

## CONIC SECTIONS

### Exercise 11.4

Q8. Find the equation of the hyperbola whose foci is  $(0, \pm 8)$  and vertices  $(0, \pm 5)$ .

**Solution:** Given

$$\mathbf{F} = \begin{pmatrix} 0 \\ \pm 8 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 0 \\ \pm 5 \end{pmatrix} \quad (1)$$

1. We know the vertex is given as

$$\mathbf{V} = \pm \begin{pmatrix} 0 \\ \sqrt{\frac{f_0}{\lambda_2}} \end{pmatrix} = \pm \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad (2)$$

$$\implies f_0 = 25\lambda_2 \quad (3)$$

2. We know the Focii is given as

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)(e^2)\sqrt{\frac{\lambda_1}{f_0}}}{\frac{\lambda_1}{f_0}} \mathbf{e}_2 \quad (4)$$

$$= \frac{\frac{e}{\sqrt{1-e^2}}}{\sqrt{\frac{\lambda_1}{f_0}}} \mathbf{e}_2 \quad (5)$$

Substituting (3) we get

$$\mathbf{F} = 5e\mathbf{e}_2 \quad (6)$$

$$\begin{pmatrix} 0 \\ 8 \end{pmatrix} = 5e\mathbf{e}_2 \quad (7)$$

$$\implies e = \frac{8}{5} \quad (8)$$

3. Now we know the eccentricity is given as

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} \quad (9)$$

$$\implies \frac{\lambda_2}{\lambda_1} = -\frac{39}{25} \quad (10)$$

4. Now we know from the standard equation

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (11)$$

Calculating  $\mathbf{n}$  and  $c$

$$\mathbf{n} = \sqrt{\frac{\lambda_1}{f_0}} \mathbf{e}_2 = \frac{1}{5} \sqrt{\frac{\lambda_1}{\lambda_2}} \mathbf{e}_2 \quad (12)$$

$$= \frac{1}{\sqrt{-39}} \mathbf{e}_2 \quad (13)$$

$$c = \frac{1}{e\sqrt{1-e^2}} = \frac{25}{8\sqrt{-39}} \quad (14)$$

Now

$$\|\mathbf{n}\|^2 = -\frac{1}{39} \quad (15)$$

$$\|\mathbf{F}\|^2 = 64 \quad (16)$$

Substituting all the values in (11) we get

$$f = -\left(\frac{1}{39}\right)(64) + \left(\frac{25}{8}\right)^2 \left(\frac{1}{39}\right) \left(\frac{64}{25}\right) \quad (17)$$

$$= -1 \quad (18)$$

$$f_0 = -f = 1 \quad (19)$$

substituting (19) in (3) we get

$$\lambda_2 = \frac{1}{25} \quad (20)$$

Substituting (20) in (10) we get

$$\lambda_1 = -\frac{1}{39} \quad (21)$$

Therefore the equation of the hyperbola is given as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (22)$$

where

$$\mathbf{V} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{39} & 0 \\ 0 & \frac{1}{25} \end{pmatrix} \quad (23)$$

$$\mathbf{u} = \mathbf{0} \quad (24)$$

$$f = -1 \quad (25)$$

The corresponding is shown in Figure 1

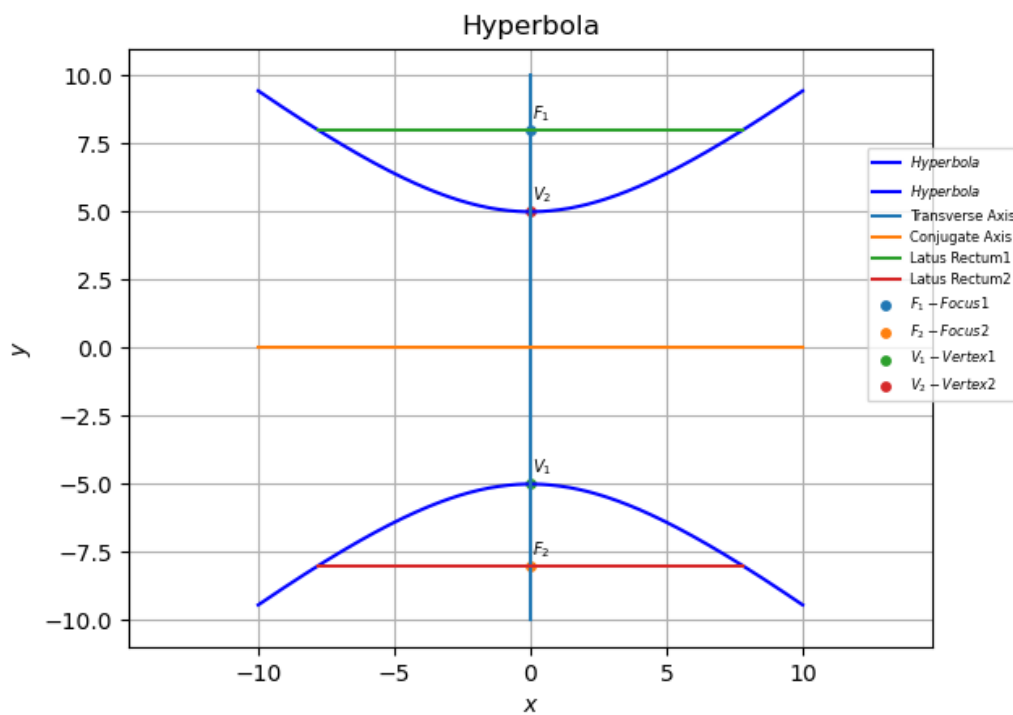


Figure 1: