Excercise 10.2

Q2.In fig 1, if TP and TQ are two tangents to a circle with centre O so that $\angle POQ = 110°$ then $\angle PTQ$ is equal to.

Solution: Let $\mathbf{T} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let $\mathbf{O}$ be the centre of the circle such that TP and TQ are tangents to the circle from $\mathbf{T}$.

Input Parameters	Value
$\mathbf{T}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
TO	1.743 cm
radius	1 cm
$\mathbf{O}$	$\begin{pmatrix} 1.743 \\ 0 \end{pmatrix}$

Table 1:

We have to find points $\mathbf{P}$ and $\mathbf{Q}$. We know that the equation of circle is given as

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top \mathbf{u} + f = 0 \tag{1}$$

where,

$$\mathbf{u} = -\mathbf{O} = -\begin{pmatrix} 1.743 \\ 0 \end{pmatrix} \tag{2}$$

$$f = \|\mathbf{O}\|^2 - r^2 = 2.039 \tag{3}$$

$$\boldsymbol{\Sigma} = (\mathbf{T} + \mathbf{u})(\mathbf{T} + \mathbf{u})^\top - \left(\|\mathbf{T}\|^2 + 2\mathbf{u}^\top \mathbf{T} + f\right)\mathbf{I} \tag{4}$$

$$= \begin{pmatrix} 3.308 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2.039 & 0 \\ 0 & 2.039 \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -2.039 \end{pmatrix} \tag{6}$$

From (6), we can deduce Eigen pairs as follows

$$\lambda_1 = 1, \lambda_2 = -2.039 \tag{7}$$

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8}$$

Then

$$\mathbf{n}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 1 \\ 1.427 \end{pmatrix} \tag{9}$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 1 \\ -1.427 \end{pmatrix} \tag{10}$$

The points of contact of a tangent on a circle from an external point is given by

$$\mathbf{q}_{ij} = \left(\pm r \frac{\mathbf{n}_j}{\|\mathbf{n}_j\|} - \mathbf{u} \right) \quad \text{i,j} = 1, 2 \tag{11}$$

$$\mathbf{q}_{i1} = \left(\pm r \frac{\mathbf{n}_1}{\|\mathbf{n}_1\|} - \mathbf{u} \right) \tag{12}$$

$$= \left(\pm \frac{1}{1.742} \begin{pmatrix} 1 \\ 1.427 \end{pmatrix} + \begin{pmatrix} 1.742 \\ 0 \end{pmatrix} \right) \tag{13}$$

$$= \begin{pmatrix} 2.316 \\ 0.8191 \end{pmatrix}, \begin{pmatrix} 1.168 \\ -0.8191 \end{pmatrix} \tag{14}$$

$$\mathbf{q}_{i2} = \left(\pm r \frac{\mathbf{n}_2}{\|\mathbf{n}_2\|} - \mathbf{u} \right) \tag{15}$$

$$= \left(\pm \frac{1}{1.742} \begin{pmatrix} 1 \\ -1.427 \end{pmatrix} + \begin{pmatrix} 1.742 \\ 0 \end{pmatrix} \right) \tag{16}$$

$$= \begin{pmatrix} 2.316 \\ -0.8191 \end{pmatrix}, \begin{pmatrix} 1.168 \\ 0.8191 \end{pmatrix} \tag{17}$$

$$\text{Hence } \mathbf{P} = \mathbf{q}_{22} = \begin{pmatrix} 1.168 \\ 0.8191 \end{pmatrix} \tag{18}$$

$$\mathbf{Q} = \mathbf{q}_{12} = \begin{pmatrix} 1.168 \\ -0.8191 \end{pmatrix} \tag{19}$$

Now for calculating the $\angle PTQ$, calculating the normal vectors

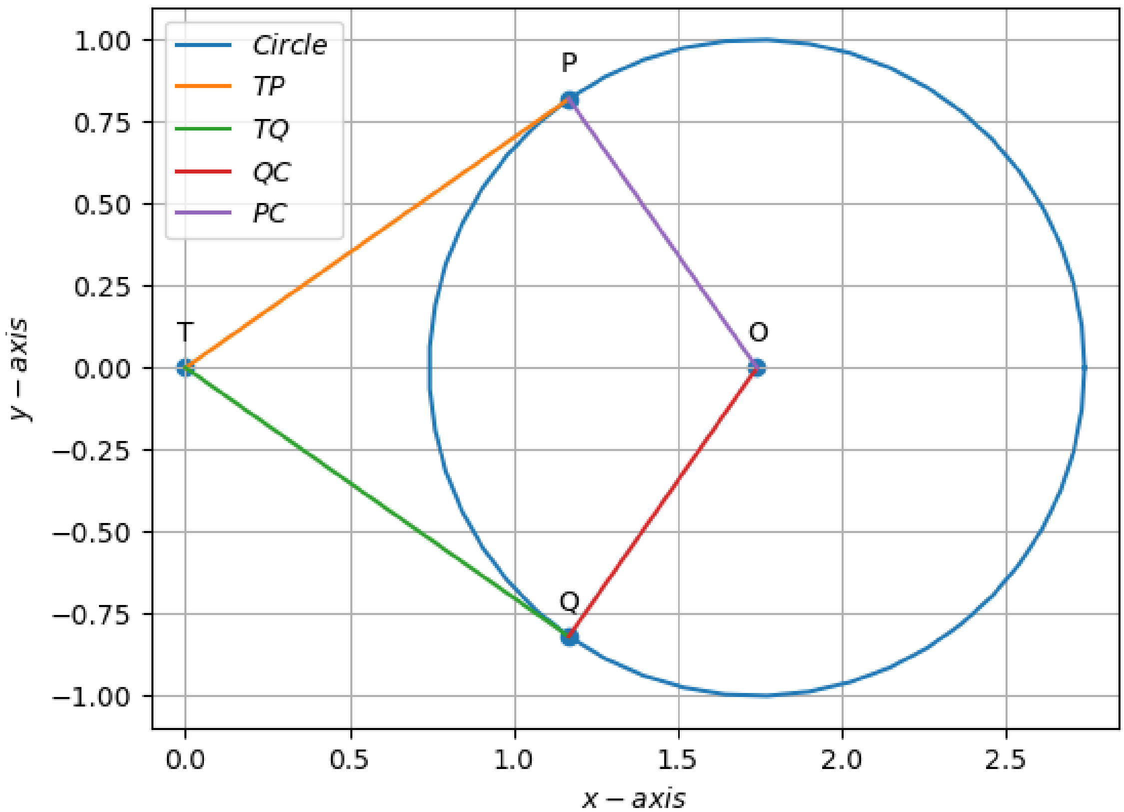

Figure 1:

$$\mathbf{n}_1 = \mathbf{P} - \mathbf{O} = \begin{pmatrix} -0.575 \\ 0.8191 \end{pmatrix} = \begin{pmatrix} 1 \\ -1.424 \end{pmatrix} \tag{20}$$

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0.7019 \end{pmatrix} \tag{21}$$

$$\mathbf{n}_2 = \mathbf{Q} - \mathbf{O} = \begin{pmatrix} -0.575 \\ -0.8191 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.424 \end{pmatrix} \tag{22}$$

$$\mathbf{m}_2 = \begin{pmatrix} 1 \\ -0.7019 \end{pmatrix} \tag{23}$$

Now the angle betweeen two lines with slope $\mathbf{m}_1$ and $\mathbf{m}_2$ is given as

$$\cos\theta = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\left\|\mathbf{m}_1\right\| \left\|\mathbf{m}_2\right\|} \tag{24}$$

$$= \frac{\begin{pmatrix} 1 & 0.7019 \end{pmatrix} \begin{pmatrix} 1 \\ -0.7019 \end{pmatrix}}{(1.221)^2} \tag{25}$$

$$\implies \theta = 70^\circ \tag{26}$$

Hence, $\angle PTQ = 70^\circ$