Excercise 6.6

Q4. Find the equation of normal to the curve $x^2 = 4y$ which passes through the point (4,-2)
Solution: The given equation of the curve can be written as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{3}$$

$$f = 0 \tag{4}$$

We are given that

$$\mathbf{h} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{5}$$

This can be formulated as optimization problem as follows:

$$\min_{\mathbf{x}} \quad \mathrm{f}(\mathbf{x}) = \|\mathbf{x} - \mathbf{h}\|^2 \tag{6}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{7}$$

First we check whether (7) is convex or not. Assume $\mathbf{x}_1$ and $\mathbf{x}_2$ that satisfy $g(\mathbf{x}) = 0$. Then,

$$g(\mathbf{x}_1) = \mathbf{x}_1^\top \mathbf{V}\mathbf{x}_1 + 2\mathbf{u}^\top \mathbf{x}_1 + f = 0 \tag{8}$$

$$g(\mathbf{x}_2) = \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_2 + 2\mathbf{u}^\top \mathbf{x}_2 + f = 0 \tag{9}$$

$$\tag{10}$$

Then, for any $0 \le \lambda \le 1$, we substitute

$$\mathbf{x}_\lambda = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \tag{11}$$

into (7), we get

$$g(\mathbf{x}_\lambda) = (\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2)^\top \mathbf{V}(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) + 2\mathbf{u}^\top(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) + f \tag{12}$$

$$\implies \left(\lambda \mathbf{x}_1^\top + (1-\lambda)\mathbf{x}_2^\top\right)(\lambda \mathbf{V}\mathbf{x}_1 + (1-\lambda)\mathbf{V}\mathbf{x}_2) + 2\lambda \mathbf{u}^\top \mathbf{x}_1 + 2(1-\lambda)\mathbf{u}^\top \mathbf{x}_2 + f \tag{13}$$

$$\begin{aligned}\implies \lambda^2 \mathbf{x}_1^\top \mathbf{V}\mathbf{x}_1 + \lambda \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_1 - \lambda^2 \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_1 + \lambda \mathbf{x}_1^\top \mathbf{V}\mathbf{x}_2 + \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_2 - \lambda \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_2 - \lambda^2 \mathbf{x}_1^\top \mathbf{V}\mathbf{x}_2 \\ - \lambda \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_2 + \lambda^2 \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_2 + 2\lambda \mathbf{u}^\top \mathbf{x}_1 + 2\mathbf{u}^\top \mathbf{x}_2 - 2\lambda \mathbf{u}^\top \mathbf{x}_2 + f\end{aligned} \tag{14}$$

Multiplying (8) by λ and (9) by $(1-\lambda)$ and adding

$$\begin{aligned}\lambda g(\mathbf{x}_1) + (1-\lambda)\, g(\mathbf{x}_2) = \lambda \mathbf{x}_1^\top \mathbf{V}\mathbf{x}_1 + 2\lambda \mathbf{u}^\top \mathbf{x}_1 + \lambda f + \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_2 + 2\mathbf{u}^\top \mathbf{x}_2 + f \\ - \lambda \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_2 + 2\lambda \mathbf{u}^\top \mathbf{x}_2 - \lambda f = 0 \\ \implies f = -\lambda \mathbf{x}_1^\top \mathbf{V}\mathbf{x}_1 - 2\lambda \mathbf{u}^\top \mathbf{x}_1 - \mathbf{x}_2^\top \mathbf{V}\mathbf{x_2} - 2\mathbf{u}^\top \mathbf{x}_2 \\ + \lambda \mathbf{x}_2^\top \mathbf{V}\mathbf{x}_2 - 2\lambda \mathbf{u}^\top \mathbf{x}_2 \end{aligned} \tag{15}$$

Substituting the value of f from (15) in (12) and simplifying

$$(12) \implies (\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{V} (\mathbf{x}_1 - \mathbf{x}_2) \tag{16}$$

Since $\mathbf{V}$ is a semi-definite matrix,the value of (16) will be ≥ 0 contracdicting the equality in (7). Hence, the optimization problem is nonconvex. However, by relaxing the constraint in (7) as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f \leq 0 \tag{17}$$

the optimization problem can be made convex. Applying convexity property to (17) and simplifying, (16) yields to

$$(\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{V} (\mathbf{x}_1 - \mathbf{x}_2) \geq 0 \tag{18}$$

Hence the revised constraint makes it a convex optimization problem. Solving the problem using *cvxpy* we get the following point on the curve

$$\mathbf{X} = \begin{pmatrix} 1.695 \\ 0.718 \end{pmatrix} \tag{19}$$

See figure 1

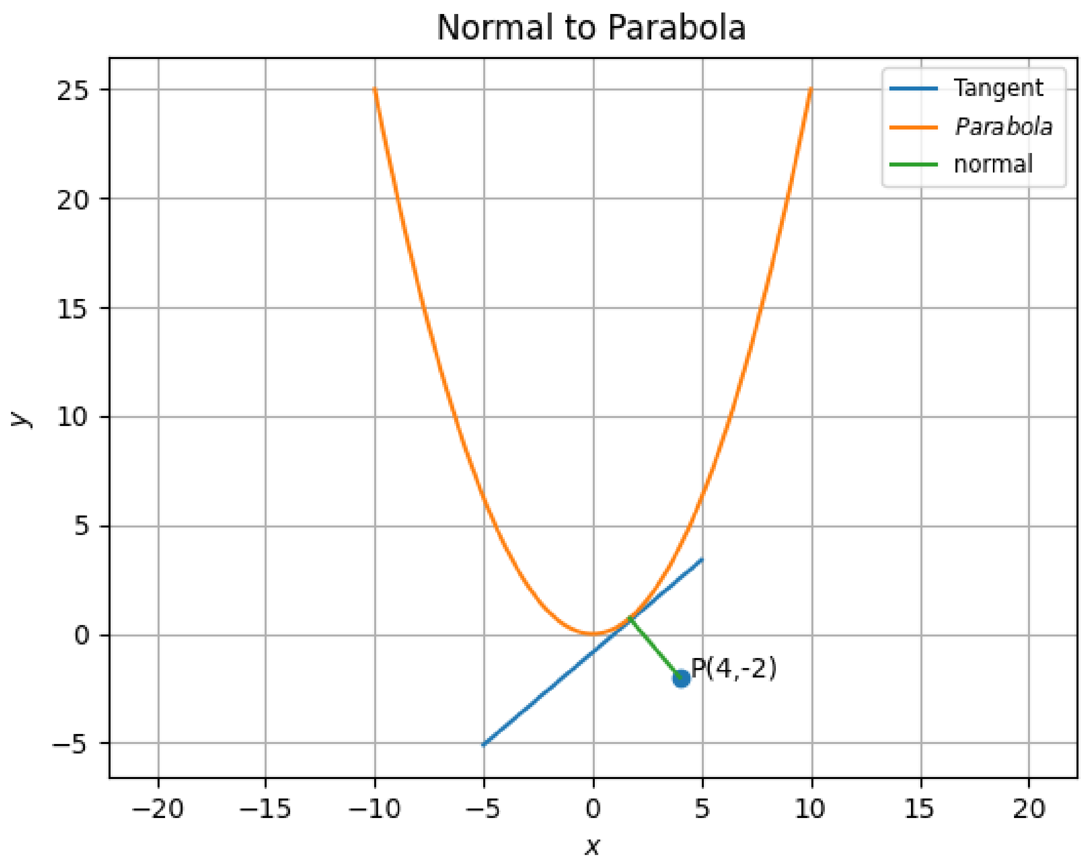
Normal to Parabola
Tangent
Parabola
normal
P(4,-2)
25
20
15
10
5
0
−5
y
−20
−15
−10
−5
0
5
10
15
20
x

Figure 1: