

CONIC SECTIONS

Exercise 10.2

Q2. In fig 1, if TP and TQ are two tangents to a circle with centre O so that $\angle POQ = 110^\circ$ then $\angle PTQ$ is equal to.

Solution: Let $\mathbf{T} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let \mathbf{O} be the centre of the circle such that TP and TQ are tangents to the circle from \mathbf{T} . We have to find points \mathbf{P} and \mathbf{Q} .

Input Parameters	value
\mathbf{T}	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
TO	1.743cm
radius	1cm
\mathbf{O}	$\begin{pmatrix} 1.743 \\ 0 \end{pmatrix}$

Table 1:

We know that the equation of circle is given as

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top \mathbf{u} + f = 0 \quad (1)$$

where,

$$\mathbf{u} = -\mathbf{O} = -(1.743) \quad (2)$$

$$f = \|\mathbf{O}\|^2 - r^2 = 2.039 \quad (3)$$

$$\Sigma = (\mathbf{T} + \mathbf{u})(\mathbf{T} + \mathbf{u})^\top - (\|\mathbf{T}\|^2 + 2\mathbf{u}^\top \mathbf{T} + f) \mathbf{I} \quad (4)$$

$$= \begin{pmatrix} 3.308 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2.039 & 0 \\ 0 & 2.039 \end{pmatrix} \quad (5)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -2.039 \end{pmatrix} \quad (6)$$

From (6), we can deduce Eigen pairs as follows

$$\lambda_1 = 1, \lambda_2 = -2.039 \quad (7)$$

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8)$$

Then

$$\mathbf{n}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 1 \\ 1.427 \end{pmatrix} \quad (9)$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 1 \\ -1.427 \end{pmatrix} \quad (10)$$

The points of contact of a tangent on a circle from an external point is given by

$$\mathbf{q}_{ij} = \left(\pm r \frac{\mathbf{n}_j}{\|\mathbf{n}_j\|} - \mathbf{u} \right) \quad i, j = 1, 2 \quad (11)$$

$$\mathbf{q}_{i1} = \left(\pm r \frac{\mathbf{n}_1}{\|\mathbf{n}_1\|} - \mathbf{u} \right) \quad (12)$$

$$= \left(\pm \frac{1}{1.742} \begin{pmatrix} 1 \\ 1.427 \end{pmatrix} + \begin{pmatrix} 1.742 \\ 0 \end{pmatrix} \right) \quad (13)$$

$$= \begin{pmatrix} 2.316 \\ 0.8191 \end{pmatrix}, \begin{pmatrix} 1.168 \\ -0.8191 \end{pmatrix} \quad (14)$$

$$\mathbf{q}_{i2} = \left(\pm r \frac{\mathbf{n}_2}{\|\mathbf{n}_2\|} - \mathbf{u} \right) \quad (15)$$

$$= \left(\pm \frac{1}{1.742} \begin{pmatrix} 1 \\ -1.427 \end{pmatrix} + \begin{pmatrix} 1.742 \\ 0 \end{pmatrix} \right) \quad (16)$$

$$= \begin{pmatrix} 2.316 \\ -0.8191 \end{pmatrix}, \begin{pmatrix} 1.168 \\ 0.8191 \end{pmatrix} \quad (17)$$

$$\text{Hence } \mathbf{P} = \mathbf{q}_{22} = \begin{pmatrix} 1.168 \\ 0.8191 \end{pmatrix} \quad (18)$$

$$\mathbf{Q} = \mathbf{q}_{12} = \begin{pmatrix} 1.168 \\ -0.8191 \end{pmatrix} \quad (19)$$

Now for calculating the $\angle PTQ$, calculating the normal vectors

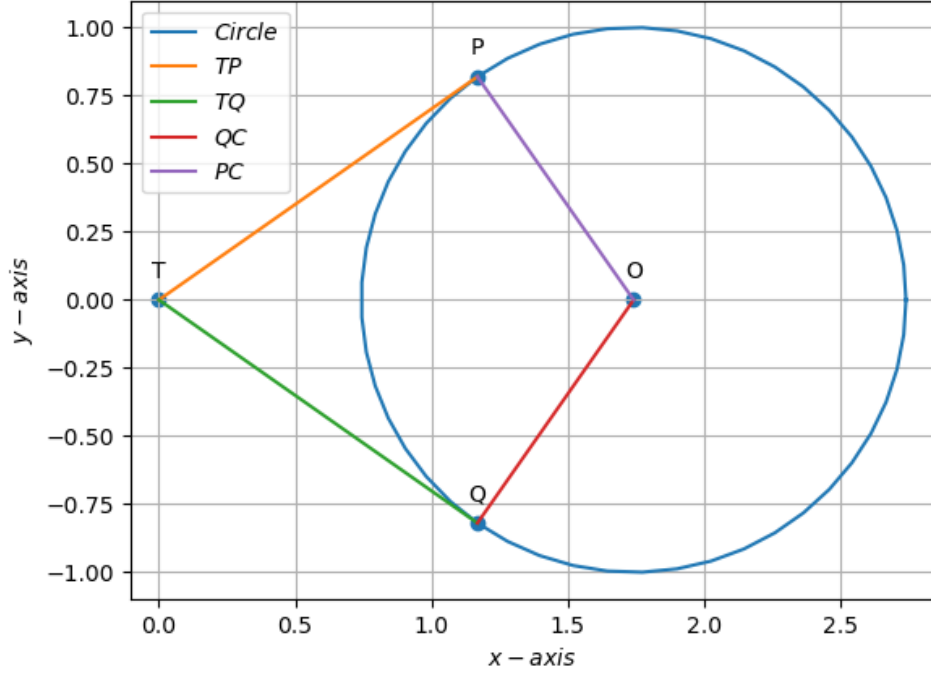


Figure 1:

$$\mathbf{n}_1 = \mathbf{P} - \mathbf{O} = \begin{pmatrix} -0.575 \\ 0.8191 \end{pmatrix} = \begin{pmatrix} 1 \\ -1.424 \end{pmatrix} \quad (20)$$

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0.7019 \end{pmatrix} \quad (21)$$

$$\mathbf{n}_2 = \mathbf{Q} - \mathbf{O} = \begin{pmatrix} -0.575 \\ -0.8191 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.424 \end{pmatrix} \quad (22)$$

$$\mathbf{m}_2 = \begin{pmatrix} 1 \\ -0.7019 \end{pmatrix} \quad (23)$$

Now the angle between two lines with slope \mathbf{m}_1 and \mathbf{m}_2 is given as

$$\cos \theta = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \quad (24)$$

$$= \frac{\begin{pmatrix} 1 & 0.7019 \end{pmatrix} \begin{pmatrix} 1 \\ -0.7019 \end{pmatrix}}{(1.221)^2} \quad (25)$$

$$\implies \theta = 70^\circ \quad (26)$$

Hence, $\angle PTQ = 70^\circ$