Excercise 11.4

Q8.Find the equation of the hyperbola whose foci is $(0, \pm 8)$ and vertices $(0, \pm 5)$.

Solution: Given

$$\mathbf{F} = \begin{pmatrix} 0 \\ \pm 8 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 0 \\ \pm 5 \end{pmatrix} \tag{1}$$

We know the vertex is given as

$$\mathbf{V} = \pm \begin{pmatrix} 0 \\ \sqrt{\frac{f_0}{\lambda_2}} \end{pmatrix} = \pm \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{2}$$

$$\implies f_0 = 25\lambda_2 \tag{3}$$

We know the Focii is given as

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)(e^2)\sqrt{\frac{\lambda_1}{f_0}}}{\frac{\lambda_1}{f_0}} \mathbf{e}_2 \tag{4}$$

$$= \frac{\frac{e}{\sqrt{1-e^2}}}{\sqrt{\frac{\lambda_1}{f_0}}} \mathbf{e}_2 \tag{5}$$

Substituting (3) we get

$$\mathbf{F} = 5e\mathbf{e}_2 \tag{6}$$

$$\begin{pmatrix} 0 \\ 8 \end{pmatrix} = 5e\mathbf{e}_2 \tag{7}$$

$$\implies e = \frac{8}{5} \tag{8}$$

Now we know the eccentricity is given as

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} \tag{9}$$

$$\implies \frac{\lambda_2}{\lambda_1} = -\frac{39}{25} \tag{10}$$

Now we know from the standard equation

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{11}$$

Calculating $\mathbf{n}$ and c

$$\mathbf{n} = \sqrt{\frac{\lambda_1}{f_0}}\mathbf{e}_2 = \frac{1}{5}\sqrt{\frac{\lambda_1}{\lambda_2}}\mathbf{e}_2 \tag{12}$$

$$= \frac{1}{\sqrt{-39}}\mathbf{e}_2 \tag{13}$$

$$c = \frac{1}{e\sqrt{1-e^2}} = \frac{25}{8\sqrt{-39}} \tag{14}$$

Now

$$\|\mathbf{n}\|^2 = -\frac{1}{39} \tag{15}$$

$$\|\mathbf{F}\|^2 = 64 \tag{16}$$

Substituting all the values in (11) we get

$$f = -\left(\frac{1}{39}\right)(64) + \left(\frac{25}{8}\right)^2 \left(\frac{1}{39}\right)\left(\frac{64}{25}\right) \tag{17}$$

$$= -1 \tag{18}$$

$$f_0 = -f = 1 \tag{19}$$

substituting (19) in (3) we get

$$\lambda_2 = \frac{1}{25} \tag{20}$$

Substituting (20) in (10) we get

$$\lambda_1 = -\frac{1}{39} \tag{21}$$

Therefore the equation of the hyperbola is given as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{22}$$

where

$$\mathbf{V} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{39} & 0 \\ 0 & \frac{1}{25} \end{pmatrix} \tag{23}$$

$$\mathbf{u} = \mathbf{0} \tag{24}$$

$$f = -1 \tag{25}$$

The corresponding is shown in Figure 1

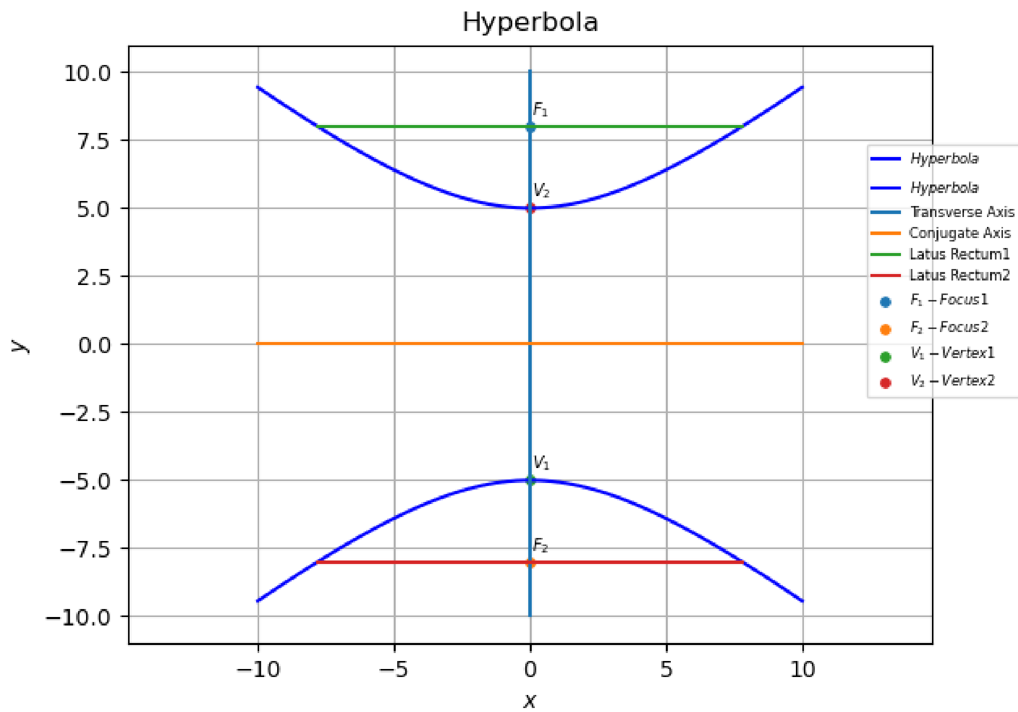
Hyperbola
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
y
−10
−5
0
5
10
x
F_1
V_2
V_1
F_2
$Hyperbola$
$Hyperbola$
Transverse Axis
Conjugate Axis
Latus Rectum1
Latus Rectum2
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$

Figure 1: