

MATH 2600–Problem Set 2

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1 Reduced Row Echelon Form Uniqueness

Let A denote an arbitrary matrix. Then, the Reduced Row Echelon form (RREF) of A is unique. That is, after performing some elementary row operations on A , there exists no two different matrices that both represent the RREF of A .

Assume two matrices, P and Q , represent the RREF of A . By definition, both P and Q have the same solutions as each other and A if they represent augmented matrices of a system. This implies that by performing elementary row operations on P , we can compute Q .

Call P_i and Q_i the i -th row of P and Q . Then, P_1 and Q_1 must be non-zero, since zero rows must be below non-zero rows. Say that the pivot entry of P_1 is at column j , call this P_{1j} , and the pivot entry of Q_1 is at column k , call this Q_{1k} . If $j = k - 1$, then it would no longer be possible to perform elementary row operations on P to get to Q due to the 0 at P_{1k} or Q_{1j} , implying that P is *not* row-equivalent to Q . In general, $j \neq k$, and similarly, $j \neq k$ for the same reason. Thus, $j = k$; the pivot columns of P_1 and Q_1 must be equal.

We can apply this same logic to the rest of the rows. That is, the pivot column of P_i must be equal to that of Q_i . Earlier, we established that by performing elementary row operations, we can get from P to Q . And since every pivot entry of P and Q are equal and in the same column, the i -th row of P is simply the i -th row of Q multiplied by the constant 1. Therefore, $P = Q$, and no two *different* matrices can represent the RREF of an arbitrary matrix. ■

2 Equal Reduced Row Echelon Forms

Let A and B be two matrices that have the same RREF R . It is possible to transform A into B by performing a sequence of elementary row operations.

By performing i elementary row operations on A , we can obtain R . By performing j elementary row operations on B , we can also obtain R . That is,

$$(O_1, O_2, \dots, O_{i-1}, O_i)(A) = R$$

$$(P_1, P_2, \dots, P_{j-1}, P_j)(B) = R$$

Therefore,

$$(O_1, O_2, \dots, O_{i-1}, O_i)(A) = (P_1, P_2, \dots, P_{j-1}, P_j)(B) = R$$

For all O_1, \dots, O_i and P_1, \dots, P_j , there exists $O_1^{-1}, \dots, O_i^{-1}$ and $P_1^{-1}, \dots, P_j^{-1}$ respectively, that reverse their original elementary row operation.

There are just 3 elementary row operations possible on any matrix M . Let M_i be the i -th row of M and M_j the j -th of M . If E_1, E_2, E_3 are...

1. switching M_i with M_j
2. adding a multiple c of M_j to M_i (that is, $M_i + c * M_j$)
3. multiplying M_i by some non-zero constant k

then the elementary row operations $E_1^{-1}, E_2^{-1}, E_3^{-1}$ represent...

1. switching M_i with M_j again
2. adding the negated multiple c of M_j to M_i (that is, $M_i - c * M_j$)
3. multiplying M_i by the constant $\frac{1}{k}$

respectively. This implies that...

$$(E_1^{-1}, E_2^{-1}, E_3^{-1})[(E_1, E_2, E_3)(M)] = M$$

Applying this logic to A and B ...

$$(O_1, \dots, O_i)(A) = (P_1, \dots, P_j)(B)$$

$$(P_1^{-1}, \dots, P_j^{-1})[(O_1, \dots, O_i)(A)] = (P_1^{-1}, \dots, P_j^{-1})[(P_1, \dots, P_j)(B)]$$

$$(P_1^{-1}, \dots, P_j^{-1})[(O_1, \dots, O_i)(A)] = B$$

$(P_1^{-1}, \dots, P_j^{-1})$ and (O_1, \dots, O_i) are all elementary row operations on A , and we are able to transform A to B . ■