Excercise 10.3

Q.3.2 Reduce the equation $y - 2 = 0$ into normal form. Find the perpendicular distances from the origin and angle between perpendicular and the positive x-axis.
Solution: The given equation can be written as

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1}$$
$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$
$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

Here, $\mathbf{A}$ is a point on the given line. We choose

$$\mathbf{A} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{5}$$
$$(4) \implies \mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6}$$

Let $\mathbf{O}$ be the origin. The perpendicular distance will be the minimum distance from $\mathbf{O}$ to the line. Let $\mathbf{P}$ be the foot of perpendicular. This problem can be formulated as an optimization problem as follows:

$$\min_{\mathbf{x}} \left\| \mathbf{x} - \mathbf{O} \right\|^2 \tag{7}$$
$$\implies \min_{\lambda} \left\| \mathbf{A} + \lambda \mathbf{m} - \mathbf{O} \right\|^2 \tag{8}$$
$$\implies \min_{\lambda} \left\| \mathbf{A} + \lambda \mathbf{m} \right\|^2 \tag{9}$$
$$\implies f(\lambda) = \left\| \mathbf{A} + \lambda \mathbf{m} \right\|^2 \tag{10}$$
$$= (\mathbf{A} + \lambda \mathbf{m})^\top (\mathbf{A} + \lambda \mathbf{m}) \tag{11}$$
$$= \left\| \mathbf{A} \right\|^2 + \mathbf{A}^\top (\lambda \mathbf{m}) + (\lambda \mathbf{m})^\top \mathbf{A} + (\lambda \mathbf{m})^\top (\lambda \mathbf{m}) \tag{12}$$
$$= \left\| \mathbf{A} \right\|^2 + \lambda \mathbf{A}^\top \mathbf{m} + \lambda \mathbf{m}^\top \mathbf{A} + \lambda^2 \left\| \mathbf{m} \right\|^2 \tag{13}$$
$$= \left\| \mathbf{m} \right\|^2 \lambda^2 + \left(\mathbf{A}^\top \mathbf{m} + \mathbf{m}^\top \mathbf{A} \right) \lambda + \left\| \mathbf{A} \right\|^2 \tag{14}$$
$$= \lambda^2 + 4\lambda + 8 \tag{15}$$

$\because$ the coefficient of $\lambda^2 > 0$, equation (15) is a convex function

$$f'(\lambda) = 2\lambda + 4 \tag{16}$$

Computing λ_{min} using Gradient Descent method:

$$\lambda_{n+1} = \lambda_n - \alpha \nabla f(\lambda_n) \tag{17}$$

$$\lambda_{n+1} = (1-2\alpha)\lambda_n - 4\alpha \tag{18}$$

Taking one-sided Z-transform on both sides of (18),

$$z\Lambda(z) = (1-2\alpha)\Lambda(z) - \frac{4\alpha}{1-z^{-1}} \tag{19}$$

$$\Lambda(z) = -\frac{4\alpha z^{-1}}{\left(1-(1-2\alpha)z^{-1}\right)\left(1-z^{-1}\right)} \tag{20}$$

$$= 2\left(\frac{1}{\left(1-(1-2\alpha)z^{-1}\right)} - \frac{1}{1-z^{-1}}\right) \tag{21}$$

$$= 2\sum_{k=0}^{\infty}\left((1-2\alpha)^k - 1\right)z^{-k} \tag{22}$$

from (22), the ROC is

$$|z| > \max\{1, |1-2\alpha|\} \tag{23}$$

$$\implies 0 < |1-2\alpha| < 1 \tag{24}$$

$$\implies 0 < \alpha < \frac{1}{2} \tag{25}$$

Thus, if α satisfies (25), then from (22)

$$\lim_{n\to\infty}\lambda_n = -2 \tag{26}$$

Choosing

1. $\alpha = 0.001$
2. precision = 0.0000001
3. n = 10000000
4. $\lambda_0 = 4$

$$\lambda_{min} = -2 \tag{27}$$

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + (-2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{28}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{29}$$

$$OP = \|\mathbf{P} - \mathbf{O}\| \tag{30}$$

$$= \left\| \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \tag{31}$$

$$= 2 \tag{32}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1} \left(\frac{2}{0} \right) \tag{33}$$

$$= 90^\circ \tag{34}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 90^\circ & \sin 90^\circ \end{pmatrix} \mathbf{x} = 0 \tag{35}$$

See figure 1 and figure 2

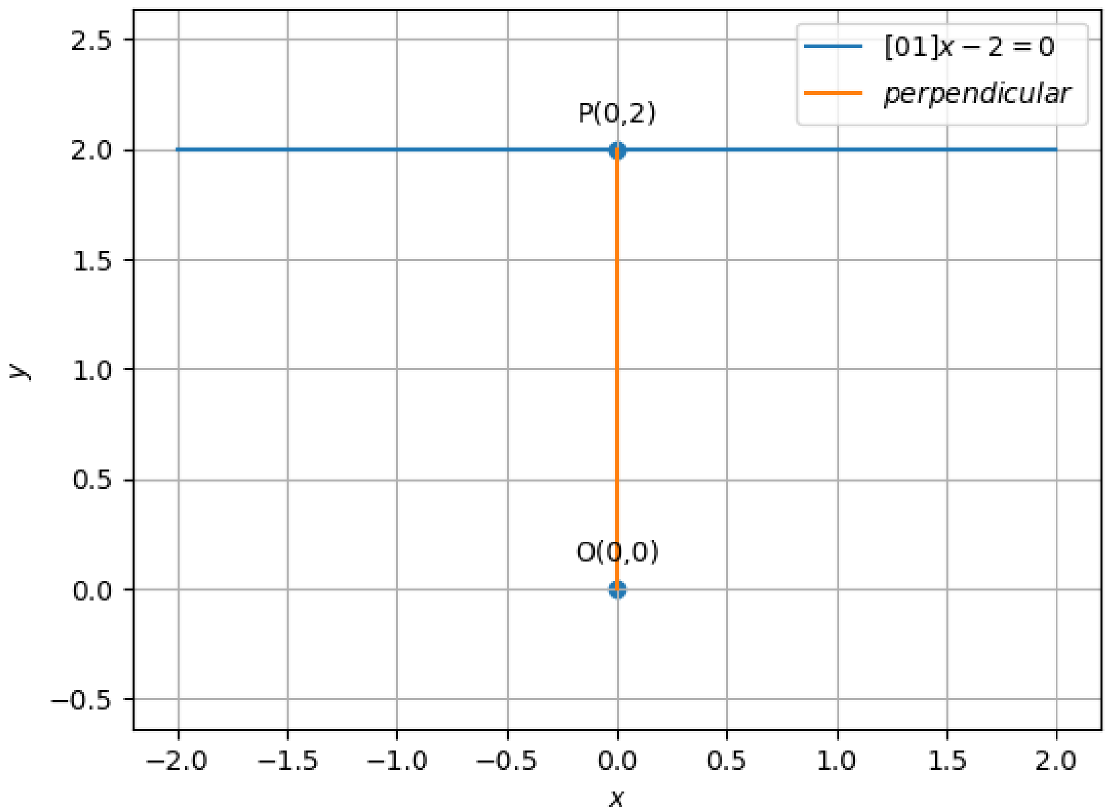
[01]$x-2=0$
perpendicular
P(0,2)
O(0,0)
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
y
x

Figure 1:

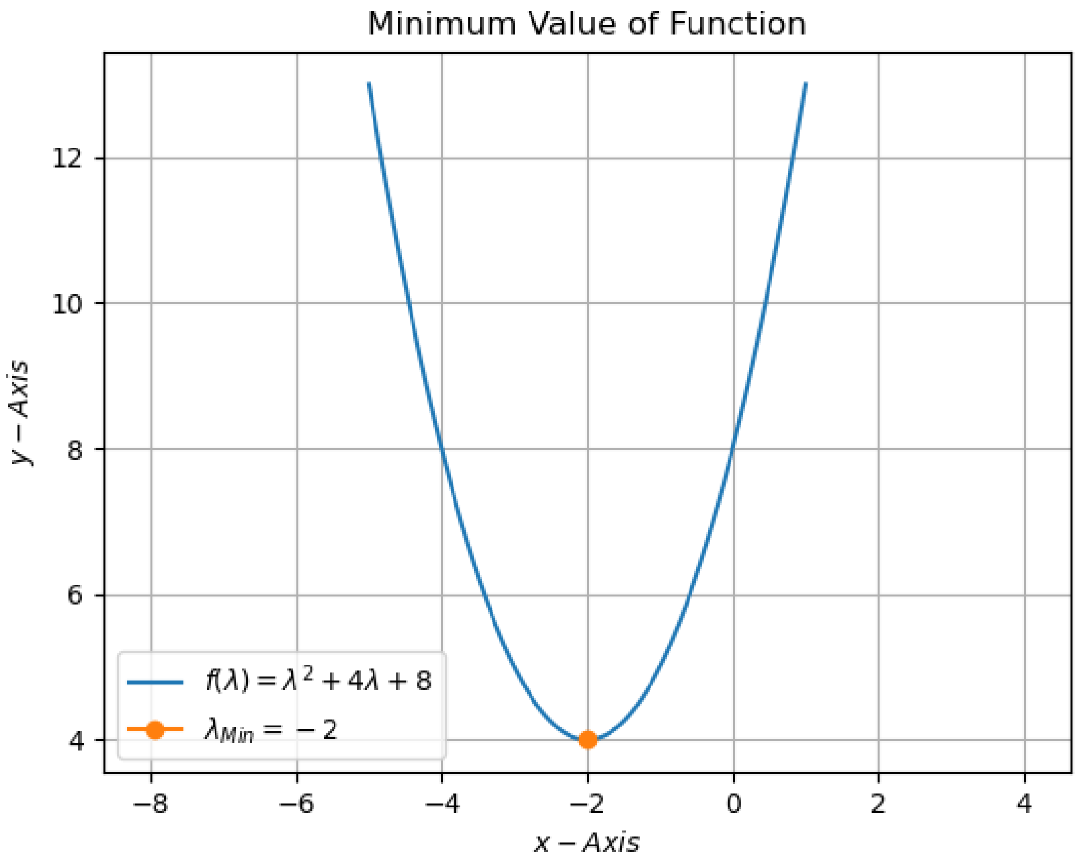
Minimum Value of Function
12
10
8
6
4
−8
−6
−4
−2
0
2
4
$x - Axis$
$y - Axis$
$f(\lambda) = \lambda^2 + 4\lambda + 8$
$\lambda_{Min} = -2$

Figure 2: