Excercise 10.2

Q2.In fig 1, if TP and TQ are two tangents to a circle with centre O so that $\angle POQ = 110°$ then $\angle PTQ$ is equal to.

Solution: Let the output angle be ϕ. The input parameters are given as

Input Parameters	Value	Description
O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre of the circle
r	1cm	radius of the circle
θ	$110°$	$\angle POQ$

Table 1:

Any point **X** on the circle is given as

$$\mathbf{X} = \mathbf{O} + r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{1}$$

So points **P** and **Q** can be calculated as

$$\mathbf{P} = \mathbf{O} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2}$$

$$\mathbf{Q} = \mathbf{e}_1 \tag{3}$$

For tangent TP

$$\mathbf{n}_1 = \mathbf{P} - \mathbf{O} \tag{4}$$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} \tag{5}$$

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ -\cot\theta \end{pmatrix} \tag{6}$$

For tangent TQ

$$\mathbf{n}_2 = \mathbf{e}_1 - \mathbf{O} \tag{7}$$

$$= \mathbf{e}_1 \tag{8}$$

$$\mathbf{m}_2 = \mathbf{e}_2 \tag{9}$$

The equation of TP is given as

$$\mathbf{n}_1^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{10}$$

$$\mathbf{n}_1^\top \left(\mathbf{x} - \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right) = 0 \tag{11}$$

$$\begin{pmatrix} \cos \theta & \sin \theta \end{pmatrix} \mathbf{x} = 1 \tag{12}$$

The equation of TQ is given as

$$\mathbf{n}_2^\top (\mathbf{x} - \mathbf{e}_1) = 0 \tag{13}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \tag{14}$$

The tangent point can be calculated by solving (12) and (14)

$$\begin{pmatrix} \cos \theta & \sin \theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{15}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \tan \frac{\theta}{2} \end{pmatrix} \tag{16}$$

Hence, the tangent point is

$$\mathbf{T} = \begin{pmatrix} 1 \\ \tan 55° \end{pmatrix} = \begin{pmatrix} 1 \\ 1.428 \end{pmatrix} \tag{17}$$

The angle between two lines with slope $\mathbf{m}_1$ and $\mathbf{m}_2$ s given as

$$\cos \phi = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \, \|\mathbf{m}_2\|} \tag{18}$$

$$= \frac{\begin{pmatrix} 1 & -\cot \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(\csc \theta)(1)} \tag{19}$$

$$= -\cos \theta \tag{20}$$

$$\implies \cos \phi = -\cos \theta \tag{21}$$

Hence,

$$\phi = \cos^{-1} \left(\cos \left(180° - \theta \right) \right) \tag{22}$$

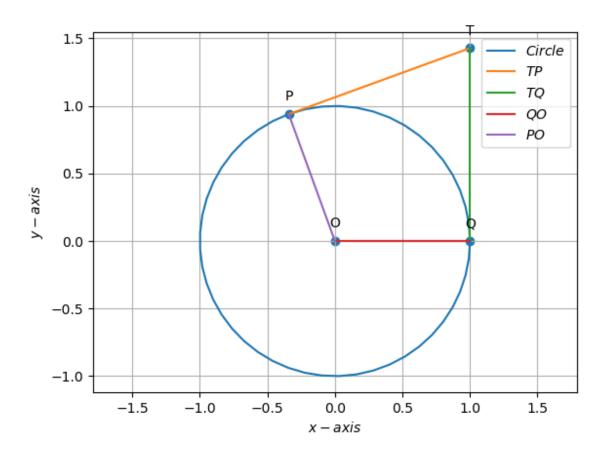$$= 180° - \theta = 70° \tag{23}$$

Hence, $\angle PTQ = 70°$. See Fig 1

Figure 1: