Excercise 8.2

Q2.Find the area bounded by the curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$.
Solution: The general equation of a conic is given as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

The first curve equation can be rearranged as

$$x^2+y^2-2x=0 \tag{2}$$

Comparing (1) and (2) we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{4}$$

$$f = 0 \tag{5}$$

The second curve equation can be rearranged as

$$x^2+y^2-1=0 \tag{6}$$

Comparing (1) and (6) we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8}$$

$$f = -1 \tag{9}$$

The intersection of conics is obtained as

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu\mathbf{V}_2)\mathbf{x} + 2(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2) = 0 \tag{10}$$

The locus of the intersection is a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2) & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{11}$$

On substituting values we get

$$\begin{vmatrix} 1+\mu & 0 & -1 \\ 0 & 1+\mu & 0 \\ -1 & 0 & -\mu \end{vmatrix} = 0 \tag{12}$$

solving the determinant we get

$$\mu^3 + 2\mu^2 + 2\mu + 1 = 0 \tag{13}$$
$$\implies \mu = -1 \tag{14}$$

Thus, the parametrs for straight line cann be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{15}$$
$$\mathbf{u} = \mathbf{u}_1 + \mu\mathbf{u}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{16}$$
$$f = 1 \tag{17}$$

Substituting these values we get

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{18}$$
$$\begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} = -1 \tag{19}$$

Therefore

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{h} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \tag{20}$$

Now intersection of line with a conic is given by

$$\mathbf{x}_i = \mathbf{h} + \mu_i\mathbf{m} \tag{21}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2 - g\left(\mathbf{h}\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}\right) \tag{22}$$

Now

$$g(\mathbf{h}) = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} - 1 \tag{23}$$

$$= -\frac{3}{4} \tag{24}$$

$$\mathbf{m}^\top \mathbf{V} \mathbf{m} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \tag{25}$$

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{26}$$

substituting in (22) we get

$$\mu_i = \pm \frac{\sqrt{3}}{2} \tag{27}$$

Hence the point of intersection are

$$\mathbf{a}_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{28}$$

The desired area of region is given as

$$= 2 \left(\int_0^{\frac{1}{2}} \sqrt{1 - (x-1)^2} dx + \int_{\frac{1}{2}}^{1} \sqrt{1 - x^2} dx \right) \tag{29}$$

$$\begin{aligned} &= 2 \left[\frac{1}{2} (x-1) \sqrt{1 - (x-1)^2} + \frac{1}{2} \sin^{-1} (x-1) \right]_0^{\frac{1}{2}} \\ &+ 2 \left[\frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x \right]_{\frac{1}{2}}^{1} \end{aligned} \tag{30}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \tag{31}$$

See figure 1

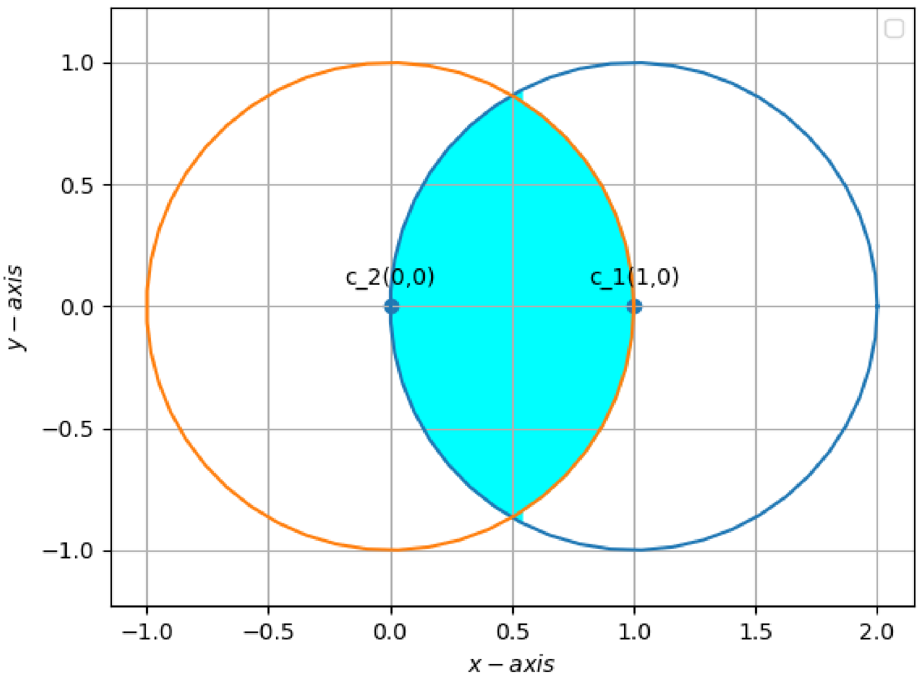

Figure 1: