

CONIC SECTIONS

Exercise 11.5

Q8. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Solution: The given equation of the parabola can be rearranged as

$$y^2 - 4ax = 0 \quad (1)$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (2)$$

Comparing the coefficients of both equations (1) and (2)

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3)$$

$$\mathbf{u} = - \begin{pmatrix} -2a \\ 0 \end{pmatrix} \quad (4)$$

$$f = 0 \quad (5)$$

Let length of side of triangle = r

Since, triangle is inscribed in the parabola and one of the vertex of the triangle lies at the vertex of the parabola, the other vertices are given as

$$\mathbf{P} = \begin{pmatrix} r \cos 30^\circ \\ r \sin 30^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}r \\ \frac{1}{2}r \end{pmatrix} \quad (6)$$

$$\mathbf{Q} = \begin{pmatrix} r \cos 30^\circ \\ -r \sin 30^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}r \\ -\frac{1}{2}r \end{pmatrix} \quad (7)$$

Now these point will satisfy the equation of the parabola. So substituting

(6) in (2)

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} + 2\mathbf{u}^\top \mathbf{P} + f = 0 \quad (8)$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2}r & \frac{1}{2}r \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2}r \\ \frac{1}{2}r \end{pmatrix} + 2 \begin{pmatrix} -2a & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2}r \\ \frac{1}{2}r \end{pmatrix} = 0 \quad (9)$$

$$\frac{1}{4}r^2 - 2a\sqrt{3}r = 0 \quad (10)$$

$$r \left(\frac{r}{4} - 2a\sqrt{3} \right) = 0 \quad (11)$$

Hence, the length of side of triangle is

$$r = 8a\sqrt{3} \quad (12)$$

The result will be the same for substituting \mathbf{Q} . The corresponding is shown in Figure 1.

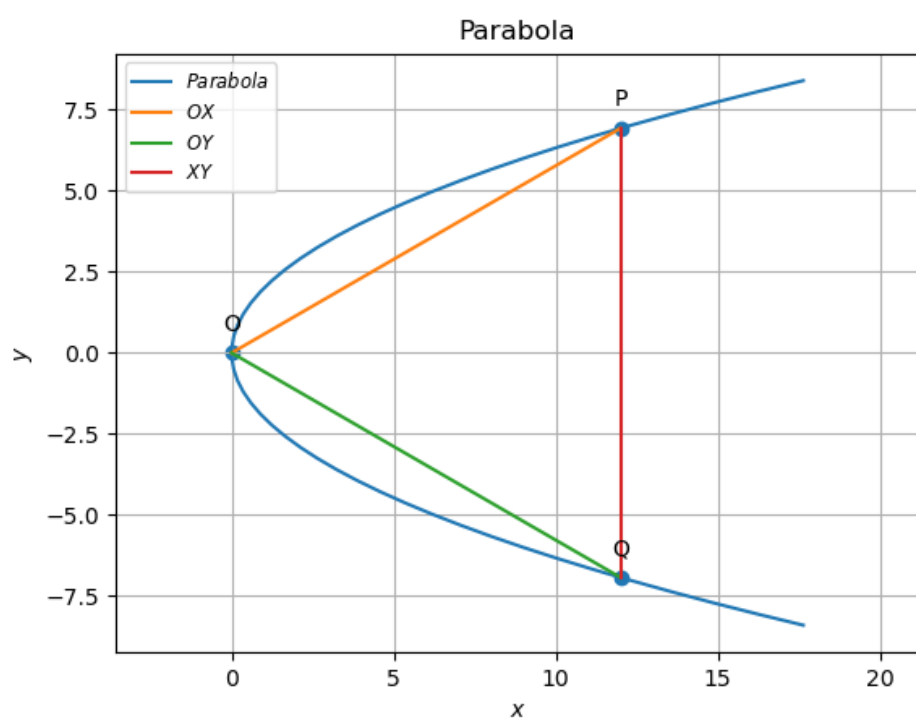


Figure 1: