

## OPTIMIZATION

### Excercise 6.6

Q4. Find the equation of normal to the curve  $x^2 = 4y$  which passes through the point (4,-2)

**Solution:** The given equation of the curve can be written as

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad (3)$$

$$f = 0 \quad (4)$$

We are given that

$$\mathbf{h} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad (5)$$

This can be formulated as optimization problem as below:

$$\min_{\mathbf{x}} \quad f(\mathbf{x}) = \|\mathbf{x} - \mathbf{h}\|^2 \quad (6)$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (7)$$

It is already proved that the optimization problem is non-convex. However, by relaxing the constraint in (7) as

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f \leq 0 \quad (8)$$

the optimization problem can be made convex. We will use Lagrange multipliers method to find the optimum value. Define

$$H(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \quad (9)$$

and we find that

$$\nabla f(\mathbf{x}) = 2(\mathbf{x} - \mathbf{h}) \quad (10)$$

$$\nabla g(\mathbf{x}) = 2(\mathbf{V}\mathbf{x} + \mathbf{u}) \quad (11)$$

We have to find  $\lambda \in \mathbb{R}$  such that

$$\nabla H(\mathbf{x}, \lambda) = 0 \quad (12)$$

$$\implies 2(\mathbf{x} - \mathbf{h}) - 2\lambda(\mathbf{V}\mathbf{x} + \mathbf{u}) = 0 \quad (13)$$

$$\implies \mathbf{x} - \mathbf{h} = \lambda(\mathbf{V}\mathbf{x} + \mathbf{u}) \quad (14)$$

$$\implies (\mathbf{I} - \lambda\mathbf{V})\mathbf{x} = \lambda\mathbf{u} + \mathbf{h} \quad (15)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \lambda \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad (16)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ -2\lambda - 2 \end{pmatrix} \quad (17)$$

Writing augmented matrix,

$$\begin{pmatrix} 1-\lambda & 0 & 4 \\ 0 & 1 & -2\lambda - 2 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{1-\lambda}} \begin{pmatrix} 1 & 0 & \frac{4}{1-\lambda} \\ 0 & 1 & -2\lambda - 2 \end{pmatrix} \quad (18)$$

Then, we get

$$\mathbf{x}_m = \begin{pmatrix} \frac{4}{1-\lambda} \\ -2\lambda - 2 \end{pmatrix} \quad (19)$$

Substituting this value in (7)

$$\left(\frac{4}{1-\lambda} \quad -2 - 2\lambda\right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{4}{1-\lambda} \\ -2 - 2\lambda \end{pmatrix} + 2 \begin{pmatrix} 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{4}{1-\lambda} \\ -2 - 2\lambda \end{pmatrix} = 0 \quad (20)$$

$$\frac{16}{(1-\lambda)^2} + 8(\lambda + 1) = 0 \quad (21)$$

$$\lambda^3 - \lambda^2 - \lambda + 3 = 0 \quad (22)$$

$$\implies \lambda = -1.3593 \quad (23)$$

Substituting the value of  $\lambda$  in (19)

$$\mathbf{x}_m = \begin{pmatrix} 1.695 \\ 0.718 \end{pmatrix} \quad (24)$$

See figure 1

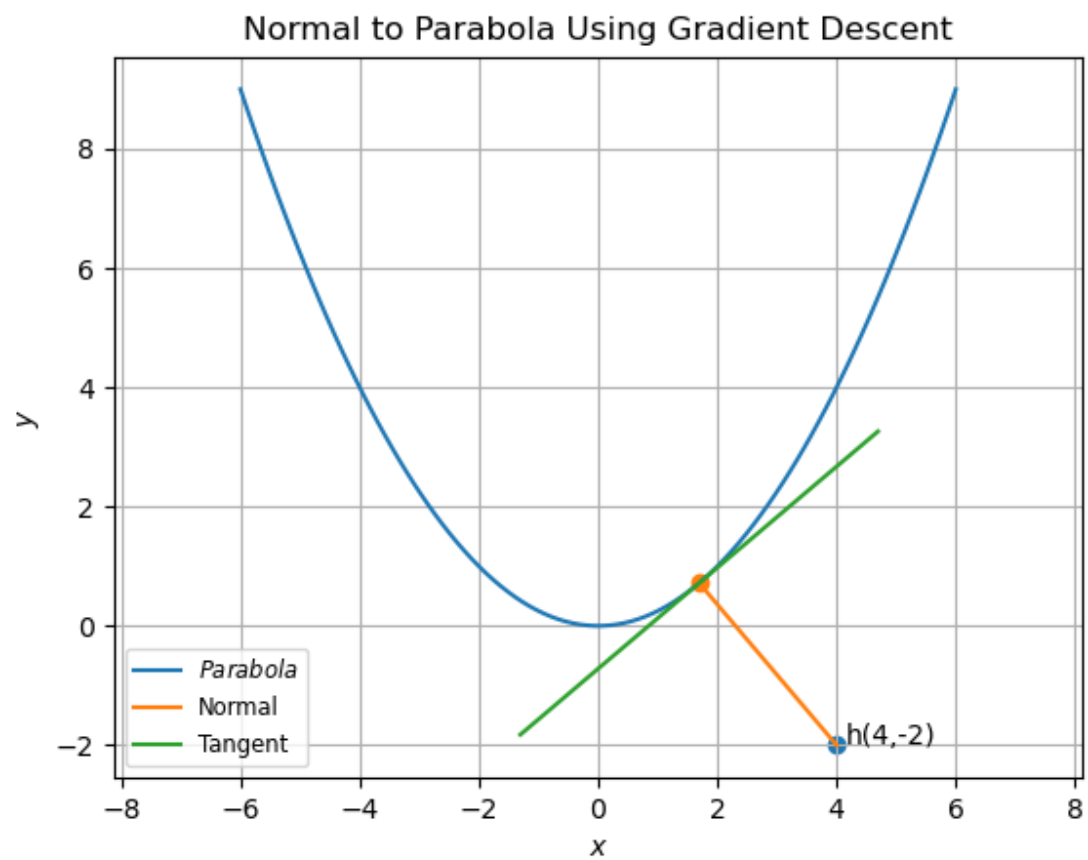


Figure 1: