6/Nov/2017

Introduction to complex no.

Euler's Formula

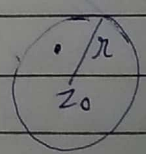
$$\boxed{e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots}$$

$$\text{Arg } z = \begin{cases} \tan^{-1} y/x & \text{if } x \geq 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \text{ \& } y > 0 \\ \tan^{-1}(y/x) - \pi & \text{if } x < 0 \text{ \& } y < 0 \end{cases}$$

Roots $\quad r^{1/n} e^{i(\theta/n + 2\pi k/n)} \leftarrow$ $1/n$ root of $z = re^{i\theta}$

$$1 + d + d^2 + \dots d^{n-1} = \frac{1-d^n}{1-d}$$

Neighbourhood: $\quad |z - z_0| < r$



$S \to$ Set

Interior Point: $\quad z_0$ interior if $\exists\ \delta$ s.t. $|z - z_0| < \delta$
if $z \in S$ $\quad$ (Boundary point is not an interior point)

Boundary Point: Nbd belong to S and does not belong to S

Exterior Point: Neither interior nor boundary.

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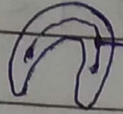
Open set :→ Only interior point.

connected set →



Path is combination of small straight line segment.

Function → ~~Not~~ May not give unique values.

$$e^{x+iy} = r\,e^{i\phi}$$

$$x = \ln r \qquad y = \phi + 2k\pi$$

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

Hyperbolic function

$$\cos iy = \cosh y$$
$$\sin iy = i \sinh y$$

$$\cancel{\cosh y = \frac{e^{\theta} + e^{-\theta}}{2}}$$

$$\cosh y = \frac{e^{y} + e^{-y}}{2} \qquad \sinh y = \frac{e^{y} - e^{y}}{2}$$

Function
↓
Limit
↓
Continuity
↓
differentiability
↓
Analytic

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Limit : for the no δ $\exists$ ϵ s.t
$|f(z) - \cancel{f(z_0} \omega_0| < \delta$ whenere $|z - z_0| < \delta$

$$\lim_{z \to z_0} f(z) = \omega_0.$$

Cauchy - Riemann Eq^n

$$f(z) = u(x, y) + i\, v(x, y)$$
$$u_x = v_y \quad \& \quad u_y = -v_x.$$

CR $\not\to$ Differentiable at (x_0, y_0)
$\times$CR $\longrightarrow$ $\times$ Differentiable.

Theorem $f(z) = u + iv$
$$f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$$

learn $f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$

if $\epsilon_1\, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ $f(x, y)$ cont. at (x_0, y_0)
$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$f(re^{i\theta}) = u(r, \theta) + i\, v(r, \theta)$$
$$u_r = \frac{1}{r} v_\theta \qquad v_r = -\frac{1}{r} u_\theta$$

Harmonic Function
$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$$

Theorem

$f(z)$ analytic $\longrightarrow$ Harmonic u and v
$\times f(z)$ analytic $\to$ Can't say
Not Harmonic $\to$ Not Analytic
Harmonic $\not\Rightarrow$ Analytic

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u_x and u_y continuous implies $u_{xy} = u_{yx}$

Analytic function $\longrightarrow$ Integral depends upon end points only.

ML Inequality

$$\left| \int_C f(z)\,dz \right| \leq ML$$

L is the length of the curve C and $|f(z)| \leq M$ everywhere on C.

Green's Theorem

$$\oint_C L\,dx + M\,dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\,dy$$

Cauchy's Theorem (Cauchy-Goursat Theorem)

$$\oint_C f(z)\,dz = 0;$$

$f(z) \rightarrow$ analytic in domain D (within given closed curve)

Independent Path

If $f(z)$ is analytic in domain D then integral of $f(z)$ is independent of path in D.

Singularity

If $f(z)$ not analytic at that point but analytic at its nbd.

Isolated Singularity Non Isolated Singularity

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Date
Page

Pole: Isolated Singularity.

$\lim_{z \to z_0} (z-z_0)^n f(z) \neq 0$ (finite)

↳ ~~Pole of order n.~~

Zero: A point $z=z_0$ is zero if $f(z_0)=0$.

Removable Singularity: $\lim_{z \to z_0} f(z)$ exists.

Essential Singularity: Neither pole, nor removable

Branch Point: Function can never be analytic at branch point. (common to all branch cut.)

$$\left.\begin{array}{ll} f(z)=z^{1/2} & z=0 \\ =(z-3)^{1/3} & z=3 \\ \log z & z=0 \\ \log(z+4-\sqrt{2}i) & z=-4+\sqrt{2}i \end{array}\right\} \text{Branch Points.}$$

Branch Cut: $f(z)=z^{1/2}$ → Circle cut in 2 parts.
$f(z)=z^{1/3}$ → Circle cut in 3 parts.

~~Intersection~~ Intersection of cuts → Branch Point.

Cauchy Theorem:
(Cauchy Integral Formula)

$$\boxed{\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)}$$ ↳ anticlockwise.

Here C encloses z_0.

$$\boxed{2\pi i f^n(z_0) = n! \oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

Cauchy Inequality

$$|f^n(z_0)| \leq \frac{M\, n!}{r^n}$$

$|f(z)| \leq M$ for all z inside and on C.

r is radius of open disk C

Morera's Theorem

$$\oint_C f(z)\,dz = 0$$

Liouville's Theorem

- $f(z)$ is analytic Everywhere in C
- $f(z)$ is bounded $|f(z)| \le m$.

$f(z)$ is constant.

Fundamental Theorem of Algebra

Every non-constant polynomial has atleast one root in C.

Argument Theorem

$f(z)$ analytic inside and on a simple closed curve C except a pole $z=a$ of order n inside C and zero at $z=b$ of order m

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \sum m_i - \sum n_i$$

zero of order m_i

Pole of order n_i

Rouche's Theorem

- $f(z)$ and $g(z)$ analytic inside C (simple closed curve)
- $|f(z)| > |g(z)|$ at each point on C.

Then $f(z)$ and $f(z)+g(z)$ have same number of zeroes counting multiplicity in C.

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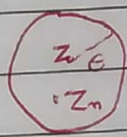
7/Nov/2017

Sequence and ~~Convergence~~ Series

Infinite Sequence of complex no. has a limit z if for every $\epsilon > 0$ $\exists$ +ve no n_0 s.t

$$|z_n - z| < \epsilon \quad \text{for } n > n_0$$



$$\lim_{n \to \infty} z_n = z$$

Theorem: $z_n = x_n + i y_n$

$\cancel{z_n} \longrightarrow \lim_{n\to\infty} z_n = z$ if and only if

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Theorem: $\sum_{n=1}^{\infty} z_n \longrightarrow$ converges $\Rightarrow z_n \longrightarrow$ converges to zero.

Absolute Convergent: $\sum_{n=1}^{\infty} |z_n|$ converges.

* Terms of convergent series are bounded.

Absolute Convergence (series) $\longrightarrow$ Convergence (series).

Conditionally Convergent:

$\sum_{n=1}^{\infty} z_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots\ldots \longrightarrow$ Convergent.

$\sum_{n=1}^{\infty} |z_n| = \Sigma \frac{1}{n} \rightarrow$ not convergent

Remainder Term: $R_N = \sum_{n=N+1}^{\infty} z_n = z_{N+1} + z_{N+2} \ldots$

$\sum_{n=1}^{\infty} z_n = S$ $\qquad S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 \ldots z_n$

$R_N \rightarrow$ Remainder Term

$S = S_N + R_N;$

$|R_N| \rightarrow$ Error Term.

$S_N \rightarrow S$ when $R_N \rightarrow 0$

Test for Convergence (Cauchy)

Theorem

$\epsilon > 0 \quad \exists N > 0 \quad s.t$

$$|z_{N+1} + z_{N+2} \ldots z_{N+p}| < \epsilon \quad \forall \; n > N$$

and $p = 1, 2, \ldots\ldots$

Comparison Test: $|z_n| \leq b_n \quad \forall \; n = 1, 2, \ldots\ldots$

s.t $\sum b_n$ converges.

$\Downarrow$

$\sum z_n$ converges Absolutely.

Ratio Test: $\lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = L$

$L < 1 \longrightarrow$ Converges

$L > 1 \longrightarrow$ diverges

Test fails $\longleftarrow L = 1$

Root Test: $\lim_{n \to \infty} |z_n|^{\frac{1}{n}} = L$

$L < 1 \longrightarrow$ Converges

$L > 1 \longrightarrow$ diverges

$L = 1 \longrightarrow$ Test fails.

Power Series: $\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots\ldots$

* Every power series converges at its centre.

Radius of Convergence: $|z - z_0| = R \rightarrow$ circle of convergence

Smallest circle with centre at z_0 that includes all the points at which power series converges.

Page

Cauchy - Hardmard : $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{L} = R$

$$\lim_{n\to\infty} \left| \frac{1}{\sqrt[n]{a_n}} \right| = \frac{1}{L} = R$$

* $\sum_{n=0}^{\infty} a_n z^n$ — non-zero radius of convergence $R>0$ then the sum function is a function of z.

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ is continuous in $|z|<R$ (if convergent)

$$\sum z^n = 1 + z + z^2 + \dots = \frac{1}{1-z}$$

Power Series Properties ★ We can differentiate and integrate the power series term wise and the resulting series has same radius of convergence as your original series.

★ A power series with $R>0$ represent an analytic function at every point interior to its circle of convergence i.e $|z-z_0|<R$

Taylor Series

Power Series Representation $\rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ $\quad |z-z_0|<R_0$

$$a_n = \frac{f^n(z_0)}{n!}, \quad n = 0, 1, 2, \dots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$$

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2 f''(z_0)}{2} + \dots + \frac{(z-z_0)^n f^n(z_0)}{n!} + R_n(z)$$

$R_n(z)$ ↓ Reminder

Date
Page

Maclauren Series $\frac{1}{1-z} = 1 + z + z^2 +$

Laurent Series

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad n = 0, \pm 1,$$

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-z_0)^n \qquad R_1 < |z-z_0| < R_2$$

where the closed curve
C lie in the region
$R_1 < |z-z_0| < R_2$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{Principal part}}$$

(~~n~~ -ve power of z)

Singularity of f(z) at $z = z_0$
→ Isolated Singularity
→ Isolated Essential Singularity
↓
Principal part of Laurent contain ∞ terms

Residue

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz \longrightarrow \text{Residue of } f(z) \text{ at } z = z_0$$

$f(z)$ singularity at
$z = z_0$ inside C
$|z-z_0| < R$

$$\oint f(z)\, dz = 2\pi i\, b_1$$

+ve anticlockwise c
-ve clockwise c

* Choose region wisely

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$$b_1 = \lim_{z \to z_0} (z-z_0) f(z)$$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

analytic $p(z_0) \neq 0$

$q(z_0) = 0$

zeros of $q(z_0) = z_0$

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-z_0)^m f(z) \right\}$$

Cauchy Residue Theorem

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

$$\boxed{\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + \cdots}$$

$0 < |x| < \pi$

Classification of PDEs

Quasilinear PDE : Highest order derivatives are linear

Semi-linear PDE : Quasilinear PDE with ~~non~~ coeff of Highest order derivative independent of dependent variable

Linear Eq^n : Semi-linear PDE that is linear in the dependent variable and its derivatives

Non-linear : Not Quasilinear.

$$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)}$$

~~$p = F_a$~~ $\quad p = z_x \quad q = z_y$

$$\frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Envelope of one-parameter

The surface determined by eliminating the parameter 'c'

$$f(x,y,z,c) = 0 \quad \& \quad \frac{\partial}{\partial c} f(x,y,z,c) = 0$$

one-parameter system

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Date
Page

Envelop of two-parameter system:

Surface obtained by eliminating a and b from eq^n

$$f(x,y,z,a,b)=0,\quad \frac{\partial f}{\partial a}=0 \text{ and } \frac{\partial f}{\partial b}=0$$

* solution of $f(x,y,z,p,q)=0$ is called integral surface of PDE.

Classification based on solution of PDE

1. Complete Integral or Complete sol^n.

Any such relation which contain 2 arbitary const a & b and is a sol^n of 1^{st} order PDE

2. $f(u,v)=0$ provides a sol^n of 1^{st} order PDE

~~General Integral.~~

$$\frac{\partial(u,v)}{\partial(z,x)}q+\frac{\partial(u,v)}{\partial(y,z)}p=\frac{\partial(u,v)}{\partial(x,y)}$$

Singular Integral.

$$\underbrace{z=F(x,y,a,b) \quad \frac{\partial F}{\partial a}=0 \quad \frac{\partial F}{\partial b}=0}$$

Envelop of 2 parameter. sol^n of this & known as Singular Integral.

Langrage Method (for solving Quasilinear Problem)

$\frac{\dot{x}}{P} = \frac{\dot{y}}{Q} = \frac{\dot{z}}{R}$ — characteristic eq^n of PDE

General sol^n

$u(x,y,z) = c_1$
$v(x,y,z) = c_2$ — Two independent sol^n

$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$

$$\boxed{\frac{M}{N} = \frac{P}{Q} = \frac{M-P}{N-Q}}$$

$F(u,v) = 0$ (or $u = G(v)$ or $v = H(\omega)$)

$F(u,v) = 0 \longrightarrow$ Implicit Form
$u = G(v)$ or $v = H(u) \longrightarrow$ Explicit Form.

$u(x,y,z) = c_1$ ~~v~~ $= v(x,y,z) = c_2$ are two independent sol^n of characteristic sol^n.

$$\frac{\dot{x}}{P} = \frac{\dot{y}}{Q} = \frac{\dot{z}}{R}$$

Aeternating Series Test

$$\sum_{n\geq 1} (-1)^n a_n \quad \text{where } a_n \geq 0$$

1.) $\{a_n\}$ is decreasing;

2.) $\lim_{n \to \infty} a_n = 0$

$$\Downarrow$$

$\sum_{n\geq 1} (-1)^n a_n$ is convergent.

25/Nov/2017

Partial Differential Equation

→ 1^{st} order PDE $\quad f(x, y, z, z_x, z_y) = 0$

→ 2^n order PDE $\quad f(x, y, z(x,y), z_{xx}, z_{yy}, z_{xy}, z_{...}$

Order of PDE: Highest order partial Derivative which appears in the P...

Classification of PDEs:

Quasilinear PDE

Semi-linear PDE

Linear

Non-Linear

$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$$

$$\downarrow \qquad\qquad \downarrow$$

$$p = z_x \qquad q = z_y$$

Quasilinear PDE: p & q have to ~~to~~ be linear

Semilinear PDE:

$$\underbrace{P(x,y)\,p + Q(x,y)\,q}_{\text{Not a fun of } z} = R(x,y,z)$$

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Date
Page

Linear PDE :

$$P(x,y)\,p + Q(x,y)\,q = R(x,y) + zG(x,y)$$

Non linear PDE : Non of the above.

$$\boxed{\frac{\partial(u,v)}{\partial(y,z)}\,p + \frac{\partial(u,v)}{\partial(z,x)}\,q = \frac{\partial(u,v)}{\partial(x,y)}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Jacobian of u, v
wrt x, y

Second Order Partial Differential Equation

Cannonical (Normal) form ~~of 2nd order PDE~~

$$ax^2 + bxy + cx^2$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0.$$

$$A^2 + B^2 + C^2 \neq 0$$

$B^2 - 4AC > 0$ Hyperbolic PDE

$B^2 - 4AC = 0$ Parabolic PDE

$B^2 - 4AC < 0$ Elliptic PDE

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$Au_{xx} + Bu_{xy} + Cu_{yy}$

If it is C then $\frac{dy}{dx} + \lambda_1 = 0$

Hyperbola $u_{\xi\eta}$

Parabola $u_{\eta\eta}$

Ellipse

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