Excercise 10.2

Q2.In fig 1, if TP and TQ are two tangents to a circle with centre O so that $\angle POQ = 110°$ then $\angle PTQ$ is equal to.

Solution: Let the output angle be ϕ. The input parameters are given as

Input Parameters	Value	Description
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre of the circle
r	1cm	radius of the circle
θ	110°	$\angle POQ$

Table 1:

Any point $\mathbf{X}$ on the circle is given as

$$\mathbf{X} = \mathbf{O} + r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{1}$$

So points $\mathbf{P}$ and $\mathbf{Q}$ can be calculated as

$$\mathbf{P} = \mathbf{O} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2}$$

$$\mathbf{Q} = \mathbf{e}_1 \tag{3}$$

For tangent TP

$$\mathbf{n}_1 = \mathbf{P} - \mathbf{O} \tag{4}$$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} \tag{5}$$

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ -\cot\theta \end{pmatrix} \tag{6}$$

For tangent TQ

$$\mathbf{n}_2 = \mathbf{e}_1 - \mathbf{O} \tag{7}$$

$$= \mathbf{e}_1 \tag{8}$$

$$\mathbf{m}_2 = \mathbf{e}_2 \tag{9}$$

The equation of TP is given as

$$\mathbf{n}_1^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{10}$$

$$\mathbf{n}_1^\top \left(\mathbf{x} - \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right) = 0 \tag{11}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \mathbf{x} = 1 \tag{12}$$

The equation of TQ is given as

$$\mathbf{n}_2^\top (\mathbf{x} - \mathbf{e}_1) = 0 \tag{13}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \tag{14}$$

The tangent point can be calculated by solving (31) and (14)

$$\begin{pmatrix} \cos\theta & \sin\theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{15}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \tan\frac{\theta}{2} \end{pmatrix} \tag{16}$$

Now, $\mathbf{T}$ =(16), since it is the intersection of TP and TQ. Hence, it is given as

$$\mathbf{T} = \begin{pmatrix} 1 \\ \tan 55^\circ \end{pmatrix} = \begin{pmatrix} 1 \\ 1.428 \end{pmatrix} \tag{17}$$

The angle between two lines with slope $\mathbf{m}_1$ and $\mathbf{m}_2$ is given as

$$\cos\phi = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \, \|\mathbf{m}_2\|} \tag{18}$$

$$= \frac{\begin{pmatrix} 1 & -\cot\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(\csc\theta)(1)} \tag{19}$$

$$= -\cos\theta \tag{20}$$

$$\implies \cos\phi = -\cos\theta \tag{21}$$

Hence,

$$\phi = \cos^{-1}\left(\cos\left(180^\circ - \theta\right)\right) \tag{22}$$

$$= 180^\circ - \theta = 70^\circ \tag{23}$$

Hence, $\angle PTQ = 70^\circ$. See Fig 1

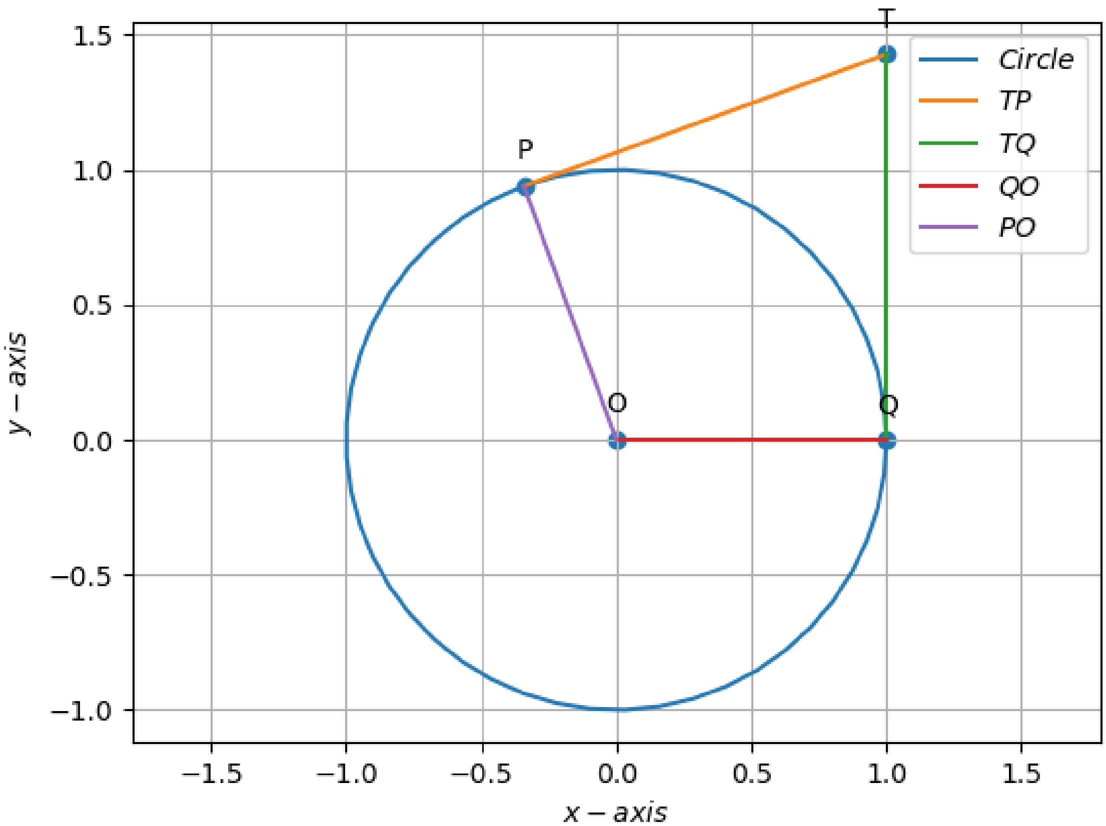

Figure 1:

Now considering the tangent point is known and verifying that the point of contacts are actually $\mathbf{P}$ and $\mathbf{Q}$.Given

$$\mathbf{T} = \begin{pmatrix} 1 \\ 1.428 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{24}$$

Transforming(rotating) the tangent point using clockwise rotation matrix by $\theta = 55°$

$$\mathbf{T}' = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 1.428 \end{pmatrix} = \begin{pmatrix} 1.743 \\ 0 \end{pmatrix} \tag{25}$$

We know that the equation of circle is given as

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top \mathbf{u} + f = 0 \tag{26}$$

where,

$$\mathbf{u} = -\mathbf{O} = -\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{27}$$

$$f = \|\mathbf{O}\|^2 - r^2 = -1 \tag{28}$$

$$\mathbf{\Sigma} = (\mathbf{T}' + \mathbf{u})(\mathbf{T}' + \mathbf{u})^\top - \left(\|\mathbf{T}'\|^2 + 2\mathbf{u}^\top\mathbf{T}' + f\right)\mathbf{I} \tag{29}$$

$$= \begin{pmatrix} 3.308 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2.039 & 0 \\ 0 & 2.039 \end{pmatrix} \tag{30}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -2.039 \end{pmatrix} \tag{31}$$

From (31), we can deduce Eigen pairs as follows

$$\lambda_1 = 1, \lambda_2 = -2.039 \tag{32}$$

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{33}$$

Then

$$\mathbf{n}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 1 \\ 1.427 \end{pmatrix} \tag{34}$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 1 \\ -1.427 \end{pmatrix} \tag{35}$$

The points of contact of a tangent on a circle from an external point is given by

$$\mathbf{q}_{ij} = \left(\pm r\frac{\mathbf{n}_j}{\|\mathbf{n}_j\|} - \mathbf{u}\right) \text{ i,j} = 1, 2 \tag{36}$$

$$\mathbf{q}_{i1} = \left(\pm r\frac{\mathbf{n}_1}{\|\mathbf{n}_1\|} - \mathbf{u}\right) \tag{37}$$

$$= \left(\pm\frac{1}{1.742}\begin{pmatrix}1\\1.427\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}\right) \tag{38}$$

$$= \begin{pmatrix}0.574\\0.819\end{pmatrix} \tag{39}$$

$$\mathbf{q}_{i2} = \left(\pm r\frac{\mathbf{n}_2}{\|\mathbf{n}_2\|} - \mathbf{u}\right) \tag{40}$$

$$= \left(\pm\frac{1}{1.742}\begin{pmatrix}1\\-1.427\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}\right) \tag{41}$$

$$= \begin{pmatrix}0.574\\-0.8191\end{pmatrix} \tag{42}$$

Transforming(rotating) the coordinates back to the original using ACW rotation matrix

$$\mathbf{P} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}0.574\\0.819\end{pmatrix} = \begin{pmatrix}-0.342\\0.939\end{pmatrix} \tag{43}$$

$$\mathbf{Q} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}0.574\\-0.819\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \tag{44}$$

From (43) and (44) we see that the tangent point of contact are same as given before. Hence, the result is verified.