MATH 2600–Problem Set 2

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1 Reduced Row Echelon Form Uniqueness

Let A denote an arbitrary matrix. Then, the Reduced Row Echelon form (RREF) of A is unique. That is, after performing some elementary row operations on A, there exists no two different matrices that both represent the RREF of A.

Assume two matrices, P and Q, represent the RREF of A. By definition, both P and Q have the same solutions as each other and A if they represent augmented matrices of a system. This implies that by performing elementary row operations on P, we can compute Q.

Call P_i and Q_i the i-th row of P and Q. Then, P_1 and Q_1 must be non-zero, since zero rows must be below non-zero rows. Say that the pivot entry of P_1 is at column j, call this P_{1j}, and the pivot entry of Q_1 is at column k, call this Q_{1k}. If $j = k - 1$, then it would no longer be possible to perform elementary row operations on P to get to Q due to the 0 at P_{1k} or Q_{1j}, implying that P is *not* row-equivalent to Q. In general, $j \not< k$, and similarly, $j \not> k$ for the same reason. Thus, $j = k$; the pivot columns of P_1 and Q_1 must be equal.

We can apply this same logic to the rest of the rows. That is, the pivot column of P_i must be equal to that of Q_i. Earlier, we established that by performing elementary row operations, we can get from P to Q. And since every pivot entry of P and Q are equal and in the same column, the i-th row of P is simply the i-th row of Q multiplied by the constant 1. Therefore, $P = Q$, and no two *different* matrices can represent the RREF of an arbitrary matrix. $\blacksquare$

2 Equal Reduced Row Echelon Forms

Let A and B be two matrices that have the same RREF R. It is possible to transform A into B by performing a sequence of elementary row operations.

By performing i elementary row operations on A, we can obtain R. By performing j elementary row operations on B, we can also obtain R. Let O_i and P_j represent the i-th and j-th elementary row operation on A and B respectively. Then,

$$(O_1, O_2, \ldots, O_{i-1}, O_i)(A) = R$$

$$(P_1, P_2, \ldots, P_{j-1}, P_j)(B) = R$$

Therefore,

$$(O_1, O_2, \ldots, O_{i-1}, O_i)(A) = (P_1, P_2, \ldots, P_{j-1}, P_j)(B) = R$$

For all $O_1, \ldots, O_i$ and $P_1, \ldots, P_j$, there exists $O_1^{-1}, \ldots, O_i^{-1}$ and $P_1^{-1}, \ldots, P_j^{-1}$ respectively, that reverse their original elementary row operation.

There are just 3 elementary row operations possible on any matrix M. Let M_i be the i-th row of M and M_j the j-th of M. If E_1, E_2, E_3 are...

1. switching M_i with M_j
2. adding a multiple c of M_j to M_i (that is, $M_i + c * M_j$)
3. multiplying M_i by some non-zero constant k

then let the elementary row operations $E_1^{-1}, E_2^{-1}, E_3^{-1}$ represent...

1. switching M_i with M_j again
2. adding the negated multiple c of M_j to M_i (that is, $M_i - c * M_j$)
3. multiplying M_i by the constant $\frac{1}{k}$

respectively. This implies that...

$$(E_3^{-1}, E_2^{-1}, E_1^{-1})[(E_1, E_2, E_3)(M)] = M$$

Applying this logic to A and B...

$$(O_1, \ldots, O_i)(A) = (P_1, \ldots, P_j)(B)$$

$$(P_j^{-1}, \ldots, P_1^{-1})[(O_1, \ldots, O_i)(A)] = (P_j^{-1}, \ldots, P_1^{-1})[(P_1, \ldots, P_j)(B)]$$

$$(P_j^{-1}, \ldots, P_1^{-1})[(O_1, \ldots, O_i)(A)] = B$$

$(P_j^{-1}, \ldots, P_1^{-1})$ and $(O_1, \ldots, O_i)$ are elementary row operations on A, and we can transform A to B. ■