Linear Algebra

Q=1 Find the rank of the matrix A by reducing in Row reduced echlon form.

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}.$$

Solⁿ: Applying $R_2 \to R_2 - 2R_1$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

again applying $R_3 \to R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

applying $R_4 \to R_4 - 6R_1$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q=2 Let W be the vector space of all symmetric 2x2 matrix and let $T: W \to P_2$ be the linear transformation defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a-b) + (b-c)x + (c-a)x^2.$$ find rank & nulity of T

Solⁿ: Let W be a V.S. of sym. matrix, then it can be repⁿ as.

$$\begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix}, \quad \therefore \text{Dim}^n(W) = 3$$

Now, nulity = dimⁿ of null space, means subspace of W which gives 0 as output from P_2.

for zero polynomial.

$a-b=0 \to a=b$

$b-c=0 \to b=c$

$c-a=0 \to c=a$

$\therefore$ matrix can be written by one variable.

$$K\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \text{null space.}$$

$\therefore$ Nulity = Dimⁿ (Null space)

$= 1$

Q=3 Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, Find the eigen values & eigen vectors of A^{-1} & $A+4I$.

Solⁿ: for A^{-1}, let $B = A^{-1}$

$$\therefore B = \frac{1}{|A|}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{4-1}\cdot\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Now, $(B - \lambda I)x = 0$

$$\begin{bmatrix} \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-\lambda \end{bmatrix} = 0$$

$$\left(\frac{2}{3}-\lambda\right)^2 - \frac{1}{9} = 0$$

$$\frac{2}{3} - \lambda = \pm\frac{1}{3}$$

$$\Rightarrow \lambda = \frac{1}{3}, -1 \Big] \text{ eigen values on } A^{-1}$$

for $\lambda = 1$

$$\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

we get, $-\frac{1}{3}x_1 + \frac{1}{3}x_2 = 0$

$\Rightarrow x_1 = x_2 = k$ (let)

$\therefore$ eigen vector

$$k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda = \frac{1}{3}$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\frac{1}{3}x_1 + \frac{1}{3}x_2 = 0$

$\Rightarrow x_1 = -x_2 = k$ (let)

$\therefore$ eigen vector

$$k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now, for $A+4I$, let $C = A+4I$ $= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$$C = \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix}$$

Now $[C - \lambda I]x = 0$

$$\begin{bmatrix} 6-\lambda & -1 \\ -1 & 6-\lambda \end{bmatrix} = 0$$

$(6-\lambda)^2 - 1 = 0$

$6 - \lambda = \pm 1$

$\Rightarrow \lambda = 5, 7$ } eigen values for $A+4I$

Now for $\lambda = 5$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2 = k$ (let)

$\therefore k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now for $\lambda = 7$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-x_1 - x_2 = 0$

$\Rightarrow x_1 = -x_2 = k$ (let)

$\therefore$ eigen vector

$$k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Q=4 Solve by gauss - seidal Method (3 iterations).

$$3x - 0.1y - 0.2z = 7.85$$
$$0.1x + 7y - 0.3z = -19.3$$
$$0.3x - 0.2y + 10z = 71.4$$

with initial values $x(0) = 0$, $y(0) = 0$, $z(0) = 0$

Sol^n: Iteration 1: $x = y = z = 0$ (initially)

$$3x = 7.85$$
$$x = \frac{7.85}{3}$$
$$= 2.6167$$

Now, $0.1x + 7y - 0.3z = -19.3$

$= 0.1 \times 2.6167 + 7y - 0 = -19.3$

$= 7y = -19.3 - 0.26167 = -19.03$

$\Rightarrow y = -2.7945.$

Now, $0.3 \times 2.6167 - 0.2 \times -2.7945 + 10z = 71.4$

$10z = 70.056$

$z = 7.0056$

Iteration 2: $x = 2.6167$; $y = -2.7945$; $z = 7.0056$

Now, leaving x & putting other two,

$3x - 0.1 \times -2.7945 - 0.2 \times 7.0056 = 7.85$

$\rightarrow 3x + 0.27945 - 1.40112 = 7.85$

$\rightarrow 3x = 8.97167$

$x = 2.990$

Now, $0.1 \times 2.99 + 7y - 0.3 \times 7.0056 = -19.3$

$\rightarrow 7y = -17.49732$

$y = -2.4996$

Now, $0.3 \times 2.99 - 0.2 \times -2.4996 + 10z = 71.4$

$\Rightarrow 10z = 70.00308$

$z = 7.0003$

Iteration 3: $x = 2.99$, $y = -2.4996$, $z = 7.0003$

Now $3x - 0.1 \times -2.4996 - 0.2 \times 7.0003 = 7.85$

$\rightarrow 3x = 9.0001$

$\Rightarrow x = 3$ (approx)

Now, $0.1 \times 3 + 7y - 0.3 \times 7.0003 = -19.3$

$\Rightarrow 7y = -17.4999$

$y = -2.5$ (approx)

Now, $0.3 \times 3 - 0.2 \times 2.5 + 10z = 71.4$

$10z = 70$

$z = 7$

hence, $x = 3$, $y = -2.5$, $z = 7$.

Q-5 Define consistent and inconsistent system of equations. Hence solve the following sys. of eqn. if consistent.

$$x + 3y + 2z = 0 \quad -(1)$$
$$2x - y + 3z = 0 \quad -(2)$$
$$x + 17y + 4z = 0 \quad -(3)$$
$$3x - 5y + 4z = 0 \quad -(4)$$

as per question.

Soln:

Consistent sys. of eqn. → having unique or many solutions.

↳ condn: $\rho(A) = \rho(A:B)$ for the sys. of linear eqn. as $AX = B$

Inconsistent sys of eqn → having no. solution.

↳ condition : $\rho(A) \neq \rho(A:B)$ for the sys. of linear eqn. as $AX = B$

Now for the given eq^n:

$$A: \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}_{4\times3} \overset{\to X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times1} = \overset{\to B}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{4\times1}$$

As it is a homogenuous eq^n, $\therefore$ It is always consistent as $\rho(A)$ will always equals to $\rho(A:B)$

Now, echlon form of A:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2$$
$$R_4 \to R_4 + 2R_2$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $\rho(A) = 2$ (No. of non-zero rows is 2)

But $\rho(A) \neq 3$ (dim^n)

$\therefore$ ∞ many sol^n possible.

Now, $x + 3y + 2z = 0$

$-7y - z = 0$

$z = -7y = k$ let.

Now $x + 3\left(-\frac{k}{7}\right) + 2k = 0$

$x + \frac{11k}{7} = 0$

$x = \frac{-11k}{7}$

Sol^n: $k \begin{bmatrix} -\frac{11}{7} \\ \frac{1}{7} \\ 1 \end{bmatrix}$

Q=6 Show whether a fun. $T: P_2 \to P_2$ is a linear transformation or not where $T(a+bx+cx^2) = (a+1)+(b+1)x+(c+1)x^2$.

Soln: for linear transformation two condn. required

1. $T(a+b) = T(a) + T(b)$
2. $T(cU) = cT(U)$

for ① let's take two vectors as

$$a_1 + b_1x + c_1x^2$$
$$a_2 + b_2x + c_2x^2$$

$$T((a_1+b_1x+c_1x^2)+(a_2+b_2x+c_2x^2))$$

$$T\{(a_1+a_2)+(b_1+b_2)x+(c_1+c_2)x^2\}$$

$$= (a_1+a_2+1)+(b_1+b_2+1)x+(c_1+c_2+1)x^2$$

$$= (a_1+1)+(b_1+1)x+(c_1+1)x^2 + a_2 + b_2x + c_2x^2$$

$$= T(a_1+b_1x+c_1x^2) + a_2 + b_2x + c_2x^2$$

$$= T(a_1+b_1x+c_1x^2) + \{(a_2+1)+(b_2+1)x+(c_2+1)x^2\} -1 - x - x^2$$

$$= T(a_1+b_1x+c_1x^2) + T(a_2+b_2x+c_2x^2) -1-x-x^2$$

$$\neq T(a_1+b_1x+c_1x^2) + T(a_2+b_2x+c_2x^2)$$

hence, it is not a linear transformation

Q=7 Determine whether the set $S = \{(1,2,3),(3,1,0),(-2,1,3)\}$ is basis of $V_3(R)$. In case S is not a basis determine the dimn and the basis of the sub space spanned by 3.

Soln:

$$S = \{\overset{v_1}{(1,2,3)}, \overset{v_2}{(3,1,0)}, \overset{v_3}{(-2,1,3)}\}$$

Now if $V_3(R)$ can be written as. $c_1v_1 + c_2v_2 + c_3v_3$ then S spans $V_3(R)$

so for spanning. $c_1V_1 + c_2V_2 + c_3V_3 = 0$ [To check L.I condition]

$$\Rightarrow c_1(1,2,3) + c_2(3,1,0) + c_3(-2,1,3) = 0$$

$$c_1 + 3c_2 - 2c_3 = 0$$
$$2c_1 + c_2 + c_3 = 0$$
$$3c_1 + 0c_2 + 3c_3 = 0$$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 0 & 3 \end{bmatrix} = 0$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \\ 0 & -9 & 9 \end{bmatrix}$$

$R_3 \to R_3 - \frac{9}{5} R_2$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

as Rank $= 2 \neq \dim^n(3)$.

$\therefore$ lin. dep.

$\therefore$ Not a basis.

Now, $\dim^n$ of S $=$ No. of uniques vectors.
$= 3.$

& basis of subspace spanned by S $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q=8 Using Jacobi's method (perform 3 iteration) solve,

$$3x - 6y + 2z = 23$$
$$-4x + y - z = -15$$
$$x - 3y + 7z = 16$$

with initial values $x_0 = 1$
$y_0 = 1$
$z_0 = 1$

Sol^n:

Iteration 1:

$x = \dfrac{23 + 6(1) - 2(1)}{3}$; $y = -15 + 4(1) + 1$; $z = \dfrac{16 - 1 + 3}{7}$

$= 9$; $= -10$; $= 2.57$

Iteration 2:

$x = \dfrac{23 + 6(-10) - 2(2.57)}{3}$; $y = -15 + 4(9) + 2.57$; $z = 16 - 9 + 3(-10)$

$= -14.04$; $= 23.57$; $= -3.28$

Iteration 3:

$$x = \frac{23 + 6 \times 23.57 - 2 \times -3.28}{3} \quad ; \quad y = -15 + 4 \times -14.04 + (-3.28)$$

$$= 57 \qquad\qquad = -74.4$$

$$z = 16 - (14.04) + 3 \times 23.57$$

$$= 100.75$$

Q-9 Explain one application of matrix operations in image processing with examples.

Soln: One application of matrix operations in image processing is convolution, where a small matrix is applied to each pixel in the image to perform operations like blurring, sharpening or edge detection.

for e.g. A blur filter kernel could be used to blur an image by averaging pixel values in the neighbourhood. This process is applied to every pixel.

Let's say we have a grayscale image represented by matrix of pixel values.

We want to apply simple 3X3 blur filter to image.

So, Blur kernel $= \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Q-10 Give a brief description of linear transformation for computer vision for rotating 2-D image.

Solⁿ: Rotating a 2-D image involves applying a linear transformation known as rotation matrix. This matrix operates on the coordinates of each pixel in the image. transforming them to new co-ordinates that represent the rotated image.

The rotation matrix for a 2-D rotation looks like this.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ represent the angle of rotation in radians. This matrix describes how each point in the original image is mapped to a new position after rotation.

1. Iterate over each pixel in the original image.
2. for each pixel, apply the rotation matrix to its co-ordinates.
3. Round the resulting coordinates to obtain the nearest integer values (since pixel are discrete).
4. Copy the pixel value from the original image to the new position in the rotated image.

By applying this process to every pixel in the original image you obtain a rotated version of the image.