Excercise 10.3

Q.3.2 Reduce the equation $y-2=0$ into normal form. Find the perpendicular distances from the origin and angle between perpendicular and the positive x-axis.
Solution: The given equation can be written as

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 2 \tag{1}$$
$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$
$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{4}$$

Here, **A** is a point on the given line. We choose

$$\mathbf{A} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{5}$$
$$(4) \implies \mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6}$$

Let **O** be the origin. The perpendicular distance will be the minimum distance from **O** to the line. Let **P** be the foot of perpendicular. This problem can be formulated as an optimization problem as follows:

$$\min_{\mathbf{x}} \left\|\mathbf{x} - \mathbf{O}\right\|^2 \tag{7}$$
$$\implies \min_{\lambda} \left\|\mathbf{A} + \lambda\mathbf{m} - \mathbf{O}\right\|^2 \tag{8}$$
$$\implies \min_{\lambda} \left\|\mathbf{A} + \lambda\mathbf{m}\right\|^2 \tag{9}$$

$$\begin{aligned}
\implies f(\lambda) &= \left\|\mathbf{A} + \lambda\mathbf{m}\right\|^2 && (10)\\
&= \left(\mathbf{A} + \lambda\mathbf{m}\right)^\top\left(\mathbf{A} + \lambda\mathbf{m}\right) && (11)\\
&= \left\|\mathbf{A}\right\|^2 + \mathbf{A}^\top\left(\lambda\mathbf{m}\right) + \left(\lambda\mathbf{m}\right)^\top\mathbf{A} + \left(\lambda\mathbf{m}\right)^\top\left(\lambda\mathbf{m}\right) && (12)\\
&= \left\|\mathbf{A}\right\|^2 + \lambda\mathbf{A}^\top\mathbf{m} + \lambda\mathbf{m}^\top\mathbf{A} + \lambda^2\left\|\mathbf{m}\right\|^2 && (13)\\
&= \left\|\mathbf{m}\right\|^2\lambda^2 + \left(\mathbf{A}^\top\mathbf{m} + \mathbf{m}^\top\mathbf{A}\right)\lambda + \left\|\mathbf{A}\right\|^2 && (14)\\
&= \lambda^2 + 4\lambda + 8 && (15)
\end{aligned}$$

$\because$ the coefficient of $\lambda^2 > 0$, equation (15) is a convex function

$$f'(\lambda) = 2\lambda + 4 \tag{16}$$

1. Computing λ_{min} using Derative method:

$$f''(\lambda) = 2 \tag{17}$$
$$\because f''(\lambda) > 0, f\prime(\lambda_{min}) = 0, for \lambda_{min} \tag{18}$$
$$f' = 2\lambda + 4 = 0 \tag{19}$$
$$\therefore \lambda_{min} = -2 \tag{20}$$

2. Computing λ_{min} using Gradient Descent method:

$$\lambda_{n+1} = \lambda_n - \alpha \nabla f(\lambda_n) \tag{21}$$

Choosing

(a) $\alpha = 0.001$

(b) precision $= 0.0000001$

(c) n $= 10000000$

(d) $\lambda_0 = 4$

$$\lambda_{min} = -2 \tag{22}$$

Both methods yield same value of λ_{min}. Substituting this value in equation (6)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + (-2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{23}$$
$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{24}$$
$$OP = \|\mathbf{P} - \mathbf{O}\| \tag{25}$$
$$= \left\| \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \tag{26}$$
$$= 2 \tag{27}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1}\left(\frac{2}{0}\right) \tag{28}$$
$$= 90^\circ \tag{29}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 90^\circ & \sin 90^\circ \end{pmatrix} \mathbf{x} = 0 \tag{30}$$

See figure 1 and figure 2

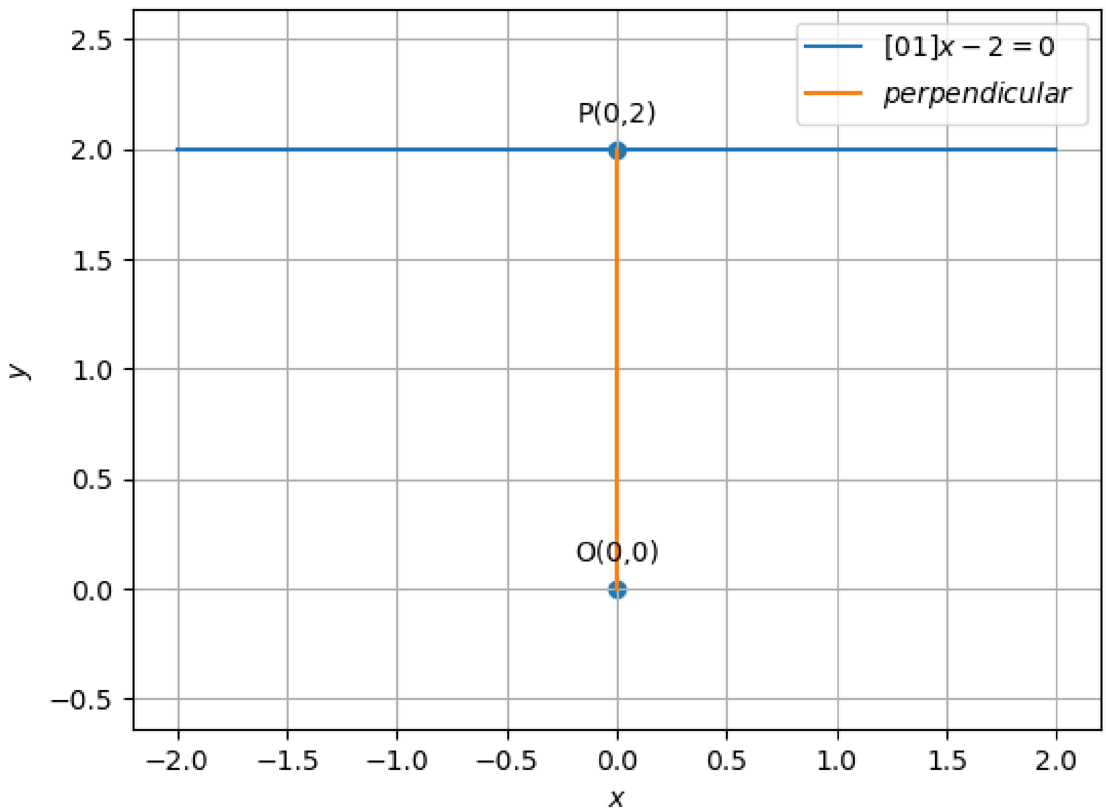
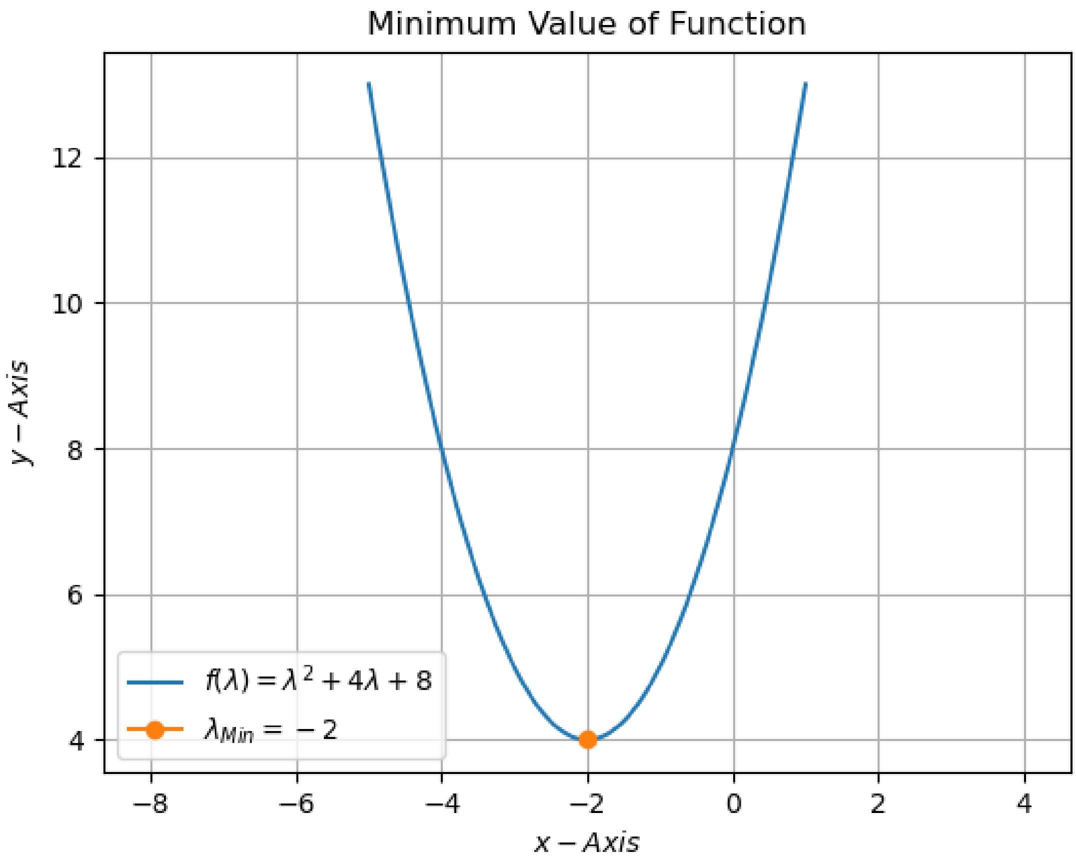

[01]x − 2 = 0
perpendicular
P(0,2)
O(0,0)
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
y
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x

Figure 1:



Minimum Value of Function
$f(\lambda) = \lambda^2 + 4\lambda + 8$
$\lambda_{Min} = -2$
12
10
8
6
4
$y - Axis$
−8
−6
−4
−2
0
2
4
$x - Axis$

Figure 2: