Excercise 10.3

Q.3.2 Reduce the equation $y - 2 = 0$ into normal form. Find the perpendicular distances from the origin and angle between perpendicular and the positive x-axis.
Solution: The given equation can be written as

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1}$$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

Here, $\mathbf{A}$ is a point on the given line. We choose

$$\mathbf{A} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{5}$$

$$(4) \implies \mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6}$$

Let $\mathbf{O}$ be the origin. The perpendicular distance will be the minimum distance from $\mathbf{O}$ to the line. Let $\mathbf{P}$ be the foot of perpendicular. This problem can be formulated as an optimization problem as follows:

$$\min_{\mathbf{x}} \left\| \mathbf{x} - \mathbf{O} \right\|^2 \tag{7}$$

$$\implies \min_{\lambda} \left\| \mathbf{A} + \lambda \mathbf{m} - \mathbf{O} \right\|^2 \tag{8}$$

$$\implies \min_{\lambda} \left\| \mathbf{A} + \lambda \mathbf{m} \right\|^2 \tag{9}$$

$$\implies f(\lambda) = \left\| \mathbf{A} + \lambda \mathbf{m} \right\|^2 \tag{10}$$

$$= (\mathbf{A} + \lambda \mathbf{m})^\top (\mathbf{A} + \lambda \mathbf{m}) \tag{11}$$

$$= \left\| \mathbf{A} \right\|^2 + \mathbf{A}^\top (\lambda \mathbf{m}) + (\lambda \mathbf{m})^\top \mathbf{A} + (\lambda \mathbf{m})^\top (\lambda \mathbf{m}) \tag{12}$$

$$= \left\| \mathbf{A} \right\|^2 + \lambda \mathbf{A}^\top \mathbf{m} + \lambda \mathbf{m}^\top \mathbf{A} + \lambda^2 \left\| \mathbf{m} \right\|^2 \tag{13}$$

$$= \left\| \mathbf{m} \right\|^2 \lambda^2 + \left(\mathbf{A}^\top \mathbf{m} + \mathbf{m}^\top \mathbf{A} \right) \lambda + \left\| \mathbf{A} \right\|^2 \tag{14}$$

$$= \lambda^2 + 4\lambda + 8 \tag{15}$$

$\because$ the coefficient of $\lambda^2 > 0$, equation (15) is a convex function

$$f'(\lambda) = 2 \left\| \mathbf{m} \right\|^2 \lambda + \left(\mathbf{A}^\top \mathbf{m} + \mathbf{m}^\top \mathbf{A} \right) \tag{16}$$

1. Computing λ_{min} using Derative method:

$$f''(\lambda) = 2 \tag{17}$$
$$\because f''(\lambda) > 0, f'(\lambda_{min}) = 0, \text{ for } \lambda_{min} \tag{18}$$
$$f'(\lambda_{min}) = 2\left\|\mathbf{m}\right\|^2 \lambda_{min} + \left(\mathbf{A}^\top\mathbf{m} + \mathbf{m}^\top\mathbf{A}\right) \tag{19}$$
$$\therefore \lambda_{min} = -\frac{\left(\mathbf{A}^\top\mathbf{m} + \mathbf{m}^\top\mathbf{A}\right)}{2\left\|\mathbf{m}\right\|^2} = -2 \tag{20}$$

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + (-2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{21}$$
$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{22}$$
$$OP = \left\|\mathbf{P} - \mathbf{O}\right\| \tag{23}$$
$$= \left\|\begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\| \tag{24}$$
$$= 2 \tag{25}$$

2. Solving using cvxpy, with

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{26}$$
$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{27}$$
$$c = 2 \tag{28}$$
$$\min_{\mathbf{x}} \left\|\mathbf{x} - \mathbf{O}\right\|^2 = 2, \mathbf{x}_{min} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{29}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1}\left(\frac{2}{0}\right) \tag{30}$$
$$= 90^\circ \tag{31}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 90^\circ & \sin 90^\circ \end{pmatrix} \mathbf{x} = 0 \tag{32}$$

See figure 1 and figure 2

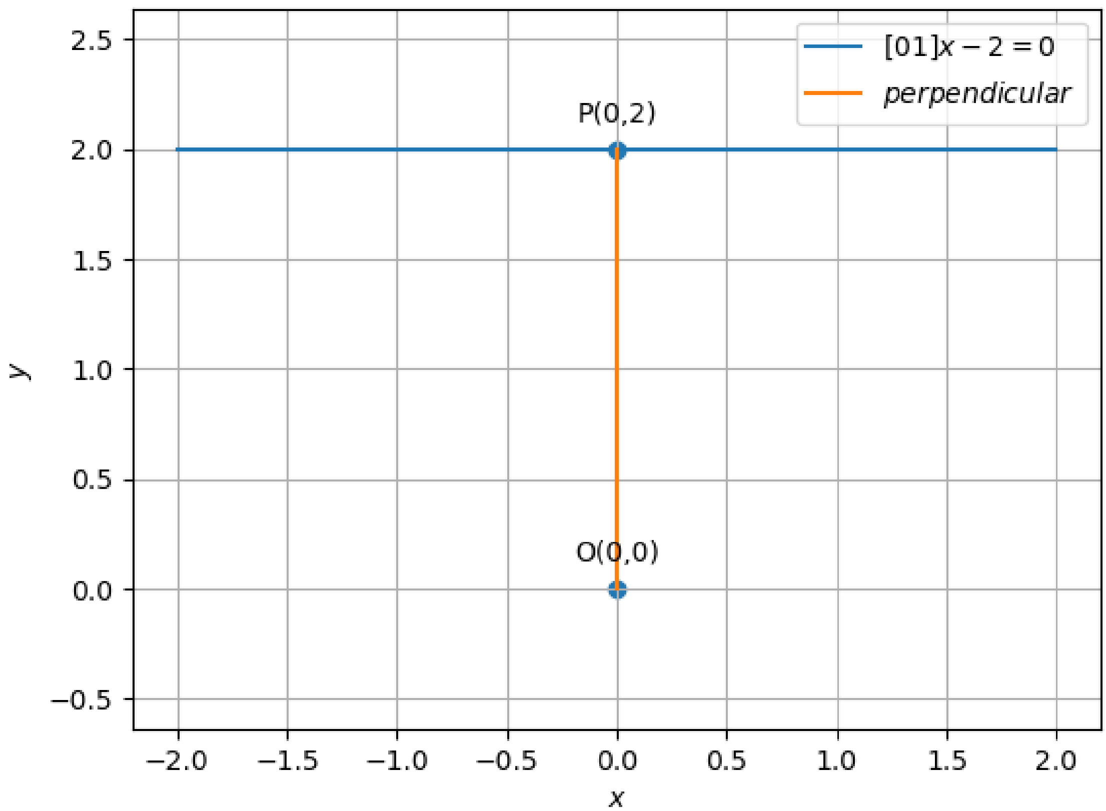
[01]x − 2 = 0
perpendicular
P(0,2)
O(0,0)
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x
y

Figure 1:

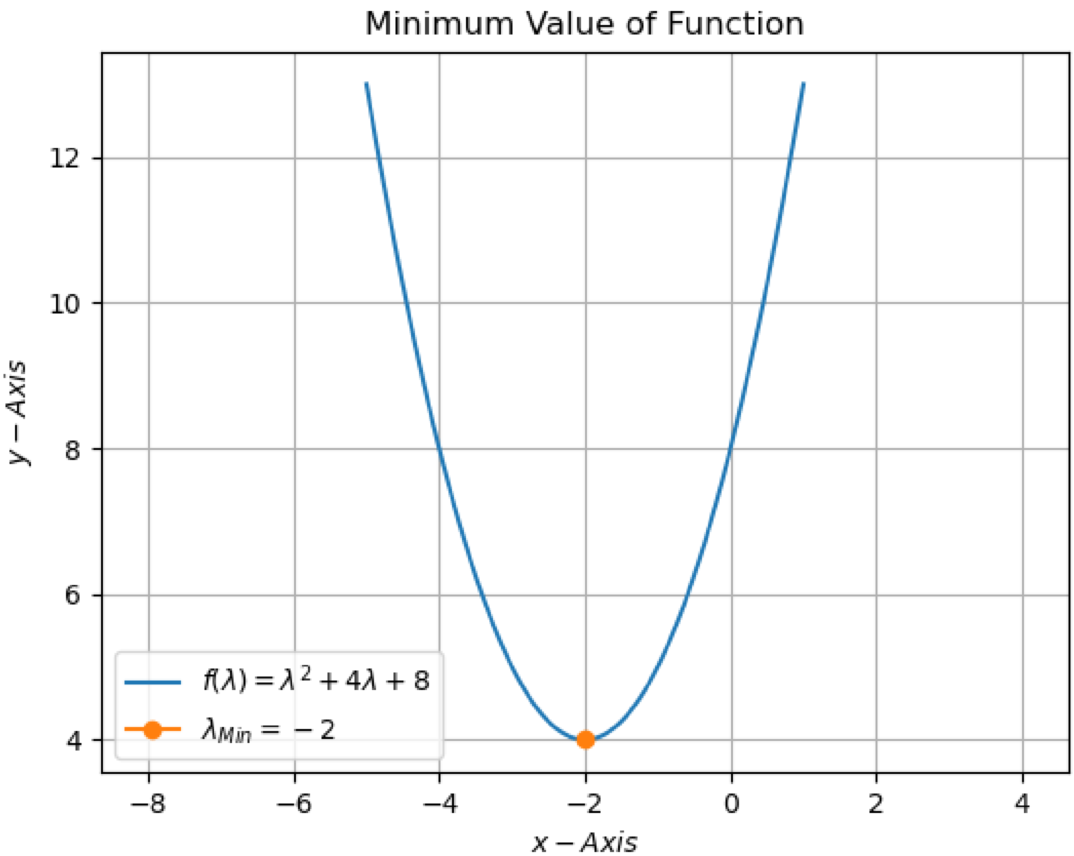
Minimum Value of Function
12
10
8
6
4
−8
−6
−4
−2
0
2
4
$y-Axis$
$x-Axis$
$f(\lambda) = \lambda^2 + 4\lambda + 8$
$\lambda_{Min} = -2$

Figure 2: