# MATH 2600–Problem Set 3

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## 1. All Possible Products

A is of shape $3 \times 2$, B is of shape $2 \times 4$, C is of shape $4 \times 3$, and D is of shape $3 \times 3$. Thus, the possible products are: $AB$, $BC$, $CD$, $DA$, $CA$.

$$AB = \begin{pmatrix} 2(1)+5(9) & 2(7)+5(2) & 2(2)+5(7) & 2(9)+5(1) \\ 1(1)+4(9) & 1(7)+4(2) & 1(2)+4(7) & 1(9)+4(1) \\ 2(1)+1(9) & 2(7)+1(2) & 2(2)+1(7) & 2(9)+1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 47 & 24 & 39 & 23 \\ 37 & 15 & 30 & 13 \\ 11 & 16 & 11 & 19 \end{pmatrix}$$

$$BC = \begin{pmatrix} [1(1)+7(2)+2(1)+9(3)] & [1(0)+7(1)+2(1)+9(2)] & [1(4)+7(3)+2(5)+9(1)] \\ [9(1)+2(2)+7(1)+1(3)] & [9(0)+2(1)+7(1)+1(2)] & [9(4)+2(3)+7(5)+1(1)] \end{pmatrix}$$

$$= \begin{pmatrix} 44 & 27 & 44 \\ 23 & 11 & 78 \end{pmatrix}$$

$$CD = \begin{pmatrix} [1(1)+0(2)+4(1)] & [1(0)+0(1)+4(3)] & [1(7)+0(2)+4(0)] \\ [2(1)+1(2)+3(1)] & [2(0)+1(1)+3(3)] & [2(7)+1(2)+3(0)] \\ [1(1)+1(2)+5(1)] & [1(0)+1(1)+5(3)] & [1(7)+1(2)+5(0)] \\ [3(1)+2(2)+1(1)] & [3(0)+2(1)+1(3)] & [3(7)+2(2)+1(0)] \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 12 & 7 \\ 7 & 10 & 16 \\ 8 & 16 & 9 \\ 8 & 5 & 25 \end{pmatrix}$$

$$DA = \begin{pmatrix} [1(2)+0(1)+7(2)] & [1(5)+0(4)+7(1)] \\ [2(2)+1(1)+2(2)] & [2(5)+1(4)+2(1)] \\ [1(2)+3(1)+0(2)] & [1(5)+3(4)+0(1)] \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 12 \\ 9 & 16 \\ 5 & 17 \end{pmatrix}$$

$$CA = \begin{pmatrix} [1(2)+0(1)+4(2)] & [1(5)+0(4)+4(1)] \\ [2(2)+1(1)+3(2)] & [2(5)+1(4)+3(1)] \\ [1(2)+1(1)+5(2)] & [1(5)+1(4)+5(1)] \\ [3(2)+2(1)+1(2)] & [3(5)+2(4)+1(1)] \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 9 \\ 11 & 17 \\ 13 & 14 \\ 10 & 24 \end{pmatrix}$$

## 2. $AB$ vs. $BA$

**2b.** $\mathbf{A} = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 0 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & 6 \\ 1 & 5 \end{pmatrix}$

A is of shape $2 \times 3$ and B is of shape $3 \times 2$. Both $AB$ and $BA$ are thus defined. $AB$ is of shape $2 \times 2$ and $BA$ is of shape $3 \times 3$, so they do not have the same number of rows and columns.

**2d.** $\mathbf{A} = \begin{pmatrix} 3 & 1 & -4 \\ -2 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

A is of shape $3 \times 3$ and B is of shape $3 \times 3$. Both $AB$ and $BA$ are thus defined. $AB$ is of shape $3 \times 3$ and $BA$ is of shape $3 \times 3$, so they *do* have the same number of rows and columns.

$$AB = \begin{pmatrix} [3(2) + 1(0) + -4(0)] & [3(0) + 1(5) + -4(0)] & [3(0) + 1(0) + -4(-1)] \\ [-2(2) + 0(0) + 5(0)] & [-2(0) + 0(5) + 5(0)] & [-2(0) + 0(0) + 5(-1)] \\ [1(2) + -2(0) + 3(0)] & [1(0) + -2(5) + 3(0)] & [1(0) + -2(0) + 3(-1)] \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 5 & 4 \\ -4 & 0 & -5 \\ 2 & -10 & -3 \end{pmatrix}$$

Since both $A$ and $B$ are square matrices and $AB \neq I_3$, $B \neq A^{-1}$, and thus $BA$ is not going to equal to $AB$. To check...

$$BA = \begin{pmatrix} [2(3) + 0(-2) + 0(1)] & [2(1) + 0(0) + 0(-2)] & [2(-4) + 0(5) + 0(3)] \\ [0(3) + 5(-2) + 0(1)] & [0(1) + 5(0) + 0(-2)] & [0(-4) + 5(5) + 0(3)] \\ [0(3) + 0(-2) + -1(1)] & [0(1) + 0(0) + -1(-2)] & [0(-4) + 0(5) + -1(3)] \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 2 & -8 \\ -10 & 0 & 25 \\ -1 & 2 & -3 \end{pmatrix}$$

Indeed, $AB \neq BA$.

**3.** $A = \begin{pmatrix} 2 & -5 \\ 3 & 1 \end{pmatrix}$

**3a. Evaluate $A^2$**

$$\begin{aligned} A^2 &= \begin{pmatrix} [2(2) + -5(3)] & [2(-5) + -5(1)] \\ [3(2) + 1(3)] & [3(-5) + 1(1)] \end{pmatrix} \\ &= \begin{pmatrix} -11 & -15 \\ 9 & -14 \end{pmatrix} \end{aligned}$$

**3b. $\alpha$, $\beta$, $\gamma$**

Find $\alpha$, $\beta$, $\gamma \in \mathbb{R}$, such that $\alpha I + \beta A + \gamma A^2 = \mathbf{0}$ and $\alpha$, $\beta$, $\gamma$ not all 0.

$$\alpha I_2 + \beta A + \gamma A^2 = \mathbf{0}$$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 2\beta & -5\beta \\ 3\beta & \beta \end{pmatrix} + \begin{pmatrix} -11\gamma & 15\gamma \\ 9\gamma & -14\gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha + 2\beta - 11\gamma & -5\beta + 15\gamma \\ 3\beta + 9\gamma & \alpha + \beta - 14\gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\alpha + 2\beta - 11\gamma = 0$$

$$-5\beta + 15\gamma = 0$$

$$3\beta + 9\gamma = 0$$

$$\alpha + \beta - 14\gamma = 0$$

From the 2nd and 3rd equations, solving for $\gamma$...

$$\beta = -3\gamma$$

Plugging that into the 4th equation...

$$\alpha - 17\gamma = 0$$

$$\alpha = 17\gamma$$

$$\gamma = \gamma$$

There are infinitely many solutions. If not all zero, one solution is $\alpha = 17$, $\beta = -3$, $\gamma = 1$.

## 5. Proof

If $A$ and $B$ are matrices s.t. $I - AB$ is invertible, then the inverse of $I - BA$ is given by the formula $(I - BA)^{-1} = I + B(I - AB)^{-1}A$.
Let's denote $C$ as $(I - AB)^{-1}$. Then,

$$(I - BA)^{-1} = I + BCA$$

$$(I - BA)(I - BA)^{-1} = (I - BA)(I + BCA)$$

We will prove that $(I - BA)(I + BCA) = I$, since $(I - BA)(I - BA)^{-1} = I$.

$$(I - BA)(I + BCA) = I$$

$$I^2 + IBCA - BAI + BABCA = I$$

$$I + BCA - BA + BABCA = I$$

$$I + B(C - I - ABC)A = I$$

$$I + B(C - ABC - I)A = I$$

$$I + B[(I - AB)C - I]A = I$$

Since $C = (I - AB)^{-1}$, $C^{-1} = (I - AB)$, because $I - AB$ is assumed to be invertible.

$$I + B(C^{-1}C - I)A = I$$

$$I + B(I - I)A = I$$

$$I + B\mathbf{0}A = I$$

$$I + \mathbf{0} = I$$

$$I = I$$

Thus, the inverse of $I - BA$ is given by $I + B(I - AB)^{-1}A$. $\blacksquare$

## 6.

**6a.**

**6b.**

**6e.**

## 7.

**7a.**

**7b.**