Applied Linear Algebra in Data Analysis

Concepts in Vector Spaces

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

Vectors

- **Vectors** are ordered list of numbers (scalars). $\mathbf{v} = \begin{bmatrix} 1.2 \\ -0.1 \\ \vdots \\ -1.24 \end{bmatrix}$.

  Note: Small bold letter will represent vectors. e.g. $\mathbf{a}, \mathbf{x}, \dots$

- Scalars can be any *field* $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{Q}$. Scalars will be represented using lower case normal font, e.g. $x, y, \alpha, \beta, \dots$

Vectors

- Addition/multiplication operations performed on vectors will follow the rules of addition/multiplication of the corresponding scalar fields.

- We will typically encounter only $\mathbb{R}$ and $\mathbb{C}$ in this course.

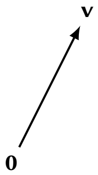
Vectors

- Individual elements of a vector $\mathbf{v}$ are indexed. The i^{th} element of $\mathbf{v}$ is referred to as v_i.
- *Dimension* or *size* of a vector is number of elements in the vector.
- Set of n-real vectors is denoted by $\mathbb{R}^n$ (similarly, $\mathbb{C}^n$)
- Vectors $\mathbf{a}$ and $\mathbf{b}$ are equal, if
  - both have the same size; and
  - $a_i = b_i, i \in \{1, 2, 3, \ldots n\}$

Vectors

- **Unit vector** $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **Zero vector** $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **One vector** $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

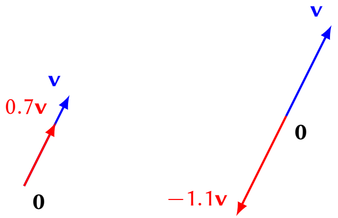
- Geometrically, real n-vectors can be thought of as points in $\mathbb{R}^n$ space.



Vector

- **Vector scaling**: Multiplication of a scalar and a vector.

$$\mathbf{w} = a\mathbf{v} = a \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \\ av_3 \\ \vdots \\ av_n \end{bmatrix} \quad a \in \mathbb{R};\ \mathbf{w}, \mathbf{v} \in \mathbb{R}^n$$



Properties of Vector Scaling

- Scalar multiplication is *commutative.*

$$\alpha\mathbf{v} = \mathbf{v}\alpha$$

- Scalar multiplication is *associative.*

$$(\alpha\beta)\,\mathbf{v} = \alpha\,(\beta\mathbf{v})$$

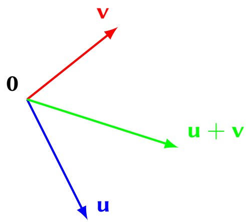
- Scalar multiplication is *distributive.*

$$(\alpha + \beta)\,\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$$

Vectors

- **Vector addition**: Adding two vectors of the same dimension, element by element.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$



Properties of Vector Addition

- Vector addition is *commutative.*

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

- Vector addition is *associative.*

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

- Zero vector has no effect.

$$\mathbf{a} + \mathbf{0} = \mathbf{a}$$

- Subtraction of vectors.

$$\mathbf{a} + (-1)\mathbf{a} = \mathbf{a} - \mathbf{a} = \mathbf{0}$$

Vector spaces

Definition: Vector Space

A set of vectors V that is closed under **vector addition** and **vector scaling**.

$$\forall \mathbf{x}, \mathbf{y} \in V, \quad \mathbf{x} + \mathbf{y} \in V$$

$$\forall \mathbf{x} \in V,\ \alpha \in F, \quad \alpha \mathbf{x} \in V$$

Vector spaces

For a set to be a vector space, it must satisfy the followng properties: $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$

- *Commutativity*: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- *Associativity of vector addition*: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
- *Additive identity*: $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}\ (0 \in V)$
- *Additive inverse*: $\exists - \mathbf{x} \in V, \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
- *Associativity of scalar multiplication*: $\alpha\,(\beta\mathbf{x}) = (\alpha\beta\mathbf{x})$
- *Distributivity of scalar sums*: $(\alpha + \beta)\,\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
- *Distributivity of vector sums*: $\alpha\,(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$
- *Scalar multiplication identity*: $1\mathbf{x} = \mathbf{x}$

We will mostly deal with $\mathbb{R}^n$ and $\mathbb{C}^n$ vectors spaces in this course.

Subspaces

Definition: Subspace

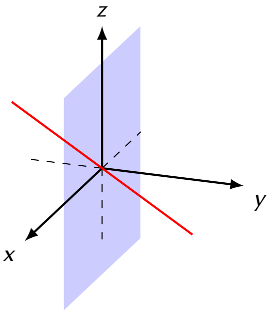
A **subspace** S of a vector space V is a subset of V and is itself a vector space.

$$S \subset V, \quad \forall \mathbf{x}, \mathbf{y} \in S, \alpha\mathbf{x} + \beta\mathbf{y} \in S, \quad \alpha, \beta \in F$$

Subspaces

- The zero vector is called the **trivial subspace** of a vector space V.

- For example, in $\mathbb{R}^3$ all planes and lines passing through the origin are subspaces of $\mathbb{R}^3$.



Linear independence

Definition: Linear Independence

A collection of vectors is *linearly independent* if it is **not** *linearly dependent.*

$$\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 \ldots = \alpha_n = 0$$

Linear independence

Another way to state this: A collection of vectors is *linearly dependent* if at least one of the vectors in the collection can be expressed as a linear combination of the other vectors in the collection, i.e.

$$\mathbf{x}_i = -\sum_{j=1, j\neq i}^{n} \left(\frac{\alpha_j}{\alpha_i}\right) \mathbf{x}_j$$

- **Linearly dependent**: At least one vector in the collection can be expressed as a linear combination of the other vectors in the collection.

- **Linearly independent**: No vector in the collection can be expressed as a linear combination of the other vectors in the collection.

Span of a set of vectors

Consider a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \ldots \mathbf{v}_r\}$ where $\mathbf{v}_i \in \mathbb{R}^n, 1 \leq i \leq r$.

Definition: Span of set of vectors

The **span** of the set S is defined as the set of all linear combinations of the vectors $\mathbf{v}_i$,

$$span\,(S) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r\}\,,\;\; \alpha_i \in \mathbb{R}$$

Is $span\,(S)$ a subspace of $\mathbb{R}^n$?

We say that the subspace $span\,(S)$ is spanned by the *spanning set S*. $\longrightarrow$ *S spans* $span\,(S)$.

Span of a set of vectors

Definition: Sum of Two Subspaces

Sum of subspaces X, Y is defined as the sum of all possible vectors from X and Y.

$$X + Y = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in X, \mathbf{y} \in Y\}$$

Sum of two subspaces is also a subspace.

(Standard) Inner Product

Definition: Standard Inner Product

The **standard inner product** is defined as the following,

$$\mathbf{x}^\top \mathbf{y} = \sum_{i=1}^{n} x_i y_i, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

For complex vectors: $\mathbf{x}^* \mathbf{y} = \sum_{i=1}^{n} \overline{x}_i y_i, \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$

(Standard) Inner Product

Properties of the standard inner product

- $\mathbf{x}^\top \mathbf{x} > 0, \;\; \forall \mathbf{x} \neq 0$ and $\mathbf{x}^\top \mathbf{x} = 0 \Leftrightarrow \mathbf{x} = 0$
- *Commutative*: $\mathbf{x}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{x}$
- *Associativity with scalar multiplication*: $(\alpha \mathbf{x})^\top \mathbf{y} = \alpha \left(\mathbf{x}^\top \mathbf{y}\right)$
- *Distributivity with vector addition*: $(\mathbf{x} + \mathbf{y})^\top \mathbf{z} = \mathbf{x}^\top \mathbf{z} + \mathbf{y}^\top \mathbf{z}$

Norm

Norm is a measure of the size of a vector.

Euclidean norm of a n-vector $\mathbf{x} \in \mathbb{R}^n$ is defined as,

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$\|\mathbf{x}\|_2$ is a measure of the length of the vector $\mathbf{x}$.

Any function of the form $\|\bullet\| : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is a valid norm, provided it satisfies the following properties.

Norm

Properties of Norms

- *Definiteness.* $\|\mathbf{x}\| = 0 \iff x = 0$

- *Non-negativity.* $\|\mathbf{x}\| \geq 0$

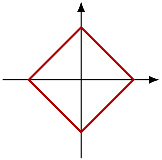
- *Non-negative homogeneity.* $\|\beta\mathbf{x}\| = |\beta| \, \|\mathbf{x}\| \, , \, \beta \in \mathbb{R}$

- *Triangle inequality.* $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

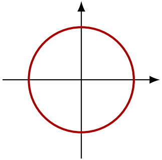
p-Norm

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

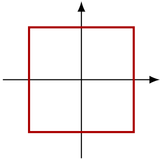
$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$



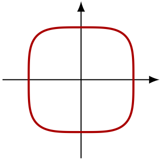
$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$



$\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$



$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$



Orthogonality

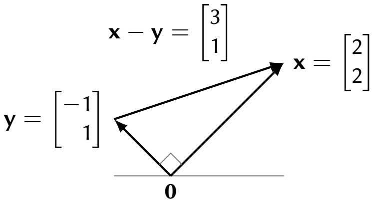
Orthogonality is the idea of two vectors being perpendicular, $\mathbf{x} \perp \mathbf{y}$.

Definition: Orthogonality

Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are orthogonal, if and only if,

$$\mathbf{x}^\top \mathbf{y} = \sum_{i=1}^{n} x_i y_i = 0$$

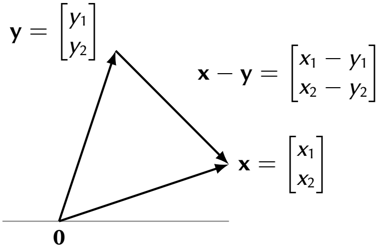
Orthogonality



Using the Pythagorean theorem, $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x}^\top \mathbf{y} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \implies \mathbf{x}^\top \mathbf{y} = 0$$

Angle between vectors



The standard inner product is also a measure of similarity between two vectors, $\cos(\theta) = \frac{\mathbf{x}^\top \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$

Cauchy-Bunyakovski-Schwartz Inequality:

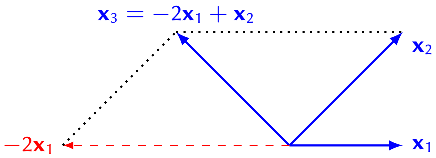
$$\left|\mathbf{x}^\top \mathbf{y}\right| \leq \|\mathbf{x}\| \|\mathbf{y}\|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

Basis

Consider a vector $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. What can we say about the coefficients α_is when the collection $\{\mathbf{x}_i\}_{i=1}^{n}$ is,

- linearly independent $\implies$ α_is are *unique*.
- linearly dependent $\implies$ α_is are not *unique*.

Consider $\mathbb{R}^2$ vector space. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

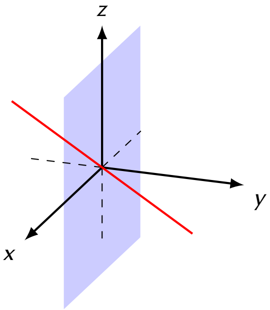


Basis

Independence-Dimension inequality: What is the maximum possible size of a linearly independent collection?

A linearly independent collection of n-vectors can at most have n vectors.

How many vectors can we choose from the following vectors before the set becomes linearly dependent?



Basis

Definition: Basis

A linearly independent set of n-vectors from $\mathbb{R}^n$, of size n, is called a *basis* for $\mathbb{R}^n$.

Any n-vector from $\mathbb{R}^n$ can be represented as a *unique* linear combination of the elements of the basis.

Consider the basis $\{\mathbf{x}_i\}_{i=1}^{n}$, $\mathbf{x}_i \in \mathbb{R}^n$. Any vector $\mathbf{y} \in \mathbb{R}^n$ can be represented as a linear combination of $\mathbf{x}_i$s, $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. This is called the *expansion of* $\mathbf{y}$ in the $\{\mathbf{x}_i\}_{i=1}^{n}$ basis.

Basis

The numbers α_i are called the *coefficients* of the expansion of $\mathbf{y}$ in the $\{\mathbf{x}_i\}_{i=1}^n$ basis.

Orthogonal vectors: A set of vectors $\{\mathbf{x}_i\}_{i=1}^n$ is *(mutually) orthogonal* if $\mathbf{x}_i \perp \mathbf{x}_j$ for all $i, j \in \{1, 2, 3, \dots n\}$ and $i \neq j$.

This set is called **orthonormal** if its elements are all of unit length $\|\mathbf{x}_i\|_2 = 1$ for all $i \in \{1, 2, 3, \dots n\}$.

$$\mathbf{x}_i^\top \mathbf{x}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Representing a Vector in an Orthonormal Basis

An orthonormal collection of vectors is linearly independent.

Consider an orthonormal basis $\{\mathbf{x}_i\}_{i=1}^{n}$. The expansion of a vector $\mathbf{y}$ is given by,

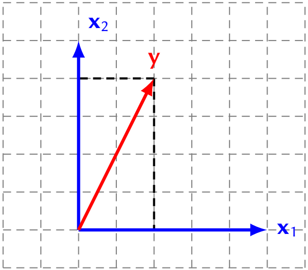
$$\mathbf{y} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \cdots + \alpha_n\mathbf{x}_n$$

$$\mathbf{x}_i^\top\mathbf{y} = \alpha_1\mathbf{x}_i^\top\mathbf{x}_1 + \alpha_2\mathbf{x}_i^\top\mathbf{x}_2 + \alpha_3\mathbf{x}_i^\top\mathbf{x}_3 + \cdots + \alpha_n\mathbf{x}_i^\top\mathbf{x}_n = \alpha_i$$

Representing a Vector in an Orthonormal Basis

Thus, we can rewrite this as,

$$\mathbf{y} = \left(\mathbf{y}^\top \mathbf{x}_1\right) \mathbf{x}_1 + \left(\mathbf{y}^\top \mathbf{x}_2\right) \mathbf{x}_2 + \left(\mathbf{y}^\top \mathbf{x}_3\right) \mathbf{x}_3 + \cdots + \left(\mathbf{y}^\top \mathbf{x}_n\right) \mathbf{x}_n$$



Dimension of a Vector Space

There an infinite number of bases for a vector space.

There is one thing that is common among all these bases – the number of bases vectors.

Definition: Dimension of a Vector Space

The number of vector in any basis for a vector space V is the dimension of the vector space V.

This number is a property of the vector space, and represents the "degrees of freedom" of the space.

Dimension of a Vector Space

- A subspace of dimension m can have at most m independent vectors.

- Notice that the word "dimension" of a vector space is different from the "dimension" of a vector.

- E.g. Vectors from $\mathbb{R}^3$ are three dimensional vectors. But the yz-plane in $\mathbb{R}^3$ is a 2 dimensional subspace of $\mathbb{R}^3$.

Linear Functions

Definition: Linear Function

A function f that maps vectors from $\mathbb{R}^n$ to $\mathbb{R}$, i.e. $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, which satisfies the superposition principle is called a *linear function*.

$$\textbf{Superposition} : f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$$

where $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Linear Functions

Inner product is a linear function in one of the arguments.

$$f(x) = \mathbf{w}^\top \mathbf{x} = w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n$$

Any linear function can be represented in the form $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$ with an appropriately chosen $\mathbf{w}$.