightarrow$	5	3	0	0	M	M	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	Min Ratio x_B/x_1
0	s_1	12	2	4	1	0	0	0	12/2 = 6
M	A_1	10	2	2	0	0	1	0	10/2 = 5
M	A_2	10	5	2	0	−1	0	1	10/5 = 2 →
$Z = 20M$		z_j	$7M$	$4M$	0	$-M$	M	M	
		$c_j - z_j$	$5 - 7M$ ↑	$3 - 4M$	0	M	0	0	

Table 4.26 Initial Solution

Iteration 1: Introduce variable x_1 into the basis and remove A_2 from the basis by applying the following row operations.

R_3 (new) $\rightarrow R_3$ (old) ÷ 5 (*key element*); R_2 (new) $\rightarrow R_2$ (old) − $2R_3$ (new).

R_1 (new) $\rightarrow R_1$ (old) − $2R_3$ (new).

The improved basic feasible solution is shown in Table 4.27.

		$c_j \rightarrow$	5	3	0	0	M	
Basic Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	*Min Ratio* x_B/x_2
0	s_1	8	0	16/5	1	2/5	0	$8/(16/5) = 5/2 \rightarrow$
M	A_1	6	0	6/5	0	2/5	1	$6/(6/5) = 5$
5	x_1	2	1	2/5	0	$-1/5$	0	$2/(2/5) = 5$
$Z = 10 + 6M$		z_j	5	$(6M/5) + 2$	0	$(2M/5) - 1$	M	
		$c_j - z_j$	0	$(-6M/5) + 1$	0	$(-2M/5) + 1$	0	
				↑				

Table 4.27 Improved Solution

Iteration 2: Since the value of $c_2 - z_2$ in Table 4.27 is the largest negative value, variable x_2 is chosen to replace basic variable s_1 in the basis. Thus, to get an improved basic feasible solution, apply, the following row operations:

R_1 (new) $\rightarrow R_1$ (old) $\times$ 5/16 (*key element*); R_2 (new) $\rightarrow R_2$ (old) – (6/5) R_1 (new).

R_3 (new) $\rightarrow R_3$ (old) – (2/5) R_1 (new).

The new solution is shown in Table 4.28.

		$c_j \rightarrow$	5	3	0	0	M	
Basic Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	*Min Ratio* x_B/s_2
3	x_2	5/2	0	1	5/16	1/8	0	$(5/2)/(1/8) = 40$
M	A_1	3	0	0	$-3/8$	1/4	1	$3/(1/4) = 12 \rightarrow$
5	x_1	1	1	0	$-1/8$	$-1/4$	0	
$Z = 25/2 + 3M$		z_j	5	3– 3M/8 + 5/16		$M/4 - 7/8$	M	
		$c_j - z_j$	0	03M/8 – 5/16		$-M/4 + 7/8$	0	
						↑		

Table 4.28 Improved Solution

Iteration 3: Since $c_4 - z_4 < 0$ (negative) in s_2-column, the current solution is not optimal. Thus, non-basic variable s_2 is chosen to replace artificial variable A_1 in the basis. To get an improved basic feasible solution, apply the following row operations:

R_2 (new) $\rightarrow R_2$ (old) $\times$ 4 (*key element*); R_1 (new) $\rightarrow R_1$ (old) – (1/8) R_2 (new)

R_3 (new) $\rightarrow R_3$ (old) + (1/4) R_2 (new).

The improved basic feasible solution is shown in Table 4.29.

		$c_j \rightarrow$	5	3	0	0
Basic Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* $b(= x_B)$	x_1	x_2	s_1	s_2
3	x_2	1	0	1	1/2	0
0	s_2	12	0	0	$-3/2$	1
5	x_1	4	1	0	$-1/2$	0
$Z = 23$		z_j	5	3	-1	0
		$c_j - z_j$	0	0	1	0

Table 4.29 Optimal Solution

In Table 4.24, all $c_j - z_j \geq 0$. Thus an optimal solution is arrived at with the value of variables as: $x_1 = 4$, $x_2 = 1$, $s_1 = 0$, $s_2 = 12$ and Min $Z = 23$.

Example 4.8 Use penalty (Big-M) method to solve the following LP problem.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

subject to the constraints

(i) $x_1 + 2x_2 + 3x_3 = 15$, (ii) $2x_1 + x_2 + 5x_3 = 20$, (iii) $x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_1, x_2, x_3, x_4 \geq 0$ [*Calicut, BTech. (Engg), 2000; Bangalore BE (Mech.), 2000; AMIE, 2009*]

Solution Since all constraints of the given LP problem are equations, therefore only artificial variables A_1 and A_2 are added in the constraints to convert given LP problem to its standard form. The standard form of the problem is stated as follows:

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

subject to the constraints

(i) $x_1 + 2x_2 + 3x_3 + A_1 = 15$, (ii) $2x_1 + x_2 + 5x_3 + A_2 = 20$

(iii) $x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_1, x_2, x_3, x_4, A_1, A_2 \geq 0$

An initial basic feasible solution is given in Table 4.30.

		$c_j \rightarrow$	1	2	3	−1	−M	−M	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	x_3	x_4	A_1	A_2	Min Ratio x_B/x_3
$-M$	A_1	15	1	2	3	0	1	0	15/3 = 5
$-M$	A_2	20	2	1	5	0	0	1	20/5 = 4 →
−1	x_4	10	1	2	1	1	0	0	10/1 = 10
$Z = -35M - 10$		z_j	$-3M-1$	$-3M-2$	$-8M-1$	−1	$-M$	$-M$	
		$c_j - z_j$	$3M+2$	$3M+4$	$8M+4$	0	0	0	
					↑				

Table 4.30
Initial Solution

Since value of $c_3 - z_3$ in Table 4.30 is the largest positive value, the non-basic variable x_3 is chosen to replace the artificial variable A_2 in the basis. Thus, to get an improved solution, apply the following row operations.

R_2 (new) $\rightarrow R_2$ (old) $\div$ 5 (*key element*) ; R_1 (new) $\rightarrow R_1$ (old) $-$ 3 R_2 (new).

R_3 (new) $\rightarrow R_3$ (old) $-$ R_2 (new)

The improved basic feasible solution is shown in Table 4.31.

		$c_j \rightarrow$	1	2	3	−1	−M	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	x_3	x_4	A_1	Min Ratio x_B/x_2
$-M$	A_1	3	−1/5	7/5	0	0	1	$\frac{3}{7/5} = \frac{15}{7}$ →
3	x_3	4	2/5	1/5	1	0	0	$\frac{4}{1/5} = 20$
−1	x_4	6	3/5	9/5	0	1	0	$\frac{6}{9/5} = \frac{30}{9}$
$Z = -3M + 6$		z_j	$M/5 + 3/5$	$-7M/5 - 6/5$	3	−1	$-M$	
		$c_j - z_j$	$-M/5 - 2/5$	$7M/5 + 16/5$	0	0	0	
				↑				

Table 4.31
Improved Solution

Since value of $c_2 - z_2 > 0$ (positive) in Table 4.31, the non-basic variable x_2 is chosen to replace the artificial variable A_1 in the basis. Thus, to get an improved basic feasible solution, apply the following row operations:

R_1 (new) $\to R_1$ (old) $\times$ (5/7) (*key element*); R_2 (new) $\to R_2$ (old) $-$ (1/5) R_1 (new).

R_3 (new) $\to R_3$ (old) $-$ (9/5) R_1 (new).

The improved basic feasible solution is shown in Table 4.32.

		$c_j \to$	1	2	3	-1	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	x_3	x_4	Min Ratio x_B/x_1
2	x_2	15/7	$-1/7$	1	0	0	–
3	x_3	25/7	3/7	0	1	0	25/7 × 7/3 = 25/3
-1	x_4	15/7	6/7	0	0	1	15/7 × 7/6 = 15/6 →
$Z = 90/7$		z_j	1/7	2	3	-1	
		$c_j - z_j$	6/7	0	0	0	
			↑				

Table 4.32 Improved Solution

Once again, the solution shown in Table 4.32 is not optimal as $c_1 - z_1 > 0$ in x_1-column. Thus, applying the following row operations for replacing non-basic variable x_1 in the basis with basic variable x_4:

R_3 (new) $\to R_3$ (old) $\times$ (7/6) (*key element*) ; R_1 (new) $\to R_1$ (old) + (1/7) R_3 (new).

R_2 (new) $\to R_2$ (old) $-$ (3/7) R_3 (new).

The improved basic feasible solution is shown in Table 4.33.

		$c_j \to$	1	2	3	-1
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	x_3	x_4
2	x_2	15/6	0	1	0	1/6
3	x_3	15/6	0	0	1	$-3/6$
1	x_1	15/6	1	0	0	7/6
$Z = 15$		z_j	1	2	3	0
		$c_j - z_j$	0	0	0	-1

Table 4.33 Optimal Solution

In Table 4.28, since all $c_j - z_j \le 0$, therefore, an optimal solution is arrived at with value of variables as: $x_1 = 15/6$, $x_2 = 15/6$, $x_3 = 15/6$ and Max $Z = 15$.

Example 4.9 ABC Printing Company is facing a tight financial squeeze and is attempting to cut costs wherever possible. At present it has only one printing contract, and luckily, the book is selling well in both the hardcover and the paperback editions. It has just received a request to print more copies of this book in either the hardcover or the paperback form. The printing cost for the hardcover books is Rs 600 per 100 books while that for paperback is only Rs 500 per 100. Although the company is attempting to economize, it does not wish to lay off any employee. Therefore, it feels obliged to run its two printing presses – I and II, at least 80 and 60 hours per week, respectively. Press I can produce 100 hardcover books in 2 hours or 100 paperback books in 1 hour. Press II can produce 100 hardcover books in 1 hour or 100 paperbacks books in 2 hours. Determine how many books of each type should be printed in order to minimize costs.

Solution Let x_1 and x_2 be the number of batches containing 100 hard cover and paperback books, to be printed respectively. The LP problem can then be formulated as follows.

Minimize $Z = 600x_1 + 500x_2$

subject to the constraints

(i) $2x_1 + x_2 \ge 80$, (ii) $x_1 + 2x_2 \ge 60$ (Printing press hours)

and

$x_1, x_2 \ge 0$

Standard form Adding surplus variables s_1, s_2 and artificial variables A_1, A_2 in the the constraints, the standard form of the LP problem becomes

Minimize $Z = 600x_1 + 500x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

subject to the constraints

(i) $2x_1 + x_2 - s_1 + A_1 = 80$, (ii) $x_1 + 2x_2 - s_2 + A_2 = 60$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

Solution by simplex method The initial basic feasible solution: $A_1 = 80$, $A_2 = 60$ and Min $Z = 80M + 60M = 140M$ obtained by putting $x_1 = x_2 = s_1 = s_2 = 0$ is shown in Table 4.34.

ble 4.34
ial Solution

		$c_j \rightarrow$	600	500	0	0	M	M	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	Min Ratio x_B/x_2
M	A_1	80	2	1	-1	0	1	0	80/1
M	A_2	60	1	2	0	-1	0	1	60/2 →
$Z = 140M$		z_j	$3M$	$3M$	$-M$	$-M$	M	M	
		$c_j - z_j$	$600 - 3M$	$500 - 3M$	M	M	0	0	
				↑					

Since $c_2 - z_2$ value in x_2-column of Table 4.34 is the largest negative, therefore non-basic variable x_2 is chosen to replace basic variable A_2 in the basis. For this, apply following row operations.

R_2 (new) = R_2 (old) ÷ 2 (*key element*); R_1 (new) → R_1 (old) – R_2 (new)

to get an improved basic feasible solution as shown in Table 4.35.

able 4.35
nproved Solution

		$c_j \rightarrow$	600	500	0	0	M	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	Min Ratio x_B/x_1
M	A_1	50	3/2	0	-1	1/2	1	100/3 →
500	x_2	30	1/2	1	0	$-1/2$	0	60
$Z = 15{,}000 + 50M$		z_j	$3M/2 + 250$	500	$-M$	$M/2 - 250$	M	
		$c_j - z_j$	$350 - 3M/2$	0	M	$250 - M/2$	0	
			↑					

Again, since value $c_1 - z_1$ of in x_1-column of Table 4.35 is the largest negative, non-basic variable x_1 is chosen to replace basic variable A_1 in the basis. For this, apply following row operations:

R_1 (new) → R_1 (old) × (2/3) (*key element*) ; R_2 (new) → R_2 (old) – (1/2) R_1 (new)

to get an improved basic feasible solution as shown in Table 4.36.

Table 4.36
Optimal Solution

		$c_j \rightarrow$	600	500	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(= x_B)$	x_1	x_2	s_1	s_2
600	x_1	100/3	1	0	$-2/3$	1/3
500	x_2	40/3	0	1	1/3	$-2/3$
$Z = 80{,}000/3$		z_j	600	500	$-700/3$	$-400/3$
		$c_j - z_j$	0	0	700/3	400/3

In Table 4.36, all $c_j - z_j \geq 0$ and no artificial variable is present in the basis (solution mix). Hence, an optimum solution is arrived at with $x_1 = 100/3$ batches of hardcover books, $x_2 = 40/3$ batches of paperback books, at a total minimum cost, Z = Rs. 80,000/3.

Example 4.10 An advertising agency wishes to reach two types of audiences: Customers with annual income greater than Rs 15,000 (target audience A) and customers with annual income less than Rs 15,000 (target audience B). The total advertising budget is Rs 2,00,000. One programme of TV advertising costs Rs 50,000; one programme on radio advertising costs Rs 20,000. For contract reasons, at least three programmes ought to be on TV, and the number of radio programmes must be limited to five. Surveys indicate that a single TV programme reaches 4,50,000 customers in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in target audience B. Determine the media mix to maximize the total reach.

[*Delhi Univ., MBA, 2000*]

Solution Let x_1 and x_2 be the number of insertions in TV and radio, respectively. The LP problem can then be formulated as follows:

Maximize (total reach) $Z = (4{,}50{,}000 + 50{,}000)\, x_1 + (20{,}000 + 80{,}000)\, x_2$
$= 5{,}00{,}000\, x_1 + 1{,}00{,}000\, x_2 = 5x_1 + x_2$

subject to the constraints

(i) $50{,}000\, x_1 + 20{,}000\, x_2 \le 2{,}00{,}000$ or $5x_1 + 2x_2 \le 20$ (Advt. budget)

(ii) $x_1 \ge 3$ (Advt. on TV) (iii) $x_2 \le 5$ (Advt. on Radio)

and $x_1, x_2 \ge 0.$

Standard form Adding slack/surplus and/or artificial variables in the constraints of LP problem. Then standard form of given LP problem becomes

Maximize $Z = 5x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 - MA_1$

subject to the constraints

(i) $5x_1 + 2x_2 + s_1 = 20,$ (ii) $x_1 - s_2 + A_1 = 3,$ (iii) $x_2 + s_3 = 5$

and $x_1, x_2, s_1, s_2, s_3, A_1 \ge 0$

Solution by simplex method An initial basic feasible solution: $s_1 = 20, A_1 = 3, s_3 = 5$ and Max $Z = -3M$ obtained by putting $x_1 = x_2 = s_2 = 0$ is shown in simplex Table 4.37.

	$c_j \rightarrow$		5	1	0	0	0	$-M$	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(= x_B)$	x_1	x_2	s_1	s_2	s_3	A_1	Min Ratio x_B/x_1
0	s_1	20	5	2	1	0	0	0	$20/5 = 4$
$-M$	A_2	3	1	0	0	-1	0	1	$3/1 = 3 \rightarrow$
0	s_3	5	0	1	0	0	1	0	—
$Z = -3M$		z_j	$-M$	0	0	M	0	$-M$	
		$c_j - z_j$	$M + 5$ ↑	1	0	$-M$	0	0	

Table 4.37 Initial Solution

The $c_1 - z_1$ value in x_1-column of Table 4.37 is the largest positive value. The non-basic variable x_1 is chosen to replace basic variable A_1 in the basis. For this apply following row operations

R_2 (new) $\rightarrow R_2$ (old) $\div$ 1 (*key element*); R_1 (new) $\rightarrow R_1$ (old) $- 5R_2$ (new)

to get the improved basic feasible solution as shown in Table 4.38.

	$c_j \rightarrow$		5	1	0	0	0	
Basic Variables Coefficient c_B	Basic Variable B	Basic Variables Value $b(= x_B)$	x_1	x_2	s_1	s_2	s_3	Min Ratio x_B/s_2
0	s_1	5	0	2	1	5	0	$5/5 = 1 \rightarrow$
5	x_1	3	1	0	0	-1	0	—
0	s_3	5	0	1	0	0	1	—
$Z = 15$		z_j	5	0	0	-5	0	
		$c_j - z_j$	0	1	0	5 ↑	0	

Table 4.38 Improved Solution

Again in Table 4.38, $c_4 - z_4$ value in s_2-column is the largest positive. Thus non-basic variable s_2 is chosen to replace basic variable s_1 into the basis. For this apply the following row operations:

R_1 (new) $\rightarrow R_1$ (old) $\div$ 5 (*key element*); R_2 (new) $\rightarrow R_2$ (old) $+ R_1$ (new)

to get the improved basic feasible solution as shown in Table 4.39.

		$c_j \rightarrow$	5	1	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	s_1	s_2	s_3
0	s_2	1	0	2/5	1/5	1	0
5	x_1	4	1	2/5	1/5	0	0
0	s_3	5	0	1	0	0	1
$Z = 20$		z_j	5	2	1	0	0
		$c_j - z_j$	0	−1	−1	0	0

Table 4.39
Optimal Solution

Since all $c_j - z_j \geq 0$ in Table 4.39, the total reach of target audience cannot be increased further. Hence, the optimal solution is: $x_1 = 4$ insertions in TV and $x_2 = 0$ in radio with Max (total audience) $Z = 20{,}00{,}000$.

Example 4.11 An Air Force is experimenting with three types of bombs P, Q and R in which three kinds of explosives, viz., A, B and C will be used. Taking the various factors into account, it has been decided to use the maximum 600 kg of explosive A, at least 480 kg of explosive B and exactly 540 kg of explosive C. Bomb P requires 3, 2, 2 kg, bomb Q requires 1, 4, 3 kg and bomb R requires 4, 2, 3 kg of explosives A, B and C respectively. Bomb P is estimated to give the equivalent of a 2 ton explosion, bomb Q, a 3 ton explosion and bomb R, a 4 ton explosion respectively. Under what production schedule can the Air Force make the biggest bang?

Solution Let x_1, x_2 and x_3 be the number of bombs of type P, Q and R to be experimented, respectively. Then the LP problem can be formulated as:

Maximize $Z = 2x_1 + 3x_2 + 4x_3$

subject to the constraints

(i) Explosive A : $3x_1 + x_2 + 4x_3 \leq 600$, (ii) Explosive B : $2x_1 + 4x_2 + 2x_3 \geq 480$,

(iii) Explosive C : $2x_1 + 3x_2 + 3x_3 = 540$,

and $x_1, x_2, x_3 \geq 0$.

Standard form Adding slack, surplus and artificial variables in the constraints of LP problem, the standard form of LP problem becomes:

Maximize $Z = 2x_1 + 3x_2 + 4x_3, + 0s_1 + 0s_2 - MA_1 - MA_2$

subject to the constraints

(i) $3x_1 + x_2 + 4x_3 + s_1 = 600$, (ii) $2x_1 + 4x_2 + 2x_3 - s_2 + A_1 = 480$, (iii) $2x_1 + 3x_2 + 3x_3 + A_2 = 540$,

and $x_1, x_2, x_3 s_1, s_2, A_1, A_2 \geq 0$.

Solution by simplex method The initial basic feasible solution: $s_1 = 600$, $A_1 = 480$, $A_2 = 540$. Max $Z = -1{,}020$ M obtained by putting basic variables $x_1 = x_2 = s_2 = 0$ is shown in Table 4.40.

		$c_j \rightarrow$	2	3	4	0	0	$-M$	$-M$	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	x_3	s_1	s_2	A_1	A_2	Min Ratio x_B/x_2
0	s_1	600	3	1	4	1	0	0	0	600/1
$-M$	A_1	480	2	4	2	0	−1	1	0	480/4 →
$-M$	A_2	540	2	3	3	0	0	0	1	540/3
$Z = 1{,}020\ M$		z_j	$-4M$	$-7M$	$-5M$	0	M	$-M$	$-M$	
		$c_j - z_j$	$2 + 4M$	$3 + 7M$ ↑	$4 + 5M$	0	$-M$	0	0	

Table 4.40
Initial Solution

Since in Table 4.40, $c_2 - z_2$ value in x_2–column is largest positive, non-basic variable x_2 is chosen to replace basic variable A_1 in the basis. For this, apply the following row operations:

$R_2(\text{new}) \rightarrow R_2(\text{old}) \times (1/4)$ (*key element*); $R_1(\text{new}) \rightarrow R_1(\text{old}) - R_2(\text{new})$;

$R_3(\text{new}) \rightarrow R_3(\text{old}) - 3R_2(\text{new})$

to get the improved basic feasible solution as shown in Table 4.41.

		$c_j \rightarrow$	2	3	4	0	0	$-M$	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\,(=x_B)$	x_1	x_2	x_3	s_1	s_2	A_1	Min Ratio x_B/x_3
0	s_1	480	5/2	0	7/2	1	1/4	0	960/7
3	x_2	120	1/2	1	1/2	0	−1/4	0	240
$-M$	A_1	180	1/3	0	3/2	0	3/4	1	120 →
$Z = 360 - 180M$		z_j	$3/2 - M/2$	3	$3/2 - 3M/2$	0	$-3/4 - 3M/4$	$-M$	
		$c_j - z_j$	$1/2 + M/2$	0	$5/2 + 3M/2$ ↑	0	$3/4 + 3M/4$	0	

Table 4.41 Improved Solution

In Table 4.41, $c_3 - z_3$ value in x_3-column is largest positive, non-basic variable x_3 is chosen to replace basic variable A_2 into the basis. For this, apply following row operations:

R_3(new) $\rightarrow R_3$(old) × (2/3) (*key element*); $\quad R_1$(new) $\rightarrow R_1$(old) − (7/2) R_3(new);

R_2(new) $\rightarrow R_2$(old) − (1/2) R_3(new).

to get the improved basic feasible solution as shown in Table 4.42.

		$c_j \rightarrow$	2	3	4	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\,(=x_B)$	x_1	x_2	x_3	s_1	s_2
0	s_1	60	4/3	0	0	1	−3/2
3	x_2	60	1/3	1	0	0	−1/2
4	x_3	120	1/3	0	1	0	1/2
$Z = 660$		z_j	7/3	3	4	0	1/2
		$c_j - z_j$	−1/3	0	0	0	−1/2

Table 4.42 Optimal Solution

In Table 4.42, all $c_j - z_j \leq 0$ and artificial variables A_1 and A_2 have been removed from the basis (solution mix). Thus, an optimal solution is arrived at with $x_1 = 0$ bombs of type P, $x_2 = 60$ bombs of type Q, $x_3 = 120$ bombs of type R, at largest benefit of Z = 660.

SELF PRACTICE PROBLEMS A

1. A television company has three major departments for manufacturing two of its models – A and B. The monthly capacities of the departments are given as follows:

	Per Unit Time Requirement (hours)		Hours Available this Month
	Model A	Model B	
Department I	4.0	2.0	1,600
Department II	2.5	1.0	1,200
Department III	4.5	1.5	1,600

The marginal profit per unit from model A is Rs 400 and from model B is Rs 100. Assuming that the company can sell any quantity of either product due to favourable market conditions, determine the optimum output for both the models, the highest possible profit for this month and the slack time in the three departments.

2. A manufacturer of leather belts makes three types of belts A, B and C which are processed on three machines M_1, M_2 and M_3. Belt A requires 2 hours on machine M_1 and 3 hours on machine M_2 and 2 hours on machine M_3. Belt B requires 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3 and Belt C requires 5 hours on machine M_2 and 4 hours on machine M_3. There are 8 hours of time per day available on machine M_1, 10 hours of time per day available on machine M_2 and 15 hours of time per day available on machine M_3. The profit gained from belt A is Rs 3.00 per unit, from Belt B is Rs 5.00 per unit, from belt C is Rs 4.00 per unit. What should be the daily production of each type of belt so that the products yield the maximum profit?

3. A company produces three products A, B and C. These products require three ores O_1, O_2 and O_3. The maximum quantities of the ores O_1, O_2 and O_3 available are 22 tonnes, 14 tonnes and 14 tonnes, respectively. For one tonne of each of these products, the ore requirements are:

	A	B	C
O_1	3	–	3
O_2	1	2	3
O_3	3	2	3
Profit per tonne (Rs in thousand)	1	4	5

The company makes a profit of Rs 1,000, 4,000 and 5,000 on each tonne of the products A, B and C, respectively. How many tonnes of each product should the company produce in order to maximize its profits.

4. A manufacturing firm has discontinued the production of a certain unprofitable product line. This has created considerable excess production capacity. Management is considering to devote this excess capacity to one or more of three products; call them product 1, 2 and 3. The available capacity on the machines that might limit output is summarized in the following table:

Machine Type	Available Time (in Machine-hours per Week)
Milling Machine	250
Lathe	150
Grinder	50

The number of machine-hours required for each unit of the respective product is as follows:

Machine Type	Productivity (in Machine-hours per Unit)		
	Product 1	Product 2	Product 3
Milling Machine	8	2	3
Lathe	4	3	0
Grinder	2	–	1

The profit per unit would be Rs 20, Rs 6 and Rs 8, respectively for product 1, 2 and 3. Find how much of each product the firm should produce in order to maximize its profit.
[Delhi Univ., MBA 2006]

5. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabean. Each acre of corn costs Rs 100 for preparation, requires 7 men-days of work and yields a profit of Rs 30. An acre of wheat costs Rs 120 to prepare, requires 10 men-days of work and yields a profit of Rs 40. An acre of soyabean costs Rs 70 to prepare, requires 8 men-days of work and yields a profit of Rs 20. If the farmer has Rs 1,00,000 for preparation and can count on 8,000 men-days of work, determine how many acres should be allocated to each crop in order to maximize profits? [Delhi Univ., MBA, 2004]

6. The annual handmade furniture show and sale is supposed to take place next month and the school of vocational studies is also planning to make furniture for this sale. There are three wood-working classes – I year, II year and III year, at the school and they have decided to make styles of chairs – *A*, *B* and *C*. Each chair must receive work in each class. The time in hours required for each chair in each class is:

Chair	I Year	II Year	III Year
A	2	4	3
B	3	3	2
C	2	1	4

During the next month there will be 120 hours available to the I year class, 160 hours to the II year class, and 100 hours to the III year class for producing the chairs. The teacher of the wood-working classes feels that a maximum of 40 chairs can be sold at the show. The teacher has determined that the profit from each type of chair will be: A, Rs 40; B, Rs 35 and C, Rs. 30. How many chairs of each type should be made in order to maximize profits at the show and sale?

7. Mr Jain, the marketing manager of ABC Typewriter Company is trying to decide how he should allocate his salesmen to the company's three primary markets. Market I is in the urban area and the salesman can sell, on the average, 40 typewriters a week. Salesmen in the other two markets, II and III can sell, on the average, 36 and 25 typewriters per week, respectively. For the coming week, three of the salesmen will be on vacation, leaving only 12 men available for duty. Also because of lack of company care, a maximum of 5 salesmen can be allocated to market area I. The selling expenses per week for salesmen in each area are Rs 800 per week for area I, Rs 700 per week for area II, and Rs 500 per week for area III. The budget for the next week is Rs 7,500. The profit margin per typewriter is Rs 150. Determine how many salesmen should be assigned to each area in order to maximize profits?

8. Three products – A, B and C – are produced in three machine centres X, Y and Z. All three products require a part of their manufacturing operation at each of the machine centres. The time required for each operation on various products is indicated in the following table. Only 100, 77 and 80 hours are available at machine centres X, Y and Z, respectively. The profit per unit from A, B and C is Rs 12, Rs 3 and Re 1, respectively.

Products	Machine Centres			Profit per Unit (Rs)
	X	Y	Z	
A	10	7	2	12
B	2	3	4	3
C	1	2	1	1
Available hours	100	77	80	

(a) Determine suitable product mix so as to maximize the profit. Comment on the queries (b) and (c) from the solution table obtained.

(b) Satisfy that full available hours of X and Y have been utilized and there is surplus hours of Z. Find out the surplus hours of Z.

(c) Your aim is to utilize surplus capacity of Z. Can you say from the table that the introduction of more units of Y is required?

9. A certain manufacturer of screw fastenings found that there is a market for packages of mixed screw sizes. His market research data indicated that two mixtures of three screw types (1, 2 and 3), properly priced, could be most acceptable to the public. The relevant data is:

Mixture	Specifications	Selling Price (Rs/kg)
A	≥ 50% type 1 ≤ 30% type 2 and quantity of type 3	5
B	≥ 35% type 1 ≤ 45% type 2 and quantity of type 3	4

For these screws, the plant capacity and manufacturing cost are as follows:

Screw Type	Plant Capacity (kg/day × 100)	Manufacturing Cost (Rs/kg)
1	10	4.50
2	10	3.50
3	6	2.70

What production shall this manufacturer schedule for greatest profit, assuming that he can sell all that he manufactures?

10. A blender of whisky imports three grades A, B and C. He mixes them according to the recipes that specify the maximum or minimum percentages of grades A and C in each blend. These are shown in the table below.

Blend	Specification	Price per Unit (Rs)
Blue Dot	Not less than 60% of A Not more than 20% of C	6.80
Highland Fling	Not more than 60% of C Not less than 15% of A	5.70
Old Frenzy	Not more than 50% of C	4.50

Following are the supplies of the three whiskies along with their cost.

Whisky	Maximum Quantity Available per Day	Cost per Unit (Rs)
A	2,000	7.00
B	2,500	5.00
C	1,200	4.00

Show how to obtain the first matrix in a simplex computation of a production policy that will maximize profits.

11. An animal feed company must produce on a daily basis 200 kg of a mixture that consists ingredients x_1 and x_2 ingredient. x_1 costs Rs 3 per kg and x_2 costs Rs 8 per kg. Not more than 80 kg of x_1 can be used and at least 60 kg of x_2 must be used. Find out how much of each ingredient should be used if the company wants to minimize costs.

12. A diet is to contain at least 20 ounces of protein and 15 ounces of carbohydrate. There are three foods A, B and C available in the market, costing Rs 2, Re 1 and Rs 3 per unit, respectively. Each unit of A contains 2 ounces of protein and 4 ounces of carbohydrate; each unit of B contains 3 ounces of protein and 2 ounces of carbohydrate; and each unit of C contains 4 ounces of protein and 2 ounces of carbohydrate. How many units of each food should the diet contain so that the cost per unit diet is minimum?

13. A person requires 10, 12 and 12 units of chemicals A, B and C, respectively for his garden. A typical liquid product contains 5, 2 and 1 unit of A, B and C, respectively per jar. On the other hand a typical dry product contains 1, 2 and 4 units of A, B and C per unit. If the liquid product sells for Rs 3 per jar and the dry product for Rs 2 per carton, how many of each should be purchased in order to minimize the cost and meet the requirement?

14. A scrap metal dealer has received an order from a customer for a minimum of 2,000 kg of scrap metal. The customer requires that at least 1,000 kg of the shipment of metal be of high quality copper that can be melted down and used to produce copper tubing. Furthermore, the customer will not accept delivery of the order if it contains more than 175 kg metal that he deems unfit for commercial use, i.e. metal that contains an excessive amount of impurities and cannot be melted down and refined profitably.

The dealer can purchase scrap metal from two different suppliers in unlimited quantities with following percentages (by weight) of high quality copper and unfit scrap.

	Supplier A	Supplier B
Copper	25%	75%
Unfit scrap	5%	10%

The cost per kg of metal purchased from supplier A and B are Re 1 and Rs 4, respectively. Determine the optimal quantities of metal to be purchased for the dealer from each of the two suppliers.

15. A marketing manager wishes to allocate his annual advertising budget of Rs 2,00,000 to two media vehicles – A and B. The unit cost of a message in media A is Rs 1,000 and that of B is Rs 1,500. Media A is a monthly magazine and not more than one insertion is desired in one issue, whereas at least five messages should appear in media B. The expected audience for unit messages in media A is 40,000 and that of media B is 55,000. Develop an LP model and solve it for maximizing the total effective audience.

16. A transistor radio company manufactures four models A, B, C and D. Models A, B and C, have profit contributions of Rs 8, Rs 15 and Rs 25 respectively and has model D a loss of Re 1. Each type of radio requires a certain amount of time for the manufacturing of components, for assembling and for packing. A dozen units of model A require one hour for manufacturing, two hours for assembling and one hour for packing. The corresponding figures for a dozen units of model B are 2, 1 and 2, and for a dozen units of C are 3, 5 and 1. A dozen units of model D however, only require 1 hour of packing. During the forthcoming week, the company will be able to make available 15 hours of manufacturing, 20 hours of assembling and 10 hours of packing time. Determine the optimal production schedule for the company.

17. A transport company is considering the purchase of new vehicles for providing transportation between the Delhi Airport and hotels in the city. There are three vehicles under consideration: Station wagons, minibuses and large buses. The purchase price would be Rs 1,45,000 for each station wagon, Rs 2,50,000 for each minibus and Rs 4,00,000 for each large bus. The board of directors has authorized a maximum amount of Rs 50,00,000 for these purchases. Because of the heavy air travel, the new vehicles would be utilized at maximum capacity, regardless of the type of vehicles purchased. The expected net annual profit would be Rs 15,000 for the station wagon, Rs 35,000 for the minibus and Rs 45,000 for the large bus. The company has hired 30 new drivers for the new vehicles. They are qualified drivers for all three types of vehicles. The maintenance department has the capacity to handle an additional 80 station wagons. A minibus is equivalent to 1.67 station wagons and each large bus is equivalent to 2 station wagons in terms of their use of the maintenance department. Determine the number of each type of vehicle that should be purchased in order to maximize profit.

18. Omega Data Processing Company performs three types of activities: Payroll, accounts receivables, and inventories. The profit and time requirement for keypunch, computation and office printing, for a standard job, are shown in the following table:

Job	Profit/ Standard Job (Rs)	Time Requirement (Min)		
		Keypunch	Computation	Printing
Payroll	275	1,200	20	100
A/c Receivable	125	1,400	15	60
Inventory	225	800	35	80

Omega guarantees overnight completion of the job. Any job scheduled during the day can be completed during the day or night. Any job scheduled during the night, however, must be completed during the night. The capacities for both day and night are shown in the following table:

Capacity (Min)	Keypunch	Computation	Print
Day	4,200	150	400
Night	9,200	250	650

Determine the mixture of standard jobs that should be accepted during the day and night.

19. A furniture company can produce four types of chairs. Each chair is first made in the carpentry shop and then furnished, waxed and polished in the finishing shop. The man-hours required in each are:

	Chair Type			
	1	2	3	4
Carpentry shop	4	9	7	10
Finishing shop	1	1	3	40
Contribution per chair (Rs)	12	20	18	40

The total number of man-hours available per month in carpentry and finishing shops are 6,000 and 4,000, respectively.

Assuming an abundant supply of raw material and an abundant demand for finished products, determine the number of each type of chairs that should be produced for profit maximization.

20. A metal products company produces waste cans, filing cabinets, file boxes for correspondence, and lunch boxes. Its inputs are sheet metal of two different thickness, called A and B, and manual labour. The input-output relationship for the company are shown in the table given below:

	Waste Cans	Filing Cabinets	Correspondence Boxes	Lunch Boxes
Sheet metal A	6	0	2	3
Sheet metal B	0	10	0	0
Manual labour	4	8	2	3

The sales revenue per unit of waste cans, filing cabinets, correspondence boxes and lunch boxes are Rs 20, Rs 400, Rs 90 and Rs 20, respectively. There are 225 units of sheet metal A available in the company's inventory, 300 of sheet metal B, and a total of 190 units of manual labour. What is the company's optimal sales revenue?

[Delhi Univ., MBA, 2003, 2005]

21. A company mined diamonds in three locations in the country. The three mines differed in terms of their capacities, number, weight of stones mined, and costs. These are shown in the table below. Due to marketing considerations, a monthly production of exactly 1,48,000 stones was required. A similar requirement called for at least 1,30,000 carats (The average stone size was at least 130/148 = 0.88 carats). The capacity of each mine is measured in cubic meter. The mining costs are not included from the treatment costs and assume to be same at each mine. The problem for the company was to meet the marketing requirements at the least cost.

Mine	Capacity (M^3 of earth processed)	Treatment Costs (Rs. per M^3)	Crude (Carats per M^3)	Stone Count (Number of stone per M^3)
Plant 1	83,000	0.60	0.360	0.58
Plant 2	3,10,000	0.36	0.220	0.26
Plant 3	1,90,000	0.50	0.263	0.21

Formulate a linear programming model to determine how much should be mined at each location.

HINTS AND ANSWERS

1. Let x_1 and x_2 = units of models A and B to be manufactured, respectively.

$$\text{Max } Z = 400x_1 + 400x_2$$

subject to
$$4x_1 + 2x_2 \le 1{,}600$$
$$5x_1/2 + x_2 \le 1{,}200$$
$$9x_1/2 + 3x_2/2 \le 1{,}600$$
and
$$x_1, x_2 \ge 0$$

Ans. $x_1 = 355.5$, $x_2 = 0$ and Max $Z = 1{,}42{,}222.2$.

2. Let x_1, x_2 and x_3 = units of types A, B and C belt to be manufactured, respectively.

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

subject to
$$2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
and
$$x_1, x_2, x_3 \ge 0$$

Ans. $x_1 = 89/41$, $x_2 = 50/41$ $x_3 = 64/41$ and Max $Z = 775/41$.

3. Let x_1, x_2 and x_3 = quantity of products A, B and C to be produced, respectively.

$$\text{Max } Z = x_1 + 4x_2 + 5x_3$$

subject to
$$3x_1 + 3x_2 \le 22$$
$$x_1 + 2x_2 + 3x_3 \le 14$$
$$3x_1 + 2x_2 \le 14$$
and
$$x_1, x_2, x_3 \ge 0$$

Ans. $x_1 = 0$, $x_2 = 7$, $x_3 = 0$ and Max Z = Rs 28,000.

4. Let x_1, x_2 and x_3 = number of units of products 1, 2 and 3 to be produced per week, respectively.

$$\text{Max } Z = 20x_1 + 6x_2 + 8x_3$$

subject to
$$8x_1 + 2x_2 + 3x_3 \le 250$$
$$4x_1 + 3x_2 \le 150$$
$$2x_1 + x_3 \le 50$$
and
$$x_1, x_2, x_3 \ge 0$$

Ans. $x_1 = 0$, $x_2 = 50$, $x_3 = 50$ and Max $Z = 700$.

5. Let x_1, x_2 and x_3 = acreage of corn, wheat and soyabean, respectively.

$$\text{Max } Z = 30x_1 + 40x_2 + 20x_3$$

subject to
$$10x_1 + 12x_2 + 7x_3 \le 10{,}000$$
$$7x_1 + 10x_2 + 8x_3 \le 8{,}000$$
$$x_1 + x_2 + x_3 \le 1{,}000$$
and
$$x_1, x_2, x_3 \ge 0$$

Ans. $x_1 = 250$, $x_2 = 625$, $x_3 = 0$ and Max Z = Rs 32,500.

6. Let x_1, x_2 and x_3 = number of units of chair of styles A, B and C, respectively.

$$\text{Max } Z = 40x_1 + 35x_2 + 30x_3$$

subject to
$$2x_1 + 3x_2 + 2x_3 \le 120$$
$$4x_1 + 3x_2 + x_3 \le 160$$
$$3x_1 + 2x_2 + 4x_3 \le 100$$
$$x_1 + x_2 + x_3 \le 40$$
and
$$x_1, x_2, x_3 \ge 0$$

Ans. $x_1 = 20$, $x_2 = 20$, $x_3 = 0$ and Max Z = Rs 1,500

7. Let x_1, x_2 and x_3 = salesman assigned to area, 1, 2 and 3, respectively.

$$\text{Max. } Z = 40 \times 150x_1 + 36 \times 150x_2 + 25 \times 150x_3 - (800x_1 + 700x_2 + 500x_3)$$

subject to (i) $x_1 + x_2 + x_3 \le 12$; (ii) $x_1 \le 5$; (iii) $800x_1 + 700x_2 + 500x_3 \le 7{,}500$

and $x_1, x_2, x_3 \ge 0$

11. Let x and y = number of kg of ingredients x_1, x_2, respectively.

Min (total cost) $Z = 3x + 8y$

subject to (i) $x + y = 200$; (ii) $x \le 80$; (iii) $y \ge 60$

and $x, y \ge 0$

Ans. $x = 80$, $y = 120$ and Min Z = Rs 1,200.

12. Let x_1, x_2 and x_3 = number of units of food A, B and C, respectively which a diet must contain.

Min (total cost) $Z = 2x_1 + x_2 + x_3$

subject to $2x_1 + 2x_2 + 4x_3 \ge 20$

$4x_1 + 2x_2 + 2x_3 \ge 15$

and $x_1, x_2, x_3 \ge 0$

13. Let x_1 and x_2 = number of units of liquid and dry product produced, respectively.

Min (total cost) $Z = 3x_1 + 2x_2$

subject to (i) $5x_1 + x_2 \ge 10$; (ii) $2x_1 + 2x_2 \ge 12$; (iii) $x_1 + 4x_2 \ge 12$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 1$, $x_2 = 5$ and Min $Z = 13$.

14. Let x_1 and x_2 = number of scrap (in kg) purchased from suppliers A and B, respectively.

Min (total cost) $Z = x_1 + 4x_2$

subject to $2.25x_1 + 0.75x_2 \ge 1{,}000$

$0.05x_1 + 0.10x_2 \ge 175$

$x_1 + x_2 \ge 2{,}000$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 2{,}500$, $x_2 = 500$ and Min $Z = 4{,}500$.

15. Let x_1 and x_2 = number of insertions messages for media A and B, respectively.

Min (total effective audience) $Z = 40{,}000x_1 + 55{,}000x_2$

subject to (i) $1{,}000x_1 + 1500x_2 \le 2{,}00{,}000$; (ii) $x_1 \le 12$; (iii) $x_2 \ge 5$

and $x_1, x_2 \ge 0$

Ans. $x_1 = 12$, $x_2 = 16/3$ and Max $Z = 77{,}3333.33$.

16. Let x_1, x_2, x_3 and x_4 = unit of models A, B, C and D to be produce, respectively.

Max (total income) $Z = 8x_1 + 15x_2 + 25x_3 - x_4$

subject to (i) $x_1 + 2x_2 + 3x_3 = 15$; (ii) $2x_1 + x_2 + 5x_3 = 20$; (iii) $x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_1, x_2, x_3, x_4 \ge 0$

Ans. $x_1 = 5/2$, $x_2 = 5/2$, $x_3 = 5/2$, $x_4 = 0$ and Max $Z = 120$.

17. Let x_1, x_2 and x_3 = number of station wagons, minibuses and large buses to be purchased, respectively.

Max $Z = 15{,}000x_1 + 35{,}000x_2 + 45{,}000x_3$

subject to $x_1 + x_2 + x_3 \le 30$

$1{,}45{,}000x_1 + 2{,}50{,}000x_2 + 4{,}00{,}000x_3 \le 50{,}00{,}000$

$x_1 + 1.67x_2 + 0.5x_3 \le 80$

and $x_1, x_2, x_3 \ge 0$

18. Let x_{ij} represents ith job and jth activity

Max $Z = 275\,(x_{11} + x_{12}) + 125(x_{21} + x_{22}) + 225\,(x_{31} + x_{32})$

subject to

$1{,}200\,(x_{11} + x_{12}) + 1{,}400\,(x_{21} + x_{22}) + 800\,(x_{31} + x_{32}) \le 13{,}400$

$20\,(x_{11} + x_{12}) + 15\,(x_{21} + x_{22}) + 35\,(x_{31} + x_{32}) \le 400$

$100\,(x_{11} + x_{12}) + 60\,(x_{21} + x_{22}) + 80\,(x_{31} + x_{32}) \le 1{,}050$

$1{,}200x_{12} + 1{,}400x_{22} + 800x_{32} \le 9{,}200$

$20x_{12} + 15x_{22} + 35x_{32} \le 250$

$100x_{12} + 60x_{22} + 80x_{32} \le 650$

and $x_{ij} \ge 0$ for all i, j

19. Let x_1, x_2, x_3 and x_4 = chair types 1, 2, 3 and 4 to be produced, respectively.

Max $Z = 12x_1 + 20x_2 + 18x_3 + 40x_4$

subject to $4x_1 + 9x_2 + 7x_3 + 10x_4 \le 6{,}000$

$x_1 + x_2 + 3x_3 + 40x_4 \le 4{,}000$

and $x_1, x_2, x_3, x_4 \ge 0$

Ans. $x_1 = 4{,}000/3$, $x_2 = x_3 = 0$, $x_4 = 200/3$ and Max Z = Rs 56,000/3.

21. Let x_1, x_2 and x_3 = cubic metric of earth processed at plants 1, 2 and 3, respectively.

Min $Z = 0.60x_1 + 0.36x_2 + 0.50x_3$

subject to

$0.58x_1 + 0.26x_2 + 0.21x_3 = 1{,}48{,}000$ (Stone count requirement)

$0.36x_1 + 0.22x_2 + 0.263x_3 \le 1{,}30{,}000$ (Carat requirement)

$x_1 \le 83{,}000$; $x_2 \le 3{,}10{,}000$;

$x_3 \le 1{,}90{,}000$ (Capacity requirement)

Ans. $x_1 = 61{,}700$; $x_2 = 3{,}10{,}000$; $x_3 = 1{,}50{,}500$ and Min Z = Rs 2,23,880.

4.5 SOME COMPLICATIONS AND THEIR RESOLUTION

In this section, some of the complications that may arise in applying the simplex method for solving both maximization and minimization LP problems and their resolution are discussed.

4.5.1 Unrestricted Variables

In actual practice, decision variables, x_j ($j = 1, 2, \ldots, n$) should have non-negative values. However, in many situations, one or more of these variables may have either positive, negative or zero value. Variables that can

subject to the constraints

$$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \le c_1$$
$$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \le c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \le c_n$$

and $$y_1, y_2, \ldots, y_m \ge 0$$

In general, the primal-dual relationship between a pair of LP problems can be expressed as follows:

Dual variables represent the potential value of resources

Primal	Dual
Max $Z_x = \sum_{j=1}^{n} c_j x_j$	Min $Z_y = \sum_{i=1}^{m} b_i y_i$
subject to the constraints	subject to the constraints
$\sum_{j=1}^{n} a_{ij} x_j \le b_i;\ i = 1, 2, \ldots, m$	$\sum_{i=1}^{m} a_{ji} y_i \ge c_j;\ j = 1, 2, \ldots, n$
$a_{ij} = a_{ji}$	
and $x_j \ge 0;\quad j = 1, 2, \ldots, n$	and $y_i \ge 0;\quad i = 1, 2, \ldots, m$

A summary of the general relationships between primal and dual LP problems is given in Table 5.1.

If Primal	Then Dual
(i) Objective is to maximize	(i) Objective is to minimize
(ii) jth primal variable, x_j	(ii) jth dual constraint
(iii) ith primal constraint	(iii) ith dual variable, y_i
(iv) Primal variable x_j unrestricted in sign	(iv) Dual constraint j is = type
(v) Primal constraint i is = type	(v) Dual variable y_i is unrestricted in sign
(vi) Primal constraints $\le$ type	(vi) Dual constraints $\ge$ type

Table 5.1 Primal-Dual Relationship

5.2.2 Economic Interpretation of Dual Variables

In the maximization primal LP model, we may define each parameter as follows:

Z = return
x_j = number units of variable j
b_i = maximum units of resource, i available
c_j = profit (or return) per unit of variable (activity) x_j
a_{ij} = units of resource, i consumed (required) per unit of variable j

The new variables introduced in the dual problem are Z_y and y_i (dual variables). When both the primal and the dual solutions are optimal, the value of objective function satisfies the strict equality, i.e. $Z_x = Z_y$. The interpretation associated with the dual variables y_i ($i = 1, 2, \ldots, m$) is discussed below. Rewrite the primal LP problem as follows:

Primal LP Problem

Maximize (return) $Z_x = \sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n}$ (Profit per unit of variable x_j) (Units of variable x_j)

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i$$

or

$\sum_{j=1}^{n}$ (Units of resource i, consumed per unit of variable, x_j) (Units of variable x_j) $\le$ Units of resource, i available

and

$x_j \ge 0$, for all j

Dual LP Problem

Minimize (cost) $Z_y = \sum_{i=1}^{m} b_i y_i = \sum_{i=1}^{m}$ (Units of resource, i) (Cost per unit of resource, i)

subject to the constraints

$$\sum_{i=1}^{m} a_{ji} y_i \geq c_j$$

or $\sum_{i=1}^{m}$ (Units of a resource, i consumed per unit of variable y_i) (Cost per unit of resource, i) $\geq$ Profit per unit for each variable, x_j

and $y_i \geq 0$, for all i

From these expressions of parameters of both primal and dual problems, it is clear that for the unit of measurement to be consistent, *the dual variable* (y_i) *must be expressed in terms of return (or worth) per unit of resource i*. This is called *dual price* (*simplex multiplier* or *shadow price*) of resource *i*. In other words, optimal value of a dual variable associated with a particular primal constraint indicates the *marginal change (increase, if positive or decrease, if negative) in the optimal value of the primal objective function.* For example, if $y_2 = 5$, then this indicates that for every additional unit (up to a certain limit) of resource 2 (resource associated with constraint 2 in the primal), the objective function value will increase by 5 units. The value $y_2 = 5$ is also called the marginal (or shadow or implicit) price of resource 2.

Similarly, for *feasible* solutions of both primal and dual LP problems, objective function value satisfy the inequality $Z_x \leq Z_y$. This inequality is interpreted as: *Profit* $\leq$ *Worth of resources*. Thus, so long as the total profit (return) from all activities is less than the worth of the resources, the feasible solution of both primal and dual is not optimal. The optimality (maximum profit or return) is reached only when the resources have been completely utilized. This is only possible if the worth of the resources (i.e. input) is equal to profit (i.e. output).

5.2.3 Economic Interpretation of Dual Constraints

As stated earlier, the dual constraints are expressed as:

$$\sum_{i=1}^{m} a_{ji} y_i - c_j \geq 0$$

Since coefficients a_{ji} represents the amount of resource b_i consumed by per unit of activity x_j, and the dual variable y_i represents shadow price per unit of resource b_i, the quantity $\Sigma\, a_{ji} y_i$ $(= z_j)$ should be total shadow price of all resources required to produce one unit of activity x_j.

For maximization LP problem, if $c_j - z_j > 0$ value corresponds to any non-basic variable, then the value of objective function can be increased. This implies that the value of variable, x_j can be increased from zero to a positive level provided its unit profit (c_j) is more than its shadow price, i.e.

$$c_j \geq \sum_{i=1}^{m} a_{ji} y_i$$

Profit per unit of activity $x_j \geq$ Shadow price of resources used per unit of activity, x_j.

5.2.4 Rules for Constructing the Dual from Primal

The rules for constructing the dual from the primal and vice-versa using the symmetrical form of LP problem are:

1. A dual variable is defined corresponds to each constraint in the primal LP problem and vice versa. Thus for a primal LP problem with m constraints and n variables, there exists a dual LP problem with m variables and n constraints and vice-versa.
2. The right-hand side constants $b_1, b_2, \ldots, b_m$ of the primal LP problem becomes the coefficients of the dual variables $y_1, y_2, \ldots, y_m$ in the dual objective function Z_y. Also the coefficients $c_1, c_2, \ldots, c_n$ of the primal variables $x_1, x_2, \ldots, x_n$ in the objective function become the right-hand side constants in the dual LP problem.

3. For a maximization primal LP problem with all ≤ (less than or equal to) type constraints, there exists a minimization dual LP problem with all ≥ (greater than or equal to) type constraints and vice versa. Thus, the inequality sign is reversed in all the constraints except the non-negativity conditions.
4. The matrix of the coefficients of variables in the constraints of dual is the transpose of the matrix of coefficients of variables in the constraints of primal and vice versa.
5. If the objective function of a primal LP problem is to be maximized, the objective function of the dual is to be minimized and vice versa.
6. If the ith primal constraint is = (equality) type, then the ith dual variables is unrestricted in sign and vice versa.

The primal-dual relationships may also be memorized by using the following table:

Dual Variables	*Primal Variables*						*Maximize Z_x*
	x_1	x_2	$\ldots$	x_j	$\ldots$	x_n	
y_1	a_{11}	a_{12}	$\ldots$	a_{1j}	$\ldots$	a_{1n}	$\leq b_1$
y_2	a_{21}	a_{22}	$\ldots$	a_{2j}	$\ldots$	a_{2n}	$\leq b_2$
$\vdots$						$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\ldots$	a_{mj}	$\ldots$	a_{mn}	$\leq b_m$
Minimize Z_y	$\geq c_1$	$\geq c_2$	$\ldots$	$\geq c_j$	$\ldots$	$\geq c_n$	↑ Dual objective function coefficients

↑ jth dual constraint

The primal constraints should be read across the rows, and the dual constraints should be read across the columns.

Example 5.1 Write the dual to the following LP problem.

Maximize $Z = x_1 - x_2 + 3x_3$

subject to the constraints

(i) $x_1 + x_2 + x_3 \leq 10$, (ii) $2x_1 - x_2 - x_3 \leq 2$, (iii) $2x_1 - 2x_2 - 3x_3 \leq 6$

and $x_1, x_2, x_3 \geq 0$

Solution In the given LP problem there are $m = 3$ constraints and $n = 3$ variables. Thus, there must be $m = 3$ dual variables and $n = 3$ constraints. Further, the coefficients of the primal variables, $c_1 = 1$, $c_2 = -1$, $c_3 = 3$ become right-hand side constants of the dual. The right-hand side constants $b_1 = 10$, $b_2 = 2$, $b_3 = 6$ become the coefficients in the dual objective function. Finally, the dual must have a minimizing objective function with all ≥ type constraints. If y_1, y_2 and y_3 are dual variables corresponding to three primal constraints in the given order, the resultant dual is

Minimize $Z_y = 10y_1 + 2y_2 + 6y_3$

subject to the constraints

(i) $y_1 + 2y_2 + 2y_3 \geq 1$, (iii) $y_1 - y_2 - 2y_3 \geq -1$, (iii) $y_1 - y_2 - 3y_3 \geq 3$

and $y_1, y_2, y_3 \geq 0$

Example 5.2 Write the dual of the following LP problem.

Minimize $Z_x = 3x_1 - 2x_2 + 4x_3$

subject to the constraints

(i) $3x_1 + 5x_2 + 4x_3 \geq 7$, (ii) $6x_1 + x_2 + 3x_3 \geq 4$, (iii) $7x_1 - 2x_2 - x_3 \leq 10$

(iv) $x_1 - 2x_2 + 5x_3 \geq 3$, (v) $4x_1 + 7x_2 - 2x_3 \geq 2$

and $x_1, x_2, x_3 \geq 0$

Solution Since the objective function of the given LP problem is of minimization, the direction of each inequality has to be changed to ≥ type by multiplying both sides by – 1. The standard primal LP problem so obtained is:

Minimize $Z_x = 3x_1 - 2x_2 + 4x_3$

subject to the constraints

(i) $3x_1 + 5x_2 + 4x_3 \geq 7$, (ii) $6x_1 + x_2 + 3x_3 \geq 4$, (iii) $-7x_1 + 2x_2 + x_3 \geq -10$

(iv) $x_1 - 2x_2 + 5x_3 \geq 3$, (v) $4x_1 + 7x_2 - 2x_3 \geq 2$

and $x_1, x_2, x_3 \geq 0$

If y_1, y_2, y_3, y_4 and y_5 are dual variables corresponding to the five primal constraints in the given order, the dual of this primal LP problem is stated as:

Maximize $Z_y = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

subject to the constraints

(i) $3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$, (ii) $5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$

(iii) $4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$

and $y_1, y_2, y_3, y_4, y_5 \geq 0$

Example 5.3 Obtain the dual LP problem of the following primal LP problem:

Minimize $Z = x_1 + 2x_2$

subject to the constraints

(i) $2x_1 + 4x_2 \leq 160$, (ii) $x_1 - x_2 = 30$, (iii) $x_1 \geq 10$

and $x_1, x_2 \geq 0$

Solution Since the objective function of the primal LP problem is of minimization, change all $\leq$ type constraints to $\geq$ type constraints by multiplying the constraint on both sides by -1. Also write $=$ type constraint equivalent to two constraints of the type $\geq$ and $\leq$. Then the given primal LP problem can be rewritten as:

Minimize $Z_x = x_1 + 2x_2$

subject to the constraint

(i) $-2x_1 - 4x_2 \geq -160$, (ii) $x_1 - x_2 \geq 30$

(iii) $x_1 - x_2 \leq 30$ or $-x_1 + x_2 \geq -30$, (iv) $x_1 \geq 10$

and $x_1, x_2 \geq 0$

Let y_1, y_2, y_3 and y_4 be the dual variables corresponding to the four constraints in the given order. The dual of the given primal LP problem can then be formulated as follows:

Maximize $Z_y = -160y_1 + 30y_2 - 30y_3 + 10y_4$

subject to the constraints

(i) $-2y_1 + y_2 - y_3 + y_4 \leq 1$, (ii) $-4y_1 - y_2 + y_3 \leq 2$

and $y_1, y_2, y_3, y_4 \geq 0$

Let $y = y_2 - y_3$ ($y_2, y_3 \geq 0$). The above dual problem then reduces to the form

Maximize $Z_y = -160y_1 + 30y + 10y_4$

subject to the constraints

(i) $-2y_1 + y + y_4 \leq 1$, (ii) $-4y_1 - y \leq 2$

and $y_1, y_4 \geq 0$; y unrestricted in sign

Remark Since second constraint in the primal LP problem is equality, therefore as per rule 6 corresponding second dual variable y $(= y_2 - y_3)$ should be unrestricted in sign.

Example 5.4 Obtain the dual LP problem of the following primal LP problem:

Minimize $Z_x = x_1 - 3x_2 - 2x_3$

subject to the constraints

(i) $3x_1 - x_2 + 2x_3 \le 7$, (ii) $2x_1 - 4x_2 \ge 12$, (iii) $-4x_1 + 3x_2 + 8x_3 = 10$

and $x_1, x_2 \ge 0$; x_3 unrestricted in sign.

Solution Let y_1, y_2 and y_3 be the dual variables corresponding to three primal constraints in the given order. As the given LP problem is of minimization, all constraints can be converted to $\ge$ type by multiplying both sides by -1, i.e., $-3x_1 + x_2 - 2x_3 \ge -7$. Since the third constraint of the primal is an equation, the third dual variable y_3 will be unrestricted in sign. The dual of the given LP primal can be formulated as follows:

Maximize $Z_y = -7y_1 + 12y_2 + 10y_3$

subject to the constraints

(i) $-3y_1 + 2y_2 - 4y_3 \le 1$, (ii) $y_1 - 4y_2 + 3y_3 \le -3$, (iii) $-2y_1 + 8y_3 \le -2$

and $y_1, y_2 \ge 0$; y_3 unrestricted in sign.

Example 5.5 Obtain the dual of the following primal LP problem

Maximize $Z_x = x_1 - 2x_2 + 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 = 1$

and $x_1, x_2, x_3 \ge 0$

Solution Since both the primal constraints are of the equality type, the corresponding dual variables y_1 and y_2, will be unrestricted in sign. Following the rules of duality formulation, the dual of the given primal LP problem is

Minimize $Z_y = 2y_1 + y_2$

subject to the constraints

(i) $-2y_1 + 2y_2 \ge 1$, (ii) $y_1 + 3y_2 \ge -2$, (iii) $3y_1 + 4y_2 \ge 3$

and y_1, y_2 unrestricted in sign.

Example 5.6 Write the dual of the following primal LP problem

Maximize $Z = 3x_1 + x_2 + 2x_3 - x_4$

subject to the constraints

(i) $2x_1 - x_2 + 3x_3 + x_4 = 1$, (ii) $x_1 + x_2 - x_3 + x_4 = 3$

and $x_1, x_2 \ge 0$ and x_3, x_4 unrestricted in sign.

Solution Here we may apply the following rules of forming a dual of given primal LP problem.

(i) The x_3 and x_4 variables in the primal are unrestricted in sign, third and fourth constraints in the dual shall be equalities.

(ii) The given primal LP problem is of maximization; the first two constraints in the dual LP problem will therefore be $\ge$ type constraints.

(iii) Since both the constraints in the primal are equalities, the corresponding dual variables y_1 and y_2 will be unrestricted in sign.

If y_1 and y_2 are dual variables corresponding to the two primal constraints in the given order, the dual of the given primal can be written as:

Minimize $Z_y = y_1 + 3y_2$

subject to the constraints

(i) $2y_1 + y_2 \ge 3$, (ii) $-y_1 + y_2 \ge 1$, (iii) $3y_1 - y_2 = 2$

(iv) $y_1 + y_2 = -1$

and y_1, y_2 unrestricted in sign.

SELF PRACTICE PROBLEMS A

Write the dual of the following primal LP problems

1. Max $Z_x = 2x_1 + 5x_2 + 6x_3$
   subject to (i) $5x_1 + 6x_2 - x_3 \le 3$
   (ii) $-2x_1 + x_2 + 4x_3 \le 4$
   (iii) $x_1 - 5x_2 + 3x_3 \le 1$
   (iv) $-3x_1 - 3x_2 + 7x_3 \le 6$
   and $x_1, x_2, x_3 \ge 0$
   [Sambalpur MSc (Maths), 1996]
2. Min $Z_x = 7x_1 + 3x_2 + 8x_3$
   subject to (i) $8x_1 + 2x_2 + x_3 \ge 3$
   (ii) $3x_1 + 6x_2 + 4x_3 \ge 4$
   (iii) $4x_1 + x_2 + 5x_3 \ge 1$
   (iv) $x_1 + 5x_2 + 2x_3 \ge 7$
   and $x_1, x_2, x_3 \ge 0$
3. Max $Z_x = 2x_1 + 3x_2 + x_3$
   subject to (i) $4x_1 + 3x_2 + x_3 = 6$, (ii) $x_1 + 2x_2 + 5x_3 = 4$
   and $x_1, x_2, x_3 \ge 0$
4. Max $Z_x = 3x_1 + x_2 + 3x_3 - x_4$
   subject to (i) $2x_1 - x_2 + 3x_3 + x_4 = 1$
   (ii) $x_1 + x_2 - x_3 + x_4 = 3$
   and $x_1, x_2, x_3, x_4 \ge 0$
5. Min $Z_x = 2x_1 + 3x_2 + 4x_3$
   subject to (i) $2x_1 + 3x_2 + 5x_3 \ge 2$
   (ii) $3x_1 + x_2 + 7x_3 = 3$
   (iii) $x_1 + 4x_2 + 6x_3 \le 5$
   and $x_1, x_2, \ge 0, x_3$ is unrestricted
6. Min $Z_x = x_1 + x_2 + x_3$
   subject to (i) $x_1 - 3x_2 + 4x_3 = 5$, (ii) $x_1 - 2x_2 \le 3$
   (ii) $2x_2 - x_3 \ge 4$
   and $x_1, x_2 \ge 0$, x_3 is unrestricted.
   [Meerut, MSc (Maths), 2005]
7. Min $Z_x = 8x_1 + 3x_2$
   subject to (i) $x_1 - 6x_2 \ge 2$, (ii) $5x_1 + 7x_2 = -4$
   and $x_1, x_2 \ge 0$.
8. Max $Z_x = 8x_1 + 8x_2 + 8x_3 + 12x_4$
   subject to (i) $30x_1 + 20x_2 + 25x_3 + 40x_4 \le 800$
   (ii) $25x_1 + 10x_2 + 7x_3 + 15x_4 \le 250$
   (iii) $4x_1 - x_2 = 0$
   (iv) $x_3 \ge 5$
   and $x_1, x_2, x_3, x_4 \ge 0$
9. Min $Z_x = 18x_1 + 10x_2 + 11x_3$
   subject to (i) $4x_1 + 6x_2 + 5x_3 \ge 480$
   (ii) $12x_1 + 10x_2 + 10x_3 \ge 1{,}200$
   (iii) $10x_1 + 15x_2 + 7x_3 \le 1{,}500$
   (iv) $x_3 \ge 50$
   (v) $x_1 - x_2 \le 0$
   and $x_1, x_2, x_3 \ge 0$
10. Min $Z_x = 2x_1 - x_2 + 3x_3$
    subject to (i) $x_1 + 2x_2 + x_3 \ge 12$
    (ii) $x_2 - 2x_3 \ge -6$
    (iii) $6 \le x_1 + 2x_2 + 4x_3 \ge 24$
    and $x_1, x_2 \ge 0$, x_3 unrestricted.

HINTS AND ANSWERS

1. Min $Z_y = 3y_1 + 4y_2 + y_3 + 6y_4$
   subject to (i) $5y_1 - 2y_2 + y_3 - 3y_4 \ge 2$
   (ii) $6y_1 + y_2 - 5y_3 - 3y_4 \ge 5$
   (iii) $-y_1 + 4y_2 + 3y_3 + 7y_4 \ge 6$
   and $y_1, y_2, y_3, y_4 \ge 0$
2. Max $Z_y = 3y_1 + 4y_2 + y_3 + 7y_4$
   subject to (i) $8y_1 + 3y_2 + 4y_3 + y_4 \le 7$
   (ii) $2y_1 + 6y_2 + y_3 + 5y_4 \le 3$
   (iii) $y_1 + 4y_2 + 5y_3 + 2y_4 \le 8$
   and $y_1, y_2, y_3, y_4 \ge 0$
3. Min $Z_y = 6y_1 + 4y_2$
   subject to (i) $4y_1 + y_2 \ge 2$, (ii) $3y_1 + 2y_2 \ge 3$
   (iii) $y_1 + 5y_2 \ge 1$
   and y_1, y_2 unrestricted in sign.
4. Min $Z_y = y_1 + 3y_2$
   subject to (i) $2y_1 + y_2 \ge 3$
   (ii) $-y_1 + y_2 \ge 1$
   (iii) $3y_1 - y_2 \ge 3$
   (iv) $y_1 + y_2 \ge -1$
   and y_1, y_2 unrestricted in sign.
5. Max $Z_y = 2y_1 + 3y_2 - 5y_3$
   subject to $2y_1 + 3y_2 - y_3 \le 2$
   $3y_1 + y_2 - 4y_3 \le 3$
   $5y_1 + 7y_2 - 6y_3 = 4$
   and $y_1, y_3 \ge 0$ and y_2 unrestricted.
6. Max $Z_y = -5y_1 - 3y_2 + 4y_3$
   subject to (i) $-y_1 - y_2 \le 1$, (ii) $-3y_1 + 2y_2 + 2y_3 \le 1$
   $4y_1 - y_3 \le 1$
   and $y_2, y_3 \ge 0$ and y_1 is unrestricted.
7. Max $Z_y = 2y_1 - 4y_2$
   subject to (i) $y_1 + 5y_2 \le 8$, (ii) $-6y_1 + 7y_2 \le 3$
   and $y_1 \ge 0$ and y_2 is unrestricted.
8. Min $Z_y = 800y_1 + 250y_2 + y_3 + 5y_4$
   subject to $30y_1 + 25y_2 + 4y_3 \ge 8$
   $20y_1 + 10y_2 - y_3 \ge 8$
   $25y_1 + 7y_2 + y_4 \ge 8$
   $40y_1 + 15y_2 \ge 12$
   $y_1, y_2 \ge 0$ and y_3, y_4 are unrestricted.