An application of the Markov Chains in Digital Communication

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Abstract

- This contribution shows an application of Markov chains in digital communication.
- A random sequence of the symbols 0 and 1 is analyzed by a state machine.
- The state machine switches to state "0" after detecting an unbroken sequence of w zero symbols (w being a fixed integer), and to state "1" after detecting an unbroken sequence of w ones.
- The task to find the probabilities of each of these two states after n time steps leads to a Markov chain.
- We show the construction of the transition matrix and determine the steady-state probabilities for the time-homogeneous case.

Markov Chain

- Let X_n be a random variable that characterizes the state of the studied system at discrete points in time n = 0, 1, 2, ... The family of random variables X_n forms a stochastic process. This stochastic process is called a Markov process. if the occurrence of a future state depends only on the immediately preceding state, i.e., if

$$\begin{aligned} \Pr(X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, ..., X_2 = x_2, X_1 = x_1) \\ = \Pr(X_{n+1} = x_{n+1} | X_n = x_n) \end{aligned}$$

- Let p_{ij}(n) is the probability of moving from state i at time n to state j at time n+1, then

$$p_{ij}(n) = \Pr(X_{n+1} = j | X_n = i), \quad i, j = 1, 2, 3, ...m$$

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- The transition matrix is defined as :

$$\mathbf{P}(n) = \begin{pmatrix} p_{11} & p_{12} & p_{13} & ...... & p_{1m} \\ p_{21} & p_{22} & p_{23} & ...... & P_{2m} \\ p_{31} & P_{32} & P_{33} & ...... & p_{3m} \\ .... & ..... & .... & .... & .... \\ p_{m1} & p_{m2} & p_{m3} & ...... & p_{mm} \end{pmatrix}$$

- Let define

$$a_j(n) = \Pr(X_n = j), \quad j = 1, 2, 3, , m, \quad n = 0, 1, 2, 3, 4, ...$$

Given the initial probabilities $\mathbf{a}(0) = (a_1(0), a_2(0), a_3(0), ..., a_m(0))$ and the transition matrix $\mathbf{P}(\mathrm{n})$ the probabilities $\mathbf{a}(n) = (a_1(n), a_2(n), a_3(n), ..., a_m(n))$ is computed as follows:

$$\mathbf{a}(1) = \mathbf{a}(0)\mathbf{P}(0)$$
$$\mathbf{a}(2) = \mathbf{a}(1)\mathbf{P}(1) = \mathbf{a}(0)\mathbf{P}(0)\mathbf{P}(1)$$

Continue...

and recursively

$$\mathbf{a}(n) = \mathbf{a}(0) \prod_{k=0}^{n} \mathbf{P}(k) \quad n = 1, 2, 3, 4, 5, 6, ...$$

Formulation of the problems

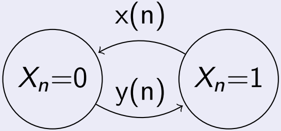
- A random sequence of 0's and 1's is analysed.The state machine switches to state "0" after detecting an unbroken sequence of w zero symbols, and to state "1" after detecting an unbroken sequence of w ones. Let the random variables X_n, n = 0, 1, 2, . . . denotes this process. In this case, X_n can assume only the values 0 and 1.
- Denote the probability of state "0" after n time step as x(n):

$$x(n) = \Pr(X_n = 0)$$

- Denote the probability of state "1" after n time step as y(n):

$$y(n) = \Pr(X_n = 1)$$

- For determining the x(n) and y(n) a markov chain can be used



Construction of the transition matrix

- Unfortunately, we cannot use the Markov chains directly for computing the desired probabilities x(n) and y(n). This is because the fact that if, e.g., the current state of the system is "0", then the probability that it will become "1" in the next time step depends upon various factors, not only on the current state and the last symbol in the random sequence R_n.Let us explain it on the example of w = 3, i.e., on the case where the state switches after the occurrence of 3 equal symbols.

Example

In this case, if the state at time n was "1" and at time n + 1 it has become "0" (which means that the end of the determining sequence of symbols is . . . , 0, 0, 0), then it cannot return to state "1" at times n + 2 and n + 3, but first at time n + 4. Thus, markov property does not hold here.

- It shows that to overcome this problem, it is convenient to consider more states than just only "0" and "1".

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- We will take into account how close to switching the situation is, i.e., how long the unbroken line of ones (if the current state is “0”) or the unbroken line of zeros (if the current state is “1”) is.
- Let us define new random variable Y_n with 2w possible values as follows:

  $Y_n = 1$ (state “0-0”) if $X_n = 0$ and the last symbol that came is 0,
  $Y_n = 2$ (state “0-01”) if $X_n = 0$ and the last two symbols are 0, 1,
  $Y_n = 3$ (state “0-011”) if $X_n = 0$ and the last three symbols are 0, 1, 1,

  ...

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  $Y_n =$ w (state “0-011 . . . 1”)(w-1 1’s) if $X_n = 0$ and the last w symbols are 0,1, 1, . . . , 1,(w-1, 1’s)

continue...

Y_n = w + 1 (state "1-1") if $X_n = 1$ and the last symbol that came is 1,
Y_n = w + 2 (state "1-10") if $X_n = 1$ and the last two symbols are 1, 0,
Y_n = w + 3 (state "1-100") if $X_n = 1$ and the last three symbols are 1, 0, 0,

...

...

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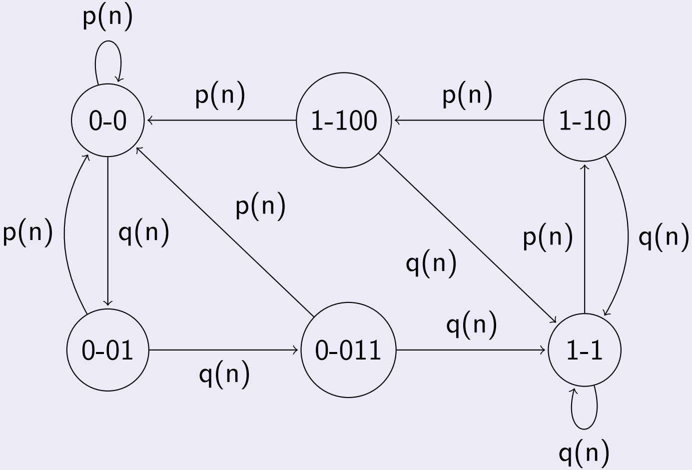
Y_n = 2w (state "1-100 . . . 0")(w-1 0's) if $X_n = 1$ and the last w symbols are 1,0, 0, . . . , 0,(w-1, 0's)

Note

Notice that if $Y_n = w$ and 1 comes as the next symbol, then X_{n+1} will be equal to 1 and Y_{n+1} will be equal to w + 1. And if $Y_n = 2w$ and the next symbol coming is zero, then $X_{n+1} = 0$ and $Y_{n+1} = 1$.

Markov chain for w=3



where

p(n)= Probability of occurrence of 0

q(n) = Probability of occurrence of 1

Steady State Probabilities

- Now we will study the asymptotic behaviour of the solution of the above system in the time-homogeneous case, i.e., in the case that the functions p(n) and q(n) are constant. For such p(n) and q(n) we are able to determine the limits (so called steady-state probabilities)

$$\lim_{n\to\infty} p(n) \quad \textit{and} \quad \lim_{n\to\infty} q(n)$$

Theorem

Suppose that function p(n) and q(n) are constant,
$p \equiv p \in (01) and q \equiv q = 1 - p. Then$

$$\lim_{n\to\infty} x(n) = \frac{p^{w-1}(1-q^w)}{p^{w-1}(1-q^w) + q^{w-1}(1-p^w)}$$

$$\lim_{n\to\infty} y(n) = \frac{q^{w-1}(1-p^w)}{p^{w-1}(1-q^w) + q^{w-1}(1-p^w)}$$

where $w =$ *length of the string of 0's and 1's*

Proof.

First, we compute the limits of the auxiliary functions a_j , j = 1,..., 2w. We will denote them as

$$\pi_j = \lim_{n \to \infty} a_j(n), \;\; j = 1, 2, 3, 4, ...2w$$

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, ...\pi_{2w})$$

The vector $\boldsymbol{\pi}$ is the solution of the system :

$$\boldsymbol{\pi} = \boldsymbol{\pi P}$$

with the condition that :

$$\sum_{j=1}^{2w} \pi_j = 1$$

where **P** is the transition matrix. □

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Since

$$x(n) = \sum_{j=1}^{w} a_j(n) \;\; and \;\; y(n) = \sum_{j=w+1}^{2w} a_j(n)$$

$$\Rightarrow$$

$$\lim_{n\to\infty} x(n) = \sum_{j=1}^{w} a_j(n) \;\; and \;\; \lim_{n\to\infty} y(n) = \sum_{j=w+1}^{2w} a_j(n)$$

For simlicity I am considering w=3

Since

$$\boldsymbol{\pi}(\boldsymbol{n}+1) = \boldsymbol{\pi}(\boldsymbol{n})\boldsymbol{P}$$

In the limiting case

$$\boldsymbol{\pi}(\boldsymbol{n}+1) = \boldsymbol{\pi}(\boldsymbol{n}) = \boldsymbol{\pi}$$

So

continue

$$\begin{aligned}
\boldsymbol{\pi} &= \boldsymbol{\pi}\boldsymbol{P} \\
\boldsymbol{\pi}^T &= (\boldsymbol{\pi}\boldsymbol{P})^T \\
\boldsymbol{\pi}^T &= \boldsymbol{P}^T\boldsymbol{\pi}^T \\
(\boldsymbol{P}^T - \boldsymbol{I})\boldsymbol{\pi}^T &= \boldsymbol{o} \\
\boldsymbol{P}^T - \boldsymbol{I} &= \begin{pmatrix} p-1 & p & p & 0 & 0 & p \\ q & -1 & 0 & 0 & 0 & 0 \\ 0 & q-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & q-1 & q & q \\ 0 & 0 & 0 & p & -1 & 0 \\ 0 & 0 & 0 & 0 & p & -1 \end{pmatrix}
\end{aligned}$$

Continue...

$$\boldsymbol{P}^T - \boldsymbol{I} \sim \begin{pmatrix} -q & p & p & 0 & 0 & p \\ 0 & -q & p & 0 & 0 & p \\ 0 & 0 & -q & 0 & 0 & p \\ 0 & 0 & 0 & -p & q & 1 \\ 0 & 0 & 0 & 0 & -p & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

the rank of the above matrix is 5=2w-1 for w=3 and we have another equation :

$$\sum_{j=1}^{2w} \pi_j = 1$$

Hence the above system has the unique solution.

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$$\pi_5 = \frac{\pi_6}{p}$$

$$\pi_4 = \frac{\pi_5}{p} = \frac{\pi_6}{p^2}$$

$$\pi_3 = \frac{p}{q}\pi_6$$

$$\pi_2 = \frac{1}{q}\pi_3 = \frac{p}{q^2}\pi_6$$

$$\pi_1 = \frac{1}{q}\pi_2 = \frac{p}{q^3}\pi_6$$

Since

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1$$

$$\pi_6 \left(\frac{p}{q^3} + \frac{p}{q^2} + \frac{p}{q} + \frac{1}{p^2} + \frac{1}{p} + 1 \right) = 1$$

Continue...

$$
\begin{aligned}
\pi_6 &= \frac{p^2q^3}{p^3(1+q+q^2)+q^3(1+p+p^2)} \\
\lim_{n\to\infty} x(n) &= \pi_1+\pi_2+\pi_3 \\
&= \frac{p}{q^3}\pi_6 + \frac{p}{q^2}\pi_6 + \frac{p}{q}\pi_6 \\
&= \left(\frac{p}{q^3} + \frac{p}{q^2} + \frac{p}{q}\right)\pi_6 \\
&= \left(\frac{p+pq+pq^2}{q^3}\right)\left(\frac{p^2q^3}{p^3(1+q+q^2)+q^3(1+p+p^2)}\right) \\
&= \frac{p^2(1-q^3)}{p^2(1-q^3)+q^2(1-q^3)}
\end{aligned}
$$

Similar expression can be obtained for $\lim_{n\to\infty} y(n)$

Further Research

- When modelling the data and clock recovery, the functions p(n) and q(n) are not constant but they are periodic functions of the argument n.
- Investigation of the asymptotic properties of the solution in such case is a much more complicated task and will be subject to further research.