4.3. Countable and Uncountable Sets:

- Any finite set is said to be countable if in the sense that all its elements can be indexed in the same way those are described.

- Not all infinite sets are countable.

  An infinite set is said to be countable if it can be put into one-to-one correspondence with N (natural num), that is, if there is a bijection from N to the set (or vice-versa), if its elements can be indexed.

- Examples of countable infinite sets (countably infinite sets)

①
$$\begin{array}{cccccccc} E = \{ & 0, & 2, & 4, & 6, & 8, & \dots & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ N = \{ & 0, & 1, & 2, & 3, & 4, & \dots & \} \end{array}$$

$f : N \to E$ $\qquad$ $\boxed{\text{I} \wedge \text{S}}$

$f(n) = 2n$

② $O = \{1, 3, 5, 7, \dots\}$

$N = \{0, 1, 2, 3, \dots\}$

$f : N \to O$ $\qquad$ $\boxed{\text{I} \wedge \text{S}}$

$f(n) = 2n+1$

✓ Examples of uncountable (uncountably infinite) sets:

① $Z^N = P(N) = \{ \emptyset,$

$\{0\}, \{1\}, \{2\}, \{3\}, \ldots$

$\{0,1\}, \{0,2\}, \{0,3\}, \ldots, \{1,2\}, \{1,3\}, \ldots$

$\{0,1,2\}, \{0,1,3\}, \{0,1,4\}, \ldots,$

$\ldots$

$\ldots$

$N \}$

— can not be indexed

Power set of an infinite set is uncountable.

② $\{ x \mid x \in R \wedge x > 0.0001 \wedge x < 0.0002 \}$

* Can the prime number be indexed?

✓ A positive integer can be expressed in terms of prime or the sequence of prime in non-decreasing order.

$2, 3, 4 = 2 \times 2, 5, 6 = 2 \times 3, 7, \ldots, 20 = 2 \times 2 \times 5, \ldots, 31,$

$\ldots, 35 = 5 \times 7$

✓ Primality:

Divisibility: 2 (1st prime), $3, 5, 7, \ldots, m$

where $m \leq \sqrt{n}$

✓ Prime-1 = 2, Prime-2 = 3, ...

P_1 P_2

$P_1 \times P_2 \times P_3 \times \dots \times P_m + 1$ is a prime.

[otherwise, $P_1 \times P_2 \times P_3 \times \dots \times P_m$ and 1 have common prime factor, which is not possible]

✓ If all primes in the sequence $P_1, P_2, P_3, \dots, P_n$ are known, then a new prime may be found, which is,

$$P_1 \times P_2 \times P_3 \times \dots \times P_n + 1$$

* There are infinitely many primes!

✓ Not all primes can be written in the form

$$P_1 \times P_2 \times P_3 \times \dots \times P_n + 1$$

For example: 5, 11, 13, 17, 19, 23, 29 | can't be indexed so far.

* There is always a scope of discovering a new prime!