Excercise 11.2

Q2.Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum of a parabola whose equation is given by $x^2 = 6y$.

Solution: The given equation of the parabola can be rearranged as

$$x^2 - 6y = 0 \tag{1}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

Comparing the coefficients of both equations (1) and (2)

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{4}$$

$$f = 0 \tag{5}$$

1. From equation (3), since $\mathbf{V}$ is already diagonalized, the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 0 \tag{6}$$

$$\lambda_2 = 1 \tag{7}$$

The Eigen vector $\mathbf{p}_1$ corresponding to Eigen value λ_1 is computed as shown below

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{9}$$

We get

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10}$$

x_2 is free variable and $x_1 = 0$,So

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1 \tag{12}$$

$$= \sqrt{1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{14}$$

Now,

$$c = \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}} \tag{15}$$

Substituting values of $\mathbf{u}, \mathbf{n}, \lambda_2$ and f in (15)

$$c = \frac{3^2 - 1\,(0)}{-2\begin{pmatrix} 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = -\frac{3}{2} \tag{16}$$

The focus $\mathbf{F}$ of parabola is expressed as

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{17}$$

$$= \frac{-\frac{3}{2}\,(1)^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}}{1} \tag{18}$$

$$= \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} \tag{19}$$

2. Equation of directrix is given as

$$\mathbf{n}^\top \mathbf{x} = c \tag{20}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = -\frac{3}{2} \tag{21}$$

3. The equation for the axis of parabola passing through $\mathbf{F}$ and orthogonal to the directrix is given as

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{22}$$

where $\mathbf{m}$ is the normal vector to the axis and also the slope of the directrix. Now since

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{23}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{24}$$

Substituting in (22)

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} \right) = 0 \tag{25}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{26}$$

4. The latus rectum of a parabola is given by

$$l = \frac{\eta}{\lambda_2} \tag{27}$$

$$= \frac{2\mathbf{u}^\top \mathbf{p}_1}{\lambda_2} \tag{28}$$

$$= \frac{2 \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{1} \tag{29}$$

$$= 6 \text{ units} \tag{30}$$

The relevant diagram is shown in Figure 1

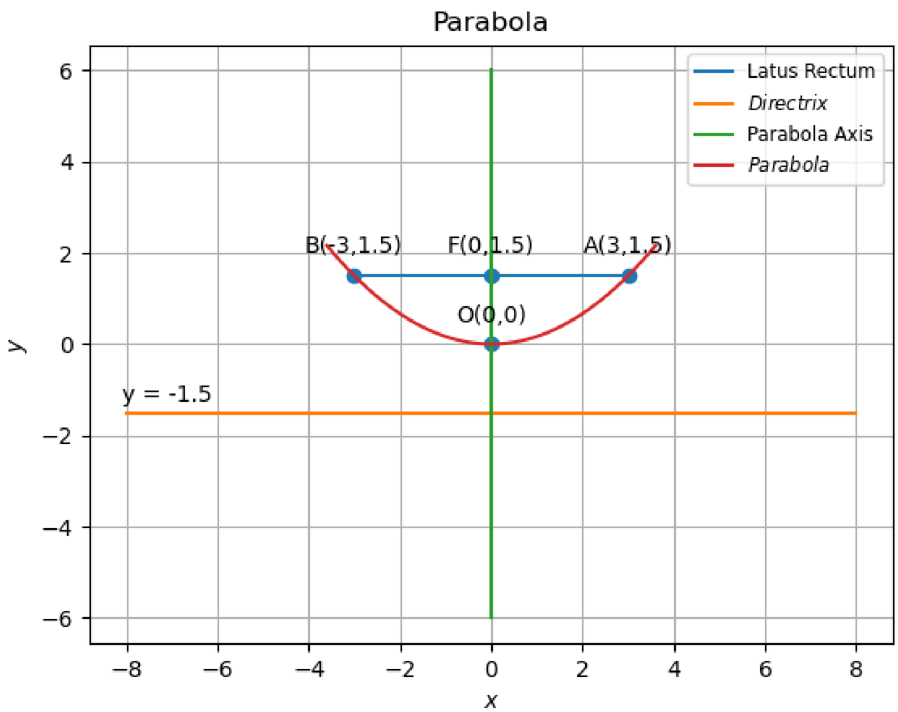
Parabola
Latus Rectum
Directrix
Parabola Axis
Parabola
B(-3,1.5)
F(0,1.5)
A(3,1.5)
O(0,0)
y = -1.5
6
4
2
0
−2
−4
−6
−8
−6
−4
−2
0
2
4
6
8
x
y

Figure 1: