Excercise 11.4

Q2.Find the coordinates of the focii, the vertices, the eccentricity and the length of the latus rectum of a hyperbola whose equation is given by $\frac{y^2}{9} - \frac{x^2}{27} = 1$. **Solution:** The equation of the hyperbola can be rearranged as

$$-x^2 + 3y^2 - 27 = 0 \tag{1}$$

The above equation can be equaded to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

Comparing coefficients of both equations (1) and (2)

$$\mathbf{V} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = \mathbf{0} \tag{4}$$

$$f = -27 \tag{5}$$

From equation (3), since $\mathbf{V}$ is already diagonalized, the eigen values λ_1 and λ_2 are given as

$$\lambda_1 = -1 \tag{6}$$

$$\lambda_2 = 3 \tag{7}$$

1. The eccentricity of the ellipse is given as

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} = \sqrt{1 + \frac{3}{1}} \tag{8}$$

$$= 2 \tag{9}$$

2. For the standard hypebola, the coordinates of Focii are given as

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_1}{f_0}}}{\frac{\lambda_1}{f_0}} \mathbf{e}_2 \tag{10}$$

where

$$
\begin{aligned}
f_0 &= -f && (11)\\
(10) \implies &= \pm\frac{\left(\frac{1}{2\sqrt{1-4}}\right)(4)\sqrt{\frac{-1}{27}}}{\frac{-1}{27}}\mathbf{e}_2 && (12)\\
&= \pm\begin{pmatrix}0\\6\end{pmatrix} && (13)
\end{aligned}
$$

3. The vertices of the ellipse are given by

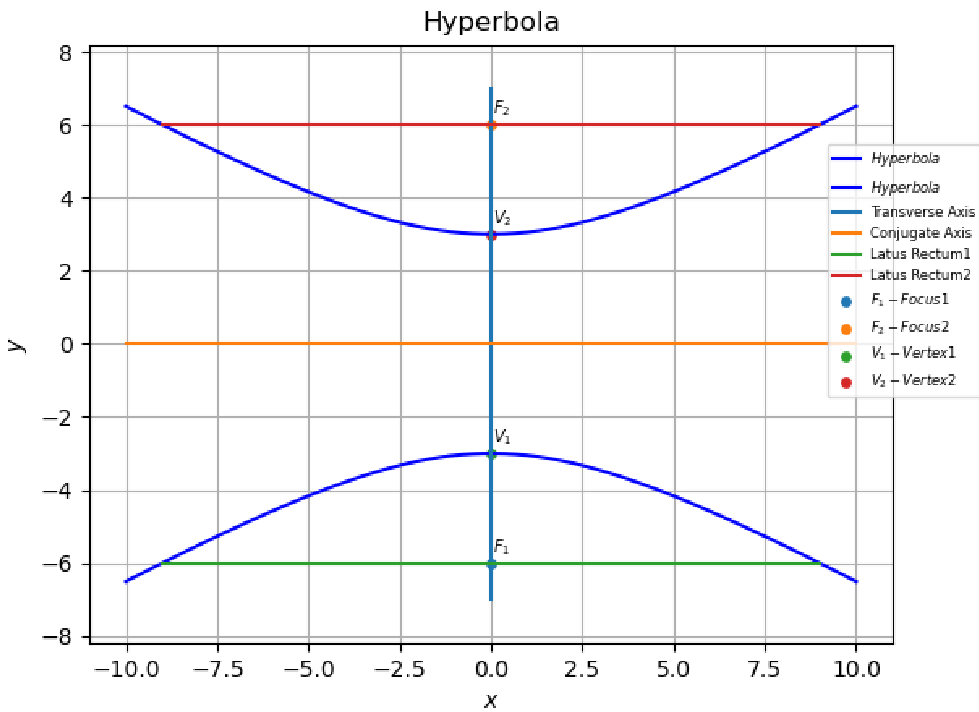
$$
\pm\begin{pmatrix}0\\ \sqrt{\left|\frac{f_0}{\lambda_2}\right|}\end{pmatrix} = \pm\begin{pmatrix}0\\3\end{pmatrix} \tag{14}
$$

4. The length of latus rectum is given as

$$
\begin{aligned}
2\frac{\sqrt{|f_0\lambda_2|}}{\lambda_1} &= 2\frac{\sqrt{|27\,(3)|}}{-1} && (15)\\
&= 18 && (16)
\end{aligned}
$$

as length cannot be negative

The corresponding is shown in Figure 1



Hyperbola
8
6
4
2
0
−2
−4
−6
−8
y
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x
F_2
V_2
V_1
F_1
$Hyperbola$
$Hyperbola$
Transverse Axis
Conjugate Axis
Latus Rectum1
Latus Rectum2
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$

Figure 1: