Excercise 10.3

Q.3.2 Reduce the equation $y - 2 = 0$ into normal form. Find the perpendicular distances from the origin and angle between perpendicular and the positive x-axis.
Solution: The given equation can be written as

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1}$$

Let $\mathbf{O}$ be the point from where we have to find the perpendicular distance. The perpendicular distance will be the minimum distance from $\mathbf{O}$ to theline. Let $\mathbf{P}$ be the foot of perpendicular. This can be formulated as an optimization problem as follows:

$$\min_{\mathbf{x}} f(\mathbf{x}) = \|\mathbf{x} - \mathbf{O}\|^2 \tag{2}$$

$$\text{s.t. } g(\mathbf{x}) = \mathbf{n}^\top \mathbf{x} - c = 0 \tag{3}$$

where

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

$$c = 2 \tag{6}$$

Now we define

$$H(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{7}$$

and we find that

$$\nabla f(\mathbf{x}) = 2(\mathbf{x} - \mathbf{O}) \tag{8}$$

$$\nabla g(\mathbf{x}) = \mathbf{n} \tag{9}$$

We have to find $\lambda \in \mathbb{R}$ such that

$$\nabla H(\mathbf{x}, \lambda) = 0 \tag{10}$$

$$\implies 2(\mathbf{x} - \mathbf{O}) - \lambda \mathbf{n} = 0 \tag{11}$$

$$\implies \mathbf{x} = \frac{\lambda}{2}\mathbf{n} + \mathbf{O} \tag{12}$$

Substituting (12) in (1)

$$\mathbf{n}^\top \left(\frac{\lambda}{2}\mathbf{n} + \mathbf{O}\right) - c = 0 \tag{13}$$

$$\implies \lambda = \frac{2\left(c - \mathbf{n}^\top \mathbf{O}\right)}{\|\mathbf{n}\|^2} = 4 \tag{14}$$

When we consider the auxillary function as

$$\nabla H(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x}) \tag{15}$$

The value of λ comes out to be negative but the value of $\mathbf{x}_{min}$ will remain same.

$$\lambda = -\frac{2\left(c - \mathbf{n}^\top \mathbf{O}\right)}{\|\mathbf{n}\|^2} = -4 \tag{16}$$

$$\mathbf{x} = -\frac{\lambda}{2}\mathbf{n} + \mathbf{O} \tag{17}$$

Substituting the value of λ in (12),

$$\mathbf{x}_{min} = \mathbf{P} = \mathbf{O} + \frac{\mathbf{n}\left(c - \mathbf{n}^\top \mathbf{O}\right)}{\|\mathbf{n}\|^2} \tag{18}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\left(2 - \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)}{1} \tag{19}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{21}$$

$$OP = \|\mathbf{P} - \mathbf{O}\|^2 \tag{22}$$

$$= \sqrt{0^2 + 2^2} = 2 \tag{23}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1}\left(\frac{2}{0}\right) \tag{24}$$

$$= 90^\circ \tag{25}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 90^\circ & \sin 90^\circ \end{pmatrix} \mathbf{x} = 0 \tag{26}$$

See figure 1

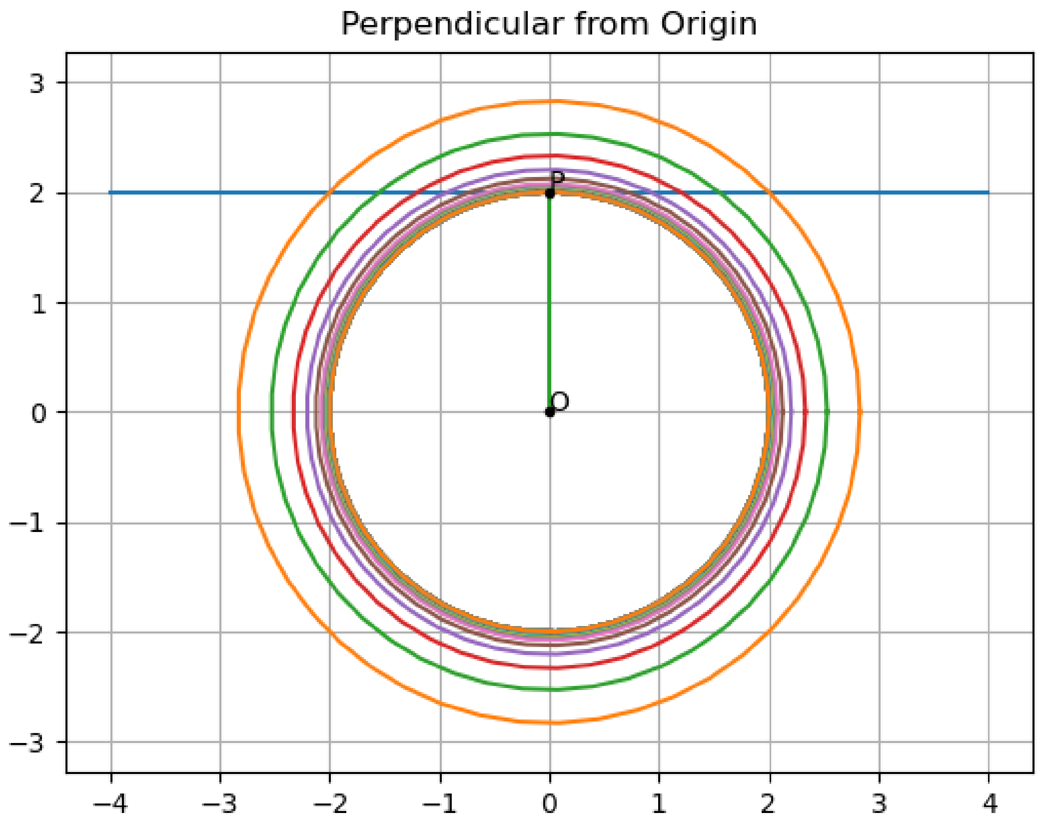

Figure 1: