

CHAPTER-7
COORDINATE GEOMETRY

Exercise 7.4

Q4. The two opposite vertices of a square are $(-1, 2)$ and $(3, 2)$. Find the coordinates of the other two vertices.

Solution:

Given points

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (1)$$

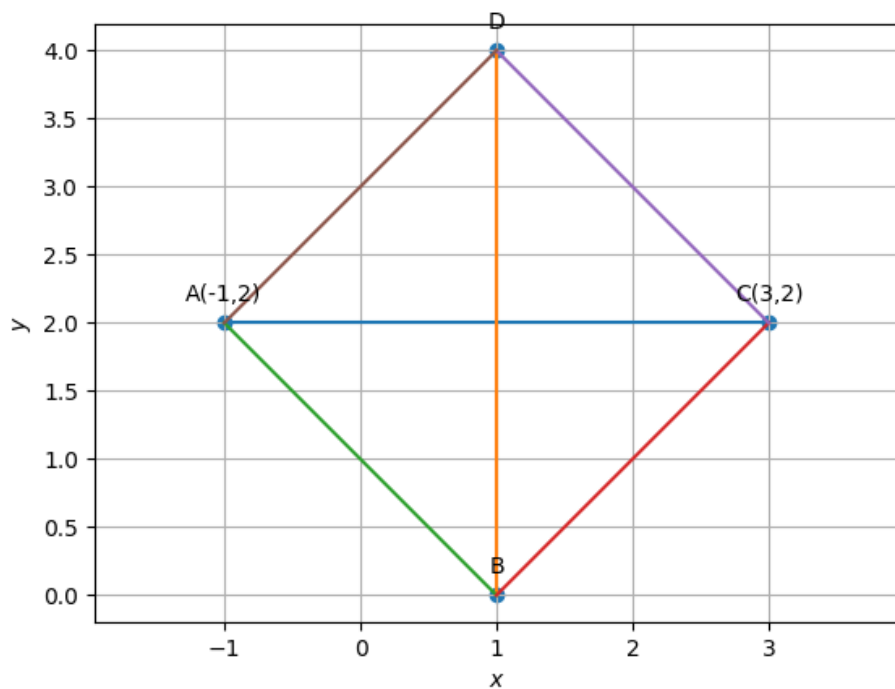


Figure 1:

Shifting \mathbf{A} to origin with reference to Figure 2

$$\mathbf{A}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C}' = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (2)$$

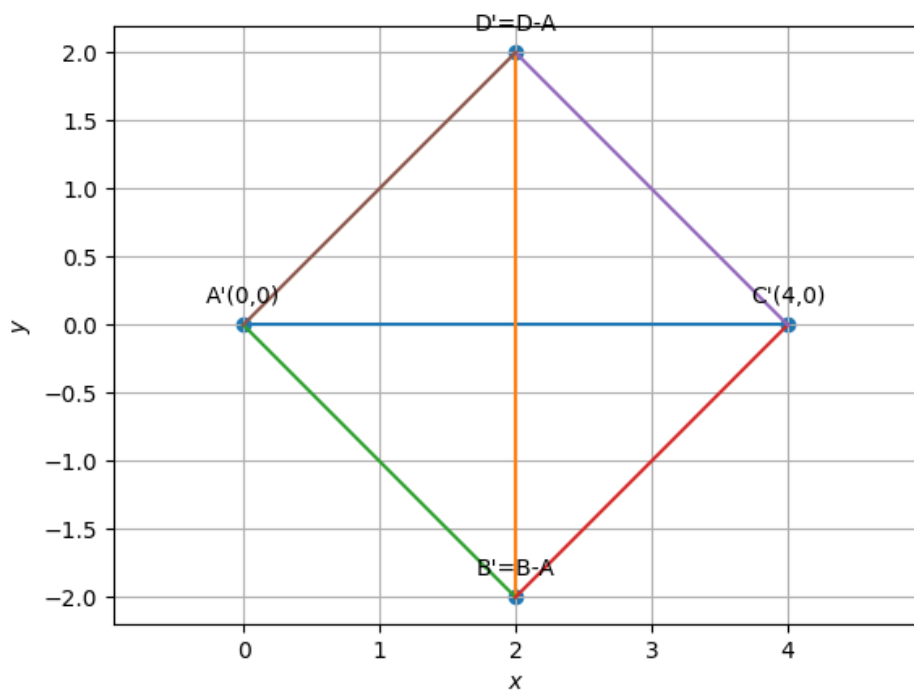


Figure 2:

For a generalised case, the angle made by AC on the x-axis will be $\beta = \theta + 45^\circ$, where θ is the angle made by AB on the x-axis. Hence, we can calculate the angle made by AB on the x-axis.

direction vector is given as

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3)$$

However in this particular case, angle made by AC on the x-axis

$$\tan \beta = \frac{0}{4} \implies \sin \beta = \frac{0}{4} \quad (4)$$

$$\beta = 0^\circ \quad (5)$$

Therefore, here $\theta = -45^\circ$

We know the rotation matrix for CW rotation is given as

$$\mathbf{P}^\top = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (6)$$

Since the angle here is negative so ultimately the coordinates will be rotated in ACW direction. Now the transformed(rotated) coordinates are with refrence to Figure 3.

$$\mathbf{C}'' = \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{pmatrix} \quad (7)$$

$$\mathbf{B}'' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{C}'' = \begin{pmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix}, \mathbf{D}'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{C}'' = \begin{pmatrix} 0 \\ \frac{4}{\sqrt{2}} \end{pmatrix} \text{ and } \mathbf{A}'' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (8)$$

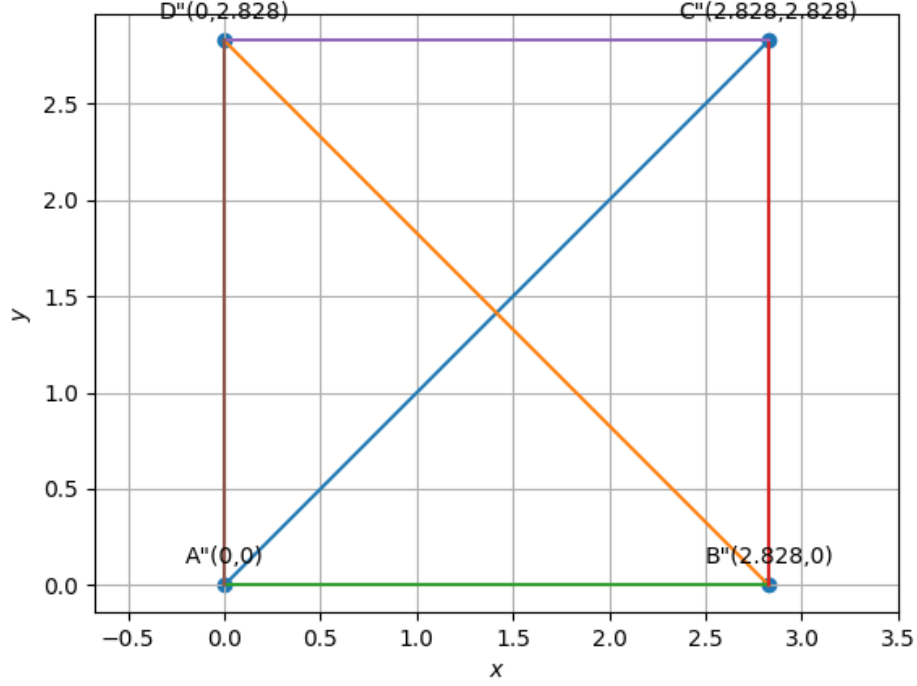


Figure 3:

Again transforming(rotating) the coordinates back to the original axis.
We know for anti-clockwise direction the rotation matrix is given as

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (9)$$

Again we know that the angle is negative so the rotation will be in clock-wise direction. So now the transformed(rotated) coordinates \mathbf{B} and \mathbf{D} are with reference to Figure 4

$$\mathbf{B}' = \mathbf{P}\mathbf{B}'' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad (10)$$

$$\mathbf{D}' = \mathbf{P}\mathbf{D}'' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{4}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (11)$$

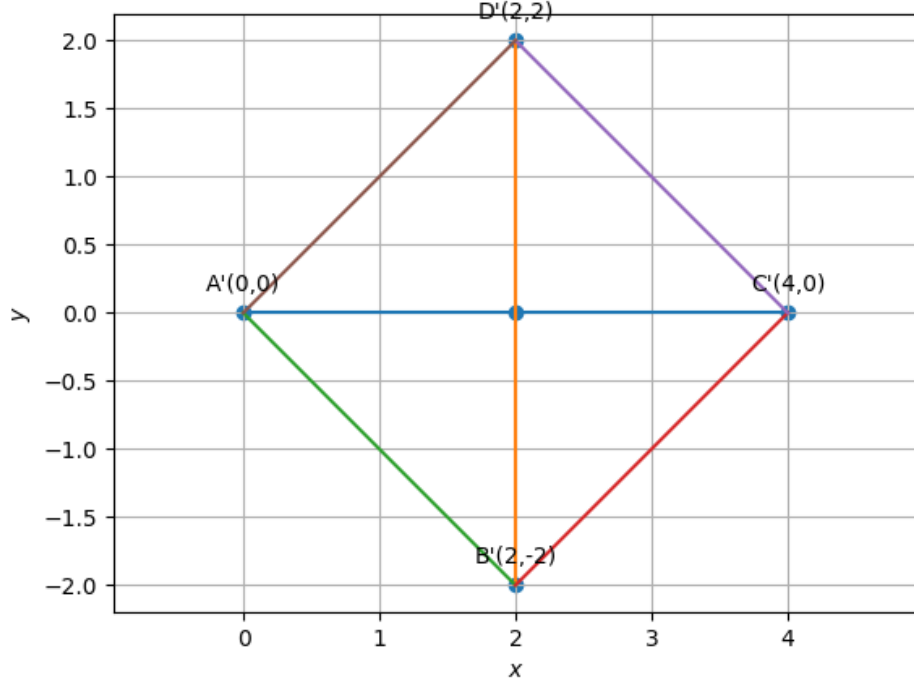


Figure 4:

Again transforming(shifting) the axis back to the original with reference to Figure 5

$$\mathbf{B} = \mathbf{B}' + \mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (12)$$

$$\mathbf{D} = \mathbf{D}' + \mathbf{A} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad (13)$$

Hence, the other two vertices are $\mathbf{B}(1, 0)$ and $\mathbf{D}(1, 4)$

The direct formula for calculation of the vertices is:

$$\mathbf{B} = \mathbf{A} + \mathbf{P} \begin{pmatrix} \mathbf{e}_1 & 0 \end{pmatrix} \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (14)$$

$$\mathbf{D} = \mathbf{A} + \mathbf{P} \begin{pmatrix} 0 & \mathbf{e}_2 \end{pmatrix} \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (15)$$

$$(16)$$

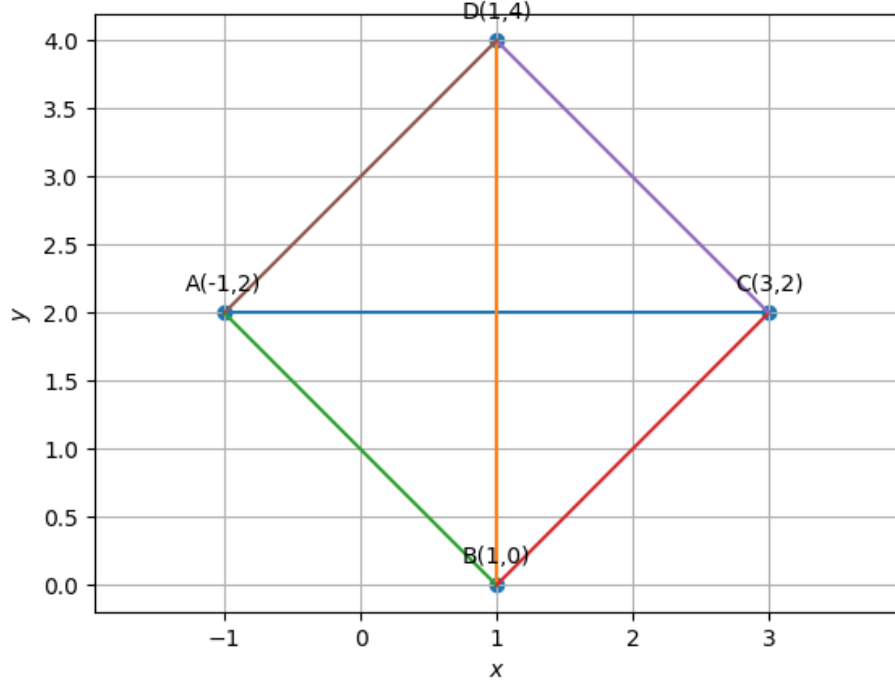


Figure 5:

where \mathbf{P} is the rotation matrix and \mathbf{A} and \mathbf{C} are the position vectors of opposite vertices.

Derivation of the above formulas:

We know that after shifting the axis and rotating by the required angle any arbitrary square will be aligned with the x and y axis so that we can directly get the vectors \mathbf{B} and \mathbf{D} as follows

$$\mathbf{C}'' = \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (17)$$

$$\mathbf{B}'' = (\mathbf{e}_1 \ 0) \mathbf{C}'' = (\mathbf{e}_1 \ 0) \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (18)$$

$$\mathbf{B}' = \mathbf{P} \mathbf{B}'' = \mathbf{P} (\mathbf{e}_1 \ 0) \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (19)$$

$$\mathbf{B} = \mathbf{A} + \mathbf{B}' \quad (20)$$

$$\mathbf{B} = \mathbf{A} + \mathbf{P} (\mathbf{e}_1 \ 0) \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (21)$$

Similarly for D it can be derived as

$$\mathbf{C}'' = \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (22)$$

$$\mathbf{D}'' = (\mathbf{0} \ \mathbf{e}_2) \mathbf{C}'' = (\mathbf{0} \ \mathbf{e}_2) \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (23)$$

$$\mathbf{D}' = \mathbf{P} \mathbf{D}'' = \mathbf{P} (\mathbf{0} \ \mathbf{e}_2) \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (24)$$

$$\mathbf{D} = \mathbf{A} + \mathbf{D}' \quad (25)$$

$$\mathbf{D} = \mathbf{A} + \mathbf{P} (\mathbf{0} \ \mathbf{e}_2) \mathbf{P}^\top (\mathbf{C} - \mathbf{A}) \quad (26)$$

Verification of the above formula for the given question

$$\mathbf{B} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (27)$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{pmatrix} \quad (28)$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix} \quad (29)$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad (30)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (31)$$

$$\mathbf{D} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (32)$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{pmatrix} \quad (33)$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{4}{\sqrt{2}} \end{pmatrix} \quad (34)$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (35)$$

$$= \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad (36)$$