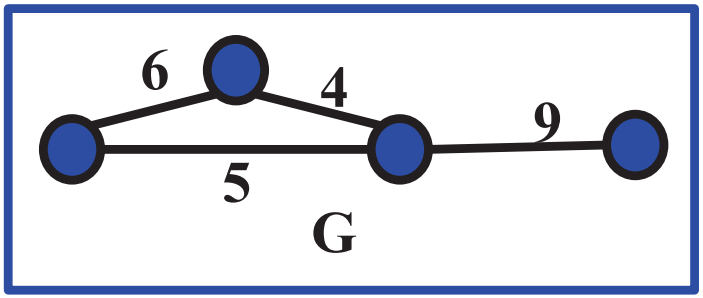
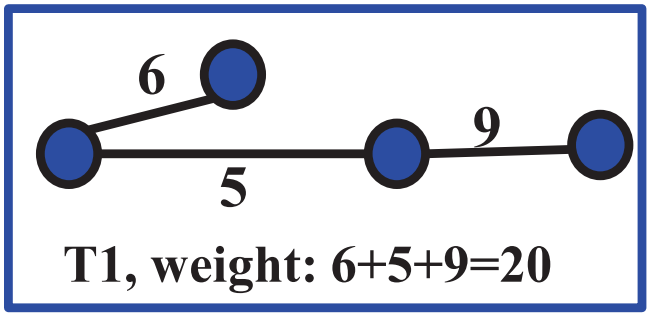
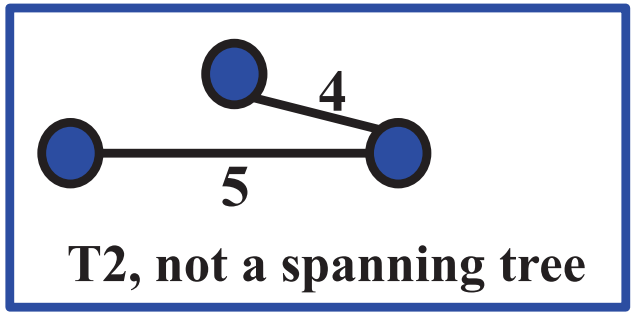
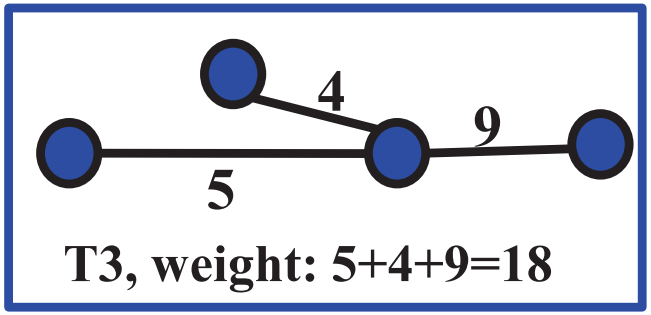
Spanning Tree and Minimum Spanning Tree

- **Spanning tree** of a connected graph: A tree that connects ***all*** nodes of the graph.
- **Minimum Spanning tree** of a connected weighted graph: A tree that connects all nodes of the graph with total edge weight lowest possible.
- **Example:** For the graph G,
  - **T1, T3 are spanning trees**
  - **T2 is not a spanning tree, because it does not connect all nodes of G**
  - **T1 is not minimum, because its total weight is 20**
  - **T2 is minimum with lowest possible total weight 19**



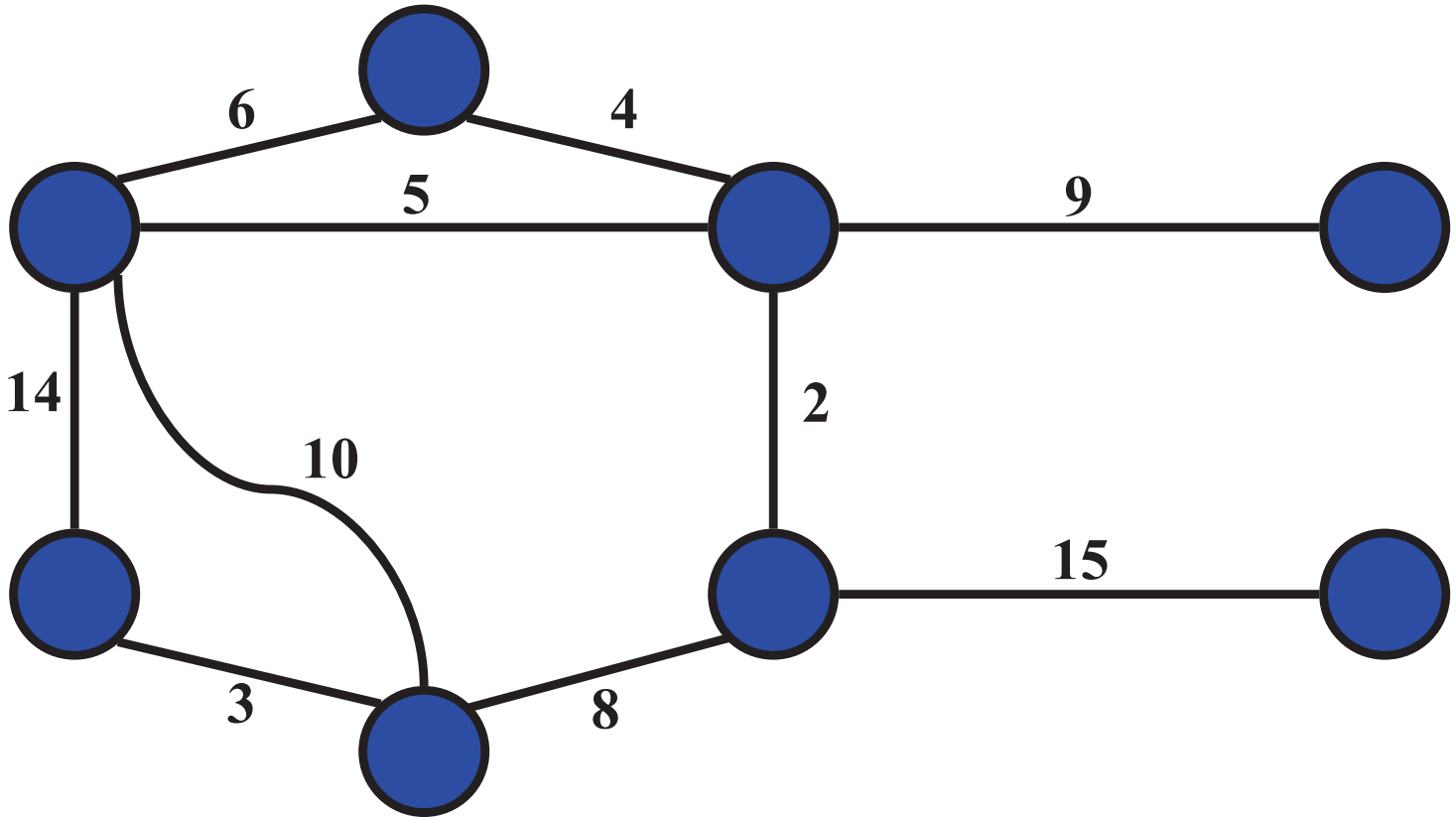






Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *minimum spanning tree*



Two Algorithm

- Krushkal's algorithm
- Prim's algorithm

Kruskal's Algorithm

- Take the minimum weight edge one after another if there is no cycle
- Use disjoint set data structure
- MakeSet() for each element
- Maintain two sets:
  - tree edges
  - other remaining edges
- Use FindSet() to check for cycle (If same set, then must be cycle, because if you connect any two vertices of one set, then it makes a cycle.)

Symbols used:
E: Edge set, V: Vertex set, T: Resulting tree, n: number of vertices

Kruskal's Algorithm

```
Kruskal()
{
    T = ∅;
    for each v ∈ V
        MakeSet(v);
    sort E by increasing edge weight w
    for each (u,v) ∈ E (in sorted order)
        if FindSet(u) ≠ FindSet(v) //no cycle
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            Union(FindSet(u), FindSet(v));
}
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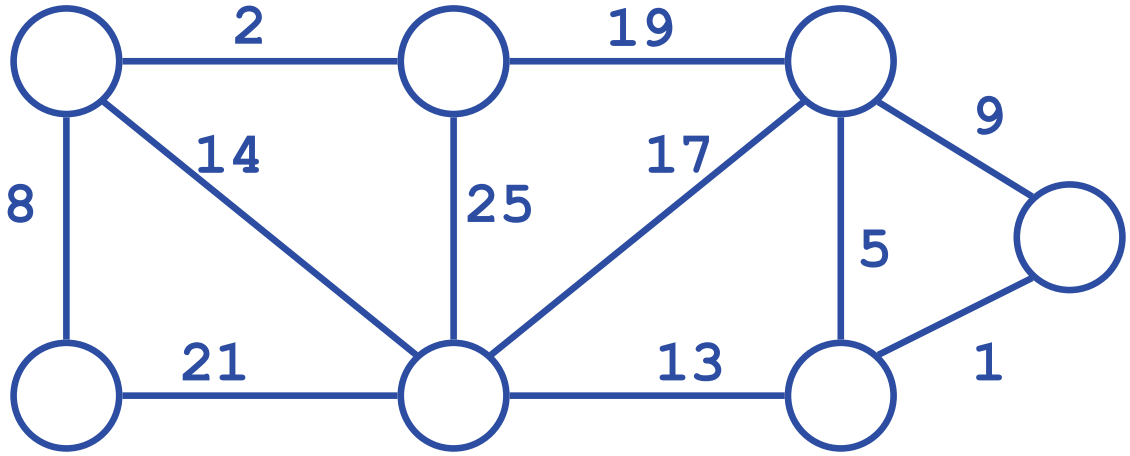

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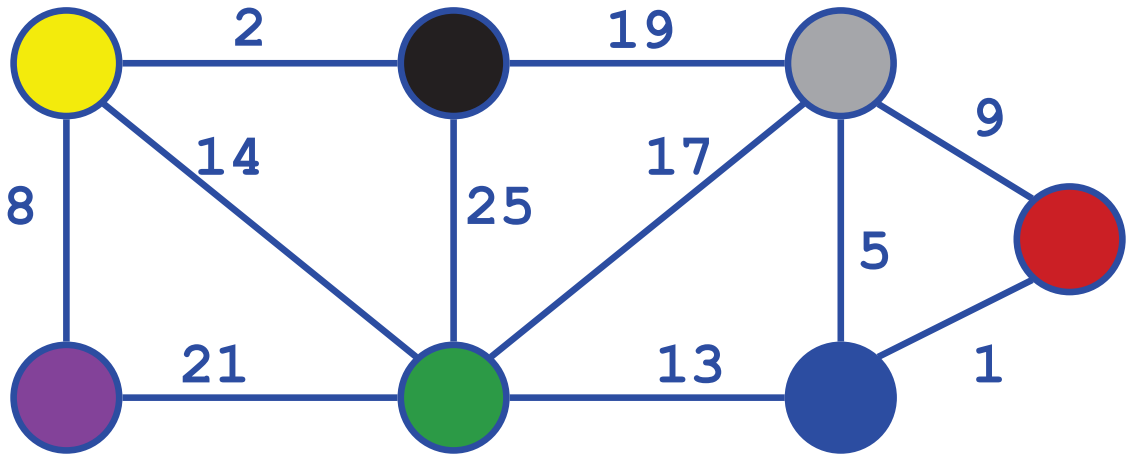

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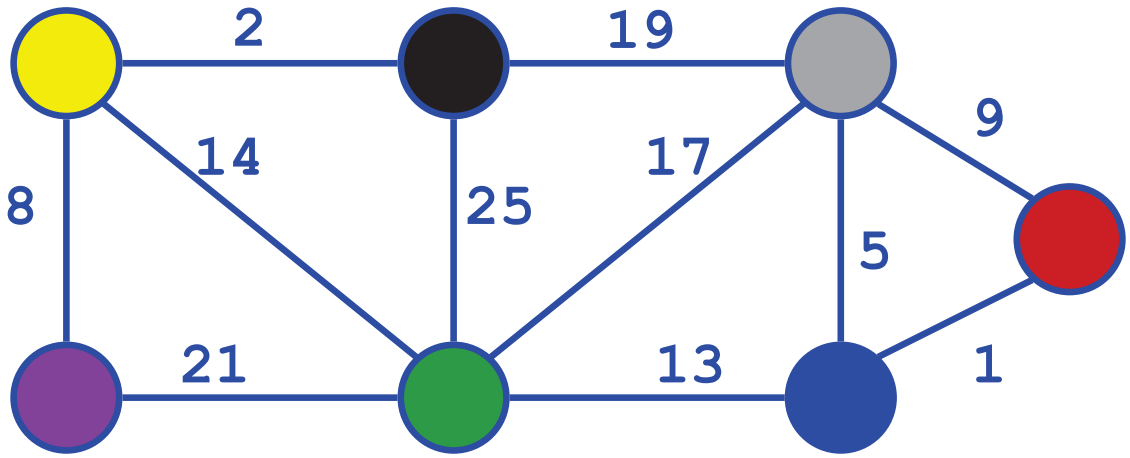

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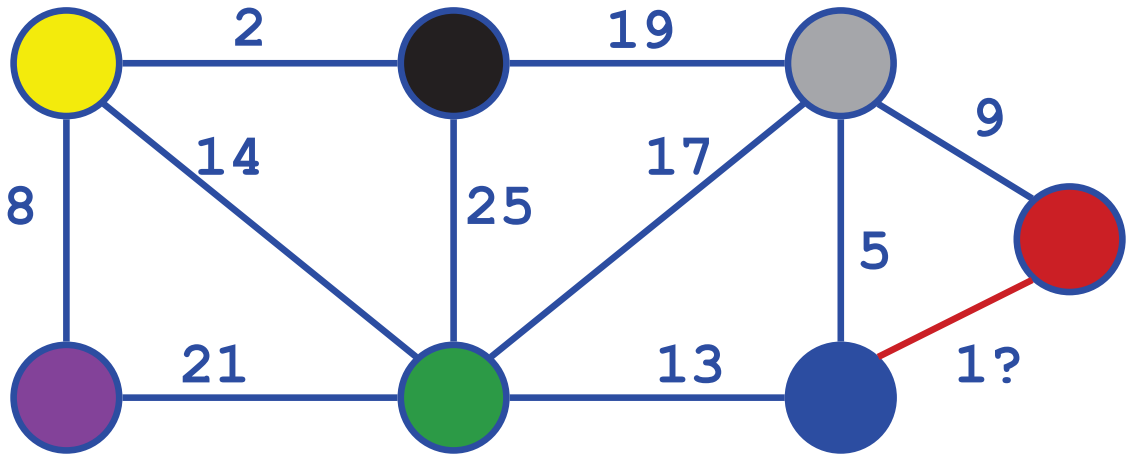

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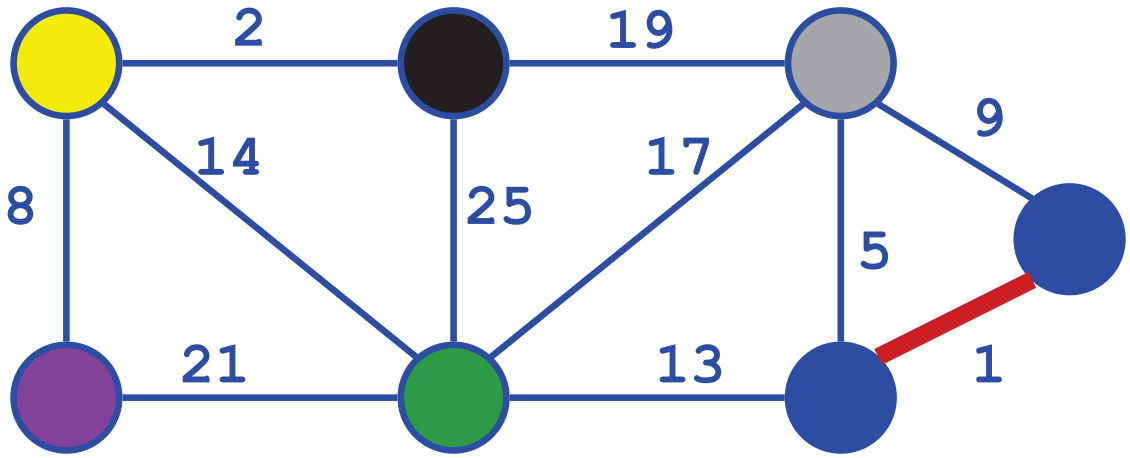

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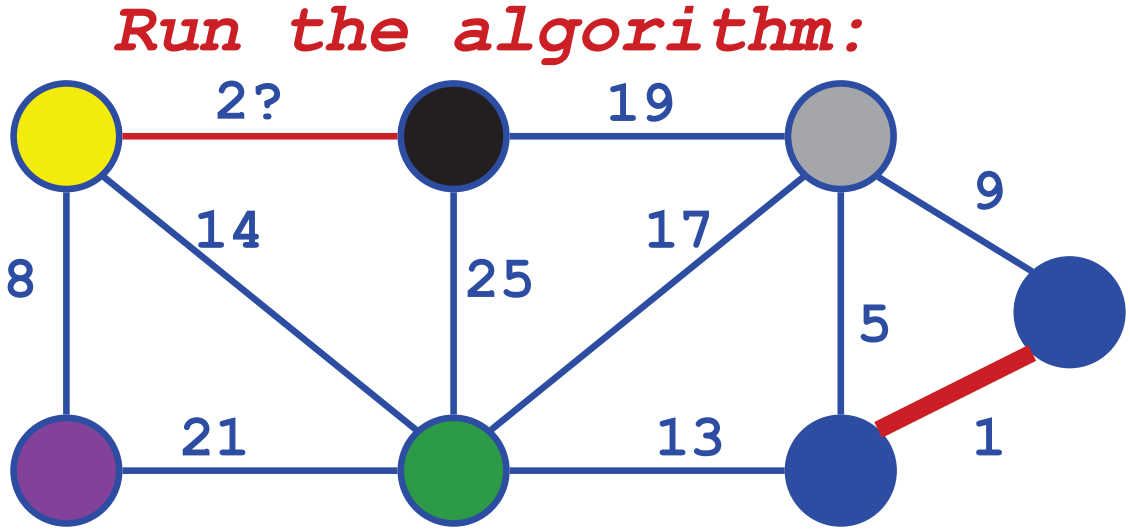

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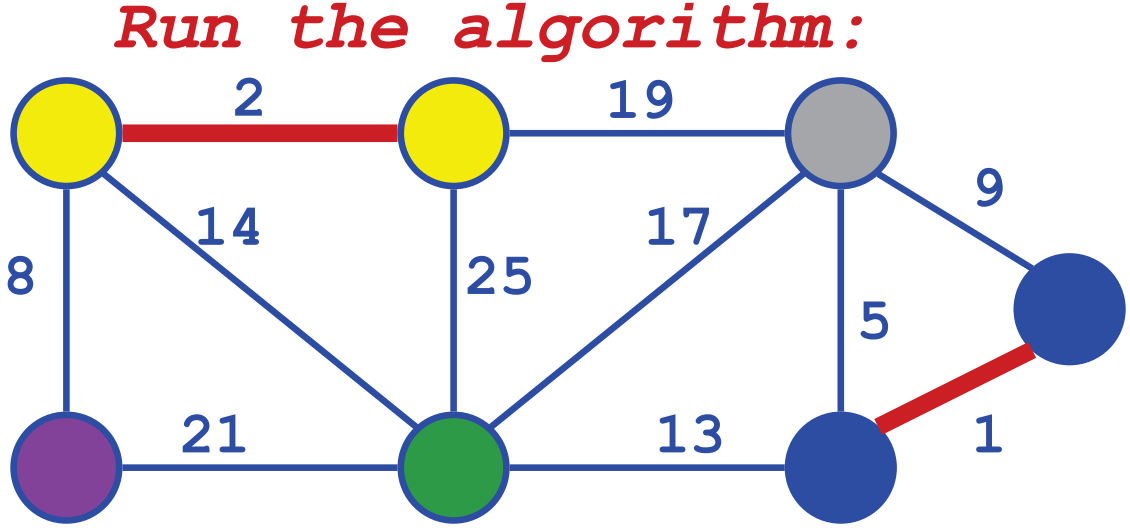

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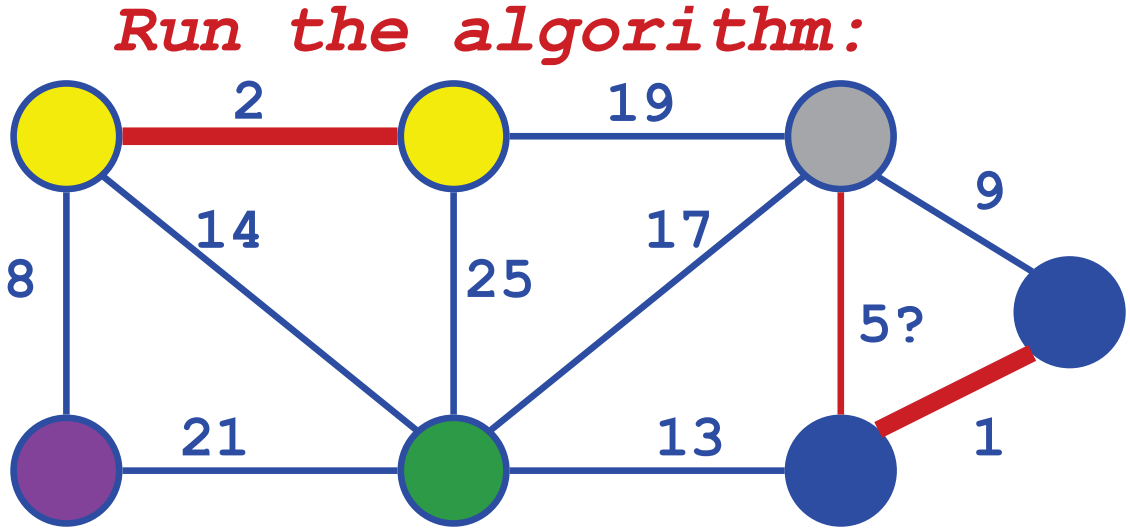

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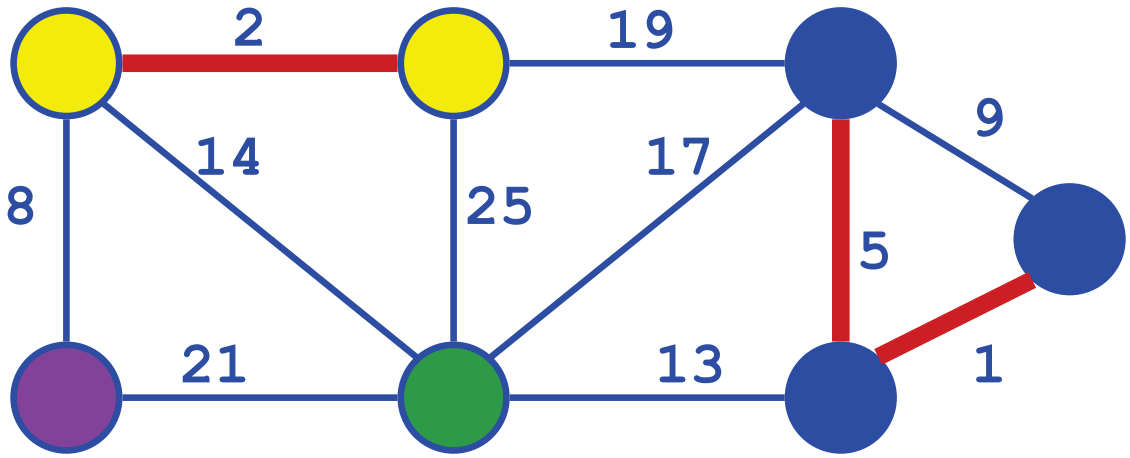

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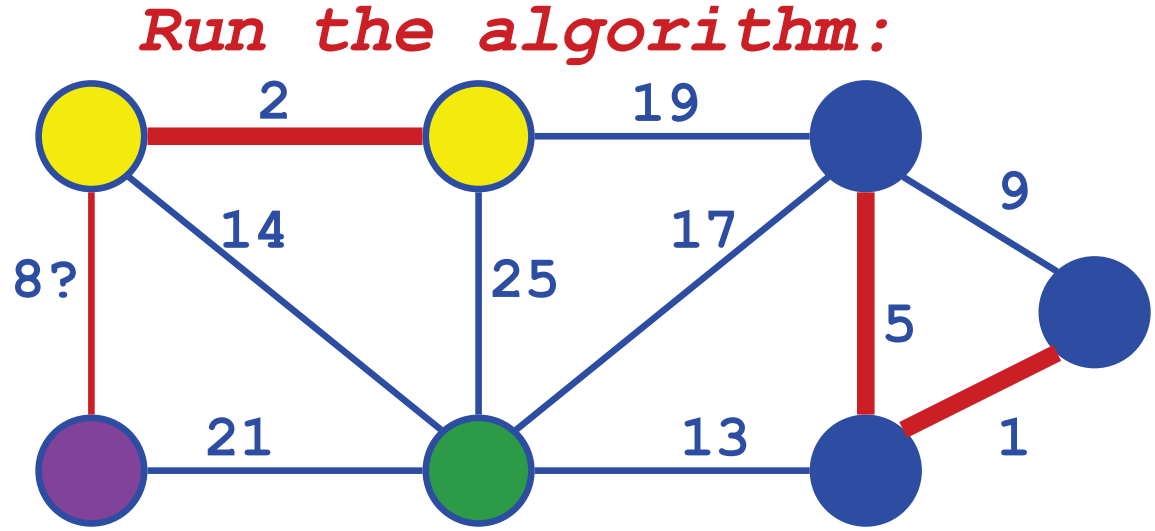

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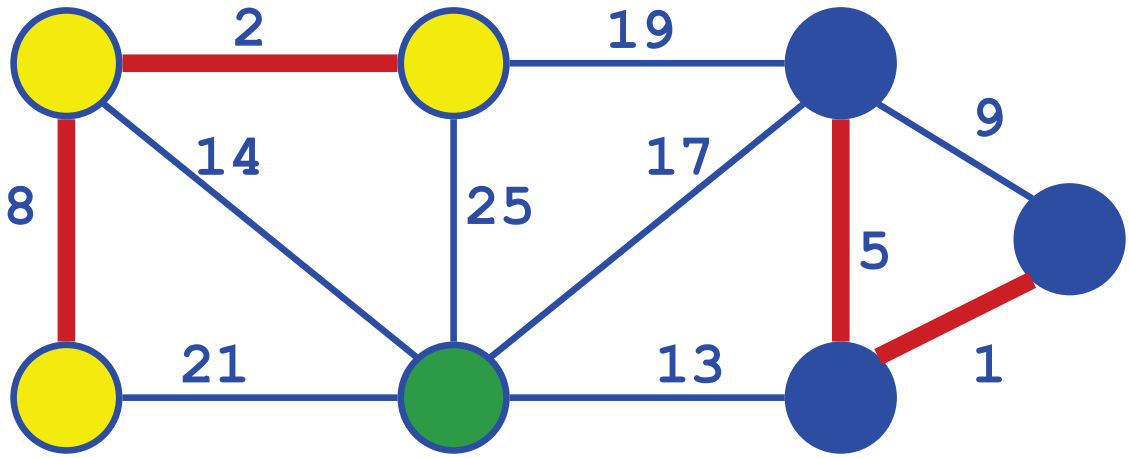

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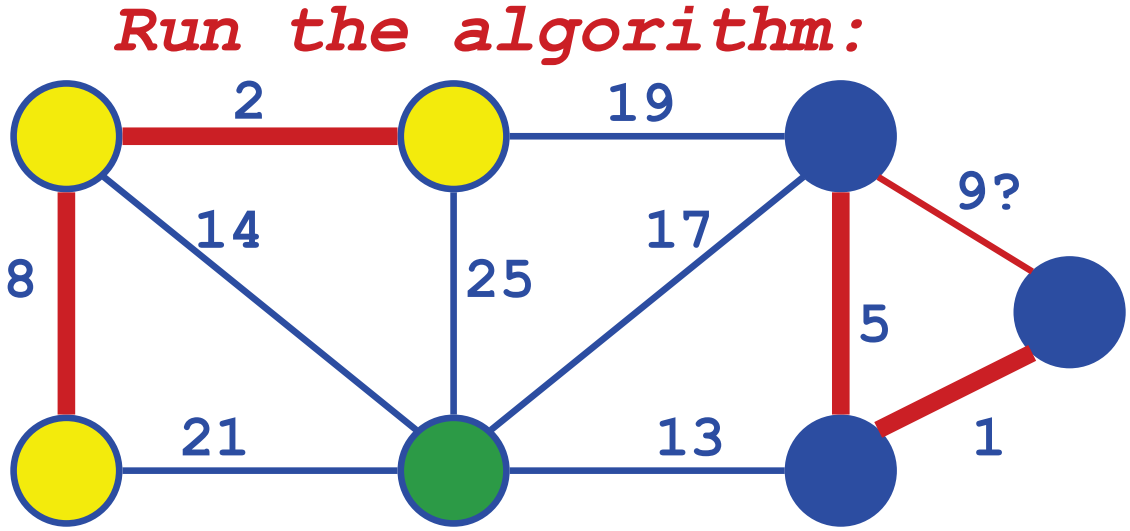

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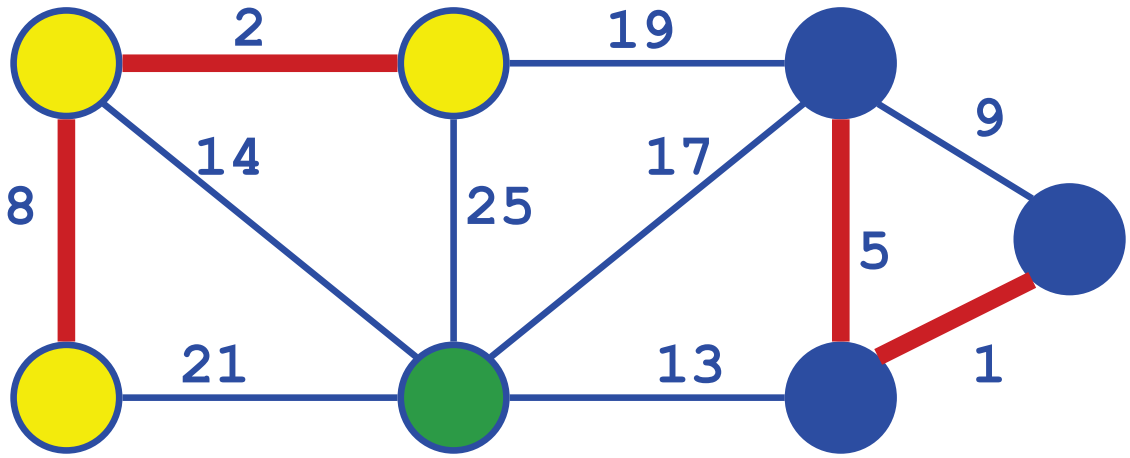

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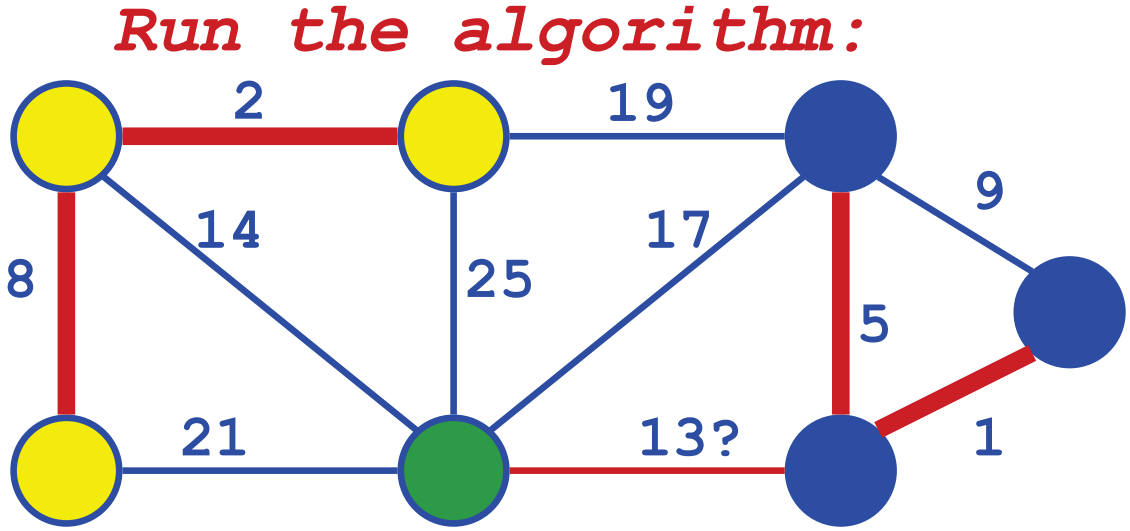

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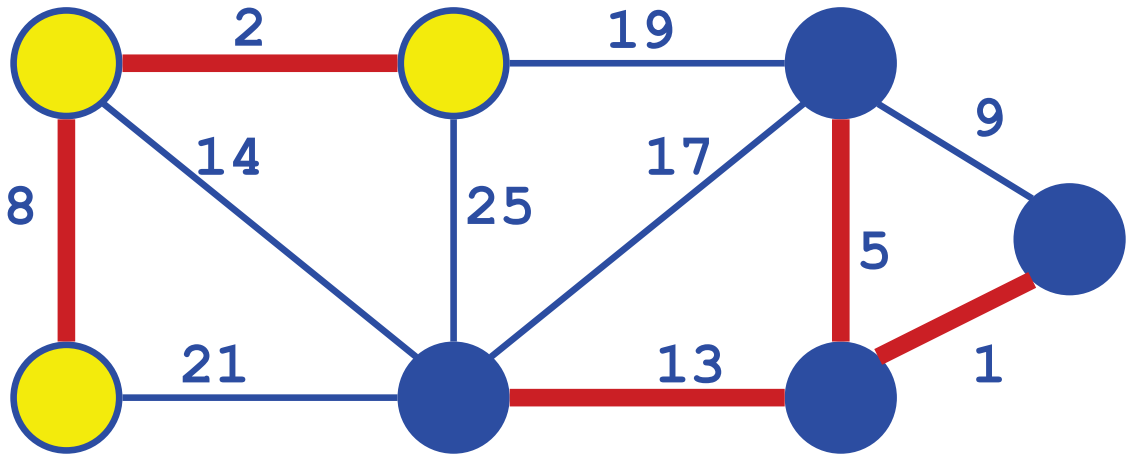

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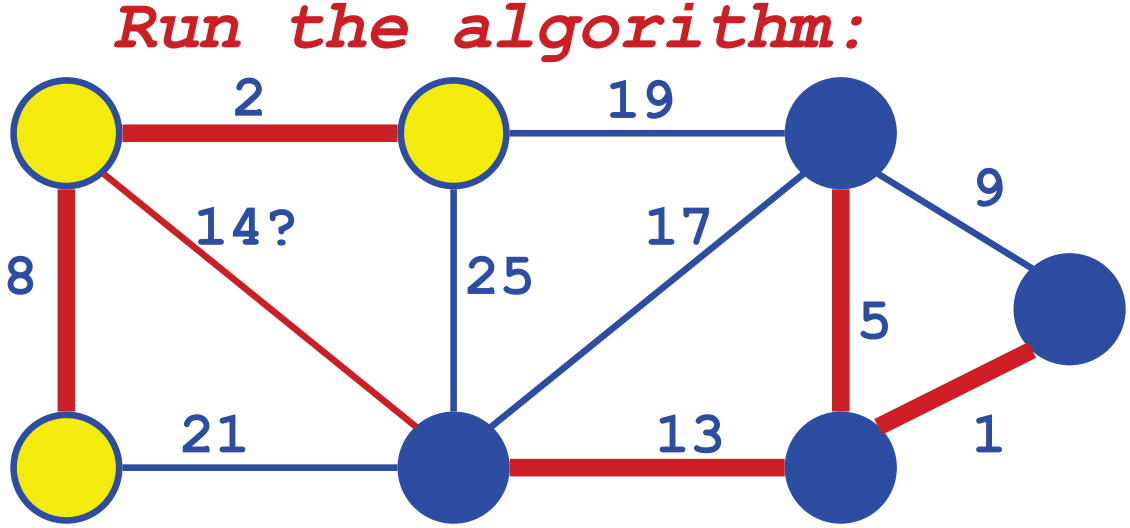

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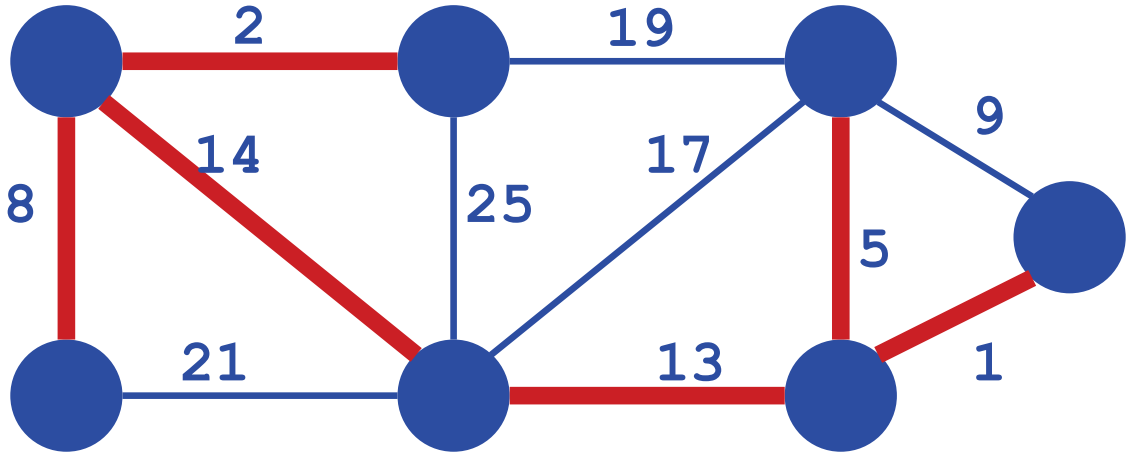

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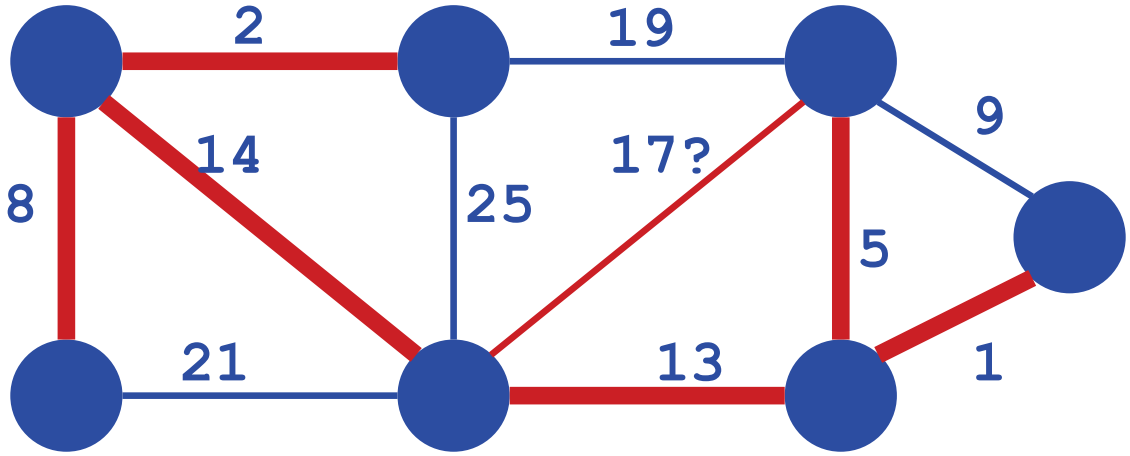

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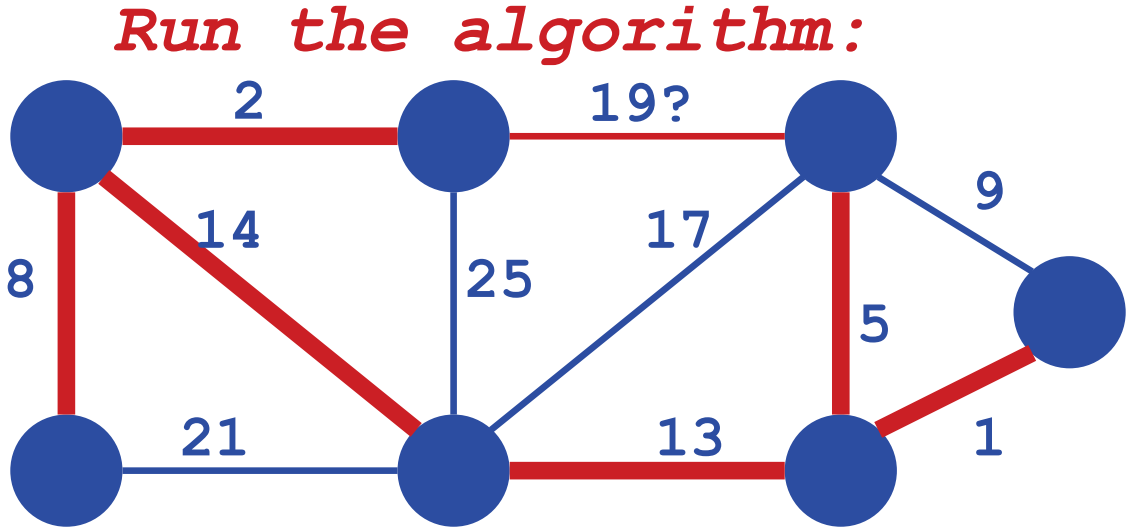

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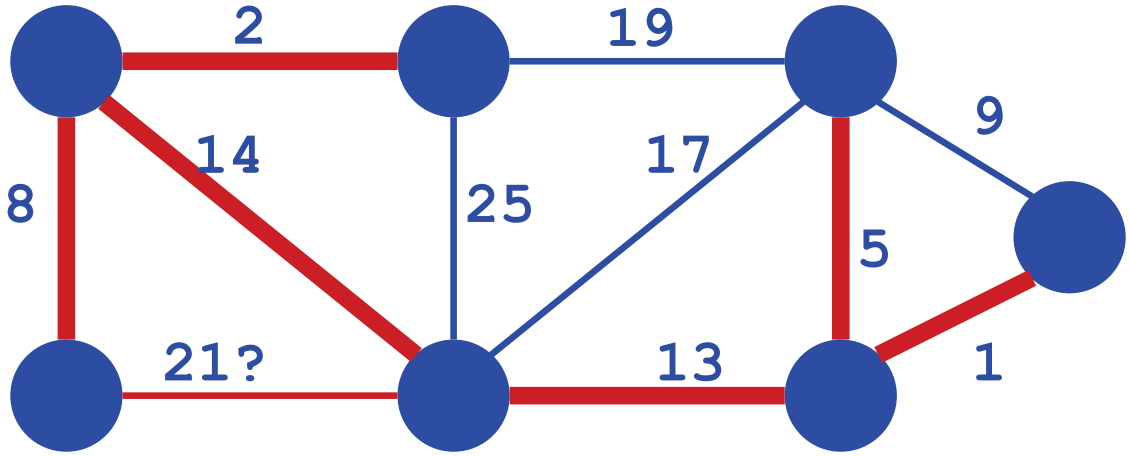

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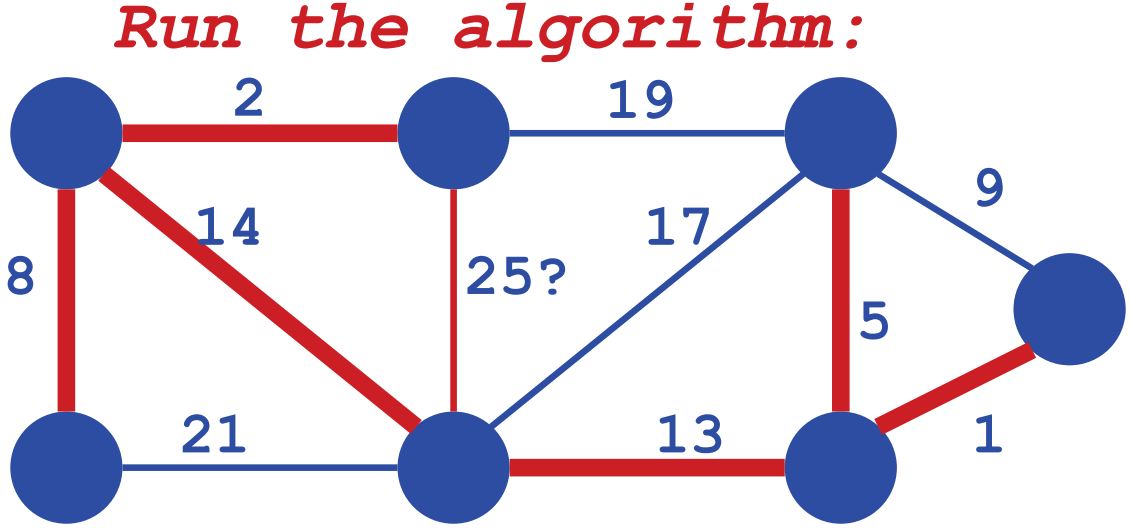

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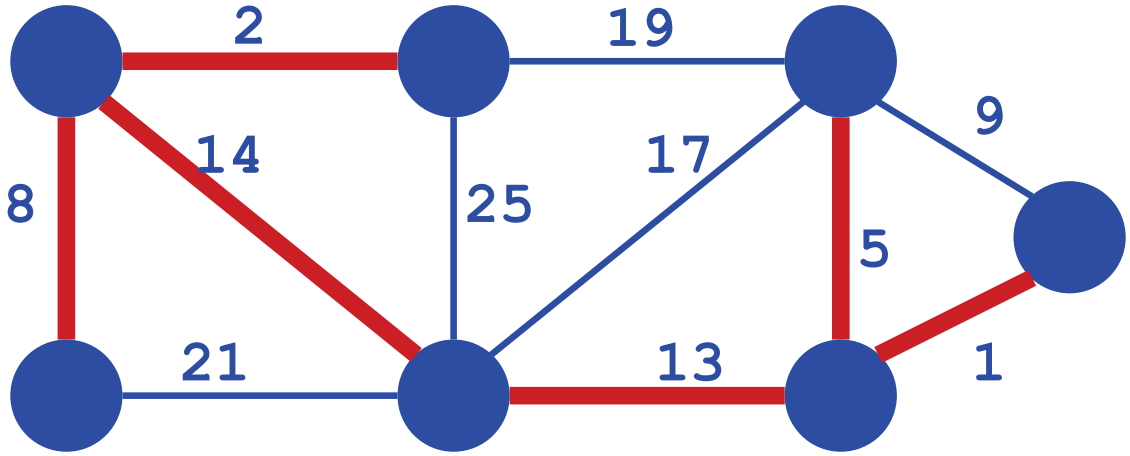

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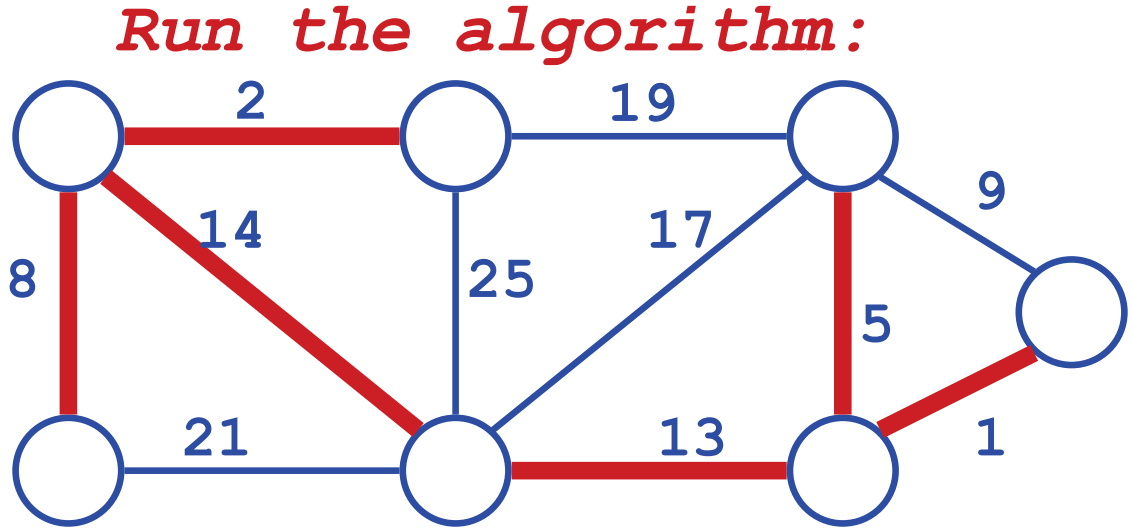

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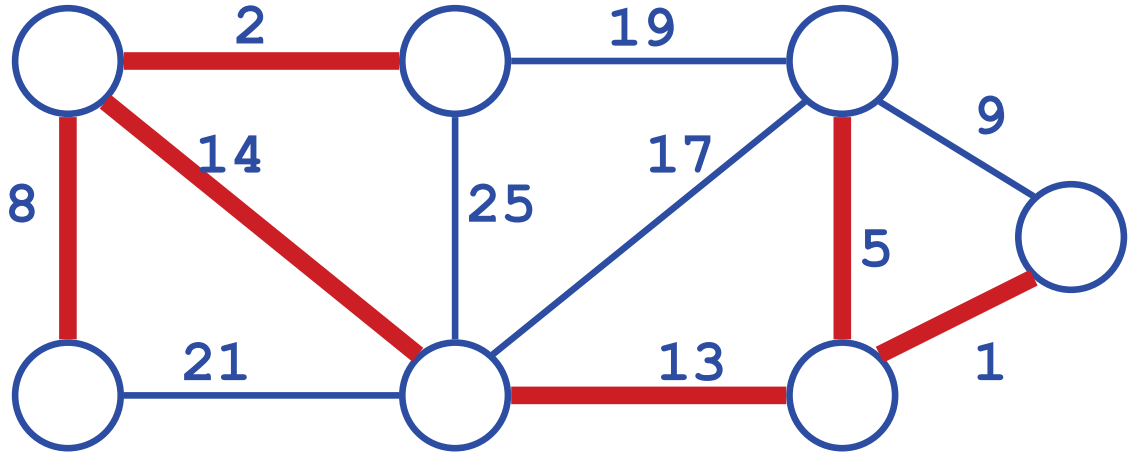
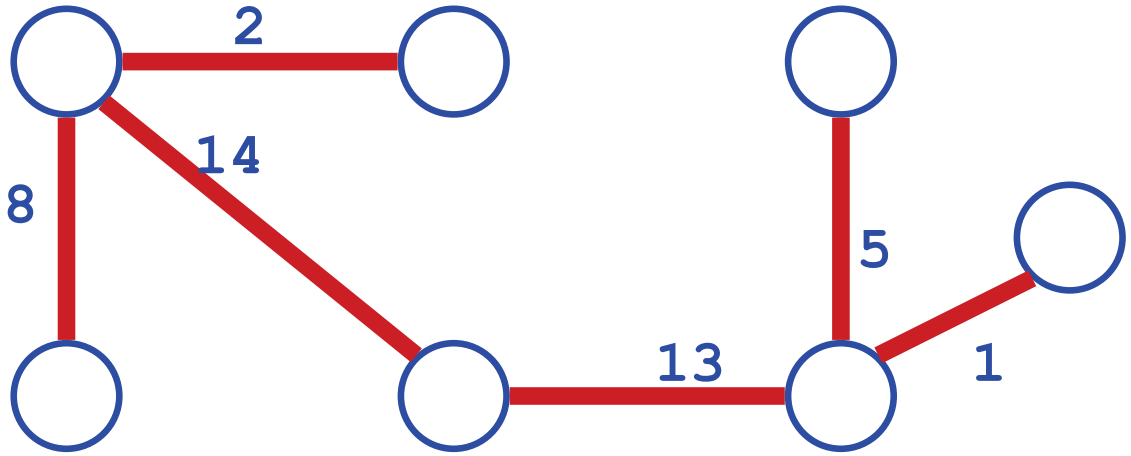
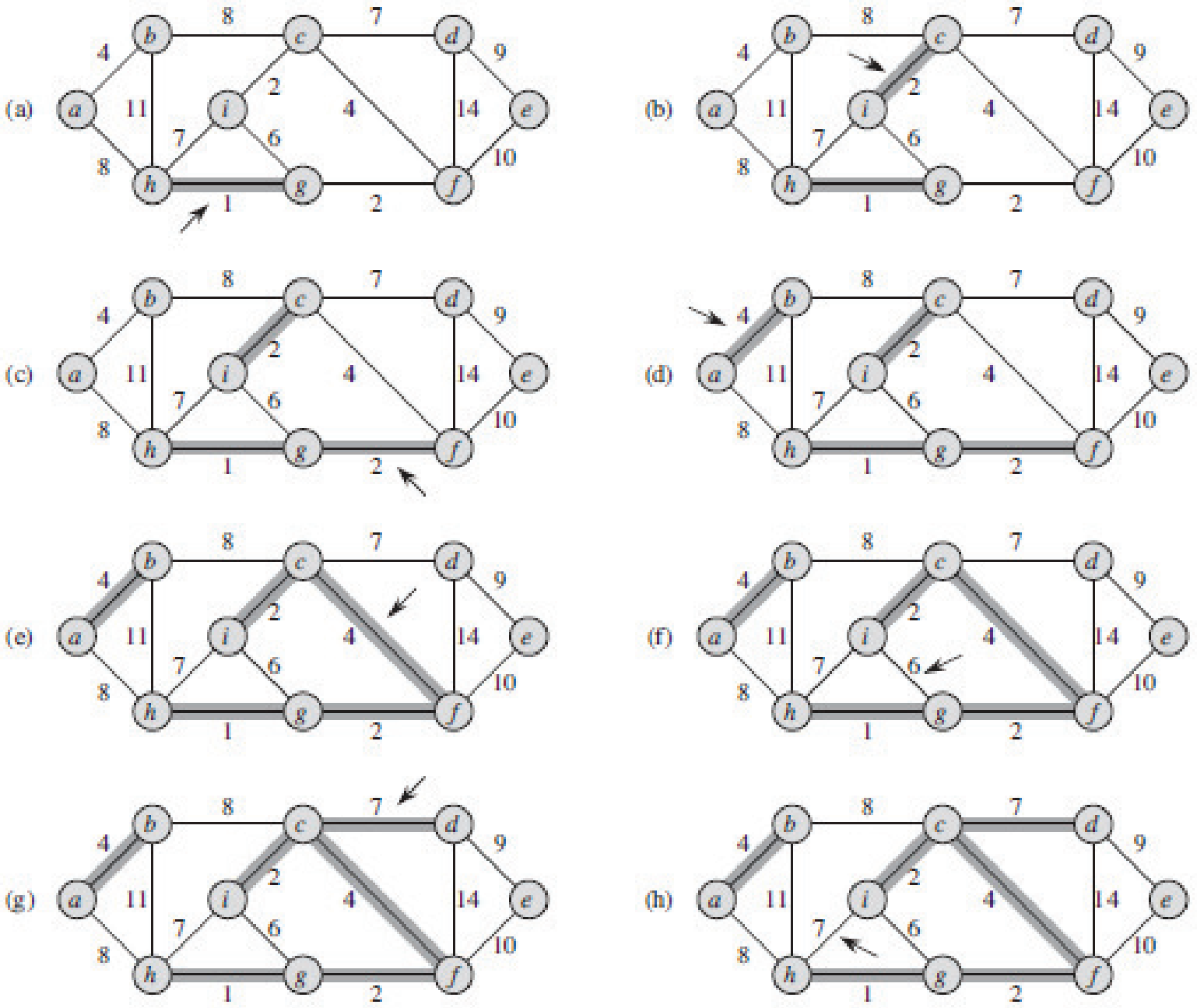




Another Example: Krushkal's Algorithm



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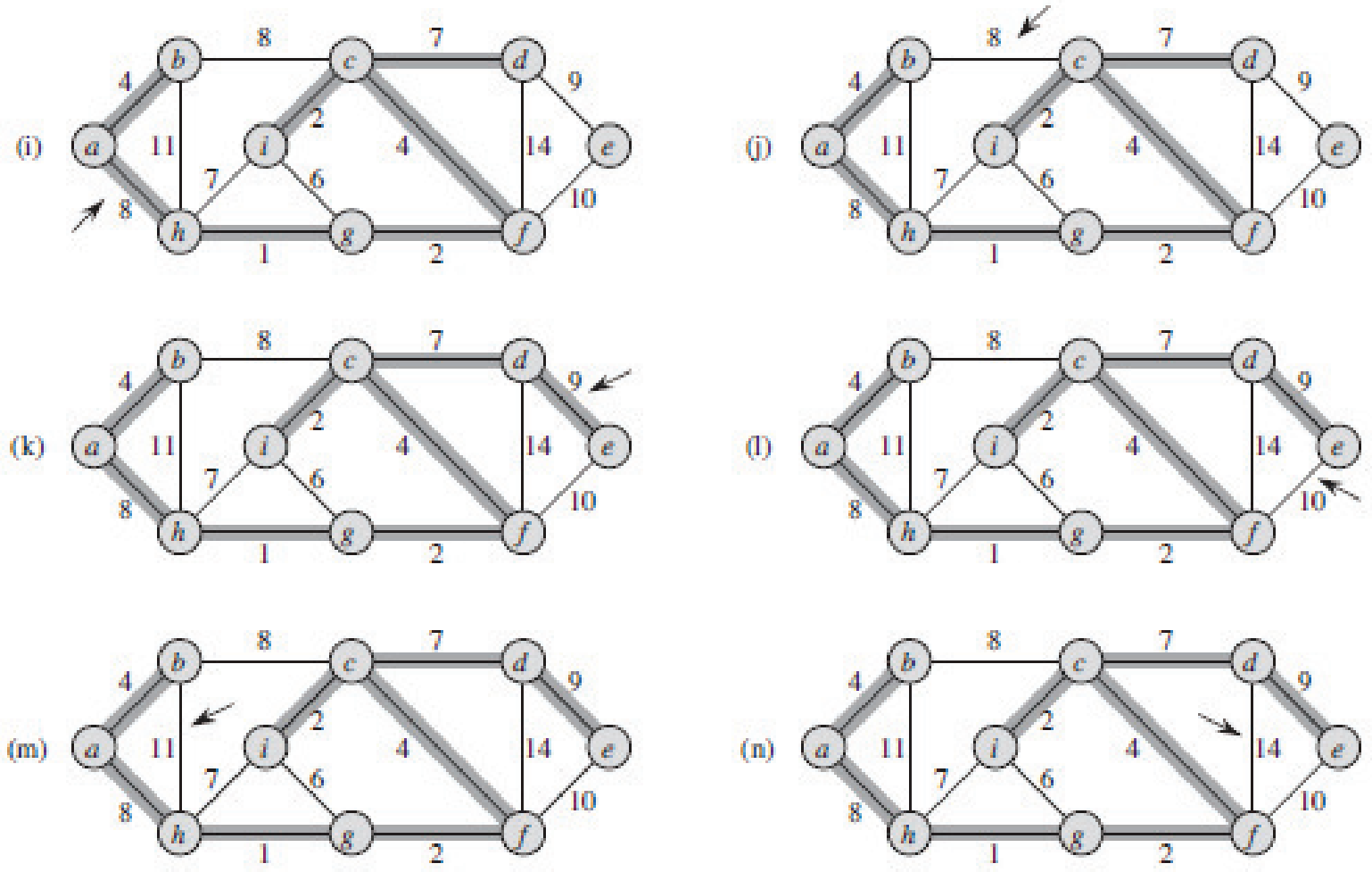

Prim's Algorithm

1. Initially, all node have weight ∞
2. Keep all of them in a queue (heap)
3. Take minimum node from heap (ExtractMin())
4. For each neighbor of this node, if edge weight from this node is smaller than current weight , then update weight (DecreaseKey()) and add that edge in the tree.
5. Repeat step 3, 4 until finished.

n: number of vertex

Prim's Algorithm

Q: heap, V: vertex set, E: Edge set, p: parent, w: edge weight

```
MST-Prim
    for all node u
        weight[u] = ∞;
    Q = all node;
    weight[root] = 0;
    p[root] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adjacent[u]
            if (v ∈ Q and w(u,v) < weight[v])
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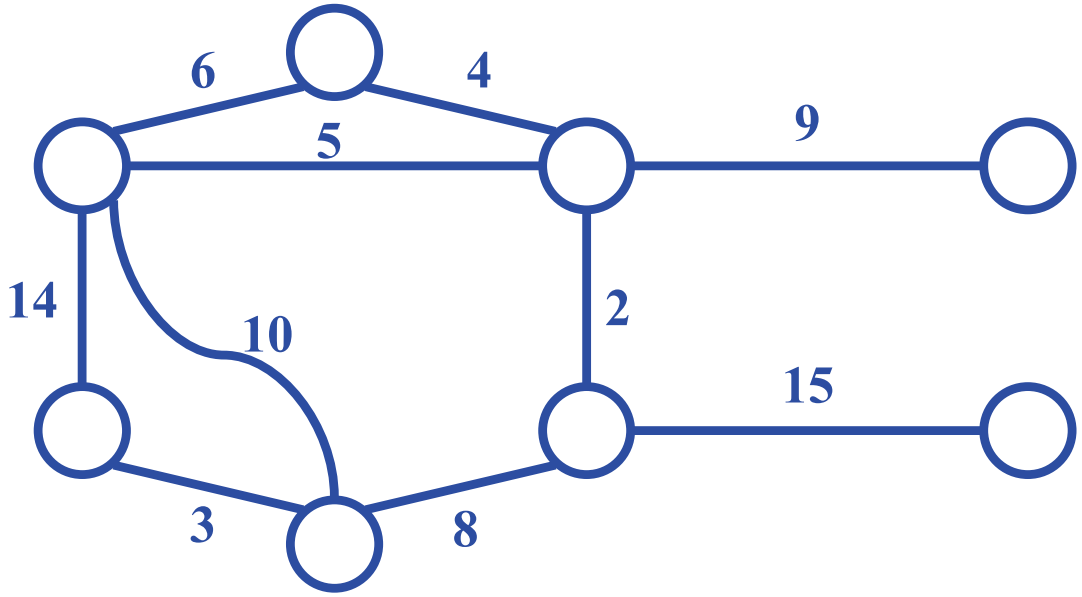

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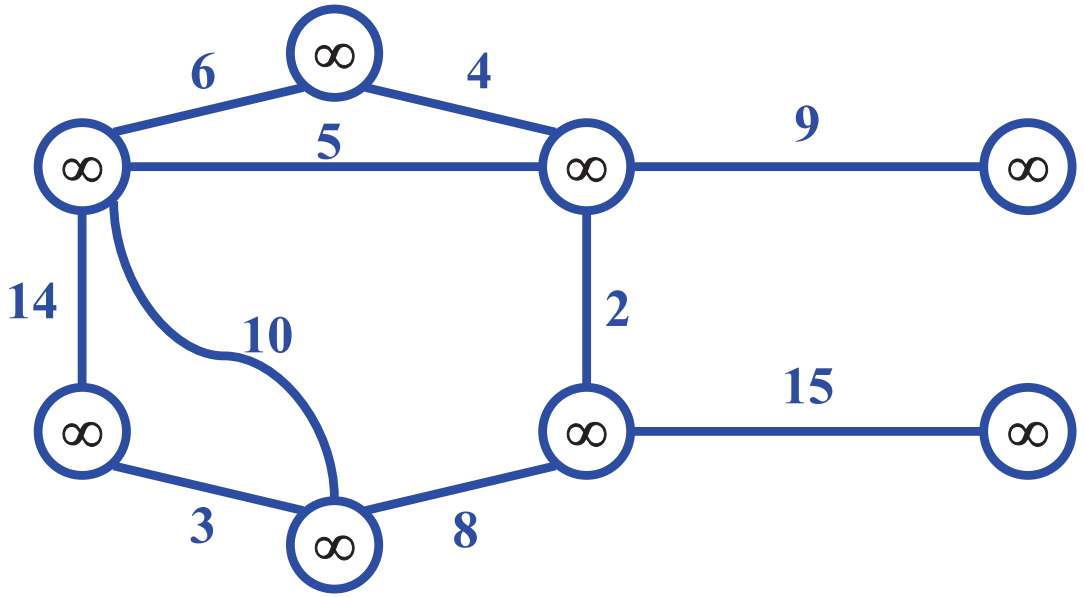

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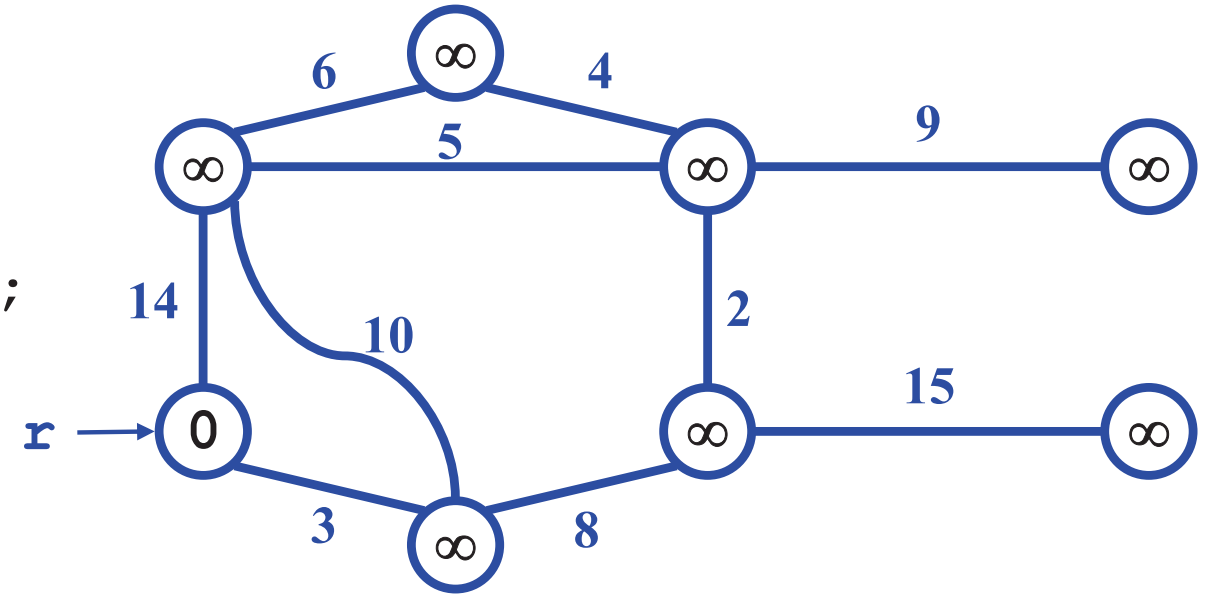
     weight[root] = 0;
     p[root] = NULL;
     while (Q not empty)
          u = ExtractMin(Q);
          for each v ∈ Adjacent[u]
               if (v ∈ Q and w(u,v) < weight[v])
                    p[v] = u;
                    weight[v] = w(u,v);
```



Prim's Algorithm

Q: heap, V: vertex set, E: Edge set, p: parent, w: edge weight

```
MST-Prim
     for all node u
          weight[u] = ∞;
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               if (v ∈ Q and w(u,v) < weight[v])
                    p[v] = u;
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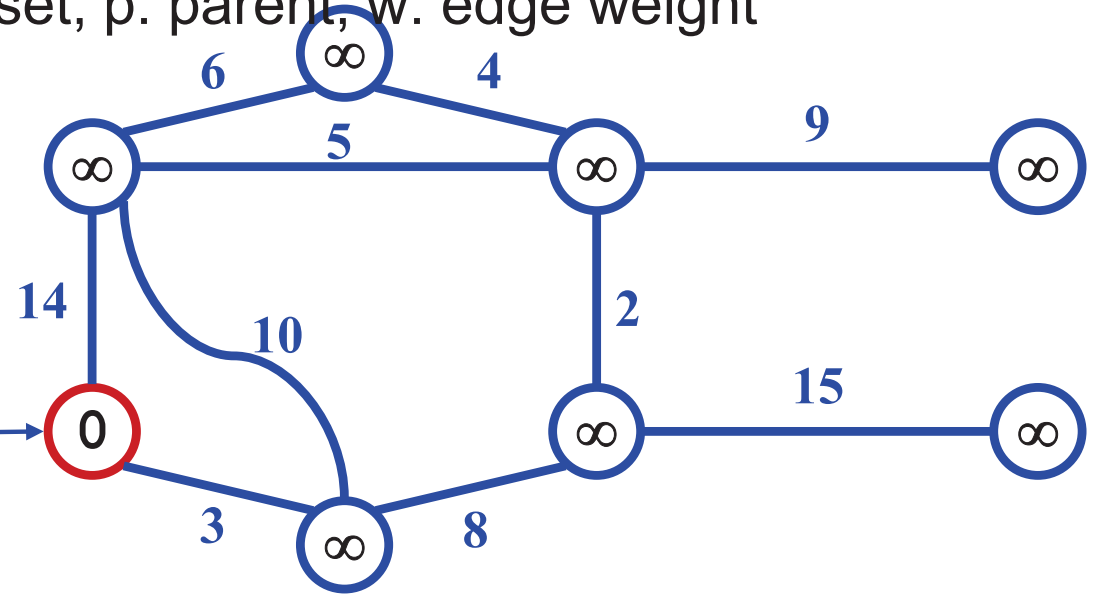
```



Red vertices have been removed from Q

Prim's Algorithm

Q: heap, V: vertex set, E: Edge set, p: parent, w: edge weight

```
MST-Prim
     for all node u
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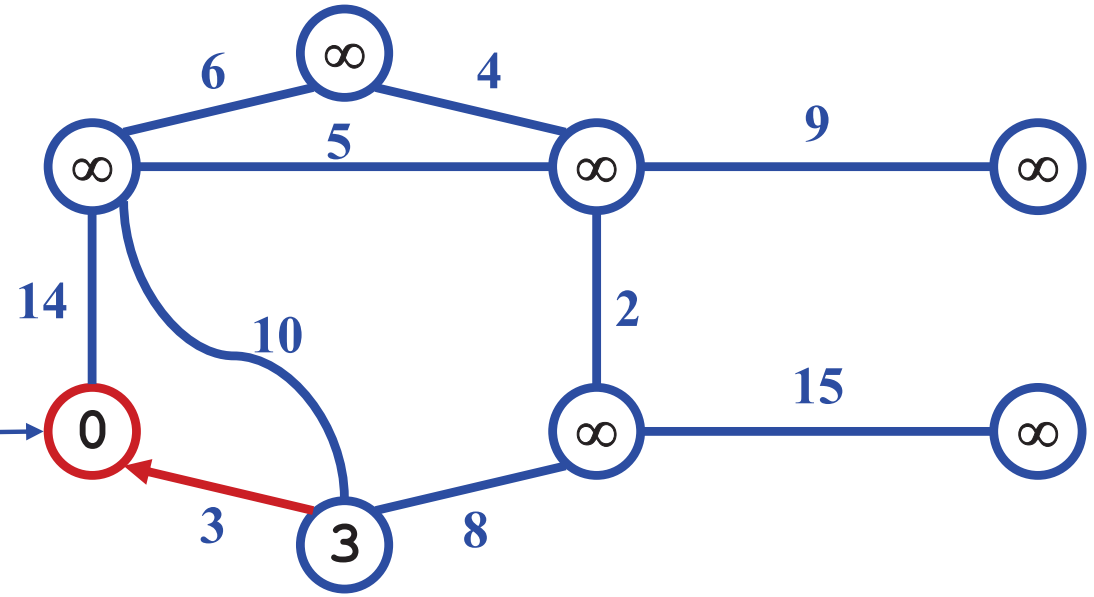

Red arrows indicate parent pointers

Prim's Algorithm

Q: heap, V: vertex set, E: Edge set, p: parent, w: edge weight

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MST-Prim
     for all node u
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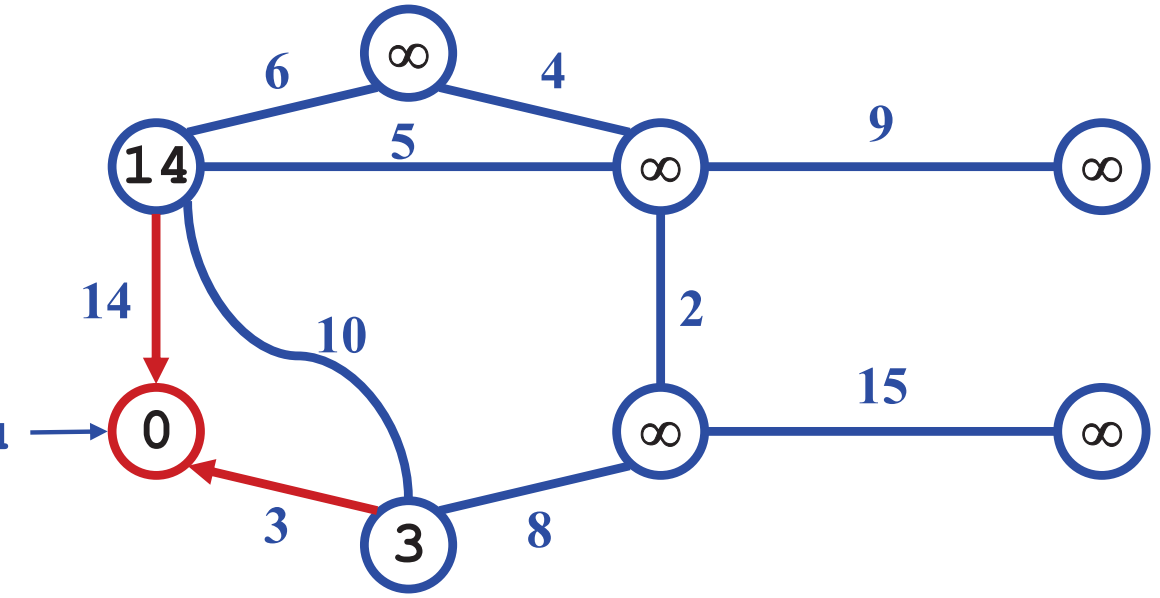

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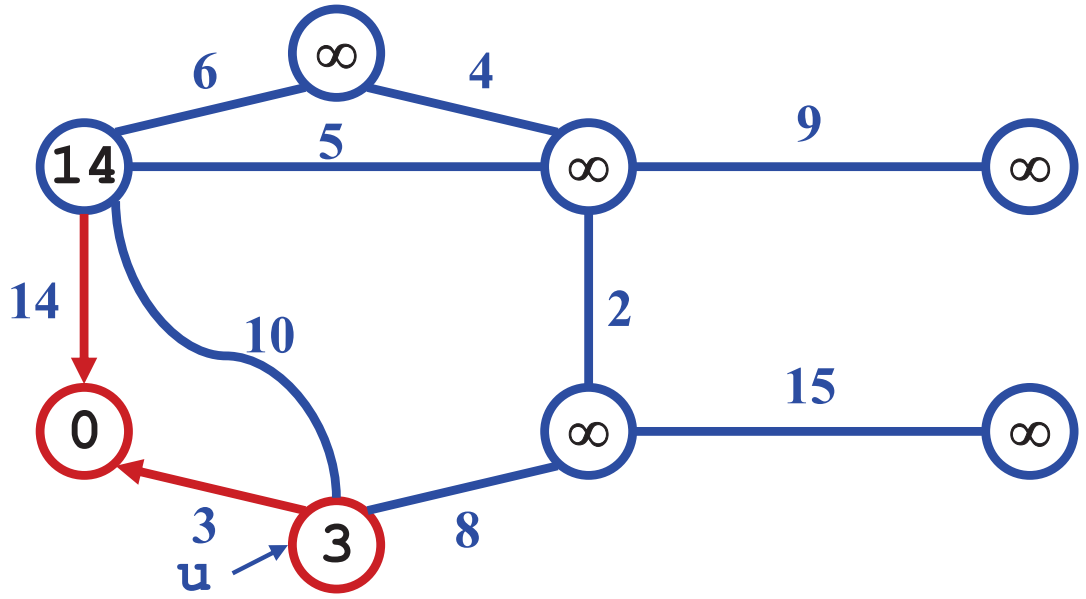

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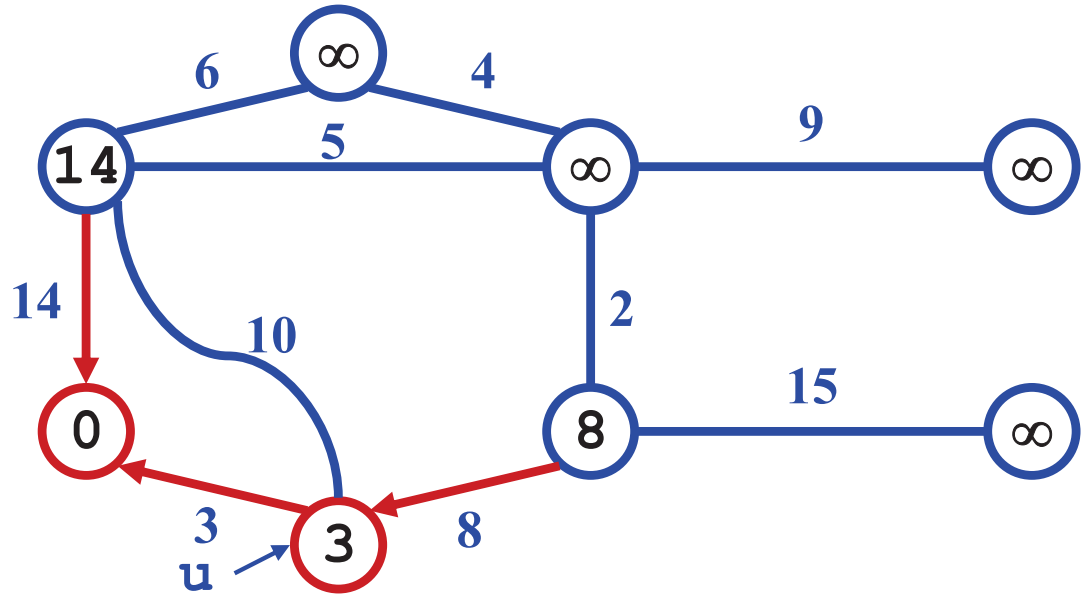

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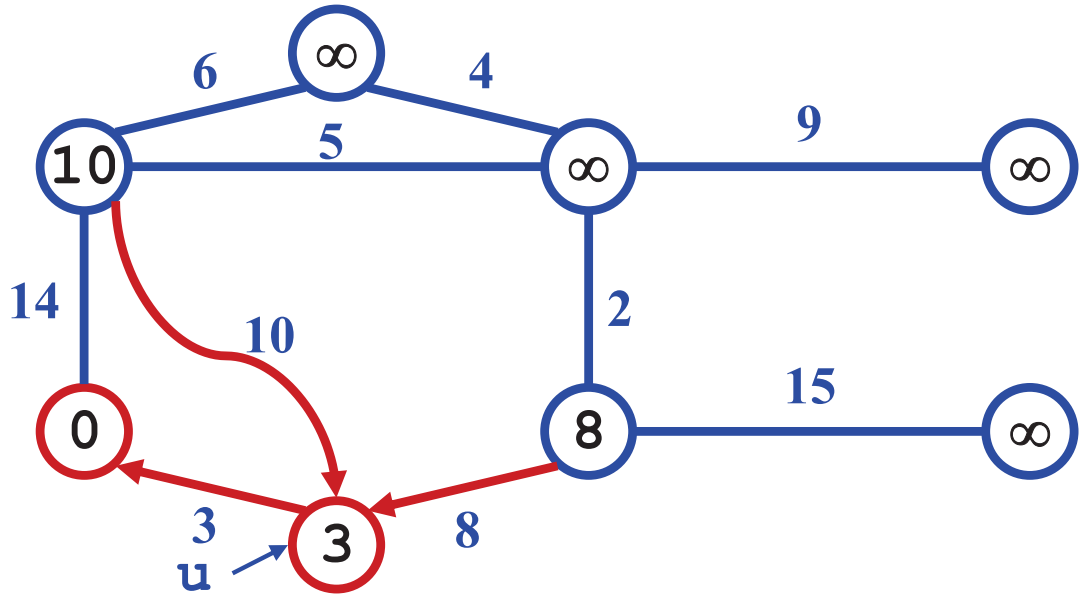

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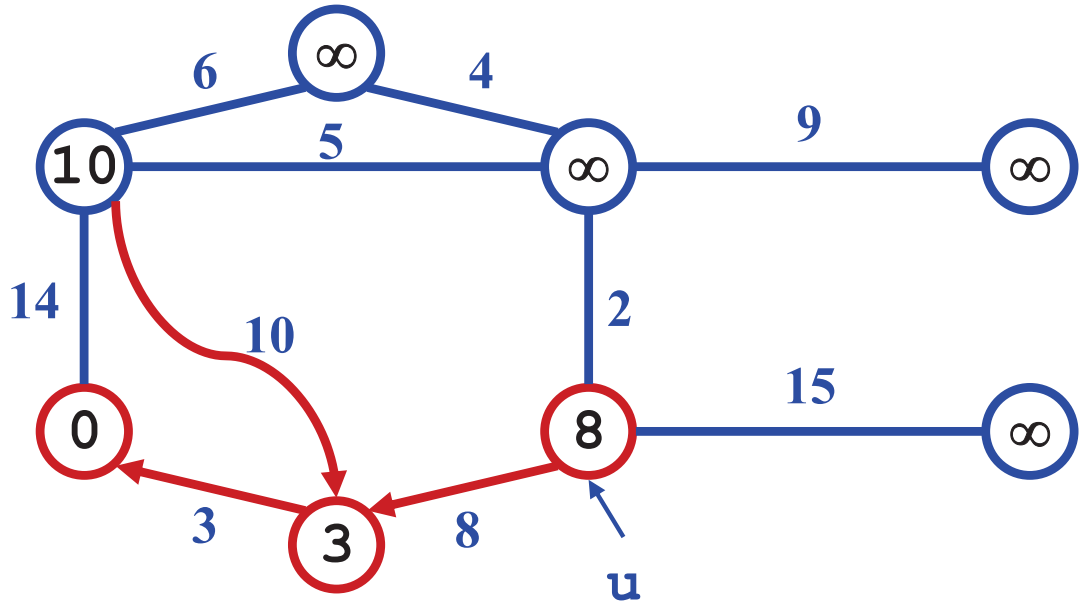

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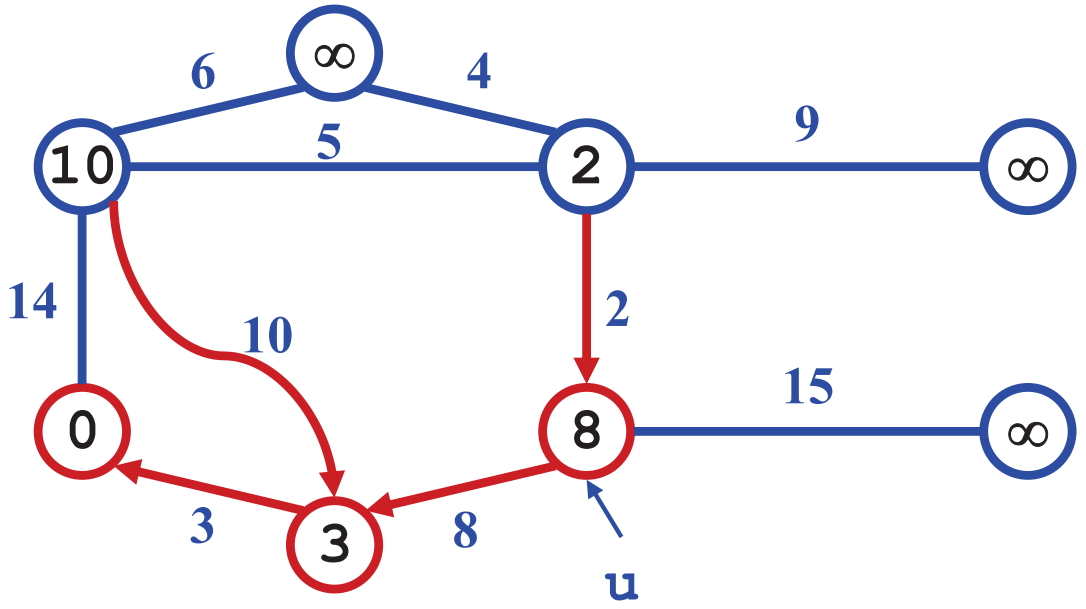

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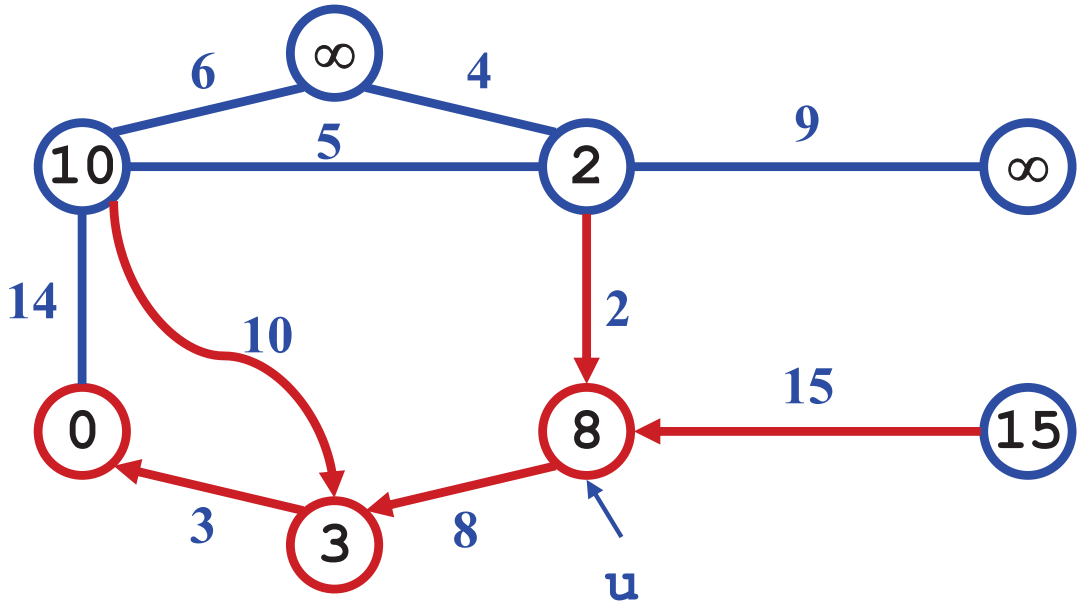

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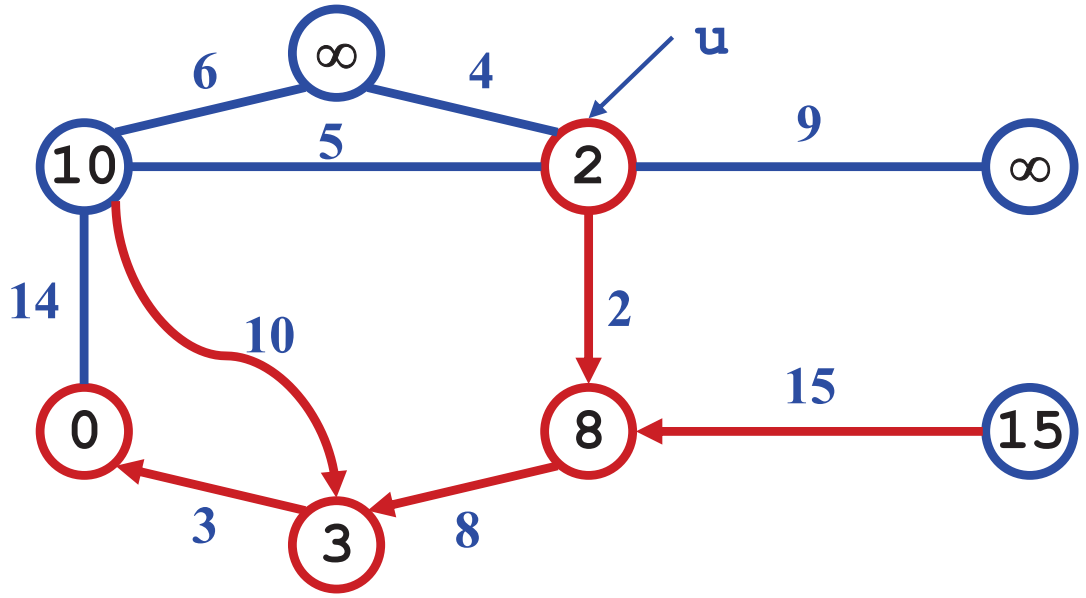

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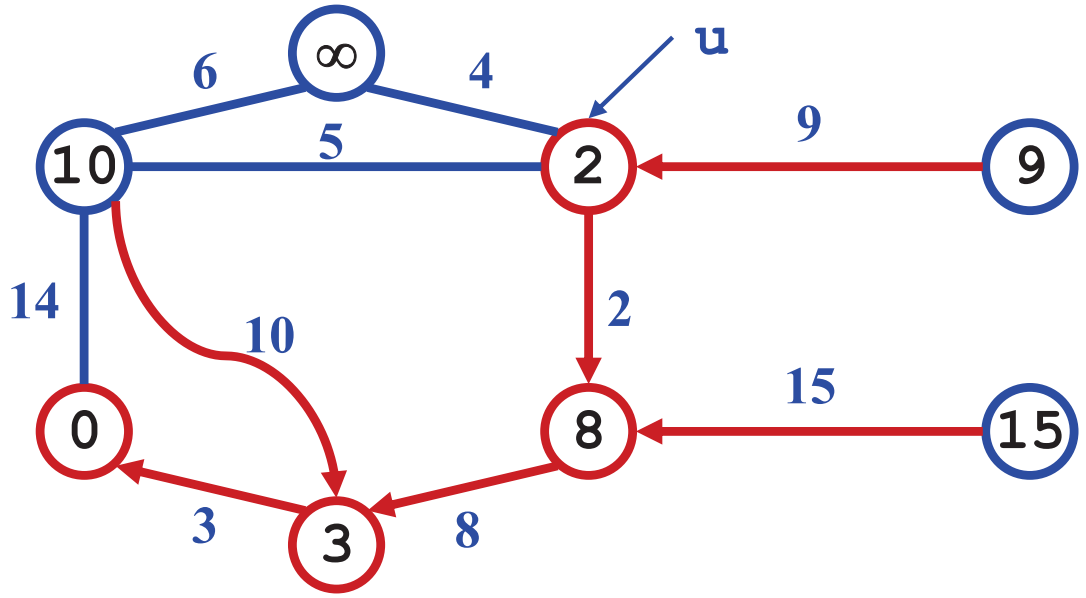

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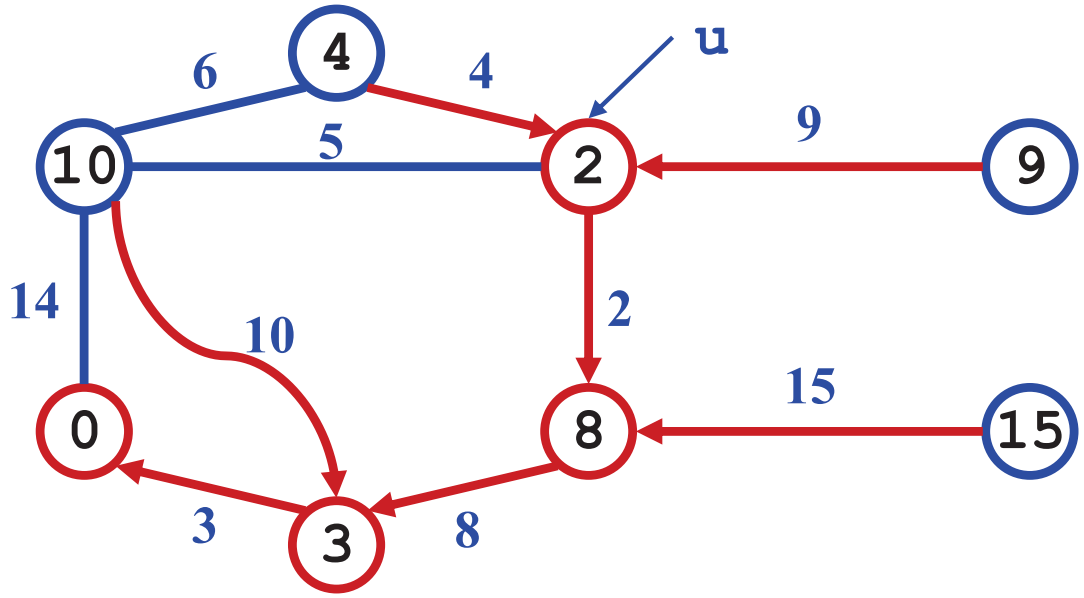

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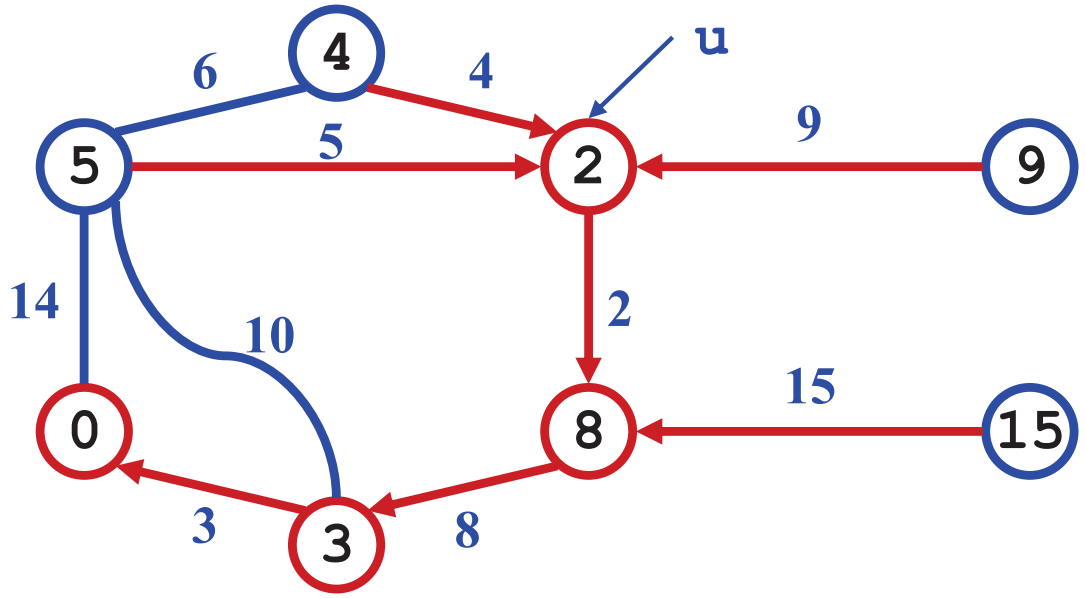

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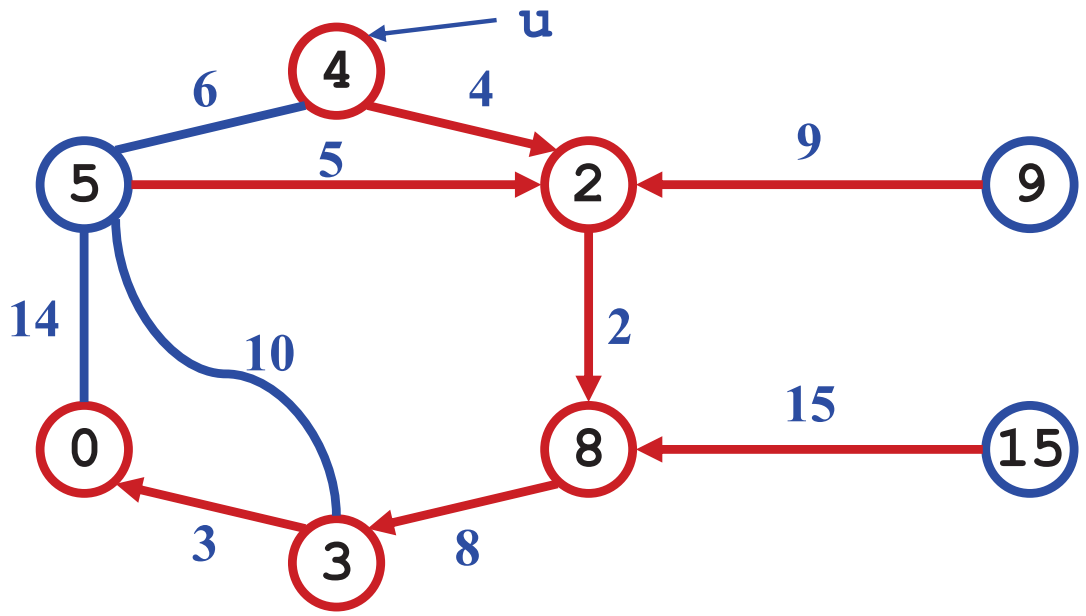


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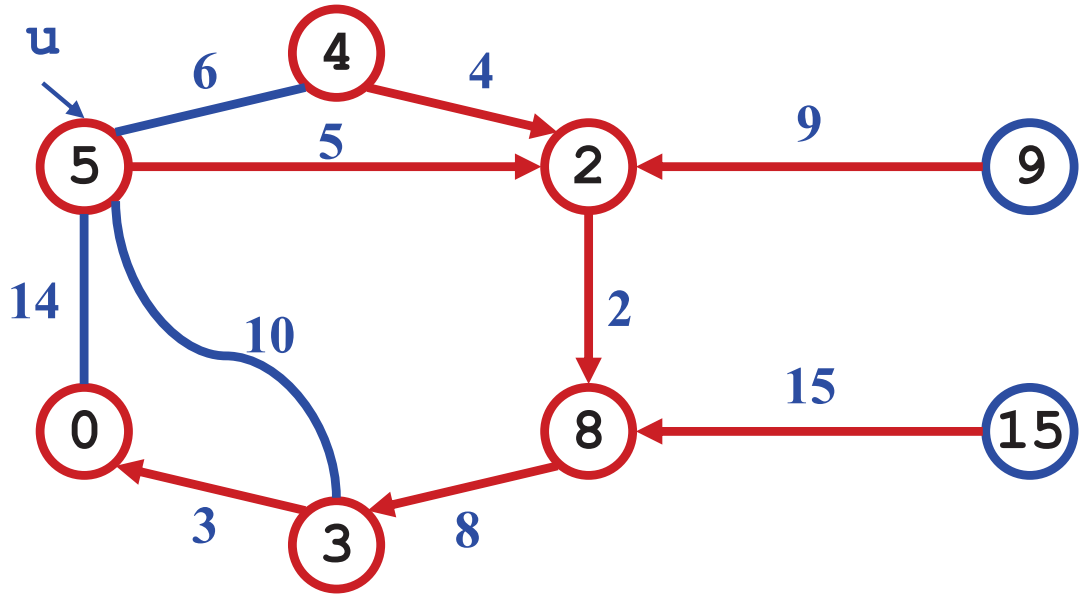


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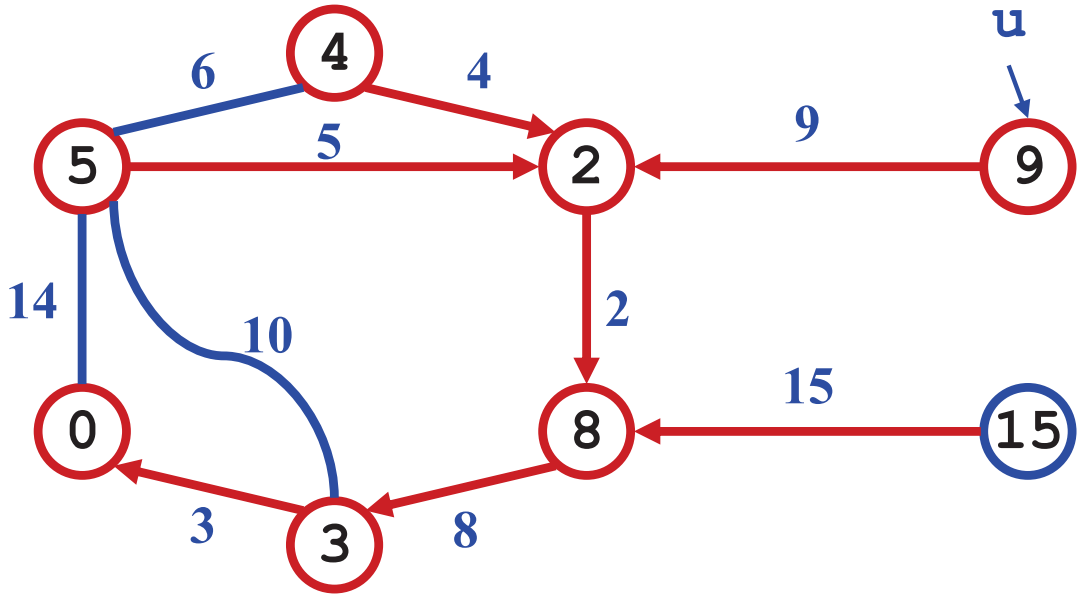

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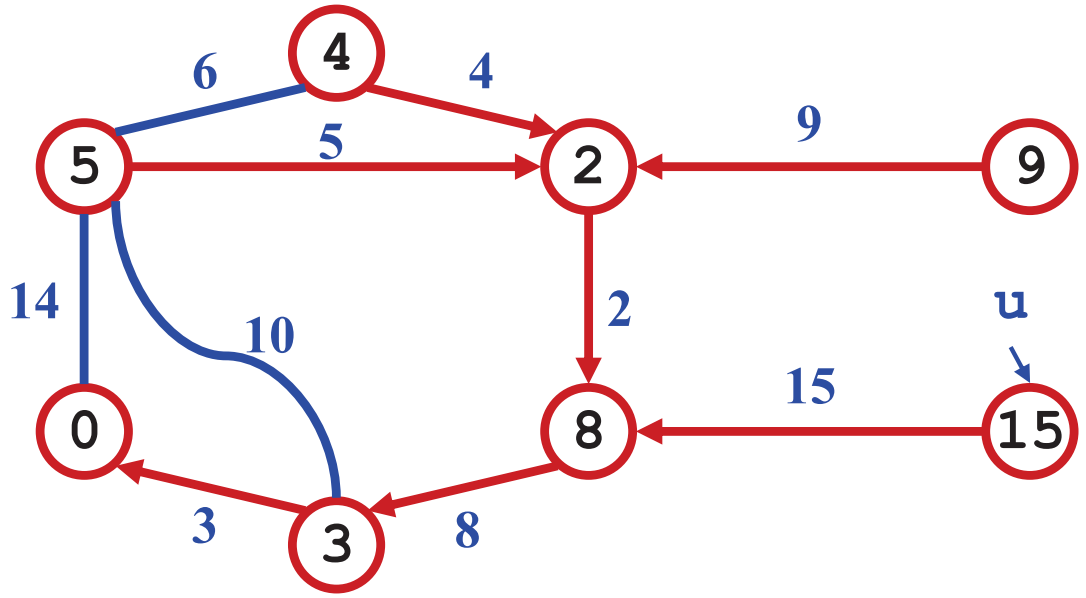

Prim's Algorithm