Excercise 6.6

Q4. Find the equation of normal to the curve $x^2 = 4y$ which passes through the point (4,-2)
Solution: The given equation of the curve can be written as

$$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{3}$$

$$f = 0 \tag{4}$$

We are given that

$$\mathbf{h} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{5}$$

This can be formulated as optimization problem as below:

$$\min_{\mathbf{x}} \quad \mathrm{f}(\mathbf{x}) = \|\mathbf{x} - \mathbf{h}\|^2 \tag{6}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{7}$$

It is already proved that the optimization problem is non-convex. However, by relaxing the constraint in (7) as

$$g(\mathbf{x}) = \mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f \le 0 \tag{8}$$

the optimization problem can be made convex. We will use Lagrange multipliers method to find the optimum value. Define

$$H(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{9}$$

and we find that

$$\nabla f(\mathbf{x}) = 2(\mathbf{x} - \mathbf{h}) \tag{10}$$

$$\nabla g(\mathbf{x}) = 2(\mathbf{V}\mathbf{x} + \mathbf{u}) \tag{11}$$

We have to find $\lambda \in \mathbb{R}$ such that

$$
\begin{aligned}
&\nabla H\left(\mathbf{x}, \lambda\right) = 0 && (12)\\
&\implies 2\left(\mathbf{x} - \mathbf{h}\right) - 2\lambda\left(\mathbf{V}\mathbf{x} + \mathbf{u}\right) = 0 && (13)\\
&\implies \mathbf{x} - \mathbf{h} = \lambda\left(\mathbf{V}\mathbf{x} + \mathbf{u}\right) && (14)\\
&\implies \left(\mathbf{I} - \lambda\mathbf{V}\right)\mathbf{x} = \lambda\mathbf{u} + \mathbf{h} && (15)\\
&\implies \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \lambda\begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} && (16)\\
&\implies \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 4 \\ -2\lambda - 2 \end{pmatrix} && (17)
\end{aligned}
$$

Writing augmented matrix,

$$
\begin{pmatrix} 1-\lambda & 0 & 4 \\ 0 & 1 & -2\lambda - 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{1-\lambda}} \begin{pmatrix} 1 & 0 & \frac{4}{1-\lambda} \\ 0 & 1 & -2\lambda - 2 \end{pmatrix} \tag{18}
$$

Then, we get

$$
\mathbf{x}_m = \begin{pmatrix} \frac{4}{1-\lambda} \\ -2\lambda - 2 \end{pmatrix} \tag{19}
$$

Substituting this value in (7)

$$
\begin{aligned}
\begin{pmatrix} \frac{4}{1-\lambda} & -2-2\lambda \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{4}{1-\lambda} \\ -2-2\lambda \end{pmatrix} + 2\begin{pmatrix} 0 & -2 \end{pmatrix}\begin{pmatrix} \frac{4}{1-\lambda} \\ -2-2\lambda \end{pmatrix} &= 0 && (20)\\
\frac{16}{\left(1-\lambda\right)^2} + 8\left(\lambda + 1\right) &= 0 && (21)\\
\lambda^3 - \lambda^2 - \lambda + 3 &= 0 && (22)\\
\implies \lambda &= -1.3593 && (23)
\end{aligned}
$$

Substituting the value of λ in (19)

$$
\mathbf{x}_m = \begin{pmatrix} 1.695 \\ 0.718 \end{pmatrix} \tag{24}
$$

This result is same as we obtained using gradient descent. Hence, it is the point of normal.