

TANGENTS AND NORMALS

Exercise 10.2

Q2. In fig 1, if TP and TQ are two tangents to a circle with centre O so that $\angle POQ = 110^\circ$ then $\angle PTQ$ is equal to.

Solution: Let the output angle be ϕ . The input parameters are given as

Input Parameters	Value	Description
\mathbf{O}	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre of the circle
r	1cm	radius of the circle
θ	110°	$\angle POQ$

Table 1:

Any point \mathbf{X} on the circle is given as

$$\mathbf{X} = \mathbf{O} + r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (1)$$

So points \mathbf{P} and \mathbf{Q} can be calculated as

$$\mathbf{P} = \mathbf{O} + \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (2)$$

$$\mathbf{Q} = \mathbf{e}_1 \quad (3)$$

For tangent TP

$$\mathbf{n}_1 = \mathbf{P} - \mathbf{O} \quad (4)$$

$$= \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} 1 \\ \tan \theta \end{pmatrix} \quad (5)$$

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ -\cot \theta \end{pmatrix} \quad (6)$$

For tangent TQ

$$\mathbf{n}_2 = \mathbf{e}_1 - \mathbf{O} \quad (7)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (8)$$

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{e}_2 \quad (9)$$

The equation of TP is given as

$$\mathbf{n}_1^\top (\mathbf{x} - \mathbf{P}) = 0 \quad (10)$$

$$\mathbf{n}_1^\top \left(\mathbf{x} - \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right) = 0 \quad (11)$$

$$\begin{pmatrix} \cos \theta & \sin \theta \end{pmatrix} \mathbf{x} = 1 \quad (12)$$

The equation of TQ is given as

$$\mathbf{n}_2^\top (\mathbf{x} - \mathbf{e}_1) = 0 \quad (13)$$

$$\mathbf{n}_2^\top \left(\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 0 \quad (14)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \quad (15)$$

The tangent point can be calculated by solving (12) and (15)

$$\begin{pmatrix} \cos \theta & \sin \theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (16)$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1.428 \end{pmatrix} \quad (17)$$

The angle between two lines with slope \mathbf{m}_1 and \mathbf{m}_2 s given as

$$\cos \phi = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \quad (18)$$

$$= \frac{\begin{pmatrix} 1 & -\cot \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(\csc \theta) (1)} \quad (19)$$

$$= -\cos \theta \quad (20)$$

$$\Rightarrow \cos \phi = -\cos \theta \quad (21)$$

Hence,

$$\phi = \cos^{-1}(\cos(180^\circ - \theta)) \quad (22)$$

$$= 180^\circ - \theta = 70^\circ \quad (23)$$

Hence, $\angle PTQ = 70^\circ$. See Fig 1

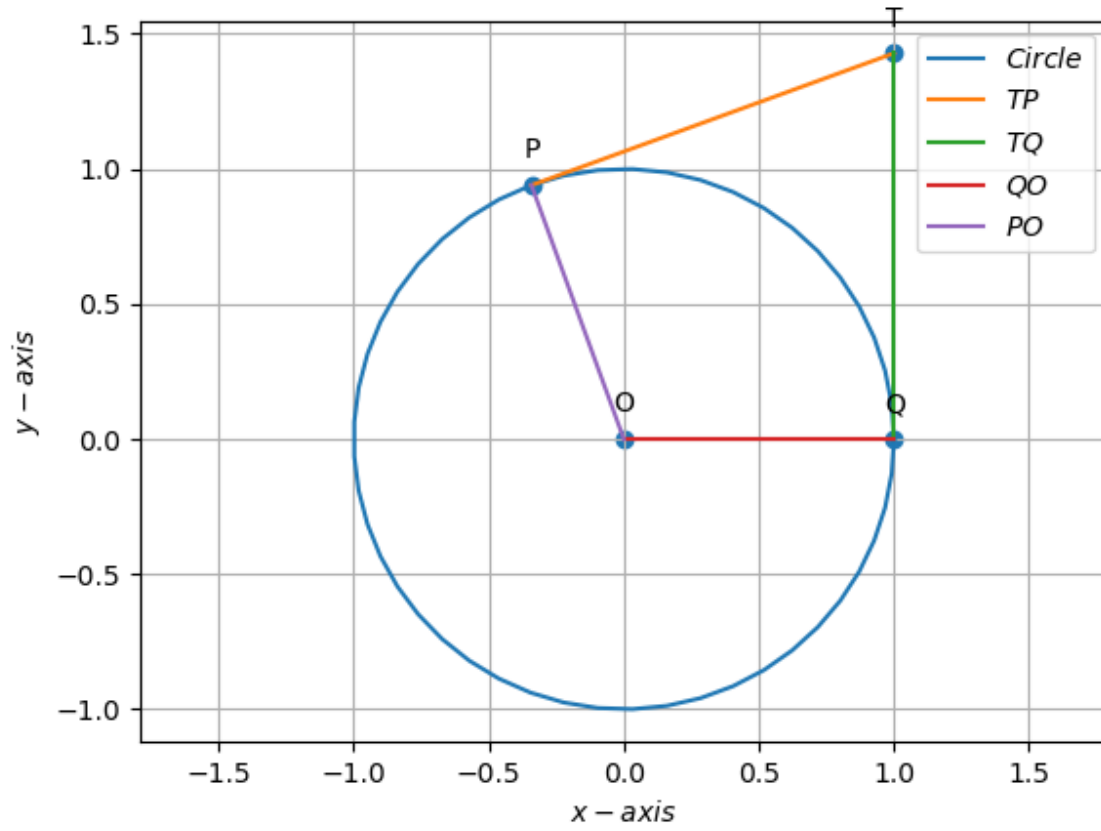


Figure 1: