APPENDIX - 2 Vector Space and Linear Transformations

[*For the EE & CS streams in II Semester*]

2.0 Preamble: Set Theory

A well defined collection of objects without repetition is called a *Set*, usually denoted by A, B, C .... and these objects are called elements of the set. If every element of a set A is also an element of a set B, then A is called a subset of B, denoted as $A \subset B$. The set, with no elements is called a *null set* or '*empty set*' denoted by ϕ.

- **Cartesian product of two sets**

The *cartesian product* of two non empty sets A and B denoted by $A \times B$ is defined by $A \times B = \{(a,b) / a \in A, b \in B\}$

Evidently, $(x,y) \in A \times B \Leftrightarrow x \in A, y \in B$

In general $(a,b) \neq (b,a)$ and $(a,b) = (c,d)$ *iff* $a = c$ and $b = d$.

- **Binary operations**

The word *binary* can be understood as a composition /involvement of two things. Also, the word *operation* in a general perspective can be regarded as an organized action.

If A is a nonempty set, then a function $f : A \times A \to A$ is called a *binary operation* on A. $*$ is a commonly used symbol for the binary operation. We present a few properties of binary operation where A is a non empty set and x, y, z are elements of A.

(i) *Closure property* : $x * y \in A$.

(ii) *Commutative property* : $x * y = y * x$

(iii) *Associative property* : $x * (y * z) = (x * y) * z$

Remark : *If R is the set of real numbers, evidently the basic operations + and × are commutative and associative, but − and ÷ are not so.*

(iv) *Identity property* : Binary operation $*$ is said to possess identity element, if $\exists$ an element $e \in A$ such that $x * e = e * x = x, \forall x \in A$.

Evidently '0' is the identity for addition and '1' for multiplication.

(v) *Inverse property* : For any element $x \in A$, if there exists an element $x' \in A$ such that $x * x' = x' * x = e$, the element x' usually denoted by x^{-1} is called

the inverse of the element x. $(x * x^{-1} = x^{-1} * x = e)$

If R is the set of real numbers without '0' then $(1/x)$ is the inverse element $\forall x \in R$.

$$x \times (1/x) = (1/x) \times x = 1.$$

Remark : *If G is a non empty set and $*$ is the binary operation, we say that $(G, *)$ is a 'Group' if the closure, associative, identity and the inverse properties are satisfied. In addition to these if the commutative property is also satisfied then $(G, *)$ is called an 'Abelian Group'.*

Note : *Distributive property for the two basic operations '+' and 'x' :*

$$x \times (y + z) = (x \times y) + (x \times z)$$

Field : A field is a set F, containing atleast two elements, on which two basic operations of addition (+) and multiplication $(\cdot)$ are defined so that for every pair of elements $x, y \in F$, there are, unique elements $x + y$ and $x \cdot y$ also belong to F.

Important Note : *The following contents already discussed in the topic : Linear Algebra is a necessary pre-requisite for our future content presentation.*

- *Elementary transformations associated with a matrix.*
- *Row echelon form and rank of a matrix.*
- *Consistency and solution of a system of equations.*

2.1 Vector Spaces

An ordered n tuple : $v = (x_1, x_2, \ldots x_n)$ is called a *vector*. Let F be a field and V be a non empty set. If $x, y \in V$ and c an arbitrary scalar belong to F, the unique elements $x + y$ and $c \cdot x$ also belong to V.

The set V or the system $(V, +, \cdot)$ is called a *Vector Space* over the field F represented as $V(F)$ if the following axioms are satisfied for every x, y, z belonging to V.

(i) $x + (y + z) = (x + y) + z$

(ii) $\exists$ an element '0' in V such that $x + 0 = x$

(iii) $\forall x \in V, \exists -x \in V$ such that $x + (-x) = 0 = -x + x$

(iv) $x + y = y + x$

(v) $c\cdot(x+y)=c\cdot x+c\cdot y$ and $(c_1+c_2)\cdot x=c_1\cdot x+c_2\cdot y$ where $c, c_1, c_2 \in F$.

(vi) $(c_1 c_2)\cdot x = c_1\cdot(c_2 x)$

(vii) $1\cdot x = x$

Remark : *$V(F)$ is a vector space if V is an abelian group under '+' and conditions* (v), (vi) *and* (vii) *are also satisfied.*

Examples.

(1) Let V be the set of all ordered pair of real numbers in the form (x, y). If $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ then,

$v_1 + v_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c\cdot(x,y) = (cx, cy)$

Evidently all the axioms for a vector space are satisfied. '0' is the zero vector $(0, 0)$ and $(-x, -y) \in V$ gives $(x,y)+(-x,-y) = (0,0)$.

(2) The set of all real numbers is a vector space w.r.t usual addition and multiplication.

(3) The set M representing matrices of order $m\times n$, having real numbers as their elements is a vector space under the usual addition of matrices and matrix multiplication with a scalar. Null matrix of order $m\times n$ is the zero vector and $-A$, A being a matrix is $(-1)A$, will give $A+(-A) = [0]$ being the null matrix.

2.11 Subspace

Let $V(F)$ be a vector space. A subset V' of V is called a 'subspace' of V if V' is also a vector space over F.

Remark : *Every vector space V has a trivial sub space namely V itself.*

Property : A non empty subset of V' of $V(F)$ is a sub space V if and only if (*iff*) the following conditions are satisified.

(i) $\forall x, y \in V'$, $x+y \in V'$ (ii) $c\cdot x \in V'$ where $c \in F$.

2.12 Linear combination and Linear span

Let $V(F)$ be a vector space and $x_1, x_2, \cdots x_n \in V$. A new vector in the form $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ where $c_1, c_2 \cdots c_n \in F$ is called a *'linear combination'* of $x_1, x_2, \cdots x_n$. Next, let V' be a non empty finite subset of V. The set of all

linear combinations of the elements of V' is called *linear span* of V' denoted by $L(V')$

Property : $L(V')$ is a sub space of $V(F)$.

Examples : (1) An ordered pair (x, y) is a linear combination of $(1, 0)$ and $(0, 1)$ as we can write $(x, y) = x(1,0) + y(0,1)$

(2) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We can write $A = a M_1 + b M_2 + c M_3 + d M_4$

WORKED PROBLEMS

[1] *Which of the following vectors :*

(i) $v_1 = (3,3,3)$ **(ii)** $v_2 = (1,5,6)$ **(iii)** $v_3 = (4,2,6)$

is a linear combination of the vectors $x_1 = (1,-1,3)$ and $x_2 = (2,4,0)$?

☞ (i) Let $c_1 x_1 + c_2 x_2 = v_1$

ie., $c_1(1,-1,3) + c_2(2,4,0) = (3,3,3)$

or $(c_1 + 2c_2, -c_1 + 4c_2, 3c_1) = (3,3,3)$

$\Rightarrow$ $c_1 + 2c_2 = 3, -c_1 + 4c_2 = 3, 3c_1 = 3$ or $c_1 = 1$

$c_1 = 1$ gives $c_2 = 1$ from the other equations.

$\therefore$ $(3,3,3) = 1(1,-1,3) + 1(2,4,0)$ **is the linear combination.**

(ii) Let $c_1 x_1 + c_2 x_2 = v_2 = (1,5,6)$

ie., $(c_1 + 2c_2 - c_1 + 4c_2, 3c_1) = (1,5,6)$

$\Rightarrow$ $c_1 + 2c_2 = 1, -c_1 + 4c_2 = 5, 3c_1 = 6$ or $c_1 = 2$

But $c_1 = 2$ gives $c_2 = -1/2$ and $c_2 = 7/4$ respectively from the first two equations which is unacceptable.

$\therefore \quad v_2 = (1,5,6)$ **cannot be a linear combination of x_1 and x_2.**

(iii) Let $c_1 x_1 + c_2 x_2 = v_3 = (4,2,6)$

or $\quad c_1 + 2c_2 = 4, -c_1 + 4c_2 = 2, 3c_1 = 6$ or $c_1 = 2$

But $c_1 = 2$ gives $c_2 = 1$ from the first and second equations.

$\therefore \quad (4,2,6) = 2(1,-1,3) + 1(2,4,0)$ **is a linear combinations.**

[2] *Show that the vector,* (5, 9, 5) *is a linear combination of the vectors* (2, 1, 4), $(1,-1,3)$ *and* (3, 2, 5).

☞ Let $c_1(2,1,4) + c_2(1,-1,3) + c_3(3,2,5) = (5,9,5)$

ie., $\quad (2c_1 + c_2 + 3c_3, c_1 - c_2 + 2c_3, 4c_1 + 3c_2 + 5c_3) = (5,9,5)$

$\Rightarrow \quad 2c_1 + c_2 + 3c_3 = 5,\ c_1 - c_2 + 2c_3 = 9,\ 4c_1 + 3c_2 + 5c_3 = 5$

We shall solve this system of equations by Gauss-elimination matrix method. The augmented matrix [$A : B$] on interchanging the first two equations (*for convenience to perform row transformations*) is as follows.

$$[A:B] = \begin{bmatrix} 1 & -1 & 2 & : & 9 \\ 2 & 1 & 3 & : & 5 \\ 4 & 3 & 5 & : & 5 \end{bmatrix} \begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -4R_1 + R_3 \\ \sim \end{matrix} \begin{bmatrix} 1 & -1 & 2 & : & 9 \\ 0 & 3 & -1 & : & -13 \\ 0 & 7 & -3 & : & -31 \end{bmatrix}$$

$$R_3 \to -7R_2 + 3R_3 \underset{\sim}{} \begin{bmatrix} 1 & -1 & 2 & : & 9 \\ 0 & 3 & -1 & : & -13 \\ 0 & 0 & -2 & : & -2 \end{bmatrix}$$

$\Rightarrow \quad c_1 - c_2 + 2c_3 = 9,\ 3c_2 - c_3 = 13,\ -2c_3 = -2$ or $c_3 = 1$

Further, we obtain $c_2 = -4$ and $c_1 = 3$.

Thus, $3(2,1,4) - 4(1,-1,3) + 1(3,2,5) = (5,9,5)$ **is the linear combination.**

[3] *Show that the vector* (2, 2, 3) *is in the linear span of the vectors* (2, 1, 4), $(1,-1,3)$ *and* $(3,2,5)$.

☞ The three given vectors are same as in the previous problem and let S be the set of these vectors.

We need to show that (2, 2, 3) is a linear combination of the vectors in S. Proceeding as in **[2]** we have the system of equations :

$$2c_1 + c_2 + 3c_3 = 2,\ c_1 - c_2 + 2c_3 = 2,\ 4c_1 + 3c_2 + 5c_3 = 3$$

The associated augmented matrix on interchanging the first two equations is as follows.

$$[A:B] = \begin{bmatrix} 1 & -1 & 2 & : & 2 \\ 2 & 1 & 3 & : & 2 \\ 4 & 3 & 5 & : & 3 \end{bmatrix} \underset{R_3 \to -4R_1 + R_3}{R_2 \to -2R_1 + R_2} \sim \begin{bmatrix} 1 & -1 & 2 & : & 2 \\ 0 & 3 & -1 & : & -2 \\ 0 & 7 & -3 & : & -5 \end{bmatrix}$$

$$\underset{\sim}{R_3 \to -7R_2 + 3R_3} \begin{bmatrix} 1 & -1 & 2 & : & 2 \\ 0 & 3 & -1 & : & -2 \\ 0 & 0 & -2 & : & -1 \end{bmatrix}$$

$$\Rightarrow \quad c_1 - c_2 + 2c_3 = 2,\ 3c_2 - c_3 = -2,\ -2c_3 = -1 \text{ or } c_3 = 1/2$$

Further we obtain $c_2 = -1/2$ and $c_1 = 1/2$

Thus, $\frac{1}{2}(2,1,4) - \frac{1}{2}(1,-1,3) + \frac{1}{2}(3,2,5) = (2,2,3)$

and hence we conclude that $(2,2,3) \in L(S)$ being the linear span of S.

[4] *Find the value of 'k' for which the vector $(1,-2,k)$ is a linear combination of the vectors $(3,0,-2)$ and $(2,-1,-5)$.*

☞ Let, $c_1(3,0,-2) + c_2(2,-1,-5) = (1,-2,k)$

ie., $(3c_1 + 2c_2, -c_2, -2c_1 - 5c_2) = (1,-2,k)$

$\Rightarrow \quad 3c_1 + 2c_2 = 1,\ -c_2 = -2,\ -2c_1 - 5c_2 = k \ \therefore c_2 = 2$

Using $c_2 = 2$ in $3c_1 + 2c_2 = 1$, we get $c_1 = -1$ and also by using $c_1 = -1,\ c_2 = 2$ in $-2c_1 - 5c_2 = k$ we get $\boxed{k = -8}$

[5] *Show that the vector (5, 9, 5)is not in the linear span of*

$$S = \{(5,3,7),(3,26,2),(7,2,10)\}$$

☞ Let $c_1(5,3,7) + c_2(3,26,2) + c_3(7,2,10) = (5,9,5)$

ie., $(5c_1+3c_2+7c_3, 3c_1+26c_2+2c_3, 7c_1+2c_2+10c_3) = (5,9,5)$

$\Rightarrow \quad 5c_1+3c_2+7c_3 = 5,\ 3c_1+26c_2+2c_3 = 9,\ 7c_1+2c_2+10c_3 = 5$

We have, $[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \begin{matrix} R_2 \to -3R_1+5R_2 \\ R_3 \to -7R_1+5R_3 \end{matrix} \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & -11 & 1 & : & -10 \end{bmatrix}$

$$\underset{\sim}{R_3 \to R_2+11R_3} \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & 0 & 0 & : & -80 \end{bmatrix}$$

Observing the last row, reflecting an absurd situation as $0 = -80$, we conclude that there does not exist a solution for the system of equations. Equivalently, we conclude that (5, 9, 5) cannot be a linear combinations of the members of S.

Thus (5, 9, 5) is not in the linear span S : L (S).

[6] *Express the polynomial* $f(x) = x^2+4x-3$ *as a linear combination of the polynomials* $f_1(x) = x^2-2x+5, f_2(x) = 2x^2-3x$ *and* $f_3(x) = x+3$.

☞ Let $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = f(x)$

ie., $c_1(x^2-2x+5) + c_2(2x^2-3x) + c_3(x+3) = x^2+4x-3$

or $\quad (c_1+2c_2)x^2 + (-2c_1-3c_2+c_3)x + (5c_1+3c_3) = x^2+4x-3$

$\Rightarrow \quad c_1+2c_2 = 1,\ -2c_1-3c_2+c_3 = 4,\ 5c_1+3c_3 = -3$

We have $[A:B] = \begin{bmatrix} 1 & 2 & 0 & : & 1 \\ -2 & -3 & 1 & : & 4 \\ 5 & 0 & 3 & : & -3 \end{bmatrix} \begin{matrix} R_2 \to 2R_1+R_2 \\ R_3 \to -5R_1+R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 & : & 1 \\ 0 & 1 & 1 & : & 6 \\ 0 & -10 & 3 & : & -8 \end{bmatrix}$

$$\underset{\sim}{R_3 \to 10R_2+R_3} \begin{bmatrix} 1 & 2 & 0 & : & 1 \\ 0 & 1 & 1 & : & 6 \\ 0 & 0 & 13 & : & 52 \end{bmatrix} \quad \begin{matrix} \Rightarrow c_1+2c_2 = 1 \\ c_2+c_3 = 6 \\ 13c_3 = 52 \text{ or } c_3 = 4;\ c_2 = 2,\ c_1 = -3 \end{matrix}$$

Thus, $-3(x^2-2x+5) + 2(2x^2-3x) + 4(x+3) = x^2+4x-3.$

[7] *Decide whether or not the vector* $M = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ *in the vector space of second order square matrices is a linear combination of the following matrices.*

$$M_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},\ M_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } M_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Let $c_1 M_1 + c_2 M_2 + c_3 M_3 = M$, where c_1, c_2, c_3 are scalars.

$$\therefore \quad c_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} c_1 + c_2 + c_3, & c_1 + c_2 - c_3 \\ -c_2 & -c_1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$$

$$\Rightarrow \quad c_1 + c_2 + c_3 = 3,\ c_1 + c_2 - c_3 = -1,\ -c_2 = 1,\ -c_1 = -2$$

$\therefore \quad c_1 = 2,\ c_2 = -1$. Also $c_3 = 2$ satisfy the first two equations.

Thus, $2M_1 - M_2 + 2M_3 = M$ **is the linear combination.**

[8] *Does the vectors* $v_1 = (1,1,1)$, $v_2 = (0,1,1)$ *and* $v_3 = (0,0,1)$ *span the vector space* $V_3(R)$, R *being the set of real numbers ?*

Let, $c_1 v_1 + c_2 v_2 + c_3 v_3 = (x, y, z)$ say.

ie., $\quad c_1(1,1,1) + c_2(0,1,1) + c_3(0,0,1) = (x, y, z)$

or $\quad (c_1, c_1 + c_2, c_1 + c_2 + c_3) = (x, y, z)$

$\Rightarrow \quad c_1 = x,\ c_1 + c_2 = y,\ c_1 + c_2 + c_3 = z$

$\therefore \quad c_1 = x, c_2 = y - x,\ c_3 = z - y$

We conclude that every x, y, z in $V_3(R)$ can be expressed as a linear combination of v_1, v_2, v_3.

Thus the vectors v_1, v_2, v_3 **span of the vector space** $V_3(R)$.

[9] *Show that the vectors* $(1,2,1)$, $(1,-1,2)$ *and* $(2,1,0)$ *span* $V_3(R)$.

☞ Let, $c_1(1,2,1)+c_2(1,-1,2)+c_3(2,1,0)=(x,y,z)$

ie., $(c_1+c_2+2c_3, 2c_1-c_2+c_3, c_1+2c_2)=(x,y,z)$

$\Rightarrow$ $c_1+c_2+2c_3=x,\ 2c_1-c_2+c_3=y,\ c_1+2c_2=z$

We have $[A:B]=\begin{bmatrix}1 & 1 & 2 & : & x\\ 2 & -1 & 1 & : & y\\ 1 & 2 & 0 & : & z\end{bmatrix} \underset{\sim}{\begin{matrix}R_2\to -2R_1+R_2\\ R_3\to -R_1+R_3\end{matrix}} \begin{bmatrix}1 & 1 & 2 & : & x\\ 0 & -3 & -3 & : & (y-2x)\\ 0 & 1 & -2 & : & (z-x)\end{bmatrix}$

$$\underset{\sim}{R_2\leftrightarrow R_3}\begin{bmatrix}1 & 1 & 2 & : & x\\ 0 & 1 & -2 & : & (z-x)\\ 0 & -3 & -3 & : & (y-2x)\end{bmatrix} \underset{\sim}{R_3\to 3R_2+R_3} \begin{bmatrix}1 & 1 & 2 & : & x\\ 0 & 1 & -2 & : & (z-x)\\ 0 & 0 & -9 & : & (-5x+y+3z)\end{bmatrix}$$

This form of matrix clearly indicates that the system of equations possess unique solution for c_1, c_2, c_3. Every (x, y, z) in $V_3(R)$ can be expressed as a linear combination of the given three vectors.

Thus the given vectors span the vector space $V_3(R)$.

[10] *Show that the polynomial* x^2+2x+2 *is in the linear span of the polynomials* $(4x^2+x+2)$, $(3x^2-x+1)$ *and* $(5x^2+2x+3)$.

☞ Let, $c_1(4x^2+x+2)+c_2(3x^2-x+1)+c_3(5x^2+2x+3)=x^2+2x+2$

ie., $(4c_1+3c_2+5c_3)x^2+(c_1-c_2+2c_3)x+(2c_1+c_2+3c_3)=x^2+2x+2$

$\Rightarrow$ $4c_1+3c_2+5c_3=1,\ c_1-c_2+2c_3=2,\ 2c_1+c_2+3c_3=2$

By solving this system of equations as in the earlier problems we obtain

$c_1=-9/2,\ c_2=1/2$ and $c_3=7/2$

$$\therefore\ -\frac{9}{2}(4x^2+x+2)+\frac{1}{2}(3x^2-x+1)+\frac{7}{2}(5x^2+2x+3)=x^2+2x+2$$

Thus the polynomial (x^2+2x+2) is in the linear span of the three given polynomials.

2.13 Linearly independent and dependent sets

A set of vectors $\{v_1, v_2, \cdots v_n\}$ of $V(F)$ is said to be *linearly independent* if there exists scalars $c_1, c_2 \cdots c_n \in F$, such that $c_1 v_1 + c_2 v_2 + \cdots c_n v_n = 0$ implies $c_1 = 0,\ c_2 = 0, \cdots c_n = 0$.

Otherwise, if atleast one of the c_i's ($i = 1$ to n) is not equal to 0, the set of vectors $\{v_i\}$ $i = 1$ to n are said to be *linearly dependent.*

Examples : (1) Consider, $A = \{(1,0,0),(0,1,0),(0,0,1)\}$

Now, $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = 0 = (0,0,0)$ will give us

$(c_1, c_2, c_3) = (0,0,0) \Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0$

$\therefore$ the set of vector in A are **linearly independent.**

(2) Consider $B = \{(1,2,3),(2,3,4)(3,4,5)\}$

Now, $c_1(1,2,3) + c_2(2,3,4) + c_3(3,4,5) = (0,0,0)$ will give us

$(c_1 + 2c_2 + 3c_3, 2c_1 + 3c_2 + 4c_3, 3c_1 + 4c_2 + 5c_3) = (0,0,0)$

or $c_1 + 2c_2 + 3c_3 = 0, 2c_1 + 3c_2 + 4c_3 = 0, 3c_1 + 4c_2 + 5c_3 = 0$

The augmented matrix $[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 2 & 3 & 4 & : & 0 \\ 3 & 4 & 5 & : & 0 \end{bmatrix}$

$$[A:B] \sim \begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & -1 & -2 & : & 0 \\ 0 & -2 & -4 & : & 0 \end{bmatrix} \underset{\sim}{R_3 \to -2R_2 + R_3} \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & -1 & -2 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Hence, $c_1 + 2c_2 + 3c_3 = 0, -c_2 + 2c_3 = 0$ or $c_2 + 2c_3 = 0$

One of the scalars can be chosen arbitrarily, , say $c_3 = k \neq 0$.

$\therefore \quad c_2 = -2k$ and $c_1 = k$

That is $k(1,2,3) - 2k(2,3,4) + k(3,4,5) = (0,0,0)$ is the linear combination, where $k \neq 0$ is arbitrary.

Hence we conclude that the set of vectors in B are **linearly dependent.**

2.14 Basis and Dimension

A subset of S of a vector space $V(F)$ is called a ***basis*** if S is a linearly independent set and linear span of $S : L(S) = V$.

We can say that a basis of a vector space is a linearly independent subset which spans V.

Further the number of elements present in any basis of V is called the ***dimension*** of V denoted as dim V or $d(V)$.

Remarks : (1) Referring to Example - 1, we can say that the set $A = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $V_3(R)$. The members of A spans $V_3(R)$. Further, this basis is called the standard basis of $V_3(R)$ and they are usually named as $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Also the dimension is equal to 3.

(2) Referring to Example - 2, we can say that the set B is not a basis of $V_3(R)$ since the members of B are linearly dependent.

Further, the dimension of the sub space is equal to 2 being the number of linearly independent members of B. Basis of the sub space is $\{(1,2,3),(0,-1,-2)\}$.

Note for problems : In problems, linear independence or dependence of vectors is concluded through matrix approach as follows.

* *We form the matrix of the given vectors and reduce this matrix to row echelon form.*
* *If all the rows are non zero, the vectors are linearly independent. If there exists a zero row the vectors are linearly dependent.*
* *The number of non zero rows is equal to the dimension of the vector space and these non zero rows constitute a basis of the vector space.*
* *Further it may be noted that in the case of formation of a square matrix say A, non zero determinant ensures linear independence of the vectors.*

WORKED PROBLEMS

[11] *Check the following vectors for linear independence.*

(i) $\{(1,1,-1),(2,-3,5),(-2,1,4)\}$ **(ii)** $\{(1,3,2),(1,-7,-8),(2,1,-1)$

☞ **(i)** Let A represent the given set of vectors.

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -3 & 5 \\ -2 & 1 & 4 \end{vmatrix} = 1(-12-5)-1(8+10)-1(2-6)$$
$$= -17-18+4 = -31 \neq 0 \; ; |A| \neq 0$$

Thus the set of vectors are linearly independent.

(ii) Let B represent the given set of vectors.

$$|B| = \begin{vmatrix} 1 & 3 & 2 \\ 1 & -7 & -8 \\ 2 & 1 & -1 \end{vmatrix} = 1(7+8)-3(-1+16)+2(1+14)$$
$$= 15-45+30 = 0. \; |A| = 0.$$

Thus the set of vectors are linearly dependent.

[12] *Show that the set of vectors* $S = \{(1,1,2)\ (-3,1,0)\ (1,-1,1)\ (1,2,-3)\}$ *are linearly dependent.*

☞ We form the matrix of the given set of vectors and aim to reduce into row echelon form.

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix} \underset{\sim}{\begin{matrix} R_2 \to 3R_1 + R_2 \\ R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4 \end{matrix}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 6 \\ 0 & -2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \underset{\sim}{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 2 & -1 \\ 0 & 4 & 6 \end{bmatrix}$$

$$\underset{\sim}{\begin{matrix} R_3 \to -2R_2 + R_3 \\ R_4 \to -4R_2 + R_4 \end{matrix}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 9 \\ 0 & 0 & 26 \end{bmatrix} \underset{R_4 \to -26R_3 + R_4}{1/9 \cdot R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has a zero row and hence **we conclude that the given set of vectors are linearly dependent.**

Remark : *The dimension of the subspace spanned by S is 3 and a basis of the subspace is* $\{(1,1,2)\ (0,1,-5)\ (0,0,1)\}$.

[13] *Given the set* $S = \{(0,2,3,4)\ (2,3,5,4)\ (4,8,13,12)\}$ *and* $T = \{(1,2,2,1), (2,1,1,2), (3,2,2,3), (0,1,1,0)\}$ *show that both S and T are linearly dependent.*

☞ Consider $[S] = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$

$$\underset{\sim}{R_3 \to -2R_1 + R_3} \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} \underset{\sim}{R_3 \to -R_2 + R_3} \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The row echelon form of [S] has a zero row and hence we conclude that the **set S is linearly dependent.**

Next, Consider $[T] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{bmatrix} \underset{\sim}{\begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \end{matrix}} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$-1/3 \cdot R_2$ and later $R_3 \to 4R_2 + R_3$, $R_4 \to -R_2 + R_4$

$[T] \sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ [T] in the row echelon form has two zero rows.

We conclude that the **set T is linearly dependent.**

[14] *Is the set of vectors* $\{(4,0,2,1)\ (2,1,3,4)\ (2,3,4,7)\ (2,3,1,4)\}$ *linearly independent ?*

☞ Let $[A] = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix} \underset{\sim}{R_2 \leftrightarrow R_1} \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$

$$\underset{\substack{R_2 \to -2R + R_2 \\ R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4}}{\sim} \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & -2 & 0 \end{bmatrix} \underset{\substack{R_3 \to R_2 + R_3 \\ R_4 \to R_2 + R_4}}{\sim} \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -6 & -7 \end{bmatrix}$$

$$\underset{R_4 \to -2R_3 + R_4}{\sim} \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ The row echelon form has no zero rows.

Hence **we conclude that the given set of vectors are linearly independent.**

[15] *Investigate the values of λ and μ such that the set*

$S = \{(1,1,1,6), (1,2,3,10), (1,2,\lambda,\mu)\}$ *is*

(a) *linearly independent* **(b)** *linearly dependent*

Consider $[S] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \underset{\substack{R_2 \to -R_1 + R_2 \\ R_3 \to -R_1 + R_3}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & (\lambda-1) & (\mu-6) \end{bmatrix}$

$$\underset{R_3 \to -R_2 + R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & (\lambda-3) & (\mu-10) \end{bmatrix}$$

a) $(\lambda-3) \neq 0$ ensures that all 3 rows in the row echelon form of $[S]$ is non zero.

Hence $\lambda \neq 3$ **is the condition for the set S to be linearly independent.**

b) $(\lambda-3) = 0$ and $(\mu-10) = 0$, will make the last row zero.

Hence $\lambda = 3$ and $\mu = 10$ **is the condition for the set S to be linearly dependent.**

[16] *Which of the following set of vectors form a basis of a vector space ?*

(i) $A = \{(5,1,3), (2,5,2)\ (3,1,2)\}$

(ii) $B = \{(5,3,7,4), (3,26,2,9)(7,2,10,5)\}$

☞ **(i)** Consider $[A] = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{matrix} R_2 \to -2R_1 + 5R_2 \\ R_3 \to -3R_1 + 5R_3 \end{matrix} \underset{\sim}{} \begin{bmatrix} 5 & 1 & 3 \\ 0 & 23 & 4 \\ 0 & 2 & 1 \end{bmatrix}$

$$\underset{\sim}{R_3 \to -2R_2 + 23R_3} \begin{bmatrix} 5 & 1 & 3 \\ 0 & 23 & 4 \\ 0 & 0 & 15 \end{bmatrix}$$

The row echelon form of $[A]$ has no zero rows and hence the set A of vectors is linearly independent. (*We can also show that* $|A| \neq 0$)

We conclude that the set of A of vectors form a basis.

(ii) Consider $[B] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix} \begin{matrix} R_2 \to -3R_1 + 5R_2 \\ R_3 \to -7R_1 + 5R_3 \end{matrix} \underset{\sim}{} \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{bmatrix}$

$$\underset{\sim}{R_3 \to R_2 + 11R_3} \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The row echelon form of $[B]$ has one zero row and hence the set B of vectors is linearly dependent.

We conclude that the set B of vectors do not form a basis.

Find the basis and dimension of the subspace of V spanned by the following vectors.

[17] $\{(1,2,3,2),\ (2,3,5,1),\ (1,3,4,5)\}$

[18] $\{(2,-1,-3,-1),\ (1,2,3,-1),\ (1,0,1,1),\ (0,1,1,-1)\}$

[19] $\{(0,1,-3,-1),\ (1,0,1,1),\ (3,1,0,2),\ (1,1,-2,0)\}$

[20] $\{(1,1,1,9),\ (2,5,7,52),\ (2,1-1,0)\}$

[17] Consider $[A] = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \underset{\sim}{\begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -R_1 + R_3 \end{matrix}} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

$$\underset{\sim}{R_3 \to R_2 + R_3} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The row echelon form of $[A]$ has 2 non zero rows and one zero row. The non zero rows of this form a basis of the subspace S spanned by V and the dimension of S is 2.

Thus, **Basis of the sub space is** $\{(1, 2, 3, 2), (0, -1, -1, 3)\}$ and dimension of S : **dim S = 2.**

[18] Consider $[B] = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$$\underset{\sim}{\begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -R_1 + R_3 \end{matrix}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -9 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \underset{\sim}{\begin{matrix} R_2 \leftrightarrow R_4 \\ R_3 \to 1/2 \cdot R_3 \end{matrix}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -5 & -9 & 1 \end{bmatrix}$$

$$\underset{\sim}{\begin{matrix} R_3 \to R_2 + R_3 \\ R_4 \to 5R_2 + R_4 \end{matrix}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \end{bmatrix} \underset{\sim}{\begin{matrix} R_3 \leftrightarrow R_4 \\ R_3 \to -1/4 \cdot R_3 \end{matrix}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ **Basis of the subspace S spanned by V is**

$\{(1, 2, 3, -1), (0, 1, 1, -1) \ (0, 0, 1, 1)\}$ **and dimension of $S = 3$.**

[19] Let $[C] = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$$\begin{matrix} R_3 \to -3R_1 + R_3 \\ R_4 \to -R_1 + R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \underset{\sim}{\begin{matrix} R_3 \to -R_2 + R_3 \\ R_4 \to -R_2 + R_4 \end{matrix}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ **Basis of the subspace S spanned by V is** $\{(1,0,1,1),(0,1,-3,-1)\}$ **and dimension $S = 2$.**

[20] Let $[D] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix} \underset{\sim}{\begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -2R_1 + R_3 \end{matrix}} \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{bmatrix}$

$$\underset{\sim}{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 3 & 5 & 34 \end{bmatrix} \underset{\sim}{R_3 \to 3R_2 + R_3} \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{bmatrix}$$

$\therefore$ **Basis of the sub space S spanned by V is**

$\{(1,1,1,9) \ (0,-1,-3,-18)(0,0,-4,20)\}$ **and dimension of $S = 3$.**

2.2 Linear transformations (LT)

Let V & W be two vector spaces over the same field F. The mapping $T: V \to W$ is said to be a *linear transformation* if

(i) $T(x+y) = T(x) + T(y)$ (ii) $T(c \cdot x) = c \cdot T(x)$

where $x, y \in V$ and $c \in F$.

Note : A linear transformation $T: V \to V$ is called a *linear map* on V.

Examples : (1) Consider $T : V_1(R) \to V_3(R)$ defined by $T(x) = (x, 2x, 3x)$.

We shall check conditions (i) and (ii) for a LT.

$T(x+y) = (x+y, 2\overline{x+y}, 3\overline{x+y}) = (x+y, 2x+2y, 3x+3y)$

or $T(x+y) = (x, 2x, 3x) + (y, 2y, 3y) = T(x) + T(y)$

Also, $T(c \cdot x) = (cx, 2cx, 3cx) = c \cdot (x, 2x, 3x) = c \cdot T(x)$

Both the required conditions are satisfied. **Hence T is a linear transformation.**

(2) Consider $T: V_1(R) \to V_3(R)$ defined by $T(x) = (x, x^2, x^3)$

Now, $T(x+y) = \left((x+y), (x+y)^2, (x+y)^3\right)$

Also, $T(x) + T(y) = (x, x^2, x^3) + (y, y^2, y^3) = (x+y, x^2+y^2, x^3+y^3)$

Evidently, $T(x+y) \neq T(x) + T(y)$

Hence T is not a linear transformation.

2.21 Algebra of linear transformations

Let $T_1 : V \to W$ and $T_2 : V \to W$ be any two linear transformations. Then the '*sum*' of T_1 and T_2 denoted by $T_1 + T_2$ is defined as

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \ \forall x \in V.$$

Also if 'c' is any scalar then the *scalar multiple* of T denoted by cT is defined as $(cT)(x) = cT(x), \ \forall x \in V$.

Further the *composition* of T_1 and T_2 denoted by $T_1 o T_2$ is defined by

$$T_1 o T_2(x) = T_1(T_2(x)), \ \forall x \in V.$$

It may be noted that $T_1 o T_2$ is denoted by $T_1 T_2$ and $T o T$ is denoted by T^2.

Examples : (1) Let T_1 and T_2 be two LTs from $V_3(R)$ to $V_2(R)$ defined by

$T_1(x,y,z) = (x+y, y+z)$ and $T_2(x,y,z) = (y+z, z+x)$

We shall find $T_1 + T_2$, $2T_1 + 3T_2$.

Now, $(T_1 + T_2)(x,y,z)$

$$= T_1(x,y,z) + T_2(x,y,z)$$

$$= (x+y, y+z) + (y+z, z+x) = (x+2y+z,\ x+y+2z)$$

Also, $(2T_1 + 3T_2)(x,y,z)$

$$= 2T_1(x,y,z) + 3T_2(x,y,z)$$

$$= 2(x+y, y+z) + 3(y+z, z+x) = (2x+5y+3z, 3x+2y+5z)$$

(2) Let T_1 and T_2 be two LTs from $V_2(R) \to V_2(R)$ defined by

$T_1(x,y) = (x+y, x-y)$, $T_2(x,y) = (x-y, x+y)$

We shall find $T_1 o T_2$, T_1^2 and T_2^2

Now, $T_1 o T_2 = T_1(T_2(x,y)) = T_1(x-y, x+y) = (x-y+x+y, x-y-\overline{x-y})$

ie., $T_1 o T_2 = (2x, -2y)$

Next, $T_1^2 = T_1(T_1(x,y)) = T_1(x+y, x-y) = (x+y+\overline{x-y},\ x+y-\overline{x-y})$

ie., $T_1^2 = (2x, 2y)$

Also, $T_2^2 = T_2(T_2(x,y)) = T_2(x-y, x+y) = (x-y-\overline{x+y}, x-y+\overline{x+y})$

ie., $T_2^2 = (-2y, 2x)$

2.22 Matrix of a linear transformation

Let V and W be two vector spaces of dimensions m and n respectively over the same field F. Also let $X = \{x_1, x_2, \cdots x_m\}$ and $Y = \{y_1, y_2 \cdots y_n\}$ be ordered basis of V and W respectively.

Further, let $T:V \to W$ be a linear transformation.

Evidently, $T(x_1), T(x_2) \cdots T(x_m) \in W$ and each one of these can be expressed as a linear combination of $y_1, y_2 \cdots y_n$

Let, $T(x_1) = a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n$

$T(x_2) = a_{21} y_1 + a_{22} y_2 + \cdots + a_{2n} y_n$

.....

$T(x_m) = a_{m1} y_1 + a_{m2} y_2 + \cdots + a_{mn} y_n$

If M is the coefficient matrix associated with the RHS, then the transpose of M is called the matrix of the linear transformation relative / w.r.t the bais X and Y denoted by M_T or $M(T)$.

That is, $M_T = M(T) = \begin{bmatrix} a_{11} & a_{21} \cdots & a_{m1} \\ a_{12} & a_{22} \cdots & a_{m2} \\ a_{1n} & a_{2n} \cdots & a_{mn} \end{bmatrix}$

Examples : (1) Let $T:V_2(R) \to V_2(R)$ be defined as $T(x,y) = (-x,-y)$ w.r.t the standard basis. We shall find the matrix of T.

Standard basis $e_1 = (1,0)$ and $e_2 = (0,1)$; $T(x,y) = (-x,-y)$

$\therefore \quad T(e_1) = T(1,0) = (-1,0)$ and $T(e_2) = T(0,1) = (0,-1)$

Coefficient matrix $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and matrix of the LT is $M' = M_T$.

$M_T = M(T) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ Here $M = M_T$.

(2) Let $T:V_3(R) \to V_2(R)$ be defined as $T(x,y,z) = (x+y, y+z)$ and we shall find the matrix of T w.r.t the baiss $\{(1,1,1)\ (1,0,0)\ (1,1,0)\}$ and also w.r.t the standard basis.

Case - (i) $T(x,y,z) = (x+y, y+z)$

$\therefore \quad T(1,1,1) = (2,2)$; $T(1,0,0) = (1,0)$; $T(1,1,0) = (2,1)$

Coefficient matrix $M = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$ and hence the required $M_T = M'$.

$$\therefore \quad M_T = M(T) = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

Case - (ii) Standard basis is $\{(1,0,0),(0,1,0)(0,0,1)\}$

$$\therefore \quad T(1,0,0) = (1,0);\ T(0,1,0) = (1,1);\ T(0,0,1) = (0,1)$$

Coefficient matrix $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $M' = M_T = M(T) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

2.23 Change of coordinates / basis

Let $X = \{x_1, x_2 \cdots x_n\}$ be an ordered basis of V and $x \in V$. We can write

$x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

where $c_1, c_2, \cdots c_n$ are scalars, called as the *coordinates* of $x \in V$ relative to basis X, denoted by $[x_n] = (c_1, c_2, \cdots c_n)$

This is called *coordinate vector* of x relative to basis X.

Let $X = \{x_1, x_2 \cdots x_n\}$ and $Y = \{y_1, y_2 \cdots y_n\}$ be two ordered basis of a finite dimensional vector space V.

We define a linear transformation :

$$W : V \to V \text{ by } W(x_i) = y_i, i = 1, 2, \cdots n$$

Let $[W] = [a_{ij}]$ be the matrix of this linear transformation and hence

$$y_j = W(x_j) = \sum_{i=1}^{n} a_{ij} x_i,\ j = 1, 2, \cdots n.$$

The matrix $[W] = [a_{ij}]$ is called the *change of basis* matrix of the transformation matrix from the basis X to Y. The matrix $[a_{ij}]$ is called *transition matrix* from the basis X to Y.

Example : Let $X = \{(1,3),(3,8)\}$ and $Y = \{(1,-2),(3,-4)\}$ be the two basis of $V_2(R)$. We shall find the transition matrix W from X to Y.

Firstly, let, $(1,-2) = x_1(1,3) + x_2(3,8)$ and $(3,-4) = y_1(1,3) + y_2(3,8)$

$$\left.\begin{aligned} x_1 + 3x_2 &= 1 \\ 3x_1 + 8x_2 &= -2 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} y_1 + 3y_2 &= 3 \\ 3y_1 + 8y_2 &= -4 \end{aligned}\right\}$$

On solving, we obtain $x_1 = -14,\ x_2 = 5$ and $y_1 = -36,\ y_2 = 13$

Transition matrix $[W] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} -14 & -36 \\ 5 & 13 \end{bmatrix}$

WORKED PROBLEMS

[21] *Show that the transformation $T: V_2(R) \to V_2(R)$ defined by $T(x,y) = (x+y, x-y)$ is a linear transformation.*

Let $a = (a_1, b_1)$ and $b = (a_2, b_2) \in V_2(R)$

$\therefore \quad T(a) = T(a_1, b_1) = (a_1 + b_1, a_1 - b_1)$

$T(b) = T(a_2, b_2) = (a_2 + b_2, a_2 - b_2)$

Consider, $T(a+b) = T(a_1 + a_2,\ b_1 + b_2)$

ie., $= (a_1 + a_2 + b_1 + b_2,\ a_1 + a_2 - b_1 - b_2)$

$= (a_1 + b_1,\ a_1 - b_1) + (a_2 + b_2, a_2 - b_2) = T(a) + T(b)$

Hence, $T(a+b) = T(a) + T(b)$

Next, consider $T(c \cdot a)$ where $c \in F$.

$T(c \cdot a) = T(ca_1, cb_1) = (ca_1 + cb_1, ca_1 - cb_1) = c(a_1 + b_1, a_1 - b_1) = c \cdot T(a)$

Thus from conclude that T is a linear transformation.

[22] *If S and T are two transformations from $V_2(R)$ to $V_2(R)$ defined by $S(x,y) = (3x + 2y, 3x - 4y)$ and $T(x,y) = (x+3, y-2)$, shwo that S is linear but T is not linear.*

☞ Let $a = (a_1, b_1)$ and $b = (a_2, b_2) \in V_2(R)$. We have,

$$S(a) = S(a_1, b_1) = (3a_1 + 2b_1, 3a_1 - 4b_1)$$

$$S(b) = S(a_2, b_2) = (3a_2 + 2b_2, 3a_2 - 4b_2)$$

$$S(a+b) = S(a_1 + a_2, b_1 + b_2)$$

ie., $= [3(a_1 + a_2) + 2(b_1 + b_2), 3(a_1 + a_2) - 4(b_1 + b_2)]$

$$= [3a_1 + 3a_2 + 2b_1 + 2b_2, 3a_1 + 3a_2 - 4b_1 - 4b_2]$$

$$S(a+b) = (3a_1 + 2b_1, 3a_1 - 4b_1) + (3a_2 + 2b_2, 3a_2 - 4b_2) = S(a) + S(b)$$

Next, consider $T(c \cdot a)$ where $c \in F$.

$$T(c \cdot a) = T(ca_1, cb_1) = (3ca_1 + 2cb_1, 3ca_1 - 4cb_1)$$

ie., $T(c \cdot a) = c(3a_1 + 2b_1, 3a_1 - 4b_1) = c \cdot T(a)$

Thus we conclude that S is linear.

Next, consider $T(x, y) = (x + 3, y - 2)$

$$T(a) = T(a_1, b_1) = (a_1 + 3, b_1 - 2)\ ;\ T(b) = T(a_2, b_2) = (a_2 + 3, b_2 - 2)$$

Now, $T(a+b) = T(a_1 + a_2, b_1 + b_2)$

ie., $T(a+b) = (a_1 + a_2 + 3,\ b_1 + b_2 - 2)$

Also, $T(a) + T(b) = (a_1 + 3, b_1 - 2) + (a_2 + 3, b_2 - 2)$

ie., $T(a) + T(b) = (a_1 + a_2 + 6,\ b_1 + b_2 - 4) \neq T(a+b)$

Thus we conclude that T is not linear.

[23] *Show that the transformation* $T : V_3(R) \to V_2(R)$ *defined by* $T(x, y, z) = (y - x, y - z)$ *is a linear transformation.*

☞ *Let* $a = (a_1, b_1, c_1)$ and $b = (a_2, b_2, c_2) \in V_3(R)$.

$\therefore$ $T(a) = (b_1 - a_1, b_1 - c_1)$ and $T(b) = (b_2 - a_2, b_2 - c_2)$

Now, $T(a+b) = T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$

ie., $T(a+b) = (b_1 + b_2 - a_1 - a_2,\ b_1 + b_2 - c_1 - c_2)$

$$T(a+b) = (b_1 - a_1, b_1 - c_1) + (b_2 - a_2, b_2 - c_2) = T(a) + T(b)$$

Next, consider $T(k \cdot a)$ where $k \in F$.

$T(k \cdot a) = T(ka_1, kb_1, kc_1) = (kb_1 - ka_1, kb_1 - kc_1) = k \cdot (b_1 - a_1, b_1 - c_1) = k \cdot T(a)$

Thus we conclude that T is a linear transformation.

[24] *If T is a transformation from $V_2(R)$ to $V_2(R)$ defined by $T(a,b) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$, decide whether T is linear or not.*

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in V_2(R)$.

Consider $T(x+y) = T(x_1 + y_1, x_2 + y_2)$

$T(x+y) = [(x_1 + y_1)\cos\theta - (x_2 + y_2)\sin\theta, (x_1 + y_1)\sin\theta + (x_2 + y_2)\cos\theta]$

$= [(x_1\cos\theta - x_2\sin\theta) + (y_1\cos\theta - y_2\sin\theta),$

$(x_1\sin\theta + x_2\cos\theta) + (y_1\sin\theta + y_2\cos\theta)]$

$= (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta)$

$+ (y_1\cos\theta - y_2\sin\theta, y_1\sin\theta + y_2\cos\theta)$

$= T(x_1, x_2) + T(y_1, y_2) = T(x) + T(y)$; $T(x+y) = T(x) + T(y)$

Next, $T(c \cdot x) = T(cx_1, cx_2) = (cx_1\cos\theta - cx_2\sin\theta, cx_1\sin\theta + cx_2\cos\theta)$

$T(c \cdot x) = c \cdot (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta) = c \cdot T(x)$

Thus we conclude that T is a linear transformation.

[25] *Find the linear map $f: R^2 \to R^2$ such that*

$f(1,2) = (3,0)$ *and* $f(2,1) = (1,2)$.

The elements $(1, 2)$ and $(2, 1)$ of R^2 form a basis of it and let $(x, y) \in R^2$.

$\therefore \quad (x,y) = c_1(1,2) + c_2(2,1)$ or $(x,y) = (c_1 + 2c_2, 2c_1 + c_2)$

$\Rightarrow c_1 + 2c_2 = x$ and $2c_1 + c_2 = y$ gives $c_1 = \frac{1}{3}(2y - x)$ and $c_2 = \frac{1}{3}(2x - y)$

$$\therefore \quad (x,y) = \left[\frac{1}{3}(2y - x)\right](1,2) + \left[\frac{1}{3}(2x - y)\right](2,1)$$

Hence, $f(x,y) = \left[\frac{1}{3}(2y - x)\right] f(1,2) + \left[\frac{1}{3}(2x - y)\right] f(2,1)$

ie., $f(x,y) = \left[\frac{1}{3}(2y-x)\right](3,0) + \left[\frac{1}{3}(2x-y)\right] f(1,2)$

Thus the required $f(x,y) = \left[\frac{1}{3}(5y-x), \frac{2}{3}(2x-y)\right]$

[26] *Determine the linear transformation* $T : V_3(R) \to V_2(R)$ *such that* $T(e_1) = (-1,0)$; $T(e_2) = (1,1)$ and $T(e_3) = (0,-1)$ *where* e_1, e_2, e_3 *is the standard basis of* $V_3(R)$.

☞ Let $(x,y,z) \in R^3$ be arbitrary.

$\therefore \quad (x,y,z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

or $\quad (x,y,z) = (c_1, c_2, c_3) \Rightarrow c_1 = x, c_2 = y, c_3 = z$

We now have, $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

Hence, $T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$

ie., $T(x,y,z) = xT(e_1) + yT(e_2) + zT(e_3)$

ie., $T(x,y,z) = x(-1,0) + y(1,1) + z(0,-1)$

Thus the required LT : $T(x,y,z) = (y-x, y-z)$

[27] *If* T_1 *and* T_2 *are two LTs from* $V_3(R)$ *to* $V_2(R)$ *defined by* $T_1(x,y,z) = (2x+y, y+2z)$ and $T_2(x,y,z) = (y+2z, 2z+x)$, *determine* $T_1 + T_2$ and $3T_1 - 2T_2$.

☞ $(T_1 + T_2)(x,y,z) = T_1(x,y,z) + T_2(x,y,z)$

ie., $= (2x+y, y+2z) + (y+2z, 2z+x)$

$= (2x+y+y+2z, y+2z+2z+x) = (2x+2y+2z, x+y+4z)$

Next, $(3T_1 - 2T_2)(x,y,z) = 3T_1(x,y,z) - 2T_2(x,y,z)$

ie., $= 3(2x+y, y+2z) - 2(y+2z, 2z+x)$

$= (6x+3y, 3y+6z) - (2y+4z, 4z+2x)$

$= (6x+3y-2y-4z, 3y+6z-4z-2x) = (6x+y-4z, -2x+3y+2z)$

Thus, $T_1 + T_2 = (2x + 2y + 2z, x + y + 4z)$ and

$$3T_1 - 2T_2 = (6x + y - 4z, -2x + 3y + 2z)$$

[28] *If T_1, T_2 are linear operators from $V_2(R)$ to $V_2(R)$ defined by $T_1 = (x, y) = (x + y, x - y)$, $T_2(x, y) = (x - y, x + y)$.*

Compute T_1T_2, T_2T_1, T_1^2, T_2^2.

$(T_1T_2)(x, y) = T_1[T_2(x, y)] = T_1(x - y, x + y) = (x - y + \overline{x + y}, x - y - \overline{x + y})$

$\therefore \quad T_1T_2 = (2x, -2y)$ . . . (i)

Next, $(T_2T_1)(x, y) = T_2[T_1(x, y)] = T_2(x + y, x - y) = (x + y - \overline{x - y}, x + y + \overline{x - y})$

$\therefore \quad T_2T_1 = (2y, 2x)$ . . . (ii)

Remark : *Comparing* (i) *and* (ii), $T_1T_2 \neq T_2T_1$ (*Not commutative*)

Also, $T_1^2(x, y) = T_1[T_1(x, y)] = T_1(x + y, x - y) = (x + y + \overline{x - y}, x + y - \overline{x - y})$

$\therefore \quad T_1^2 = (2x, 2y)$. Similarly $T_2^2 = (-2y, 2x)$

Thus, $T_1T_2 = (2x, -2y)$; $T_2T_1 = (2y, 2x)$; $T_1^2 = (2x, 2y)$; $T_2^2 = (-2y, 2x)$

[29] *Find the matrix of the linear transformation $T: V_2(R) \to V_2(R)$ defined by $T(x, y) = (x, -y)$ relative to the basis :*

(i) $X = \{e_1, e_2\}$ *and* $Y = \{(1,1)\ (1,-1)\}$

(ii) $X = \{(1,1)(1,0\}$ *and* $Y = \{(2,3)\ (4,5)\}$

(i) We have $e_1 = (1,0)$ and $e_2 = (0,1)$.

$T(e_1) = T(1,0) = (1,0)$ and $T(e_2) = T(0,1) = (0,-1)$

We need to express the vectors $(1,0)$ and $(0,-1)$ as a linear combination of the elements in Y.

That is, $(1,0) = c_1(1,1) + c_2(1,-1)$ and $(0,-1) = c_1'(1,1) + c_2'(1,-1)$

or $\quad (1,0) = (c_1 + c_2, c_1 - c_2)$ and $(0,-1) = (c_1' + c_2', c_1' - c_2')$

$\Rightarrow \quad c_1 + c_2 = 1,\ c_1 - c_2 = 0$ and $c_1' + c_2' = 0,\ c_1' - c_2' = -1$

On solving we get, $c_1 = 1/2 = c_2$ and $c_1' = -1/2,\ c_2' = 1/2$

Hence we have, $T(1,0) = (1,0) = 1/2(1,1) + 1/2(1,-1)$

and $\quad T(0,1) = (0,-1) = -1/2(1,1) + 1/2(1,-1)$

$\therefore \quad$ The coeffcient matrix $M = \begin{bmatrix} c_1 & c_2 \\ c_1' & c_2' \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

Thus the required matrix $M_T = M(T) = M' = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

(ii) Now consider, $X = \{(1,1),(1,0)\}$ and $Y = \{(2,3)(4,5)\}$

$T(1,1) = (1,-1)$ and $T(1,0) = (1,0)$

Let, $\quad (1,-1) = c_1(2,3) + c_2(4,5)$ and $(1,0) = c_1'(2,3) + c_2'(4,5)$

or $\quad (1,-1) = (2c_1 + 4c_2, 3c_1 + 5c_2)$ and $(1,0) = (2c_1' + 4c_2', 3c_1' + 5c_2')$

$\Rightarrow \quad 2c_1 + 4c_2 = 1,\ 3c_1 + 5c_2 = -1$ and $2c_1' + 4c_2' = 1,\ 3c_1' + 5c_2' = 0$

On solving we get, $c_1 = -9/2, c_2 = 5/2$ and $c_1' = -5/2,\ c_2' = 3/2$

$\therefore \quad$ The coefficient matrix $M = \begin{bmatrix} c_1 & c_2 \\ c_1' & c_2' \end{bmatrix} = \begin{bmatrix} -9/2 & 5/2 \\ -5/2 & 3/2 \end{bmatrix}$

Thus the required $M_T = M(T) = M' = \begin{bmatrix} -9/2 & -5/2 \\ 5/2 & 3/2 \end{bmatrix}$

[30] *Determine the matrix of the linear transformation* $T: V_2(R) \to V_3(R)$ *defined by* $T(x,y) = (-x+2y, y, 3x+3y)$ *relative to the basis :*

$X = \{(1,1)(-1,1)\}$ *and* $Y = \{(1,1,1)\ (1,-1,1)(0,0,1)\}$

☞ By the definition of the given $T(x, y)$ we have,

$T(1,1) = (1,1,0)$ and $T(-1,1) = (3,1,6)$

Let, $\quad (1,1,0) = c_1(1,1,1) + c_2(1,-1,1) + c_3(0,0,1)$

and $(3,1,6) = c_1'(1,1,1) + c_2'(1,-1,1) + c_3'(0,0,1)$

or $(1,1,0) = (c_1 + c_2, c_1 - c_2, c_1 + c_2 + c_3)$

and $(3,1,6) = (c_1' + c_2', c_1' - c_2', c_1' + c_2' + c_3')$

$\Rightarrow$ $c_1 + c_2 = 1,\ c_1 - c_2 = 1, c_1 + c_2 + c_3 = 0$

and $c_1' + c_2' = 3,\ c_1' - c_2' = 1,\ c_1' + c_2' + c_3' = 6$

On solving we get, $c_1 = 1,\ c_2 = 0,\ c_3 = -1$ and $c_1' = 2,\ c_2' = 1,\ c_3' = 3$

$\therefore$ The coefficient of matrix $M = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_1' & c_2' & c_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Thus the required matrix $M_T = M(T) = M' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}$

[31] *Find the linear transformation* $T : R^2 \to R^2$ *satisfying*

$T(1,1) = (0,1)$ *and* $T(-1,1) = (3,2)$.

☞ Let $(x,y) \in V_2(R)$ be arbitrary.

$\therefore$ $(x,y) = c_1(1,1) + c_2(-1,1)$ or $(x,y) = (c_1 - c_2, c_1 + c_2)$

Hence $c_1 - c_2 = x$ and $c_1 + c_2 = y$.

On solving, $c_1 = (x+y)/2$ and $c_2 = (-x+y)/2$

$$\therefore \quad (x,y) = \left[\frac{(x+y)}{2}\right](1,1) + \left[\frac{(-x+y)}{2}\right](1,-1)$$

$$\Rightarrow \quad T(x,y) = \left[\frac{(x+y)}{2}\right]T(1,1) + \left[\frac{(-x+y)}{2}\right]T(1,-1)$$

$$\text{or} \quad T(x,y) = \left[\frac{(x+y)}{2}\right](0,1) + \left[\frac{(-x+y)}{2}\right](3,2)$$

Thus the required $T(x,y) = \left(\frac{-3x}{2} + \frac{3y}{2}, \frac{-x}{2} + \frac{3y}{2}\right)$

[32] *Given* $A = \begin{bmatrix} -1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$ *and the basis.*

$X = \{(1,1,1), (1,2,3), (1,1,0)\}$; $Y = \{(1,0),(2,-1)\}$

determine the linear transformation T from $V_3(R)$ to $V_2(R)$.

☞ T with reference to the given matrix A is defined by

$T(1,1,1) = -1(1,0)+3(2,-1) = (5,-3)$

$T(1,2,3) = -1(1,0)+1(2,-1) = (1,-1)$

$T(1,1,0) = 2(1,0)+0(2,-1) = (2,0)$

Let $(x,y,z) \in V_3(R)$ be arbitrary.

$\therefore \quad (x,y,z) = c_1(1,1,1)+c_2(1,2,3)+c_3(1,1,0) \quad \ldots (1)$

or $\quad (x,y,z) = (c_1+c_2+c_3,\ c_1+2c_2+c_3,\ c_1+3c_2)$

$\Rightarrow \quad c_1+c_2+c_3 = x,\ c_1+2c_2+c_3 = y,\ c_1+3c_2 = z$

On solving we get, $c_1 = 3x-3y+z,\ c_2 = y-x,\ c_3 = -x+2y-z.$

With reference to (i), along with the expression for c_1, c_2, c_3 we have,

$T(x,yz) = (3x-3y+z)T(1,1,1)+(y-x)T(1,2,3)+(-x+2y-z)T(1,1,0)$

ie., $T(x,yz) = (3x-3y+z)(5,-3)+(y-x)(1,-1)+(-x+2y-z)(2,0)$

Thus, $T(x,yz) = (12x-10y+3z, -8x+8y-3z)$

[33] *Determine the linear transformation from* $V_3(R)$ to $V_2(R)$ *with reference to the standard basis in these and the matrix A as in the previous problems.*

☞ The standard basis are respectively $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{(1,0),(0,1)\}$.

Hence, $T(1,0,0) = -1(1,0)+3(0,1) = (-1,3)$

$T(0,1,0) = -1(1,0)+1(0,1) = (-1,1)$

$T(0,0,1) = 2(1,0)+0(0,1) = (2,0)$

Let $(x,y,z) \in V_3(R)$ be arbitrary.

$\therefore \quad (x,y,z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) \quad \ldots$ (i)

or $\quad (x,y,z) = (c_1, c_2, c_3) \Rightarrow c_1 = x,\ c_2 = y,\ c_3 = z$

With reference to (i) we have,

$T(x,y,z) = c_1 T(1,0,0) + c_2 T(0,1,0) + c_3(0,0,1)$

Thus, $T(x,y,z) = x(-1,3) + y(-1,1) + z(2,0) = (-x-y+2z, 3x+y)$

[34] *If* $X = \{(1,3),(3,8)\}$ *and* $Y = \{(1,-2),(3,-4)\}$ *are two basis of* $V_2(R)$, *find the transition matrix* **(i)** *from* X *to* Y **(ii)** *from* Y *to* X.

(i) Let $(1,-2) = x_1(1,3) + x_2(3,8)$ and $(3,-4) = y_1(1,3) + y_2(3,8)$

or $\quad (1,-2) = (x_1 + 3x_2, 3x_1 + 8x_2)$ and $(3,-4) = (y_1 + 3y_2, 3y_1 + 8y_2)$

$$\Rightarrow \left.\begin{aligned} x_1 + 3x_2 &= 1 \\ 3x_1 + 8x_2 &= -2 \end{aligned}\right\} \text{ and } \left.\begin{aligned} y_1 + 3y_2 &= 3 \\ 3y_1 + 8y_2 &= -4 \end{aligned}\right\}$$

On solving, $x_1 = -14, x_2 = 5$ and $y_1 = -36,\ y_2 = 13$.

Thus the requuired transition matrix [A] from X to Y is

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix};\ [A] = \begin{bmatrix} -14 & -36 \\ 5 & 13 \end{bmatrix}$$

(ii) Let $(1,3) = x_1'(1,-2) + x_2'(3,-4)$ and $(3,8) = y_1'(1,-2) + y_2'(3,-4)$

or $\quad (1,3) = (x_1' + 3x_2' - 2x_1' - 4x_2')$ and $(3,8) = (y_1' + 3y_2', -2y_1' - 4y_2')$

$\Rightarrow \quad x_1' + 3x_2' = 1,\ -2x_1' - 4x_2' = 3$ and $y_1' + 3y_2' = 3,\ -2y_1' - 4y_2' = 8$

On solving, $x_1' = -13/2,\ x_2' = 5/2$ and $y_1' = -18,\ y_2' = 7$

Thus the required transition matrix [B] from Y to X is

$$\begin{bmatrix} x_1' & y_1' \\ x_2' & y_2' \end{bmatrix};\ [B] = \begin{bmatrix} -13/2 & -18 \\ 5/2 & 7 \end{bmatrix}$$

2.3 Rank and Nullity of a linear transformation

Let $T: V \to W$ be a lienar transformation. The *Range* of T denoted by $R(T)$ is the set $R(T) = \{ T(x) / x \in V \}$ which being the set of all images of the elements of V under the transformation T.

The dimension of $R(T)$ is called the *rank* of T denoted $r(T)$.

Further *kernel* of T denoted by $N(T)$ is the set

$$N(T) = \{ x \in V / T(x) = 0 \}$$

The dimension of $N(T)$ is called the *Nullity* of T denoted by $n(T)$.

2.31 Rank-Nullity theorem

Let V be a finite dimensional vector space and $T: V \to W$ be a linear transformation. The theorem states that the sum of rank and nullity of T is equal to the dimension of V.

That is, $r(T) + n(T) = d(V)$

Working procedure for problems

- Given a defined transformation say $T(x, y, z)$ we have to find $T(e_1), T(e_2), T(e_3)$ and write down the associated matrix $[A]$.
- $[A]$ is reduced to the row echelon form. The number of non zero rows is the *rank* and the number of zero rows is the *nullity*. The sum of these two should be equal to the dimension of the domain which verifies the associated theorem.
- Range space $R(T)$ will be the linear span of the non zero rows in the row echelon form of A.
- Kernel of $T: N(T)$ will be the solution of the system of equations from $T(x, y, z) = 0$.

WORKED PROBLEMS

Find the range, null space, rank, nullity in the case of the following linear transformations. Also verify the rank, nullity theorem.

[35] $T: V_3(R) \to V_2(R)$ *defined by* $T(x, y, z) = (y - x, y - z)$

[36] $T: V_3(R) \to V_3(R)$ *defined by* $T(x, y, z) = (x + y, x - y, 2x + z)$

[37] $T: V_3(R) \to V_3(R)$ *defined by* $T(x, y, z) = (x + 2y + z, z - x, y + z)$

[38] $T: V_3(R) \to V_4(R)$ *defined by* $T(e_1) = (0,1,0,2), T(e_2) = (0,1,1,0)$
$T(e_3) = (0,1,-1,4)$

[39] $T: V_2(R) \to V_3(R)$ *defined by* $T(1,2) = (15,-20,5)$ *and*
$T(2,1) = (12,-16,4)$

[35] $T(x,y,z) = (y-x, y-z)$

$\therefore$ $T(e_1) = T(1,0,0) = (-1,0); T(e_2) = T(0,1,0) = (1,1);$
$T(e_3) = T(0,0,1) = (0,-1)$

Hence, $[A] = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \underset{\sim}{R_2 \to R_1 + R_2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \underset{\sim}{R_3 \to R_2 + R_3} \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Rank of $T = r(T) = 2$, being the number of non zero rows.

Nullity of $T = n(T) = 1$, being the number of zero rows.

$r(T) + n(T) = 2 + 1 = 3 = d[V_3(R)]$ being 3.

Hence the theorem is verified.

Next, $R(T) = L[S]$ being the linear span of $S = \{(-1,0),(0,1)\}$

That is, $\{x_1(-1,0) + x_2(0,1) = (-x_1, x_2)\}$

$\therefore$ $R(T) = \{(-x_1, x_2) / x_1, x_2 \in R\}$

Next, to find $N(T)$, consider $T(x,y,z) = 0$

That is $y - x = 0$, $y - z = 0$ or $y = x$, $y = z \Rightarrow x = y = z$

Let $z = k$ be arbitrary.

$\therefore$ $N(T) = \{(k,k,k) / k \in R\}$

[36] $T(x,y,z) = (x+y, x-y, 2x+z)$

$T(1,0,0) = (1,1,2): T(0,1,0) = (1,-1,0); T(0,0,1) = (0,0,1)$

Consider, $[A] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_2 \to -1/2 \cdot R_1}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \therefore r(T) = 3 \text{ and } n(T) = 0$$

$r(T) + n(T) = 3 + 0 = 3 = d[V_3(R)]$. **Theorem is verified.**

$R(T) = L(S) = \{x_1(1,1,2) + x_2(0,1,1) + x_3(0,0,1)\}$

$\therefore \quad R(T) = \{(x_1, x_1 + x_2, 2x_1 + x_2 + x_3) / x_1, x_2, x_3 \in R\}$

Next, consider $T(x, y, z) = 0$.

That is, $x + y = 0,\ x - y = 0,\ 2x + z = 0$

On solving $x = 0, y = 0, z = 0 \quad \therefore \ N(T) = \{(0,0,0)\}$

[37] $T(x, y, z) = (x + 2y + z, z - x, y + z)$

$T(1,0,0) = (1,-1,0)$; $T(0,1,0) = (2,0,1)$; $T(0,0,1) = (1,1,1)$

$$[A] = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -R_1 + R_3 \\ \sim \end{matrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} R_3 \to -R_2 + R_1 \\ \sim \end{matrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad r(T) = 2,\ n(T) = 1$

$r(T) + n(T) = 2 + 1 = 3 = d[V_3(R)]$

Theorem is verified.

$R(T) = L(S) = \{x_1(1,-1,0) + x_2(0,2,1)\}$

$\therefore \ R(T) = \{(x_1, -x_1 + 2x_2, x_2) / x_1, x_2, x_3 \in R\}$

Next consider, $T(x, y, z) = 0$.

That is, $x + 2y + z = 0,\ z - x = 0, y + z = 0$.

$\Rightarrow \quad x = z,\ y = -z$ and let $z = k$ be a arbitrary.

$\therefore \quad N(T) = \{(k, -k, k) / k \in R\}$

[38] $T(e_1) = (0,1,0,2);\ T(e_2) = (0,1,1,0);\ T(e_3) = (0,1,-1,4)$

Consider, $[A] = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{bmatrix} \begin{matrix} R_2 \to -R_1 + R_2 \\ R_3 \to -R_1 + R_3 \end{matrix} \sim \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} R_3 \to R_2 + R_3 \sim$

$[A] \sim \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \therefore\ r(T) = 2, n(T) = 1$

$r(T) + n(T) = 2 + 1 = d[V_3(R)] = 3$ **The theorem is verified.**

$R(T) = L(S) = \{x_1(0,1,0,2) + x_2(0,0,1,-2)\}$

$\therefore \quad R(T) = \{(0, x_1, x_2, 2(x_1 - x_2) / x_1, x_2 \in R\}$

Next consider, $T(x,y,z) = 0$ or $T(xe_1 + ye_2 + ze_3) = (0,0,0,0)$

or $xT(e_1) + yT(e_2) + zT(e_3) = (0,0,0)$

ie., $x(0,1,0,2) + y(0,1,1,0) + z(0,1,-1,4) = (0,0,0,0)$

or $(0, x+y+z, y-z, 2x+4z) = (0,0,0,0)$

$\Rightarrow \quad x+y+z = 0, y-z = 0,\ 2x+4z = 0$ or $x = -2z, y = z$

Let $z = k$ be arbitrary. $\quad \therefore\ N(T) = \{(-2k, k, k / k \in R\}$

[39] $T(1,2) = (15,-20,5);\ T(2,1) = (12,-16,4)$

We have, $(1,2) = 1(1,0) + 2(0,1) \Rightarrow T(1,2) = 1 \cdot T(e_1) + 2 \cdot T(e_2)$

That is, $T(e_1) + 2T(e_2) = (15,-20,5)$

and $2T(e_1) + T(e_2) = (12,-16,4)$

On solving we get, $T(e_1) = (3,-4,1)$ and $T(e_2) = (6,-8,2)$

Consider $[A] = \begin{bmatrix} 3 & -4 & 1 \\ 6 & -8 & 2 \end{bmatrix} R_2 \to -2R_1 + R_2 \sim \begin{bmatrix} 3 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \quad r(T) = 1, n(T) = 1, r(T) + n(T) = 1 + 1 = 2 = d[V_2(R)]$

The theorem is verified.

$R(T) = L(S) = \{x_1(3,-4,1)\} = \{(3x_1, -4x_1, x_1) / x_1 \in R\}$

Next consider, $T(x,y) = 0$ or $T(xe_1 + ye_2) = (0,0)$

That is, $xT(e_1) + yT(e_2) = 0$

or $x(3,-4,1) + y(6,-8,2) = (0,0)$

or $(3x+6y, -4x-8y, x+2y) = (0,0)$

$\Rightarrow \quad 3x+6y = 0, -4x-8y = 0, x+2y = 0$

Equivalently we have $x+2y = 0$ only. If $y = k$ (say) then $x = -2k$

$\therefore \quad N(T) = \{(-2k, k / k \in R\}$

2.4 Inner product spaces and Orthogonality

Let V be a vector space over the field of real numbers. An inner product on V is a function from $V \times V$ to R.

If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in V$, then the *length* of a denoted by $|a|$ or $\|a\|$ is defined by $\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Also the *inner product* of a and b denoted by $\langle a,b \rangle$ is defined by

$$\langle a,b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Evidently, $\langle a,a \rangle = a_1^2 + a_2^2 + a_3^2 = \|a\|^2$

Definition : An inner product on V being a function from $V \times V$ to R satisfying the following conditions is called an *inner product space.*

(i) $\langle a,a \rangle = 0$ *iff* $a = 0$ and $\langle a,a \rangle > 0$ *iff* $a \neq 0$. (*Positiveness property*)

(ii) $\langle a,b \rangle = \langle b,a \rangle \ \forall \ a,b \in V$ (*Symmetric property*)

(iii) $\langle a+b, c \rangle = \langle a,c \rangle + \langle b, c \rangle \ \forall \ a,b,c \in V$

and $\langle ka, b \rangle = k\langle a,b \rangle$ where $k \in F$ (*Linearity property*)

Further a and b are said to be *orthogonal* if their inner product $\langle a,b \rangle = 0$.

Also in addition if $\|a\| = 1 = \|b\|$, then a and b are said to be *orthonormal.*

WORKED PROBLEMS

Show that the following vectors are mutually orthogonal... **[40] to [42]**

[40] $a = (2,1,3)$; $b = (-6/7, 15/14, 3/14)$; $c = (1/3, 1/3, -1/3)$

[41] $a = (2,0,0,0)$; $b = (0,3,3,0)$; $c = (0,-1,1,1)$

[42] $a = (1,1,0,0)$; $b = (-1/2, 1/2, 2, 0) = c = (2/3, -2/3, 1/3, 4)$

We have to show that the inner product of a, b ; b, c ; c, a are equal to 0.

[40] $\langle a,b \rangle = (2)(-6/7) + (1)(15/14) + (3)(3/14)$

ie., $\langle a,b \rangle = -\dfrac{12}{7} + \dfrac{15}{14} + \dfrac{9}{14} = \dfrac{-24+15+9}{14} = 0$

Next, $\langle b,c \rangle = (-6/7)(1/3) + (15/14)(1/3) + (3/14)(-1/3)$

ie., $\langle b,c \rangle = \dfrac{-6}{21} + \dfrac{15}{42} - \dfrac{3}{42} = \dfrac{-12+15-3}{42} = 0$

Lastly, $\langle c,a \rangle = (1/3)(2) + (1/3)(1) + (-1/3)(3)$

ie., $\langle c,a \rangle = \dfrac{2}{3} + \dfrac{1}{3} - \dfrac{3}{3} = \dfrac{2+1-3}{3} = 0$

Since $\langle a,b \rangle = 0 = \langle b,c \rangle = \langle c,a \rangle$ **we conclude that *a, b, c* are mutually orthogonal.**

[41] $\langle a,b \rangle = (2)(0) + (0)(3) + (0)(3) + (0)(0) = 0$

$\langle b,c \rangle = (0)(0) + (3)(-1) + (3)(1) + (0)(1) = 0$

$\langle c,a \rangle = (0)(2) + (-1)(0) + (1)(0) + (1)(0) = 0$

Thus *a, b, c* are mutually orthogonal.

[42] $\langle a,b \rangle = (1)(-1/2) + (1)(1/2) + (0)(2) + (0)(0) = -1/2 + 1/2 = 0$

$\langle b,c \rangle = (-1/2)(2/3) + (1/2)(-2/3) + (2)(1/3) + (0)(4) = -1/3 - 1/3 + 2/3 = 0$

$\langle c,a \rangle = (2/3)(1) + (-2/3)(1) + (1/3)(0) + (4)(0) = 2/3 - 2/3 + 0 + 0 = 0$

Thus *a, b, c* are mutually orthogonal.

Show that the following pairs of vectors are orthonormal **[43] to [45].**

[43] $a = (1,0,0,0)$; $b = (0, 1/\sqrt{5}, 2/\sqrt{5}, 0)$

[44] $a = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0)$; $b = (-\sqrt{2}/6, \sqrt{2}/6, 0, 2\sqrt{2}/3)$

[45] $a = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0)$; $b = (2/3, -2/3, 0, 1/3)$

>> We have to show that $\langle a, b \rangle = 0$ and $\| a \| = 1 = \| b \|$

[43] $\langle a, b \rangle = (1)(0) + (0)(1/\sqrt{5}) + (0)(2/\sqrt{5}) + (0)(0) = 0$

$\| a \| = \sqrt{1^2 + 0^2 + 0^2 + 0^2} = 1$; $\| b \| = \sqrt{0 + 1/5 + 4/5 + 0} = 1$

Thus *a, b* is orthonormal.

[44] $\langle a, b \rangle = (1/\sqrt{2})(-\sqrt{2}/6) + (1/\sqrt{2})(\sqrt{2}/6) + (0)(0) + (0)(2\sqrt{2}/3)$

ie., $\langle a, b \rangle = -1/6 + 1/6 + 0 + 0 = 0$

$\| a \| = \sqrt{1/2 + 1/2 + 0 + 0} = 1$; $\| b \| = \sqrt{2/36 + 2/36 + 8/9} = 1$

Thus *a, b, c* is orthonormal.

[45] $\langle a, b \rangle = (1/\sqrt{2})(2/3) + (1/\sqrt{2})(-2/3) + (0)(0) + (0)(1/3)$

ie., $\langle a, b \rangle = \sqrt{2}/3 - \sqrt{2}/3 + 0 + 0 = 0$

$\| a \| = \sqrt{1/2 + 1/2 + 0 + 0} = 1$; $\| b \| = \sqrt{4/9 + 4/9 + 1/9} = 1$

Thus *a, b, c* is orthonormal.

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APPENDIX - 4 : LAPLACE TRANSFORMS

Many physical problems when analysed assumes the form of an ordinary differential equation subjected to a set of initial conditions or boundary conditions. Such problems are referred to as initial value problems and boundary value problems respectively.

Laplace Transforms serves as a very useful tool in solving these problems without actually finding the general solution of the differential equation by various known methods.

The application of Laplace transforms is highly significant in engineering problems associated with electric circuits.

4.1 Definition of Laplace Transform

If $f(t)$ is a real valued function defined for all $t \geq 0$ then the ***Laplace transform*** of $f(t)$ denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\,dt$$

provided the integral exists. On integration of the indefinite integral we will be having a function of s and t. When this is evaluated between the limits $t = 0$ and $t = \infty$ we will be left with a function of s only and we shall denote it by $\bar{f}(s)$ where s is a parameter, real or complex. Symbolically,

$$L[f(t)] = \bar{f}(s)$$

Equivalently we can express this in the form,

$$L^{-1}[\bar{f}(s)] = f(t)$$

and is called the ***inverse Laplace transform.***

Linearity property : Since linearity property holds good for an integral we can obviously infer that $L[c_1 f_1(t) \pm c_2 f_2(t)] = c_1 L[f_1(t)] \pm c_2 L[f_2(t)]$ where c_1 and c_2 are constants.

4.2 Laplace transform of elementary functions

1. $L(a)$ *where 'a' is a constant.*

$$L(a) = \int_0^{\infty} e^{-st} \cdot a\,dt = a\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{-a}{s}(0-1) = \frac{a}{s} \quad (e^{-x} \to 0 \text{ as } x \to \infty)$$

Thus, $L(a) = \frac{a}{s}$ where $s > 0$. If $a = 1$ then $L(1) = \frac{1}{s}$

2. $L(e^{at})$

$$L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$L(e^{at}) = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = 0 - \frac{1}{-(s-a)} = \frac{1}{s-a}$$

Thus, $L(e^{at}) = \frac{1}{s-a}$, where $s > a$.

3. $L(\cosh at)$

$$L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right) = \frac{1}{2}\{L(e^{at}) + L(e^{-at})\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} \text{ since, } L(e^{at}) = \frac{1}{s-a}$$

$$= \frac{1}{2}\left\{\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right\} = \frac{1}{2} \cdot \frac{2s}{s^2 - a^2} = \frac{s}{s^2 - a^2}$$

Thus, $L(\cosh at) = \frac{s}{s^2 - a^2}$ where $s > a$.

4. $L(\sinh at)$

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}\{L(e^{at}) - L(e^{-at})\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{1}{2}\left\{\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right\} = \frac{a}{s^2 - a^2}$$

Thus, $L(\sinh at) = \frac{a}{s^2 - a^2}$ where $s > a$.

5. $L(\cos at)$

$$L(\cos at) = \int_0^\infty e^{-st} \cos at \; dt$$

Using, $\int e^{at} \cos bt \; dt = \dfrac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$ we have,

$$L(\cos at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{1}{s^2+a^2}\left[e^{-st}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{1}{s^2+a^2}\left[0 - e^0(-s\cos 0 + a\sin 0)\right] = \frac{s}{s^2+a^2}$$

Thus, $L(\cos at) = \dfrac{s}{s^2+a^2}$ where $s > 0$.

6. $L(\sin at)$

$$L(\sin at) = \int_0^\infty e^{-st} \sin at \; dt$$

Using, $\int e^{at} \sin bt \; dt = \dfrac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$ we have,

$$L(\sin at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\sin at - a\cos at)\right]_{t=0}^{\infty}$$

$$L(\text{Sin}\, at) = \frac{-1}{s^2+a^2}\left[e^{-st}(s\sin at + a\cos at)\right]_0^\infty = \frac{-1}{s^2+a^2}(0-a) = \frac{a}{s^2+a^2}$$

Thus, $L(\sin at) = \dfrac{a}{s^2+a^2}$ where $s > 0$.

7. $L(t^n)$

Note : *Definition and properties of gamma function is a pre-requisite.*

$$L(t^n) = \int_0^\infty e^{-st} t^n \, dt$$

Put $st = x$ $\therefore dt = dx/s$ and x varies from 0 to ∞.

Now, $L(t^n) = \int_{x=0}^{\infty} e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^{\infty} e^{-x} x^n \, dx = \frac{\Gamma(n+1)}{s^{n+1}}$

Thus, $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ where n is a constant.

Remarks :

(i) *We know that $\Gamma(n+1)$ exists if n is a positive real number or n is a non negative integer. Hence the expression for $L(t^n)$ is valid for n belonging to these categories of n.*

(ii) *We know that $\Gamma(n+1) = n!$ if n is a positive integer.*

$\therefore \quad L(t^n) = \frac{n!}{s^{n+1}}$ *if n is a positive integer.*

We shall establish this result without the involvement of gamma functions.

8. $L(t^n)$ where n is a positive integer.

$$L(t^n) = \int_0^{\infty} e^{-st} t^n \, dt = \int_0^{\infty} t^n e^{-st} \, dt$$

Integrating by parts we have,

$$L(t^n) = \left[t^n \frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s}(n t^{n-1}) \, dt = 0 + \frac{n}{s}\int_0^{\infty} e^{-st} t^{n-1} \, dt$$

$$\therefore \quad L(t^n) = \frac{n}{s} L(t^{n-1})$$

Similarly, $L(t^{n-1}) = \frac{n-1}{s} L(t^{n-2})$, $L(t^{n-2}) = \frac{n-2}{s} L(t^{n-3})$ etc.

Also, $L(t^2) = \frac{2}{s} L(t^1)$, $L(t^1) = \frac{1}{s} L(t^0)$

Using all these results by back substitution we have,

$$L(t^n) = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} L(t^0)$$

$$= \frac{n!}{s^n} L(1) = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

Thus, $L(t^n) = \frac{n!}{s^{n+1}}$ where n is a positive integer.

Table of Laplace Transforms

	$f(t)$	$L[f(t)] = \bar{f}(s)$		$f(t)$	$L[f(t)] = \bar{f}(s)$
1.	a	$\frac{a}{s}$	5.	$\sinh at$	$\frac{a}{s^2 - a^2}$
2.	e^{at}	$\frac{1}{s-a}$	6.	$\sin at$	$\frac{a}{s^2 + a^2}$
3.	$\cosh at$	$\frac{s}{s^2 - a^2}$	7.	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$
4.	$\cos at$	$\frac{s}{s^2 + a^2}$	8.	t^n $n = 1, 2, 3, \cdots$	$\frac{n!}{s^{n+1}}$

Observe the following illustrations based on the table of Laplace transforms.

1. $L(4) = \frac{4}{s}$

2. $L(e^{2t}) = \frac{1}{s-2}$

3. $L(e^{-t}) = \frac{1}{s+1}$

4. $L(2\cosh 2t) = \frac{2s}{s^2 - 4}$

5. $L(3\cos 4t) = \frac{3s}{s^2 + 16}$

6. $L(4t^3) = 4 \cdot \frac{3!}{s^4} = \frac{24}{s^4}$

7. $L(\sqrt{t}) = L(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{1/2 \cdot \Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

WORKED PROBLEMS

Find the Laplace transform of the following functions

1. $\cosh^2 3t$
2. $e^{-2t} \sinh 4t$
3. $\sin 5t \cdot \cos 2t$
4. $\cos t \cdot \cos 2t \cdot \cos 3t$ **[June 2018]**
5. $\sin^2(2t+1)$
6. $(3t+4)^3 + 5^t$
7. $3\sqrt{t} + \frac{4}{\sqrt{t}}$
8. $t^{-5/2} + t^{5/2}$

☞ **Solutions**

[1] Let, $f(t) = \cosh^2 3t = \left[\dfrac{e^{3t}+e^{-3t}}{2}\right]^2 = \dfrac{1}{4}(e^{6t}+e^{-6t}+2)$

Thus, $$\boxed{L[f(t)] = \frac{1}{4}\left[\frac{1}{s-6}+\frac{1}{s+6}+\frac{2}{s}\right]}$$

[2] Let, $f(t) = e^{-2t}\cdot\sinh 4t = e^{-2t}\cdot\left(\dfrac{e^{4t}-e^{-4t}}{2}\right) = \dfrac{1}{2}(e^{2t}-e^{-6t})$

Thus, $$\boxed{L[f(t)] = \frac{1}{2}\left[\frac{1}{s-2}-\frac{1}{s+6}\right] = \frac{4}{(s-2)(s+6)}}$$

[3] Let, $f(t) = \sin 5t\cos 2t = \dfrac{1}{2}\{\sin(5t+2t)+\sin(5t-2t)\}$

ie., $f(t) = \dfrac{1}{2}\{\sin 7t+\sin 3t\}$

Thus, $$\boxed{L[f(t)] = \frac{1}{2}\left(\frac{7}{s^2+49}+\frac{3}{s^2+9}\right)}$$

[4] Let, $f(t) = \cos t\cos 2t\cos 3t$

Now, $\cos t\cdot\cos 2t = \dfrac{1}{2}\{\cos(t+2t)+\cos(t-2t)\} = \dfrac{1}{2}(\cos 3t+\cos t)$

$\therefore$ $\cos t\cos 2t\cos 3t = \dfrac{1}{2}\{\cos 3t\cos 3t+\cos 3t\cos t\}$

ie., $f(t) = \dfrac{1}{2}\left\{\dfrac{1}{2}(\cos 6t+\cos 0)+\dfrac{1}{2}(\cos 4t+\cos 2t)\right\}$

$f(t) = \dfrac{1}{4}(\cos 6t+1+\cos 4t+\cos 2t)$

Thus, $$\boxed{L[f(t)] = \frac{1}{4}\left[\frac{s}{s^2+36}+\frac{1}{s}+\frac{s}{s^2+16}+\frac{s}{s^2+4}\right]}$$

[5] Let, $f(t) = \sin^2(2t+1) = \frac{1}{2}\{1-\cos 2(2t+1)\}$

ie., $f(t) = \frac{1}{2}\{1-\cos(4t+2)\}$

ie., $f(t) = \frac{1}{2}\{1-\cos 4t \cdot \cos 2 + \sin 4t \sin 2\}$

Thus, $$\boxed{L[f(t)] = \frac{1}{2}\left\{\frac{1}{s} - \frac{s\cos 2}{s^2+16} + \frac{4\sin 2}{s^2+16}\right\}}$$

[6] Let, $f(t) = (3t+4)^3 + 5^t$

ie., $f(t) = (27t^3 + 108t^2 + 144t + 64) + e^{\log 5 \cdot t}$

$$\therefore \quad L[f(t)] = 27 \cdot \frac{3!}{s^4} + 108 \cdot \frac{2!}{s^3} + 144 \cdot \frac{1!}{s^2} + \frac{64}{s} + \frac{1}{s - \log 5}$$

Thus, $$\boxed{L[f(t)] = \frac{162}{s^4} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{64}{s} + \frac{1}{s-\log 5}}$$

[7] Let, $f(t) = 3\sqrt{t} + \frac{4}{\sqrt{t}} = 3t^{1/2} + 4t^{-1/2}$

$$\therefore \quad L[f(t)] = 3\frac{\Gamma(3/2)}{s^{3/2}} + 4\frac{\Gamma(1/2)}{s^{1/2}} = \frac{3\sqrt{\pi}}{2s^{3/2}} + \frac{4\sqrt{\pi}}{\sqrt{s}}$$

Here, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = 1/2 \cdot \Gamma(1/2)$

Thus, $$\boxed{L[f(t)] = \sqrt{\frac{\pi}{s}}\left[\frac{3}{2s} + 4\right]}$$

[9] Let, $f(t) = t^{-5/2} + t^{5/2}$

$$\therefore \quad L[f(t)] = \frac{\Gamma(-3/2)}{s^{-3/2}} + \frac{\Gamma(7/2)}{s^{7/2}} \quad \ldots (1)$$

But $$\Gamma(-3/2) = \frac{\Gamma(-1/2)}{-3/2} = \frac{-2}{3} \cdot \frac{\Gamma(1/2)}{-1/2} = \frac{4\sqrt{\pi}}{3}$$

$$\Gamma(7/2) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{15\sqrt{\pi}}{8}$$

Substituting these values in (1) we get,

$$L[f(t)] = \frac{4\sqrt{\pi}}{3s^{-3/2}} + \frac{15\sqrt{\pi}}{8s^{7/2}}$$

Thus,
$$\boxed{L[f(t)] = \sqrt{\pi}\left[\frac{4}{3}s^{3/2} + \frac{15}{8}\frac{1}{s^{7/2}}\right]}$$

4.3 Properties of Laplace transforms

Property - 1 If $L[f(t)] = \bar{f}(s)$ then $L[f(at)] = (1/a)\,\bar{f}(s/a)$

Proof : $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$\therefore \quad L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$

Put $at = u \therefore dt = (1/a)\,du$

Hence, $L[f(at)] = \int_0^\infty e^{-s(u/a)} f(u)\,\frac{du}{a}$

or $\quad L[f(at)] = (1/a)\bar{f}(s/a)$

Remark : *This property is known as scaling property.*

Property - 2 If $L[f(t)] = \bar{f}(s)$ then $L[e^{at} f(t)] = \bar{f}(s-a)$

Proof : We have by the definition,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s) \qquad \ldots (1)$$

$$\therefore \quad L[e^{at} f(t)] = \int_0^\infty e^{-st}[e^{at} f(t)]\,dt = \int_0^\infty e^{-st+at} f(t)\,dt$$

ie., $\quad L[e^{at} f(t)] = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

Thus $\quad L[e^{at} f(t)] = \bar{f}(s-a)$

Remark : *This property is known as the shifting property.* (*s domain shift*)

Property-3 If $L[f(t)] = \bar{f}(s)$, then

$L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}[\bar{f}(s)]$ where n is a positive integer.

Proof : We establish the result by the principle of mathematical induction.

We have, $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

Differentiating w.r.t. 's' on both sides we have,

$$\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}] f(t)\,dt.$$

$$\therefore \quad \frac{d}{ds}[\bar{f}(s)] = \int_0^\infty e^{-st}(-t) f(t)\,dt$$

or $$(-1)\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty e^{-st}[t f(t)]\,dt = L[t f(t)] \quad \ldots (1)$$

This verifies the result for $n = 1$.

Let us assume the result to be true for $n = k$.

ie., $$(-1)^k \frac{d^k}{ds^k}[\bar{f}(s)] = L[t^k f(t)] \quad \ldots (2)$$

or $$(-1)^k \frac{d^k}{ds^k}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^k f(t)]\,dt$$

Differentiating w.r.t. 's' again we get,

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})[t^k f(t)]\,dt$$

ie., $$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty e^{-st}(-t)[t^k f(t)]\,dt$$

Multiplying by (-1) we get,

$$(-1)^{k+1}\frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^{k+1}f(t)]dt$$

ie., $$(-1)^{k+1}\frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = L[t^{k+1}f(t)] \quad \ldots (3)$$

Comparing (2) and (3) we conclude that the result is true for $n = k + 1$. Hence by the principle of mathematical induction the result is true for all positive integral values of n.

Thus $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)] = (-1)^n \bar{f}^{(n)}(s)$

Remark : *This property is called the* ***derivative of the transform*** *property.* *(Differentiation in the s domain)*

Property - 4 If $L[f(t)] = \bar{f}(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)ds$

Proof : We have, $\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$

$$\therefore \quad \int_s^\infty \bar{f}(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right]ds$$

ie., $$= \int_0^\infty \int_s^\infty e^{-st} f(t)ds\,dt,$$ on changing the order of integration.

$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)dt = \int_0^\infty \left[0 - \frac{e^{-st}}{-t}\right] f(t)dt$$

ie., $$\int_s^\infty \bar{f}(s)ds = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]dt = L\left[\frac{f(t)}{t}\right]$$

Thus $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)ds$ *(Integral of the transform property)*

Properties at a glance

If $L[f(t)] = \bar{f}(s)$ then we have,

1. $L[f(at)] = (1/a)\bar{f}(s/a)$
2. $L[e^{at} f(t)] = \bar{f}(s-a)$
3. $L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}[\bar{f}(s)]$

In particular, $L[t f(t)] = -\dfrac{d}{ds}[\bar{f}(s)]$, $L[t^2 f(t)] = \dfrac{d^2}{ds^2}[\bar{f}(s)]$

4. $L\left[\dfrac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

WORKED PROBLEMS

Find the Laplace transform of the following functions

9. $e^{-2t}(2\cos 5t - \sin 5t)$
10. $e^{-t}\cos^2 3t$
11. $(1+3t\,e^{2t})^2$
12. $\sinh at \sin at$

☞ **Solutions**

[9] Let, $f(t) = 2\cos 5t - \sin 5t$

$$\therefore \quad L[f(t)] = 2\cdot\frac{s}{s^2+5^2} - \frac{5}{s^2+5^2} = \frac{2s-5}{s^2+25}$$

Now, $L[e^{-2t} f(t)] = \left\{\dfrac{2s-5}{s^2+25}\right\}_{s\to s+2} = \dfrac{2(s+2)-5}{(s+2)^2+25}$

Thus, $$\boxed{L[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2s-1}{s^2+4s+29}}$$

[10] Let, $f(t) = \cos^2 3t = \dfrac{1+\cos 6t}{2}$

$$\therefore \quad L[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

Now, $L[e^{-t}\cos^2 3t] = \dfrac{1}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+36}\right]_{s\to(s+1)}$

Thus, $$\boxed{L[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s+1} + \frac{(s+1)}{(s+1)^2+36}\right]}$$

[11] Let, $f(t) = (1+3t\,e^{2t})^2 = 1+6t\,e^{2t}+9t^2\,e^{4t}$

$\therefore \quad L[f(t)] = L(1)+6L(e^{2t}\cdot t)+9L(e^{4t}\cdot t^2)$

$$= \frac{1}{s}+6\{L(t)\}_{s\to(s-2)}+9\{L(t^2)\}_{s\to(s-4)}$$

But, $L(t) = \dfrac{1}{s^2}$ and $L(t^2) = \dfrac{2}{s^3}$

Thus, $$\boxed{L[(1+3t\,e^{2t})^2] = \frac{1}{s}+\frac{6}{(s-2)^2}+\frac{18}{(s-4)^3}}$$

[12] Let, $f(t) = \sinh at \sin at = \dfrac{e^{at}-e^{-at}}{2}\cdot\sin at$

ie., $f(t) = \dfrac{1}{2}(e^{at}\sin at - e^{-at}\sin at)$

$\therefore \quad L[f(t)] = \dfrac{1}{2}\left\{L(\sin at)_{s\to s-a} - L(\sin at)_{s\to s+a}\right\}$

But, $L(\sin at) = \dfrac{a}{s^2+a^2}$

Hence, $$L(\sinh at\sin at) = \frac{1}{2}\left\{\frac{a}{(s-a)^2+a^2}-\frac{a}{(s+a)^2+a^2}\right\}$$

$$L(\sinh at\sin at) = \frac{a}{2}\left\{\frac{1}{s^2+2a^2-2as}-\frac{1}{s^2+2a^2+2as}\right\}$$

$$= \frac{a}{2}\left\{\frac{4as}{(s^2+2a^2)^2-4a^2s^2}\right\}$$

Thus, $\boxed{L(\sinh at\sin at) = \dfrac{2a^2s}{s^4+4a^4}}$ on simplification.

Find the Laplace transform of the following functions

13. $t\cos at$ **[Dec 2018]** 14. $t^2\sin at$

15. $t^3\cos ht$ 16. $te^{-2t}\sin 4t$

Solutions

[13] Let, $f(t) = \cos at \quad \therefore \; L[f(t)] = \dfrac{s}{s^2 + a^2}$

Now, $L[t f(t)] = \dfrac{-d}{ds}\left[\dfrac{s}{s^2 + a^2}\right] = -\left\{\dfrac{(s^2 + a^2)1 - s \cdot 2s}{(s^2 + a^2)^2}\right\}$

Thus, $$\boxed{L(t \cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2}}$$

[14] Let, $f(t) = \sin at \quad \therefore \; L[f(t)] = \dfrac{a}{s^2 + a^2}$

Now, $L[t^2 f(t)] = \dfrac{d^2}{ds^2}\left[\dfrac{a}{s^2 + a^2}\right] = \dfrac{d}{ds}\left\{\dfrac{d}{ds}\left(\dfrac{a}{s^2 + a^2}\right)\right\}$

$$L[t^2 \sin at] = \frac{d}{ds}\left\{\frac{-2as}{(s^2 + a^2)^2}\right\}$$

$$L[t^2 \sin at] = \frac{(s^2 + a^2)^2(-2a) + 2as \cdot 2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4}$$

$$= \frac{2a(s^2 + a^2)\{-(s^2 + a^2) + 4s^2\}}{(s^2 + a^2)^4}$$

Thus, $$\boxed{L[t^2 \sin at] = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}}$$ *(On simplification)*

[15] Let, $f(t) = t^3 \cosh t$

Note : *Here we should not prefer to work the problem similar to the previous problem as we have cosh t which can be converted to the form $(e^t + e^{-t})/2$ so that it will be highly convenient to apply the shifting property.*

$$f(t) = t^3\left(\frac{e^t + e^{-t}}{2}\right) = \frac{1}{2}\{e^t t^3 + e^{-t} t^3\}$$

$$L[f(t)] = \frac{1}{2}\{L(t^3)_{s \to s-1} + L(t^3)_{s \to s+1}\}$$

But, $L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$

Thus, $\boxed{L\left[t^3 \cosh t\right] = \frac{1}{2}\left\{\frac{6}{(s-1)^4} + \frac{6}{(s+1)^4}\right\} = 3\left\{\frac{1}{(s-1)^4} + \frac{1}{(s+1)^4}\right\}}$

[16] Let, $f(t) = t\,e^{-2t}\sin 4t$

$$L(\sin 4t) = \frac{4}{s^2+16} \quad \therefore\ L[e^{-2t}\sin 4t] = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}$$

Hence, $L[t\,e^{-2t}\sin 4t] = \frac{-d}{ds}\left\{\frac{4}{s^2+4s+20}\right\} = \frac{4(2s+4)}{(s^2+4s+20)^2}$

Thus, $\boxed{L[t\,e^{-2t}\sin 4t] = \frac{8(s+2)}{(s^2+4s+20)^2}}$

[17] *Show that* $\int_0^\infty t\,e^{-2t}\sin 4t\,dt = \frac{1}{25}$

☞ We have, $\int_0^\infty e^{-st}\,t\sin 4t\,dt = L(t\sin 4t)$ ... (1)

Now, $L(t\sin 4t) = \frac{-d}{ds}L(\sin 4t) = \frac{-d}{ds}\left(\frac{4}{s^2+16}\right) = \frac{8s}{(s^2+16)^2}$

Hence (1) becomes $\int_0^\infty e^{-st}\,t\sin 4t\,dt = \frac{8s}{(s^2+16)^2}$

Thus by putting $s = 2$ we get, $\boxed{\int_0^\infty e^{-2t}\,t\sin 4t\,dt = \frac{16}{400} = \frac{1}{25}}$

Find the Laplace transform of the following functions.

18. $\frac{1-e^{-at}}{t}$ **19.** $\frac{\sin^2 t}{t}$ **20.** $\frac{\sin at}{t}$ **[Dec 2018]**

☞ **Solutions**

[18] Let, $f(t) = 1 - e^{-at} \quad \therefore\ \bar{f}(s) = L[f(t)] = \frac{1}{s} - \frac{1}{s+a}$

We have, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

Hence, $L\left\{\dfrac{1-e^{-at}}{t}\right\} = \int_s^\infty \left(\dfrac{1}{s} - \dfrac{1}{s+a}\right) ds$

$$= [\log s - \log(s+a)]_s^\infty = \left[\log\left(\frac{s}{s+a}\right)\right]_s^\infty$$

$$= \left[\lim_{s\to\infty} \log\left(\frac{s}{s+a}\right)\right] - \log\left(\frac{s}{s+a}\right)$$

ie.,

$$L\left\{\frac{1-e^{-at}}{t}\right\} = \lim_{s\to\infty} \log\left(\frac{s}{s(1+a/s)}\right) - \log\left(\frac{s}{s+a}\right) = \log 1 - \log\left(\frac{s}{s+a}\right)$$

Thus, $$\boxed{L\left[\frac{1-e^{-at}}{t}\right] = \log\left(\frac{s+a}{s}\right)}$$

[19] Let, $f(t) = \sin^2 t = \dfrac{1}{2}(1-\cos 2t)$

$\therefore \quad \bar{f}(s) = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right)$

Hence, $L\left[\dfrac{f(t)}{t}\right] = \dfrac{1}{2}\int_s^\infty \left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right) ds$

ie., $L\left[\dfrac{\sin^2 t}{t}\right] = \dfrac{1}{2}\left[\log s - \dfrac{1}{2}\log(s^2+4)\right]_s^\infty = \dfrac{1}{2}\log\left[\dfrac{s}{\sqrt{s^2+4}}\right]_s^\infty$

ie., $$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s}{s\sqrt{1+(4/s^2)}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

Thus, $$\boxed{L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)}$$

[20] Let, $f(t) = \sin at \quad \therefore \ \bar{f}(s) = \dfrac{a}{s^2 + a^2}$

Hence, $L\left[\dfrac{f(t)}{t}\right] = \int_s^\infty \dfrac{a}{s^2 + a^2}\, ds$

ie., $L\left[\dfrac{\sin at}{t}\right] = a \cdot \dfrac{1}{a}\left[\tan^{-1}(s/a)\right]_s^\infty = \tan^{-1}(\infty) - \tan^{-1}(s/a)$

Thus, $$\boxed{L\left[\dfrac{\sin at}{t}\right] = \pi/2 - \tan^{-1}(s/a) = \cot^{-1}(s/a)}$$

Evaluate the following integrals using Laplace transforms.

21. $\int_0^\infty \dfrac{\cos 6t - \cos 4t}{t}\, dt$ **22.** $\int_0^\infty \dfrac{e^{-at} - e^{-bt}}{t}\, dt$ **23.** $\int_0^\infty \dfrac{e^{-t}\sin t}{t}\, dt$

☞ **Solutions**

[21] We know that, $\int_0^\infty e^{-st}\left[\dfrac{\cos 6t - \cos 4t}{t}\right] dt = L\left[\dfrac{\cos 6t - \cos 4t}{t}\right]$

We can obtain,

$$L\left[\dfrac{\cos 6t - \cos 4t}{t}\right] = \log\sqrt{(s^2 + 16)/(s^2 + 36)}$$

ie., $\int_0^\infty e^{-st}\left[\dfrac{\cos 6t - \cos 4t}{t}\right] dt = \log\sqrt{(s^2 + 16)/(s^2 + 36)}$

Thus by putting $s = 0$ we get,

$$\boxed{\int_0^\infty \dfrac{\cos 6t - \cos 4t}{t}\, dt = \log\sqrt{16/36} = \log(2/3)}$$

[22] Proceeding as in **[21]**, we can obtain the answer $\log(b/a)$

[23] We shall first find $L\left[\dfrac{e^{-t}\sin t}{t}\right]$

Now, $L\left[\dfrac{e^{-t}\sin t}{t}\right] = \int_s^\infty L(e^{-t}\sin t)\, ds = \int_s^\infty \{L(\sin t)_{s \to s+1}\}\, ds$

$$= \int_s^\infty \frac{1}{(s+1)^2+1}\,ds = \left[\tan^{-1}(s+1)\right]_s^\infty = \tan^{-1}(\infty) - \tan^{-1}(s+1)$$

i.e.,

$$L\left[\frac{e^{-t}\sin t}{t}\right] = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

$\therefore$

$$\int_0^\infty e^{-st}\cdot\frac{e^{-t}\sin t}{t}\,dt = \cot^{-1}(s+1)$$

i.e.,

Thus by putting $s = 0$ we get, $\boxed{\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \cot^{-1}(1) = \frac{\pi}{4}}$

4.4 Laplace transform of periodic function

Definition : A function $f(t)$ is said to be a periodic function of period $T > 0$ if $f(t+nT) = f(t)$ where $n = 1, 2, 3 \ldots$

Example : $\sin t$, $\cos t$ are periodic functions of period 2π because

$$\sin(t+2n\pi) = \sin t,\ \cos(t+2n\pi) = \cos t.$$

Theorem : *If $f(t)$ is a periodic function of period T, then*

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

Proof : We have by the definition,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-su} f(u)\,du$$

$$= \int_{u=0}^{T} e^{-su} f(u)\,du + \int_{u=T}^{2T} e^{-su} f(u)\,du + \cdots + \int_{u=nT}^{(n+1)T} e^{-su} f(u)\,du + \cdots$$

$$L[f(t)] = \sum_{n=0}^{\infty}\int_{u=nT}^{(n+1)T} e^{-su} f(u)\,du \qquad \ldots (1)$$

Now put, $u = t + nT \quad \therefore \quad du = dt$

If $\quad u = nT \quad$ then $\quad t + nT = nT \Rightarrow t = 0$

$\quad u = (n+1)T \quad$ then $\quad t + nT = nT + T \Rightarrow t = T$

Further, $f(u) = f(t+nT) = f(t)$ by the periodic property.

Using these results in the RHS of (1) we obtain,

$$L[f(t)] = \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-s(t+nT)} f(t)\,dt$$

$$L[f(t)] = \sum_{n=0}^{\infty} e^{-snT} \int_{t=0}^{T} e^{-st} f(t)\,dt \qquad \ldots (2)$$

But, $\sum_{n=0}^{\infty} e^{-snT} = \sum_{n=0}^{\infty} (e^{-sT})^n = 1 + (e^{-sT}) + (e^{-sT})^2 + \cdots$

Putting $r = e^{-sT}$ the series involved is a geometric series of the form $1 + r + r^2 + \cdots$ whose sum to infinity is known to be $1/1-r$ where $r < 1$.

Hence, $\sum_{n=0}^{\infty} e^{-snT} = \dfrac{1}{1-e^{-sT}}$ $\qquad \ldots (3)$

Now using (3) in the RHS of (2) we have,

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

WORKED PROBLEMS

[24] *Find the Laplace transform of the full wave rectifier* $f(t) = E\sin wt,\ 0 < t < \pi/w$ *having period* π/w. **[Dec 2017, 18]**

☞ We have, $L[f(t)] = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

Now, $L[f(t)] = \dfrac{1}{1-e^{-s(\pi/w)}} \int_0^{\pi/w} e^{-st} E\sin wt\,dt$

$$= \frac{E}{1-e^{-\pi s/w}} \left[\frac{e^{-st}}{s^2+w^2} (-s\sin wt - w\cos wt) \right]_0^{\pi/w}$$

$$= \frac{E}{(1-e^{-\pi s/w})(s^2+w^2)} [e^{-\pi s/w} \cdot w - (-w)]$$

$\therefore \quad L[f(t)] = \dfrac{Ew(1+e^{-\pi s/w})}{(s^2+w^2)(1-e^{-\pi s/w})}$. We can simplify further.

Multiplying both the numerator and the denominator by $e^{\pi s/2w}$ in RHS, the expression assumes the form,

$$L[f(t)] = \frac{Ew}{s^2+w^2}\cdot\frac{(e^{\pi s/2w}+e^{-\pi s/2w})}{(e^{\pi s/2w}-e^{-\pi s/2w})} = \frac{Ew}{s^2+w^2}\cdot\frac{2\cosh(\pi s/2w)}{2\sinh(\pi s/2w)}$$

Thus,

$$\boxed{L[f(t)] = \frac{Ew}{s^2+w^2}\coth(\pi s/2w)}$$

[25] *Given* $f(t) = \begin{cases} E, & 0<t<a/2 \\ -E, & a/2<t<a \end{cases}$ *where* $f(t+a) = f(t)$,

show that $L[f(t)] = E/s\cdot\tanh(as/4)$.

☞ The given function is periodic with period $T = a$.

We have, $L[f(t)] = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st}f(t)\,dt$

$$= \frac{1}{1-e^{-sa}}\int_0^a e^{-st}f(t)\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-as}}\left\{\int_0^{a/2} e^{-st}E\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} + \left[\frac{e^{-st}}{s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left\{-[e^{-st}]_0^{a/2} + [e^{-st}]_{a/2}^{a}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\{-e^{-as/2}+1+e^{-as}-e^{-as/2}\}$$

$$= \frac{E}{s(1-e^{-as})}(1-2e^{-as/2}+e^{-as}) = \frac{E(1-e^{-as/2})^2}{s(1-e^{-as})}$$

$$L[f(t)] = \frac{E(1-e^{-as/2})^2}{s(1-e^{-as/2})(1+e^{-as/2})} = \frac{E(1-e^{-as/2})}{s(1+e^{-as/2})}$$

Multiplying both the numerator and denominator by $e^{as/4}$ we get,

$$L[f(t)] = \frac{E(e^{as/4} - e^{-as/4})}{s(e^{as/4} + e^{-as/4})} = \frac{E \cdot 2\sinh(as/4)}{s \cdot 2\cosh(as/4)}$$

Thus, $$\boxed{L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)}$$

Note : Similar Problem [*Square wave function*]

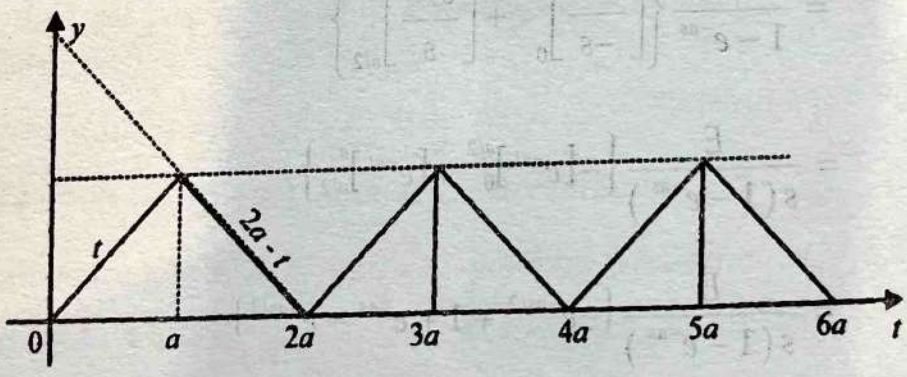
If $f(t) = \begin{cases} E, & 0<t<a \\ -E, & a<t<2a \end{cases}$ *show that* $L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$ **[June 2017]**

[26] *If* $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a - t, & a \le t \le 2a, \end{cases}$ $f(t+2a) = f(t)$

(i) *Sketch the graph of* $f(t)$ *as a periodic function*

(ii) *Show that* $L[f(t)] = \frac{1}{s^2}\tanh(as/2)$ **[June & Dec 2016]**

☞ (i) Let $f(t) = y$ and $y = t$ is a straight line passing through the origin making an angle 45^0 with the t-axis. $y = 2a - t$ or $y + t = 2a$ or $t/2a + y/2a = 1$ is a straight line passing through the points $(2a, 0)$ and $(0, 2a)$.
The graph of $y = f(t)$ is as follows.



The periodic function $f(t)$ is called the *triangular wave function.*

(ii) We have, $T = 2a$ and $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left\{\int_0^a t e^{-st}\,dt + \int_a^{2a}(2a-t)e^{-st}\,dt\right\}$$

Applying Bernoulli's rule to each of the integrals we have,

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left\{\left[t\cdot\frac{e^{-st}}{-s}-(1)\frac{e^{-st}}{s^2}\right]_0^a + \left[(2a-t)\frac{e^{-st}}{-s}-(-1)\frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left\{\frac{-1}{s}(ae^{-as}-0)-\frac{1}{s^2}(e^{-as}-1)-\frac{1}{s}(0-ae^{-as})+\frac{1}{s^2}(e^{-2as}-e^{-as})\right\}$$

$$L[f(t)] = \frac{1}{s^2(1-e^{-2as})}(-e^{-as}+1+e^{-2as}-e^{-as})$$

$$= \frac{1}{s^2(1-e^{-2as})}(1-2e^{-as}+e^{-2as}) = \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$L[f(t)] = \frac{(1-e^{-as})}{s^2(1+e^{-as})} = \frac{e^{as/2}-e^{-as/2}}{s^2(e^{as/2}+e^{-as/2})}$$

where we have multiplied both the numerator and denominator by $e^{as/2}$.

$$\therefore \quad L[f(t)] = \frac{2\sinh(as/2)}{s^2\cdot 2\cosh(as/2)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

Thus, $\boxed{L[f(t)] = 1/s^2\cdot\tanh(as/2)}$

Note: Similar Problems

(i) *Find* $L[f(t)]$ *if* $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \end{cases}$ **[June 2015]**

(ii) *Find* $L[f(t)]$ *if* $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$ **[June 2018]**

[27] *A periodic function of period* $2\pi/\omega$ *is defined by*

$$f(t) = \begin{cases} E\sin\omega t, & 0 \le t \le \pi/\omega \\ 0, & \pi/\omega \le t \le 2\pi/\omega \end{cases}$$ *where E and* ω *are constants.*

(*Half wave rectifier*)

Show that $L[f(t)] = \dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

(Half Wave Rectifier)

☞ We have for a periodic function $f(t)$,

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt. \text{ Here, } T = 2\pi/\omega.$$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left\{\int_0^{\pi/\omega} e^{-st} E\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\cdot 0\,dt\right\}$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{-E}{(s^2+w^2)(1-e^{-2\pi s/\omega})}\left\{e^{-s\pi/\omega}(s\sin\pi + \omega\cos\pi) - e^0(s\sin 0 + \omega\cos 0)\right\}$$

$$= \frac{-E}{(s^2+w^2)(1-e^{-2\pi s/\omega})}\left\{-\omega e^{-s\pi/\omega} - \omega\right\} = \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+w^2)(1-e^{-2\pi s/\omega})}$$

$$L[f(t)] = \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+w^2)(1-e^{-\pi s/\omega})(1+e^{-\pi s/\omega})}$$

Thus, $$L[f(t)] = \frac{E\omega}{(s^2+w^2)(1-e^{-\pi s/\omega})}$$

4.5 Unit step function (Heaviside function)

Definition : The ***unit step function*** $u(t-a)$ or ***Heaviside function*** $H(t-a)$ is defined as follows.

$$u(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases} \quad \text{where } a \text{ is a constant.}$$

Properties associated with the unit step function

(i) $L[u(t-a)] = \dfrac{e^{-as}}{s}$

(ii) $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$ where $L[f(t)] = \overline{f}(s)$

Proof : (i) $L[u(t-a)] = \int_0^\infty e^{-st}\,u(t-a)\,dt$

$$L[u(t-a)] = \int_0^a e^{-st}\,u(t-a)\,dt + \int_a^\infty e^{-st}\,u(t-a)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = 0 - \frac{e^{-as}}{-s} = \frac{e^{-as}}{s}$$

Thus, $L[u(t-a)] = \dfrac{e^{-as}}{s}$

(ii) $L[f(t-a)u(t-a)] = \int_0^\infty e^{-st}\,f(t-a)\,u(t-a)\,dt$

$$= \int_0^a e^{-st}\,f(t-a)\cdot 0\,dt + \int_a^\infty e^{-st}\,f(t-a)\cdot 1\,dt$$

ie., $L[f(t-a)u(t-a)] = \int_a^\infty e^{-st}\,f(t-a)\,dt$

Put $t-a = v \therefore dt = dv$ If $t = a,\ v = 0\ ;\ t = \infty,\ v = \infty$

Hence, $L[f(t-a)u(t-a)] = \int_{v=0}^{\infty} e^{-s(a+v)}\,f(v)\,dv$

$$= e^{-as}\int_{v=0}^{\infty} e^{-sv}\,f(v)\,dv = e^{-as}\,\overline{f}(s)$$

Thus, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

Remarks

1. *The result (i) follows as a particular case of (ii) when* $f(t-a)=1$ *as we have* $f(t)$ *also equal to* 1 *and hence* $L[f(t)] = 1/s$

Hence (ii) becomes $L[u(t-a)] = e^{-as}/s$

2. *It is possible to express a discontinuous function* $f(t)$ *interms of unit step function and inturn we can find its Laplace transform by using properties (i) and (ii). The following two results (iii) and (iv) which can be easily verified will be highly useful.*

(iii) If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

(iv) If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

Working procedure for problems

Type-1 : *To find* $L[F(t)u(t-a)]$ *where* $F(t)$ *is a polynomial in t.*

Step-1 Let $F(t) = f(t-a)$ which implies that $F(t+a) = f(t)$

That is to replace t by $t+a$ to obtain $f(t)$ and find $L[f(t)] = \bar{f}(s)$

Step-2 $L[F(t)u(t-a)] = L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$ by property (ii).

Type-2 : *Given* $f(t)$ *as a discontinuous function, to find* $L[f(t)]$ *by expressing* $f(t)$ *in terms of unit step function*

Step-1 We express $f(t)$ in terms of unit step function by directly making use of the result (iii) or (iv) as the case may be.

Step-2 We find $L[f(t)]$ as in Type-1

WORKED PROBLEMS

Find the Laplace transform of the following functions

28. $(3t^2 + 4t + 5)u(t-3)$

29. $(t^3 + t^2 + t + 1)u(t+1)$

☞ **Solutions**

[28] Let, $f(t-3) = 3t^2 + 4t + 5$

$\Rightarrow \quad f(t) = 3(t+3)^2 + 4(t+3) + 5 = 3t^2 + 22t + 44$

$\therefore \quad \bar{f}(s) = 3 \cdot \frac{2!}{s^3} + 22 \cdot \frac{1!}{s^2} + \frac{44}{s} = \frac{6}{s^3} + \frac{22}{s^2} + \frac{44}{s}$

We have, $L[f(t-3)u(t-3)] = e^{-3s}\bar{f}(s); (a = 3)$

Thus, $\boxed{L[(3t^2 + 4t + 5)u(t-3)] = e^{-3s}[6/s^3 + 22/s^2 + 44/s]}$

[29] Let, $f(t+1) = t^3 + t^2 + t + 1$

$\Rightarrow \quad f(t) = (t-1)^3 + (t-1)^2 + (t-1) + 1$

$= (t^3 - 3t^2 + 3t - 1) + (t^2 - 2t + 1) + (t-1) + 1$

ie., $f(t) = t^3 - 2t^2 + 2t \quad \therefore \bar{f}(s) = \frac{3!}{s^4} - 2 \cdot \frac{2!}{s^3} + 2 \cdot \frac{1!}{s^2}$

We have, $L[f(t+1)u(t+1)] = e^{s}\bar{f}(s); (a = -1)$

Thus, $\boxed{L[(t^3 + t^2 + t + 1)u(t+1)] = e^{s}\left(\frac{6}{s^4} - \frac{4}{s^3} + \frac{2}{s^2}\right)}$

Express the following functions interms of Heaviside unit step function and hence find their Laplace transform.

30. $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$

31. $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$ **[Dec.2018]**

32. $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ t^2, & t > 2 \end{cases}$ **[June 2016, 18]**

33. $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$

☞ **Solutions**

[30] $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$

$f(t) = t + (5-t)u(t-4)$ by a property.

$L[f(t)] = L(t) + L[(5-t)u(t-4)] \quad \ldots (1)$

We have, $L(t) = 1/s^2$

Let, $F(t-4) = (5-t) \quad \therefore \; F(t) = 5-(t+4) = 1-t$

Hence, $\bar{F}(s) = L[F(t)] = \dfrac{1}{s} - \dfrac{1}{s^2}$

But, $L[F(t-4)u(t-4)] = e^{-4s}\,\bar{F}(s)$

ie., $L[(5-t)u(t-4)] = e^{-4s}\left(1/s - 1/s^2\right)$

We shall use these results in (1).

Thus, $$\boxed{L[f(t)] = \frac{1}{s^2} + e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right) = \frac{1}{s}e^{-4s} + \frac{1}{s^2}(1-e^{-4s})}$$

[31] $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ \sin t, & t>\pi \end{cases}$

$f(t) = \cos t + (\sin t - \cos t)u(t-\pi)$ by a property.

$$L[f(t)] = L(\cos t) + L[(\sin t - \cos t)u(t-\pi)] \qquad \ldots (1)$$

Now, let $F(t-\pi) = \sin t - \cos t$

$\Rightarrow \quad F(t) = \sin(t+\pi) - \cos(t+\pi) = -\sin t + \cos t$

$\therefore \quad \bar{F}(s) = \dfrac{-1}{s^2+1} + \dfrac{s}{s^2+1} = \dfrac{s-1}{s^2+1}$

But $\quad L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\,\bar{F}(s) = \dfrac{e^{-\pi s}(s-1)}{s^2+1}$

We shall use these results in (1).

Thus, $$\boxed{L[f(t)] = \frac{s}{s^2+1} + \frac{e^{-\pi s}(s-1)}{s^2+1} = \frac{s+e^{-\pi s}(s-1)}{s^2+1}}$$

[32] $f(t) = \begin{cases} 1, & 0<t\le 1 \\ t, & 1<t\le 2 \\ t^2, & t>2 \end{cases}$

$f(t) = 1 + (t-1)u(t-1) + (t^2-t)u(t-2)$ by a property.

$$L[f(t)] = L(1) + L[(t-1)u(t-1)] + L[(t^2-t)u(t-2)] \quad \ldots (1)$$

Let, $F(t-1) = (t-1)$; $G(t-2) = t^2 - t$

$\Rightarrow$ $F(t) = t$; $G(t) = (t+2)^2 - (t+2) = t^2 + 3t + 2$

$\therefore$ $\bar{F}(s) = \dfrac{1}{s^2}$; $\bar{G}(s) = \dfrac{2}{s^3} + \dfrac{3}{s^2} + \dfrac{2}{s}$

$$L[F(t-1)u(t-1)] = e^{-s}\bar{F}(s) \text{ and } L[G(t-2)u(t-2)] = e^{-2s}\bar{G}(s)$$

ie., $L[(t-1)u(t-1)] = \dfrac{e^{-s}}{s^2}$ and

$$L[(t^2-t)u(t-2)] = e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$$

We shall use these results in (1).

Thus, $$\boxed{L[f(t)] = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)}$$

[33] $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$

$$f(t) = \cos t + (\cos 2t - \cos t)u(t-\pi) + (\cos 3t - \cos 2t)u(t-2\pi)$$

$$L[f(t)] = L(\cos t) + L[(\cos 2t - \cos t)u(t-\pi)] + L[(\cos 3t - \cos 2t)u(t-2\pi)] \quad \ldots (1)$$

Let $F(t-\pi) = \cos 2t - \cos t$; $G(t-2\pi) = \cos 3t - \cos 2t$

$\Rightarrow$ $F(t) = \cos 2(t+\pi) - \cos(t+\pi)$ and

$G(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$

ie., $F(t) = \cos 2t + \cos t$; $G(t) = \cos 3t - \cos 2t$

$\therefore$ $\bar{F}(s) = \dfrac{s}{s^2+4} + \dfrac{s}{s^2+1}$; $\bar{G}(s) = \dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}$

But $L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{F}(s)$ and

$L[G(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\bar{G}(s)$

ie., $L[(\cos 2t - \cos t)u(t-\pi)] = e^{-\pi s}\left(\dfrac{s}{s^2+4} + \dfrac{s}{s^2+1}\right)$

and $L[(\cos 3t - \cos 2t)u(t-2\pi)] = e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$

Hence (1) becomes

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Thus $\boxed{L[f(t)] = \frac{s}{s^2+1} + s\,e^{-\pi s}\left(\frac{1}{s^2+4} + \frac{1}{s^2+1}\right) - \frac{5s\,e^{-2\pi s}}{(s^2+4)(s^2+9)}}$

[34] *Define Heaviside unit step function. Using unit step function find the Laplace transform of*

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \ge 2\pi \end{cases}$$ **[Dec. 2016]**

☞ Proceeding as in **[33]** we can obtain,

$$\boxed{L[f(t)] = \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]}$$

EXERCISE

Find the Laplace transform of the following functions

1. $\sin 3t \sin 2t \sin t$ **2.** $t\sqrt{t} + 4t^3 + 3^t$ **3.** $e^{-t}\cos^2 3t$

4. $(1 + t\,e^t)^3$ **5.** $\frac{\cosh at - \cos bt}{t}$ **6.** $\frac{1 - \cos at}{t}$

Find the Laplace transform for the following functions

7. $f(t) = \begin{cases} a, & 0 \le t \le a \\ -a, & a < t < 2a \end{cases}$ where $f(t+2a) = f(t)$

8. $f(t) = E\sin\omega t$ in $0 < t < \pi/\omega$

Express the following functions interms of Heaviside unit step function and hence find its Laplace transform

9. $f(t) = \begin{cases} t^2, & 1 < t \le 2 \\ 4t, & t > 2 \end{cases}$ **10.** $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ \sin 3t, & t > 2\pi \end{cases}$

ANSWERS

1. $\frac{1}{2}\left[\frac{1}{s^2+4}-\frac{3}{s^2+36}+\frac{2}{s^2+16}\right]$

2. $\frac{3\sqrt{\pi}}{4s^{5/2}}+\frac{24}{s^4}+\frac{1}{s-\log 3}$

3. $\frac{s^2+2s+19}{(s+1)(s^2+2s+37)}$

4. $\frac{1}{s}+\frac{3}{(s-1)^2}+\frac{6}{(s-2)^3}+\frac{6}{(s-3)^4}$

5. $\log\sqrt{(s^2+b^2)/(s^2-a^2)}$

6. $\log\left(\sqrt{s^2+a^2/s^2}\right)$

7. $a/s\cdot\tanh(as/2)$

8. $\frac{E\omega}{s^2+\omega^2}\coth(\pi s/2\omega)$

9. $\frac{2}{s^3}+\left(\frac{4}{s}-\frac{2}{s^3}\right)e^{-2s}$

10. $\frac{1}{s^2+1}+e^{-\pi s}\left(\frac{2}{s^2+4}+\frac{1}{s^2+1}\right)+e^{-2\pi s}\left(\frac{3}{s^2+9}-\frac{2}{s^2+4}\right)$

4.6 Unit Impulse function

The *unit impulse function* or the *Dirac delta function* $\delta(t-a)$ is defined as follows.

$$\delta(t-a)=\lim_{\varepsilon\to 0}\delta_\varepsilon(t-a)\ ;\ a\ge 0$$

$$\text{where } \delta_\varepsilon(t-a)=\begin{cases}\frac{1}{\varepsilon} & \text{if } a\le t\le a+\varepsilon\\ 0, & \text{otherwise}\end{cases}$$

Laplace transform of the unit impulse function

We shall first find the Laplace transform of $\delta_\varepsilon(t-a)$

$$L[\delta_\varepsilon(t-a)]=\int_0^\infty e^{-st}\,\delta_\varepsilon(t-a)\,dt$$

$$=\int_0^a e^{-st}\,\delta_\varepsilon(t-a)\,dt+\int_a^{a+\varepsilon} e^{-st}\,\delta_\varepsilon(t-a)\,dt+\int_{a+\varepsilon}^\infty e^{-st}\,\delta_\varepsilon(t-a)\,dt$$

ie., $L[\delta_\varepsilon(t-a)]=\int_a^{a+\varepsilon} e^{-st}\cdot\frac{1}{\varepsilon}\,dt$, by using the definition.

$$=\frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon}=\frac{-1}{\varepsilon s}\left\{e^{-s(a+\varepsilon)}-e^{-as}\right\}$$

$$\therefore \quad L[\delta_\varepsilon(t-a)] = e^{-as}\left\{\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right\} \qquad \ldots (1)$$

But $\quad L[\delta(t-a)] = L\left\{\lim_{\varepsilon\to 0}\delta_\varepsilon(t-a)\right\} = \lim_{\varepsilon\to 0} L[\delta_\varepsilon(t-a)]$

Hence taking limit on both sides of (1) as $\varepsilon \to 0$ we have,

$$\lim_{\varepsilon\to 0} L[\delta_\varepsilon(t-a)] = e^{-as}\lim_{\varepsilon\to 0}\left\{\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right\} = e^{-as}k \qquad \ldots (2)$$

where $k = \lim_{\varepsilon\to 0}\frac{1-e^{-\varepsilon s}}{\varepsilon s}$. This being in the indeterminate form $\frac{0}{0}$, we employ L'Hospital's rule to evaluate the limit.

Hence, $\quad k = \lim_{\varepsilon\to 0}\frac{0-(-s)e^{-\varepsilon s}}{s} = \lim_{\varepsilon\to 0} e^{-\varepsilon s} = 1$

Using $k = 1$ in (2) we have, $\lim_{\varepsilon\to 0} L[\delta_\varepsilon(t-a)] = e^{-as}$

Thus $L[\delta(t-a)] = e^{-as}$; In particular if $a = 0$, $L[\delta(t)] = 1$

WORKED PROBLEMS

[35] *Find* $L[2\delta(t-1)+3\delta(t-2)+4\delta(t+3)]$

☞ We have $2L[\delta(t-1)]+3L[\delta(t-2)]+4L[\delta(t+3)]$

Since $L[\delta(t-a)] = e^{-as}$, we get $\boxed{2e^{-s}+3e^{-2s}+4e^{3s}}$

[36] *Find* $L[\cosh 3t\,\delta(t-2)]$

☞ $$\cosh 3t\,\delta(t-2) = \frac{1}{2}\{e^{3t}+e^{-3t}\}\delta(t-2)$$

$$L[\cosh 3t\,\delta(t-2)] = \frac{1}{2}\{L[e^{3t}\delta(t-2)]+L[e^{-3t}\delta(t-2)]\}$$

$$= \frac{1}{2}\{L[\delta(t-2)]_{s\to s-3} + L[\delta(t-2)]_{s\to s+3}\}$$

$$= \frac{1}{2}\{(e^{-2s})_{s\to s-3} + (e^{-2s})_{s\to s+3}\}$$

$$L[\cosh 3t\ \delta(t-2)] = \frac{1}{2}\left\{e^{-2(s-3)} + e^{-2(s+3)}\right\}$$

Thus $\boxed{L[\cosh 3t\ \delta(t-2)] = \frac{e^{-2s}}{2}\{e^{6} + e^{-6}\}}$

[37] *Find* $L[t^4\,\delta(t-3)]$

☞ $L[\delta(t-3)] = e^{-3s}$

$$\therefore\quad L[t^4\,\delta(t-3)] = (-1)^4\,\frac{d^4}{ds^4}[e^{-3s}] = (-3)^4\,e^{-3s}$$

Thus $\boxed{L[t^4\,\delta(t-3)] = 81\,e^{-3s}}$

[38] *Find* $L[(t-1)^2\,\delta(t-a)]$

☞ $(t-1)^2\,\delta(t-a) = (t^2-2t+1)\,\delta(t-a)$

Proceeding as in **[37]** we obtain

$$\boxed{L[(t-1)^2\,\delta(t-a)] = (a-1)^2\,e^{-as}}$$

[39] *Find* $L\left[\frac{2\delta(t-3)+3\delta(t-2)}{t}\right]$

$$L[2\delta(t-3)+3\delta(t-2)] = 2e^{-3s} + 3e^{-2s}$$

$$\therefore\quad L\left[\frac{2\delta(t-3)+3\delta(t-2)}{t}\right] = \int_s^{\infty}(2e^{-3s}+3e^{-2s})\,ds$$

$$= \left[2\frac{e^{-3s}}{-3} + 3\frac{e^{-2s}}{-2}\right]_s^{\infty}$$

$$= \frac{2}{3}e^{-3s} + \frac{3}{2}e^{-2s}$$

Thus $\boxed{L\left[\frac{2\delta(t-3)+3\delta(t-2)}{t}\right] = \frac{1}{6}(4e^{-3s}+9e^{-2s})}$

4.7 Inverse Laplace Transforms

Inverse Laplace Transforms can as well be regarded as the reverse process of finding the Laplace transform of a given function. We also discuss *convolution theorem* which helps in finding the inverse Laplace transform. Finally we discuss solution of linear differential equations with a given set of initial conditions referred to as *initial value problems*. This method is highly useful in various branches of engineering. We have made a mention of this while defining the Laplace transform of a function $f(t)$.

If $L[f(t)] = \bar{f}(s)$ then $f(t)$ is called the **inverse Laplace transform of** $\bar{f}(s)$ and is denoted by $L^{-1}[\bar{f}(s)]$.

Thus we can say that,

$$L[f(t)] = \bar{f}(s) \Leftrightarrow L^{-1}[\bar{f}(s)] = f(t)$$

Observe the following illustrations.

$$L(1) = \frac{1}{s} \Rightarrow L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L(\cos at) = \frac{s}{s^2 + a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$$

We revert the table of Laplace transforms of standard functions given earlier to present the basic table of inverse Laplace transforms.

	Function	Inverse Transform		Function	Inverse Transform
1.	$\frac{1}{s}$	1	5.	$\frac{1}{s^2 + a^2}$	$\frac{\sin at}{a}$
2.	$\frac{1}{s - a}$	e^{at}	6.	$\frac{1}{s^2 - a^2}$	$\frac{\sinh at}{a}$
3.	$\frac{s}{s^2 + a^2}$	$\cos at$	7.	$\frac{1}{s^{n+1}}$ $(n > -1)$	$\frac{t^n}{\Gamma(n+1)}$
4.	$\frac{s}{s^2 - a^2}$	$\cosh at$	8.	$\frac{1}{s^{n+1}}$ $n = 1, 2, 3..,$	$\frac{t^n}{n!}$

We present a few illustrative examples based on this table of inverse Laplace transforms.

1. $L^{-1}\left(\frac{1}{s-1}\right) = e^{t}$

2. $L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

3. $L^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t$

4. $L^{-1}\left(\frac{s}{s^2-16}\right) = \cosh 4t$

5. $L^{-1}\left(\frac{1}{s^2+5}\right) = \frac{1}{\sqrt{5}}\sin(\sqrt{5}\,t)$

6. $L^{-1}\left(\frac{1}{s^2-36}\right) = \frac{1}{6}\sinh 6t$

7. $L^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{3!}$

8. $L^{-1}\left(\frac{1}{s^{3/2}}\right) = \frac{t^{1/2}}{\Gamma(3/2)} = \frac{t^{1/2}}{1/2\cdot\sqrt{\pi}} = 2\sqrt{t/\pi}$

Property : $L^{-1}[c_1\bar{f}(s)+c_2\bar{g}(s)] = c_1 L^{-1}[\bar{f}(s)]+c_2 L^{-1}[\bar{g}(s)]$

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions.

40. $\frac{2s-5}{4s^2+25}+\frac{8-6s}{16s^2+9}$

41. $\frac{2s-5}{8s^2-50}+\frac{4s}{9-s^2}$

42. $\frac{(s+2)^3}{s^6}$

43. $\frac{1}{s\sqrt{s}}+\frac{3}{s^2\sqrt{s}}-\frac{8}{\sqrt{s}}$

☞ **Solutions**

[40] $2L^{-1}\left[\frac{s}{4s^2+25}\right]-5L^{-1}\left[\frac{1}{4s^2+25}\right]+8L^{-1}\left[\frac{1}{16s^2+9}\right]-6L^{-1}\left[\frac{s}{16s^2+9}\right]$

$$= \frac{1}{2}L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right]-\frac{5}{4}L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right]$$

$$+\frac{1}{2}L^{-1}\left[\frac{1}{s^2+(3/4)^2}\right]-\frac{3}{8}L^{-1}\left[\frac{s}{s^2+(3/4)^2}\right]$$

$$= \frac{1}{2}\cos(5t/2)-\frac{5}{4}\cdot\frac{2}{5}\sin(5t/2)+\frac{1}{2}\cdot\frac{4}{3}\sin(3t/4)-\frac{3}{8}\cos(3t/4)$$

Thus the required inverse Laplace transform is

$$\boxed{1/2\cdot\cos(5t/2)-1/2\cdot\sin(5t/2)+2/3\cdot\sin(3t/4)-3/8\cdot\cos(3t/4)}$$

[41] $L^{-1}\left[\dfrac{2s-5}{2(4s^2-25)}\right]-L^{-1}\left[\dfrac{4s}{s^2-9}\right]$

$$=\frac{1}{2}L^{-1}\left[\frac{1}{2s+5}\right]-4L^{-1}\left[\frac{s}{s^2-3^2}\right]=\frac{1}{2}\cdot\frac{1}{2}L^{-1}\left[\frac{1}{s+(5/2)}\right]-4L^{-1}\left[\frac{s}{s^2-3^2}\right]$$

Thus the required inverse Laplace transform is

$$\boxed{1/4\cdot e^{-5t/2}-4\cosh 3t}$$

[42] $L^{-1}\left[\dfrac{s^3+6s^2+12s+8}{s^6}\right]$

$$=L^{-1}\left(\frac{1}{s^3}\right)+6L^{-1}\left(\frac{1}{s^4}\right)+12L^{-1}\left(\frac{1}{s^5}\right)+8L^{-1}\left(\frac{1}{s^6}\right)$$

$$=\frac{t^2}{2!}+6\cdot\frac{t^3}{3!}+12\cdot\frac{t^4}{4!}+8\cdot\frac{t^5}{5!}$$

Thus the required inverse Laplace transform is

$$\boxed{\frac{t^2}{2}+t^3+\frac{t^4}{2}+\frac{t^5}{15}}$$

[43] $L^{-1}\left(\dfrac{1}{s^{3/2}}\right)+3L^{-1}\left(\dfrac{1}{s^{5/2}}\right)-8L^{-1}\left(\dfrac{1}{s^{1/2}}\right)$

$$=\frac{t^{1/2}}{\Gamma(3/2)}\div 3\cdot\frac{t^{3/2}}{\Gamma(5/2)}-8\frac{t^{-1/2}}{\Gamma(1/2)}$$

$$=\frac{\sqrt{t}}{1/2\cdot\Gamma(1/2)}+3\cdot\frac{t\sqrt{t}}{3/2\cdot 1/2\cdot\Gamma(1/2)}-\frac{8}{\sqrt{t}\,\Gamma(1/2)}$$

$$=\frac{2\sqrt{t}}{\sqrt{\pi}}+4\frac{t\sqrt{t}}{\sqrt{\pi}}-\frac{8}{\sqrt{t}\,\sqrt{\pi}}$$

Thus the required inverse Laplace transform is

$$\boxed{\frac{2}{\sqrt{\pi}}\left[\sqrt{t}+2t\sqrt{t}-\frac{4}{\sqrt{t}}\right]}$$

4.71 Computation of the inverse transform of $e^{-as}\bar{f}(s)$

We have proved that $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$\therefore \quad L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)u(t-a)$

Working procedure for problems

Step-1 In the given function we should observe the presence of e^{-as} first and identify the remaining part of the function to be called as $\bar{f}(s)$.

Step-2 Taking the inverse of $\bar{f}(s)$ we obtain $f(t)$

Step-3 The required inverse of $e^{-as}\bar{f}(s)$ is obtained by replacing t by $(t-a)$ in $f(t)$ to be mulitplied by the unit step function $u(t-a)$.

WORKED PROBLEMS

Find the inverse Laplace transforms of the following.

44. $\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}$ **45.** $\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{s\,e^{-2\pi s}}{s^2+4}$

☞ **Solutions**

[44] $\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3} = \dfrac{2-2e^{-s}-e^{-2s}+e^{-3s}}{s^3}$

Now, $L^{-1}\left[\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$

$$= 2L^{-1}\left(\frac{1}{s^3}\right) - 2L^{-1}\left(\frac{e^{-s}}{s^3}\right) - L^{-1}\left(\frac{e^{-2s}}{s^3}\right) + L^{-1}\left(\frac{e^{-3s}}{s^3}\right)$$

But, $L^{-1}(1/s^3) = t^2/2$

Thus, $L^{-1}\left[\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$ is given by

$$\boxed{= t^2 - (t-1)^2 u(t-1) - \frac{(t-2)^2 u(t-2)}{2} + \frac{(t-3)^2 u(t-3)}{2}}$$

[45] $L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right)+L^{-1}\left(e^{-2\pi s}\cdot\frac{s}{s^2+4}\right)$ ... (1)

We have, $L^{-1}\left(\frac{1}{s^2+1}\right)=\sin t,\ L^{-1}\left(\frac{s}{s^2+4}\right)=\cos 2t$

Hence (1) becomes, $\sin(t-\pi)u(t-\pi)+\cos 2(t-2\pi)u(t-2\pi)$

Thus, $L^{-1}\left[\frac{e^{-\pi s}}{s^2+1}+\frac{s\,e^{-2\pi s}}{s^2+1}\right]=-\sin t\,u(t-\pi)+\cos 2t\,u(t-2\pi)$

4.72 Inverse transform by completing the square

We have the property that if $L[f(t)]=\bar{f}(s)$ then $L[e^{at}f(t)]=\bar{f}(s-a)$.

$\Rightarrow$ $L^{-1}[\bar{f}(s)]=f(t)$ ... (1) and $L^{-1}[\bar{f}(s-a)]=e^{at}f(t)$ ... (2)

Hence, (2) as a consequence of (1) becomes,

$$L^{-1}[\bar{f}(s-a)]=e^{at}L^{-1}[\bar{f}(s)] \quad \text{... (3)}$$

Working procedure for problems

Step-1 Given $\bar{f}(s)=\frac{\phi(s)}{(ps^2+qs+r)}$, we first express (ps^2+qs+r) in the form $(s-a)^2\pm b^2$ and later express $\phi(s)$ in terms of $(s-a)$ so that the given function of s reduces to a function of $(s-a)$.

Step-2 We then use (3) to obtain the result.

Step-3 However if $\bar{f}(s)=\phi(s)/\psi(s-a)$ we only need to express $\phi(s)$ in terms of $(s-a)$ to compute the inverse transform.

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions.

46. $\frac{s+5}{s^2-6s+13}$ **47.** $\frac{s^2}{(s+1)^3}$

48. $\frac{2s-1}{s^2+4s+29}$ **49.** $\frac{s+1}{s^2+6s+9}$

50. $\dfrac{s+1}{s^2+s+1}$ 51. $\dfrac{7s+4}{4s^2+4s+9}$

☞ **Solutions**

[46] $L^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right] = L^{-1}\left[\dfrac{s+5}{(s-3)^2+4}\right]$

ie., $= L^{-1}\left[\dfrac{\overline{s-3}+3+5}{(s-3)^2+2^2}\right] = L^{-1}\left[\dfrac{(s-3)+8}{(s-3)^2+2^2}\right]$

Here, $a = 3$ and $(s-3)$ changes to s.

ie., $= e^{3t}\, L^{-1}\left[\dfrac{s+8}{s^2+2^2}\right]$

$$= e^{3t}\left\{L^{-1}\left(\frac{s}{s^2+2^2}\right) + 8L^{-1}\left(\frac{1}{s^2+2^2}\right)\right\}$$

Thus, $\boxed{L^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right] = e^{3t}(\cos 2t + 4\sin 2t)}$

[47] Here we need to express s^2 in terms of $(s+1)$

$$s^2 = (s+1)^2 - 2s - 1 = (s+1)^2 - 2(s+1) + 2 - 1$$

ie., $s^2 = (s+1)^2 - 2(s+1) + 1$

$\therefore$ $L^{-1}\left[\dfrac{s^2}{(s+1)^3}\right] = L^{-1}\left[\dfrac{(s+1)^2 - 2(s+1) + 1}{(s+1)^3}\right]$

Here, $a = -1$ and $(s+1)$ changes to s. Hence RHS becomes,

$$e^{-t}\, L^{-1}\left[\frac{s^2-2s+1}{s^3}\right] = e^{-t}\left\{L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s^3}\right)\right\}$$

Thus, $\boxed{L^{-1}\left[\dfrac{s^2}{(s+1)^3}\right] = e^{-t}\left(1 - 2t + \dfrac{t^2}{2}\right)}$

[48] $L^{-1}\left[\dfrac{2s-1}{s^2+4s+29}\right] = L^{-1}\left\{\dfrac{2s-1}{(s+2)^2+25}\right\} = L^{-1}\left\{\dfrac{2(s+2)-5}{(s+2)^2+25}\right\}$

ie., $= e^{-2t}\,L^{-1}\left\{\dfrac{2s-5}{s^2+5^2}\right\} = e^{-2t}\left\{2L^{-1}\left(\dfrac{s}{s^2+5^2}\right) - L^{-1}\left(\dfrac{5}{s^2+5^2}\right)\right\}$

Thus, $\boxed{L^{-1}\left[\dfrac{2s-1}{s^2+4s+29}\right] = e^{-2t}\,(2\cos 5t - \sin 5t)}$

[49] $L^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = L^{-1}\left[\dfrac{(s+3)-2}{(s+3)^2}\right] = e^{-3t}\,L^{-1}\left[\dfrac{s-2}{s^2}\right]$

ie., $= e^{-3t}\left\{L^{-1}\left(\dfrac{1}{s}\right) - 2L^{-1}\left(\dfrac{1}{s^2}\right)\right\}$

Thus, $\boxed{L^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = e^{-3t}\,(1-2t)}$

[50] $s^2+s+1 = (s+1/2)^2 - 1/4 + 1 = (s+1/2)^2 + (\sqrt{3}/2)^2$

$$L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = L^{-1}\left[\frac{(s+1/2)+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] = e^{-t/2}\,L^{-1}\left[\frac{s+1/2}{s^2+(\sqrt{3}/2)^2}\right]$$

ie., $= e^{-t/2}\left\{L^{-1}\left[\dfrac{s}{s^2+(\sqrt{3}/2)^2}\right] + \dfrac{1}{2}L^{-1}\left[\dfrac{1}{s^2+(\sqrt{3}/2)^2}\right]\right\}$

Thus, $\boxed{L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right] = e^{-t/2}\left\{\cos(\sqrt{3}\,t/2) + 1/\sqrt{3}\cdot\sin(\sqrt{3}\,t/2)\right\}}$

[51] $4s^2+4s+9 = 4(s^2+s+9/4) = 4\left\{(s+1/2)^2+2\right\}$

Hence, $7s+4 = 7\,(s+1/2)+1/2$

Now, $L^{-1}\left[\dfrac{7s+4}{4s^2+4s+9}\right] = \dfrac{1}{4}L^{-1}\left[\dfrac{7\,(s+1/2)+1/2}{(s+1/2)^2+2}\right]$

ie., $= \dfrac{e^{-t/2}}{4}\left\{7L^{-1}\left[\dfrac{s}{s^2+(\sqrt{2})^2}\right]+\dfrac{1}{2}L^{-1}\left[\dfrac{1}{s^2+(\sqrt{2})^2}\right]\right\}$

Thus, $\boxed{L^{-1}\left[\dfrac{7s+4}{4s^2+4s+9}\right] = \dfrac{e^{-t/2}}{4}\left\{7\cos(\sqrt{2}\,t)+\dfrac{1}{2\sqrt{2}}\sin(\sqrt{2}\,t)\right\}}$

[52] *Find* $L^{-1}\left[\dfrac{s}{s^4+4a^4}\right]$

☞ $s^4+4a^4 = (s^2+2a^2)^2-4a^2s^2$

ie., $s^4+4a^4 = (s^2+2a^2+2as)(s^2+2a^2-2as)$ . . . (1)

Also, $4as = (s^2+2a^2+2as)-(s^2+2a^2-2as)$

Now, $\dfrac{s}{s^4+4a^4} = \dfrac{1}{4a}\left\{\dfrac{(s^2+2a^2+2as)-(s^2+2a^2-2as)}{(s^2+2a^2+2as)(s^2+2a^2-2as)}\right\}$

ie., $\dfrac{s}{s^4+4a^4} = \dfrac{1}{4a}\left\{\dfrac{1}{s^2+2a^2-2as}-\dfrac{1}{s^2+2a^2+2as}\right\}$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{4a}\left\{L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right]-L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right\}$$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{4a}\left\{e^{at}L^{-1}\left[\frac{1}{s^2+a^2}\right]-e^{-at}L^{-1}\left[\frac{1}{s^2+a^2}\right]\right\}$$

$$= \frac{1}{4a}\left\{e^{at}\frac{\sin at}{a}-e^{-at}\frac{\sin at}{a}\right\}$$

Thus, $\boxed{L^{-1}\left[\dfrac{s}{s^4+4a^4}\right] = \dfrac{\sin at}{2a^2}\left\{\dfrac{e^{at}-e^{-at}}{2}\right\} = \dfrac{\sin at \sinh at}{2a^2}}$

4.73 Inverse transform by the method of partial fractions

We know that, the method of partial fractions is a technique of converting an algebraic fraction $\phi(s)/\psi(s)$ [*where degree of* $\phi(s)$ *is less than that of* $\psi(s)$] into a sum. Depending on the nature of terms in $\psi(s)$ we have to split into a sum of various terms with constants $A, B, C, D, \ldots$ which can be determined. Later the inverse is found term by term.

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions.

53. $\dfrac{1}{s(s+1)(s+2)(s+3)}$ **54.** $\dfrac{4s+5}{(s+1)^2(s+2)}$

55. $\dfrac{(3s+1)}{(s-1)(s^2+1)}$

☞ **Solutions**

[53] Let, $\dfrac{1}{s(s+1)(s+2)(s+3)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2} + \dfrac{D}{s+3}$

Multiplying by $s(s+1)(s+2)(s+3)$ we get,

$$1 = A(s+1)(s+2)(s+3) + Bs(s+2)(s+3) + Cs(s+1)(s+3) + Ds(s+1)(s+2)$$

Put $s = 0$: $1 = A(6)$ $\therefore A = 1/6$

Put $s = -1$: $1 = B(-2)$ $\therefore B = -1/2$

Put $s = -2$: $1 = C(2)$ $\therefore C = 1/2$

Put $s = -3$: $1 = D(-6)$ $\therefore D = -1/6$

Now, $L^{-1}\left[\dfrac{1}{s(s+1)(s+2)(s+3)}\right]$

$$= \frac{1}{6}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

Thus, $\boxed{L^{-1}\left[\dfrac{1}{s(s+1)(s+2)(s+3)}\right] = \dfrac{1}{6} - \dfrac{1}{2}e^{-t} + \dfrac{1}{2}e^{-2t} - \dfrac{1}{6}e^{-3t}}$

[54] Let, $\dfrac{4s+5}{(s+1)^2(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+2}$

Multiplying by $(s+1)^2(s+2)$ we get,

$$4s+5 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \quad \ldots (1)$$

Put $s = -1$: $1 = B(1)$ $\therefore B = 1$

Put $s = -2$: $-3 = C(1)$ $\therefore C = -3$

Equating the coefficient of s^2 on both sides of (1) we get,

$0 = A + C$ $\therefore A = 3$

Now, $L^{-1}\left[\dfrac{4s+5}{(s+1)^2(s+2)}\right]$

$$= 3L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right] - 3L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 3e^{-t} + e^{-t}L^{-1}[1/s^2] - 3e^{-2t}$$

Thus, $\boxed{L^{-1}\left[\dfrac{4s+5}{(s+1)^2(s+2)}\right] = 3e^{-t} + e^{-t}\cdot t - 3e^{-2t}}$

[55] Let, $\dfrac{3s+1}{(s-1)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1}$

or $3s+1 = A(s^2+1) + (Bs+C)(s-1) \quad \ldots (1)$

Put $s = 1$: $4 = A(2)$ $\therefore A = 2$

Put $s = 0$: $1 = A - C$ $\therefore C = 1$

Equating the coefficient of s^2 on both sides of (1) we get,

$0 = A + B$ $\therefore B = -2$.

Now, $L^{-1}\left[\dfrac{3s+1}{(s-1)(s^2+1)}\right] = 2L^{-1}\left[\dfrac{1}{s-1}\right] - 2L^{-1}\left[\dfrac{s}{s^2+1}\right] + L^{-1}\left[\dfrac{1}{s^2+1}\right]$

Thus, the required inverse is, $\boxed{2e^t - 2\cos t + \sin t}$

4.74 Inverse transform of logarithmic functions and inverse functions

Given $\bar{f}(s)$ we need to find $L^{-1}[\bar{f}(s)] = f(t)$

We have the property : $L[t\,f(t)] = -\bar{f}'(s)$

Equivalently, $L^{-1}[-\bar{f}'(s)] = t\,f(t)$ $\quad \ldots (1)$

Working procedure for problems

Step-1 In the case of logarithmic functions we apply the properties of logarithms and then differentiate w.r.t. s to obtain $\bar{f}'(s)$.

Step-2 We then multiply by –1 and take inverse on both sides.

Step-3 LHS becomes $t\,f(t)$ by (1) and inverses are also found for the terms in RHS with the result we obtain the required $f(t)$.

Step-4 If logarithmic function persists in $\bar{f}'(s)$ we differentiate again w.r.t s to obtain $\bar{f}''(s)$ and use the property that

$L^{-1}[\bar{f}''(s)] = t^2 f(t)$ since $L[t^2 f(t)] = \bar{f}''(s)$

Step-5 In the cases of inverse functions we simply differentiate the given $\bar{f}(s)$ and use the result (1) to obtain $f(t)$.

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions.

56. $\log\left(\dfrac{s+a}{s+b}\right)$ **57.** $\log\left(1-\dfrac{a^2}{s^2}\right)$

58. $\cot^{-1}(s/a)$ **59.** $\log\sqrt{(s^2+1)/(s^2+4)}$

☞ **Solutions**

[56] Let, $\bar{f}(s) = \log\left(\dfrac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$

$$\therefore \quad -\bar{f}'(s) = -\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = \frac{1}{s+b} - \frac{1}{s+a}$$

Now $L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\dfrac{1}{s+b}\right) - L^{-1}\left(\dfrac{1}{s+a}\right)$

ie., $t\,f(t) = e^{-bt} - e^{-at}$

Thus, $\boxed{f(t) = \dfrac{e^{-bt} - e^{-at}}{t}}$

[57] Let, $\bar{f}(s) = \log\left(1 - \dfrac{a^2}{s^2}\right) = \log\left(\dfrac{s^2 - a^2}{s^2}\right)$

ie., $\bar{f}(s) = \log(s^2 - a^2) - 2\log s$

$\therefore \quad -\bar{f}'(s) = -\left\{\dfrac{1}{s^2 - a^2}\cdot 2s - \dfrac{2}{s}\right\} = 2\left(\dfrac{1}{s} - \dfrac{s}{s^2 - a^2}\right)$

Now $L^{-1}[-\bar{f}'(s)] = 2\left\{L^{-1}\left(\dfrac{1}{s}\right) - L^{-1}\left(\dfrac{s}{s^2 - a^2}\right)\right\}$

ie., $t\,f(t) = 2(1 - \cosh at)$

Thus, $\boxed{f(t) = \dfrac{2(1 - \cosh at)}{t}}$

[58] Let, $\bar{f}(s) = \cot^{-1}(s/a)$

Differentiate w.r.t s and multiply with -1.

$\therefore \quad \bar{f}'(s) = \dfrac{-1}{1 + (s/a)^2}\cdot\dfrac{1}{a}$ and $-\bar{f}'(s) = \dfrac{a}{a^2 + s^2}$

Taking inverse, $L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\dfrac{a}{s^2 + a^2}\right)$

$t\,f(t) = \sin at$

Thus, $\boxed{f(t) = \dfrac{\sin at}{t}}$

[59] Let, $\bar{f}(s) = \log\sqrt{s^2 + 1/s^2 + 4}$

$= \dfrac{1}{2}\{\log(s^2 + 1) - \log(s^2 + 4)\}$

$\therefore \quad -\bar{f}'(s) = -\dfrac{1}{2}\left\{\dfrac{2s}{s^2 + 1} - \dfrac{2s}{s^2 + 4}\right\} = \dfrac{s}{s^2 + 4} - \dfrac{s}{s^2 + 1}$

Now, $L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\frac{s}{s^2+2^2}\right) - L^{-1}\left(\frac{s}{s^2+1^2}\right)$

ie., $t\,f(t) = \cos 2t - \cos t$

Thus, $$\boxed{f(t) = \frac{\cos 2t - \cos t}{t}}$$

EXERCISE

Find the inverse Laplace transform of the following functions.

1. $\frac{s^2-3s+4}{s^3}$
2. $\frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2}$
3. $\frac{s-3}{s^2-6s+13}$
4. $\frac{2s-3}{s^2-2s+5}$
5. $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$
6. $\frac{4s+3}{(s-1)^2(s+2)}$
7. $\log\left(1+\frac{a^2}{s^2}\right)$
8. $\log\left[\frac{s^2+1}{s(s+1)}\right]$

ANSWERS

1. $1-3t+2t^2$
2. $\frac{1}{2}\{\cos(5t/2)-\sin(5t/2)\} - 4\cosh 3t + 6\sinh 3t$
3. $e^{3t}\cos 2t$
4. $e^t/2\cdot(4\cos 2t - \sin 2t)$
5. $\frac{e^t}{2} - e^{2t} + \frac{5}{2}e^{3t}$
6. $\frac{5}{9}e^t + \frac{7}{3}te^t - \frac{5}{9}e^{-2t}$
7. $\frac{2(1-\cos at)}{t}$
8. $\frac{1+e^{-t}-2\cos t}{t}$

4.8 Convolution

Definition : The convolution of two functions $f(t)$ and $g(t)$ usually denoted by $f(t) * g(t)$ is defined in the form of an integral as follows.

$$f(t)*g(t) = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

4.81 Convolution theorem

If $L^{-1}[\bar{f}(s)] = f(t)$ and $L^{-1}[\bar{g}(s)] = g(t)$ then

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

- **Computation of the inverse transform by using convolution theorem**

Working procedure for problems

Step-1 The given function is expressed as the product of two functions say $\bar{f}(s)$ and $\bar{g}(s)$.

Step-2 We find $L^{-1}[\bar{f}(s)] = f(t)$ and $L^{-1}[\bar{g}(s)] = g(t)$.

Step-3 We apply the convolution theorem in one of the form :

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

Step-4 We evaluate the convolution integral to obtain the required inverse.

WORKED PROBLEMS

Using convolution theorem obtain the inverse Laplace transform of the following functions

60. $\dfrac{1}{s(s^2+a^2)}$ **[June 2016, 18]**

61. $\dfrac{s}{(s^2+a^2)^2}$ **[Dec 2016]**

62. $\dfrac{s^2}{(s^2+a^2)^2}$

63. $\dfrac{1}{(s^2+a^2)^2}$ **[June 2017]**

64. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

65. $\dfrac{1}{(s-1)(s^2+1)}$ **[Dec 2017]**

☞ **Solutions**

[60] Let, $\bar{f}(s) = \dfrac{1}{s}$; $\bar{g}(s) = \dfrac{1}{s^2+a^2}$

Taking inverse,

$$f(t) = L^{-1}\left[\frac{1}{s}\right] = 1 \; ; \; g(t) = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$$

We have convolution theorem,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

$$\therefore \quad L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_{u=0}^{t} 1\cdot\frac{\sin(at-au)}{a}\,du$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \left[\frac{\cos(at-au)}{a^2}\right]_{u=0}^{t} = \frac{1}{a^2}(1-\cos at)$$

Thus, $$\boxed{L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}(1-\cos at)}$$

[61] Let, $\bar{f}(s) = \dfrac{1}{s^2+a^2}$; $\bar{g}(s) = \dfrac{s}{s^2+a^2}$

$$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \frac{\sin at}{a}\ ;\ g(t) = L^{-1}[\bar{g}(s)] = \cos at$$

We have convolution theorem,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

$$\therefore \quad L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a}\cdot\cos(at-au)\,du$$

$$= \frac{1}{2a}\int_{u=0}^{t}[\sin(au+at-au)+\sin(au-at+au)]\,du$$

$$= \frac{1}{2a}\int_{u=0}^{t}[\sin at+\sin(2au-at)]\,du$$

$$= \frac{1}{2}\left\{\sin at\,[u]_{u=0}^{t} - \left[\frac{\cos(2au-at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2}\left\{\sin at\,(t-0) - \frac{1}{2a}(\cos at-\cos at)\right\} = \frac{t\sin at}{2a}$$

Thus, $$\boxed{L^{-1} = \left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}}$$

[62] Let, $\bar{f}(s) = \dfrac{s}{s^2 + a^2} = \bar{g}(s)$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \cos at = g(t)$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \cos au \cos(at - au)\, du$$

$$= \frac{1}{2}\int_{u=0}^{t} [\cos(au + at - au) + \cos(au - at + au)]\, du$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\int_{u=0}^{t} [\cos at + \cos(2au - at)]\, du$$

$$= \frac{1}{2}\left\{\cos at\,[u]_0^t + \left[\frac{\sin(2au - at)}{2a}\right]_0^t\right\}$$

$$= \frac{1}{2}\left\{\cos at\,(t - 0) + \frac{1}{2a}(\sin at + \sin at)\right\} = \frac{1}{2}\left\{t\cos at + \frac{\sin at}{a}\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}(at\cos at + \sin at)}$$

[63] Let, $\bar{f}(s) = \dfrac{1}{s^2 + a^2}$; $\bar{g}(s)$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \dfrac{\sin at}{a} = g(t)$

Proceeding as in **[62]** by using the basic formula of sin A sin B, we can obtain

$$\boxed{L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)}$$

[64] Let, $\bar{f}(s) = \dfrac{s}{s^2 + a^2}$; $\bar{g}(s) = \dfrac{s}{s^2 + b^2}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \cos at$; $g(t) = L^{-1}[\bar{g}(s)] = \cos bt$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_{u=0}^{t} \cos au \cdot \cos(bt-bu)\,du$$

Note : *The integral can be evaluated as in* **[62]**.

Thus, $$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2},\ a \neq b$$

[65] Let, $\bar{f}(s) = \dfrac{1}{s-1}$; $\bar{g}(s) = \dfrac{1}{s^2+1}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = e^t$; $g(t) = L^{-1}[\bar{g}(s)] = \sin t$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \int_{u=0}^{t} e^u \cdot \sin(t-u)\,du$$

But, $\int e^{ax}\sin(bx+c)\,dx = \dfrac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)]$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \left[\frac{e^u}{1+1}\{\sin(t-u)+\cos(t-u)\}\right]_0^t$$

$$= \frac{1}{2}\{e^t(0+1) - 1(\sin t + \cos t)\}$$

Thus, $$\boxed{L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{1}{2}(e^t - \sin t - \cos t)}$$

Note : Similar problem

Find $L^{-1}\left[\dfrac{1}{(s+1)(s^2+1)}\right]$ *using Convolution theorem.* **[June 2018]**

4.9 Solution of linear differential equations using Laplace transforms (Initial value problems)

- **Laplace transform of the derivatives.**

We derive an expression for $L[y'(t)]$ and hence deduce the expressions for

$L[y'(t)], L[y'''(t)]\ldots$

$$L[y'(t)] = \int_0^\infty e^{-st}\, y'(t)\,dt$$

Integrating by parts we have,

$$L[y'(t)] = \left[e^{-st}y(t)\right]_{t=0}^{\infty} - \int_0^\infty y(t)\cdot e^{-st}(-s)\,dt$$

$$L[y'(t)] = [0 - 1\cdot y(0)] + s\int_0^\infty e^{-st}\, y(t)\,dt$$

$$L[y'(t)] = -y(0) + sL[\ (t)]$$

$$\therefore \quad L[y'(t)] = sL[\ y(t)] - y(0) \qquad \ldots (1)$$

Now, $L[y''(t)] = L[\{y'(t)\}']$ and applying (1) we have,

$$= sL[y'(t)] - y'(0)$$

$$= s\{sL[y(t)] - y(0)\} - y'(0) \text{ by using (1).}$$

$$\therefore \quad L[y''(t)] = s^2 L[y(t)] - sy(0) - y'(0) \qquad \ldots (2)$$

Also, $L[y'''(t)] = s^3 L[y(t)] - s^2 y(0) - sy'(0) - y''(0) \qquad \ldots (3)$

and so on.

We have already said that a differential equation with a set of initial conditions is called an initial value problem. However if the boundary conditions are given the problem is called a boundary value problem.

Laplace transform serves as a useful tool in solving such problems.

Working procedure for problems

Step-1 The given differential equation is expressed in the notation :

$y'(t), y''(t), y'''(t)\ldots$ for the derivatives.

Step-2 We take Laplace transform on both sides of the given equation.

Step-3 We use the expressions for $L[y'(t)], L[y''(t)]\ldots$

Step-4 We substitute the given initial conditions and simplify to obtain $L[y(t)]$ as a function of s.

Step-5 We find the inverse to obtain $y(t)$.

WORKED PROBLEMS

[66] *Solve by using Laplace transforms :* $\frac{d^2y}{dt^2} + k^2y = 0$ *given that* $y(0) = 2,\ y'(0) = 0.$ **[Dec 2018]**

☞ The given equation is $y''(t) + k^2y(t) = 0.$

Taking Laplace transform on both sides we have,

$$L[y''(t)] + k^2 L[y(t)] = L(0)$$

ie., $\{s^2 L[y(t)] - s\,y(0) - y'(0)\} + k^2 L[y(t)] = 0$

Using the given initial conditions we obtain,

$$(s^2 + k^2)L[y(t)] - 2s = 0 \text{ or } L[y(t)] = \frac{2s}{s^2 + k^2}$$

$$\therefore \quad y(t) = 2L^{-1}\left[\frac{s}{s^2 + k^2}\right] = 2\cos kt.$$

Thus, $\boxed{y(t) = 2\cos kt}$

[67] *Solve* $y''' + 2y'' - y' - 2y = 0$ *given* $y(0) = y'(0) = 0$ *and* $y''(0) = 6$ *by using Laplace transform method.* **[Dec 2016]**

☞ Taking Laplace transform on both sides of the given equation,

$$L[y'''(t)] + 2L[y''(t)] - L[y'(t)] - 2L[y(t)] = L(0)$$

ie., $\{s^3 L[y(t)] - s^2 y(0) - s\,y'(0) - y''(0)\} + 2\{s^2 L[y(t)] - s\,y(0) - y'(0)\}$
$- \{s L[y(t)] - y(0)\} - 2L[y(t)] = 0$

Using the given initial conditions we obtain,

$$L[y(t)]\{s^3 + 2s^2 - s - 2\} - 6 = 0$$

ie., $L[y(t)]\{s^2(s+2) - 1(s+2)\} = 6$

or $L[y(t)]\{(s+2)(s^2 - 1)\} = 6$

or $L[y(t)] = \dfrac{6}{(s+2)(s-1)(s+1)}$

$$\therefore \quad L^{-1}\left[\frac{s^3}{s^4-k^4}\right] = \frac{1}{4}L^{-1}\left[\frac{1}{s-k}\right] + \frac{1}{4}L^{-1}\left[\frac{1}{s+k}\right] + \frac{1}{2}\left[\frac{s}{s^2+k^2}\right]$$

ie., $$y(t) = \frac{1}{4}e^{kt} + \frac{1}{4}e^{-kt} + \frac{1}{2}\cos kt = \frac{1}{4}(e^{kt}+e^{-kt}) + \frac{1}{2}\cos kt$$

Thus, $$\boxed{y(t) = \frac{1}{2}(\cosh kt + \cos kt)}$$

[69] *Solve the following initial value problem by using Laplace transforms :*

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}\ ;\ y(0) = 0,\ y'(0) = 0$$ **[June 2016]**

☞ The given equation is $y''(t) + 4y'(t) + 4y(t) = e^{-t}$

Taking Laplace transform on both sides we have,

$$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L(e^{-t})$$

ie., $$\{s^2L[y(t)] - sy(0) - y'(0)\} + 4\{sL[y(t)] - y(0)\} + 4L[y(t)] = \frac{1}{s+1}$$

Using the given initial conditions we obtain,

$$L[y(t)]\{s^2+4s+4\} = \frac{1}{s+1} \text{ or } L[y(t)] = \frac{1}{(s+1)(s+2)^2}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$ and by partial fractions,

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} + \frac{-1}{s+2} + \frac{-1}{(s+2)^2}$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

ie., $$y(t) = e^{-t} - e^{-2t} - e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

Thus, $$\boxed{y(t) = e^{-t} - e^{-2t} - e^{-2t}t = e^{-t} - (1+t)e^{-2t}}$$

$$\therefore \quad y(t) = L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\}$$

By resolving into partial fractions,

$$\frac{6}{(s+2)(s-1)(s+1)} = \frac{2}{s+2} + \frac{1}{s-1} + \frac{-3}{s+1}$$

$$\therefore \quad L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\} = 2L^{-1}\left(\frac{1}{s+2}\right) + L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left(\frac{1}{s+1}\right)$$

Thus, $$\boxed{y(t) = 2e^{-2t} + e^{t} - 3e^{-t}}$$

[68] *Solve* $\frac{d^4y}{dt^4} - k^4y = 0$ *given* $y(0) = 1$ *and* $y'(0) = 0 = y''(0) = y'''(0)$.

☞ The given equation is $y^{(4)}(t) - k^4y(t) = 0$.

Taking Laplace transform on both sides we have,

$$L[y^{(4)}(t)] - k^4L[y(t)] = L(0)$$

ie., $$\{s^4L[y(t)] - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\} - k^4L[y(t)] = 0$$

Using the given initial conditions we obtain,

$$L[y(t)][s^4-k^4] - s^3\cdot 1 = 0 \text{ or } L[y(t)] = \frac{s^3}{s^4-k^4}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{s^3}{s^4-k^4}\right]$$

Now, $s^4 - k^4 = (s^2-k^2)(s^2+k^2) = (s-k)(s+k)(s^2+k^2)$

Let, $$\frac{s^3}{(s-k)(s+k)(s^2+k^2)} = \frac{A}{(s-k)} + \frac{B}{(s+k)} + \frac{Cs+D}{s^2+k^2}$$

On computation of A, B, C, D we have,

$$\frac{s^3}{(s-k)(s+k)(s^2+k^2)} = \frac{1}{4}\frac{1}{s-k} + \frac{1}{4}\frac{1}{s+k} + \frac{1}{2}\frac{s}{s^2+k^2}$$

EXERCISE

Applying convolution theorem find the inverse Laplace transform of the following functions.

1. $\dfrac{s}{(s^2+4)^2}$

2. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

Solve the following differential equations by using Laplace transforms.

3. $x''(t)-2x'(t)+x(t)=e^t$; $x(0)=2,\ x'(0)=-1$

4. $x''(t)+4x'(t)+4x(t)=4e^{-2t}$; $x(0)=-1,\ x'(0)=4$

5. $y''(t)+y'(t)=e^{2t}$; $y(0)=0=y'(0)=y''(0)$

ANSWERS

1. $\dfrac{t\sin 2t}{4}$

2. $\dfrac{\cos bt-\cos at}{a^2-b^2}$

3. $x(t)=e^t[2-3t+(t^2/2)]$

4. $x(t)=e^{-2t}(2t^2+2t-1)$

5. $y(t)=\dfrac{-1}{2}+\dfrac{e^{2t}}{10}+\dfrac{2}{5}\cos t-\dfrac{1}{5}\sin t$

[70] *Solve* $y''+6y'+9y=12t^2e^{-3t}$ *subject to the conditions,* $y(0)=0=y'(0)$ *by using the Laplace transforms.* **[Dec. 2017]**

☞ The given equation is $y''(t)+6y'(t)+9y(t)=12t^2e^{-3t}$

Taking Laplace transform on both sides we have,

$$L[y''(t)]+6L[y'(t)]+9L[y(t)]=12L[e^{-3t}t^2]$$

ie., $$\{s^2L[y(t)]-sy(0)-y'(0)\}+6\{sL[y(t)]-y(0)\}+9L[y(t)]=\frac{12\cdot 2!}{(s+3)^3}$$

Using the given initial conditions we obtain,

$$(s^2+6s+9)L[y(t)]=\frac{24}{(s+3)^3}$$

ie., $$(s+3)^2L[y(t)]=\frac{24}{(s+3)^3} \quad \text{or} \quad L[y(t)]=\frac{24}{(s+3)^5}$$

$$\therefore \quad y(t)=L^{-1}\left[\frac{24}{(s+3)^5}\right]$$

$$y(t)=24e^{-3t}L^{-1}\left(\frac{1}{s^5}\right)=24e^{-3t}\frac{t^4}{4!}=e^{-3t}t^4$$

Thus, $\boxed{y(t)=e^{-3t}t^4}$

6 INTEGRAL CALCULUS

[For the EE stream in I Sem & Civ, Mech, CS streams in II Sem]

6.1 Multiple Integrals - Introduction

We are already conversant with the indefinite and definite integrals of a function of a single independent variable along with applications.

As a continuation, the concept is discussed for a function of two and three independent variables along with the applications.

Further we also discuss two special functions 'Beta function' and 'Gamma function' defined in the form of definite integrals.

Basic integration, definition and properties of definite integrals along with the following *two* established ***Reduction Formulae*** is an essential pre-requisite.

(1) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\times k$$

where n is a positive integer and $k = 1$ if n is odd and $k = \pi/2$ if n is even.

(2) $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)\cdots}\times k$$

where m, n are positive integers and $k = \pi/2$ only when m and n are even. Otherwise $k = 1$.

Ex - (i) : $$\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15} = \int_0^{\pi/2} \cos^5 x\,dx$$

Ex - (ii) : $$\int_0^{\pi/2} \cos^6 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\times\frac{\pi}{2} = \frac{5\pi}{32} = \int_0^{\pi/2} \sin^6 x\,dx$$

Ex - (iii) : $$\int_0^{\pi/2} \sin^5 x \cos^4 x\,dx = \frac{[(4)(2)][(3)(1)]}{9\cdot7\cdot5\cdot3\cdot1} = \frac{8}{315}$$

Ex - (iv) : $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx = \dfrac{[(6)(4)(2)][(4)(2)]}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} = \dfrac{1}{120}$

Ex - (v) : $\int_0^{\pi/2} \sin^8 x \cos^6 x \, dx = \dfrac{[(7)(5)(3)(1)][(5)(3)]}{14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \times \dfrac{\pi}{2} = \dfrac{5\pi}{4096}$

6.2 Evaluation of Double and Triple integrals

In this topic we discuss a repeated process of integration of a function of two and three variables referred to as *double integrals* : $\iint f(x,y)\,dx\,dy$ and *triple integrals* : $\iiint f(x,y,z)\,dx\,dy\,dz$.

The principle of partial differentiation is adopted here in the process of integration.

Ex - (i) $\iint (x+y)\,dx\,dy = \int \left(\dfrac{x^2}{2} + y \cdot x\right) dy = \dfrac{x^2}{2} \cdot y + \dfrac{y^2}{2} \cdot x = \dfrac{xy}{2}(x+y)$

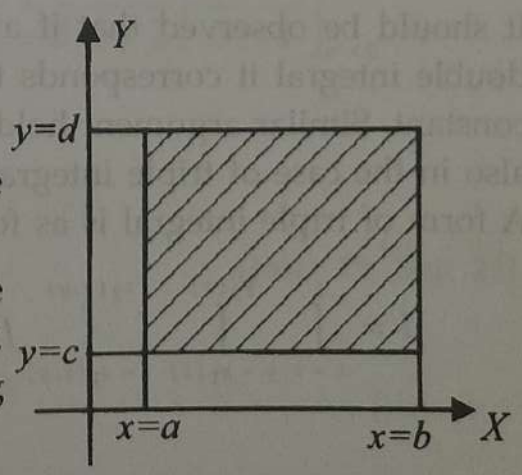
Ex - (ii) $\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} xyz\,dz\,dy\,dx$

ie., $= \int_{x=0}^{1} \int_{y=0}^{2} xy \left[\dfrac{z^2}{2}\right]_0^3 dy\,dx = \int_{x=0}^{1} \int_{y=0}^{2} xy\left(\dfrac{9}{2} - 0\right) dy\,dx = \dfrac{9}{2} \int_{x=0}^{1} \int_{y=0}^{2} xy\,dy\,dx$

ie., $= \dfrac{9}{2} \int_{x=0}^{1} x \left[\dfrac{y^2}{2}\right]_0^2 dx = \dfrac{9}{2} \int_{x=0}^{1} 2x\,dx = 9\left[\dfrac{x^2}{2}\right]_0^1 = \dfrac{9}{2}$

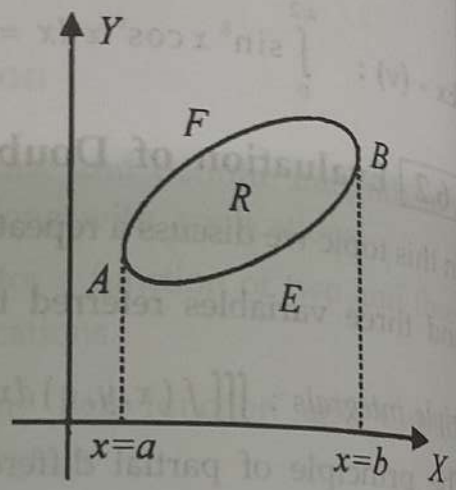
6.21 Geometrical meaning

$\int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dy\,dx$ can be regarded as the integral over the region bounded by the rectangle with sides $x = a$, $x = b$, $y = c$, $y = d$. The integral can also be evaluated by writing



in the form $\int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\,dx\,dy$ and the value will be the same.

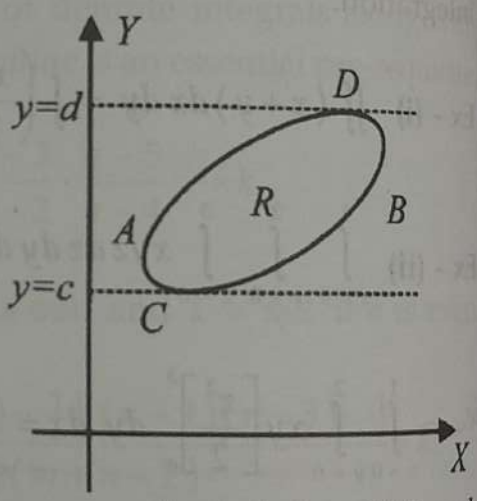
That is to say that when limits are constants the integral can be evaluated in either way.



If R is a region of the $x - y$ plane bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and the lines $x = a$, $x = b$ we have,

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$$

where $y = y_1(x)$ and $y = y_2(x)$ are the equations of the lower and upper part of the boundary curve respectively being AEB and AFB.



Further the integral can also be expressed in the form

$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d} \int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$$

where $x = x_1(y)$, $x = x_2(y)$ are the equations of the left and right part of the boundary curve respectively being CAD and CBD.

It should be observed that if a function of x is involved as a limit in the double integral it corresponds to y in which case the limits for x will be constant. Similar argument holds good for a function of y in the limit and also in the case of triple integral involving three variables x, y, z.
A form of triple integral is as follows.

$$I = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

WORKED PROBLEMS

Direct evaluation of double and triple integrals

[1] *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

☞ We have, $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy\, dy\, dx$

$$I = \int_{x=0}^{1} x \left[\frac{y^2}{2}\right]_x^{\sqrt{x}} dx = \int_{x=0}^{1} \frac{x}{2}\left[\left(\sqrt{x}\right)^2 - x^2\right] dx$$

ie., $I = \frac{1}{2}\int_0^1 x\,(x - x^2)\, dx = \frac{1}{2}\int_0^1 (x^2 - x^3)\, dx$

$$I = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left[\left(\frac{1}{3} - \frac{1}{4}\right) - 0\right] = \boxed{\frac{1}{24}}$$

[2] *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

☞ We have, $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

$$I = \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3}\right]_{y=x}^{\sqrt{x}} dx = \int_{x=0}^{1} \left[x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\right] dx$$

$$I = \int_{x=0}^{1} \left[x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3}\right] dx = \left[\frac{x^{7/2}}{7/2} + \frac{1}{3}\,\frac{x^{5/2}}{5/2} - \frac{4}{3}\cdot\frac{x^4}{4}\right]_{x=0}^{1}$$

$$I = \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{9}{105} = \boxed{\frac{3}{35}}$$

[3] *Evaluate* $\int_{-c}^{c} \int_{-b}^{-a} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dz\, dy\, dx$ **[June 19, Sep 21]**

☞ $I = \int_{x=-c}^{c} \int_{y=-b}^{b} \int_{z=-a}^{a} (x^2 + y^2 + z^2)\, dz\, dy\, dx$

$$= \int_{x=-c}^{c} \int_{y=-b}^{b} \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_{z=-a}^{a} dy\, dx$$

$$= \int_{x=-c}^{c} \int_{y=-b}^{b} \left[x^2 (a+a) + y^2 (a+a) + (a^3/3 + a^3/3) \right] dy\, dx$$

$$I = \int_{x=-c}^{c} \left[2a x^2 y + 2a (y^3/3) + (2a^3/3)(y) \right]_{y=-b}^{b} dx$$

$$= \int_{x=-c}^{c} \left[2a x^2 (b+b) + 2a (b^3/3 + b^3/3) + (2a^3/3)(b+b) \right] dx$$

$$= \left[(4ab)(x^3/3) + (4ab^3/3)(x) + (4a^3 b/3)(x) \right]_{-c}^{c}$$

$$= (4ab)(c^3/3 + c^3/3) + (4ab^3/3)(c+c) + (4a^3 b/3)(c+c)$$

$$I = \frac{8abc^3}{3} + \frac{8ab^3c}{3} + \frac{8a^3bc}{3} = \boxed{\frac{8abc(a^2+b^2+c^2)}{3}}$$

[4] *Evaluate* $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$ **[Dec 16, 18 June 18, Jan 21]**

☞ $$I = \int_{z=-1}^{1} \int_{x=0}^{z} \int_{y=x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left[xy + \frac{y^2}{2} + zy \right]_{y=x-z}^{x+z} dx\, dz$$

$$I = \int_{z=-1}^{1} \int_{x=0}^{z} \left\{ x(\overline{x+z} - \overline{x-z}) + \frac{1}{2}\left[(x+z)^2 - (x-z)^2 \right] + z(\overline{x+z} - \overline{x-z}) \right\} dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} (2xz + 2xz + 2z^2)\, dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} (4xz + 2z^2)\, dx\, dz$$

$$I = 1\int_{z=-1}^{1}\left[z(2x^2)+2z^2(x)\right]_{x=0}^{z} = \int_{z=-1}^{1}(2z^3+2z^3)\,dz$$

$$I = \int_{z=-1}^{1} 4z^3\,dz = \left[z^4\right]_{z=-1}^{1} = \boxed{0}$$

[5] *Evaluate* $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$ **[June 18, Dec 19]**

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$$

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_{z=0}^{\sqrt{1-x^2-y^2}} dy\,dx = \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dy\,dx$$

$$I = \frac{1}{2}\int_{x=0}^{1}\left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\frac{1}{4}\left[2xy^2 - 2x^3y^2 - xy^4\right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8}\int_{x=0}^{1}\left[2x(1-x^2) - 2x^3(1-x^2) - x(1-x^2)^2\right]dx$$

$$= \frac{1}{8}\int_{x=0}^{1}(2x - 2x^3 - 2x^3 + 2x^5 - x + 2x^3 - x^5)\,dx$$

$$I = \frac{1}{8}\int_{x=0}^{1}(x^5 - 2x^3 + x)\,dx = \frac{1}{8}\left[\frac{x^6}{6} - \frac{x^4}{2} + \frac{x^2}{2}\right]_{x=0}^{1}$$

$$I = \frac{1}{8}\left[\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right] = \boxed{\frac{1}{48}}$$

Note : ***Similar problem***

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\, dz\, dy\, dx$ [Dec 17]

☞ Proceeding on the same lines, we can obtain $\boxed{I = a^6/48}$

[6] *Evaluate* $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$ [June 18]

☞ $$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2 - z^2}}$$

We shall first integrate w.r.t z by treating x, y as constants and let $k = \sqrt{1-x^2-y^2}$ for convenience.

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{k} \frac{dz}{\sqrt{k^2-z^2}}\, dy\, dx = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{k}\right]_{z=0}^{k} dy\, dx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\, dx = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0\right) dy\, dx$$

$$I = \frac{\pi}{2} \int_{x=0}^{1} [y]_{y=0}^{\sqrt{1-x^2}}\, dx = \frac{\pi}{2} \int_{x=0}^{1} \sqrt{1-x^2}\, dx$$

But, $$\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$$

Hence, $$I = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_{x=0}^{1}$$

$$I = \frac{\pi}{2}\left[0 + \frac{1}{2}(\sin^{-1} 1 - \sin^{-1} 0)\right] = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \boxed{\frac{\pi^2}{8}}$$

Note : *Similar problem*

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\ dy\ dx}{\sqrt{a^2-x^2-y^2-z^2}}$ **[June 17]**

☞ Proceeding on the same lines, we can obtain $\boxed{I = \pi^2 a^2/8}$

[7] *Evaluate* $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

☞ We have, $I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y}\, e^z\, dz\, dy\, dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left[e^z\right]_{z=0}^{x+y} dy\, dx = \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left[e^{x+y} - 1\right] dy\, dx$$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \left(e^{2x} \cdot e^{2y} - e^x \cdot e^y\right) dy\, dx = \int_{x=0}^{a} \left\{ e^{2x} \left[\frac{e^{2y}}{2}\right]_{y=0}^{x} - e^x \left[e^y\right]_{y=0}^{x} \right\} dx$$

$$I = \int_{x=0}^{a} \left\{ \frac{e^{2x}}{2}\left(e^{2x} - 1\right) - e^x\left(e^x - 1\right) \right\} dx = \int_{x=0}^{a} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x \right) dx$$

$$I = \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x \right]_{x=0}^{a} = \left(\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a \right) - \left(\frac{1}{8} - \frac{3}{4} + 1 \right)$$

Thus, $\boxed{I = \frac{1}{8}\left(e^{4a} - 6e^{2a} + 8e^a - 3\right)}$

[8] *Evaluate* $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{(a^2-r^2)/a} r\, dr\, d\theta\, dz$

☞ $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} \int_{z=0}^{(a^2-r^2)/a} r\, dz\, dr\, d\theta = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} r\left[z\right]_{z=0}^{(a^2-r^2)/a} dr\, d\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} r \cdot \frac{a^2 - r^2}{a}\, dr\, d\theta$$

$$I = \frac{1}{a}\int_{\theta=0}^{\pi/2}\left\{a^2\left[\frac{r^2}{2}\right]_{r=0}^{a\sin\theta} - \left[\frac{r^4}{4}\right]_{r=0}^{a\sin\theta}\right\}d\theta = \frac{1}{a}\int_{\theta=0}^{\pi/2}\left(\frac{a^4}{2}\sin^2\theta - \frac{a^4}{4}\sin^4\theta\right)$$

Using, $\int_{\theta=0}^{\pi/2}\sin^n\theta\, d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdot\frac{1}{2}\cdot\frac{\pi}{2}$ (*where n is even*)

$$I = \frac{1}{a}\left\{\frac{a^4}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{a^4}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\} = \frac{\pi a^4}{8a}\left(1-\frac{3}{8}\right) = \boxed{\frac{5\pi a^3}{64}}$$

[9] *Evaluate* $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$

☞ Proceeding as in earlier problems we get $\boxed{I = 8\pi}$

[10] *Evaluate* $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{dz\, dy\, dx}{(1+x+y+z)^3}$

☞ Let $I = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}\frac{dz\, dy\, dx}{(1+x+y+z)^3}$

$$I = \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{-1}{2(1+x+y+z)^2}\right]_{z=0}^{1-x-y} dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[-\frac{1}{8}+\frac{1}{2(1+x+y)^2}\right]dy\, dx$$

$$= \int_{x=0}^{1}\left[-\frac{1}{8}y-\frac{1}{2(x+y+1)}\right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1}\left[-\frac{1}{8}(1-x)-\frac{1}{4}+\frac{1}{2(x+1)}\right]dx$$

$$I = \int_{x=0}^{1}\left[-\frac{3}{8}+\frac{x}{8}+\frac{1}{2(x+1)}\right]dx$$

$$I = \left[-\frac{3x}{8}+\frac{x^2}{16}+\frac{1}{2}\log(x+1)\right]_{x=0}^{1}$$

$$= -\frac{3}{8} + \frac{1}{16} + \frac{1}{2}\log 2 = \frac{-5}{16} + \log\sqrt{2}$$

Thus, $\boxed{I = \log\sqrt{2} - (5/16)}$

6.22 Evaluation of $\iint_R f(x,y)\,dx\,dy$ over the specific region R

We need to draw the befitting figure from the given description to identify the specific region R. We have to then express

$$I = \iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx \qquad \ldots (1)$$

or $$I = \iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d} \int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy \qquad \ldots (2)$$

I is obtained by the evaluation of (1) or (2)

Remark : *Carefully take a note of the content in article* **6.21.**

6.23 Evaluation of double integral by change of order of integration

The method is illustrated stepwise.

Step-1 : Given the integral in either of the forms as in article **6.22,** say (1) we have to identify the region of integration R by writing the figure and express (1) in the form (2).

Step-2 : The evaluation of (2) will be the value of (1) on changing the order of integration. This can be vice versa also.

Remark : *The advantage of this procedure is that, some times the double integral which is difficult to be evaluated in the existing form becomes easy for evaluation on changing the order of integration.*

6.24 Evaluation of double integral by changing into polar coordinates

The method is illustrated stepwise.

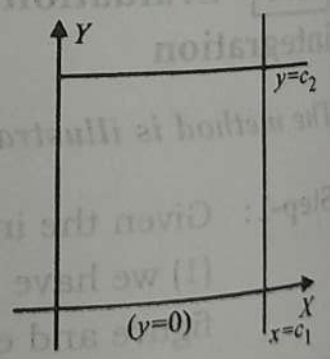
Step-1 : Given a double integral with limits, we use the well known polar form of substitution $x = r\cos\theta$, $y = r\sin\theta$. This will give us $x^2 + y^2 = r^2$, $y/x = \tan\theta$ and it should be noted that $dx\,dy = J\,dr\,d\theta$ where J is the Jacobian of the transformation given by

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Step-2 : Hence, $dx\,dy = r\,dr\,d\theta$ and we need to change the limits of integration to r, θ suitably for the purpose of evaluation.

Remark: *The method might be advantageous if the terms of the form $x^2 + y^2$ are involved in $f(x, y)$ and terms like $\sqrt{a^2 - y^2}$, $\sqrt{a^2 - x^2}$ etc. are involved in limits.*

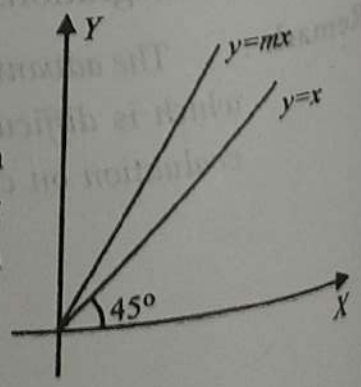
Note : *Some of the important and standard curves along with their equations and shape is given below as it will be highly useful for working problems.*



1. *Straight lines*

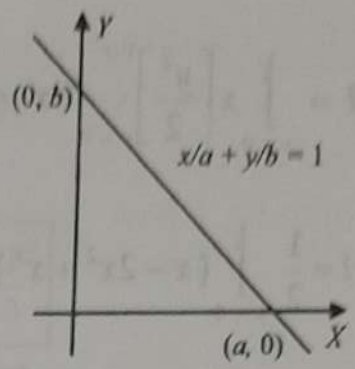
(i) $x = 0$ and $y = 0$ are respectively the equations of y and x - axis.

(ii) $x = c_1$ and $y = c_2$ are respectively the equations of a line parallel to y-axis and a line parallel to x - axis.

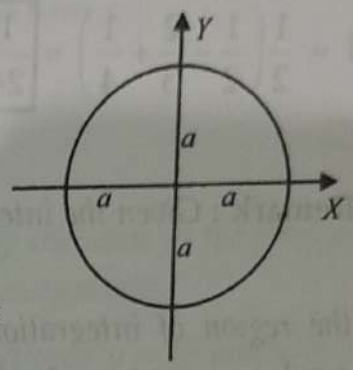


(iii) $y = mx$ is a straight line passing through the origin and in particular $y = x$ is a straight line passing through the origin subtending an angle 45^0 with the x - axis.

(iv) $x/a + y/b = 1$ is a straight line having x intercept a and y intercept b, being a straight line passing through $(a, 0)$ and $(0, b)$



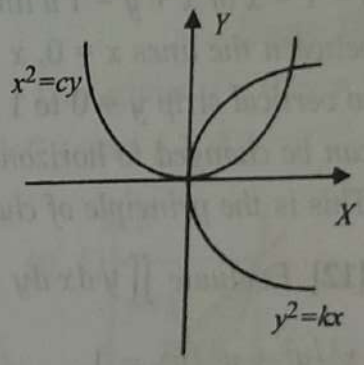
2. *Circle* $x^2 + y^2 = a^2$ is a circle with centre origin and radius a.



ut

3. *Parabola* $y^2 = kx$ is a parabola symmetrical about the x - *axis.*

$x^2 = cy$ is a parabola symmetrical about the y - *axis.*



WORKED PROBLEMS

Type-1 : *Evaluation of double integral over a given region*

[11] *Evaluate* $\iint_R xy\,dx\,dy$ *where R is the region bounded by the coordinate axes and the line* $x + y = 1$.

☞ R is the region bounded by $x = 0$, $y = 0$ being the coordinate axes and $x + y = 1$ being the straight line through $(1, 0)$ and $(0, 1)$.

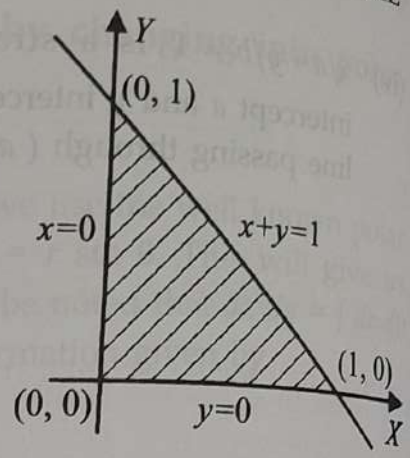
Shaded portion in the figure is the region R. [*Figure in the next page*]
From the figure we have,

$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{1} \int_{y=0}^{1-x} xy\,dy\,dx \quad \text{or} \quad \int_{y=0}^{1} \int_{x=0}^{1-y} xy\,dx\,dy$$

$$I = \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_{y=0}^{1-x} dx = \int_{x=0}^{1} \frac{x}{2}(1-x)^2\, dx$$

$$I = \frac{1}{2}\int_{x=0}^{1} (x - 2x^2 + x^3)\, dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$I = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \boxed{\frac{1}{24}}$$



Remark : *Given the integral* $\int_{x=0}^{1}\int_{y=0}^{1-x} xy\, dy\, dx$ *we can write the figure to identify the region of integration. This being the region bounded by* $y = 0$ *(x - axis),* $y = 1 - x$ *or* $x + y = 1$ *a line passing through the points* $(1, 0)$ *and* $(0, 1)$ *embedded between the lines* $x = 0$, $x = 1$. $x = 0$ *to* 1 *being the horizontal strip can be changed to vertical strip* $y = 0$ *to* 1 *(constant limits)* $y = 0$ *to* $(1 - x)$ *being the vertical strip can be changed to horizontal strip* $x = 0$ *to* $1 - y$ *(variable limits).*
This is the principle of changing the order of integration of a given double integral.

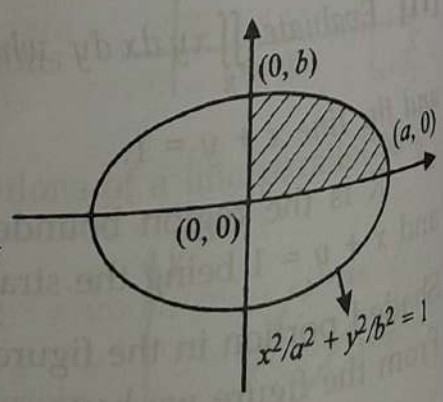
[12] *Evaluate* $\iint y\, dx\, dy$ *over the region bounded by the first quadrant of the ellipse* $x^2/a^2 + y^2/b^2 = 1$. **[Dec 19]**

☞ Shaded portion in the figure is the region (R) of integration. We observe that x varies from 0 to a and we need to express $x^2/a^2 + y^2/b^2 = 1$ in the form $y = f(x)$.

ie., $y^2/b^2 = 1 - (x^2/a^2) = (a^2 - x^2)/a^2$

or $y = (b/a)\sqrt{a^2 - x^2}$.

Since $y = 0$ is the equation of x axis, we can say that y varies from 0 to $(b/a)\sqrt{a^2 - x^2}$



$$\therefore \quad I = \iint_R y\, dx\, dy = \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2-x^2}} y\, dy\, dx \quad \ldots (1)$$

$$I = \int_{x=0}^{a} \left[\frac{y^2}{2} \right]_{y=0}^{(b/a)\sqrt{a^2-x^2}} dx = \frac{b^2}{2a^2} \int_{x=0}^{a} (a^2 - x^2)\, dx$$

$$I = \frac{b^2}{2a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a = \frac{b^2}{2a^2} \left[\left(a^3 - \frac{a^3}{3} \right) - 0 \right] = \boxed{\frac{a b^2}{3}}$$

Note: *From the figure, on a similar argument we can also have*

$$\iint_R y\, dx\, dy = \int_{y=0}^{b} \int_{x=0}^{(a/b)\sqrt{b^2-y^2}} y\, dx\, dy = \frac{ab^2}{3} \qquad \ldots (2)$$

Remark : *Given the double integral in the form (1) writing the same in the form as in (2) with the help of the figure is the change of order of integration.*

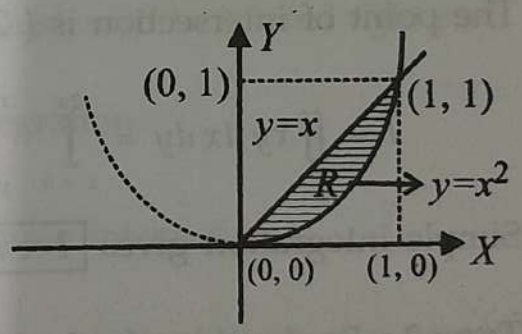
[13] *Evaluate* $\iint xy(x+y)\, dy\, dx$ *taken over the area between* $y = x^2$ *and* $y = x$ **[June 18]**

☞ Now $x^2 = x$ or $x(x-1) = 0 \Rightarrow x = 0, x = 1$. This gives $y = 0, y = 1$ and hence the two curves intersect at the points $(0, 0)$ and $(1, 1)$

$$I = \iint_R xy(x+y)\, dy\, dx$$

$$I = \int_{x=0}^{1} \int_{y=x^2}^{x} (x^2 y + xy^2)\, dy\, dx$$

On simple integration we obtain $\boxed{I = 3/56}$



[14] *Evaluate* $\iint_R x^2 y\, dx\, dy$ *where R is the region bounded by the lines* $y = x$, $y + x = 2$ *and* $y = 0$.

☞ The lines $y = x$ and $y + x = 2$ intersect at $(1, 1)$. [*Figure in the next page*]

$$I = \iint_R x^2 y\, dx\, dy = \int_{y=0}^{1} \int_{x=y}^{2-y} x^2 y\, dx\, dy = \int_{y=0}^{1} y \left[\frac{x^3}{3} \right]_{x=y}^{2-y} dy$$

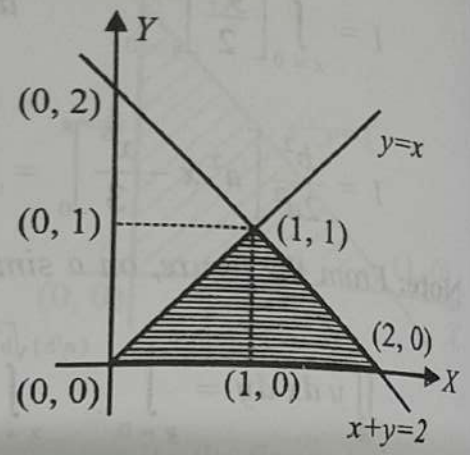
$$I = \frac{1}{3} \int_{y=0}^{1} y \left\{ (2-y)^3 - y^3 \right\} dy$$

$$= \frac{1}{3}\int_{y=0}^{1} y\,(8 - 12y + 6y^2 - y^3 - y^3)\,dy$$

$$I = \frac{1}{3}\int_{y=0}^{1} (8y - 12y^2 + 6y^3 - 2y^4)\,dy$$

$$I = \frac{1}{3}\left[4y^2 - 4y^3 + \frac{3}{2}y^4 - 2\frac{y^5}{5}\right]_{y=0}^{1}$$

$$I = \frac{1}{3}\left(4 - 4 + \frac{3}{2} - \frac{2}{5}\right) = \boxed{\frac{11}{30}}$$

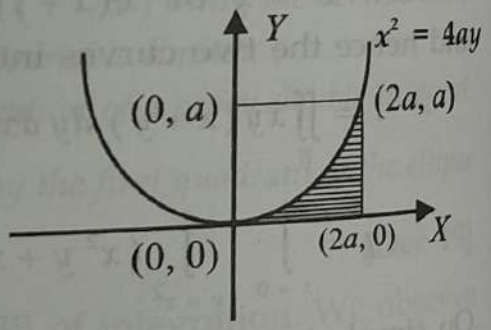


[15] *Evaluate* $\iint_R xy\,dx\,dy$ *where R is the region bounded by the x-axis, ordinate* $x = 2a$ *and the curve* $x^2 = 4ay$.

☞ $x^2 = 4ay$ is a parabola symmetrical about the y-axis. The point of intersection of this curve with $x = 2a$ is to be found.

Hence, $(2a)^2 = 4ay$ or $4a^2 = 4ay$ $\therefore$ $y = a$

The point of intersection is $(2a, a)$



$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx$$

Simple integration gives $\boxed{I = a^4/3}$

Type-2 : ***Evaluation of a double integral by changing the order of integration***

The procedure is illustrated in the article **6.23** and in **[11]** to **[12]** the alternative form of the double integral is exactly the integral by changing the order of integration.

We complete the problem by evaluating the new form of the double integral.

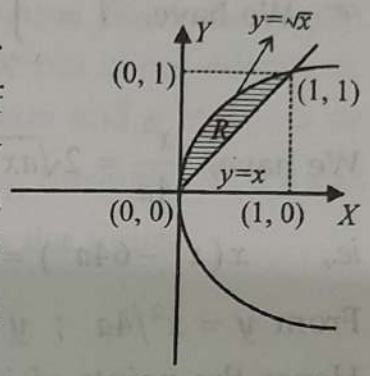
Note : *Simple integration is omitted in problems.*

[16] *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$ *by changing the order of integration.* **[June16, Dec 18]**

☞ $$I = \int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} xy\,dy\,dx$$

We need to first identify the region of integration R bounded by the curves $y = x$, $y = \sqrt{x}$ between the lines $x = 0$, $x = 1$. We shall find the points of intersection of $y = x$ and $y = \sqrt{x}$ by equating their RHS.

ie., $x = \sqrt{x} \Rightarrow x^2 = x$ or $x(x-1) = 0$ or $x = 0, 1$.



This will give us $y = 0$, $y = 1$ and hence the points of intersection are $(0, 0)$ and $(1, 1)$. Further we know that $y = x$ is a straight line passing through the origin making an angle 45^0 with the x-axis and $y = \sqrt{x}$ or $y^2 = x$ is a parabola symmetrical about the x - *axis*. The befitting figure indicating R is given.

On changing the order of integration we must have constant limits for y and variable limits for x. From the figure we observe that y varies from 0 to 1 and x varies from y^2 $(\because y = \sqrt{x})$ to y.

[It should be noted that $y = x$ and $\sqrt{x}$ are the lower and upper parts of the boundary of R and on changing the order, $x = y^2$ and $x = y$ represent the left and right parts of the boundary of R]

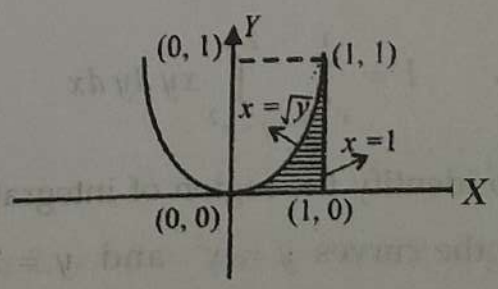
Hence, we have on changing the order of integration,

$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy\, dx\, dy = \boxed{1/24}, \text{ on simple integration.}$$

Remark : *Referring to* **[1]** *it may be seen that we have obtained the same answer by direct evaluation.*

[17] *Change the order of the integration and hence evaluate* $\int_0^1 \int_{\sqrt{y}}^1 dx\, dy$

☞ Let $I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} dx\, dy$



On changing the order of integration,

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} dy\, dx \quad (\because x = \sqrt{y} \Rightarrow x^2 = y)\ ; \boxed{I = 1/3}, \text{ on integration.}$$

[18] *Change the order of integration and hence evaluate* $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$

[Dec 16, Sep 20]

☞ We have, $I = \int_{x=0}^{4a} \int_{y=x^2/4a}^{y=2\sqrt{ax}} xy\, dy\, dx$

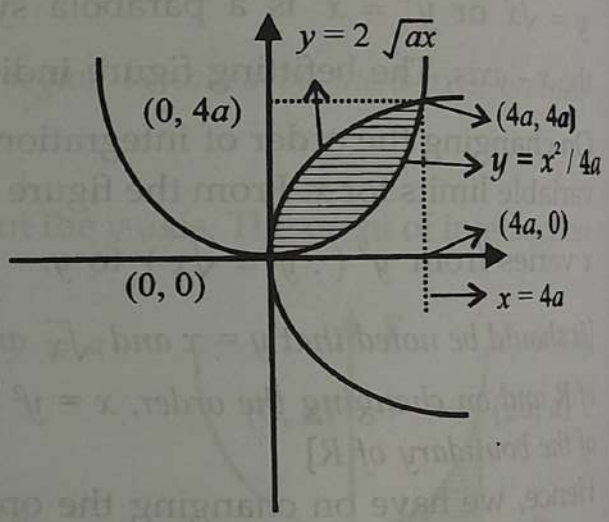
We have, $\frac{x^2}{4a} = 2\sqrt{ax}$ or $x^4 = 64a^3 x$

ie., $x(x^3 - 64a^3) = 0 \Rightarrow x = 0$ and $x = 4a$

From $y = x^2/4a$; $y = 0$ & $y = 4a$.

Hence the points of intersection of the parabolas $y = x^2/4a$ and $y = 2\sqrt{ax}$ are $(0, 0)$ and $(4a, 4a)$

On changing the order of integration we have y varying from 0 to $4a$ and x varying from $y^2/4a$ ($\because y = 2\sqrt{ax}$) to $2\sqrt{ay}$ ($\because y = x^2/4a$)



Now, $I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} xy\, dx\, dy = \boxed{64\, a^3/3}$, on integration.

Note : ***Alternative version :*** *Find the area enclosed by the parabolas* $y^2 = 4\,ax$ *and* $x^2 = 4ay$.

[Sep 21]

[19] *Evaluate* $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ *by changing the order of integration* [June 18]

☞ $I = \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy\, dy\, dx$

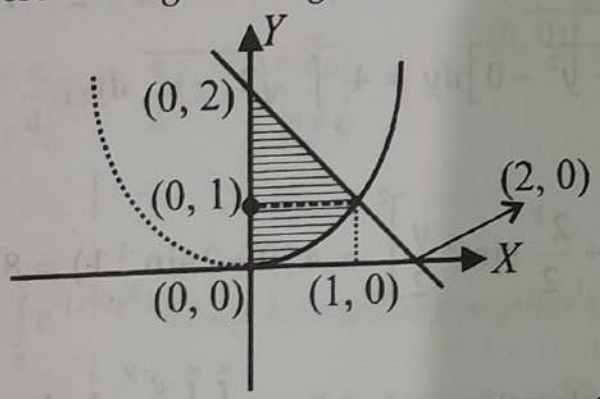
To identify the region of integration R, let us find the points of intersection of the curves $y = x^2$ and $y = 2 - x$.

That is to solve, $x^2 = 2 - x$

or $x^2 + x - 2 = 0$ or $(x-1)(x+2) = 0 \Rightarrow x = 1, x = -2$

$\therefore$ from $y = x^2$ (or $y = 2 - x$) we obtain $y = 1, y = 4$.

Hence $(1, 1)$ and $(-2, 4)$ are the points of intersection. Since the region is bounded by $x = 0$ and $x = 1$ the point $(-2, 4)$ will be out of consideration. Further $y = x^2$ is a parabola symmetrical about y-axis and $y = 2 - x$ or $x + y = 2$ or $x/2 + y/2 = 1$ is a straight line passing through $(2, 0)$ and $(0, 2)$. With these findings the region is indicated in the figure.



On changing the order we must have constant limits for y and variable limits for x to cover the same region.

From the figure we can make out that the same will be in two parts.

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\, dx\, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\, dx\, dy = I_1 + I_2 \text{ (say)}$$

On simple integration, $I = 1/6 + 5/24 = \boxed{3/8}$

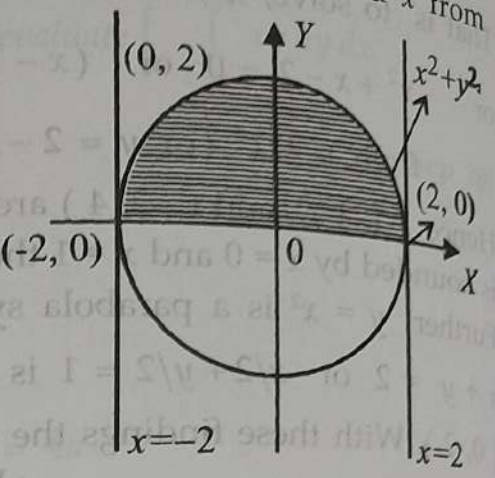
[20] *Evaluate* $\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} (2-x)\, dy\, dx$ *by changing the order of integration.* **[Dec 19]**

☞ We have $I = \int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} (2-x)\, dy\, dx$

Here, $y = \sqrt{4-x^2}$ or $x^2 + y^2 = 4$

This is a circle with centre origin and radius 2. $y = 0$ to $\sqrt{4-x^2}$ is the upper half of the circle being bounded by the lines $x = -2$ and 2.

On changing the order we must have y varying from 0 to 2 and x from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$.



$$\therefore \quad I = \int_{y=0}^{2} \int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2-x)\,dx\,dy$$

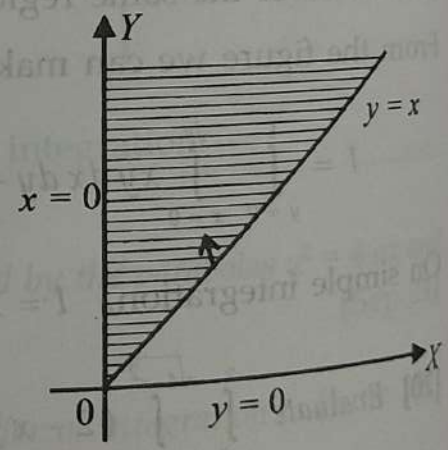
$$I = \int_{y=0}^{2} \left[2x - \frac{x^2}{2}\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$I = \int_{y=0}^{2} \left[2\cdot 2\sqrt{4-y^2} - 0\right] dy = 4\int_{y=0}^{2} \sqrt{2^2-y^2}\,dy$$

$$I = 4\left[\frac{y\sqrt{4-y^2}}{2} + \frac{2^2}{2}\sin^{-1}\frac{y}{2}\right]_0^2 = 4(0+2\sin^{-1}1) = 8\cdot\frac{\pi}{2} = \boxed{4\pi}$$

[21] *Change the order of integration and evaluate* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ **[June 18, Sep 21]**

☞ $$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dy\,dx$$



On changing the order we must have $y = 0$ to ∞ and $x = 0$ to y.

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y}[x]_0^y\,dy$$

$$I = \int_{y=0}^{\infty} \frac{e^{-y}}{y}\cdot y\,dy = \int_{y=0}^{\infty} e^{-y}\,dy = -[e^{-y}]_0^\infty = \boxed{1}$$

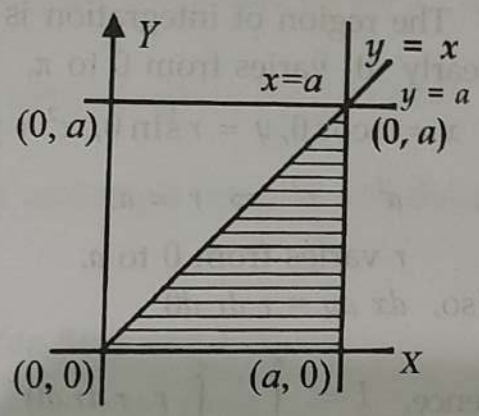
[22] *Evaluate* $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ *by changing the order of integration.*

[June, Dec. 17]

☞ The region bounded by the curves $x = y$, $x = a$ embedded between the

lines $y = 0$, $y = a$ is shown in the figure.
On changing the order x varies from 0 to a and y varies from 0 to x.

$$I = \int_{x=0}^{a} \int_{y=0}^{x} x \cdot \frac{1}{x^2+y^2}\, dy\, dx$$



$$I = \int_{x=0}^{a} x \cdot \frac{1}{x}\left[\tan^{-1}(y/x)\right]_{y=0}^{x} dx$$

$$I = \int_{x=0}^{a} (\tan^{-1} 1 - \tan^{-1} 0)\, dx$$

$$I = \int_{x=0}^{a} \frac{\pi}{4}\, dx = \frac{\pi}{4}[x]_0^a = \boxed{\frac{\pi a}{4}}$$

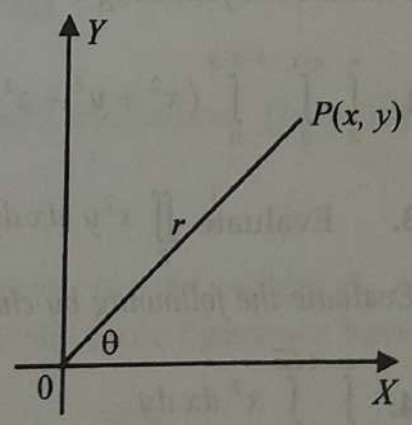
[23] *Evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ *by changing to polar coordinates.*

[June 16, 17, Dec 16, 18]

☞ In polars we have $x = r\cos\theta$, $y = r\sin\theta$. [*Refer article* **6.24**]

$\therefore$ $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$

Since x, y varies from 0 to ∞,
r also varies from 0 to ∞.
In the first quadrant θ varies from 0 to $\pi/2$.



Hence, $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta.$

Put $r^2 = t \therefore r\, dr = \frac{dt}{2}$;

t also varies from 0 to ∞.

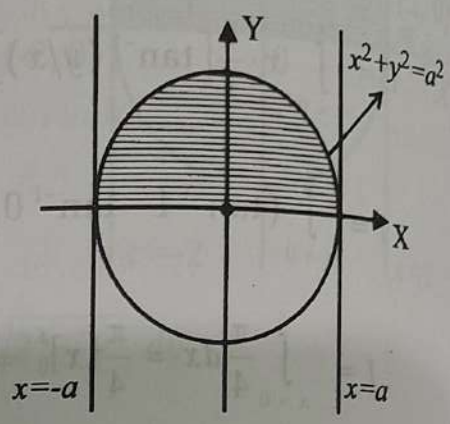
$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2}\, d\theta$$

$$I = \frac{1}{2}\int_{\theta=0}^{\pi/2} [-e^{-t}]_{t=0}^{\infty}\, d\theta = -\frac{1}{2}\int_{\theta=0}^{\pi/2} (0-1)\, d\theta = \frac{1}{2}[\theta]_0^{\pi/2} = \boxed{\frac{\pi}{4}}$$

[24] *Change the integral* $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ *into polars and hence evaluate.*

☞ The region of integration is as shown in the figure.

Clearly θ varies from 0 to π.



If $x = r\cos\theta, y = r\sin\theta, x^2+y^2 = r^2$

ie., $a^2 = r^2 \Rightarrow r = a.$

$\therefore$ r varies from 0 to a.

Also, $dx\, dy = r\, dr\, d\theta$

Hence, $I = \int_{\theta=0}^{\pi}\int_{r=0}^{a} r\cdot r\, dr\, d\theta$

$$I = \int_{\theta=0}^{\pi}\left[\frac{r^3}{3}\right]_0^a d\theta$$

$$I = \frac{a^3}{3}[\theta]_0^{\pi} = \frac{a^3}{3}(\pi - 0) = \boxed{\frac{\pi a^3}{3}}$$

EXERCISE

Evaluate the following :

1. $\int_0^a\int_0^{a-x}\int_0^{a-x-y} (x^2+y^2+z^2)\, dz\, dy\, dx$

2. $\int_0^a\int_0^{\sqrt{a^2-z^2}}\int_0^{\sqrt{a^2-y^2-z^2}} x\, dx\, dy\, dz$

3. Evaluate $\iint_R x^2 y\, dx\, dy$ over the region bounded by $y = x^2$ and $y = x$

Evaluate the following by changing the order of integration

4. $\int_0^a\int_0^{2\sqrt{ax}} x^2\, dx\, dy$

5. $\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (a-x)\, dy\, dx$

ANSWERS

1. $a^5/20$ 2. $\pi a^4/16$ 3. $1/35$ 4. $4a^4/7$ 5. $\pi a^3/2$

6.3 Applications to find Area and Volume

The reader is already familiar with the application of definite integrals of a single variable to find the area of a curve $y = f(x)$ bounded by the x - axis and the extreme ordinates $x = a$ and $x = b$ which is given by $\int_a^b f(x)\,dx$.

In this article, with the knowledge of double and triple integrals, we discuss *Area and Volume* as *Applications*.

Note the following ***Applications formulae for Area and Volume.***

1. $\iint_R dx\,dy$ = Area of the region R in the cartesian form.
2. $\iint_R r\,dr\,d\theta$ = Area of the region R in the polar form.
3. $\iiint_V dx\,dy\,dz$ = Volume of the solid in the cartesian form.
4. $\iint_A 2\pi r^2 \sin\theta\,dr\,d\theta$ = Volume of a solid obtained by the revolution of a curve enclosing an area A about the initial line in the polar form.

WORKED PROBLEMS

[25] *Find the area of ellipse* $x^2/a^2 + y^2/b^2 = 1$ *by double integration.* **[June 19]**

☞ Area $(A) = \iint_R dx\,dy$.

Referring to the figure in Problem - [14] we can write $A = 4A_1$ where A_1 is the area in the first quadrant and with reference to the same figure we have,

$$A = 4A_1 = 4\int_{x=0}^{a}\int_{y=0}^{(b/a)\sqrt{a^2-x^2}} dy\,dx = 4\int_{x=0}^{a}\left[y\right]_0^{(b/a)\sqrt{a^2-x^2}} dx$$

$$A = 4\int_{x=0}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{x=0}^{a}$$

$$A = \frac{4b}{a}\left[0 + \frac{a^2}{2}(\sin^{-1}1 - \sin^{-1}0)\right] = \frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \pi ab$$

Thus the required area $\boxed{A = \pi\, ab \text{ sq. units.}}$

Note : *The area of the circle $x^2 + y^2 = a^2$ by double integration is πa^2. This is a particular case of this problem when $b = a$.*

[26] *Find by double integration the area enclosed by the curve $r = a(1+\cos\theta)$ between $\theta = 0$ and $\theta = \pi$.* **[June 18, Dec 19]**

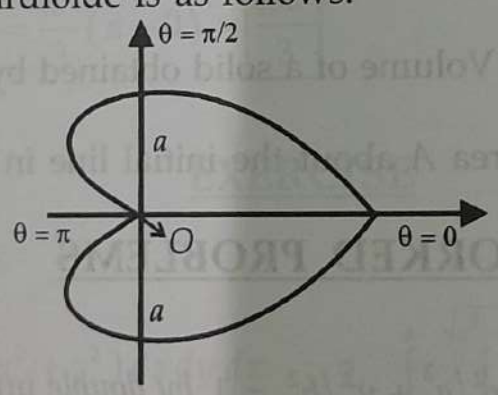
☞ Let, $f(r,\theta) = a(1+\cos\theta) = f(r,-\theta)$

$\Rightarrow$ Curve is symmetrical about the initial line.

We tabulate the value of r for certain angles θ.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
$r = a(1+\cos\theta)$	$2a$	$3a/2$	a	$a/2$	0

The shape of the cardioide is as follows.



Area $A = \iint r\,dr\,d\theta$ where r varies from 0 to $a(1+\cos\theta)$ and θ from 0 to π.

$$\therefore \quad A = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$A = \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{\theta=0}^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$A = \frac{a^2}{2}\int_{\theta=0}^{\pi}\{2\cos^2(\theta/2)\}^2\,d\theta = 2a^2\int_0^{\pi}\cos^4(\theta/2)\,d\theta$$

Put $\theta/2 = \phi$, $d\theta = 2d\phi$ and ϕ varies from 0 to $\pi/2$.

$$\therefore \quad A = 2a^2 \int_0^{\pi/2} \cos^4 \phi \cdot 2\, d\phi$$

$$= 4a^2 \int_0^{\pi/2} \cos^4 \phi \cdot d\phi = 4a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \text{ by the reduction formula.}$$

Thus the required area $\boxed{A = 3\,\pi\, a^2/4 \text{ sq. units}}$

[27] *Find the volume generated by the revolution of the cardioide $r = a\,(1 + \cos\theta)$ about the initial line.*

☞ Volume of the solid of revolution in polars is given by

$$V = \iint_A 2\,\pi\, r^2 \sin\theta\, dr\, d\theta.$$

$$V = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} 2\,\pi\, r^2 \sin\theta\, dr\, d\theta$$

$$V = \int_{\theta=0}^{\pi} 2\pi \left[\frac{r^3}{3}\right]_{r=0}^{a(1+\cos\theta)} \sin\theta\, d\theta = \frac{2\pi}{3} \int_{\theta=0}^{\pi} a^3\, (1+\cos\theta)^3 \sin\theta\, d\theta$$

Put $1 + \cos\theta = t \;\therefore\; -\sin\theta\, d\theta = dt$; If $\theta = 0, t = 2$; $\theta = \pi,\ t = 0$

$$V = \frac{2\pi a^3}{3} \int_2^0 t^3\,(-dt) = \frac{2\pi a^3}{3} \int_0^2 t^3\, dt = \frac{2\pi a^3}{3} \left[\frac{t^4}{4}\right]_0^2 = \boxed{\frac{8\pi a^3}{3}}$$

[28] *Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x/a + y/b + z/c = 1$.* **[Dec 18]**

☞ $V = \iiint dx\, dy\, dz$

$x/a + y/b + z/c = 1 \quad \therefore\; z = c\,(1 - x/a - y/b)$

If $z = 0$, then $x/a + y/b = 1 \;\therefore\; y = b\,(1 - x/a)$

If $z = 0$, $y = 0$ then $x = a$

$$\therefore \quad V = \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} \int_{z=0}^{c(1-x/a-y/b)} dz\, dy\, dx$$

On simple integration $\boxed{V = abc/6}$

Note : ***Similar problem***

Find the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0,$ $x + y + z = 1$ [June 15, Sep 20, 21]

☞ This particular case where, $a = b = c = 1$; $\boxed{V = 1/6}$

6.4 Beta and Gamma functions

In this topic we define two special functions namely *Beta function* and *Gamma function* by means of an integral and study the associated properties. These help us to evaluate certain definite integrals which are either difficult or impossible to evaluate by various known methods of integration.

6.41 Definitions

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ (m,n>0) \qquad \ldots(1)$$

is called the **Beta function.**

$$\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx,\ (n>0) \qquad \ldots(2)$$

is called the **Gamma function.**

These definitions can be put in the following alternative forms.

In (1), put $x = \sin^2\theta$. $dx = 2\sin\theta\cos\theta\,d\theta$

If, $x = 0,\ \sin^2\theta = 0 \Rightarrow \theta = 0$

$x = 1,\ \sin^2\theta = 1 \Rightarrow \theta = \pi/2$

$$\therefore \quad \beta(m,n) = \int_{\theta=0}^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\,d\theta$$

ie., $$\beta(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad \ldots(3)$$

In (2), put $x = t^2$. $\therefore$ $dx = 2t\,dt$, t also varies from 0 to ∞.

$$\therefore \quad \Gamma(n) = \int_{t=0}^{\infty} e^{-t^2}(t^2)^{n-1}\,2t\,dt \ ;\ \Gamma(n) = 2\int_0^\infty e^{-t^2}t^{2n-1}\,dt \qquad \ldots(4)$$

(3) and (4) are also regarded as definitions of Beta function and Gamma function respectively.

6.42 Properties of Beta and Gamma functions

1. $\beta(m,n) = \beta(n,m)$

Proof : We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = 1-y$ or $1-x = y$ $\therefore$ $dx = -dy$

When $x = 0$, $y = 1$ and when $x = 1$, $y = 0$.

$$\therefore \quad \beta(m,n) = \int_{y=1}^{0} (1-y)^{m-1} y^{n-1}(-dy)$$

That is, $\beta(m,n) = \int_{y=0}^{1} y^{n-1}(1-y)^{m-1}\,dy = \beta(n,m)$; $\beta(m,n) = \beta(n,m)$

2. (i) $\Gamma(n+1) = n\,\Gamma(n)$ (ii) $\Gamma(n+1) = n!$ for a positive integer n.

Proof (i) By definition $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

$\therefore$ $\Gamma(n+1) = \int_0^\infty e^{-x} x^n\,dx$ or $\int_0^\infty x^n e^{-x}\,dx$. Integrating by parts,

$$\Gamma(n+1) = \left[x^n(-e^{-x})\right]_0^\infty - \int_0^\infty (-e^{-x})\,n\,x^{n-1}\,dx$$

Note : *$x^n/e^x \to 0$ as $x \to \infty$ by L - Hospital's rule.*

$$\therefore \quad \Gamma(n+1) = (0-0) + n\int_0^\infty e^{-x} x^{n-1}\,dx = n\Gamma(n) \quad ; \Gamma(n+1) = n\Gamma(n)$$

(ii) Continuing from the above, we have similarly

$\Gamma(n) = (n-1)\Gamma(n-1), \Gamma(n-1) = (n-2)\Gamma(n-2)\ldots$

$\Gamma(3) = 2\Gamma(2),\ \Gamma(2) = 1\Gamma(1)$.

$\therefore$ $\Gamma(n+1) = n\{(n-1)\cdot\Gamma(n-1)\} = n(n-1)\{(n-2)\Gamma(n-2)\}$ etc.

Hence, $\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot \Gamma(1) = n!\Gamma(1)$

But $\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx = -\left[e^{-x} \right]_0^\infty = -(0-1) = 1$

Thus, $\boxed{\Gamma(n+1) = n!}$ for a positive integer n.

Note :

1. *This result can be remembered in the form* $\Gamma(n) = (n-1)\Gamma(n-1)$ *where* $n \neq 1$ *and* $\Gamma(n) = (n-1)!$ *where n is a positive integer.*
2. $\Gamma(n)$ *is not defined for* $n = 0$ *and also for a negative integer n.*
3. $\Gamma(n+1) = n\Gamma(n)$ *or* $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ *and this expression is used for finding* $\Gamma(n)$ *when n is a negative real number.*

6.43 Relationship between Beta and Gamma functions

$$\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$ **[June 16, 17, 18,19 Dec 17,18,19 Sep 20, 21]**

Proof : We have by the definition of Beta and Gamma functions,

$$\beta(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \quad \ldots (1)$$

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx \quad \ldots (2) \qquad \Gamma(m) = 2\int_0^\infty e^{-y^2} y^{2m-1}\, dy \quad \ldots (3)$$

$$\Gamma(m+n) = 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr \quad \ldots (4)$$

Now, $$\Gamma(m)\cdot\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\, dx\, dy \quad \ldots (5)$$

Let us evaluate RHS by changing into polars.
Putting $x = r\cos\theta$, $y = r\sin\theta$, we have, $x^2 + y^2 = r^2$.
Also $dx\, dy = r\, dr\, d\theta$. r varies from 0 to ∞, θ varies from 0 to $\pi/2$.
We now have (5) in the form,

$$\Gamma(m)\cdot\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1}\, r\, dr\, d\theta$$

$$\Gamma(m)\cdot\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\, r^{2m+2n-1}\sin^{2m-1}\theta\cos^{2n-1}\theta\, dr\, d\theta$$

$$\Gamma(m)\cdot\Gamma(n) = \left[2\int_{r=0}^{\infty} e^{-r^2}\, r^{2(m+n)-1}\, dr\right]\left[2\int_{\theta=0}^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta\right]$$

$\therefore \quad \Gamma(m)\cdot\Gamma(n) = \Gamma(m+n)\cdot\beta(m,n)$ by using (1) and (4).

Thus, $$\boxed{\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}}$$

Corollary : To show that $\Gamma(1/2) = \sqrt{\pi}$

Putting, $m = n = 1/2$ in this result we get,

$\therefore \quad \beta(1/2,1/2) = \{\Gamma(1/2)\}^2$, since $\Gamma(1) = 1$. $\quad \ldots (6)$

Now consider, $\beta(m,n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$

$$\beta(1/2,1/2) = 2\int_0^{\pi/2}\sin^0\theta\cos^0\theta\, d\theta = 2\int_0^{\pi/2} 1\, d\theta = 2[\,\theta\,]_0^{\pi/2} = \pi$$

From (6), $\pi = \{\Gamma(1/2)\}^2 \Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}}$

Note : *We can independently prove that* $\Gamma(1/2) = \sqrt{\pi}$. *The proof is as follows.*

Question : *Prove that* $\Gamma(1/2) = \sqrt{\pi}$ *using the definition of* $\Gamma(n)$. **[June 17,18]**

We have by the definition, $\Gamma(n) = 2\int_0^{\infty} e^{-x^2}\, x^{2n-1}\, dx$

$$\therefore \quad \Gamma(1/2) = 2\int_0^{\infty} e^{-x^2}\, dx = 2\int_0^{\infty} e^{-y^2}\, dy$$

Hence, $\{\Gamma(1/2)\}^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy = 4\cdot\frac{\pi}{4} = \pi \Rightarrow \Gamma(1/2) = \sqrt{\pi}$

We need to retrace the steps of **[23]**.

Note : *Duplication formula*

$$\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\Gamma(m+1/2)$$

This formula is derived by taking $n = m$ and $n = 1/2$ separately in the relationship between Beta and Gamma functions.

Corollary : Putting $m = 1/4$ in this formula we have,

$$\sqrt{\pi}\,\Gamma(1/2) = 2^{-1/2}\,\Gamma(1/4)\Gamma(3/4)$$

ie., $\sqrt{\pi}\cdot\sqrt{\pi} = 1/\sqrt{2}\cdot\Gamma(1/4)\Gamma(3/4)$ or $\boxed{\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}}$

Results to remember for problems

(i) $\Gamma(n) = (n-1)\Gamma(n-1), \Gamma(n) = (n-1)!$ if n is a positive integer.

(ii) $\beta(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

(iii) $\Gamma(1) = 1, \Gamma(1/2) = \sqrt{\pi}, \Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$

WORKED PROBLEMS

[29] *Show that* $\beta(m+1,n)+\beta(m,n+1) = \beta(m,n)$

☞ Using the relationship between beta and gamma functions in LHS we have

$$\frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$\text{LHS} = \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} + \frac{\Gamma(m)n\Gamma(n)}{(m+n)\Gamma(m+n)}, \text{ since } \Gamma(p+1) = p\Gamma(p)$$

$$\text{LHS} = \frac{\Gamma(m)\cdot\Gamma(n)}{(m+n)\Gamma(m+n)}(m+n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = \text{RHS}$$

Note : *We can also obtain the result from the basic definition of Beta function.*

[30] *Evaluate* (i) $\dfrac{\Gamma(3)\Gamma(2\cdot5)}{\Gamma(5\cdot5)}$ (ii) $\dfrac{6\Gamma(8/3)}{5\Gamma(2/3)}$

☞ (i) $\dfrac{\Gamma(3)\Gamma(2\cdot5)}{\Gamma(5\cdot5)} = \dfrac{(2!)(1\cdot5)(0\cdot5)\Gamma(0\cdot5)}{(4\cdot5)(3\cdot5)(2.5)(1\cdot5)(0\cdot5)\Gamma(0\cdot5)}$

$$\frac{\Gamma(3)\Gamma(2\cdot5)}{\Gamma(5\cdot5)} = \frac{2}{(4\cdot5)(3\cdot5)(2\cdot5)} = \frac{2}{9/2\cdot7/2\cdot5/2} = \boxed{\frac{16}{315}}$$

(ii) $$\frac{6\Gamma(8/3)}{5\Gamma(2/3)} = \frac{6\cdot5/3\cdot2/3\cdot\Gamma(2/3)}{5\Gamma(2/3)} = \boxed{\frac{4}{3}}$$

[31] *Show that* $\int_0^1 [\log(1/y)]^{n-1}\,dy = \Gamma(n)$

☞ Consider LHS and put $\log(1/y) = t$

ie., $1/y = e^t$ or $y = e^{-t}$ $\therefore$ $dy = -e^{-t}\,dt$

If $y = 0$, $e^{-t} = 0 \Rightarrow t = \infty$; If $y = 1, e^{-t} = 1 \Rightarrow t = 0$

$$\therefore \quad \textbf{LHS} = \int_{t=\infty}^{0} t^{n-1}(-e^{-t})\,dt = \int_0^\infty e^{-t}\,t^{n-1}\,dt = \Gamma(n) = \textbf{RHS}$$

[32] *Show that* $\int_0^1 x^m[\log(1/x)]^n\,dx = \dfrac{n!}{(m+1)^{n+1}}$ *where n is a positive integer.*

☞ $$\text{LHS} = \int_0^1 x^m[\log(1/x)]^n\,dx$$

Put $\log(1/x) = t$ or $1/x = e^t$ or $x = e^{-t}$ $\therefore$ $dx = -e^{-t}\,dt$

If $x = 0$, $e^{-t} = 0 \Rightarrow t = \infty$; If $x = 1$, $e^{-t} = 1 \Rightarrow t = 0$

$$\text{LHS} = \int_{t=\infty}^{0} (e^{-t})^m\,t^n\,(-e^{-t})\,dt$$

$$\text{LHS} = \int_{t=0}^{\infty} (e^{-t})^{m+1}\,t^n\,dt = \int_{t=0}^{\infty} e^{-(m+1)t}\,t^n\,dt \qquad \ldots (1)$$

Now, let $(m+1)t = u$ $\therefore$ $dt = du/(m+1)$

u also varies from 0 to ∞ and hence (1) becomes,

$$\text{LHS} = \int_{u=0}^{\infty} e^{-u}\left(\frac{u}{m+1}\right)^n \frac{du}{(m+1)} = \frac{1}{(m+1)^{n+1}}\int_0^\infty e^{-u}u^n\,du$$

$$\text{LHS} = \frac{1}{(m+1)^{n+1}} \cdot \Gamma(n+1) = \frac{n!}{(m+1)^{n+1}} = \text{RHS}$$

Evaluation of definite integrals by converting into gamma functions

Step-1 : We need to take a suitable substitution keeping in mind the definition of gamma function in two of the standard forms :

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \; ; \; \Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

Step-2 : We need to correlate the value correspondig to $(n-1)$ or $(2n-1)$ as the case may be and find n.

[33] *Show that* $\int_0^\infty \sqrt{y}\, e^{-y^2}\, dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\, dy = \frac{\pi}{2\sqrt{2}}$

☞ Let $I_1 = \int_0^\infty \sqrt{y}\, e^{-y^2}\, dy = \int_0^\infty e^{-y^2} y^{1/2}\, dy$ ... (1)

$$I_2 = \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\, dy = \int_0^\infty e^{-y^2} y^{-1/2}\, dy \quad \text{... (2)}$$

We have, $\frac{1}{2}\Gamma(n) = \int_0^\infty e^{-x^2} x^{2n-1}\, dx$ ... (3)

Comparing (1) and (3) $2n-1 = 1/2 \Rightarrow n = 3/4$ $\quad \therefore \; I_1 = 1/2 \cdot \Gamma(3/4)$

Comparing (2) and (3) $2n-1 = -1/2 \Rightarrow n = 1/4$ $\quad \therefore \; I_2 = 1/2 \cdot \Gamma(1/4)$

Hence, $I_1 \times I_2 = \frac{1}{4}\Gamma(3/4)\Gamma(1/4) = \frac{\pi\sqrt{2}}{4} = \boxed{\frac{\pi}{2\sqrt{2}}}$

Evaluate the following integrals

[34] $\int_0^\infty x^{3/2} e^{-x}\, dx$

[35] $\int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$

☞ **Solutions**

[34] Let $I = \int_0^\infty x^{3/2} e^{-x}\, dx$

We have, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$; Taking $(n-1) = 3/2$ we get $n = 5/2$.

Hence, $I = \Gamma(5/2) = 3/2 \cdot 1/2 \cdot \Gamma(1/2) = \boxed{3\sqrt{\pi}/4}$

[35] Taking $\sqrt{x} = t$ or $x = t^2$ we can obtain $\boxed{I = 3\sqrt{\pi}/2}$

Problems on converting an integral into beta function and evaluation by transforming into gamma function

We have, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

Putting, $2m-1 = p$ and $2n-1 = q$ we get, $m = (p+1)/2$, $n = (q+1)/2$

$$\therefore \quad \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta$$

$$\therefore \quad \int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \qquad \ldots (1)$$

Integrals involving trigonometric functions are converted into the above form and in turn is expressed in terms of beta function. Further some of the integrals involving algebraic functions are converted into the trigonometric form by a suitable substitution.

Two important forms along with the substitution is as follows.

(i) $(a - x^n)$: Substitution : $x^n = a \sin^2\theta$

(ii) $(a + x^n)$: Substitution : $x^n = a \tan^2\theta$

The integral expressed in terms of beta function is converted into gamma function by using the relationship between them for the purpose of evaluation.

[36] *Show that* $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

[June 15, 16, 17, 18, 19 Dec 17, 18, 19, 21, Sep 20, 21]

☞ Let, $I_1 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta$

and $I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$

We have, $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Hence $I_1 = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$

$$I_2 = \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$\therefore \quad I_1 \times I_2 = \frac{1}{4}\beta(1/4, 1/2)\cdot\beta(3/4, 1/2)$$

$$I_1 \times I_2 = \frac{1}{4}\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}\cdot\frac{\Gamma(3/4)\cdot\Gamma(1/2)}{\Gamma(5/4)}$$

This, $I_1 \times I_2 = \frac{1}{4}\Gamma(1/4)\sqrt{\pi}\cdot\frac{\sqrt{\pi}}{1/4\cdot\Gamma(1/4)} = \boxed{\pi}$

[37] *Evaluate* $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$ *by expressing in terms of gamma functions.*

☞ Let, $I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$

Hence, $I = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right)$

ie., $I = \frac{1}{2}\beta(1/4, 3/4) = \frac{1}{2}\frac{\Gamma(1/4)\cdot\Gamma(3/4)}{\Gamma(1)} = \frac{1}{2}\cdot\frac{\pi\sqrt{2}}{1} = \boxed{\frac{\pi}{\sqrt{2}}}$

[June 18]

Note : $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$ *is also equal to* $\pi/\sqrt{2}$.

Express the following integrals in terms of beta function and hence evaluate them.

[38] $\int_0^2 (4-x^2)^{3/2}\,dx$ [39] $\int_0^\infty \frac{dx}{1+x^4}$

Solutions

[38] Let $I = \int_0^2 (4-x^2)^{3/2}\,dx$

Put $x = 2\sin\theta \;\therefore\; dx = 2\cos\theta\,d\theta$

$$(4-x^2)^{3/2} = [4(1-\sin^2\theta)]^{3/2} = (2^2\cos^2\theta)^{3/2} = 8\cos^3\theta$$

If $x = 0,\ 2\sin\theta = 0$ or $\sin\theta = 0 \Rightarrow \theta = 0$

$x = 2,\ 2\sin\theta = 2$ or $\sin\theta = 1 \Rightarrow \theta = \pi/2$

Hence, $I = \int_{\theta=0}^{\pi/2} 8\cos^3\theta \cdot 2\cos\theta\,d\theta = 16\int_0^{\pi/2}\cos^4\theta\,d\theta$

Now, $I = 16\cdot\frac{1}{2}\beta\left(\frac{0+1}{2},\frac{4+1}{2}\right) = 8\beta\left(\frac{1}{2},\frac{5}{2}\right)$

$$I = 8\cdot\frac{\Gamma(1/2)\Gamma(5/2)}{\Gamma(3)} = \frac{8\cdot\sqrt{\pi}\cdot 3/2\cdot 1/2\cdot\sqrt{\pi}}{2!} = \boxed{3\pi}$$

[39] Let $I = \int_0^\infty \frac{dx}{1+x^4}$

Put $x^4 = \tan^2\theta$ or $x = \tan^{1/2}\theta;\ \therefore\ dx = 1/2\cdot\tan^{-1/2}\theta\sec^2\theta\,d\theta$

θ varies from 0 to $\pi/2$.

$$\therefore\quad I = \int_{\theta=0}^{\pi/2}\frac{1/2\cdot\tan^{-1/2}\theta\sec^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{\sin^{-1/2}\theta}{\cos^{-1/2}\theta}\,d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\theta\,\cos^{1/2}\theta\,d\theta = \frac{1}{2}\cdot\frac{1}{2}\beta\left(\frac{-1/2+1}{2},\frac{1/2+1}{2}\right)$$

$$I = \frac{1}{4}\beta(1/4,\ 3/4) = \frac{1}{4}\frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1)} = \frac{\pi\sqrt{2}}{4} = \boxed{\frac{\pi}{2\sqrt{2}}}$$

[40] *Show that* $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

☞ Consider RHS : Put $x = \tan^2\theta \;\therefore\; dx = 2\tan\theta\sec^2\theta\,d\theta\;;\; 0 \le \theta \le \pi/2$

$$\text{RHS} = \int_0^{\pi/2} \frac{(\tan^2\theta)^{m-1}\cdot 2\tan\theta\sec^2\theta}{(\sec^2\theta)^{m+n}}\,d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2m-2+1}\theta\,d\theta}{\sec^{2m+2n-2}\theta} = 2\int_0^{\pi/2} \tan^{2m-1}\theta\cdot\cos^{2m+2n-2}\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2m-1}\theta}{\cos^{2m-1}\theta}\cdot\cos^{2m+2n-2}\theta\,d\theta$$

$$\textbf{RHS} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \beta(m,n) = \textbf{LHS}.$$

[41] *Show that* $\int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}\,dx = 2^{p+q-1}\beta(p,q)$

☞ Put $x = \cos 2\theta \;\therefore\; dx = -2\sin 2\theta\,d\theta = -4\sin\theta\cos\theta\,d\theta$

If $x = -1,\ \cos 2\theta = -1 \Rightarrow 2\theta = \pi$ or $\theta = \pi/2$

$x = 1,\ \cos 2\theta = 1 \Rightarrow 2\theta = 0$ or $\theta = 0$

$$\text{LHS} = I = \int_{\pi/2}^{0} (1+\cos 2\theta)^{p-1}(1-\cos 2\theta)^{q-1}\cdot(-4\sin\theta\cos\theta)\,d\theta$$

$$I = \int_0^{\pi/2} (2\cos^2\theta)^{p-1}(2\sin^2\theta)^{q-1}\cdot 2^2\sin\theta\cos\theta\,d\theta$$

$$I = 2^{p+q}\int_0^{\pi/2} \cos^{2p-1}\theta\sin^{2q-1}\theta\,d\theta = 2^{p+q}\cdot\frac{1}{2}\beta(q,p) = \boxed{2^{p+q-1}\beta(p,q)}$$

EXERCISE

Evaluate the following :

1. $\int_0^\infty x^{-7/4}e^{-\sqrt{x}}\,dx$
2. $\int_0^{\pi/2} \sin^{1/2}x\cos^{3/2}x\,dx$
3. $\int_0^\infty \frac{x}{1+x^6}\,dx$

ANSWERS

1. $8\sqrt{\pi}/3$
2. $\pi/4\sqrt{2}$
3. $\pi/3\sqrt{3}$

7 VECTOR CALCULUS

[Common to all the streams in II Sem]

7.1 Vector Differentiation [Pre-requisite Vector Algebra]

- **Preamable and Preliminaries**

Basically *Vector* is a quantity having both magnitude and direction.
Vector quantities like force, velocity, acceleration etc, have lot of reference in physical and engineering problems.

- **Derivative of a vector valued function**

Let the position vector of a point $P(x, y, z)$ in space be

$$\vec{r} = x\,i + y\,j + z\,k$$

If x, y, z are all functions of a single parameter t, then $\vec{r}$ is said to be a vector function of t which is also referred to as a *vector point function* usually denoted as $\vec{r} = \vec{r}\,(t)$. As the parameter t varies, the point P traces a curve in space.

Hence, $$\vec{r} = x(t)\,i + y(t)\,j + z(t)\,k$$

is called as the vector equation of the curve.

Evidently, $$\frac{d\vec{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

and $$\frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k$$

7.11 Scalar and Vector fields

If to every point (x, y, z) of a region R in space there corresponds a scalar $\phi(x, y, z)$ then ϕ is called a ***scalar point function*** and we say that a ***scalar field*** ϕ is defined in R.

Examples : **1.** $\phi = x^2 + y^2 + z^2$ **2.** $\phi = xy^2z^3$

If to every point (x, y, z) of a region R in space there corresponds a vector $\vec{A}\,(x, y, z)$ then $\vec{A}$ is called a ***vector point function*** and we say that a ***vector field*** $\vec{A}$ is defined in R.

Examples : **1.** $\vec{A} = x^2 i + y^2 j + z^2 k$ **2.** $\vec{A} = xyz\, i + yz\, j + z\, k$

Operators (i) The ***vector differential operator*** ∇, ***read as "Nabla" or "Del"*** is defined by,

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \Sigma \frac{\partial}{\partial x} i$$

(ii) The ***Laplacian operator*** ∇^2 is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Sigma \frac{\partial^2}{\partial x^2}$$

7.12 Gradient, Divergence and Curl

Gradient of a scalar field

If $\phi\ (x, y, z)$ is a continuously differentiable scalar function, then the **gradient of** ϕ (grad ϕ *in precise*) is defined to be $\nabla\phi$.

ie., $$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k \quad [\textit{Vector quantity}]$$

Divergence of a vector field

If $\vec{A}(x, y, z)$ is a continuously differentiable vector function, then the **divergence of** $\vec{A}$ (div $\vec{A}$ *in precise*) is defined to be $\nabla \cdot \vec{A}$

If $\vec{A} = a_1 i + a_2 j + a_3 k$, where a_1, a_2, a_3 are all functions of x, y, z then we have

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (a_1 i + a_2 j + a_3 k)$$

ie., $$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \quad [\textit{Scalar quantity}]$$

Curl of a vector field

If $\vec{A}\ (x, y, z)$ is a continuously differentiable vector function then, **curl of** $\vec{A}$ (curl $\vec{A}$ *in precise*) is defined to be $\nabla \times \vec{A}$.

If $\vec{A} = a_1 i + a_2 j + a_3 k$ where a_1, a_2, a_3 are all functions of x, y, z then we have

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\text{curl } \vec{A} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right) \text{ [Vector quantity]}$$

Note : Laplacian : *If $\phi(x, y, z)$ is a continuously differentiable scalar function, we define the Laplacian of ϕ as follows.*

$$\text{Laplacian of } \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \text{ [Scalar quantity]}$$

Obviously Laplacian of a scalar function is a scalar quantity.

Remark : *The computation of gradient, divergence, curl and laplacian is a combination of the concept of vector algebra and partial differentiation.*

Here are few illustrations.

Ex-1. *Given $\phi = x^2 y + y^2 z + z^2 x$ let us find $\nabla\phi$ and $\nabla^2 \phi$.*

$$\nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

ie., $$\nabla\phi = (2xy + z^2)\, i + (x^2 + 2yz)\, j + (y^2 + 2xz)\, k$$

Next, $$\nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 2x + 2y + 2z$$

2. *Given $\vec{A} = x^2 yz\, i + y^2 zx\, j + z^2 xy\, k$ let us find div $\vec{A}$ and curl $\vec{A}$.*

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(x^2 yz\, i + y^2 zx\, j + z^2 xy\, k\right)$$

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial}{\partial x}(x^2 yz) + \frac{\partial}{\partial y}(y^2 zx) + \frac{\partial}{\partial z}(z^2 xy)$$

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & y^2zx & z^2xy \end{vmatrix}$$

$$= i(z^2x - y^2x) - j(z^2y - x^2y) + k(y^2z - x^2z)$$

$$\therefore \quad \text{curl } \vec{A} = x(z^2 - y^2)i + y(x^2 - z^2)j + z(y^2 - x^2)k$$

- **Geometrical meaning of the gradient**

If $\phi(x, y, z)$ is a scalar function, then grad ϕ *is a vector normal to the surface $\phi(x, y, z) = c$.*

Note : 1. *Obviously the unit vector normal $\hat{n}$ along $\nabla\phi$ is given by $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$*

2. *The angle between the two surfaces is defined to be equal to the angle between their normals. If $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$ be the equations of the two surfaces then, $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$, where θ is the angle between the normals.*

If $\theta = \pi/2$ then the surfaces are said to intersect each other orthogonally.

When $\theta = \pi/2$, $\cos\theta = \cos(\pi/2) = 0 \Rightarrow \nabla\phi_1 \cdot \nabla\phi_2 = 0$.

This is the condition for the surfaces to intersect at right angles.

- **Directional derivative**

If $\phi(x, y, z)$ is a scalar function and $\vec{d}$ is a given direction, then $\nabla\phi \cdot \hat{n}$, where $\hat{n} = \vec{d}/|\vec{d}|$ is called as the *directional derivative* of ϕ along $\hat{n}$.

- **Physical interpretation of divergence**

If $\vec{V}(x, y, z)$ represents any physical quantity, the divergence of $\vec{V}$ gives the rate at which the physical quantity is originating at that point per unit volume.

An illustration : Let us suppose that a fluid is moving such that its velocity at any point $P(x, y, z)$ is given by the vector point function $\vec{V}(x, y, z)$. Consider a small parallelopiped of volume $\delta x\,\delta y\,\delta z$ through which the fluid is passing.

If $\vec{V}(x,y,z) = v_1 i + v_2 j + v_3 k$ then $\operatorname{div}\vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$ gives the total gain in the volume of the fluid per unit volume per unit time.

$\operatorname{div}\vec{V} = 0$ is called as the continuity equation of an incompressible fluid.

A vector $\vec{V}$ whose divergence is zero is called a "solenoidal vector".

- **Physical interpretation of curl**

Curl means rotation. A vector function $\vec{V}(x,y,z)$ is said to be **"irrotational"** if curl $\vec{V} = \vec{0}$.

An illustration : Let us suppose that a rigid body is rotating about a fixed axis through a point O. If $\vec{w}$ is the constant angular velocity and $\vec{v}$ is the velocity of a particle at a point $P(x, y, z)$ of the body having the position vector $\vec{r}$ then we know that $\vec{v} = \vec{\omega} \times \vec{r}$.

We can easily show that, curl $\vec{v} = 2\vec{\omega}$ (*Refer* **[12]**)

Thus curl of the velocity vector is equal to twice the angular velocity of rotation. This is an illustration to show that "curl" is analogous to "rotation".

In general we can as well say that the curl of any vector point function will give the measure of the angular velocity at any point.

Summary of all the concepts useful for problems

1. $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \Sigma \frac{\partial}{\partial x} i$; $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Sigma \frac{\partial^2}{\partial x^2}$

2. grad $\phi = \nabla\phi$; $\operatorname{div}\vec{A} = \nabla \cdot \vec{A}$; curl $\vec{A} = \nabla \times \vec{A}$; $\nabla^2\phi = \nabla \cdot \nabla\phi$

3. $\nabla\phi$ is a vector normal to the surface $\phi(x, y, z) = c$ and $\nabla\phi/|\nabla\phi|$ is the unit vector normal to the surface.

4. Directional derivative of $\phi(x, y, z)$ along a given direction $\vec{D}$ is $\nabla\phi \cdot \hat{n}$ where $\hat{n} = \vec{D}/|\vec{D}|$ and also directional derivative is maximum along $\nabla\phi$.

WORKED PROBLEMS

Find the unit vector normal to the following surfaces at the indicated points.

1. $x^2 y + 2xz = 4$ *at* $(2,-2,3)$ **2.** $x y^3 z^2 = 4$ *at* $(-1,-1,2)$

☞ **Solutions**

[1] Let, $\phi = x^2 y + 2xz$. $\nabla\phi$ is a vector normal to the surface.

We have, $\nabla\phi = \dfrac{\partial\phi}{\partial x} i + \dfrac{\partial\phi}{\partial y} j + \dfrac{\partial\phi}{\partial z} k$

ie., $\nabla\phi = (2xy + 2z) i + x^2 j + 2x k$

$$[\nabla\phi]_{(2,-2,3)} = -2i + 4j + 4k = 2(-i + 2j + 2k)$$

The required unit vector normal, $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Thus, $\hat{n} = \dfrac{2(-i + 2j + 2k)}{\sqrt{2^2(1+4+4)}} = \boxed{\dfrac{-i + 2j + 2k}{3}}$

[2] Let, $\phi = x y^3 z^2$. $\nabla\phi$ is a vector normal to the surface,

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

ie., $\nabla\phi = y^3 z^2 i + 3xy^2 z^2 j + 2xy^3 z k$

$$[\nabla\phi]_{(-1,-1,2)} = -4i - 12j + 4k = -4(i + 3j - k)\ ;\ \hat{n} = \nabla\phi/|\nabla\phi|$$

Thus $\hat{n} = \dfrac{-4(i + 3j - k)}{\sqrt{4^2(1+9+1)}} = \boxed{\dfrac{-(i + 3j - k)}{\sqrt{11}}}$

Find the directional derivative of the following. [Dec 17]

3. $\phi = x^2 yz + 4xz^2$ *at* $(1,-2,-1)$ *along* $2i - j - 2k$

4. $\phi = 4xz^3 - 3x^2y^2z$ *at* $(2,-1,2)$ *along* $2i - 3j + 6k$

☞ **Solutions**

[3] $\phi = x^2yz + 4xz^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\nabla\phi = (2xyz + 4z^2)i + (x^2 z)j + (x^2 y + 8xz)k$$

ie., $[\nabla\phi]_{(1,-2,-1)} = 8i - j - 10k$

The unit vector in the direction of $2i - j - 2k$ is,

$\hat{n} = \dfrac{2i - j - 2k}{\sqrt{4+1+4}} = \dfrac{2i - j - 2k}{3}$; The required directional derivative is

$$\nabla\phi \cdot \hat{n} = (8i - j - 10k) \cdot \frac{(2i - j - 2k)}{3}$$

Thus $\nabla\phi \cdot \hat{n} = \dfrac{(8)(2) + (-1)(-1) + (-10)(-2)}{3} = \boxed{\dfrac{37}{3}}$

[4] $\phi = 4xz^3 - 3x^2y^2z$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

ie., $\nabla\phi = (4z^3 - 6xy^2 z)i - (6x^2yz)j + (12xz^2 - 3x^2 y^2)k$

$[\nabla\phi]_{(2,-1,2)} = 8i + 48j + 84k$

The unit vector in the direction of $2i - 3j + 6k$ is,

$\hat{n} = \dfrac{2i - 3j + 6k}{\sqrt{4+9+36}} = \dfrac{2i - 3j + 6k}{7}$; The required directional derivative is

$$\nabla\phi \cdot \hat{n} = (8i + 48j + 84k) \cdot \frac{2i - 3j + 6k}{7} = \frac{16 - 144 + 504}{7} = \boxed{\frac{376}{7}}$$

[5] *Find the angle between the surfaces* $x^2 + y^2 + z^2 = 9$ *and* $z = x^2 + y^2 - 3$ *at* (2,–1,2). **[June 17, 18, 19, Feb 21]**

☞ The angle between the surfaces is defined to be equal to the angle between their normals and we know that $\nabla\phi$ is a vector normal to the surface. We have the equation of the two surfaces given by,

$x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$

Let, $\phi_1 = x^2+y^2+z^2$ and $\phi_2 = x^2+y^2-z$

We have, $\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$\therefore \quad \nabla\phi_1 = 2xi+2yj+2zk$ and $\nabla\phi_2 = 2xi+2yj-k$

$\therefore \quad [\nabla\phi_1]_{(2,-1,2)} = 4i-2j+4k = 2(2i-j+2k)$

$[\nabla\phi_2]_{(2,-1,2)} = 4i-2j-k$

If θ is the angle between these two normals, $\cos\theta = \frac{\nabla\phi_1\cdot\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$\therefore \quad \cos\theta = \frac{2(8+2-2)}{\sqrt{36}\sqrt{21}} = \frac{8}{3\sqrt{21}} \Rightarrow \boxed{\theta = \cos^{-1}(8/3\sqrt{21})}$$

[6] *Find the angle between the normals to the surface* $xy = z^2$ *at the points* $(4,1,2)$ *and* $(3,3,-3)$.

☞ Let $\phi = xy - z^2$ and we know that $\nabla\phi$ is a vector normal to the surface.

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = yi+xj-2zk$$

$\therefore \quad [\nabla\phi]_{(4,1,2)} = i+4j-4k \cdots \vec{A}\,(say)$

$[\nabla\phi]_{(3,3,-3)} = 3i+3j+6k = 3(i+j+2k) \cdots \vec{B}\,(say)$

If θ is the angle between $\vec{A}$ and $\vec{B}$, $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$

Now, $\cos\theta = \frac{3(1+4-8)}{\sqrt{1+16+16}\sqrt{3^2(1+1+4)}} = \frac{-3}{\sqrt{33}\sqrt{6}} = \frac{-1}{\sqrt{22}}$

Thus, $\boxed{\cos\theta = -\frac{1}{\sqrt{22}} \text{ or } \theta = \pi \pm \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)}$

[7] *Find the value of the constants a and b such that the surfaces* $ax^2 - byz = (a+2)x$ *and* $4x^2y + z^3 = 4$ *are orthogonal at the point* $(1, -1, 2)$.

☞ First we have to ensure that the given point lies on both the surfaces.

Substituting $(1, -1, 2)$ onto the equation $ax^2 - byz = (a+2)x$ we obtain

$$a + 2b = a + 2 \Rightarrow 2b = 2 \text{ or } b = 1$$

Also if $(1, -1, 2)$ is substituted onto the LHS of the equation $4x^2y + z^3 = 4$ we get 4 which is equal to the RHS.

$\Rightarrow$ the given point lies on both the surfaces when $b = 1$.

In order to find 'a' we have to use the orthogonality condition $\nabla\phi_1 \cdot \nabla\phi_2 = 0$ where,

$$\phi_1 = ax^2 - byz - (a+2)x \text{ and } \phi_2 = 4x^2y + z^3$$

Now, $\nabla\phi_1 = \{2ax - (a+2)\}i + (-bz)j + (-by)k$

$$\nabla\phi_2 = 8xyi + 4x^2 j + 3z^2 k$$

$$[\nabla\phi_1]_{(1,-1,2)} = (a-2)i - 2bj + bk \; ; \; [\nabla\phi_2]_{(1,-1,2)} = -8i + 4j + 12k$$

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0 \text{ gives } -8a + 4b + 16 = 0.$$

ie., But $b = 1$ and hence $a = 5/2$.

Thus $\boxed{a = 5/2,\ b = 1}$

[8] *If* $\vec{A} = 2x^2 i - 3yz j + xz^2 k$ *and* $\phi = 2z - x^3 y$, *compute* $\vec{A} \cdot \nabla\phi$ *and* $\vec{A} \times \nabla\phi$ *at* $(1, -1, 1)$.

☞ $$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k \; ; \; \phi = 2z - x^3 y$$

$$\nabla\phi = -3x^2 y i - x^3 j + 2k \; ; \; [\nabla\phi]_{(1,-1,1)} = 3i - j + 2k$$

Now, $[\vec{A}]_{(1,-1,1)} = 2i + 3j + k$

$$\therefore \quad [\vec{A} \cdot \nabla\phi]_{(1,-1,1)} = (3)(2) + (-1)(3) + (2)(1) = \boxed{5}$$

Also, $$\vec{A} \times \nabla\phi = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix} = i(6+1) - j(4-3) + k(-2-9) = \boxed{7i - j - 11k}$$

[9] *Find* div $\vec{F}$ *and* curl $\vec{F}$ *where* $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

[June 16, 19, Dec 19]

☞ Let, $\phi = x^3 + y^3 + z^3 - 3xyz$

$$\vec{F} = \text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\therefore \quad \vec{F} = (3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k$$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \{(3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k\}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy) = \boxed{6(x + y + z)}$$

Also, $$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right\}$$

$$-j\left\{\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right\} + k\left\{\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right\}$$

$$\nabla \times \vec{F} = i\{-3x - (-3x)\} - j\{-3y - (-3y)\} + k\{-3z - (-3z)\} = \boxed{\vec{0}}$$

[10] *If* $\vec{F} = \nabla(xy^3z^2)$ *find* div $\vec{F}$ *and* curl $\vec{F}$ *at the point* $(1, -1, 1)$.

[Dec 17, June 18, Sep 20]

☞ Let $\phi = xy^3z^2$

$$\therefore \quad \vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = y^3z^2 i + 3xy^2z^2 j + 2xy^3z k$$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (y^3z^2 i + 3xy^2z^2 j + 2xy^3z k)$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(y^3 z^2) + \frac{\partial}{\partial y}(3xy^2 z^2) + \frac{\partial}{\partial z}(2xy^3 z)$$

$$\nabla\cdot\vec{F} = 2xy(3z^2 + y^2)\ ;\ \text{div}\,\vec{F} \text{ at } (1,-1,1) = \boxed{-8}$$

Also, $$\text{curl}\ \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 z^2 & 3xy^2 z^2 & 2xy^3 z \end{vmatrix}$$

$$\nabla\times\vec{F} = i(6xy^2 z - 6xy^2 z) - j(2y^3 z - 2y^3 z) + k(3y^2 z^2 - 3y^2 z^2) = \boxed{\vec{0}}$$

[11] *If* $\vec{F} = (x+y+1)i + j - (x+y)k$, *show that* $\vec{F}\cdot\text{curl}\ \vec{F} = 0$

[Dec 17, June 18]

☞ $$\text{curl}\ \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+1) & 1 & -x-y \end{vmatrix}$$

$$\text{curl}\ \vec{F} = i(-1-0) - j(-1-0) + k(0-1)$$

ie., $$\text{curl}\ \vec{F} = -i + j - k$$

$\therefore$ $$\vec{F}\cdot\text{curl}\ \vec{F} = \{(x+y+1)i + j - (x+y)k\}\cdot(-i+j-k)$$

ie., $$\vec{F}\cdot\text{curl}\ \vec{F} = (x+y+1)(-1) + (1)(1) + (x+y)(1)$$

$$\vec{F}\cdot\text{curl}\ \vec{F} = -x - y - 1 + 1 + x + y = \boxed{0}$$

[12] *If* $\vec{V} = \vec{\omega}\times\vec{r}$ *prove that* curl $\vec{V} = 2\vec{\omega}$ *where* $\vec{\omega}$ *is a constant vector.*

☞ Let $\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$; $\vec{r} = xi + yj + zk$

$$\vec{V} = \vec{\omega}\times\vec{r} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \Sigma i\ (\omega_2 z - \omega_3 y)$$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$\text{curl } \vec{V} = \Sigma i \left(\omega_1 - (-\omega_1) \right) = \Sigma 2\omega_1 i = 2(\omega_1 i + \omega_2 j + \omega_3 k) = \boxed{2\vec{\omega}}$$

Remark : *We have referred to this example while giving the physical meaning of "curl". The theoretical version of the problem is as follows :*
When a rigid body is in motion, the angular velocity is equal to half the curl of its linear velocity at any point.

If $\vec{r} = xi + yj + zk$ *and* $r = |\vec{r}|$ *prove the following.*

13. $\nabla(r^n) = n r^{n-2} \vec{r}$

14. $\nabla \cdot (r^n \vec{r}) = (n+3) r^n$

15. $\nabla \times (r^n \vec{r}) = \vec{0}$

16. $\nabla^2 (r^n) = n(n+1) r^{n-2}$ **[June 16]**

Note : *These are also a set of problems based on the aspect of symmetry and the problems are worked in the general form involving "n". Particular cases for n can also be asked. The notation and meaning concerning* $\vec{r}$ *and r are standard which need not be explicitly mentioned while asking these problems or particular cases of these.*

☞ **Solutions**

[13] $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow$ $r^2 = x^2 + y^2 + z^2$ and differentiating partially *w.r.t* x we get,

$$2r \frac{\partial r}{\partial x} = 2x \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} \text{ Also, } \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

Note : *The expression for these three partial derivatives will be used in all the problems and the same will not be worked in every problem.*

$$\nabla(r^n) = \left(\Sigma \frac{\partial}{\partial x} i \right)(r^n) = \Sigma n r^{n-1} \frac{\partial r}{\partial x} i = \Sigma n r^{n-1} \left(\frac{x}{r} \right) i$$

$$\nabla(r^n) = \Sigma n r^{n-2} x i = n r^{n-2} \Sigma x i = \boxed{n r^{n-2} \vec{r}}$$

[14] $r^n \vec{r} = r^n \Sigma xi = \Sigma (r^n x) i$

$$\nabla \cdot (r^n \vec{r}) = \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma (r^n x) i = \Sigma \frac{\partial}{\partial x} (r^n x)$$

$\therefore$

$$\nabla \cdot (r^n \vec{r}) = \Sigma \left(r^n + n r^{n-1} \frac{\partial r}{\partial x} x \right) = \Sigma \left(r^n + n r^{n-1} \frac{x}{r} x \right) = \Sigma (r^n + n r^{n-2} x^2)$$

$$\nabla \cdot (r^n \vec{r}) = (r^n + n r^{n-2} x^2) + (r^n + n r^{n-2} y^2) + (r^n + n r^{n-2} z^2)$$

$$\nabla \cdot (r^n \vec{r}) = 3r^n + n r^{n-2} (x^2 + y^2 + z^2) = 3r^n + n r^{n-2} r^2 = \boxed{(n+3) r^n}$$

[15] $r^n \vec{r} = r^n \Sigma xi = \Sigma (r^n x) i$

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix} = \Sigma i \left\{ \frac{\partial}{\partial y} (r^n z) - \frac{\partial}{\partial z} (r^n y) \right\}$$

$$\nabla \times (r^n \vec{r}) = \Sigma i \left\{ n r^{n-1} \frac{\partial r}{\partial y} z - n r^{n-1} \frac{\partial r}{\partial z} y \right\}$$

$$\nabla \times (r^n \vec{r}) = \Sigma i \left\{ n r^{n-1} \frac{y}{r} z - n r^{n-1} \frac{z}{r} y \right\}$$

$$\nabla \times (r^n \vec{r}) = \Sigma i (n r^{n-2} yz - n r^{n-2} yz) = \boxed{\vec{0}}$$

[16] $\nabla^2 (r^n) = \Sigma \frac{\partial^2}{\partial x^2} (r^n) = \Sigma \frac{\partial}{\partial x} \frac{\partial}{\partial x} (r^n)$

$$\nabla^2 (r^n) = \Sigma \frac{\partial}{\partial x} \left(n r^{n-1} \frac{\partial r}{\partial x} \right) = \Sigma \frac{\partial}{\partial x} \left\{ n r^{n-1} \left(\frac{x}{r} \right) \right\}$$

$$\nabla^2 (r^n) = \Sigma \frac{\partial}{\partial x} (n r^{n-2} x) = n \Sigma \left(r^{n-2} + (n-2) r^{n-3} \frac{\partial r}{\partial x} x \right)$$

$$\nabla^2 (r^n) = n \Sigma \left(r^{n-2} + (n-2) r^{n-3} \frac{x}{r} x \right) = n \Sigma \left(r^{n-2} + (n-2) r^{n-4} x^2 \right)$$

$$\nabla^2(r^n) = n\{(r^{n-2}+(n-2)r^{n-4}x^2)+(r^{n-2}+(n-2)r^{n-4}y^2)$$
$$+(r^{n-2}+(n-2)r^{n-4}z^2)\}$$

$$\nabla^2(r^n) = n\{3r^{n-2}+(n-2)r^{n-4}(x^2+y^2+z^2)\}$$

$$\nabla^2(r^n) = n\{3r^{n-2}+(n-2)r^{n-4}r^2\} = \boxed{n(n+1)r^{n-2}}$$

Note : *A few important particular cases of these are given and is left as an exercise for the reader to work out these independently.*

Prove with usual meaning the following :

(i) $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$ (ii) $\nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = 0$ (iii) $\nabla^2\left(\frac{1}{r}\right) = 0$

7.13 Solenoidal and Irrotational vectors.

We have already referred to these while giving the physical meaning of divergence and curl.

A vector $\vec{F}$ is said to be **solenoidal if** div $\vec{F} = 0$ and **irrotational if** curl $\vec{F} = \vec{0}$.

Irrotational vector field is also called as *conservative field* or *potential field.*

When $\vec{F}$ is irrotational there always exists a scalar point function ϕ such that $\nabla\phi = \vec{F}$. Then ϕ is called a **scalar potential** of $\vec{F}$.

WORKED PROBLEMS

[17] *Show that* $\vec{F} = (y+z)i+(z+x)j+(x+y)k$ *is irrotational. Also find a scalar function* ϕ *such that* $\vec{F} = \nabla\phi$. **[Dec 17, June 18, Feb 21]**

☞ We have to show that curl $\vec{F} = \nabla\times\vec{F} = \vec{0}$.

$$\therefore \quad curl\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$$

$$\text{curl}\,\vec{F} = i(1-1)-j(1-1)+k(1-1) = \boxed{\vec{0}}$$

Now let us consider, $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (y+z)i + (z+x)j + (x+y)k$$

ie.,

$$\frac{\partial\phi}{\partial x} = y+z \qquad \therefore \phi = \int (y+z)\,dx + f_1(y,z)$$

$$\Rightarrow \quad \phi = xy + xz + f_1(y,z) \qquad \ldots(1)$$

ie.,

$$\frac{\partial\phi}{\partial y} = z+x \qquad \therefore \phi = \int (z+x)\,dy + f_2(x,z)$$

$$\phi = yz + xy + f_2(x,z) \qquad \ldots(2)$$

ie.,

$$\frac{\partial\phi}{\partial z} = x+y \qquad \therefore \phi = \int (x+y)\,dz + f_3(x,y)$$

$$\phi = xz + yz + f_3(x,y) \qquad \ldots(3)$$

ie.,

[Now we need to suitably choose the arbitrary functions $f_1(y,z), f_2(x,z), f_3(x,y)$ such that we obtain an unique expression for ϕ. To choose $f_1(y,z)$ we look into the equations (2) and (3) and select terms which do not contain x. It can be terms with y or z or y and z. Similarly we have to choose $f_2(x,z)$ from (1) and (3), $f_3(x,y)$ from (1) and (2)] ***It is the union of (1),(2), (3) avoiding repetitions.***

Let us choose, $f_1(y,z) = yz, f_2(z,x) = xz, f_3(x,y) = xy$ from (1), (2) & (3).

Thus the required $\boxed{\phi = xy + yz + zx}$

[18] *Show that* $\vec{F} = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)j + (2y^2z + xy)k$ *is a conservative force field. Find its scalar potential.*

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy^2+yz) & (2x^2y+xz+2yz^2) & (2y^2z+xy) \end{vmatrix}$$

$$\nabla\times\vec{F} = i(4yz + x - x - 4yz) - j(y-y) + k(4xy + z - 4xy - z) = \vec{0}$$

Thus, $\boxed{\vec{F} \text{ is conservative.}}$

Now we have to find ϕ such that $\nabla\phi = \vec{F}$.

Proceeding as in **[17]** we obtain, $\boxed{\phi = x^2y^2 + y^2z^2 + xyz}$

[19] *Find the value of the constant 'a' such that the vector field,* $\vec{F} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$ *is irrotational and hence find a scalar function* ϕ *such that* $\vec{F} = \nabla\phi$. **[Sep 20]**

☞ We have to find 'a' such that curl $\vec{F} = \vec{0}$.

$$ie., \quad \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \vec{0}$$

$$ie., \quad i(0-0) - j\{(1-a)z^2 + 3z^2\} + k\{(a-2)2x - ax\} = \vec{0}$$

$$ie., \quad (a-4)z^2 j + (a-4)xk = \vec{0} \Rightarrow \boxed{a = 4}$$

The above equation is identically satisfied when $a = 4$.

Now consider, $\nabla\phi = (\vec{F})_{a=4}$

$$ie., \quad \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = (4xy - z^3)i + 2x^2 j - 3xz^2 k \text{ ; we have,}$$

$$\frac{\partial\phi}{\partial x} = 4xy - z^3 \quad \therefore \quad \phi = 2x^2y - xz^3 + f_1(y,z), \text{ on integration w.r.t.} x. \quad \ldots(1)$$

$$\frac{\partial\phi}{\partial y} = 2x^2 \quad \therefore \quad \phi = 2x^2y + f_2(x,z), \text{ on integration w.r.t.} y. \quad \ldots(2)$$

$$\frac{\partial\phi}{\partial z} = -3xz^2 \quad \therefore \quad \phi = -xz^3 + f_3(x,y) \text{ on integration w.r.t.} z. \quad \ldots(3)$$

Let, $f_1(y,z) = 0$, $f_2(x,z) = -xz^3$, $f_3(x,y) = 2x^2y$, from (1), (2) & (3).

Thus $\boxed{\phi = 2x^2y - xz^3}$

[20] *Find constants a and b such that* $\vec{F} = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ *is irrotational. Also find a scalar function* ϕ *such that* $\vec{F} = \nabla\phi$. **[Dec 19]**

☞ We have to find a and b such that curl $\vec{F} = \vec{0}$.

Proceeding as in **[19]** we obtain $\boxed{a = 6, b = 3. ; \phi = 3x^2y + xz^3 - yz}$

[21] *If* $\vec{F} = (x+y+az)i + (bx+2y-z)j + (x+cy+2z)k$ *find* a, b, c *such that* curl $\vec{F} = \vec{0}$ *and then find* ϕ *such that* $\vec{F} = \nabla\phi$. **[June 18]**

☞ $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+az) & (bx+2y-z) & (x+cy+2z) \end{vmatrix} = \vec{0}$$

ie., $$i(c+1) - j(1-a) + k(b-1) = \vec{0} \Rightarrow \boxed{a=1, b=1, c=-1}$$

Now consider. $\nabla\phi = \vec{F}$ when $a=1, b=1, c=-1$.

ie., $$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (x+y+z)i + (x+2y-z)j + (x-y+2z)k$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x} = x+y+z \quad \therefore \quad \phi = \int (x+y+z)\, dx + f_1(y,z)$$

ie., $$\phi = \frac{x^2}{2} + xy + xz + f_1(y,z) \qquad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = x+2y-z \quad \therefore \quad \phi = \int (x+2y-z)\, dy + f_2(x,z)$$

ie., $$\phi = xy + y^2 - yz + f_2(x,z) \qquad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = x-y+2z \quad \therefore \quad \phi = \int (x-y+2z)\, dz + f_3(x,y)$$

ie., $$\phi = xz - yz + z^2 + f_3(x,y) \qquad \ldots (3)$$

Thus the required, $\boxed{\phi = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2}$

EXERCISE

Find the unit vector normal to the following surfaces at the indicated points

1. $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$.
2. $x^2 y + y^2 z + z^2 x = 5$ at $(1, -1, 2)$.

Find the directional derivatives of the following :

3. $\phi = xy + yz + zx$ at $(1, 2, 3)$ along $3i + 4j + 5k$.
4. $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ along $i + 2j + 2k$.
5. If $\vec{A} = xy^2 i + 2x^2 yz\, j - 3y^2 z\, k$ find div $\vec{A}$, curl $\vec{A}$ and div (curl $\vec{A}$) at the point $(2, 1, 1)$.
6. Find the values of the constants a, b, c such that

$$\vec{F} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k \text{ is conservative.}$$

Also find its scalar potential.

ANSWERS

1. $\dfrac{-i + 3j + 2k}{\sqrt{14}}$

2. $\dfrac{2i - 3j + 5k}{\sqrt{38}}$

3. $\dfrac{46}{5\sqrt{2}}$

4. $-\dfrac{11}{3}$

5. $6,\ -11i + 4j,\ 0$

6. $a = 4,\ b = 2,\ c = -1\ ;\ \phi = \dfrac{x^2}{2} + 2xy + 4xz - \dfrac{3y^2}{2} - yz + z^2$

7.2 Vector Integration (*Not included for CS stream*)

7.21 Vector Line Integral

Consider a curve C in space which consists of infinitesimally small line elements of length dr. Then the line integral of a vector $\vec{A}(x,y,z)$ along the curve C is defined to be the sum of the scalar products of $\vec{A}$ and $d\vec{r}$ and is represented by $\int_C \vec{A}\cdot d\vec{r}$

If C is a closed curve which do not intersect any where, the line integral around C is denoted by $\int_C \vec{A}\cdot d\vec{r}$

If $\vec{F}$ is the force acted upon by a particle in displacing it along the curve C then $\int_C \vec{F}\cdot d\vec{r}$ represents the *total work done* by the force. It also represents the *circulation* of $\vec{F}$ about C where $\vec{F}$ represents the velocity of the fluid.

$\vec{F}$ is said to be *irrotational* if $\int_C \vec{F}\cdot d\vec{r} = 0$.

WORKED PROBLEMS

[22] *If* $\vec{F} = xy\,i + yz\,j + zx\,k$, *evaluate* $\int_C \vec{F}\cdot d\vec{r}$ *where C is the curve represented by* $x = t,\ y = t^2,\ z = t^3,\ -1 \le t \le 1$. **[Dec 19]**

We have $\vec{F} = xy\,i + yz\,j + zx\,k$ and $\vec{r} = x\,i + y\,j + z\,k$ will give

$$d\vec{r} = dx\,i + dy\,j + dz\,k$$

$$\therefore \quad \vec{F}\cdot d\vec{r} = xy\,dx + yz\,dy + zx\,dz$$

Since, $x = t,\ y = t^2, z = t^3$ by data, we obtain

$$dx = dt,\ dy = 2t\,dt,\ dz = 3t^2\,dt$$

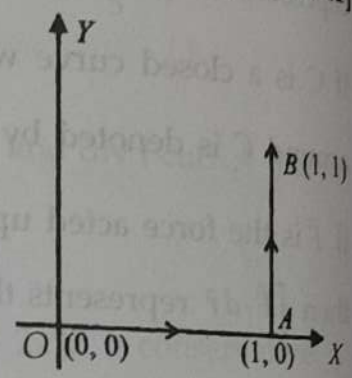
$$\therefore \quad \vec{F}\cdot d\vec{r} = t^3\,dt + t^5(2t)\,dt + t^4(3t^2)\,dt$$

ie., $\vec{F}\cdot d\vec{r} = (t^3 + 2t^6 + 3t^6)\,dt = (t^3 + 5t^6)\,dt$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{t=-1}^{1} (t^3 + 5t^6)\,dt = \left[\frac{t^4}{4}\right]_{-1}^{1} + 5\left[\frac{t^7}{7}\right]_{-1}^{1} = 0 + (10/7) = \boxed{10/7}$$

[23] *Evaluate* $\int_C \vec{F}\cdot d\vec{r}$ *where* $\vec{F} = xy\,i + (x^2 + y^2)\,j$ *along*

(i) the path of the straight line from $(0, 0)$ *to* $(1, 0)$ *and then to* $(1, 1)$. **[Feb 21]**
(ii) the straight line joining the origin and $(1, 2)$.



☞ $$\int_C \vec{F}\cdot d\vec{r} = \int_C xy\,dx + (x^2 + y^2)\,dy \quad \ldots (1)$$

(i) $$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} \quad \ldots (2)$$

Along $OA : y = 0$. $\therefore dy = 0$ and $0 \le x \le 1$

From (1), $$\int_{OA} \vec{F}\cdot d\vec{r} = 0 \quad \ldots (3)$$

Along $AB : x = 1$ $\therefore dx = 0$ and $0 \le y \le 1$. Again from (1)

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{y=0}^{1} 0 + (1 + y^2)\,dy = \left[y + \frac{y^3}{3}\right]_{y=0}^{1} = 1 + \frac{1}{3} = \frac{4}{3} \quad \ldots (4)$$

Using (3) and (4) in (2) we obtain $\int_C \vec{F}\cdot d\vec{r} = 0 + \frac{4}{3} = \boxed{\frac{4}{3}}$

(ii) C is the straight line joining $(0, 0)$ and $(1, 2)$ and it's equation is,

$$\frac{y-0}{x-0} = \frac{2-0}{1-0} \quad \text{or} \quad y = 2x \quad \therefore dy = 2\,dx \;;\; 0 \le x \le 1$$

From (1), $$\int_C \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} x\cdot 2x\,dx + (x^2 + 4x^2)\,2\,dx = \int_0^1 12x^2\,dx = \boxed{4}$$

[24] *If* $\vec{F} = (3x^2 + 6y)i - 14yz\,j + 20xz^2\,k$, *evaluate* $\int \vec{F}\cdot d\vec{r}$ *from* $(0, 0, 0)$ *to* $(1, 1, 1)$ *along the curve given by* $x = t,\ y = t^2,\ z = t^3$. **[Sep 20]**

☞ $\vec{F} = (3x^2 + 6y)i - 14yz\,j + 20\,xz^2\,k$

$d\vec{r} = dx\,i + dy\,j + dz\,k$

$\therefore \quad \vec{F}\cdot d\vec{r} = (3x^2 + 6y)\,dx - 14yz\,dy + 20\,xz^2\,dz$

Since $x = t,\ y = t^2,\ z = t^3$ we obtain $dx = dt,\ dy = 2t\,dt,\ dz = 3t^2\,dt$

$\therefore \quad \vec{F}\cdot d\vec{r} = (3t^2 + 6t^2)\,dt - (14t^5)\,2t\,dt + (20\,t^7)\,3t^2\,dt$

ie., $\quad \vec{F}\cdot d\vec{r} = (9t^2 - 28t^6 + 60t^9)\,dt\ ;\ 0 \le t \le 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt = \boxed{5} \text{ on integration.}$$

[25] *Find the total work done by the force represented by* $\vec{F} = 3xy\,i - y\,j + 2zx\,k$ *in moving a particle round the circle* $x^2 + y^2 = 4$.

☞ Total work done, $W = \int_C \vec{F}\cdot d\vec{r}$

$x^2 + y^2 = 4$ can be represented in the parametric form $x = 2\cos\theta$, $y = 2\sin\theta$ and $z = 0$. $0 \le \theta \le 2\pi$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int 3xy\,dx - y\,dy + 2zx\,dz$$

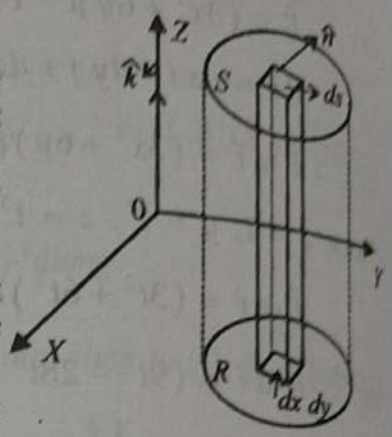
$$W = \int_{\theta=0}^{2\pi} 3(4\cos\theta\sin\theta)(-2\sin\theta)\,d\theta - \int_{\theta=0}^{2\pi} 4\sin\theta\cos\theta\,d\theta$$

$$= -24\int_0^{2\pi} \sin^2\theta\cos\theta\,d\theta - 2\int_0^{2\pi} \sin 2\theta\,d\theta$$

$$W = -24\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} - 2\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi} = \boxed{0}$$

- **Surface Integral**

Surface integral



An integral evaluated over a surface is called a surface integral. Consider a surface S and a point P on it. Let $\vec{A}$ be a vector function of x, y, z defined and continuous over S.

If $\hat{n}$ is the unit outward normal to the surface S at P then the integral of the normal component of $\vec{A}$ at P $(ie., \vec{A}\cdot\hat{n})$ *over the surface S is called the* surface integral *written as* $\iint_S \vec{A}\cdot\hat{n}\,ds$ *where ds is the small elemental area. To evaluate a surface integral we have to find the double integral over the orthogonal projection of the surface on one of the coordinate planes.* Suppose R is the orthogonal projection of S on the $x\,o\,y$ plane and $\hat{n}$ is the unit outward normal to S then it should be noted that $\hat{n}\cdot\hat{k}\,ds$ ($\hat{k}$ *being the unit vector along z-axis*) is the projection of the vectorial area element $\hat{n}\,ds$ on the $x\,o\,y$ plane and this projection is equal to $dx\,dy$ which being the area element in the $x\,o\,y$ plane. That is to say that $\hat{n}\cdot\hat{k}\,ds = dx\,dy$. Similarly we can argue to state that $\hat{n}\cdot\hat{j}\,ds = dz\,dx$ and $\hat{n}\cdot\hat{i}\,ds = dy\,dz$. All these three results hold good if we write

$$\hat{n}\,ds = dy\,dz\,i + dz\,dx\,j + dx\,dy\,k$$

Sometimes we also write, $\overrightarrow{ds} = \hat{n}\,ds = \sum dy\,dz\,i$

7.22 Integrals Theorems

- **Green's theorem in a plane**

Statement : *If R is a closed region of the x - y plane bounded by a simple closed curve C and if M and N are two continuous functions of x, y having continuous first order partial deriviatives in the region R then*

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

- **Stoke's theorem**

Statement : *If S is a surface bounded by a simple closed curve C and if $\vec{F}$ is any continuously differentiable vector function then*

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

WORKED PROBLEMS

[26] *Verify Green's theorem in a plane for* $\int (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, *where C is the boundary of the region enclosed by* $y = \sqrt{x}$ *and* $y = x^2$. **[June 18, Feb 21]**

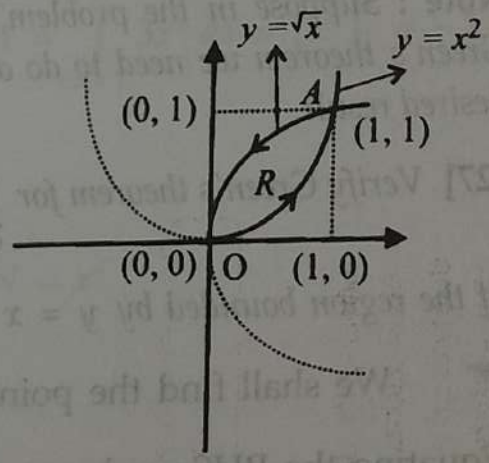
☞ We shall find the points of intersection of the parabolas $y = \sqrt{x}$ & $y = x^2$.

ie., $\sqrt{x} = x^2 \Rightarrow x = x^4$ or $x(x^3 - 1) = 0$

$\therefore$ $x = 0, 1$ and hence $y = 0, 1$. The points of are $(0, 0)$ and $(1, 1)$.

Let, $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y\,;\ \frac{\partial N}{\partial x} = -6y$$



We have Green's theorem in a plane,

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\text{LHS} = \int_C Mdx + Ndy$$

$$= \int_{OA} Mdx + Ndy + \int_{AO} Mdx + Ndy = I_1 + I_2\ (say)$$

Along OA: $y = x^2$, $dy = 2x\,dx$, x varies from 0 to 1.

$$I_1 = \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)(2x)\,dx$$

$$= \int_{x=0}^{1} (3x^2 + 8x^3 - 20x^4)\,dx = [x^3 + 2x^4 - 4x^5]_0^1 = -1$$

Along $AO: y = \sqrt{x}$ or $x = y^2 \Rightarrow dx = 2y\,dy$, y varies from 1 to 0.

$$I_2 = \int_{y=1}^{0} (3y^4 - 8y^2)(2y)\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\,dy = \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0 = \frac{5}{2}$$

Hence, LHS $= I_1 + I_2 = -1 + (5/2) = \boxed{3/2}$

Also, RHS $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (-6y + 16y)\,dy\,dx$

$$\text{RHS} = \int_{x=0}^{1} 10\left[\frac{y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx = 5\int_{x=0}^{1} (x - x^4)\,dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \boxed{\frac{3}{2}}$$

Thus we have $\boxed{\text{LHS} = 3/2 = \text{RHS}}$ and hence the theorem is verified.

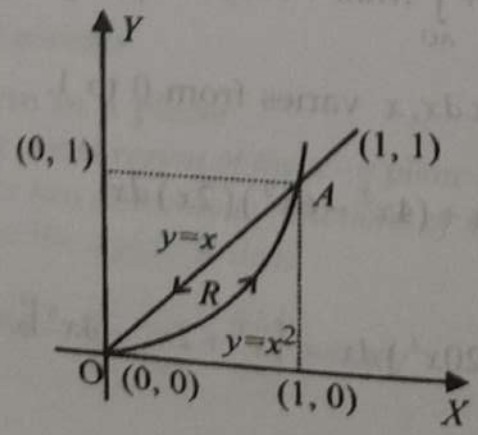
Note : *Suppose in the problem, we are asked to evaluate the line integral using Green's theorem we need to do only the RHS part of the theorem for obtaining the desired result.*

[27] *Verify Green's theorem for* $\int_C (xy + y^2)\,dx + x^2\,dy$ *where C is the closed curve of the region bounded by* $y = x$ and $y = x^2$. **[Dec 16, June 17, 19]**

☞ We shall find the points of intersection of $y = x$ and $y = x^2$.

Equating the RHS we have, $x = x^2$ or $x(1 - x) = 0 \Rightarrow x = 0, 1$

$\therefore$ $y = 0, 1$ and hence $(0, 0)$, $(1, 1)$ are the points of intersection.



We have Green's theorem in a plane,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

$$\int_C (xy + y^2)\, dx + x^2\, dy = \int_{OA} (xy + y^2)\, dx + x^2\, dy + \int_{AO} (xy + y^2)\, dx + x^2\, dy$$

Along OA we have $y = x^2 \therefore dy = 2x\, dx$ and x varies from 0 to 1.

First integral, $I_1 = \int_{x=0}^{1} (x \cdot x^2 + x^4)\, dx + x^2 \cdot 2x\, dx$

$$I_1 = \int_{x=0}^{1} (3x^3 + x^4)\, dx = 3\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Next, along AO we have $y = x \therefore dy = dx$ and x varies from 1 to 0.

Second integral, $I_2 = \int_{x=1}^{0} (x \cdot x + x^2)\, dx + x^2\, dx = \int_{x=1}^{0} 3x^2\, dx = [x^3]_1^0 = -1$

Hence, LHS $= I_1 + I_2 = \frac{19}{20} - 1 = \boxed{\frac{-1}{20}}$

To evaluate the RHS, we have $M = xy + y^2$, $N = x^2$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - (x + 2y) = x - 2y$$

R is the region bounded by $y = x^2$ and $y = x$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dy\, dx, \text{ from the figure.}$$

ie., $= \int_{x=0}^{1} \left[xy - y^2 \right]_{y=x^2}^{x} dx = \int_{x=0}^{1} (x^4 - x^3)\, dx = \boxed{\frac{-1}{20}}$

Thus the theorem is verified.

[28] *Employ Green's theorem in a plane to show that the area enclosed by a plane curve C is* $\frac{1}{2}\int_C x\, dy - y\, dx$ *and hence find the area of the ellipse* $x^2/a^2 + y^2/b^2 = 1$.

☞ We have Green's theorem in a plane,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \quad \cdots (1)$$

Area enclosed by a plane curve C is $\iint_R dx\, dy$ $\cdots (2)$

Taking $N = \frac{x}{2}$, $M = \frac{-y}{2}$ we obtain, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1.$

Hence (1) becomes,

$$\frac{1}{2}\int_C x\, dy - y\, dx = \iint_R dx\, dy = \text{Area}(A)$$

Now to find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ we have,

$x = a\cos\theta,\ y = b\sin\theta;\ 0 \le \theta \le 2\pi.\ dx = -a\sin\theta\, d\theta,\ dy = b\cos\theta\, d\theta$

$$\therefore \quad A = \frac{1}{2}\int_{\theta=0}^{2\pi} [a\cos\theta \cdot b\cos\theta - b\sin\theta(-a\sin\theta)]\, d\theta$$

$$ie., \quad A = \frac{1}{2}\int_{\theta=0}^{2\pi} ab(\cos^2\theta + \sin^2\theta)\, d\theta = \frac{ab}{2}[\theta]_0^{2\pi} = \pi ab$$

Thus the required area (A) = $\boxed{\pi\, ab}$ sq. units.

[29] *Find the area between the parabolas $y^2 = 4x$ and $x^2 = 4y$ with the help of Green's theorem in a plane.* **[June 18, Sep 20]**

☞ We have the area,

$$A = \iint dx\, dy = \frac{1}{2}\int_c x\, dy - y\, dx \quad \text{[Refer the previous problem]}$$

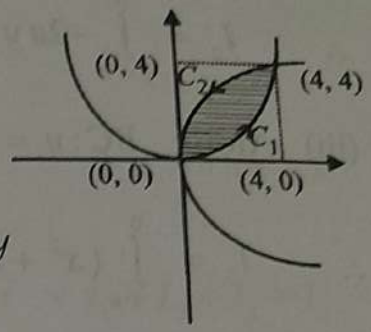
Let us find the points of intersection of $y^2 = 4x$ and $x^2 = 4y$

$$ie., \quad \left(\frac{x^2}{4}\right)^2 = 4x \text{ or } x(x^3 - 64) = 0 \Rightarrow x = 0, x = 4 \quad \therefore y = 0, 4$$

The points of intersection are (0, 0) and (4, 4).

C_1 is the curve $x^2 = 4y \ \therefore\ dy = \frac{x}{2}dx$ and $0 \le x \le 4$

C_2 is the curve $y^2 = 4x \therefore dx = \frac{y}{2}dy$ and $4 \le y \le 0$



Now, $A = \frac{1}{2}\int_{C_1} x\,dy - y\,dx + \frac{1}{2}\int_{C_2} x\,dy - y\,dx$

$$A = \frac{1}{2}\int_{x=0}^{4}\left(x\cdot\frac{x}{2} - \frac{x^2}{4}\right)dx + \frac{1}{2}\int_{y=4}^{0}\left(\frac{y^2}{4} - \frac{y^2}{2}\right)dy$$

$$A = \frac{1}{2}\int_{x=0}^{4}\frac{x^2}{4}dx - \frac{1}{2}\int_{y=0}^{4} -\frac{y^2}{4}dy$$

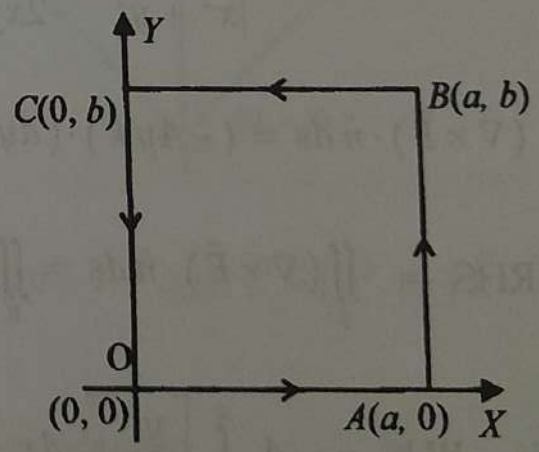
Thus, $A = \left[\frac{x^3}{24}\right]_0^4 + \left[\frac{y^3}{24}\right]_0^4 = \frac{64}{24} + \frac{64}{24} = \frac{128}{24} = \boxed{\frac{16}{3}}$ Sq. units.

Thus the required area is 16/3 sq. units.

[30] *Verify Stoke's theorem for vector* $\vec{F} = (x^2 + y^2)i - 2xy\,j$ *taken round the rectangle bounded by* $x = 0,\ x = a,\ y = 0,\ y = b$. **[June 18, Dec 19 (2nd part), Feb 21]**

☞ We have Stoke's theorem : $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\,ds$

$$\vec{F}\cdot\overrightarrow{dr} = (x^2 + y^2)\,dx - 2xy\,dy$$



$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r}$$

$$+\int_{BC}\vec{F}\cdot d\vec{r} + \int_{CO}\vec{F}\cdot d\vec{r}$$

$$\text{LHS} = \int_C \vec{F}\cdot d\vec{r} = I_1 + I_2 + I_3 + I_4\ (say)$$

(i) Along $OA: y = 0 \Rightarrow dy = 0$ and $0 \le x \le a$

$$\therefore\ I_1 = \int_{x=0}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

(ii) Along $AB: x = a \Rightarrow dx = 0$ and $0 \le y \le b$

$$\therefore \quad I_2 = \int_{y=0}^{b} -2ay\,dy = \left[-ay^2\right]_0^b = -ab^2$$

(iii) Along $BC: y = b \Rightarrow dy = 0$ and $a \le x \le 0$

$$\therefore \quad I_3 = \int_{x=a}^{0} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2 x\right]_a^0 = -\frac{a^3}{3} - ab^2$$

(iv) Along $CO: x = 0 \Rightarrow dx = 0$ and $b \le y \le 0$

$$\therefore \quad I_4 = \int_{y=b}^{0} 0\,dy = 0$$

Hence, LHS $= \int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -2ab^2$ ...(1)

Now, $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = k(-2y - 2y) = -4yk$

$$(\nabla \times \vec{F}) \cdot \hat{n}\,ds = (-4yk) \cdot (dy\,dz\,i + dz\,dx\,j + dx\,dy\,k) = -4y\,dx\,dy$$

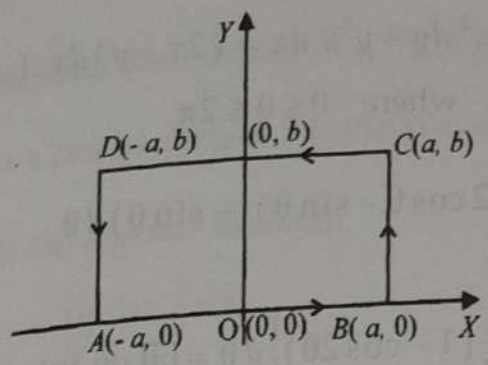
$$\text{RHS} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,ds = \iint_R -4y\,dx\,dy = -4 \int_{x=0}^{a} \int_{y=0}^{b} y\,dy\,dx$$

ie., RHS $= -4 \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_0^b dx = -2b^2 [\,x\,]_0^a = -2ab^2$ ...(2)

Thus from (1) and (2) we conclude that the theorem is verified.

[31] *Verify Stoke's theorem for* $\vec{F} = (x^2 + y^2)\,i - 2xy\,j$ *taken round the rectangle bounded by the lines* $x = \pm a$, $y = 0$ *and* $y = b$. **[Dec 16, June 17]**

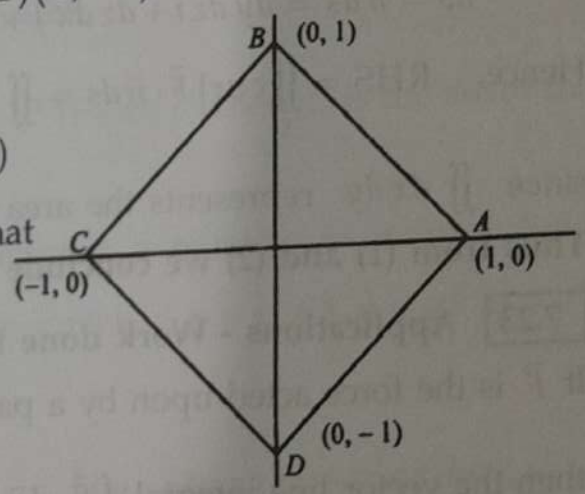
☞ $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\,ds$ [*Stoke's theorem*]



$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r} = I_1 + I_2 + I_3 + I_4$$

Proceeding as in the previous problems $\boxed{\text{LHS} = \text{RHS} = -4ab^2}$

[32] *Evaluate* $\int_C xy\,dx + xy^2\,dy$ *by Stoke's theorem where C is the square in the x-y plane with vertices* $(1,0)(-1,0)(0,1)(0,-1)$.



☞ We have Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot \overrightarrow{ds} \quad \ldots (1)$$

From the given integral it is evident that

$$\vec{F} = xy\,\hat{i} + xy^2\,\hat{j}$$

since $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Hence, $\int_C xy\,dx + xy^2\,dy = \int_C \vec{F}\cdot d\vec{r}$

RHS of (1) is to be evaluated. $-1 \le x \le 1,\ -1 \le y \le 1$; $\text{curl}\,\vec{F} = (y^2 - x)\,\hat{k}$

Proceeding as in the earlier problems,

$$\text{RHS} = \int_{x=-1}^{1} \int_{y-1}^{1} (y^2 - x)\,dy\,dx = \boxed{\frac{4}{3}} \text{ on evaluation.}$$

[33] *Verify Stoke's theorem for* $\vec{F} = (2x - y)\,i - yz^2\,j - y^2\,zk$ *where S is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$, *C is its boundary.* **[Dec 17]**

☞ $$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot \hat{n}\,ds \text{ (Stoke's theorem)}$$

C is the circle : $x^2 + y^2 = 1,\ z = 0$.

$$\vec{F}\cdot d\vec{r} = (2x-y)\,dx - yz^2\,dy - y^2z\,dz = (2x-y)\,dx \quad (\because z=0)$$

Taking, $x = \cos\theta,\ y = \sin\theta,$ where $0 \le \theta \le 2\pi$

$$\text{LHS} = \int_C \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$

$$\text{LHS} = \int_0^{2\pi} \left\{-\sin 2\theta + \frac{1}{2}(1-\cos 2\theta)\right\} d\theta = \boxed{\pi} \text{ on integration} \quad \ldots (1)$$

Also, $\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{k}$, on evaluation.

$\therefore \quad \overrightarrow{ds} = \hat{n}\,ds = dy\,dz\,i + dz\,dx\,j + dx\,dy\,k$

Hence, $\text{RHS} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds = \iint dx\,dy = \boxed{\pi}$, $\quad \ldots (2)$

since $\iint dx\,dy$ represents the area of the circle $x^2+y^2=1$ which is π.

Thus from (1) and (2) we conclude that the theorem is verified.

7.23 Applications - Work done by a force and flux

If $\vec{F}$ is the force acted upon by a particle in displacing it along the curve C then the vector line integral $\int_C \vec{F}\cdot d\vec{r}$ represents the ***total work done*** by the force.

We have Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds = \iint_S \operatorname{curl}\vec{F}\cdot\overrightarrow{ds} \quad \ldots (1)$$

Further, we know that $\operatorname{curl}\vec{F} = \vec{0} \Rightarrow \vec{F}$ is irrotational. Therefore, from (1), $\int_C \vec{F}\cdot d\vec{r} = 0$, when $\vec{F}$ is irrotational. Hence we conclude that the total work done is zero if $\vec{F}$ is irrotational and vice-versa.

Also, if $\vec{F}$ represents the velocity of a fluid particle then the ***total outward flux*** of $\vec{F}$ across a closed surface S is the surface integral $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \vec{F}\cdot\overline{ds}$.

We have Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV \qquad \ldots (2)$$

Further we know that $\operatorname{div}\vec{F} = 0 \Rightarrow \vec{F}$ is solenoidal. Therefore from (2) $\iint_S \vec{F}\cdot\hat{n}\,ds = 0$, when $\vec{F}$ is solenoidal. Hence we conclude that the flux of $\vec{F}$ across S is zero if $\vec{F}$ is solenoidal and vice-versa.

EXERCISE

1. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x^2 - y^2)i + xy\,j$ where C is the arc of the curve $y = x^3$ from $(0, 0)$ to $(2, 8)$.
2. Find the total work done in moving a particle in a force field $\vec{F} = 3xy\,i - 5z\,j + 10x\,k$ along the curve : $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$.
3. Verify Green's theorem in a plane for $\int_C (y - \sin x)\,dx + \cos x\,dy$ where C is the triangle formed by the lines $y = 0$, $x = \pi/2$ and $y = 2x/\pi$.
4. Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)i - 2xy\,j$ over the rectangle in the $x\,o\,y$ plane bounded by $x = 0, x = 2a, y = 0, y = 2b$.

ANSWERS

1. $824/21$
2. 303
3. $-(\pi/4 + 2/\pi)$
4. $-16ab^2$

9 NUMERICAL METHODS - 1

[*Common to all the streams in II Semester*]

Introduction

We come across with situations having a data comprising a set of discrete values of the dependent variable corresponding to various values of the independent variable.

Numerical Methods are discussed for the following situations.

Solution of polynomial and transcendental equations, by locating an approximate real root or an interval containing real root & subsequently improving the accuracy of the root.

Estimating to the desired degree of accuracy the value of the dependent variable corresponding to a value of the independent variable and also knowing the form of function satisfying all the conditions on hand, referred to as **Interpolation.**

Evaluation of a definite integral without the actual integration of the function, by using some discrete values of the function, referred to as **Numerical Integration.**

9.1 Solution of Algebraic and Transcendental Equations

Given an equation $f(x) = 0$ it is generally not possible to find roots x such that $f(x)$ becomes zero exactly. We discuss two numerical methods to find an approximate real root of the given equation.

Equations involving algebraic quantities like x, x^2, x^3 etc are called as ***polynomial or algebraic equations.***

Example: (i) $x^3 - 4x - 9 = 0$ (ii) $x^4 + x^3 = 80$ (iii) $x^3 + x^2 + 3x + 4 = 0$

Equations that involves non algebraic quantities like e^x, $\log x$, $\sin x$, $\tan x$ etc. are called as ***transcendental equations.***

Example : (i) $xe^x - 2 = 0$ (ii) $x\log_e x - 12 = 0$ (iii) $\tan x = 2x$

Numerical methods of finding an approximate root of the given equation is a repetitive type of process known as ***iteration process.*** In each step the result of the previous step is used and the process is carried out till we get the result to the desired accuracy. The value obtained in the succeeding step

is always better than the value of the preceeding step. All the numerical methods are only approximate techniques for the solution of any problem and computers play a great role in various numerical methods for obtaining the result to the highest degree of accuracy.

If $f(x) = 0$ is a real valued continuous function of the real variable x we have the following fundamental property :

If there exits two values a, b such that f (x) has opposite signs, say f (a) < 0, f (b) > 0, then there always exist atleast one real root in the interval (a, b).

This property is useful in locating an initial approximation for a real root of $f(x) = 0$.

It is **important** to note that ***if the values a, b of x are nearer enough then the numerical iterative methods will give the real root to the desired accuracy quickly.*** In some methods we have to locate an approximate root $x = x_0$ for starting the iterative process. In which case, if the magnitude say $f(a)$ is nearer to zero compared to $f(b)$, we prefer to take $x = x_0 = a$ as the initial approximation.

We discuss two numerical iterative methods.

9.11 Regula Falsi Method or Method of False Position

The formula to find the first approximation x_1 to the real root lying in (a, b) is as follows.

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

The process is continued till we get consistency in two consecutive approximations.

WORKED PROBLEMS

[1] *Find a real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places, by using regula-falsi method.* **[Sep 20, 21]**

☞ Let, $f(x) = x^3 - 2x - 5$

$f(0) = -5, f(1) = -6, f(2) = -1 < 0, f(3) = 16 > 0$ Root lies in $(2, 3)$.

It may be observed that the value of $f(x)$ at $x = 2$ being -1 is nearer to zero compared to $f(3) = 16$ and we expect the root in the neighbourhood

of 2. We shall have the interval (a, b) for applying the method such that $(b-a)$ is small enough.

Now, $f(2.1) = (2.1)^3 - 2(2.1) - 5 = +0.061 > 0 \quad \therefore$ the root lies in $(2, 2.1)$

I Step : $a = 2,\ f(a) = f(2) = -1$

$b = 2.1,\ f(b) = f(2.1) = +0.061$

First approximation, $x_1 = \dfrac{a f(b) - b f(a)}{f(b) - f(a)} \quad \ldots (1)$

$$x_1 = \frac{2(0.061) - (2.1)(-1)}{0.061 - (-1)} = 2.0942$$

II Step :

$f(2.0942) = (2.0942)^3 - 2(2.0942) - 5 = -0.00392 < 0$

$\therefore$ the root lies in $(2.0942, 2.1)$

Now, $a = 2.0942 \quad f(a) = -0.00392$

$b = 2.1 \quad f(b) = 0.061$

Substituting in the RHS of (1) we obtain the second approximation.

$$x_2 = \frac{(2.0942)(0.061) - (2.1)(-0.00392)}{0.061 - (-0.00392)} = 2.09455$$

x_1 and x_2 are close enough. x_2 is a better approximation than x_1.

Thus the required approximate root correct to 3 decimal places is $\boxed{2.095}$

[2] *Compute the real root of $x \log_{10} x - 1.2 = 0$, correct to four decimals by applying Regula - Falsi method.* **[June & Dec 19]**

Let, $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2,\ f(2) = -0.6 < 0,\ f(3) = 0.23 > 0$

The real root lies in $(2, 3)$ and from the values of $f(x)$ at $x = 2, 3$ we expect the root in the neighbourhood of 3 and let us find (a, b) for applying the method such that $(b - a)$ is small enough.

$$\left.\begin{aligned} f(2.7) &= 2.7 \log_{10} 2.7 - 1.2 = -0.0353 \\ f(2.8) &= 2.8 \log_{10} 2.8 - 1.2 = +0.052 \end{aligned}\right\} \text{The root lies in } (2.7, 2.8)$$

I iteration : $a = 2.7 \quad f(a) = -0.0353$

$b = 2.8 \quad f(a) = +0.052$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} \qquad \ldots (1)$$

$$\therefore \quad x_1 = \frac{(2.7)(0.052) - (2.8)(-0.0353)}{0.052 + 0.0353} = 2.7404$$

II iteration : $f(2.7404) = -0.00021 < 0 \quad \therefore$ the root lies in $(2.7404, 2.8)$

Now, $a = 2.7404 \quad f(a) = -0.00021$

$b = 2.8 \quad f(b) = 0.052$

Substituting in (1) we have,

$$x_2 = \frac{(2.7404)(0.052) + (2.8)(0.00021)}{0.052 + 0.00021} = 2.7406$$

III iteration : $f(2.7406) = -0.00004 < 0 \quad \therefore$ the root lies in $(2.7406, 2.8)$

Now $a = 2.7406 \quad f(a) = -0.00004$

$b = 2.8 \quad f(b) = 0.052$

Again substituting in (1) we have,

$$x_3 = \frac{(2.7406)(0.052) + (2.8)(0.00004)}{0.052 + 0.00004} = 2.740646$$

Thus the required approximate root correct to four decimal places is $\boxed{2.7406}$

[3] *Compute the fourth root of 12 correct to 3 decimal places, using regula-falsi method.*

☞ Let $x = \sqrt[4]{12} \quad \therefore \; x^4 = 12$ or $x^4 - 12 = 0$

Taking, $f(x) = x^4 - 12$ we have,

$$f(0) = -12 < 0, \; f(1) = -11 < 0, \; f(2) = 4 > 0$$

A real root of $f(x) = 0$ lies in $(1, 2)$ and will be in the neighbourhood of 2.

Now, $f(1.7) = -3.6479, f(1.8) = -1.5024 < 0, \; f(1.9) = 1.0321 > 0$

$\therefore$ the root in the neighbourhood of 2 lies in $(1.8, 1.9)$

I Step : Let $a = 1.8 \quad f(a) = -1.5024$

$b = 1.9 \quad f(b) = 1.0321$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = 1.8593 \text{ on substitution.} \quad \ldots (1)$$

II Step : $f(1.8593) = -0.0492 < 0 \quad \therefore$ the root lies in (1.8593, 1.9)

Now, $a = 1.8593 \quad f(a) = -0.0492$

$b = 1.9 \quad f(b) = 1.0321$

Substituting in (1) we obtain $x_2 = 1.8612$

III Step : $f(x_2) = f(1.8612) = -0.00025 < 0 \quad \therefore$ the root lies in (1.8612, 1.9)

Now, $a = 1.8612 \quad f(a) = -0.00025$

$b = 1.9 \quad f(b) = 1.0321$

Substituting in (1) we obtain $x_3 = 1.86121 \approx 1.861$

Thus the required fourth root of 12 correct to 3 decimal places is $\boxed{1.861}$

[4] *Find the real root of the equation,* $\cos x = 3x - 1$ *correct to three decimals by using regula-falsi method.*

☞ Let $f(x) = \cos x + 1 - 3x$.

In radians, $f(0) = 2 > 0, f(1) = -1.46 < 0$

A real root lies in (0, 1) and we expect the root in the neighbourhood of 1.

Consider, $f(0.6) = -0.0253 > 0, f(0.7) = -0.3352 < 0$

$\therefore$ the root lies in (0.6, 0.7)

I iteration : $a = 0.6 \quad f(a) = -0.0253 \; ; \; b = 0.7 \quad f(b) = -0.3352$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} \; ; \; \therefore \; x_1 = 0.607 \text{ on substitution.}$$

II iteration : $f(x_1) = f(0.607) = 0.00036 > 0 \quad \therefore$ the root lies in (0.607, 0.7)

Now, $a = 0.607 \quad f(a) = 0.00036 \; ; \; b = 0.7 \quad f(b) = -0.3352$

$\therefore \; x_2 = \boxed{0.607}$ on substitution.

9.12 Newton - Raphson method

In this method we locate an approximate real root x_0 of the given equation and improve its accuracy by an iterative process.

The first approximation to the root x_0 is given by the following formula.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{[Newton - Raphson formula]}$$

The second approximation is obtained by replacing x_0 by x_1 in the RHS of this expression.

The process is continued till we get consistency in two consecutive approximations.

WORKED PROBLEMS

[5] *Find a real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places, by applying Newton-Raphson method.*

☞ We shall find an interval (a, b) where a real root of the equation lies and then locate the approximate root.

Let, $f(x) = x^3 - 2x - 5$

$$f(0) = -5 < 0, f(1) = -6 < 0, f(2) = -1 < 0, f(3) = 16 > 0$$

A real root lies in (2, 3). It will be in the neighbourhood of 2 and let the approximate root $x_0 = 2$.

The first approximation is given by,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} \; ; \; f(x) = x^3 - 2x - 5, f'(x) = 3x^2 - 2$$

$$\therefore \quad x_1 = 2 - \frac{(-1)}{3(2^2) - 2} = 2 + \frac{1}{10} = 2.1$$

Again, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2.1 - \dfrac{f(2.1)}{f'(2.1)}$

$$x_2 = 2.1 - \frac{[(2.1)^3 - 2(2.1) - 5]}{[3(2.1)^2 - 2]} = 2.0946$$

Similarly, $x_3 = 2.0946 - \dfrac{[(2.0946)^3 - 2(2.0946) - 5]}{[3(2.0946)^2 - 2]} = 2.0946$

Thus the required approximate root correct to 3 decimal places is $\boxed{2.095}$

[6] *Find a real root of the equation* $x^3 + x^2 + 3x + 4 = 0$, *by performing two iterations, using Newton-Raphson method..*

☞ Let, $f(x) = x^3 + x^2 + 3x + 4$

$f(0) = 4, f(-1) = 1, f(-2) = -6 < 0$

$\therefore$ a real root lies in $(-2, -1)$ and let $x_0 = -1$; $f'(x) = 3x^2 + 2x + 3$

I iteration : $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$; $x_1 = -1 - \dfrac{f(-1)}{f'(-1)} = -1 - \dfrac{1}{4} = -1.25$

II iteration : $x_2 = -1.25 - \dfrac{f(-1.25)}{f'(-1.25)} = -1.2229 \approx \boxed{-1.223}$

[7] *Let us find cube root of 37 correct to 3 decimal places using Newton-Raphson method.* **[Aug 21]**

☞ Let, $x = \sqrt[3]{37}$ $\therefore$ $x^3 = 37$ or $x^3 - 37 = 0$

Taking, $f(x) = x^3 - 37$, we have $f(3) = -10 < 0$, $f(4) = 27 > 0$.

$\therefore$ a real root lies in (3, 4) and let $x_0 = 3$. Also $f'(x) = 3x^2$

I approximation : $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$; $x_1 = 3 - \dfrac{f(3)}{f'(3)} = 3.3704$

$$x_2 = 3.3704 - \frac{f(3.3704)}{f'(3.3704)} = 3.3704 - \frac{[(3.3704)^3 - 37]}{3(3.3704)^2} = 3.3327$$

Further on the same line, $x_3 = 3.3322$; $x_4 = 3.3322$

Thus the required $\sqrt[3]{37}$ correct to 3 decimal places is $\boxed{3.332}$

[8] *Find the real root of the equation* $x e^x - 2 = 0$ *correct to three decimal places by applying Newton Raphson method.* **[Sep 20]**

☞ Let, $f(x) = x e^x - 2$; $f(0) = -2 < 0$, $f(1) = 0.7183 > 0$

$\therefore$ The root lies in (0, 1) & let $x_0 = 1$. $f'(x) = x e^x + e^x = e^x (x+1)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{0.1783}{e^1(2)} = 0.8679$$

$$x_2 = 0.8679 - \frac{f(0.8679)}{f'(0.8679)} = 0.8679 - \frac{[0.8679\, e^{0.8679} - 2]}{e^{0.8679}(0.8679+1)} = 0.8528$$

On the same lines we get, $x_3 = 0.8526$, $\boxed{x_4 = 0.8526}$

[9] *Find the real root of the equation $x \sin x + \cos x = 0$ near $x = \pi$ using Newton-Raphson method. Carry out 3 iterations.* **[June, Dec 19, Feb 21]**

☞ Let, $f(x) = x \sin x + \cos x$

$\therefore \quad f'(x) = x\cos x + \sin x - \sin x = x\cos x$. Let $x_0 = \pi$ in radians.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}\ ;\ x_1 = \pi - \frac{f(\pi)}{f'(\pi)} = \pi - \frac{(\pi \sin \pi + \cos \pi)}{\pi \cos \pi} = \pi - \frac{1}{\pi} = 2.8233$$

$$x_2 = 2.8233 - \frac{[2.8233 \sin(2.8233) + \cos(2.8233)]}{2.8233 \cos(2.8233)} = 2.7986$$

On the same lines we get, $\boxed{x_3 = 2.7984}$

[10] *Compute the fourth root of 22 using Newton-Raphson method.*

☞ $x = \sqrt[4]{22} \Rightarrow x^4 = 22$ or $x^4 - 22 = 0$.

Proceeding as in **[7]** we obtain the solution as $\boxed{2.1657}$

9.2 Finite differences

Consider a function $y = f(x)$. Let x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, $\cdots$, $x_n = x_{n-1} + h$ be a set of points at a common interval h. Let the corresponding values of $y = f(x)$ be $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$.

We define **forward** and **backward differences** as follows.

9.21 Forward differences

The *first forward difference* of $f(x)$ denoted by $\Delta f(x)$ is defined as follows.

$$\Delta f(x) = f(x+h) - f(x).$$ 'Δ' is called the *forward difference operator*.

Now, we have for the values $x_0, x_1, x_2, \ldots, x_n$:

$\Delta f(x_0) = f(x_0+h) - f(x_0)$ or $\Delta y_0 = y_1 - y_0$

$\Delta f(x_1) = f(x_1+h) - f(x_1)$ or $\Delta y_1 = y_2 - y_1$

$\Delta f(x_2) = f(x_2+h) - f(x_2)$ or $\Delta y_2 = y_3 - y_2$ etc.,

The difference of the first forward differences are called *second forward differences*. They are as follows.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1, \text{ etc.}$$

Similarly the other higher order differences namely the third, fourth, etc., are obtained and tabulated. Such a tabular arrangement is called the **forward difference table :**

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\ldots$	$\Delta^n y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
x_2	y_2	Δy_1		$\Delta^3 y_0$		
			$\Delta^2 y_1$			
.	.	.	.	.	.	
						$\Delta^n y_0$
.	.	.	.	.	.	
.	.	.	.	$\Delta^3 y_{n-3}$		
x_{n-1}	y_{n-1}		$\Delta^2 y_{n-2}$			
		Δy_{n-1}				
x_n	y_n					

The first entries in the table namely $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots \Delta^n y_0$ are called the *leading forward differences*.

9.22 Backward differences

The *first backward difference* of $f(x)$ denoted by $\nabla f(x)$ is defined as follows.

$$\nabla f(x) = f(x) - f(x-h)$$

∇ is called the *backward difference operator.*

If, $\quad x = x_n : \nabla f(x_n) = f(x_n) - f(x_n - h)$ or $\nabla y_n = y_n - y_{n-1}$

$\because \quad x_n - h = x_{n-1}$ and $f(x_{n-1}) = y_{n-1}$

If, $\quad x = x_{n-1} : \nabla f(x_{n-1}) = f(x_{n-1}) - f(x_{n-1} - h)$

or $\quad \nabla y_{n-1} = y_{n-1} - y_{n-2}$ etc. $\nabla y_2 = y_2 - y_1$, $\nabla y_1 = y_1 - y_0$

The difference of the first backward differences are known as *second backward differences.* They are as follows.

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}, \ \nabla^2 y_{n-1} = \nabla y_{n-1} - \nabla y_{y-2}, \cdots,$$

Similarly the other higher order backward differences namely the third, fourth etc., are formed and tabulated. Such a tabular arrangement is called the **backward difference table :**

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	. . .	$\nabla^n y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2				
x_2	y_2			$\nabla^3 y_3$		
			$\nabla^2 y_3$		.	
.	.	.	.	.	.	$\nabla^n y_n$
.	.	.	.	.	.	
.	.	.	.	$\nabla^3 y_{n-3}$		
x_{n-1}	y_{n-1}		$\nabla^2 y_n$			
		∇y_n				
x_n	y_n					

The last entries in the table namely $\nabla y_n, \nabla^2 y_n, \nabla^3 y_n, \ldots \nabla^n y_n$ are called the *leading backward differences.*

ILLUSTRATIVE EXAMPLE

Let us construct a finite difference table for the function $f(x) = x^3 + x + 1$ where x takes the values 0, 1, 2, 3, 4, 5, 6 and identify the leading forward and backward differences.

$f(x) = x^3 + x + 1$ by data. From this we obtain,

$f(0) = 1, f(1) = 3, f(2) = 11, f(3) = 31, f(4) = 69,$

$f(5) = 131, f(6) = 223.$

The finite difference table is as follows.

x	$f(x) = y$	First Difference	Second Difference	Third Difference	Fourth Difference
0	1				
		2			
1	3		6		
		8		6	
2	11		12		0
		20		6	
3	31		18		0
		38		6	
4	69		24		0
		62		6	
5	131		30		
		92			
6	223				

Taking $x_0 = 0, y_0 = 1$, the first value in every column are the leading forward differences. They are as follows :

$$\Delta y_0 = 2, \Delta^2 y_0 = 6, \Delta^3 y_0 = 6, \Delta^4 y_0 = 0$$

Also by taking $x_n = 6, y_n = 223$, the last value in every column are the leading backward differences. They are as follows :

$$\nabla y_n = 92, \nabla^2 y_n = 30, \nabla^3 y_n = 6, \nabla^4 y_n = 0$$

9.3 Interpolation

If $y_0, y_1, y_2, \ldots y_n$ be a set of values of an unknown function $y = f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \ldots x_n$, the process of finding (*estimating*) the value of y for any given value of x between x_0 and x_n is called ***interpolation.*** Also the process of finding (*estimating*) the value of y outside the given range of x is called ***extrapolation.*** In general the concept of interpolation includes extrapolation also.

Hence we can say that interpolation is a technique of estimating the value of an unknown function for any intermediate value of the independent variable.

For example, if we have the data :

x	0	4	5	8	10	15
$f(x)$	6	15	17	29	40	87

the process of estimating $f(3), f(7), f(12.5), f(14)$ etc. is interpolation and estimating $f(-0.5), f(18), f(25)$ etc. is extrapolation.

We first discuss interpolation for equal intervals which will be followed with interpolation for unequal intervals.

9.31 Newton's forward interpolation formula

The value of $y = f(x)$ at $x = x_0 + rh$, that is $y_r = f(x_0 + rh)$ is approximately

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots + \frac{r(r-1)(r-2)\cdots(r-\overline{n-1})}{n!}\Delta^n y_0$$

where r is any real number.

9.32 Newton's backward interpolation formula

The value of $y = f(x)$ at $x = x_n + rh$, that is $y_r = f(x_n + rh)$ is approximately

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots + \frac{r(r+1)(r+2)\cdots(r+\overline{n-1})}{n!}\nabla^n y_n$$

- **Appropriate interpolation formula**

To estimate the value of y at a desired value of x near the beginning of the table (*first half, x is close to x_0*) forward formula is appropriate.

Similarly to estimate the value of y at a desired value fo x near the end of the table (*second half, x is close to x_n*) backward formula is appropriate.

The polynomial $y = f(x)$ satisfying the data can also be found from these formulae and it is called an interpolating polynomial.

Note : *The word appropriate is used in the sense that the computational work will involve relatively small magnitudes. Either of the formulae can be used for obtaining the required result.*

Working procedure for problems

Step - 1 : We construct the difference table in accordance with the interpolation formula.

Step - 2 : We compute the value of r where,

(a) $r = \dfrac{x - x_0}{h}$ in the case of forward interpolation formula, x_0 being the first value of x and h is the step length.

(b) $r = \dfrac{x - x_n}{h}$ in the case of backward interpolation formula, x_n being the last value of x and h is the step length.

Step - 3 : The value of r along with the value of the finite differences are substituted in the interpolation formula which results in the value of y at the desired value of x.

Note : *If no numerical value is substituted for x then $r = r(x)$.*

The associated $y = f(x)$ is called the interpolating polynomial.

WORKED PROBLEMS

[11] *Find $y(1.4)$ given that*

x	1	2	3	4	5
y	10	26	58	112	194

☞ Here we have to find y at $x = 1.4$

Since the value $x = 1.4$ is in the first half of the table near $x = 1$, Newton's forward interpolation formula is appropriate and we shall construct the forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 1$	$y_0 = 10$				
		16			
2	26		16		
		32		6	
3	58		22		0
		54		6	
4	112		28		
		82			
5	194				

We have Newton's forward interpolation formula :

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots;\ r = (x - x_0)/h$$

x = point at which y is required = 1.4 ; x_0 = first value of x = 1

h = common interval length = 1. Here $r = \dfrac{1.4-1}{1} = 0.4$

From the table, $\Delta y_0 = 16$, $\Delta^2 y_0 = 16$, $\Delta^3 y_0 = 6$, $\Delta^4 y_0 = 0$

$$\therefore\ y(1.4) = f(1.4) = 10 + (0.4)16 + \frac{(0.4)(0.4-1)}{2}(16) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(6)$$

$$y(1.4) = 10 + 6.4 + (0.4)(-0.6)8 + (0.4)(-0.6)(-1.6) = \boxed{14.864}$$

[12] *Find $u_{0.5}$ from the data :*

$$u_0 = 225,\ u_1 = 238,\ u_2 = 320,\ u_3 = 340$$

☞ The value $x = 0.5$ is near to $x = 0$ and hence Newton's forward interpolation formula is appropriate. The forward difference table. is formed ;

x	$u_x = y$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 225$			
		$\Delta y_0 = 13$		
1	238		$\Delta^2 y_0 = 69$	
		82		$\Delta^3 y_0 = -131$
2	320		−62	
		20		
3	340			

Here, $r = (x - x_0)/h$; $r = (0.5 - 0)/1 = 0.5$

Substituting in the formula as in **[11]**,

$$\boxed{u_{0.5} = 214.6875}$$

[13] *The area of a circle (A) corresponding to diameter (D) is given below.*

D	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find the area corresponding to diameter 105 using an appropriate interpolation formula. **[Feb 21]**

☞ Here we have to find A when $D = 105$

As this value 105 is near to the end value 100, Newton's backward interpolation formula is appropriate. D and A correspond to x and y. The backward difference table is formed first.

$x = D$	$y = A$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		−2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
$x_n = 100$	$y_n = 7854$				

We have Newton's backward interpolation formula :

$$y_r = y_n + r\,\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

where, $r = (x - x_n)/h = (105 - 100)/5 = 1$

From the table, $\nabla y_n = 766$, $\nabla^2 y_n = 40$, $\nabla^3 y_n = 2$, $\nabla^4 y_n = 4$

$\therefore\ f(105) = 7854 + 766 + 40 + 2 + 4 = 8666$, on substitution.

Thus the area (A) corresponding to diameter (D) = 105 is $\boxed{8666}$

[14] *The following table give the values of* tan x *for* $0.10 \le x \le 0.30$. *Find* tan (0.26).

x	0.10	0.15	0.20	0.25	0.30
tan x	0.1003	0.1511	0.2027	0.2553	0.3093

☞ Here the value $x = 0.26$ is near the end value 0.30

Hence Newton's backward interpolation formula is appropriate.

x	$f(x) = y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
$x_n = 0.30$	$y_n = 0.3093$				

Here, $r = \dfrac{x - x_n}{h}$; $r = \dfrac{0.26 - 0.30}{0.05} = -0.8$

From the table

$$\nabla y_n = 0.0540,\ \nabla^2 y_n = 0.0014,\ \nabla^3 y_n = 0.0004,\ \nabla^4 y_n = 0.0002$$

On substitution in the formula as in **[13]** we get,

$$f(0.26) = 0.26602\ ;$$

Thus, $\boxed{\tan(0.26) = 0.2660}$

[15] *Extrapolate for* 25.4 *given the data,*

x	19	20	21	22	23
y	91	100.25	110	120.25	131

☞ We shall use backward interpolatation formula.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
19	91			
		9.25		
20	100.25		0.5	
		9.75		0
21	110		0.5	
		10.25		$\nabla^3 y_n = 0$
22	120.25		$\nabla^2 y_n = 0.5$	
		$\nabla y_n = 10.75$		
$x_n = 23$	$y_n = 131$			

Here, $r = \dfrac{x - x_n}{h}$; $r = \dfrac{25.4 - 23}{1} = 2.4$; Substituting in the formula as in [13]

$$\therefore \quad f(25.4) = 131 + (2.4)10.75 + \frac{(2.4)(2.4+1)}{2}(0.5) = \boxed{158.84}$$

[16] *Given $f(40) = 184$, $f(50) = 204$, $f(60) = 226$, $f(70) = 250$, $f(80) = 276$, $f(90) = 304$, find $f(38)$ and $f(85)$ using suitable interpolation formulae.* **[June 18, Dec 19]**

☞ Here we shall find $f(38)$ using Newton's forward interpolation formula and find $f(85)$ using Newton's backward interpolation formula.

On computation, $\boxed{f(38) = 180.24 \ ; \ f(85) = 289.75}$

[17] *Given $\sin 45^0 = 0.7071$, $\sin 50^0 = 0.7660$, $\sin 55^0 = 0.8192$, $\sin 60^0 = 0.8660$, find $\sin 57^0$ using an appropriate interpolation formula.* **[June 18]**

☞ We have to find the value of $f(x) = \sin x$ at $x = 57^0$ which is near the end value $x = 60^0$ and hence Newton's backward interpolation formula is appropriate. The difference table is as follows.

x	$f(x) = y$	∇y	$\nabla^2 y$	$\nabla^3 y$
45	0.7071			
		0.0589		
50	0.7660		−0.0057	
		0.0532		−0.0007
55	0.8192		−0.0064	$(\nabla^3 y_n)$
		0.0468	$(\nabla^2 y_n)$	
$x_n = 60$	$y_n = 0.8660$	(∇y_n)		

Here, $r = \dfrac{x - x_n}{h}$; $r = \dfrac{57 - 60}{5} = -0.6$

Substituting in the formula as in **[13]** $\boxed{f(57) = 0.8387 = \sin 57^0}$

[18] *Find the interpolating polynomial* $f(x)$ *satisfying* $f(0) = 0, f(2) = 4,$ $f(4) = 56, f(6) = 204, f(8) = 496, f(10) = 980$ *and hence find* $f(3)$, $f(5)$ *and* $f(7)$

☞ The interpolating polynomial can be found from either of the two interpolation formulae. We shall use Newton's forward interpolation formula. The difference table is as follows.

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$y_0 = 0$				
		$\Delta y_0 = 4$			
2	4		$\Delta^2 y_0 = 48$		
		52		$\Delta^3 y_0 = 48$	
4	56		96		$\Delta^4 y_0 = 0$
		148		48	
6	204		144		0
		292		48	
8	496		192		
		484			
10	980				

Here, $r = \dfrac{x - x_0}{h}$ or $r = \dfrac{x}{2}$ for finding $y = f(x)$

Substituting in the formula as in **[11]**

$$y = f(x) = 0 + \frac{x}{2}(4) + \frac{\frac{x}{2}\left(\frac{x}{2} - 1\right)}{2}(48) + \frac{\frac{x}{2}\left(\frac{x}{2} - 1\right)\left(\frac{x}{2} - 2\right)}{6}(48)$$

$$y = f(x) = 2x + \frac{x}{2}\left(\frac{x-2}{2}\right)(24) + \frac{x}{2}\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)(8)$$

$$y = f(x) = 2x + x(x-2)(6) + x(x-2)(x-4) = \boxed{x^3 - 2x}$$

By putting $x = 3, 5, 7$ in $f(x)$, we obtain,

$$\boxed{f(3) = 21,\ f(5) = 115,\ f(7) = 329}$$

Note: *It may be observed that if we substitute the given values of x namely 0, 2, 4, 6, 8, 10 in the polynomial, the values of $f(x)$ coincide with the values given in the data.*

[19] *From the following table find the number of students who have obtained (a) less than 45 marks (b) between 40 and 45 marks.* **[June 17, 19]**

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

We shall reconstitute the given table with $f(x)$ representing the number of students less than x marks. That is,

less than 40 marks 31 students,

less than 50 marks 31 + 42 = 73 students,

less than 60 marks 73 + 51 = 124 students,

less than 70 marks 124 + 35 = 159 students,

less than 80 marks 159 + 31 = 190 students,

We have the new table along with the forward differences.

(a) We need to find $f(45)$ being the number of students scoring less than 45 marks.

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 40$	$y_0 = 31$				
		$\Delta y_0 = 42$			
50	73		$\Delta^2 y_0 = 9$		
		51		$\Delta^3 y_0 = -25$	
60	124		−16		$\Delta^4 y_0 = 37$
		35		12	
70	159		−4		
		31			
80	190				

We shall find $f(45)$ by applying Newton's forward interpolation formula.

Here, $r = \dfrac{x - x_0}{h} = \dfrac{45 - 40}{10} = 0.5$

On substitution in the formla $f(45) = 47.86 \approx 48$

Thus the number of students obtaining less than 45 marks is 48.

(b) We need to find $f(45) - f(40)$; But $f(40) = 31$, by data.

Hence $f(45) - f(40) = 48 - 31 = 17$

Thus the number of students scoring between 40 and 45 is 17.

[20] *Use Newton's forward interpolation formula to find* y_{35} *given,* $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$.

☞ By construction of the forward difference table $\Delta y_0 = -73$, $\Delta^2 y_0 = -20$ $\Delta^3 y_0 = 10$ and substituting these values in the forward formula along with $r = (x - x_0)/h = 1.5$, we obtain $\boxed{f(35) = y_{35} = 394.375}$

EXERCISE

Use an appropriate interpolation formula to estimate $y = f(x)$ *for the given value of* x [Problems 1 & 2]

1.

x	1.7	1.8	1.9	2.0	2.1	2.2
$f(x)$	5.474	6.050	6.686	7.389	8.166	9.025

$f(1.85) = ?$

2.

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.9	21.27

$y(218) = ?$

3. Given, $\sin 20^0 = 0.3420$, $\sin 25^0 = 0.4226$, $\sin 30^0 = 0.5000$, $\sin 35^0 = 0.5736$, $\sin 40^0 = 0.6428$. Find $\sin 24^0$ and $\sin 42^0$ using appropriate interpolation formulae.

4. From the following data, estimate the number of students who have scored less than 70 marks.

Marks	0-20	20-40	40-60	60-80	80-100
No. of Students	41	62	65	50	17

ANSWERS

1. 6.354 2. 15.99 3. 0.4068, 0.6693 4. 196

9.4 Interpolation formulae for unequal intervals

9.41 Divided differences

Let $f(x_0), f(x_1), f(x_2), \cdots f(x_n)$ be the values of an unknwon function $y = f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \ldots x_n$ at unequal intervals. The *first order divided differences* are defined as follows.

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{etc.}$$

The second order divided differences are defined as follows.

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} \quad \text{etc.,}$$

Similarly the other higher order divided differences are defined. The tabular arrangement of these values is called the **divided difference table** and is as follows.

x	$f(x)$	1st D.D	2nd D.D	..	n^{th} D.D.
x_0	$f(x_0)$				
		$f(x_0, x_1)$			
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$		
		$f(x_1, x_2)$			
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$		
		$f(x_2, x_3)$			
...	...	...	...	...	$f(x_0, x_1, x_2, \ldots x_n)$
x_{n-2}	$f(x_{n-2})$		$f(x_{n-3}, x_{n-2}, x_{n-1})$		
		$f(x_{n-2}, x_{n-1})$			
x_{n-1}	$f(x_{n-1})$		$f(x_{n-2}, x_{n-1}, x_n)$		
		$f(x_{n-1}, x_n)$			
x_n	$f(x_n)$				

Note : *The notation for divided differences should not be confused with the notation for functions of two or more variables. $f(x_1, x_2)$ does not stand for a function of two variables x_1 and x_2. Similarly $f(x_1, x_2, x_3)$ is not a function of x_1, x_2, x_3.*

9.42 Newton's divided difference formula
or
Newton's general interpolation formula

Statement : *If $f(x_0), f(x_1), f(x_2), \cdots f(x_n)$ be a set of values of an unknown function $f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \ldots x_n$ at unequal intervals, then*

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \cdots$$
$$+ (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f(x_0, x_1, x_2, \cdots x_n)$$

WORKED PROBLEMS

[21] *Use Newton's divided difference formula to find $f(4)$ given the data :*

x	0	2	3	6
$f(x)$	-4	2	14	158

[Sep 20]

☞ We shall first form the table of divided differences.

x	$f(x)$	1st D.D	2nd D.D	3rd D.D.
$x_0 = 0$	$f(x_0) = -4$			
		$f(x_0, x_1)$ $\frac{2-(-4)}{2-0} = 3$		
$x_1 = 2$	$f(x_1) = 2$		$f(x_0, x_1, x_2)$ $\frac{12-3}{3-0} = 3$	
		$f(x_1, x_2)$ $\frac{14-2}{3-2} = 12$		$f(x_0, x_1, x_2, x_3)$ $\frac{9-3}{6-0} = 1$
$x_2 = 3$	$f(x_2) = 14$		$f(x_2, x_3, x_4)$ $\frac{48-12}{6-2} = 9$	
		$f(x_2, x_3)$ $\frac{158-14}{6-3} = 48$		
$x_3 = 6$	$f(x_3) = 158$			

We have Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

$$\therefore \quad f(4) = -4 + (4-0)3 + (4-0)(4-2)3 + (4-0)(4-2)(4-3)1$$
$$= -4 + 12 + 24 + 8 = 40$$

Thus, $\boxed{f(4) = 40}$

[22] *Construct the interpolating polynomial for the data given below using Newton's general interpolation formula for divided differences.*

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

[June 17, 18]

☞ The divided difference table is as follows.

x	$f(x)$	1st D.D	2nd D.D	3rd D.D.
$x_0 = 2$	$f(x_0) = 10$			
		$f(x_0, x_1)$ $\frac{96-10}{4-2} = 43$		
$x_1 = 4$	$f(x_1) = 96$		$f(x_0, x_1, x_2)$ $\frac{100-43}{5-2} = 19$	
		$f(x_1, x_2)$ $\frac{196-96}{5-4} = 100$		$f(x_0, x_1, x_2, x_3)$ $\frac{27-19}{6-2} = 2$
$x_2 = 5$	$f(x_2) = 196$		$f(x_1, x_2, x_3)$ $\frac{154-100}{6-4} = 27$	
		$f(x_2, x_3)$ $\frac{350-196}{6-5} = 154$		$f(x_1, x_2, x_3, x_4)$ $\frac{35-27}{8-4} = 2$
$x_3 = 6$	$f(x_3) = 350$		$f(x_2, x_3, x_4)$ $\frac{259-154}{8-5} = 35$	
		$f(x_3, x_4)$ $\frac{868-350}{8-6} = 259$		$f(x_2, x_3, x_4, x_5)$ $\frac{45-35}{10-5} = 2$
$x_4 = 8$	$f(x_4) = 868$		$f(x_3, x_4, x_5)$ $\frac{439-259}{10-6} = 45$	
		$f(x_4, x_5)$ $\frac{1746-868}{10-8} = 439$		
$x_5 = 10$	$f(x_5) = 1746$			

The fourth order differences are zero as third order differences are same.
We have Newton's general interpolation formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

Now, $f(x) = 10 + (x-2)(43) + (x-2)(x-4)(19) + (x-2)(x-4)(x-5)(2)$

$$= 10 + (x-2)\{43 + (19x - 76) + (x^2 - 9x + 20)2\}$$
$$= 10 + (x-2)(2x^2 + x + 7) = 2x^3 - 3x^2 + 5x - 4.$$

Thus the required interpolating polynomial is given by

$$\boxed{f(x) = 2x^3 - 3x^2 + 5x - 4}$$

[23] *Fit an interpolating polynomial for the data.*

$$u_{10} = 355,\ u_0 = -5,\ u_8 = -21,\ u_1 = -14,\ u_4 = -125$$

by using Newton's general interpolation formula and hence find u_2.

☞ We shall arrange the data taking the values of x in the ascending order along with the corresponding values of u_x just for convenience.
(*However this arrangement is not necessary*)
The divided difference table is formed first.

x	$u_x = f(x)$	1st D.D	2nd D.D	3rd D.D.
$x_0 = 0$	$f(x_0) = -5$	$f(x_0, x_1)$ $\frac{-14-(-5)}{1-0} = -9$	$f(x_0, x_1, x_2)$ $\frac{-37-(-9)}{4-0} = -7$	$f(x_0, x_1, x_2, x_3)$ $\frac{9-(-7)}{8-0} = 2$
$x_1 = 1$	$f(x_1) = -14$	$f(x_1, x_2)$ $\frac{-125-(-14)}{4-1} = -37$	$f(x_1, x_2, x_3)$ $\frac{26-(-37)}{8-1} = 9$	$f(x_1, x_2, x_3, x_4)$ $\frac{27-9}{10-1} = 2$
$x_2 = 4$	$f(x_2) = -125$	$f(x_2, x_3)$ $\frac{-21+125}{8-4} = 26$	$f(x_2, x_3, x_4)$ $\frac{188-26}{10-4} = 27$	
$x_3 = 8$	$f(x_3) = -21$	$f(x_3, x_4)$ $\frac{355-(-21)}{10-8} = 188$		
$x_4 = 10$	$f(x_4) = 355$			

The fourth order differences are zero since the third order differences are same.
On substitution in Newton's general interpolation formula,

$$f(x) = -5 + (x)(-9) + (x)(x-1)(-7) + (x)(x-1)(x-4)(2)$$

Thus, $\boxed{f(x) = u_x = 2x^3 - 17x^2 + 6x - 5}$ on simplification.

Further, $f(2) = u_2 = 2(2)^3 - 17(2)^2 + 6(2) - 5 = \boxed{-45}$

[24] *Using Newton's divided difference formula find $f(8)$, $f(15)$ from the following data.*

x	4	5	7	10	11	13
y	48	100	294	900	1210	2028

As in earlier problems forming the divided difference table we have, $x_0 = 4$, $f(x_0) = 48$, $f(x_0, x_1) = 52$, $f(x_0, x_1, x_2, x_3) = 1$, being the lead values. On substitution in the divided difference formula along with $x = 8$ and 15 independently, we obtain $\boxed{f(8) = 448 \text{ and } f(15) = 3150}$

[25] *Determine $f(x)$ as a polynomial in x for the following data using Newton's divided difference formula.*

x	−4	−1	0	2	5
y	1245	33	5	9	1335

[Dec 18]

☞ The divided difference table is formed first.

x	$y = f(x)$	1st D.D	2nd D.D	3rd D.D.	4th D.D.
$x_0 = -4$	$f(x_0) = 1245$	$f(x_0, x_1)$ $\frac{33-1245}{-1+4} = -404$	$f(x_0, x_1, x_2)$ $\frac{-28+404}{0+4} = 94$		
$x_1 = -1$	$f(x_1) = 33$	$f(x_1, x_2)$ $\frac{5-33}{0+1} = -28$	$f(x_1, x_2, x_3)$ $\frac{2+28}{2+1} = 10$	$f(x_0, x_1, x_2, x_3)$ $\frac{10-94}{2+4} = -14$	$f(x_0, x_1, x_2, x_3, x_4)$ $\frac{13+14}{5+4} = 3$
$x_2 = 0$	$f(x_2) = 5$	$f(x_2, x_3)$ $\frac{9-5}{2-0} = 2$	$f(x_2, x_3, x_4)$ $\frac{442-2}{5-0} = 88$	$f(x_1, x_2, x_3, x_4)$ $\frac{88-10}{5+1} = 13$	
$x_3 = 2$	$f(x_3) = 9$	$f(x_3, x_4)$ $\frac{1335-9}{5-2} = 442$			
$x_4 = 5$	$f(x_4) = 1335$				

On substituting in the Newton's divided difference formula, we have,

$$\boxed{f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5}$$

EXERCISE

Use Newton's divided difference formula to compute $f(5)$ given,

1.

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

2. *Use Newton's general interpolation formula to fit an interpolating polynomial for the following data.*

x	0	1	2	5
$f(x)$	2	3	12	147

ANSWERS

1. 194 2. $x^3 + x^2 - x + 2$

9.43 Lagrange's formula for interpolation and inverse interpolation

Statement : *If $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \cdots y_n = f(x_n)$ be a set of values of an unknown function $y = f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \cdots x_n$ not necessarily at equal intervals then,*

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} + \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})y_n}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

Remarks :

1. *The special feature of this formula is that, the terms of the formula involve only the values in the variables x and y.*
2. *The values of x also need not be equally spaced, nor need they be in any order.*
3. *We can interchange the role of x and y in the formula and the same is called Lagrange's inverse interpolation formula which helps to find x for a given y.*

WORKED PROBLEMS

[26] *Use Lagrange's interpolation formula to find $f(4)$ given,*

x	0	2	3	6
$f(x)$	−4	2	14	158

☞ Let, $x_0 = 0 \quad x_1 = 2 \quad x_2 = 3 \quad x_3 = 6$
$y_0 = -4 \quad y_1 = 2 \quad y_2 = 14 \quad y_3 = 158$ $\Big\}$ $x = 4$, $y = ?$

We have Lagrange's interpolation formula for four given values,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Substituting the values we have,

$$f(4) = \frac{(4-2)(4-3)(4-6)(-4)}{(0-2)(0-3)(0-6)} + \frac{(4-0)(4-3)(4-6)2}{(2-0)(2-3)(2-6)}$$

$$+\frac{(4-0)(4-2)(4-6)14}{(3-0)(3-2)(3-6)} + \frac{(4-0)(4-2)(4-3)158}{(6-0)(6-2)(6-3)}$$

$$f(4) = \frac{(2)(1)(-2)(-4)}{(-2)(-3)(-6)} + \frac{(4)(1)(-2)2}{(2)(-1)(-4)} + \frac{(4)(2)(-2)14}{(3)(1)(-3)} + \frac{(4)(2)(1)158}{(6)(4)(3)}$$

$$f(4) = \frac{-4}{9} - 2 + \frac{224}{9} + \frac{158}{9} = \frac{382}{9} - \frac{22}{9} = \frac{360}{9} = 40$$

Thus, $\boxed{f(4) = 40}$

[27] *Use Lagrange's interpolation formula to find y at $x = 10$ given,*

x	5	6	9	11
y	12	13	14	16

[June 17]

☞ Let, $x_0 = 5 \quad x_1 = 6 \quad x_2 = 9 \quad x_3 = 11$
$y_0 = 12 \quad y_1 = 13 \quad y_2 = 14 \quad y_3 = 16$ $\Big\}$ $x = 10$, $y = ?$

Substituting these value in the Lagrange's interpolation formula as in **[26]** we have,

$$f(10) = 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{44}{3} = 14.666 \cdots \approx \boxed{14.67}$$

[28] *Use Lagrange's interpolation formula to fit a polynomial for the data.*

x	0	1	3	4
y	−12	0	6	12

[June 18]

Hence estimate y at $x = 2$.

☞ Let, $\left.\begin{array}{llll} x_0 = 0 & x_1 = 1 & x_2 = 3 & x_3 = 4 \\ y_0 = -12 & y_1 = 0 & y_2 = 6 & y_3 = 12 \end{array}\right\} y = f(x) = ?$

Using these values in the Lagrange's interpolation formula, as in **[26]**

$$y = f(x) = \frac{(x-1)(x-3)(x-4)(-12)}{(-1)(-3)(-4)} + 0$$

$$+ \frac{x(x-1)(x-4)6}{(3)(2)(-1)} + \frac{x(x-1)(x-3)12}{(4)(3)(1)}$$

$$f(x) = (x-1)(x-3)(x-4) - x(x-1)(x-4) + x(x-1)(x-3)$$

$$f(x) = (x-1)[(x^2 - 7x + 12) - (x^2 - 4x) + (x^2 - 3x)]$$

$$f(x) = (x-1)[x^2 - 6x + 12] = \boxed{x^3 - 7x^2 + 18x - 12}$$

Now, $f(2) = 2^3 - 7(2)^2 + 18(2) - 12 = \boxed{4}$

[29] *The following table gives the normal weights of babies during first eight months of life.*

Age (in Months)	0	2	5	8
Weight (in pounds)	6	10	12	16

Estimate the weight of the baby at the age of seven months using Lagrange's interpolation formula.

☞ Let, $\left.\begin{array}{llll} x_0 = 0 & x_1 = 2 & x_2 = 5 & x_3 = 8 \\ y_0 = 6 & y_1 = 10 & y_2 = 12 & y_3 = 16 \end{array}\right\} \begin{array}{l} x = 7 \\ y = ? \end{array}$

Using these values in the Lagrange's interpolation formula, as in **[26]**,

$$f(7) = \frac{3}{4} - \frac{35}{9} + \frac{28}{3} + \frac{70}{9} = 13.97 \approx 14.$$

Thus the approx. wt. of the baby at the age of 7 months is 14 pounds.

[30] *Use lagrange's formula, find the interpolating polynomial that approximates to the function described by the following table.*

x	0	1	2	5
$f(x)$	2	3	12	147

[Sep 20]

(b) *Find y at x = 4.* **[June 19]**

☞(a) Let, $\left.\begin{array}{llll} x_0 = 0 & x_1 = 1 & x_2 = 2 & x_3 = 5 \\ y_0 = 2 & y_1 = 3 & y_2 = 12 & y_3 = 147 \end{array}\right\} y = f(x) = ?$

Using these values in the Lagrange's interpolation formula as in **[26]**,

$$y = f(x) = \frac{(x-1)(x-2)(x-5)2}{(-1)(-2)(-5)} + \frac{x(x-2)(x-5)3}{(1)(-1)(-4)}$$

$$+\frac{x(x-1)(x-5)12}{(2)(1)(-3)} + \frac{x(x-1)(x-2)147}{(5)(4)(3)}$$

$$f(x) = -\frac{1}{5}(x-1)(x-2)(x-5) + \frac{3}{4}x(x-2)(x-5)$$

$$-2x(x-1)(x-5) + \frac{49}{20}x(x-1)(x-2)$$

$$f(x) = \frac{1}{20}\{-4(x^3 - 8x^2 + 17x - 10) + 15(x^3 - 7x^2 + 10x)$$

$$-40(x^3 - 6x^2 + 5x) + 49(x^3 - 3x^2 + 2x)\}$$

$$f(x) = \frac{1}{20}(20x^3 + 20x^2 - 20x + 40)$$

Thus, $\boxed{f(x) = x^3 + x^2 - x + 2}$

(b) Along with the given values, $x = 4$ also to be substituted in the formula.

We can obtain $\boxed{f(4) = 78}$

Note : *When more number of values are given at unequal intervals for interpolation without the specific mention of the formula, we must prefer Newton's general interpolation formula.*

EXERCISE

1. *Apply Lagrange's formula to find y at x = 11.*

x	2	5	8	14
y	94.8	87.9	81.3	68.7

2. *Use Lagrange's interpolation formula to fit a polynomial* $f(x)$:

x	1	2	4	5
y	14	41	197	350

ANSWERS

1. 74.925

2. $2x^3 + 3x^2 + 4x + 5$

9.5 Numerical Integration

This is the process of obtaining approximately the value of the definite integral $I = \int_a^b y\,dx$ without actually integrating the function but only using the values of y at some points of x equally spaced over $[a, b]$. We need techniques to accomplish this because, not all functions can be integrated by the various standard methods of integration. Further there are many situations where we have only some values of y corresponding to equidistant values of x.

We present three rules / formulae to obtain the value of the definite integral

$$I = \int_a^b y\,dx \text{ numerically.}$$

The following is a common step for applying any of the rules.

The interval $[a, b]$ is divided into n equal parts of width h where, $h = (b-a)/n$.

Let $a = x_0$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \cdots x_n = x_0 + nh = b$ be the points of division. Also, let $y_0 = f(x_0), y_1 = f(x_1), \cdots y_n = f(x_n)$ be the corresponding values of $y = f(x)$.

Now we have a set of values of $y = f(x)$ at equidistant points of x and the values (x, y) are tabulated.

x	$x_0 = a$	x_1	x_2	x_3	$\cdots$	$x_n = b$
y	y_0	y_1	y_2	y_3	$\cdots$	y_n

9.51 Trapezoidal rule

$$I = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$$

9.52 Simpson's one third rule

$$I = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

9.53 Simpson's three eighth rule

$$I = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})]$$

Notes : (*i*) *When we divide the interval* [*a, b*] *into n equal parts there will be* $(n+1)$ *values* $x : a = x_0, x_1, x_2, \cdots, x_n = b$. *The corresponding values of y, also* $(n+1)$ *in number are referred to as the 'ordinates'. So we can conclude that if there are* $(n+1)$ *ordinates there must be n equal divisions.*

(*ii*) ***It is very important to know that***

(*a*) *to apply Simpson's 1/3rd rule n must be a multiple of* 2.

(*b*) *to apply Simpson's 3/8th rule n must be a multiple of* 3.

Remark : *Trapezoidal rule is not preferred as the accuracy in the answer is less.*

Working procedure for problems

Step - 1 : Given the definite integral $I = \int_a^b y\,dx$ for evaluation, first divide the interval [*a, b*] into appropriate number of equal parts (*strips*) so as to apply the desired rule. $a = x_0, x_1, x_2, \cdots x_n = b$ be the points of division inclusive of the ends.

Step - 2 : Prepare a table consisting the values of x and the corresponding computed values of y denoted respectively by $y_0, y_1, y_2, \cdots y_n$

Step - 3 : Substitute values from this table into the appropriate rule to obtain the approximate value of the given definite integral.

Note : *Sometimes it is possible to deduce the value of a certain quantity by equating the theoretical value of the definite integral (when exists) with that of the numerical value obtained without integration using the rule.*

WORKED PROBLEMS

[31] *Evaluate* $\int_0^1 \frac{dx}{1+x^2}$ *by using Simpson's 1/3rd rule taking four equal strips and hence deduce an approximate value of* π.

☞ Divide [0, 1] into 4 equal strips ($n = 4$). $h = (1-0)/4 = 1/4$

The points of division are $x = 0, 1/4, 2/4 = 1/2, 3/4, 1$.

$y = \frac{1}{1+x^2}$ and we compute y at each of the values of x.

$\therefore$ At $x = 0$, $y = \frac{1}{1+0^2} = 1$; $x = \frac{1}{4}$, $y = \frac{1}{1+(1/4)^2} = \frac{16}{17}$

$x = \frac{1}{2}$, $y = \frac{1}{1+(1/2)^2} = \frac{4}{5}$; $x = \frac{3}{4}$, $y = \frac{1}{1+(3/4)^2} = \frac{16}{25}$

Now, we have the following table.

x	0	1/4	1/2	3/4	1
$y = 1/1+x^2$	1	16/17	4/5	16/25	1/2
	y_0	y_1	y_2	y_3	y_4

Simpson's 1/3rd rule for $n = 4$ is given by,

$$I = \int_a^b y\,dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2 \right]$$

$$\therefore \quad I = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/4}{3}\left[\left(1+\frac{1}{2}\right) + 4\left(\frac{16}{17}+\frac{16}{25}\right) + 2\left(\frac{4}{5}\right)\right] = \boxed{0.7854}$$

To deduce the value of π *:* We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

We must have, $\frac{\pi}{4} = 0.7854$ or $\pi = 4\,(0.7854) = 3.1416$

Thus, $\boxed{\pi = 3.1416}$ (*Actual value of* $\pi = 3.1415927\ldots$)

[32] *Evaluate* $\int_0^1 \frac{dx}{1+x}$ *taking seven ordinates by applying Simpson's 3/8 th rule. Hence deduce the value of* $\log_e 2$. *Also compute by Trapezoidal rule.*

☞ 7 ordinates means that the given interval [0, 1] must be divided into 6 equal parts. That is $n = 6$. $h = (1-0)/6 = 1/6$; Starting from 0 we keep adding 1/6 to reach the value 1.

The values of x and $y = 1/1 + x$ are tabulated.

x	0	1/6	2/6	3/6	4/6	5/6	6/6
$y = 1/1+x$	1	6/7	3/4	2/3	3/5	6/11	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Note : The values of x are 0, 1/6, 1/3, 1/2, 2/3, 5/6, 1

Simpson's $3/8^{th}$ rule for $n = 6$ is given by,

$$I = \int_a^b y\,dx = \frac{3h}{8}\left[\,(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\,\right]$$

$$\therefore \quad I = \int_0^1 \frac{1}{1+x}\,dx = \frac{1}{16}\left[\left(1+\frac{1}{2}\right) + 3\left(\frac{6}{7}+\frac{3}{4}+\frac{3}{5}+\frac{6}{11}\right) + 2\left(\frac{2}{3}\right)\right] = \boxed{0.6932}$$

To deduce the value of $\log_e 2$, we integrate $y = 1/1+x$ theoretically.

$$\int_0^1 \frac{1}{1+x}\,dx = \left[\,\log_e(1+x)\right]_0^1 = \log_e 2 - \log_e 1 = \log_e 2\,.$$

Equating this with the obtained numerical value of I, we have,

$$\boxed{\log_e 2 = 0.6932}$$

Note : *The actual value of* $\log_e 2 = 0.6931471$

By Trapezoidal rule : $I = \frac{h}{2}\left[\,(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\,\right]$

On substituting the values from the table, $\boxed{I = 0.6949}$

Remark : *Accuracy in the value is relatively less compared to the value obtained from Simpson's $3/8^{th}$ rule and the actual value.*

[33] *Use Simpson's 3/8th rule to evaluate* $\int_1^4 e^{1/x}\,dx$. **[Sep 20]**

☞ To apply Simpson's 3/8th rule, n must be a multiple of 3 and we shall take $n = 3$ itself.

$\therefore$ $h = (4-1)/3 = 1$. The rule for $n = 3$ is,

$$I = \int_a^b y\,dx = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right]$$

The table showing the values of x and the corresponding values of $y = e^{1/x}$ correct to four decimal places is as follows.

x	1	2	3	4
$y = e^{1/x}$	2.7183	1.6487	1.3956	1.2840
	y_0	y_1	y_2	y_3

Substituting these values in the rule with $h = 1$, $\boxed{I = 4.9257}$

Note : $e^{1/x}$ *cannot be integrated theoretically.*

[34] *Find the approximate value of* $\int_0^{\pi/2} \sqrt{\cos\theta}\,d\theta$ *by Simpson's* $1/3^{rd}$ *rule, by dividing* $[0, \pi/2]$ *into 6 equal parts.*

☞ Here $h = (\pi/2 - 0)/6 = \pi/12 = 15^0$

The values of θ and the corresponding values of $y = f(\theta) = \sqrt{\cos\theta}$, correct to four decimal places are tabulated.

θ^0	0^0	15^0	30^0	45^0	60^0	75^0	90^0
$\sqrt{\cos\theta}$	1	0.9828	0.9306	0.8409	0.7071	0.5087	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's 1/3 rd rule for $n = 6$ is given by,

$$I = \int_a^b y\,d\theta = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

Here, $h = \pi/12$; $\pi = 22/7$. On substitution of the values in the rule we get,

$$\boxed{\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta = 1.1873}$$

[35] *Use Simpson's 1/3 rd rule with seven ordinates to evaluate* $\int_2^8 \frac{dx}{\log_{10} x}$

[June 18]

☞ Seven ordinates means that the given interval [2, 8] must be divided into 6 equal parts. ($n = 6$) ; $h = (8-2)/6 = 1$.

The values of x and $y = 1/\log_{10} x$ are tabulated correct to 4 decimals.

x	2	3	4	5	6	7	8
$y=1/\log_{10}x$	3.3219	2.0959	1.661	1.4307	1.2851	1.1833	1.1073
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Substituting the Simpson's $1/3^{rd}$ rule for $n = 6$ as in **[34]** , we get

$$\boxed{\int_2^8 \frac{dx}{\log_{10} x} = 9.7203}$$

Note : *The function* $1/\log_{10} x$ *is not integrable by analytical methods.*

[36] *Use Simpson's 3/8th rule to obtain the approximate value of* $\int_0^{0.3} (1-8x^3)^{1/2}\, dx$ *by considering 3 equal intervals.*

[Feb 21]

☞ Here, $h = (0.3-0)/3 = 0.1$; $n = 3$.

x	0	0.1	0.2	0.3
$y = (1-8x^3)^{1/2}$	1	0.996	0.9675	0.8854
	y_0	y_1	y_2	y_3

Substituting in the Simpson's $3/8^{th}$ rule for $n = 3$ as in **[33]** , we get,

$$\boxed{\int_0^{0.3} (1-8x^3)^{1/2} = 0.2916}$$

[37] *Use Simpson's 1/3rd rule to find* $\int_0^{0.6} e^{-x^2}\,dx$ *by taking 6 sub-intervals.*

[June, Dec 17, Sep 21]

☞ Here $h = (0.6-0)/6 = 0.1$; $n = 6$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$y = e^{-x^2}$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Substituting in the Simpson's 1/3rd rule for $n = 6$ as in **[34]** we get,

$$\boxed{\int_0^{0.6} e^{-x^2}\,dx = 0.5351}$$ (e^{-x^2} *is not integrable by analytical methods.*)

[38] *Evaluate* $\int_0^1 \frac{x\,dx}{1+x^2}$ *by Simpson's 3/8th rule taking seven ordinates and hence find* $\log_e 2$. *Also evaluate by Trapezoidal rule.* **[June 18, Sep 20, 21]**

☞ Seven ordinates means that the interval $[0, 1]$ must be divided into 6 equal parts. $h = (1-0)/6 = 1/6$ and $n = 6$; $x : 0, 1/6, 1/3, 1/2, 2/3, 5/6$. (*Refer* **[32]**)

The corresponding values of $y = x/1 + x^2$ are computed and tabulated.

x	0	1/6	1/3	1/2	2/3	5/6	1
$y = x/1+x^2$	0	6/37	3/10	2/5	6/13	30/61	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Substituting these values in the Simpson's 3/8th rule for $n = 6$ as in **[32]** we have,

$$I = \frac{3(1/6)}{8}\left[(1+1/2)+3(6/37+3/10+6/13+30/61)+2(2/5)\right] \approx \boxed{0.3466}$$

Now integrating theoretically the given function we have,

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x\,dx}{1+x^2} = \frac{1}{2}\left[\log_e(1+x^2)\right]_0^1 = \frac{1}{2}(\log 2 - \log 1) = \frac{1}{2}\log 2$$

Equating this value with the numerical value obtained we have,

$$\frac{1}{2}\log_e 2 = 0.3466 \Rightarrow \boxed{\log_e 2 = 0.6932}$$

By Trapezoidal rule : Substituting the values from the table in the trapezoidal rule as in **[32]** we get $\boxed{I = 0.34425}$

[39] *Evaluate* $\int_4^{5.2} \log_e x\,dx$ *taking 6 equal strips by applying Simpson's 1/3rd rule.*

☞ Here $h = (5.2-4)/6 = 0.2$; $n = 6$.

The values of x and $y = \log_e x$ are tabulated.

x	4	4.2	4.4	4.6	4.8	5.0	5.2
y	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Substituting the values from the table in the rule as in **[34]**, we obtain the value of the integral as $\boxed{1.82785}$

[40] *Use Trapezoidal rule to compute the area bounded by the curve $y = f(x)$, x-axis and the extreme ordinates from the following table.*

x	0	1	2	3	4	5	6
y	0	2	2.5	2.3	2	1.7	1.5

☞ Area $(A) = \int_0^6 y\,dx$. Here $h = 1$ and we have by data,

$$y_0 = 0,\ y_1 = 2,\ y_2 = 2.5,\ y_3 = 2.3,\ y_4 = 2,\ y_5 = 1.7,\ y_6 = 1.5$$

Substituting in the Trapezoidal rule for $n = 6$ as in **[32]** we obtain $\boxed{A = 11.25}$

EXERCISE

Evaluate by using Simpson's 1/3rd rule :

1. $\int_0^1 e^{-x^2}dx$ *by taking* $h = 0.1$ **2.** $\int_0^1 \frac{x^2}{1+x^3}\,dx$ $(n = 7)$. Deduce the value of $\log_e 2$

Evaluate by using Simpson's 3/8th rule :

3. $\int_0^{0.3} (1-8x^3)^{3/2}\,dx$ $(h = 0.05)$ **4.** $\int_0^1 dx/(1+x)^2$ $(n = 6)$

ANSWERS

1. 0.7468 **2.** 0.23105 ; 0.69315 **3.** 0.2765 **4.** 0.5

10 NUMERICAL METHODS - 2

[*Common to all the streams in II Semester*]

10.1 Numerical Solution of Ordinary Differential Equations (ODES)

Consider a differential equation of first order and first degree in the form $\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$, that is $y = y_0$ at $x = x_0$.

This problem of finding y is called an ***initial value problem.***

We discuss five numerical methods for solving an initial value problem.

In these numerical methods, we compute $y(x)$ in the neighbourhood of the value of 'x_0'.

Equivalently, we compute $y(x_0 \pm h)$ where h is small enough. Lesser the value of h, results in greater the accuracy of $y(x_0 \pm h)$.

10.2 Taylor's series method

Consider the initial value problem : $\frac{dy}{dx} = f(x, y)$ and $y(x_0) = y_0$.

The solution $y(x)$ is approximated to a power series in $(x - x_0)$ using Taylor's series. Then we can find the value of y for various values of x in the neighbourhood of x_0.

We have Taylor's series expansion $y(x)$ about the point x_0 in the form :

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots$$

Here $y'(x_0), y''(x_0), \cdots$ denote the value of the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots$ at x_0 which can be found by making use of the data.

WORKED PROBLEMS

[1] *Use Taylor's series method to find y at $x = 0.1, 0.2, 0.3$ considering terms upto the third degree given that $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$.* **[Dec 18]**

Taylor's series expansion of $y(x)$ is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

By data, $x_0 = 0,\ y_0 = 1$ and $y' = x^2 + y^2$

$$\therefore \quad y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) \qquad \ldots (1)$$

We need to compute $y'(0), y''(0), y'''(0)$.

Consider, $y' = x^2 + y^2$; $y'(0) = 0^2 + [y(0)]^2 = 0 + 1 = 1$

Differentiating y' w.r.t x we have,

$$y'' = 2x + 2yy' \ ;$$

$$y''(0) = (2)(0) + 2 \cdot y(0) \cdot y'(0) = (2)(1)(1) = 2$$

Differentiating y'' w.r.t x we have,

$$y''' = 2 + 2[y\,y'' + (y')^2] \ ; \ y'''(0) = 2 + 2[(1)(2) + 1^2] = 8$$

Substituting these values in (1) we have,

$$y(x) = 1 + x \cdot 1 + \frac{x^2}{2} \cdot 2 + \frac{x^3}{6} \cdot 8 = 1 + x + x^2 + \frac{4x^3}{3} \qquad \ldots (2)$$

This is called as Taylor's series approximation upto the third degree and we need to put $x = 0.1, 0.2, 0.3$ in the same. Thus we obtain,

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^3}{3} = \boxed{1.1113}$$

Similarly on substituting $x = 0.2$ and 0.3 in (2), we obtain,

$$\boxed{y(0.2) = 1.2507,\ y(0.3) = 1.426}$$

[2] *Find y at $x = 1.02$ correct to five decimal places given $dy = (xy-1)dx$ and $y = 2$ at $x = 1$ applying Taylor's series method.*

☞ Taylor's series expansion is given by

$$y(x) = y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\dots$$

By data, $x_0 = 1$, $y_0 = 2$ and $y' = \dfrac{dy}{dx} = xy-1$

Since the number of derivatives for approximation is not specifically mentioned, we shall have the approximation upto third degree.

$$\therefore\ y(x) = y(1)+(x-1)y'(1)+\frac{(x-1)^2}{2!}y''(1)+\frac{(x-1)^3}{3!}y'''(1) \qquad \dots(1)$$

Consider, $y' = xy-1$; $y'(1) = (1)(2)-1 = 1$

$y'' = xy'+y$; $y''(1) = (1)(1)+2 = 3$

$y''' = xy''+y'+y'$; $y'''(1) = (1)(3)+1+1 = 5$

To find $y(1.02)$, we shall substitute these values along with $x = 1.02$ in (1).

Thus, $\boxed{y(1.02) = 2.02061}$

[3] *From Taylor's series method, find $y(0.1)$ considering upto fourth degree term if $y(x)$ satisfies the equation $\dfrac{dy}{dx} = x-y^2$, $y(0) = 1$.*

☞ $$y(x) = y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots$$

By data, $x_0 = 0$, $y_0 = 1$, $y' = x-y^2$

$$\therefore\ y(x) = y(0)+xy'(0)+\frac{x^2}{2!}y''(0)+\frac{x^3}{3!}y'''(0)+\frac{x^4}{4!}y^{(4)}(0)+\cdots \dots(1)$$

Consider, $y' = x-y^2$; $y'(0) = 0-1^2 = -1$

$y'' = 1-2yy'$; $y''(0) = 1-(2)(1)(-1) = 3$

$y''' = 0-2[yy''+(y')^2]$; $y'''(0) = -2[(1)(3)+(-1)^2] = -8$

$y^{(4)} = -2[yy'''+y''y'+2y'y''] = -2[yy'''+3y'y'']$ *(fourth derivative)*

$\therefore \quad y^{(4)}(0) = -2[(1)(-8)+(3)(-1)(3)] = 34$

To find $y(0.1)$, we shall substitute these values along with $x = 0.1$ in (1).

$$y(0.1) = 1+(0.1)(-1)+\frac{(0.1)^2}{2}(3)+\frac{(0.1)^3}{6}(-8)+\frac{(0.1)^4}{24}(34)$$

Thus, $\boxed{y(0.1) = 0.9138}$

[4] *Use Taylor's series method to obtain a power series in $(x-4)$ for the equation $5x\frac{dy}{dx}+y^2-2=0$; $x_0 = 4$, $y_0 = 1$ and use it to find y at $x = 4.1, 4.2, 4.3$ correct to four decimal places.*

☞ Taylor's series expansion is given by

$$y(x) = y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\cdots$$

Since $x_0 = 4$, $y_0 = 1$ by data, the series becomes

$$y(x) = y(4)+(x-4)y'(4)+\frac{(x-4)^2}{2}y''(4)+\cdots \quad \ldots (1)$$

Consider, $5xy'+y^2-2=0$

Substituting the initial values we obtain, [**Note :** $y' = y'(x)$]

$$(5)(4)y'(4)+1^2-2=0 \text{ or } 20y'(4) = 1 \; ; \; y'(4) = 1/20 = 0.05$$

Differentiating the given equation w.r.t. x,

$$5[xy''+y']+2yy' = 0 \quad [\textbf{ Note : } y'' = y''(x)]$$

Substituting the initial values and the value of $y'(4)$ we have,

$$5[4y''(4)+0.05]+(2)(1)(0.05) = 0 \text{ or } y''(4) = -0.0175$$

Since the value of the second derivative itself is small enough we shall approximate Taylor's series as in (1) upto second degree terms only. Substituting these values in (1) we have,

$$y(x) = 1+(x-4)(0.05)+\frac{(x-4)^2}{2}(-0.0175)$$

We now find $y(4.1)$, $y(4.2)$ and $y(4.3)$ from this expression.

Thus, $\boxed{y(4.1) = 1.0049,\; y(4.2) = 1.0097,\; y(4.3) = 1.0142}$

[5] *Use Taylor's series method to find* $y(4.1)$ *given that* $\frac{dy}{dx} = \frac{1}{x^2+y}$ *and* $y(4) = 4$. **[Dec 17]**

☞ The given DE is $(x^2+y)y = 1$; $x_0 = 4 = y_0$

Indirect diferentiation as in the previous problem is to be carried out once and on substitution of the relevant values we obtain $y'(4) = -0.05$, $y''(4) = -0.020125$. Substituting in the Taylor's series we get $\boxed{y(4.1) = 4.0049}$

[6] *Use Taylor's series method to solve* $y' = x^2 + y$ *in the range* $0 \le x \le 0.2$ *by taking step size* $h = 0.1$ *given that* $y = 10$ *at* $x = 0$ *initially considering terms upto the fourth degree.*

☞ In this problem, since the step size is specified as 0.1, the problem has to be done in two stages. We have to first find $y(0.1)$ and use this as the initial condition to compute $y(0.2)$. We have to proceed as earlier. From the I stage, $\boxed{y(0.1) = 11.052.}$ Next by taking $x_0 = 0.1$ and $y_0 = 11.052$, proceeding on the same lines we get $\boxed{y(0.2) = 12.2168}$

[7] *Employ Taylor's series method to find y at* $x = 0.1$ *given* $\frac{dy}{dx} - 2y = 3e^x$ *whose solution passes through the origin.* **[Dec 17, Aug 21]**

☞ By data, $y' = 2y + 3e^x$ and $y(0) = 0$. That is $x_0 = 0$, $y_0 = 0$.

Consider, $y' = 2y + 3e^x$; $y'(0) = 2(0) + 3e^0 = 3$

$\therefore$ $y'' = 2y' + 3e^x$; $y''(0) = 2(3) + 3 = 9$

$y''' = 2y'' + 3e^x$; $y'''(0) = 2(9) + 3 = 21$

From the Taylor's series as in **[1]** we have,

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2}y''(0) + \frac{(0.1)^3}{6}y'''(0)$$

On substitution of the values , $\boxed{y(0.1) = 0.3485}$

EXERCISE

Use Taylor's series method to solve the following initial value problems.

1. $\frac{dy}{dx} = x - y,\ y(0) = 1$. Compute $y(0.1)$ considering terms upto 4th degree.
2. $\frac{dy}{dx} = x^2 y - 1, y(0) = 1$. Compute $y\ (0.03)$.
3. $\frac{dy}{dx} = x^2 y - 1, y(0) = 1$ in the range $0 \le x \le 0.2$ taking step size 0.1

ANSWERS

1. 0.8373 2. 0.97 3. 0.9003, 0.8023

10.3 Modified Euler's Method

Consider the initial value problem $\frac{dy}{dx} = f(x, y)\ ;\ y(x_0) = y_0$.

We need to find y at $x_1 = x_0 + h$.

We first obtain $y(x_1) = y_1$ by applying ***Euler's formula*** and this value is regarded as the initial approximation for y_1 usually denoted by $y_1^{(0)}$, also called as the predicted value of y_1.

Euler's formula is given by : $y_1^{(0)} = y_0 + h f(x_0, y_0)$

Since the accuracy is poor in this formula this value y_1 is successively improved (*corrected*) to the desired degree of accuracy by the following ***Modified Euler's formula,*** where the successive approximations are denoted by $y_1^{(1)}, y_1^{(2)}, y_1^{(3)}, \cdots$ etc.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]\ ;\ y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

Each of the succeeding approximation is better than the preceeding ones.

WORKED PROBLEMS

[8] *Given* $\frac{dy}{dx} = 1 + \frac{y}{x}, y = 2$ *at* $x = 1$, *find the approximate value* y *at* $x = 1.4$ *by taking step size* $h = 0.2$ *applying modified Euler's method.* **[Dec 18, Sep 20]**

☞ The problem has to be worked in two stages.

I Stage : $x_0 = 1,\ y_0 = 2,\ f(x,y) = 1 + (y/x),\ h = 0.2$

$x_1 = x_0 + h = 1.2,\ y(x_1) = y_1 = y(1.2) = ?\ ;\ f(x_0, y_0) = 1 + (2/1) = 3$

We have Euler's formula : $y_1^{(0)} = y_0 + h f(x_0, y_0)$ . . . (1)

$\therefore \quad y_1^{(0)} = 2 + (0.2)(3) = 2.6$

Further we have modified Euler's formula :

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \quad \ldots (2)$$

$$y_1^{(1)} = 2 + (0.1)[3 + (1 + y_1^{(0)}/x_1)] = 2 + (0.1)[4 + 2.6/1.2] = 2.6167$$

Next approximation $y_1^{(2)}$ is got just by replacing the value of $y_1^{(1)}$ in place of $y_1^{(0)}$.

Now, $y_1^{(2)} = 2 + (0.1)[4 + 2.6167/1.2] = 2.6181$

$y_1^{(3)} = 2 + (0.1)[4 + 2.6181/1.2] = 2.6182\ ;\ y_1^{(4)} = 2.6182 = y(1.2)$

II Stage : We repeat the process by taking $y(1.2) = 2.6182$ as the initial condition.

$$x_0 = 1.2,\ y_0 = 2.6182\ ;\ f(x_0, y_0) = 1 + (y_0/x_0) = 3.1818$$

$$x_1 = x_0 + h = 1.4,\ y(x_1) = y_1 = y(1.4) = ?$$

From (1), $y_1^{(0)} = 2.6182 + (0.2)(3.1818) = 3.2546$

From (2), $y_1^{(1)} = 2.6182 + (0.1)[3.1818 + (1 + y_1^{(0)}/x_1)]$

ie., $y_1^{(1)} = 2.6182 + (0.1)[4.1818 + 3.2546/1.4] = 3.2689$

$$y_1^{(2)} = 2.6182 + (0.1)[4.1818 + 3.2689/1.4] = 3.2699$$

$$y_1^{(3)} = 2.6182 + (0.1)[4.1818 + 3.2699/1.4] = 3.2699$$

Thus, $\boxed{y(1.4) = 3.2699 \approx 3.27}$

Remark : *Observing the successive values, two steps in the modified Euler's method results into accurate answer.*

[9] *Using modified Euler's method find y at $x = 0.2$ given $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $y(0) = 1$ taking $h = 0.1$. Perform three iterations at each step.* **[Dec 17, Sep 20]**

☞ We need to find $y\,(0.2)$ by taking $h = 0.1$

This implies that the problem has to be done in two stages.

I Stage : By data $x_0 = 0,\ y_0 = 1,\ h = 0.1,\ f(x,y) = 3x + (y/2)$

$$f(x_0, y_0) = 0.5,\ x_1 = x_0 + h = 0.1\ ;\ y(x_1) = y_1 = y(0.1) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + h\,f(x_0, y_0)$ we obtain,

$y_1^{(0)} = 1 + (0.1)(0.5) = 1.05$; Next by modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$y_1^{(1)} = 1 + 0.05\left[0.5 + 3(0.1) + \frac{y_1^{(0)}}{2}\right] = 1.06625$$

The second iterative value is got simply by replacing $y_1^{(0)}$ by $y_1^{(1)}$.

$$y_1^{(2)} = 1 + 0.05\left[0.8 + \frac{1.06625}{2}\right] = 1.0667\ ;\ \boxed{y(0.1) = 1.0667}$$

II Stage : Now, let $x_0 = 0.1,\ y_0 = 1.0667$

We have, $f(x,y) = 3x + (y/2)$; $f(x_0, y_0) = 3(0.1) + \frac{1.0667}{2} = 0.83335$

$$x_1 = x_0 + h = 0.2\ ;\ y_1 = y(x_1) = y(0.2) = ?$$

From Euler's formula : $y_1^{(0)} = 1.0667 + 0.1(0.83335) = 1.15$

Next, from modified Euler's formula,

$$y_1^{(1)} = 1.0667 + \frac{0.1}{2}\left[0.83335 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$y_1^{(1)} = 1.0667 + 0.05\left[0.83335 + 3(0.2) + \frac{y_1^{(0)}}{2}\right] = 1.1671$$

$$y(0.2) = y_1^{(2)} = 1.0667 + 0.05\left[1.43335 + \frac{1.1671}{2}\right] = \boxed{1.167545}$$

[10] *Using modified Euler's method find $y(0.1)$ correct to four decimal places solving the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ taking $h = 0.1$* [June 18]

☞ By data $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $f(x,y) = x - y^2$

$f(x_0, y_0) = 0 - 1^2 = -1$, $x_1 = x_0 + h = 0.1$; $y(x_1) = y_1 = y(0.1) = ?$

From Euler's formula : $y_1^{(0)} = y_0 + h f(x_0, y_0)$ we obtain, $y_1^{(0)} = 0.9$

We have modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + \frac{0.1}{2}\left[-1 + x_1 - \left\{y_1^{(0)}\right\}^2\right] = 1 + 0.05[-0.9 - (0.9)^2] = 0.9145$$

$$y(0.1) = y_1^{(2)} = 1 + 0.05[-0.9 - (0.9145)^2] = \boxed{0.9132}$$

[11] *Using modified Euler's method find $y(20.2)$ and $y(20.4)$ given that $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right)$ with $y(20) = 5$ taking $h = 0.2$* [June 17]

☞ We shall first compute $y(20.2)$ and use this value to compute $y(20.4)$

I Stage : By data $x_0 = 20$, $y_0 = 5$ and $h = 0.2$

$$f(x,y) = \log_{10}\left(\frac{x}{y}\right) ; \; f(x_0, y_0) = \log_{10}(4) = 0.6021$$

$$x_1 = x_0 + h = 20.2 \; ; \; y(x_1) = y_1 = y(20.2) = ?$$

Euler's formula : $y_1^{(0)} = y_0 + h f(x_0, y_0)$; $y_1^{(0)} = 5.1204$

Next by modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 5 + \frac{0.2}{2}\left[0.6021 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$$

$$y_1^{(1)} = 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1204}\right)\right] = 5.1198$$

$$\therefore \quad y(20.2) = y_1^{(2)} = 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1198}\right)\right] = \boxed{5.1198}$$

II stage : Now, let $x_0 = 20.2,\ y_0 = 5.1198$

$$f(x,y) = \log_{10}\left(\frac{x}{y}\right);\ f(x_0, y_0) = 0.5961$$

$$x_1 = x_0 + h = 20.4,\ y(x_1) = y_1 = y(20.4) = ?$$

By Euler's formula, $y_1^{(0)} = 5.1198 + (0.2)(0.5961) = 5.239$

Now by modified Euler's formula,

$$y_1^{(1)} = 5.1198 + \frac{0.2}{2}\left[0.5961 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$$

$$y_1^{(1)} = 5.1198 + 0.1\left[0.5961 + \log_{10}\left(\frac{20.4}{5.239}\right)\right] = 5.2384$$

$$y(20.4) = y_1^{(2)} = 5.1198 + 0.1\left[0.5961 + \log_{10}\left(\frac{20.4}{5.2384}\right)\right] = \boxed{5.2385}$$

[12] *Use modified Euler's method to solve* $\frac{dy}{dx} = x + \left|\sqrt{y}\right|$ *in the range* $0 \le x \le 0.4$ *by taking* $h = 0.2$ *given that* $y = 1$ *at* $x = 0$ *initially.* **[June, 17, Dec 18, Feb 21]**

☞ We need to compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$

I Stage : By data $x_0 = 0,\ y_0 = 1,\ f(x,y) = x + \sqrt{y},\ h = 0.2$ where the modulus sign indicates that we have to take only the positive value of $\sqrt{y}$.

$$f(x_0, y_0) = 0 + \sqrt{1} = 1.\ \ x_1 = x_0 + h = 0.2\ ;\ y(x_1) = y_1 = y(0.2) = ?$$

Euler's formula : $y_1^{(0)} = y_0 + h\,f(x_0, y_0)$ gives $y_1^{(0)} = 1.2$

We have modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 1 + \frac{0.2}{2}\left[1 + x_1 + \sqrt{y_1^{(0)}}\right]$$

$$y_1^{(1)} = 1 + 0.1[1 + 0.2 + \sqrt{1.2}] = 1.2295$$

$$y(0.2) = y_1^{(2)} = 1 + 0.1[1.2 + \sqrt{1.2295}] = \boxed{1.2309}$$

II Stage : Now, let $x_0 = 0.2,\ y_0 = 1.2309$

$$f(x, y) = x + \sqrt{y}\ ;\ f(x_0, y_0) = 0.2 + \sqrt{1.2309} = 1.3095$$

$$x_1 = x_0 + h = 0.4\ ;\ y(x_1) = y_1 = y(0.4) = ?$$

Proceeding as before, $\boxed{y(0.4) = 1.5254}$

[13] *Use modified Euler's method to compute $y(0.1)$ given that $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ by taking $h = 0.05$ considering the accuracy upto two approximations in each step.* **[Feb 21]**

☞ We need to compute $y(0.05)$ first and use this value to compute $y(0.1)$. Proceeding as in earlier problems we can obtain $\boxed{y(0.05) = 1.0513}$ and subsequently $\boxed{y(0.1) = 1.1055}$

10.4 Runge-Kutta method of fourth order

Consider the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

We need to find $y(x_0 + h)$, where h is the step size.

We have to first compute k_1, k_2, k_3, k_4 by the following formulae.

$$k_1 = h f(x_0, y_0)\ ;\ k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)\ ;\ k_4 = h f(x_0 + h, y_0 + k_3)$$

The required $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

WORKED PROBLEMS

[14] *Given* $\frac{dy}{dx} = 3x + \frac{y}{2}, y(0) = 1$, *compute* $y(0.2)$ *by taking* $h = 0.2$ *using Runge-Kutta method of fourth order.* **[Feb 21]**

☞ By data, $f(x,y) = 3x + \frac{y}{2}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h f(x_0, y_0) = (0.2) f(0,1) = (0.2)\left[3 \times 0 + \frac{1}{2}\right] = 0.1$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$$

$$k_2 = (0.2) f(0.1, 1.05) = (0.2)\left[3 \times 0.1 + \frac{1.05}{2}\right] = 0.165$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right)$$

$$k_3 = (0.2) f(0.1, 1.0825) = (0.2)\left[3 \times 0.1 + \frac{1.0825}{2}\right] = 0.16825$$

$$k_4 = h f(x_0 + h, y_0 + k_3) = (0.2) f(0.2, 1.16825)$$

$$k_4 = (0.2)\left[3 \times 0.2 + \frac{1.16825}{2}\right] = 0.236825$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Thus, $y(0.2) = 1 + \frac{1}{6}(0.1 + 2 \times 0.165 + 2 \times 0.16825 + 0.236825) = \boxed{1.1672}$

[15] *Use fourth order Runge-Kutta method to solve,* $(x+y)\frac{dy}{dx} = 1, y(0.4) = 1$ *at* $x = 0.5$ *correct to four decimal places.* **[Aug 21]**

☞ We have, $\frac{dy}{dx} = \frac{1}{x+y}$ and $y = 1$ at $x = 0.4$

$$f(x,y) = \frac{1}{x+y},\ x_0 = 0.4,\ y_0 = 1,\ y(0.5) = ?$$

Here, $x_0 + h = 0.5 \ \therefore\ h = 0.5 - x_0 = 0.5 - 0.4 = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h f(x_0, y_0) = (0.1) f(0.4, 1) = (0.1)\left[\frac{1}{0.4+1}\right] = 0.0714$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0714}{2}\right)$$

$$k_2 = (0.1) f(0.45, 1.0357) = (0.1)\left[\frac{1}{0.45+1.0357}\right] = 0.0673$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1) f\ (0.45, 1.03365)$$

$$k_3 = (0.1)\left[\frac{1}{0.45+1.03365}\right] = 0.0674$$

$$k_4 = h f(x_0 + h,\ y_0 + k_3) = (0.1) f(0.5, 1.0674)$$

$$k_4 = (0.1)\left[\frac{1}{0.5+1.0674}\right] = 0.0638$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Thus, $y(0.5) = 1 + \frac{1}{6}[\,0.0714 + 2(0.0673) + 2(0.0674) + 0.0638\,] = \boxed{1.0674}$

[16] *Using Runge-Kutta method of fourth order, find y (0.2) for the equation* $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ *taking* $h = 0.2$. **[Dec 17, June 18]**

☞ By data, $f(x, y) = \frac{y-x}{y+x}, x_0 = 0,\ y_0 = 1,\ h = 0.2$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0) = (0.2)f(0,1) = (0.2)\left[\frac{1-0}{1+0}\right] = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)f(0.1, 1.1) = 0.1667$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f(0.1, 1.0835) = 0.1662$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)f(0.2, 1.1662) = 0.1414$$

$$y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \; ; \; \boxed{y(0.2) = 1.1679}$$

[17] *Use fourth order Runge-Kutta method to find y at x = 0.1 given that* **[Sep 20]**

$$\frac{dy}{dx} = 3e^x + 2y, y(0) = 0 \text{ and } h = 0.1.$$

☞ By data, $f(x,y) = 3e^x + 2y,\ x_0 = 0, y_0 = 0,\ h = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0) = (0.1)f(0,0) = (0.1)[3e^0 + 2\times 0] = 0.3$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(0.05, 0.15) = 0.3454$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.05, 0.1727) = 0.3499$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)f(0.1, 0.3499) = 0.4015$$

$$y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \; ; \; \boxed{y(0.1) = 0.3487}$$

[18] *Using Runge-Kutta method of fourth order solve* $\frac{dy}{dx} + y = 2x$ *at* $x = 1.1$ *given that* $y = 3$ *at* $x = 1$ *initially.*

☞ We have, $\frac{dy}{dx} = 2x - y, x_0 = 1,\ y_0 = 3$

$$f(x,y) = 2x - y,\ x_0 + h = 1.1 \quad \therefore h = 0.1$$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h f(x_0, y_0) = (0.1) f(1,3) = (0.1)[2(1) - 3] = -0.1$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) f\left(1.05, 3 - \frac{0.1}{2}\right) = -0.085$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1) f\left(1.05, 3 - \frac{0.085}{2}\right) = -0.08575$$

$$k_4 = h f(x_0 + h,\ y_0 + k_3) = (0.1) f(1.1, 3 - 0.08575) = -0.071425$$

$$y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\ ;\ \boxed{y(1.1) = 2.9145}$$

[19] *Solve :* $(y^2 - x^2)\,dx = (y^2 + x^2)\,dy$ *for* $x = 0.2$ *given that* $y = 1$ *at* $x = 0$ *initially, by applying Runge-Kutta method of order 4.* **[June 17]**

☞ We have $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$; $x_0 = 0,\ y_0 = 1,\ h = 0.2$

This problem is similar to **[16]**. On computation we obtain $k_1 = 0.2,\ k_2 = 0.1967 = k_3,\ k_4 = 0.1891$ and finally $\boxed{y\,(0.2) = 1.196}$

EXERCISE

Apply modified Euler's method to solve the following initial value problems by considering the accuracy upto two approximations in every step.

1. $\dfrac{dy}{dx} = -xy^2, y(0) = 2.$ Compute $y\,(0.2)$ by taking $h = 0.1$

2. $\dfrac{dy}{dx} = x + y, y(0) = 1.$ Compute $y\,(0.2)$ by taking $h = 0.1$

Use fourth order Runge-Kutta method to solve the following initial value problems.

3. $\dfrac{dy}{dx} = x + y^2, y(0) = 1.$ Compute $y\,(0.2)$ by taking $h = 0.1$

4. $\dfrac{dy}{dx} = xy, y(1) = 2.$ Compute $y\,(1.2)$ by taking $h = 0.2$.

ANSWERS

1. 1.923 2. 1.2427 3. 1.2736 4. 2.4921

10.5 Milne's Predictor and Corrector method

In this method the value of y at a desired value of x is estimated from a set of four values of y corresponding to four equally spaced values of x. The four values may be readily available or be generated using the given initial condition by Taylor's series method.

Consider the DE: $y' = \frac{dy}{dx} = f(x, y)$ with a set of 4 pre determined values of

$y : y(x_0) = y_0, y(x_1) = y_1, y(x_2) = y_2$ and $y(x_3) = y_3$.

$x : x_0, x_1, x_2, x_3$ are equidistant values with width h. $x_4 = x_3 + h$

Predictor and Corrector formulae to compute $y(x_4) = y_4$ are as follows.

- **Milne's predictor and corrector formulae**

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')\ldots \qquad \textit{(Predictor formula)}$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')\ldots \qquad \textit{(Corrector formula)}$$

Working procedure for problems

- We first prepare the table showing the values of y corresponding to four equidistant values of x and the computation of $y' = f(x, y)$.
- We compute y_4 from the predictor formula.
- We use this value of y_4 to compute $y_4' = f(x_4, y_4)$
- We apply corrector formula to obtain the corrected value of y_4.

Note : *The process can be continued till we get consistency in two consecutive values of y_4.*

WORKED PROBLEMS

[20] *Given that* $\frac{dy}{dx} = x - y^2$ *and the data* $y(0) = 0, y(0.2) = 0.02,$ $y(0.4) = 0.0795, y(0.6) = 0.1762$. *Compute* $y(0.8)$ *by applying Milne's method.*

[June 18, Sep 20, Feb 21]

☞ We prepare the following table using the given data which is essentially required for applying the predictor and corrector formulae.

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y'_0 = 0 - 0^2 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y'_1 = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y'_2 = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y'_3 = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	

We have the predictor formula : $y_4^{(P)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$

$$\therefore \quad y_4^{(P)} = 0 + \frac{4(0.2)}{3}[2(0.1996) - 0.3937 + 2(0.5689)] = 0.3049$$

Now, $y'_4 = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$

Next we have the corrector formula : $y_4^{(C)} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$

$$y(0.8) = y_4^{(c)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.707] = \boxed{0.3046}$$

[21] *Apply Milne's method to compute* $y(1.4)$ *corrector to four decimal places given* $\frac{dy}{dx} = x^2 + \frac{y}{2}$ *and following data :*

$$y(1) = 2,\ y(1.1) = 2.2156,\ y(1.2) = 2.4649,\ y(1.3) = 2.7514$$

☞ First we shall prepare the following table.

x	y	$y' = x^2 + \frac{y}{2}$
$x_0 = 1$	$y_0 = 2$	$y'_0 = 1^2 + \frac{2}{2} = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y'_1 = (1.1)^2 + \frac{2.2156}{2} = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y'_2 = (1.2)^2 + \frac{2.4649}{2} = 2.67245$
$x_3 = 1.3$	$y_3 = 2.7514$	$y'_3 = (1.3)^2 + \frac{2.7514}{2} = 3.0657$
$x_4 = 1.4$	$y_4 = ?$	

We have $y_4^{(P)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$

$$\therefore \quad y_4^{(P)} = 2 + \frac{4(0.1)}{3}[2(2.3178) - 2.67245 + 2(3.0657)] = 3.0793$$

Hence, $y'_4 = x_4^2 + \frac{y_4}{2} = (1.4)^2 + \frac{3.0793}{2} = 3.49965$

Now consider, $y_4^{(C)} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$

$$y(1.4) = y_4^{(C)} = 2.4649 + \frac{0.1}{3}[2.67245 + 4(3.0657) + 3.49965] = \boxed{3.0794}$$

[22] *Use Taylor's series method (upto third derivative term) to find y at $x = 0.1, 0.2, 0.3$ given that $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 1$.*

Apply Milne's predictor-corrector formulae to find $y(0.4)$ using the generated set of initial values.

☞ Referring to Problem - **[1]**, we have obtained $y(0.1) = 1.1113$, $y(0.2) = 1.2507$, $y(0.3) = 1.426$. Using these values along with $y(0) = 1$ initially, we prepare the following table.

x	y	$y' = x^2 + y^2$
$x_0 = 0$	$y_0 = 1$	$y'_0 = 0^2 + 1^2 = 1$
$x_1 = 0.1$	$y_1 = 1.1113$	$y'_1 = (0.1)^2 + (1.1113)^2 = 1.245$
$x_2 = 0.2$	$y_2 = 1.2507$	$y'_2 = (0.2)^2 + (1.2507)^2 = 1.6043$
$x_3 = 0.3$	$y_3 = 1.426$	$y'_3 = (0.3)^2 + (1.426)^2 = 2.1235$
$x_4 = 0.4$	$y_4 = ?$	

$y_4^{(P)} = 1.6844$ on substitution in the predictor formula.

Hence, $y'_4 = x_4^2 + y_4^2 = (0.4)^2 + (1.6844)^2 = 2.9972$

$\boxed{y\ (0.4) = y_4^{(c)} = 1.6872}$ on substitution in the corrector formula.

[23] *The following table gives the solution of $5xy' + y^2 - 2 = 0$. Find the value of y at $x = 4.4$ using Milne's predictor and corrector formulae. Use the corrector formula twice.*

x	4	4.1	4.2	4.3
y	1	1.0049	1.0097	1.0143

[June 17]

☞ By data $y' = \dfrac{2 - y^2}{5x}$ and the relavent table is prepared.

We prepare the following table.

x	y	$y' = \dfrac{2 - y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y'_0 = \dfrac{2 - 1^2}{5 \times 4} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y'_1 = \dfrac{2 - (1.0049)^2}{5 \times 4.1} = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y'_2 = \dfrac{2 - (1.0097)^2}{5 \times 4.2} = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$y'_3 = \dfrac{2 - (1.0143)^2}{5 \times 4.3} = 0.0452$

Proceeding as earlier, we get $y_4^{(P)} = 1.0561$; $y_4' = 0.0402$ and further $y_4^{(c)} = 1.0186$. Applying the corrector formula once again we get the required

$\boxed{y(4.4) = 1.0187.}$

[24] If $\dfrac{dy}{dx} = 2e^x - y$, $y(0) = 2, y(0.1) = 2.010, y(0.2) = 2.040$ and $y(0.3) = 2.090$, find $y(0.4)$ using Milne's method. [Dec 17, Aug 21]

We prepare the following table.

x	y	$y' = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$y_0' = 2e^0 - 2 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y_1' = 2e^{0.1} - 2.01 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y_2' = 2e^{0.2} - 2.04 = 0.4028$
$x_3 = 0.3$	$y_3 = 2.090$	$y_3' = 2e^{0.3} - 2.09 = 0.6097$
$x_4 = 0.4$	$y_4 = ?$	

Substituting in the Milne's predictor formula,

$$\therefore \quad y_4^{(P)} = 2 + \frac{4(0.1)}{3}[2(0.2003) - 0.4028 + 2(0.6097)] = 2.1623$$

Now $y_4' = 2e^{0.4} - 2.1623 = 0.8213$

Next, by substituting in the corrector formula,

$$y(0.4) = y_4^{(C)} = 2.04 + \frac{0.1}{3}[0.4028 + 4(0.6097) + 0.8213] = 2.1621$$

Thus $\boxed{y(0.4) = 2.1621}$

EXERCISE

1. Given $\frac{dy}{dx} + xy^2 = 0$ and the following table, compute $y(0.8)$ using Milne's method.

x	0	0.2	0.4	0.6
y	2	1.9231	1.7241	1.4706

2. Use Taylor's series method to obtain the solution as a power series in x (upto the third derivative terms) given that $\frac{dy}{dx} + y^2 = x, y(0) = 0$. Using this, generate the values of y corresponding to $x = 0.2, 0.4, 0.6$ correct to four decimal places. Then apply Milne's predictor- corrector formulae to compute y at $x = 0.8$.

ANSWERS

1. 1.22 2. 0.3046

APPENDIX - 1 CURVILINEAR COORDINATES

[*For the CS stream in II Semester*]

Definitions

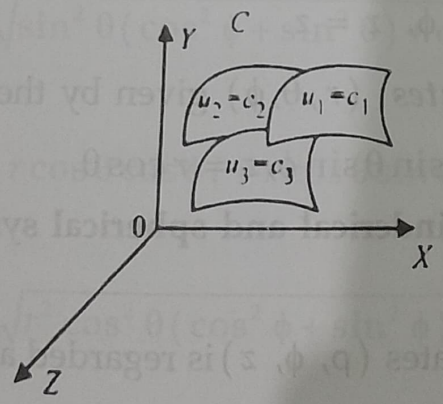
1.1 Curvilinear coordinates, curvilinear coordinate surfaces and curvilinear coordinate curves

Let the coordinates of any point P in space be (x, y, z) in the cartesian system. Suppose x, y, z are expressible interms of new coordinates (u_1, u_2, u_3), we can say that x, y, z are functions of u_1, u_2, u_3. Let us suppose that we are also in a position to express u_1, u_2, u_3 in terms of x, y, z by solving / eliminating. Then the coordinates (u_1, u_2, u_3) are known as ***curvilinear coordinates*** of the point P, where it is assumed that the correspondence between (x, y, z) and (u_1, u_2, u_3) is unique.

The surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ where c_1, c_2, c_3 are constants, are called ***coordinate surfaces*** and the intersection of each pair of these surfaces give rise to curves called ***coordinate curves***.

- **Orthogonal curvilinear coordinates**

A system of curvilinear coordinates is said to be orthogonal if at each point, the tangents to the coordinate curves are mutually perpendicular.



- **Unit vectors, scale factors and orthogonality condition**

Suppose $\vec{r} = xi + yj + zk$ be the position vector of a point in space, we have $\vec{r} = \vec{r}(u_1, u_2, u_3)$.

$\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ are called the *tangent vectors* to the coordinate curves and the *unit tangent vectors* in the same direction are respectively,

$$\hat{e}_1 = \frac{\partial \vec{r}}{\partial u_1} \Big/ \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \ \hat{e}_2 = \frac{\partial \vec{r}}{\partial u_2} \Big/ \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \ \hat{e}_3 = \frac{\partial \vec{r}}{\partial u_3} \Big/ \left|\frac{\partial \vec{r}}{\partial u_3}\right|$$

The quantities $h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$ are called *scale factors.*

For the ***orthogonality of the curvilinear coordinate system*** we must have,

$$\hat{e}_1 \cdot \hat{e}_2 = 0, \ \hat{e}_2 \cdot \hat{e}_3 = 0, \ \hat{e}_3 \cdot \hat{e}_1 = 0.$$

These are analogous to the property of basic unit vectors in the cartesian system $i \cdot j = 0, j \cdot k = 0, k \cdot i = 0$

We have, $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \ \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \ \hat{e}_3 \times \hat{e}_1 = \hat{e}_2.$

Thus $\hat{e}_1, \hat{e}_2, \hat{e}_3$ form a right handed system of vectors. If $\vec{A}$ is any vector in the orthogonal curvilinear coordinate system then,

$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$ where A_1, A_2, A_3 are scalar functions of u_1, u_2, u_3.

In addition to the well acquainted rectangular cartesian coordinates (x, y, z), we introduce two new set of coordinates.

(i) ***Cylindrical polar coordinates*** (ρ, ϕ, z) given by the transformation :

$$x = \rho\cos\phi, \ y = \rho\sin\phi, \ z = z$$

(ii) ***Spherical polar coordinates*** (r, θ, ϕ) given by the transformation :

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$$

1.2 Scale factors of the cylindcrical and spherical systems

(i) Cylindrical system

The cylindrical polar coordinates (ρ, ϕ, z) is regarded as a particular case of the general orthogonal curvilinear coordinates (u_1, u_2, u_3) by setting $(u_1, u_2, u_3) = (\rho, \phi, z)$ and are related to the catesian coordinates (x, y, z) by the transformation :

$$x = \rho\cos\phi, \ y = \rho\sin\phi, \ z = z$$

Thus, $\vec{r} = xi + yj + zk$ becomes,

$$\vec{r} = \rho\cos\phi i + \rho\sin\phi j + zk$$

We have by the definition of scale factors,

$$h_1 = \left|\frac{\partial \vec{r}}{\partial \rho}\right| = |\cos\phi\, i + \sin\phi\, j + 0k| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = |-\rho\sin\phi\, i + \rho\cos\phi\, j + 0k| = \sqrt{\rho^2(\sin^2\phi + \cos^2\phi)} = \rho$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial z}\right| = |0i + 0j + 1k| = \sqrt{0^2 + 0^2 + 1^2} = 1$$

Thus $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$ **for the cylindrical system.**

(ii) Spherical System

We have, $(u_1, u_2, u_3) = (r, \theta, \phi)$ and by the transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, we have,

$$\vec{r} = r\sin\theta\cos\phi\, i + r\sin\theta\sin\phi\, j + r\cos\theta\, k$$

$$h_1 = \left|\frac{\partial \vec{r}}{\partial r}\right| = |\sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k|$$

ie., $$h_1 = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\theta) + \cos^2\theta} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = |r\cos\theta\cos\phi\, i + r\cos\theta\sin\phi\, j - r\sin\theta k|$$

ie., $$h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta}$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = |-r\sin\theta\sin\phi\, i + r\sin\theta\cos\phi\, j + 0k|$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \sqrt{r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)} = r\sin\theta$$

Thus, $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$ **for the spherical system.**

Note : $\vec{r} = xi + yj + zk$ *will give us* $h_1 = 1 = h_2 = h_3$ *for the cartesian system.*

1.3 Orthogonality of the cylindrical system

We have for the cylindrical system $\vec{r} = \rho\cos\phi\, i + \rho\sin\phi\, j + zk$

Let $\hat{e}_\rho,\ \hat{e}_\phi,\ \hat{e}_z$ be the basic unit vectors of this system.

Further we have, $h_1 = 1,\ h_2 = \rho,\ h_3 = 1$ for the cylindrical system.

Using the definition of the unit vectors,

$$\hat{e}_\rho = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial \rho} = (\cos\phi\, i + \sin\phi\, j + 0k)$$

$$\hat{e}_\phi = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{\rho}(-\rho\sin\phi\, i + \rho\cos\phi\, j + 0k) = -\sin\phi\, i + \cos\phi\, j + 0k$$

$$\hat{e}_z = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial z} = \frac{1}{1}(0i + 0j + 1k) = 0i + 0j + 1k$$

Now, $\hat{e}_\rho \cdot \hat{e}_\phi = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0,\ \hat{e}_\phi \cdot \hat{e}_z = 0,\ \hat{e}_z \cdot \hat{e}_\rho = 0.$

Thus the cylindrical system is orthogonal.

1.4 Orthogonality of the spherical system

We have for the spherical system,

$$\vec{r} = r\sin\theta\cos\phi\, i + r\sin\theta\sin\phi\, j + r\cos\theta\, k$$

and let $\hat{e}_r,\ \hat{e}_\theta,\ \hat{e}_\phi$ be the basic unit vectors of this system.

Further we have $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$ for the spherical system.

Now, $$\hat{e}_r = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k$$

$$\hat{e}_\theta = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = \frac{1}{r}(r\cos\theta\cos\phi\, i + r\cos\theta\sin\phi\, j - r\sin\theta\, k)$$

ie., $$\hat{e}_\theta = \cos\theta\cos\phi\, i + \cos\theta\sin\phi\, j - \sin\theta\, k$$

$$\hat{e}_\phi = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{r\sin\theta}(-r\sin\theta\sin\phi\, i + r\sin\theta\cos\phi\, j + 0\, k)$$

$$\hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j + 0\, k$$

i.e.,

Now, $\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - \sin\theta\cos\theta = 0$

$$\hat{e}_r \cdot \hat{e}_\theta = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi = 0$$

$$\hat{e}_\phi \cdot \hat{e}_r = -\sin\theta\cos\phi\sin\phi + \sin\theta\cos\phi\sin\phi = 0$$

Thus the spherical system is orthogonal.

WORKED PROBLEMS

[1] *Express the vector* $\vec{A} = zi - 2x\, j + y\, k$ *in cylindrical coordinates.*

☞ Let, $\vec{A} = A_1\hat{e}_\rho + A_2\hat{e}_\phi + A_3\hat{e}_z$ . . . (1)

where A_1, A_2, A_3 are to be determined and we have from (1),

$$\vec{A}\cdot\hat{e}_\rho = A_1\ ;\ \vec{A}\cdot\hat{e}_\phi = A_2\ ;\ \vec{A}\cdot\hat{e}_z = A_3 \qquad \ldots (2)$$

In clyindrical system, we have $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$

$$\therefore\quad \vec{A} = zi - 2\rho\cos\phi\, j + \rho\sin\phi\, k \qquad \ldots (3)$$

We also have, $\hat{e}_\rho = \dfrac{1}{h_1}\dfrac{\partial\vec{r}}{\partial\rho};\ \hat{e}_\phi = \dfrac{1}{h_2}\dfrac{\partial\vec{r}}{\partial\phi};\ \hat{e}_z = \dfrac{1}{h_3}\dfrac{\partial\vec{r}}{\partial z}$

where $h_1 = 1,\ h_2 = \rho,\ h_3 = 1$ and $\vec{r} = \rho\cos\phi\, i + \rho\sin\phi\, j + zk$

Hence $\hat{e}_\rho = \cos\phi\, i + \sin\phi\, j,\ \hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j,\ \hat{e}_z = k$ . . . (4)

Hence (2) as a consequence (3) and (4) becomes,

$$A_1 = z\cos\phi - 2\rho\sin\phi\cos\phi = z\cos\phi - \rho\sin 2\phi$$

$$A_2 = -z\sin\phi - 2\rho\cos^2\phi = -(z\sin\phi + 2\rho\cos^2\phi)$$

$$A_3 = \rho\sin\phi.$$

Thus, by using these results in (1) we have,

$$\vec{A} = (z\cos\phi - \rho\sin 2\phi)\hat{e}_\rho - (z\sin\phi + 2\rho\cos^2\phi)\,\hat{e}_\phi + \rho\sin\phi\,\hat{e}_z$$

[2] *Express the vector* $\vec{A} = 2x\,i - 3y^2\,j + xz\,k$ *in the cylindrical polar coordinate system.*

☞ (This example is similar to the previous one. Common steps are omitted)

$$\vec{A} = 2\rho\cos\phi\, i - 3\rho^2\sin^2\phi\, j + \rho\cos\phi\, z\, k$$

Also, $\hat{e}_\rho = \cos\phi\, i + \sin\phi\, j,\ \hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j, \hat{e}_z = k$

$$A_1 = \vec{A}\cdot\hat{e}_\rho = 2\rho\cos^2\phi - 3\rho^2\sin^3\phi$$

$$A_2 = \vec{A}\cdot\hat{e}_\phi = -2\rho\sin\phi\cos\phi - 3\rho^2\sin^2\phi\cos\phi$$

$$A_3 = \vec{A}\cdot\hat{e}_z = \rho\cos\phi\, z$$

Thus, $\vec{A} = A_1\,\hat{e}_\rho + A_2\,\hat{e}_\phi + A_3\,\hat{e}_z$ **becomes**

$$\vec{A} = (2\rho\cos^2\phi - 3\rho^2\sin^3\phi)\,\hat{e}_\rho - (2\rho\sin\phi\cos\phi + 3\rho^2\sin^2\phi\cos\phi)\,\hat{e}_\phi + (\rho\cos\phi\, z)\,\hat{e}_z$$

[3] *Represent* $\vec{F} = y\,i - z\,j + x\,k$ *in spherical polar coordinates and hence find* F_r, F_θ, F_ϕ.

☞ $\vec{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta + F_\phi\,\hat{e}_\phi$ ... (1)

where F_r, F_θ, F_ϕ are to be determined.

$\therefore \quad F\cdot\hat{e}_r = F_r,\ \vec{F}\cdot\hat{e}_\theta = F_\theta,\ \vec{F}\cdot\hat{e}_\phi = F_\phi$ ... (2)

In terms of spherical polar coordinates, the given function

$$\vec{F} = (r\sin\theta\sin\phi)\,i - (r\cos\theta)\,j + (r\sin\theta\cos\phi)\,k \quad \text{... (3)}$$

Also we have, $\hat{e}_r = \dfrac{1}{h_1}\dfrac{\partial\vec{r}}{\partial r}$; $\hat{e}_\theta = \dfrac{1}{h_2}\dfrac{\partial\vec{r}}{\partial\theta}$; $\hat{e}_\phi = \dfrac{1}{h_3}\dfrac{\partial\vec{r}}{\partial\phi}$

where $\vec{r} = (r\sin\theta\cos\phi)\,i + (r\sin\theta\sin\phi)\,j + (r\cos\theta)\,k$ and

$h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta.$

$$\therefore \quad \hat{e}_r = \sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k$$

$$\hat{e}_\theta = \cos\theta\cos\phi\, i + \cos\theta\sin\phi\, j - \sin\theta\, k$$

$$\hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j$$

Thus we have from (2), the required F_r, F_θ, F_ϕ is as follows.

$$F_r = r\sin^2\theta\sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi + r\sin\theta\cos\theta\cos\phi$$

$$F_\theta = r\sin\theta\cos\theta\sin\phi\cos\phi - r\cos^2\theta\sin\phi - r\sin^2\theta\cos\phi$$

$$F_\phi = -r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$$

[4] *Obtain the Jacobian of the cylindrical system.*

We have, $x = \rho\cos\phi,\ y = \rho\sin\phi$ and $z = z$

$$\text{Jacobian } (J) = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$\therefore\ J = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(\rho\cos^2\phi + \rho\sin^2\phi) = \rho$$

Thus $J = \rho$ **for the cylindrical system.**

[5] *Obtain the Jacobian of the spherical system.*

We have, $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi, z = r\cos\theta$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

On expanding by last row,

$$J = \cos\theta\left[r^2\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi)\right] + r\sin\theta\left[r\sin^2\theta(\cos^2\phi + \sin^2\phi)\right]$$

$$J = r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta$$

Thus $J = r^2\sin\theta$ **for the spherical system.**