Applied Linear Algebra in Data Analysis

Matrices

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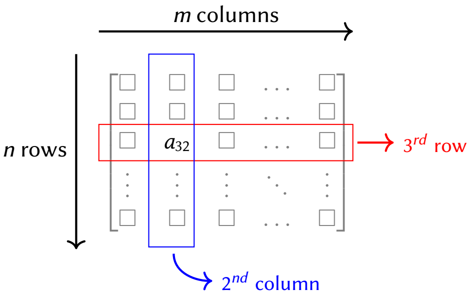
Matrices

Matrices are rectangular array of numbers. $\begin{bmatrix} 1.1 & -24 & \sqrt{2} \\ 0 & 1.12 & -5.24 \end{bmatrix}$



Matrices

Consider a matrix $\mathbf{A}$ with n rows and m columns.

$$\mathbf{A} \longrightarrow \begin{cases} \textbf{Tall/Skinny} & n > m \\ \textbf{Square} & n = m \\ \textbf{Wide/Fat} & n < m \end{cases}$$

Matrices

n-vectors can be interpreted as $n \times 1$ matrices. These are called *column vectors.*

A matrix with only one row is called a *row vector*, which can be referred to as n-row-vector. $\mathbf{x} = \begin{bmatrix} 1.45 & -3.1 & 12.4 \end{bmatrix}$

Block matrices & **Submatrices**: $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$. What are the dimensions of the different matrices?

Matrices

Matrices are also compact way to give a set of indexed column n-vectors, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \ldots \mathbf{x}_m$.

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_m \end{bmatrix}$$

$$\textbf{Zero matrix} = \mathbf{0}_{n \times m} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Identity matrix is a square $n \times n$ matrix with all zero elements, except the diagonals where all elements are 1.

$$i_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \qquad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix}$$

Matrices

Diagonal matrices is a square matrix with non-zero elements on its diagonal.

$$\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & -11 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 9.3 \end{bmatrix}$$

Triangular matrices: Are square matrices. *Upper triangular* $a_{ij} = 0, \forall i > j$; *Lower triangular* $a_{ij} = 0, \forall i < j$.

Matrix operations: Transpose

Transpose switches the rows and columns of a matrix. $\mathbf{A}$ is a $n \times m$ matrix, then its transpose is represented by $\mathbf{A}^\top$, which is a $m \times n$ matrix.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \implies \mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

Transpose converts between column and row vectors.

What is the transpose of a block matrix? $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$

Matrix operations: Matrix Addition

Matrix addition can only be carried out with matrices of same size. Like vectors we perform element wise addition.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Matrix operations: Matrix Addition

Properties of matrix addition:

- *Commutative*: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- *Associative*: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- *Addition with zero matrix*: $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$
- *Transpose of sum*: $(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$

Matrix operations: Scalar multiplication

Scalar multiplication Each element of the matrix gets multiplied by the scalar.

$$\alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}$$

We will mostly only deal with matrices with real entries. Such matrices are elements of the set $\mathbb{R}^{n \times m}$.

Given the aforementioned matrix operations and their properties, is $\mathbb{R}^{n \times m}$ a vector space?

Matrix operations: Matrix multiplication

A useful multiplication operation can be defined for matrices.

It is possible to *multiply* two matrices $\mathbf{A} \in \mathbb{R}^{n \times p}$ and $\mathbf{B} \in \mathbb{R}^{p \times m}$ through this *matrix multiplication* procedure.

The product matrix $\mathbf{C} := \mathbf{AB} \in \mathbb{R}^{n \times m}$, if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

$$c_{ij} := \sum_{k=1}^{p} a_{ik} b_{kj} \quad \forall i \in \{1, \ldots n\} \quad , j \in \{1 \ldots m\}$$

Matrix multiplication

Inner product is a special case of matrix multiplication between a *row vector* and a *column vector*.

$$\mathbf{x}^\top \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i$$

Matrix multiplication: Post-multiplication by a column vector

Consider a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ and a m-vector $\mathbf{x} \in \mathbb{R}^m$. We can multiply $\mathbf{A}$ and $\mathbf{x}$ to obtain $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^n$.

$$\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1i} x_i \\ \sum_{i=1}^m a_{2i} x_i \\ \vdots \\ \sum_{i=1}^m a_{ni} x_i \end{bmatrix} = \sum_{i=1}^m x_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} = \sum_{i=1}^m x_i \mathbf{a}_i$$

Post-multiplying a matrix $\mathbf{A}$ by a column vector $\mathbf{x}$ results in a linear combination of the columns of matrix $\mathbf{A}$.

$\mathbf{x}$ provides the column mixture.

Matrix multiplication: Pre-multiplication by a row vector

Let $\mathbf{x}^\top \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n\times m}$, then $\mathbf{y} = \mathbf{x}^\top \mathbf{A}$.

$$\mathbf{y} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^n x_i a_{i1} & \cdots & \sum_{i=1}^n x_i a_{im} \end{bmatrix} = \sum_{i=1}^n x_i \tilde{\mathbf{a}}_i^\top$$

where, $\tilde{\mathbf{a}}_i^\top = \begin{bmatrix} a_{i1} & \cdots & a_{im} \end{bmatrix}$.

Pre-multiplying a matrix $\mathbf{A}$ by a row vector $\mathbf{x}$ results in a linear combination of the rows of $\mathbf{A}$.

$\mathbf{x}^\top$ provides the row mixture.

Matrix multiplication

Multiplying two matrices $\mathbf{A} \in \mathbb{R}^{n\times p}$ and $\mathbf{B} \in \mathbb{R}^{p\times m}$ produces $\mathbf{C} \in \mathbb{R}^{n\times m}$,

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pm} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{n2} & \cdots & c_{nm} \end{bmatrix}$$

Matrix multiplication

Four interpretations of matrix multiplication.

1. Inner-Product interpretation
2. Column interpretation
3. Row interpretation
4. Outer product interpretation.

Matrix multiplication: Inner-product Interpreation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \ \mathbf{B} \in \mathbb{R}^{p \times m}, \ \mathbf{C} \in \mathbb{R}^{n \times m}$$

ij^{th} element of $\mathbf{C}$ is the inner product of the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$.

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = \tilde{\mathbf{a}}_i^\top \mathbf{b}_j$$

where, $i \in \{1 \dots n\}, j \in \{1 \dots m\}$

Matrix multiplication: Column interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \, \mathbf{B} \in \mathbb{R}^{p \times m}, \, \mathbf{C} \in \mathbb{R}^{n \times m}$$

Columns of $\mathbf{C}$ are the linear combinations of the columns of $\mathbf{A}$.

$$\mathbf{C} = \mathbf{A} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \dots & \mathbf{A}\mathbf{b}_m \end{bmatrix}$$

j^{th} column of $\mathbf{C}$ is the linear combination of the columns of $\mathbf{A}$

$$\mathbf{c}_j = \sum_{k=1}^{p} b_{kj} \mathbf{a}_k$$

Matrix multiplication: Row interpretation

$$\mathbf{C} = \mathbf{A}\mathbf{B}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \ \mathbf{B} \in \mathbb{R}^{p \times m}, \ \mathbf{C} \in \mathbb{R}^{n \times m}$$

Rows of $\mathbf{C}$ are the linear combinations of the rows of $\mathbf{B}$.

$$\mathbf{C} = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \\ \tilde{\mathbf{a}}_2^\top \\ \dots \\ \tilde{\mathbf{a}}_n^\top \end{bmatrix} \mathbf{B} = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \mathbf{B} \\ \tilde{\mathbf{a}}_2^\top \mathbf{B} \\ \dots \\ \tilde{\mathbf{a}}_n^\top \mathbf{B} \end{bmatrix}$$

i^{th} row of $\mathbf{C}$ is the linear combination of the rows of $\mathbf{B}$

$$\tilde{\mathbf{c}}_i^\top = \sum_{k=1}^{p} a_{ik} \tilde{\mathbf{b}}_k^\top$$

Matrix multiplication: Outer product interpretation

Outer product: Product between a colum vector and a row vector. Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. The *outer product* is defined as,

$$\mathbf{x}\mathbf{y}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_m \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_m \end{bmatrix} \in \mathbb{R}^{n \times m}$$

Matrix multiplication: Outer product interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \, \mathbf{B} \in \mathbb{R}^{p \times m}, \, \mathbf{C} \in \mathbb{R}^{n \times m}$$

$\mathbf{C}$ can be written as the sum of p outer products of columns of $\mathbf{A}$ and rows of $\mathbf{B}$.

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \dots & \mathbf{a}_p \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{b}}_1^\top \\ \tilde{\mathbf{b}}_2^\top \\ \tilde{\mathbf{b}}_3^\top \\ \vdots \\ \tilde{\mathbf{b}}_p^\top \end{bmatrix} = \sum_{i=1}^{p} \mathbf{a}_i \tilde{\mathbf{b}}_i^\top$$

Properties of matrix multiplication

Not commutative: $\mathbf{AB} \neq \mathbf{BA}$
The product of two matrices might not alwasys be defined. When it is defined, $\mathbf{AB}$ and $\mathbf{BA}$ need not match.

Distributive: $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{BC}$ and $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$

Associative: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

Transpose: $(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$

Scalar product: $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$

Linear transformations

Definition: Linear Transformation/Map

A *linear transformation* or *linear map* is a function $f : \mathbb{R}^m \mapsto \mathbb{R}^n$ that satisfies the following properties:

$$\textbf{Superposition} : f\left(\alpha \mathbf{x} + \beta \mathbf{y}\right) = \alpha f\left(\mathbf{x}\right) + \beta f\left(\mathbf{y}\right)$$

where $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and $f\left(\mathbf{x}\right), f\left(\mathbf{y}\right) \in \mathbb{R}^n$.

Each component of the vector $f\left(\mathbf{x}\right)$ is a linear function of $\mathbf{x}$.

$$.f\left(\mathbf{x}\right) = \begin{bmatrix} f_1\left(\mathbf{x}\right) \\ f_2\left(\mathbf{x}\right) \\ \vdots \\ f_n\left(\mathbf{x}\right) \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \mathbf{x} \\ \tilde{\mathbf{a}}_2^\top \mathbf{x} \\ \vdots \\ \mathbf{a}_n^\top \mathbf{x} \end{bmatrix} = \mathbf{A}\mathbf{x}$$

where, $\mathbf{A} \in \mathbb{R}^{n \times m}$.

Matrices can be thought of as representation of a particular linear transformation.

Why does matrix multiplication have this strange definition?

Consider the following two functions,

$$\mathbf{y} = f(\mathbf{x}) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{A}\mathbf{x}$$

$$\mathbf{v} = g(\mathbf{u}) = \begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{B}\mathbf{u}$$

What will be $\mathbf{z} = f(g(\mathbf{x}))$?

Why does matrix multiplication have this strange definition?

$$\mathbf{z} = h(\mathbf{u}) = f(g(\mathbf{u})) = f\left(\begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix}\right) = \begin{bmatrix} a\alpha u_1 + a\beta u_2 + b\gamma u_1 + b\delta u_2 \\ c\alpha u_1 + c\beta u_2 + d\gamma u_1 + d\delta u_2 \end{bmatrix}$$

$$= \begin{bmatrix} (a\alpha + b\gamma)\, u_1 + (a\beta + b\delta)\, u_2 \\ (c\alpha + d\gamma)\, u_1 + (c\beta + d\delta)\, u_2 \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{z} = h(\mathbf{u}) = f(g(\mathbf{u})) = f(\mathbf{Bu}) = (\mathbf{AB})\,\mathbf{u}$$

$$\implies \mathbf{AB} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

Matrix multiplication represents the composition of linear transformations.

Rank of a matrix A

Defintion: Rank of a matrix A

The **rank of a matrix A** is the dimension of the subspace spanned by the columns of **A** or the rows of $\mathbf{A} \in \mathbb{R}^{n \times m}$.

$$\begin{aligned} rank\left(\mathbf{A}\right) &= \dim span\left(\{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m\}\right) \rightarrow \text{Column rank} \\ &= \dim span\left(\left\{\tilde{\mathbf{a}}_1^\top, \tilde{\mathbf{a}}_2^\top, \ldots \tilde{\mathbf{a}}_n^\top\right\}\right) \rightarrow \text{Row rank} \end{aligned}$$

Column Rank is always equal to the row rank.

Rank tells us the number of independent columns/rows in the matrix.

Full rank matrix A: $rank\left(\mathbf{A}\right) = \min\left(n, m\right)$
Rank deficient matrix A: $rank\left(\mathbf{A}\right) < \min\left(n, m\right)$

Matrix Inverse

Consider the square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. $\mathbf{B} \in \mathbb{R}^{n \times n}$ is the inverse of $\mathbf{A}$, if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, and $\mathbf{B}$ is represented as $\mathbf{A}^{-1}$.

Not all matrices have inverses. A matrix with an inverse is called **non-singular**, otherwise it is called **singular**.

For a non-singular matrix $\mathbf{A}$, $\mathbf{A}^{-1}$ is unique. $\mathbf{A}^{-1}$ is both the left and right inverse.

Matrix Inverse

A matrix $\mathbf{A}$ has an inverse, if and only if $\mathbf{A}$ is full rank, i.e. *rank* $(\mathbf{A}) = n$

$\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved as follows, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. *It is never solved like this in practice.*

Inverse of product of matrices, $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$ and $\left(\mathbf{A}^{-1}\right)^{\top} = \left(\mathbf{A}^{\top}\right)^{-1}$

Complex Vectors and Matrices

Similar to $\mathbb{R}^n$, we can have $\mathbb{C}^n$. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{r1} + jx_{i1} \\ x_{r2} + jx_{i2} \\ \vdots \\ x_{rn} + jx_{in} \end{bmatrix}$.

Vector addition and scalar mulitplication are the same. The scalar is a complex number.

Additive identity, and scalar multiplication identity are the same. So is the **standard basis** $\{\mathbf{e}_i\}_{i=1}^n$

Linear independence: The set $\{\mathbf{v}_i\}_{i=1}^n$ with $\mathbf{v}_i \in \mathbb{C}^n$ is linearly independent, if $\sum_{i=1}^n c_i \mathbf{v}_i = 0, \implies c_i = 0, \ \forall 1 \leq i \leq n, \ \ c_i \in \mathbb{C}$

Complex Vectors and Matrices

Inner product: $\mathbf{x}^*\mathbf{y} = \begin{bmatrix}\overline{x}_1 & \overline{x}_2 & \cdots & \overline{x}_n\end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} \overline{x}_i y_i$

Length: $\|\mathbf{x}\|_2^2 = \mathbf{x}^*\mathbf{x} = \sum_{i=1}^{n} \overline{x}_i x_i = \sum_{i=1}^{n} |x_i|^2$

Orthogonality: $\mathbf{x}^*\mathbf{y} = 0$

Complex matrices have complex entries. $\mathbf{A} \in \mathbb{C}^{m \times n}$ such that $a_{ij} \in \mathbb{C}, \; \forall 1 \leq i \leq m, \; 1 \leq j \leq n$

Complex Vectors and Matrices

The transpose operation is generalized to conjugate transpose known as the *Hermitian*. $\mathbf{A}^* = \overline{\mathbf{A}}^\top$.

The idea of symmetric matrices $\mathbb{R}^{n\times n}$ are now generalized to $\mathbb{C}^{n\times n}$ as $\mathbf{A} = \mathbf{A}^*$. Such matrices are called **Hermitian** matrices.

Orthogonal matrices in the complex case are called **Unitary** matrices, $\mathbf{U}^*\mathbf{U} = \mathbf{I} \implies \mathbf{U}^{-1} = \mathbf{U}^*$.