Excercise 6.6

Q4. Find the equation of normal to the curve $x^2 = 4y$ which passes through the point (4,-2)
Solution: The given equation of the curve can be written as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{3}$$

$$f = 0 \tag{4}$$

We are given that

$$\mathbf{h} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{5}$$

This can be formulated as optimization problem as below:

$$\min_{\mathbf{x}} \quad \mathrm{f}(\mathbf{x}) = \|\mathbf{x} - \mathbf{h}\|^2 \tag{6}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{7}$$

Now,

$$\|\mathbf{x} - \mathbf{h}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{h}^\top \mathbf{x} + \|\mathbf{h}\|^2 \tag{8}$$

$$= \mathbf{y}^\top \mathbf{C}\mathbf{y} \tag{9}$$

where,

$$\mathbf{C} = \begin{pmatrix} \mathbf{I} & -\mathbf{h} \\ -\mathbf{h}^\top & \|h\|^2 \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \tag{10}$$

And equation (7) can be expressed as

$$\mathbf{y}^\top \mathbf{A}\mathbf{y} = 0 \tag{11}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \tag{12}$$

Using SDR(Semi Definite Relaxation), (6) can be expressed as

$$\min_{\mathbf{X}} tr\left(\mathbf{C}\mathbf{X}\right) \tag{13}$$

$$\text{s.t. } tr\left(\mathbf{A}\mathbf{X}\right) = 0 \tag{14}$$

$$\mathbf{X} \succcurlyeq \mathbf{0} \tag{15}$$

On solving it yields to the point

$$\mathbf{x} = \begin{pmatrix} 1.695 \\ 0.718 \end{pmatrix} \tag{16}$$

This is same as we obtained in Gradient Descent and Lagrange multiplier. Hence this is the required point.