

OPTIMIZATION

Exercise 10.3

Q.3.2 Reduce the equation $y - 2 = 0$ into normal form. Find the perpendicular distances from the origin and angle between perpendicular and the positive x-axis.

Solution: The given equation can be written as

$$(0 \ 1) \mathbf{x} = 2 \quad (1)$$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2)$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3)$$

Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (4)$$

Here, \mathbf{A} is a point on the given line. We choose

$$\mathbf{A} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (5)$$

$$(4) \implies \mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6)$$

Let \mathbf{O} be the origin. The perpendicular distance will be the minimum distance from \mathbf{O} to the line. Let \mathbf{P} be the foot of perpendicular. This problem can be formulated as an optimization problem as follows:

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \quad (7)$$

$$\implies \min_{\lambda} \|\mathbf{A} + \lambda \mathbf{m} - \mathbf{O}\|^2 \quad (8)$$

$$\implies \min_{\lambda} \|\mathbf{A} + \lambda \mathbf{m}\|^2 \quad (9)$$

$$\implies f(\lambda) = \|\mathbf{A} + \lambda \mathbf{m}\|^2 \quad (10)$$

$$= (\mathbf{A} + \lambda \mathbf{m})^\top (\mathbf{A} + \lambda \mathbf{m}) \quad (11)$$

$$= \|\mathbf{A}\|^2 + \mathbf{A}^\top (\lambda \mathbf{m}) + (\lambda \mathbf{m})^\top \mathbf{A} + (\lambda \mathbf{m})^\top (\lambda \mathbf{m}) \quad (12)$$

$$= \|\mathbf{A}\|^2 + \lambda \mathbf{A}^\top \mathbf{m} + \lambda \mathbf{m}^\top \mathbf{A} + \lambda^2 \|\mathbf{m}\|^2 \quad (13)$$

$$= \|\mathbf{m}\|^2 \lambda^2 + (\mathbf{A}^\top \mathbf{m} + \mathbf{m}^\top \mathbf{A}) \lambda + \|\mathbf{A}\|^2 \quad (14)$$

$$= \lambda^2 + 4\lambda + 8 \quad (15)$$

\therefore the coefficient of $\lambda^2 > 0$, equation (15) is a convex function

$$f'(\lambda) = 2\lambda + 4 \quad (16)$$

Computing λ_{min} using Gradient Descent method:

$$\lambda_{n+1} = \lambda_n - \alpha \nabla f(\lambda_n) \quad (17)$$

Choosing

1. $\alpha = 0.001$
2. precision = 0.0000001
3. n = 10000000
4. $\lambda_0 = 4$

$$\lambda_{min} = -2 \quad (18)$$

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + (-2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (19)$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (20)$$

$$OP = \|\mathbf{P} - \mathbf{O}\| \quad (21)$$

$$= \left\| \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \quad (22)$$

$$= 2 \quad (23)$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1} \left(\frac{2}{0} \right) \quad (24)$$

$$= 90^\circ \quad (25)$$

The normal form of equation for straight line is given by

$$(\cos 90^\circ \quad \sin 90^\circ) \mathbf{x} = 0 \quad (26)$$

See figure 1 and figure 2

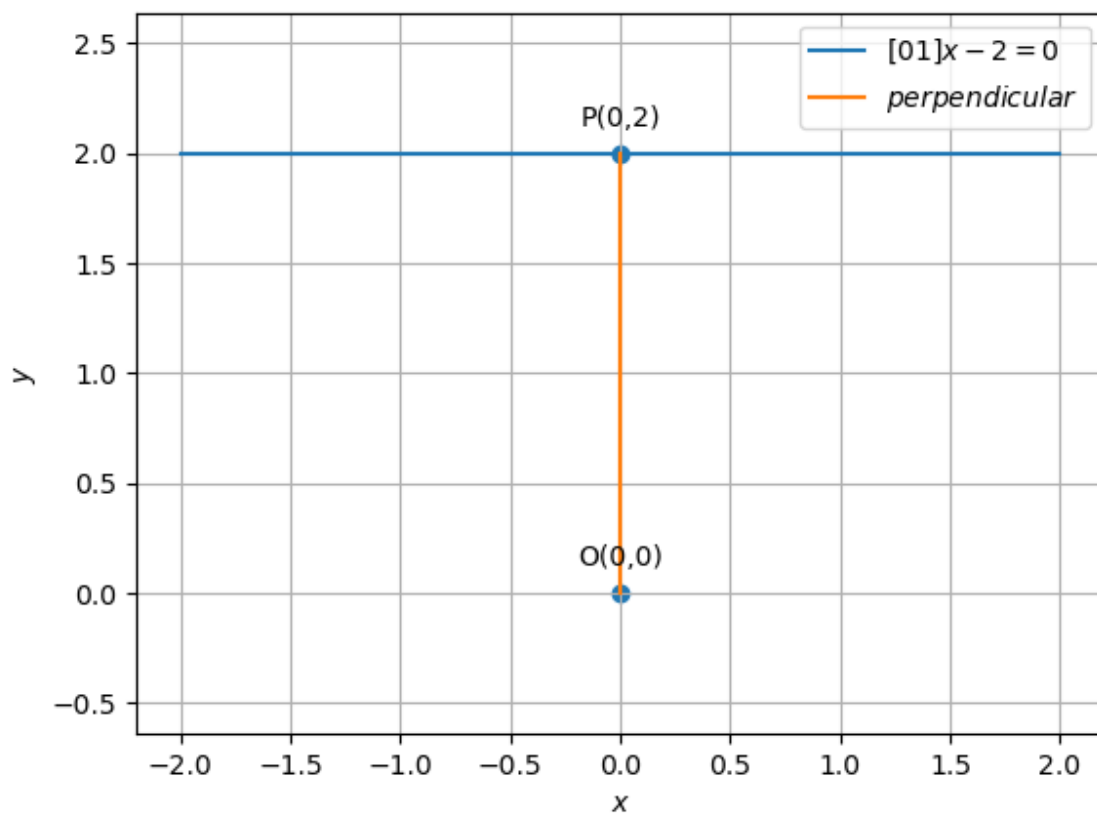


Figure 1:

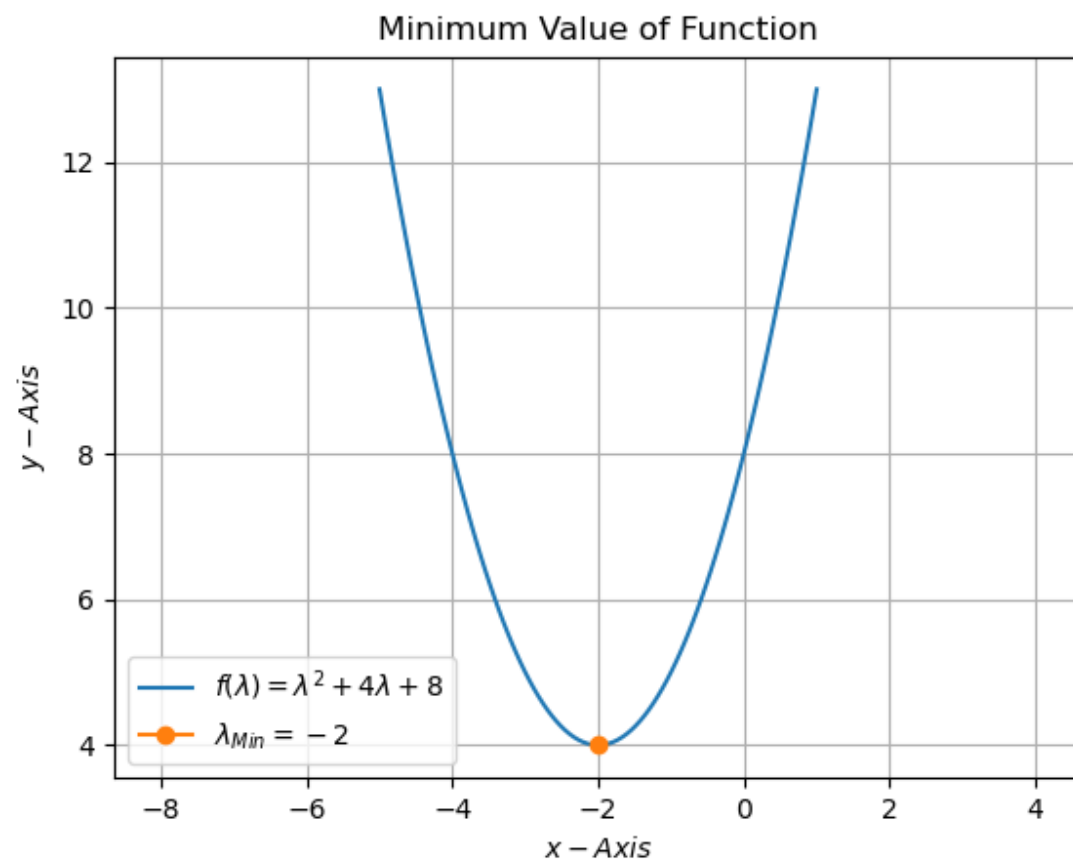


Figure 2: