Excercise 7.4

Q4.The two opposite vertices of a square are $(-1, 2)$ and $(3, 2)$. Find the coordinates of the other two vertices.

Solution:

Given points

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1}$$

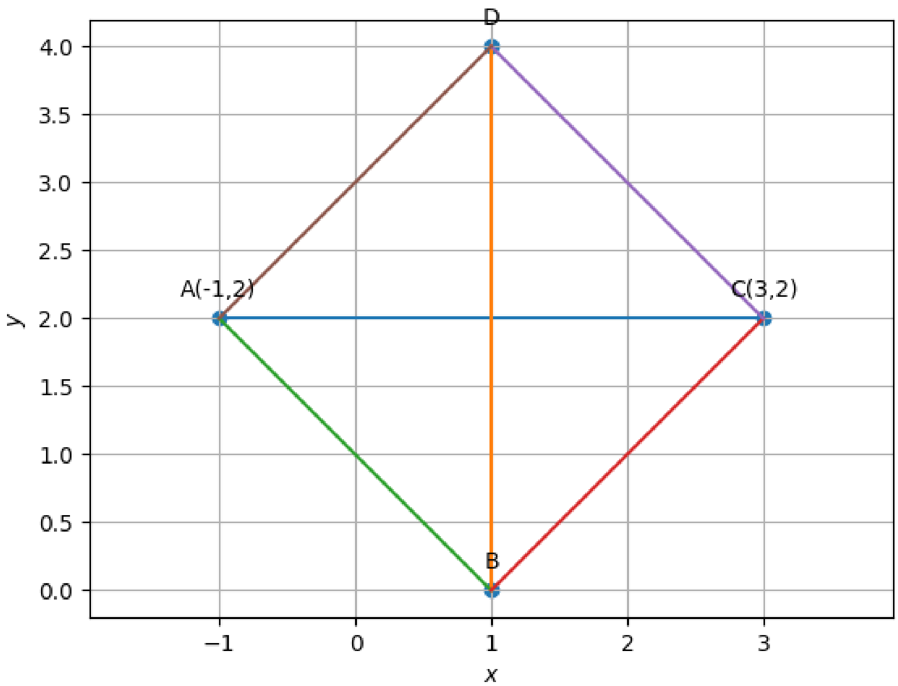

Figure 1:

Shifting $\mathbf{A}$ to origin with refrence to Figure 2

$$\mathbf{A}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C}' = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{2}$$

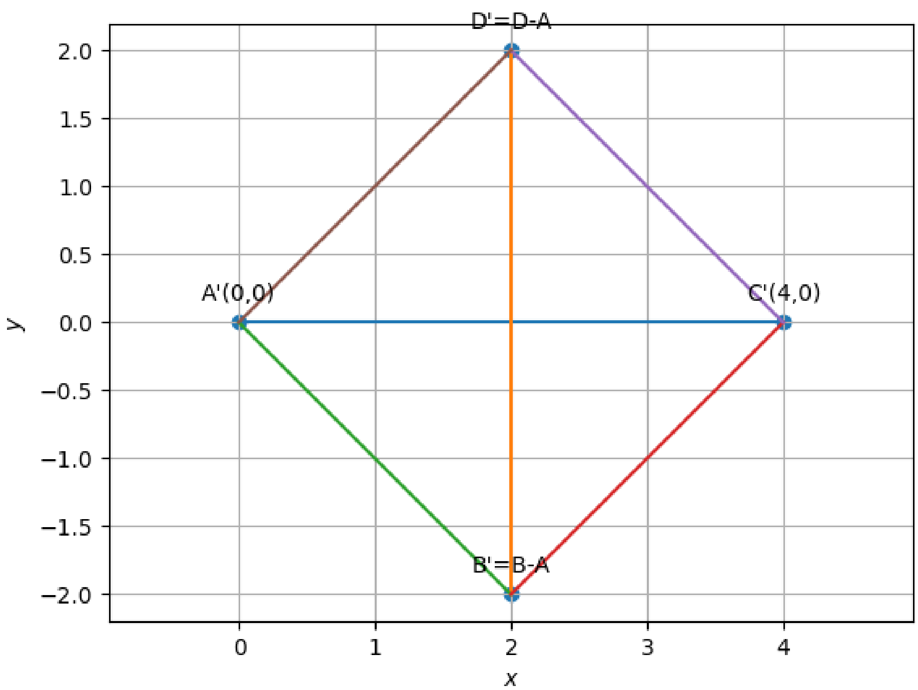

Figure 2:

For a generalised case, the angle made by AC on the x-axis will be $\beta = \theta + 45°$, where θ is the angle made by AB on the x-axis. Hence, we can calculate the angle made by AB on the x-axis.

direction vector is given as

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

However in this particular case, angle made by AC on the x-axis

$$\tan\beta = \frac{0}{4} \implies \sin\beta = \frac{0}{4} \tag{4}$$

$$\beta = 0° \tag{5}$$

Therefore, here $\theta = -45°$
We know the rotation matrix for CW rotation is given as

$$\mathbf{P}^{\top} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{6}$$

Since the angle here is negative so ultimately the coordinates will be rotated in ACW direction. Now the transformed(rotated) coordinates are with refrence to Figure 3.

$$\mathbf{C}'' = \mathbf{P}^{\top}(\mathbf{C} - \mathbf{A}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{pmatrix} \tag{7}$$

$$\mathbf{B}'' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{C}'' = \begin{pmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix}, \mathbf{D}'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{C}'' = \begin{pmatrix} 0 \\ \frac{4}{\sqrt{2}} \end{pmatrix} \text{ and } \mathbf{A}'' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8}$$

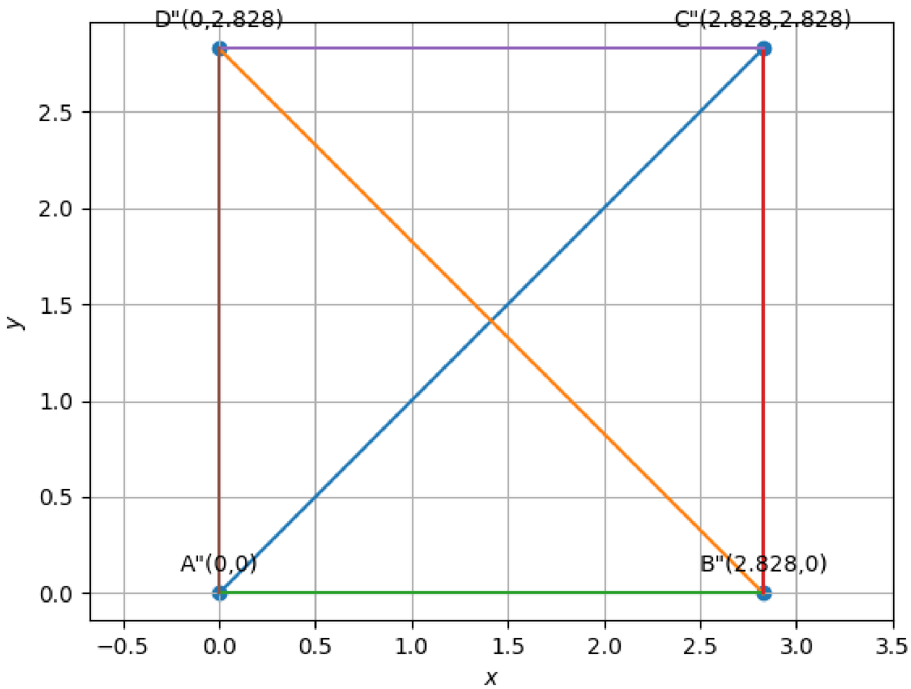

Figure 3:

Again tranforming(rotating) the coordinates back to the original axis. We know for anti-clockwise direction the rotation matrix is given as

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{9}$$

Again we know that the angle is negative so the rotation will be in clockwise direction. So now the transformed(rotated) coordinates **B** and **D** are with refrence to Figure 4

$$\mathbf{B}' = \mathbf{P}\mathbf{B}'' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \tag{10}$$

$$\mathbf{D}' = \mathbf{P}\mathbf{D}'' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{4}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{11}$$

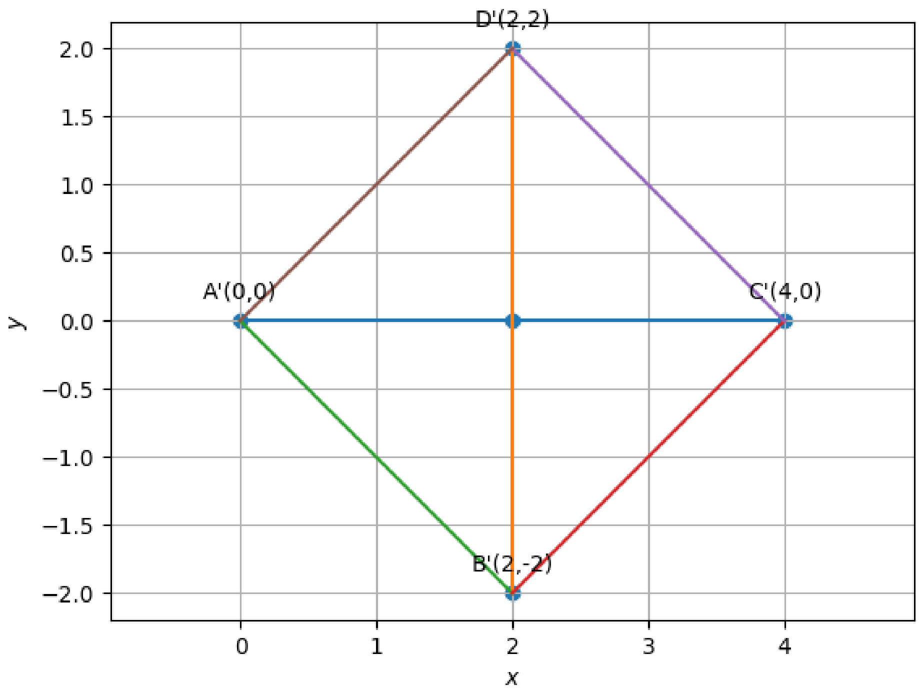

Figure 4:

Again transforming(shifting) the axis back to the original with refrence to Figure 5

$$\mathbf{B} = \mathbf{B'} + \mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12}$$

$$\mathbf{D} = \mathbf{D'} + \mathbf{A} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{13}$$

Hence, the other two vertices are $\mathbf{B}(1,0)$ and $\mathbf{D}(1,4)$
The direct formula for calculation of the vertices is:

$$\mathbf{B} = \mathbf{A} + \mathbf{P}\begin{pmatrix} \mathbf{e_1} & \mathbf{0} \end{pmatrix}\mathbf{P}^\top(\mathbf{C} - \mathbf{A}) \tag{14}$$

$$\mathbf{D} = \mathbf{A} + \mathbf{P}\begin{pmatrix} \mathbf{0} & \mathbf{e_2} \end{pmatrix}\mathbf{P}^\top(\mathbf{C} - \mathbf{A}) \tag{15}$$

$$\tag{16}$$

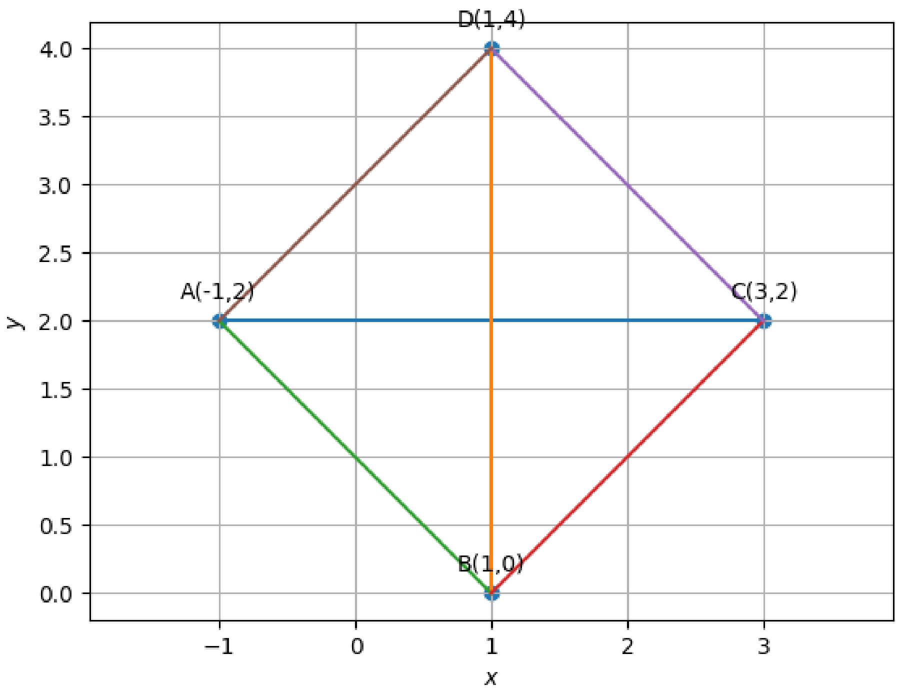

Figure 5:

where $\mathbf{P}$ is the rotation matrix and $\mathbf{A}$ and $\mathbf{C}$ are the position vectors of opposite vertices.

Derivation of the above formulas:

We know that after shifting the axis and rotating by the required angle any arbitrary square will be aligned with the x and y axis so that we can directly get the vectors $\mathbf{B}$ and $\mathbf{D}$ as follows

$$\mathbf{C''} = \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \tag{17}$$

$$\mathbf{B''} = \begin{pmatrix} \mathbf{e_1} & \mathbf{0} \end{pmatrix} \mathbf{C''} = \begin{pmatrix} \mathbf{e_1} & \mathbf{0} \end{pmatrix} \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \tag{18}$$

$$\mathbf{B'} = \mathbf{P}\mathbf{B''} = \mathbf{P} \begin{pmatrix} \mathbf{e_1} & \mathbf{0} \end{pmatrix} \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \tag{19}$$

$$\mathbf{B} = \mathbf{A} + \mathbf{B'} \tag{20}$$

$$= \mathbf{A} + \mathbf{P} \begin{pmatrix} \mathbf{e_1} & \mathbf{0} \end{pmatrix} \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \tag{21}$$

Similarly for D it can be derived as

$$\mathbf{C}'' = \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \tag{22}$$
$$\mathbf{D}'' = \begin{pmatrix}\mathbf{0} & \mathbf{e_2}\end{pmatrix} \mathbf{C}'' = \begin{pmatrix}\mathbf{0} & \mathbf{e_2}\end{pmatrix} \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \tag{23}$$
$$\mathbf{D}' = \mathbf{P}\mathbf{D}'' = \mathbf{P} \begin{pmatrix}\mathbf{0} & \mathbf{e_2}\end{pmatrix} \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \tag{24}$$
$$\mathbf{D} = \mathbf{A} + \mathbf{D}' \tag{25}$$
$$= \mathbf{A} + \mathbf{P} \begin{pmatrix}\mathbf{0} & \mathbf{e_2}\end{pmatrix} \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \tag{26}$$

Verification of the above formula for the given question

$$\mathbf{B} = \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}4\\0\end{pmatrix} \tag{27}$$
$$= \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \begin{pmatrix}\frac{4}{\sqrt{2}}\\ \frac{4}{\sqrt{2}}\end{pmatrix} \tag{28}$$
$$= \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}\frac{4}{\sqrt{2}}\\ 0\end{pmatrix} \tag{29}$$
$$= \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix} \tag{30}$$
$$= \begin{pmatrix}1\\0\end{pmatrix} \tag{31}$$
$$\mathbf{D} = \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}4\\0\end{pmatrix} \tag{32}$$
$$= \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \begin{pmatrix}\frac{4}{\sqrt{2}}\\ \frac{4}{\sqrt{2}}\end{pmatrix} \tag{33}$$
$$= \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}0\\ \frac{4}{\sqrt{2}}\end{pmatrix} \tag{34}$$
$$= \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}2\\2\end{pmatrix} \tag{35}$$
$$= \begin{pmatrix}1\\4\end{pmatrix} \tag{36}$$