JEE Maths-65/5/3

Q26.1 Find the vector equation of the line passing through $(2, 1, -1)$ and parallel to the line $\mathbf{r} = \left(\hat{i} + \hat{j}\right) + \lambda \left(2\hat{i} - \hat{j} + \hat{k}\right)$. Also, find the distance between these two lines.
Solution: The given equations can be written as

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \tag{1}$$
$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \tag{2}$$

where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \tag{3}$$

Also since (1) is parallel to (2) so

$$\mathbf{m}_1 = \mathbf{m}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{4}$$

Assume

$$\mathbf{M} = \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix} \tag{5}$$
$$\mathbf{B} - \mathbf{A} = (\lambda_2 \mathbf{m}_2 - \lambda_1 \mathbf{m}_1) + (\mathbf{x}_2 - \mathbf{x}_1) \tag{6}$$
$$= \mathbf{M}\lambda + \mathbf{k} \tag{7}$$
$$\text{where } \lambda = \begin{pmatrix} -\lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{k} = (\mathbf{x}_2 - \mathbf{x}_1) \tag{8}$$

We can formulate an unconstrained optimization problem as below

$$\min_{\lambda} \left\|\mathbf{B} - \mathbf{A}\right\|^2 \tag{9}$$

Substituting (7) in (9)

$$(9) \implies \min_{\lambda} \left\|\mathbf{M}\lambda + \mathbf{k}\right\|^2 \tag{10}$$
$$\implies f(\lambda) = (\mathbf{M}\lambda + \mathbf{k})^\top (\mathbf{M}\lambda + \mathbf{k}) \tag{11}$$
$$= \left(\lambda^\top \mathbf{M}^\top + \mathbf{k}^\top\right)(\mathbf{M}\lambda + \mathbf{k}) \tag{12}$$
$$= \lambda^\top \mathbf{M}^\top \mathbf{M}\lambda + 2\mathbf{k}^\top \mathbf{M}\lambda + \left\|\mathbf{k}\right\|^2 \tag{13}$$

(13) is a quadratic vector equation. To check whether it is convex or not, we will compute the value of $\mathbf{M}^\top\mathbf{M}$.

$$\mathbf{M}^\top\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix} \tag{14}$$

The eigen values of $\mathbf{M}^\top\mathbf{M}$ are 0 and 12, which are greater than equal to zero. Therefore $\mathbf{M}^\top\mathbf{M}$ is positive definite matrix implying (13) is convex. Solving the problem in cvxpy yields,

$$\lambda_{min} = \begin{pmatrix} 0.0833 \\ -0.0833 \end{pmatrix} \tag{15}$$

$$\mathbf{A} = \begin{pmatrix} 1.167 \\ 0.916 \\ 0.083 \end{pmatrix} \tag{16}$$

$$\mathbf{B} = \begin{pmatrix} 1.833 \\ 1.083 \\ -1.083 \end{pmatrix} \tag{17}$$

$$\|\mathbf{B} - \mathbf{A}\| = 1.354 \tag{18}$$

See figure 1

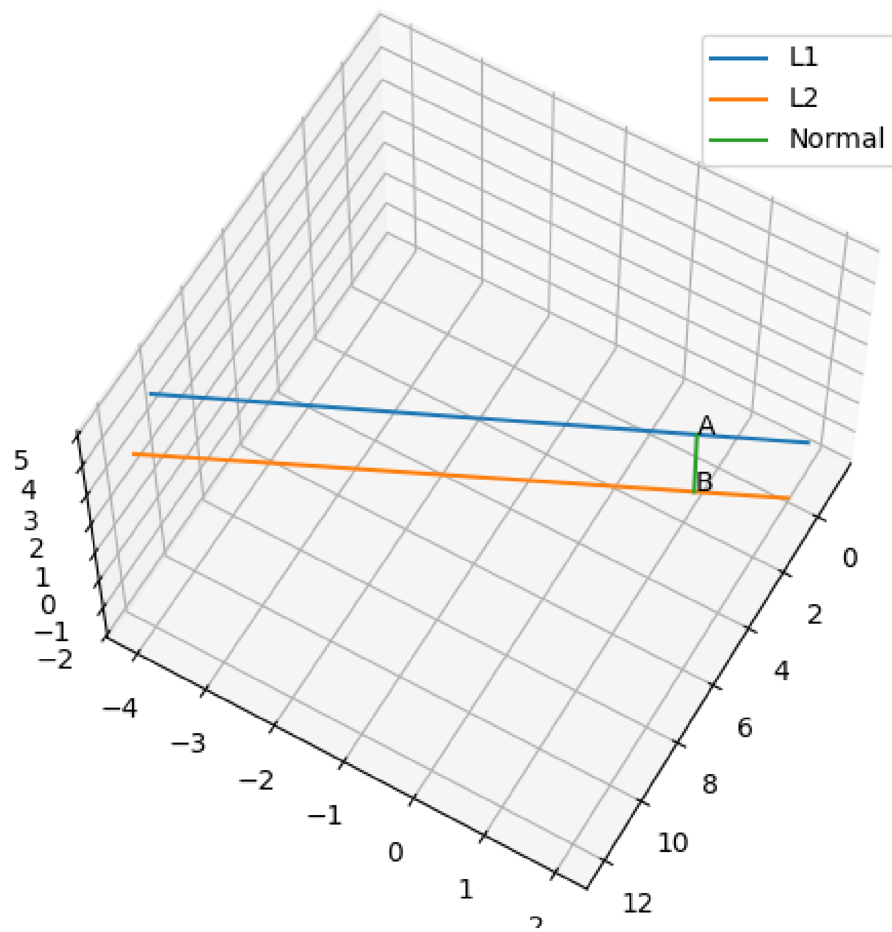

Figure 1: