Wavelet Transforms

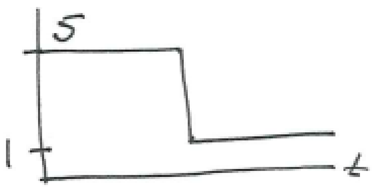
- Length N input -> wavelet transform -> length N output
- Wavelet transforms are often implemented with filter banks consisting of a low-pass and a high-pass filter
- The filter bank(s) can be represented by an (analysis) matrix
- The rows of the matrix are the basis functions
  - (sines and cosines for Fourier transform)
  - For wavelet transforms, the 1st row is a scaling function, and the remaining rows are wavelet functions
- Can have several levels of wavelet transforms (all are length N)
- Length 8 example:
  - 1 level: $[a_{00}, a_{01}, \quad a_{02}, a_{03}, \quad b_{00}, b_{01}, b_{02}, b_{03}]$
  - 2 levels: $[a_{00}, a_{01}, \quad b_{00}, b_{01}, \quad b_{10}, b_{11}, b_{12}, b_{13}]$
  - 3 levels: $[a_{00}, b_{00}, \quad b_{11}, b_{11}, \quad b_{20}, b_{21}, b_{22}, b_{23}]$

Example: Wavelet transform of signal of length 2 using the Haar wavelet

Split the function into a scaling function & wavelet using a Haar wavelet

$$f(t) = a_0\,\phi_0(t) + b_0\,\omega_0(t)$$

where $f(t) =$



Find a_0 and b_0

write problem in discrete form. $\bar{f} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

write f as a column vector

$$\bar{Y} = \bar{\bar{A}}\,\bar{f}$$

Haar wavelet

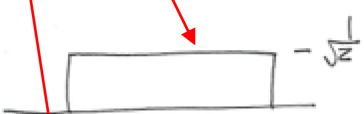
LP = $(1/\sqrt{2})[1, 1]$

HP = $(1/\sqrt{2})[1, -1]$

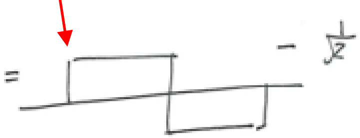
$$\bar{Y} = \frac{1}{\sqrt{2}} \begin{bmatrix} \boxed{1 \quad 1} \\ \boxed{1 \quad -1} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow \bar{Y} = \frac{1}{\sqrt{2}} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

write the WT matrix **A** and multiply by f: $a_0 = 6/\sqrt{(2)}$, $b_0 = 4/\sqrt{(2)}$

$\phi_0(t) =$



$\omega_0(t) =$



$[a_0\ b_0]$ is the WT of the signal

$\phi_0(t)$ and $\omega_0(t)$ are the basis functions

Example: Verify the inverse wavelet transform of the signal

inverse transform is the
inverse of the **A** matrix

verify

$$\bar{f} = \bar{\bar{A}}^{-1}\,\bar{y}$$

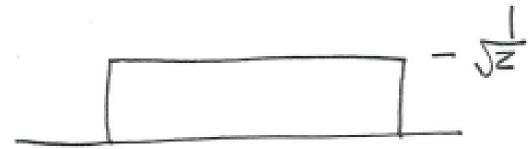
$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

true!

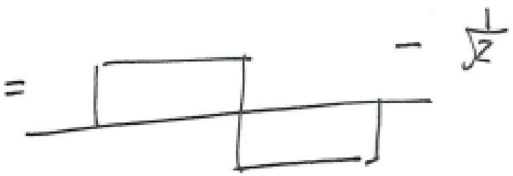
$$a_0 = \frac{6}{\sqrt{2}}$$

$$b_0 = \frac{4}{\sqrt{2}}$$

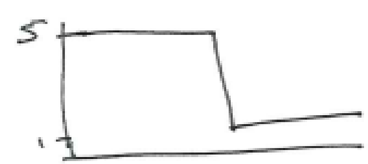
$\phi_0(t) =$



$w_0(t) =$



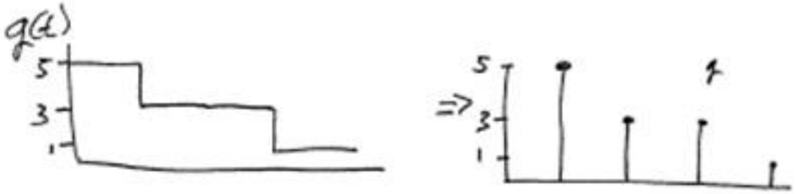
$a_0\,\phi_0(t) + b_0\,w_0(t) =$

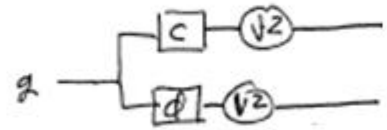


analytically

graphically

Haar expansion of example function g(t) using one level





c = LP
d = HP

Haar wavelet
LP = [r, r]
HP = [r, -r]
where r = 1/√2

matrix for first filter bank

$$\begin{bmatrix} a_{0,0} \\ a_{0,1} \\ b_{0,0} \\ b_{0,1} \end{bmatrix} = \begin{bmatrix} r & r & 0 & 0 \\ 0 & 0 & r & r \\ r & -r & 0 & 0 \\ 0 & 0 & r & -r \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \end{bmatrix} \leftarrow g \quad \text{where } r = \frac{1}{\sqrt{2}}$$

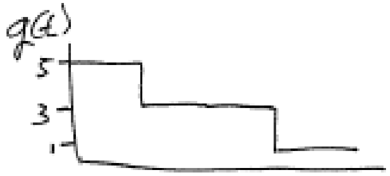
$a_{0,0} = 8/\surd 2$, $a_{0,1} = 4/\surd 2$, $b_{0,0}$ $2/\surd 2$, $b_{0,1} = 2/\surd 2$ -> $1/\surd 2[8, 4, 2, 2]$

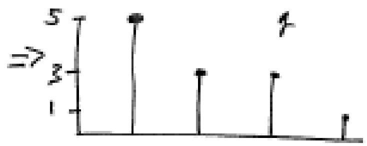
1 level of expansion

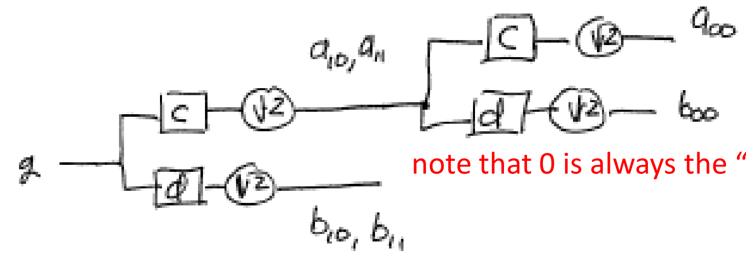
$$g(t) = 8/\surd 2\phi_{0,0}(t) + 4/\surd 2\phi_{0,1}(t) + 2/\surd 2\omega_{0,0}(t) + 2/\surd 2\omega_{0,1}(t)$$

1[st] row of matrix, 2[nd] row of matrix, …
The first indice is the scale, and the second is the shift

Haar expansion of example function g(t) using two levels







note that 0 is always the "last" scale

Haar wavelet
LP = [r, r]
HP = [r, -r]
where r = 1/√2

matrix for first filter bank

$$\begin{bmatrix} a_{10} \\ a_{11} \\ b_{10} \\ b_{11} \end{bmatrix} = \begin{bmatrix} r & r & 0 & 0 \\ 0 & 0 & r & r \\ r & -r & 0 & 0 \\ 0 & 0 & r & -r \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \end{bmatrix} \leftarrow g \qquad \text{where } r = \frac{1}{\sqrt{2}}$$

$$a_{10} = \frac{8}{\sqrt{2}}, \quad a_{11} = \frac{4}{\sqrt{2}}, \quad b_{10} = \frac{2}{\sqrt{2}}, \quad b_{11} \frac{2}{\sqrt{2}}$$

second filter bank

$$\begin{bmatrix} a_{00} \\ b_{00} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{8}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{bmatrix}$$

$$a_{00} = 6, \quad b_{00} = 2$$

1 level expansion

$$g(t) = \frac{8}{\sqrt{2}} \phi_{10}(t) + \frac{4}{\sqrt{2}} \phi_{11}(t) + \frac{2}{\sqrt{2}} w_{10}(t) + \frac{2}{\sqrt{2}} w_{11}(t)$$

where $\phi_{10}(t)$ is row 1 of the matrix
" $\phi_{11}(t)$ is row 2 " " "
" $w_{10}(t)$ " row 3 " " "
" $w_{11}(t)$ " row 4 " " "

$$\begin{bmatrix} r & r & 0 & 0 \\ 0 & 0 & r & r \\ r & -r & 0 & 0 \\ 0 & 0 & r & -r \end{bmatrix}$$

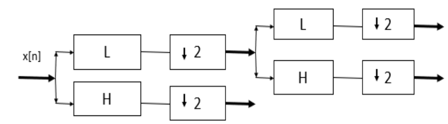
2 level expansion

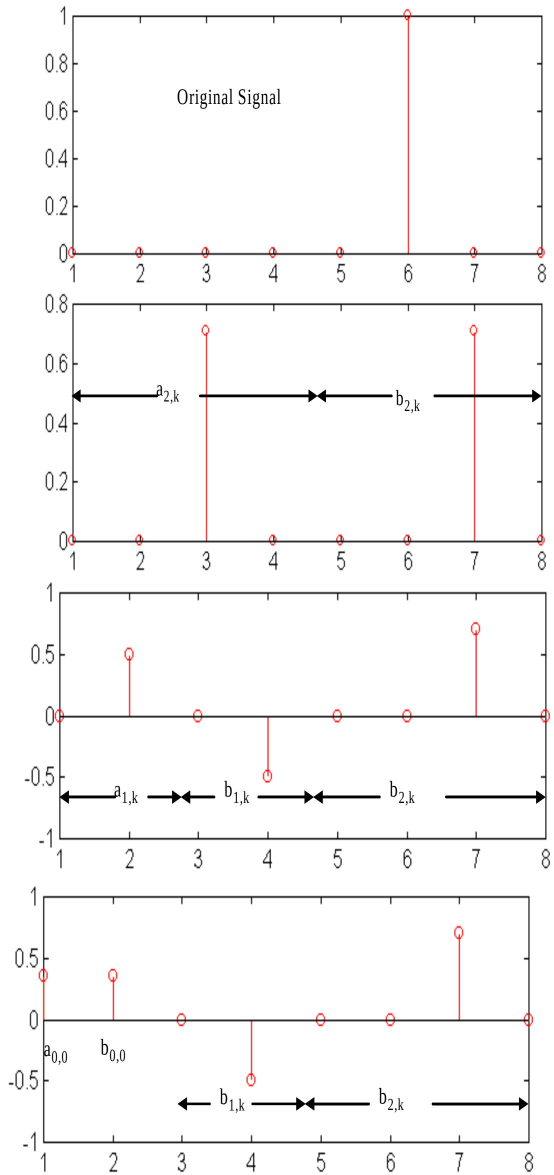
$$g(t) = 6\, \phi_{00}(t) + 2\, w_{00}(t) + \frac{2}{\sqrt{2}} w_{10}(t) + \frac{2}{\sqrt{2}} w_{11}(t)$$

where $\phi_{00}(t)$ is row 1 of the matrix
" $w_{00}(t)$ " row 2 " "
" $w_{10}(t)$ " " 3 " " "
" $w_{11}(t)$ " " 4 " " "

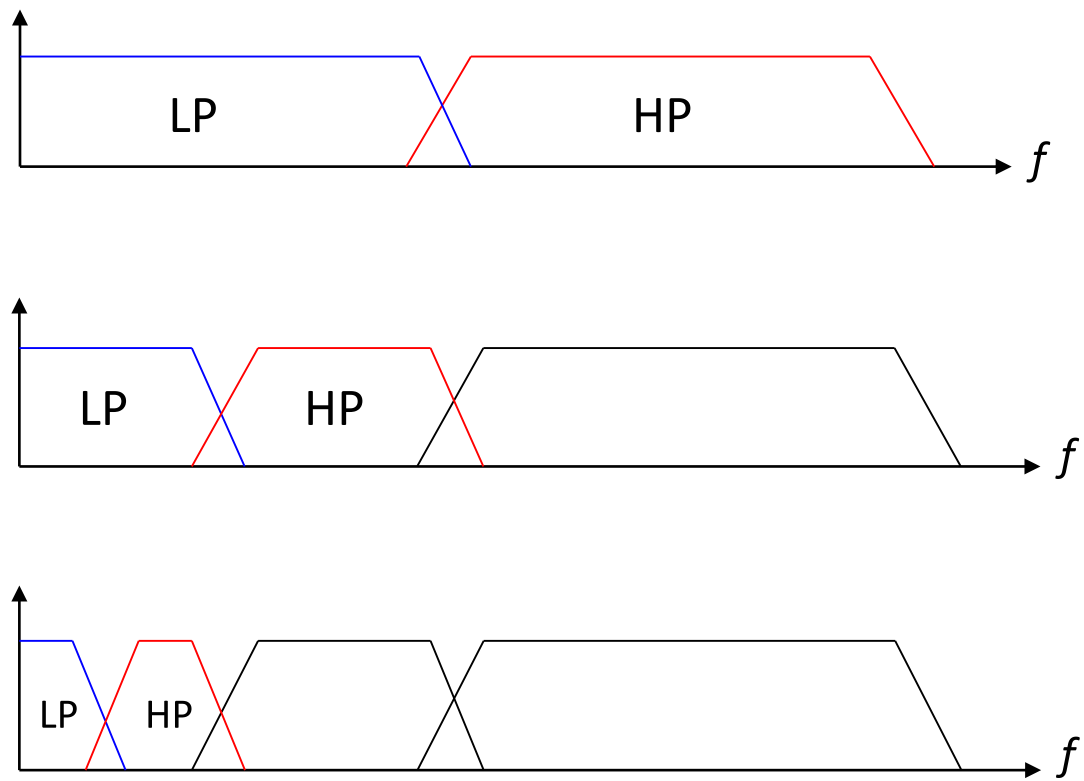
$$\begin{bmatrix} r^2 & r^2 & r^2 & r^2 \\ r^2 & r^2 & -r^2 & -r^2 \\ r & -r & 0 & 0 \\ 0 & 0 & r & -r \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} r & r \\ r & -r \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix} \begin{bmatrix} r & r & 0 & 0 \\ 0 & 0 & r & r \\ r & -r & 0 & 0 \\ 0 & 0 & r & -r \end{bmatrix}$$

Frequency Band Interpretation

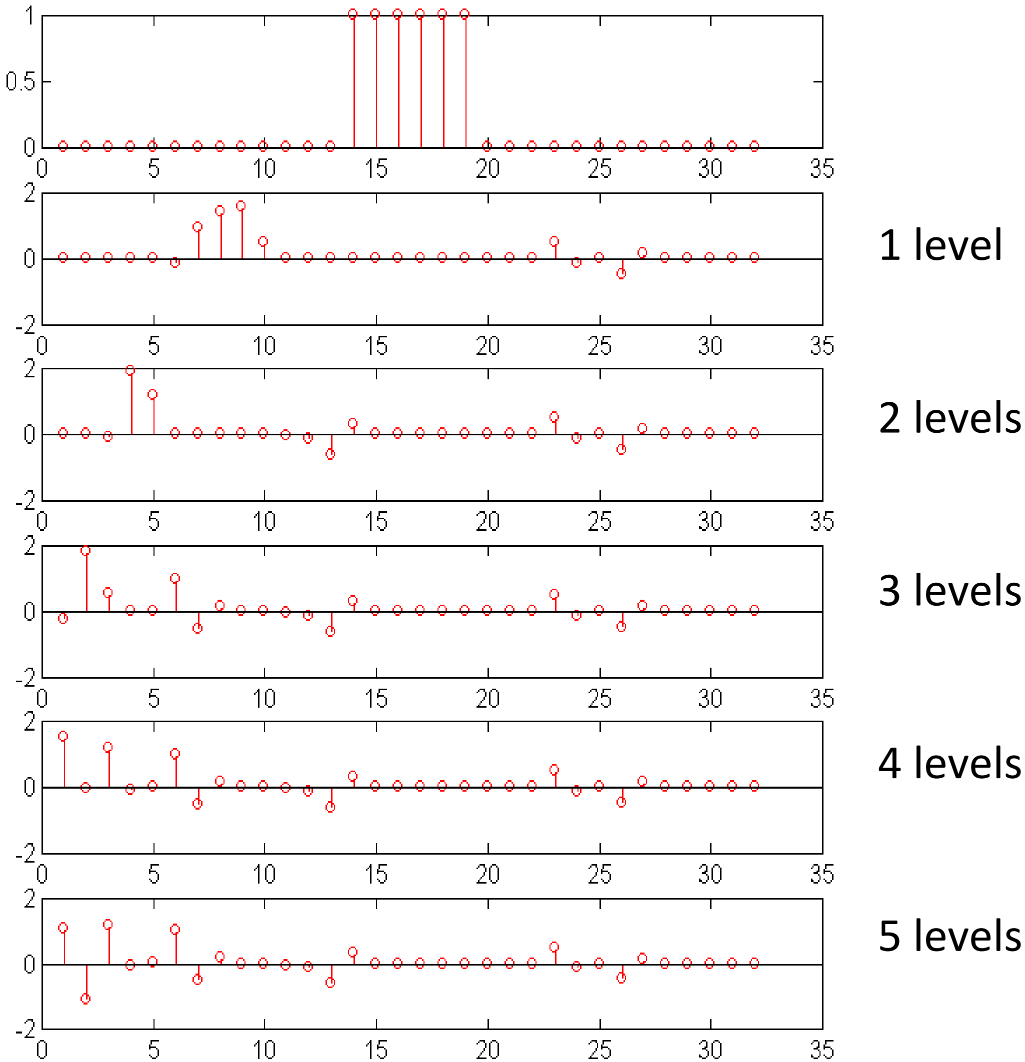




- Only filter LP for the next level
- The output of the HPs are the wavelet coefficients
- Outputs of HPs don’t change as additional levels are used
- Range of HP changes at each level due to downsampling

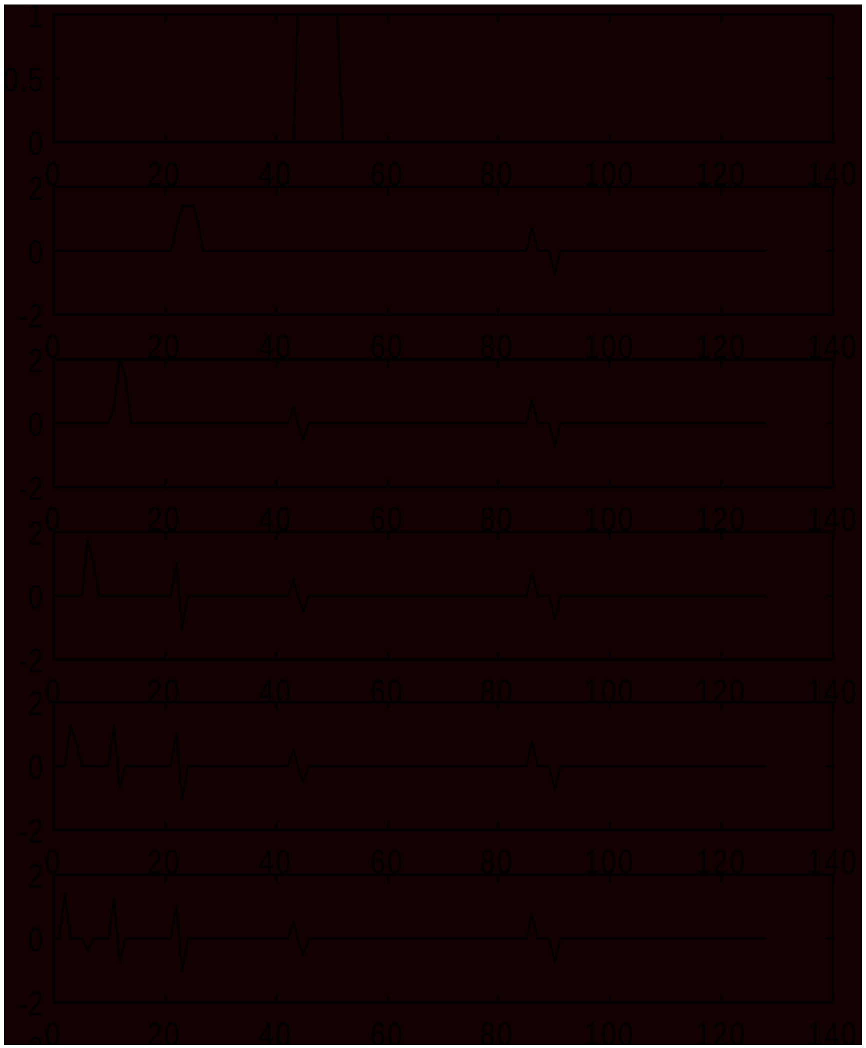


Wavelet Transforms are Sparse



- Transform data much more compactly represented than the Fourier transform

Wavelet Transforms are Sensitive to Changes
(easy to see edges)



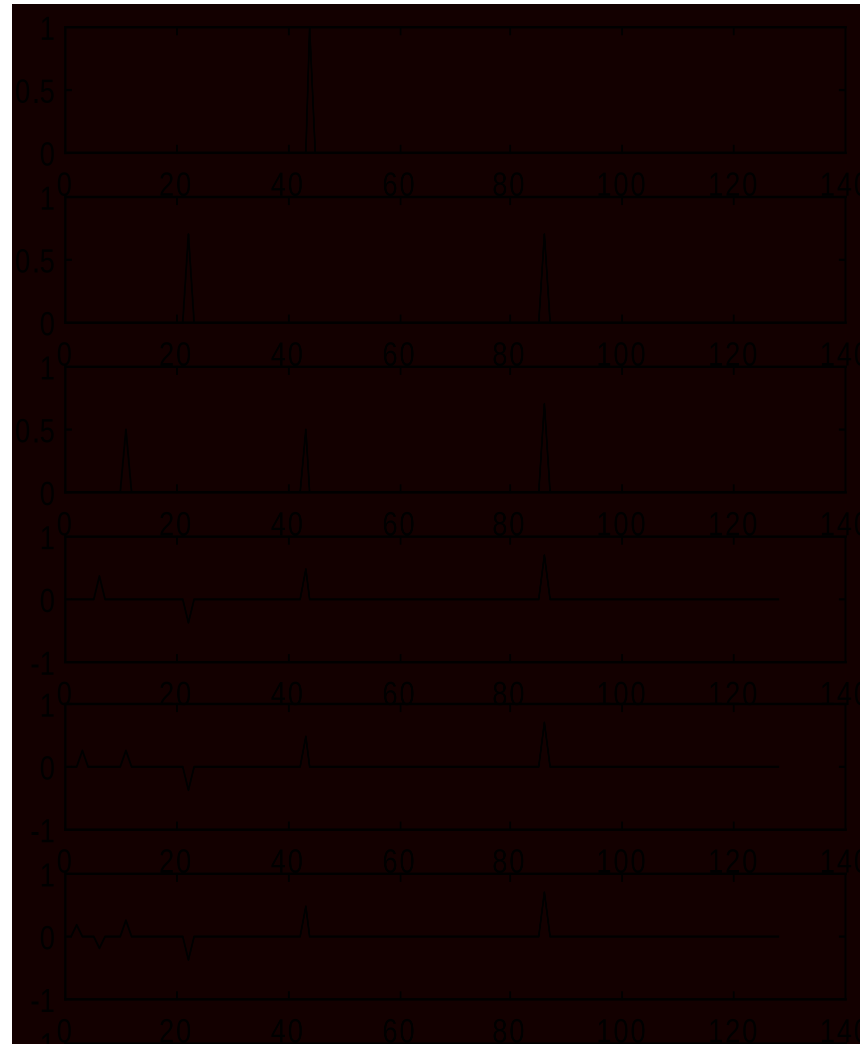
1 level

2 levels

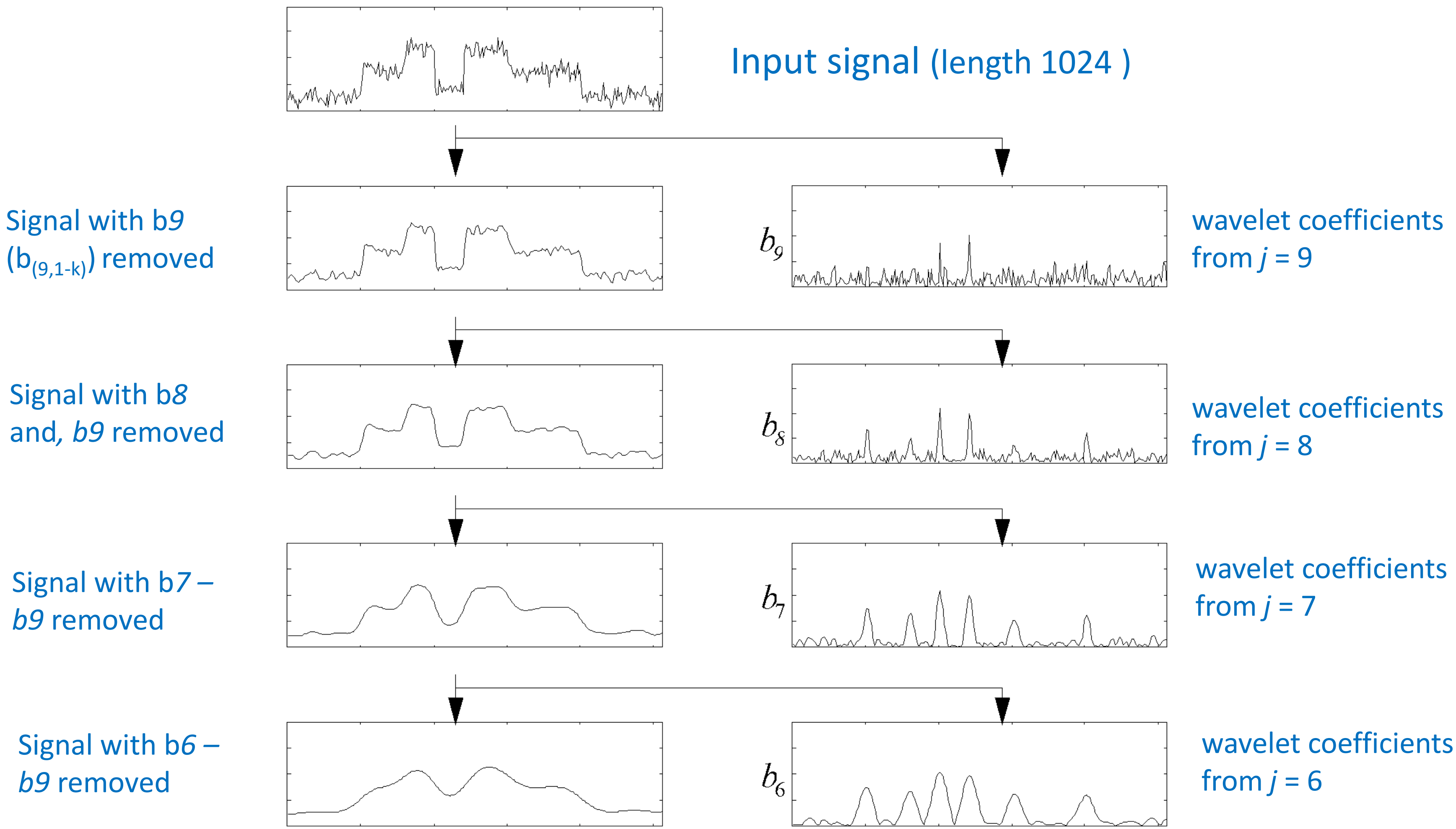
3 levels

4 levels

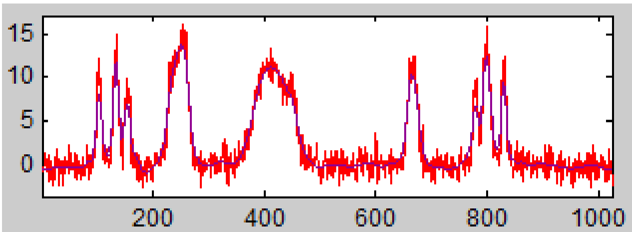
5 levels



Can Detect Features in Signals



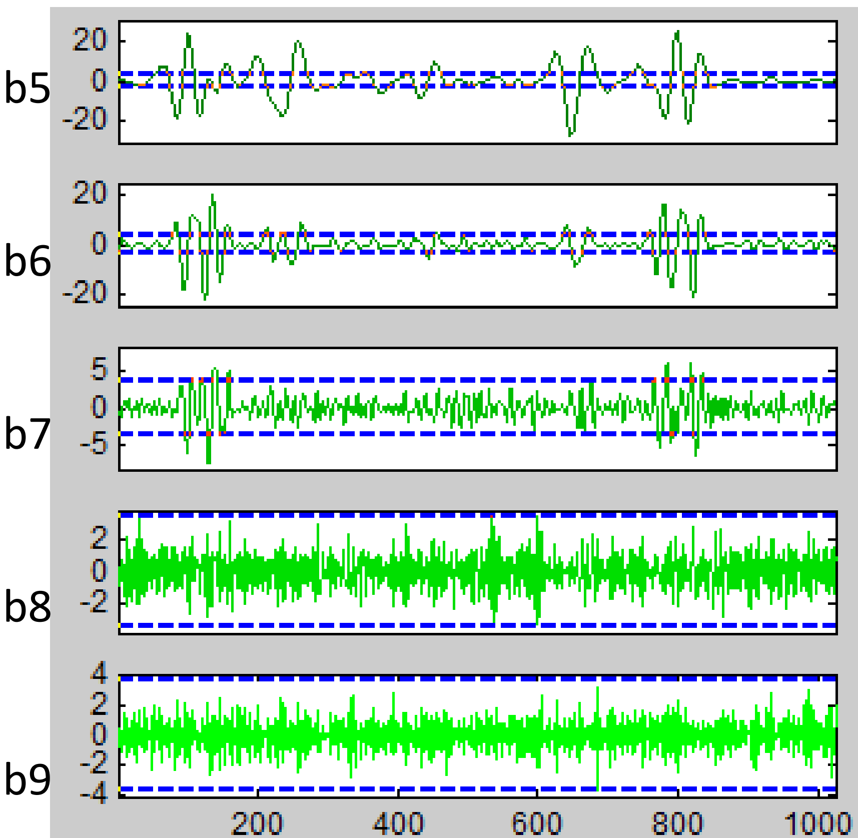
Can Remove Noise



Noisy input

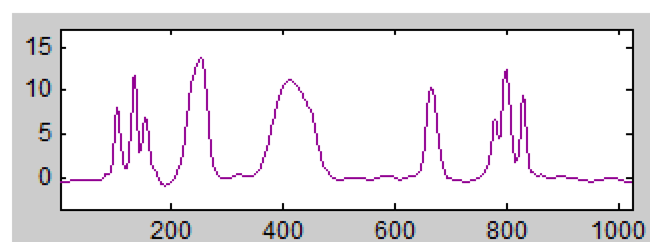
This is a signal with N = 1024
You only need:
512 coefficients for b9
256 coefficients for b8
128 coefficients for b7
64 coefficients for b6
32 coefficients for b5
etc

Set noisy coefficients to zero

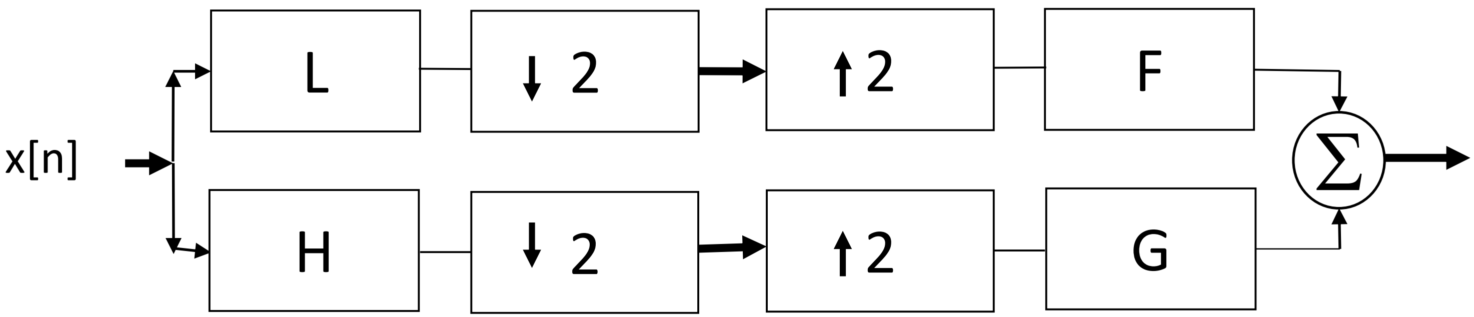


Reconstruct signal with noisy coefficients removed.

In this example, b9 set to 0, b8 set to 0. Values of blue-dotted line are subtracted from coefficients.



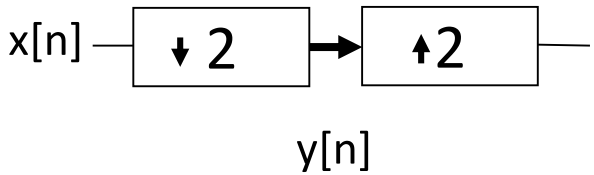
Reconstruction is the Reverse of Analysis



L and **F** are low-pass filters
H and **G** are high-pass filters

- The downsampling followed by upsampling introduces aliasing
- The filters **F** and **G** are chosen to cancel aliasing and give perfect reconstruction

Downsampling Followed by Upsampling Introduces Aliasing



$w[n] <-> \frac{1}{2}\,[X(z) + X(-z)]$

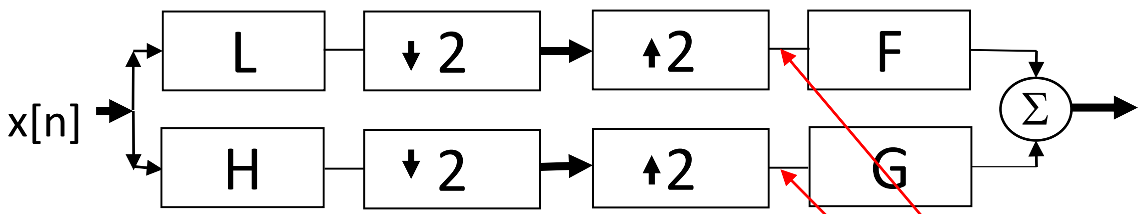
Fourier transform of downsampling, $\mathbf{y} = (\downarrow 2)\mathbf{x}$: $Y(\omega) = \frac{1}{2}\,[X(\omega/2) + X(\omega/2 + \pi)]$

Fourier transform of upsampling, $\mathbf{w} = (\uparrow 2)\mathbf{y}$: $W(\omega) = Y(2\omega)$

Fourier transform of downsampling then upsampling, $\mathbf{w} = \frac{1}{2}\,[X(\omega) + X(\omega + \pi)]$

Written in z-transform notation, $\mathbf{w} = \frac{1}{2}\,[X(z) + X(-z)]\mathbf{x}$ ($z = -1$ in z-plane $= \omega + \pi$)

Design Reconstruction Filters to Cancel Aliasing



Low-pass channel, $(\uparrow 2)\ (\downarrow 2)\mathbf{Lx}$ = ½ [L(z)X(z) + L(−z)X(-z)]

High-pass channel, $(\uparrow 2)\ (\downarrow 2)\mathbf{Hx}$ = ½ [H(z)X(z) + H(−z)X(-z)]

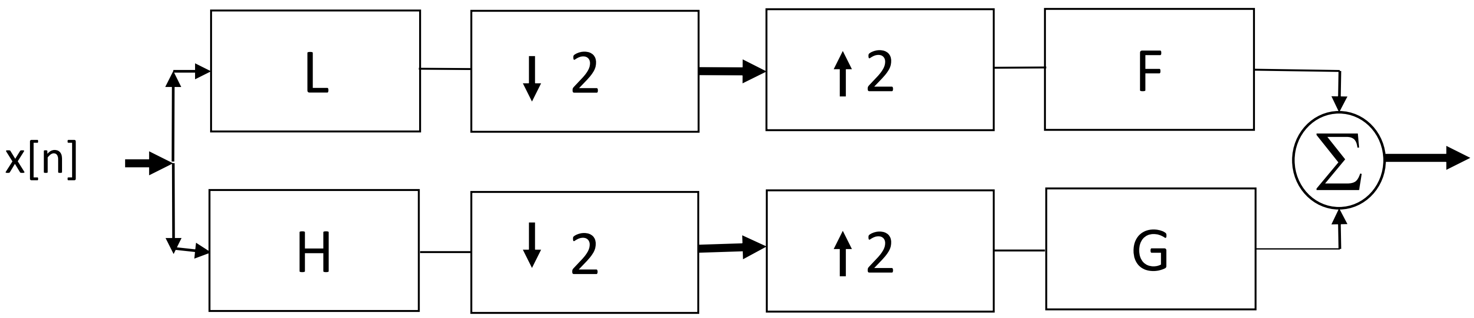
Sum of channels, $\mathbf{F}(\uparrow 2)\ (\downarrow 2)\mathbf{Lx} + \mathbf{G}(\uparrow 2)\ (\downarrow 2)\mathbf{Hx}$

- Write only the aliasing part because we want it to cancel

½[F(z)L(−z)X(-z)] + ½[G(z)H(−z)X(-z)] = 0

Aliasing cancellation condition is: F(z)L(−z) + G(z)H(−z) = 0

Reconstruction is the Reverse of Analysis



- The downsampling followed by upsampling introduces aliasing
- The filters **F** and **G** are chosen to **<u>cancel</u>** aliasing and give perfect reconstruction

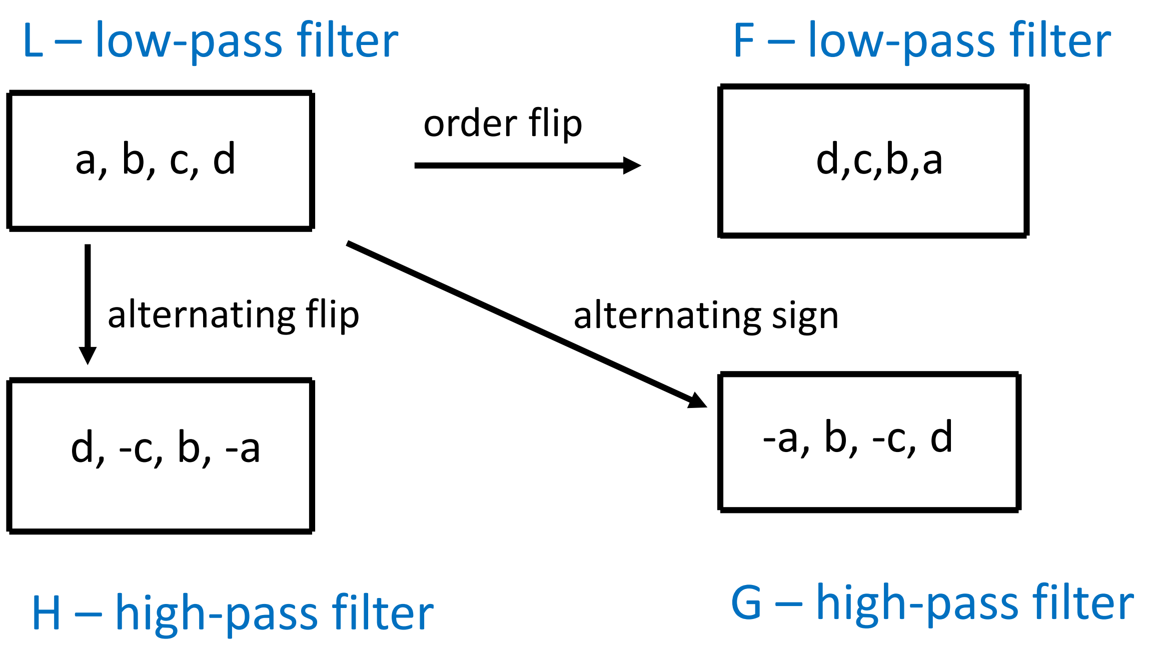
$$F(z) = H(-z)$$

Aliasing cancellation condition: $F(z)L(-z) + G(z)H(-z) = 0$

$$G(z) = -L(-z)$$

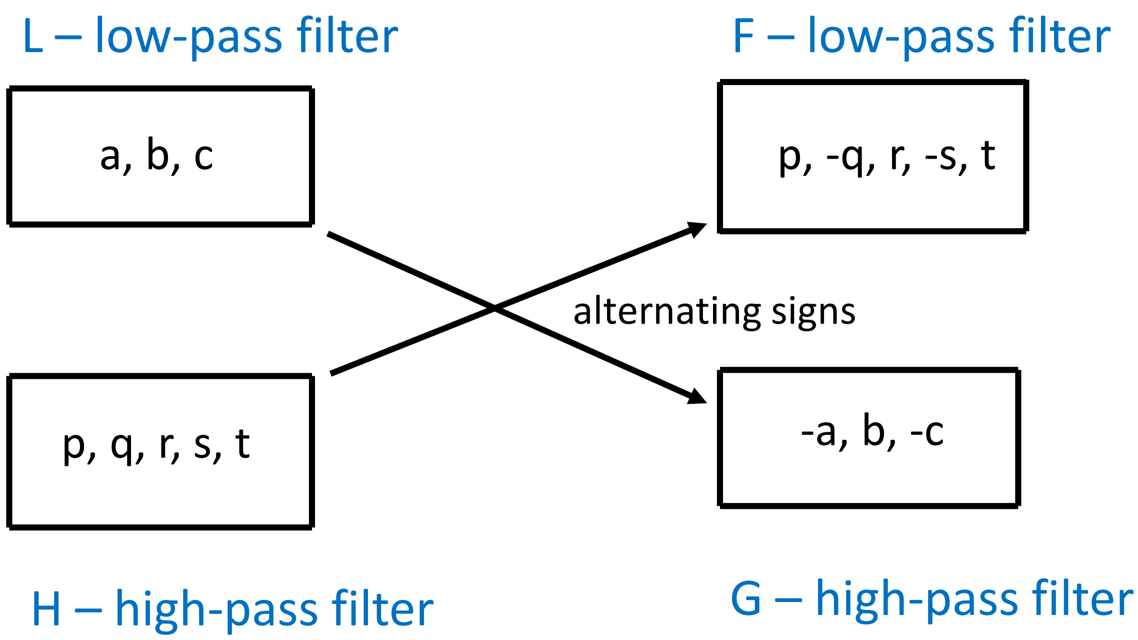
Filters Are Related

For orthogonal transforms – not linear phase



Filters Are Related

Non-orthogonal transforms (more general)



Three examples of filter banks that lead to a wavelet transform

L $h_0 = \frac{1}{8}(1\ \ 3\ \ 3\ \ 1)$ **F** $f_0 = \frac{1}{2}(-1\ \ 3\ \ 3\ \ -1)$

H $h_1 = \frac{1}{2}(-1\ \ -3\ \ 3\ \ +1)$ **G** $f_1 = (-1\ \ 3\ \ -3\ \ 1)\frac{1}{8}$

L $= \frac{1}{2}(1\ \ 2\ \ 1)$ **F** $= \frac{1}{8}(-1\ \ 2\ \ 6\ \ 2\ \ -1)$

H $\frac{1}{8}(-1\ \ -2\ \ 6\ \ -2\ \ -1)$ **G** $= \frac{1}{2}(-1\ \ 2\ \ -1)$

L $= \frac{1}{8}[-1\ \ 2\ \ 6\ \ 2\ \ -1]$ **F** $= [1\ \ 2\ \ 1]\frac{1}{2}$

H $= \frac{1}{2}(-1\ \ 2\ \ -1]$ **G** $= \frac{1}{8}[-1\ \ -2\ \ 6\ \ -2\ \ -1]$

Many Filters Can Be Used

Conditions

- **L** and **G** are quadrature mirror filters (QMFs)
- Mirror images about $\omega = \pi/2$
- for **L**, amplitude = 1 at $\omega = 0$

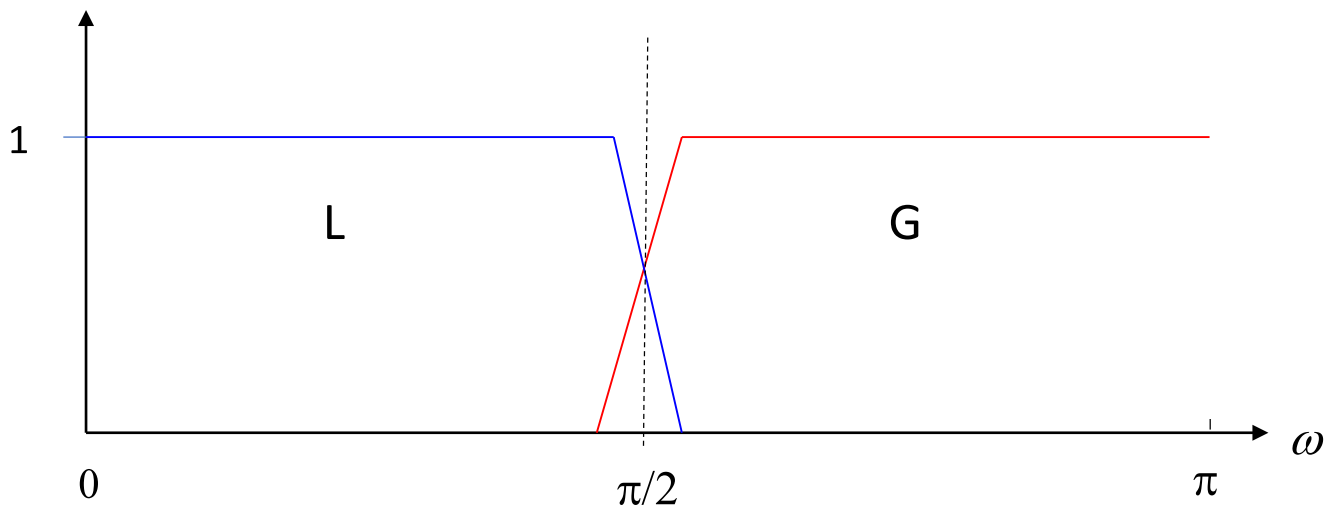
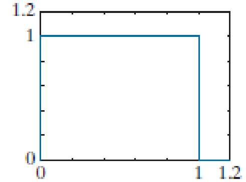
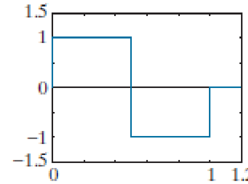
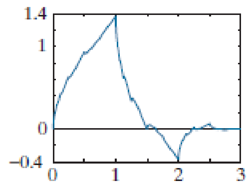
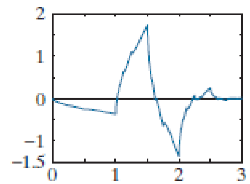
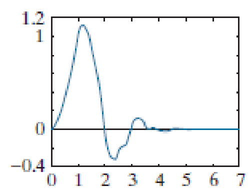
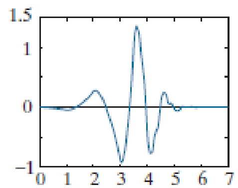
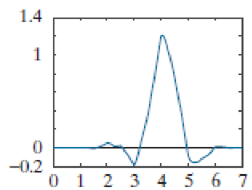
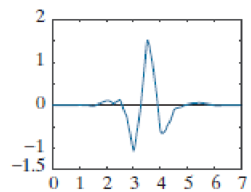
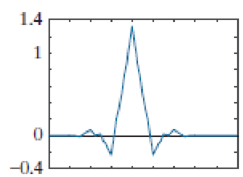
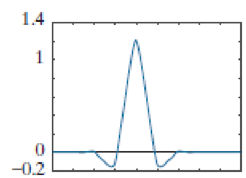
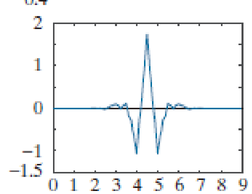
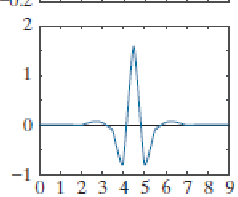


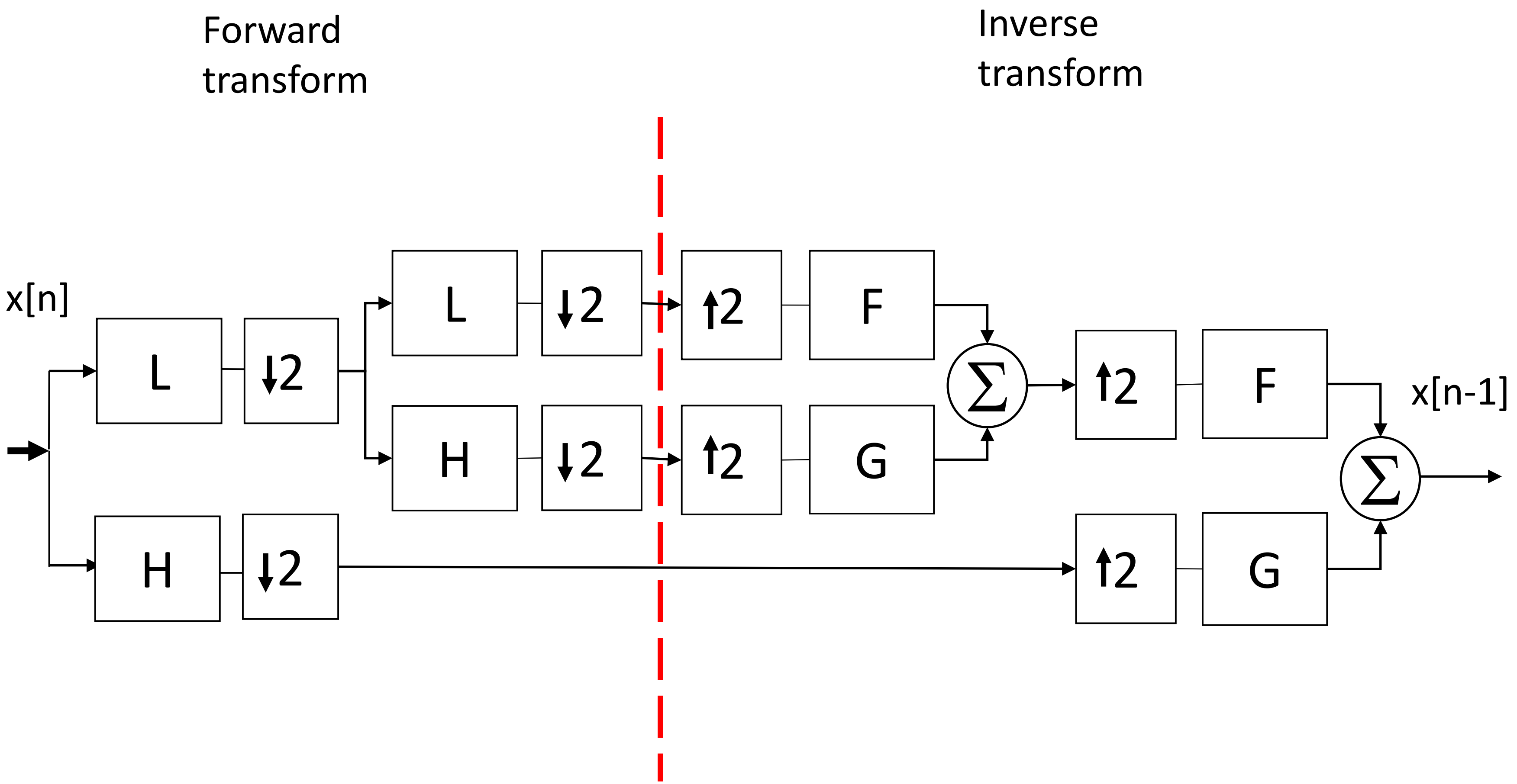
TABLE 6.1
Some representative wavelets.

Wavelet Name or Family	Scaling Function	Wavelet Function	Filter Coefficients
Haar The oldest and simplest wavelets. Orthogonal and discontinuous.			$g_0(n) = \{1/\sqrt{2},\ 1/\sqrt{2}\}$
Daubechies family Orthogonal with the most vanishing moments for a given support. Denoted db$N$, where N is the number of vanishing moments; db2 and db4 shown; db1 is the Haar of the previous row.			$g_0(n) = \{0.482963, 0.836516, 0.224144, -0.129410\}$
			$g_0(n) = \{0.230372, 0.714847, 0.630881, -0.027984, -0.187035, 0.030841, 0.032883, -0.010597\}$
Symlet family Orthogonal with the least asymmetry and most vanishing moments for a given support (sym4 or 4th order shown).			$g_0(n) = \{0.032231, -0.012604, -0.099220, 0.297858, 0.803739, 0.497619, -0.029636, -0.075766\}$
Cohen-Daubechies-Feauveau 9/7 Biorthogonal B-spline used in the irreversable JPEG2000 compression standard (see Chapter 8).			$h_0(n) = \{0.026749, -0.016864, -0.078223, 0.266864, 0.602949, 0.266864, -0.078223, -0.016864, 0.026749\}$
			$h_1(n) = \{-0.091271, -0.057544, 0.591272, -1.115087, 0.591272, 0.057544, -0.091271, 0\}$

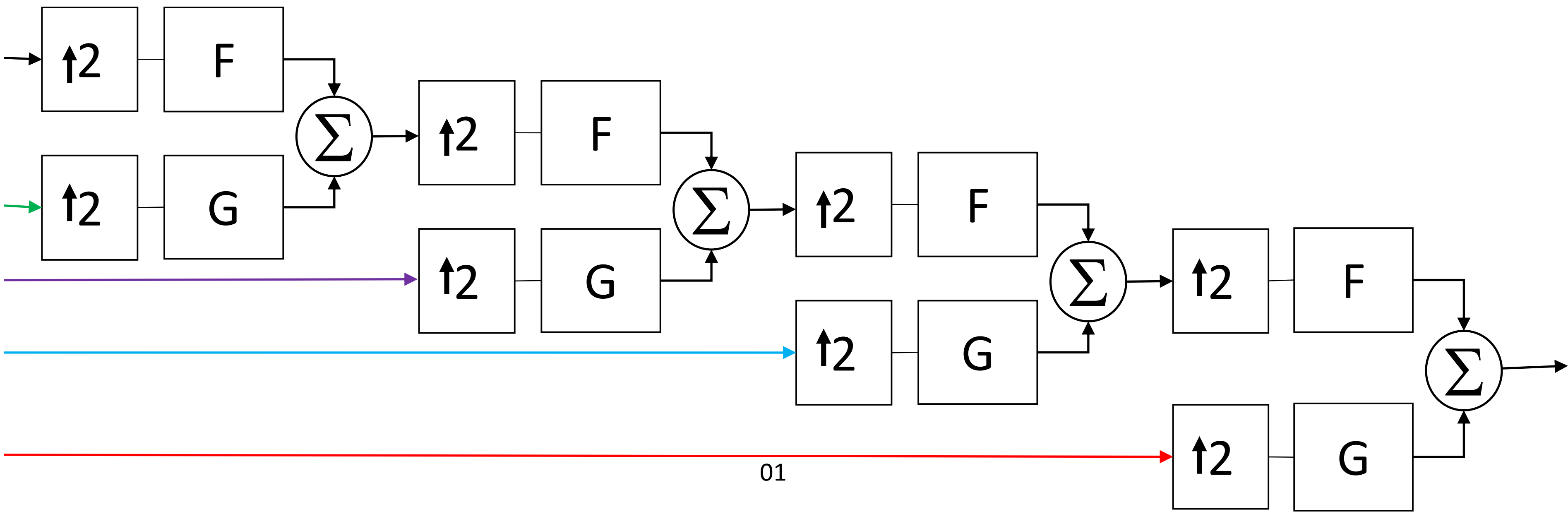
Scaling functions and wavelets calculated from passing filter coefficients through a large number of levels

Filter coefficients

Use the Same Structure for More Levels



What Do Wavelets Look Like Exactly?



Matrix form (input x is an impulse)

3rd level = $\mathbf{G}(\uparrow 2)\mathbf{x}$ (first upsample has no effect on an impulse). Wavelet = **G** (same length as G)

2^{nd} level = $\mathbf{F}(\uparrow 2)\mathbf{G}(\uparrow 2)\mathbf{x}$. Wavelet = $\mathbf{F}(\uparrow 2)\mathbf{G}$ (length(2G-1) + length F-1)

1^{st} level = $\mathbf{F}(\uparrow 2)\ \mathbf{F}(\uparrow 2)\mathbf{G}(\uparrow 2)\mathbf{x}$. $=\mathbf{F}(\uparrow 2)\mathbf{F}(\uparrow 2)\mathbf{G}$ (length(2G-1) + 2*length F-1)

0^{th} level = $\mathbf{F}(\uparrow 2)\ \mathbf{F}(\uparrow 2)\ \mathbf{F}(\uparrow 2)\mathbf{G}(\uparrow 2)\mathbf{x}$ (length(2G-1) + 3*length F-1)

0^{th} level (scaling function) = $\mathbf{F}(\uparrow 2)\ \mathbf{F}(\uparrow 2)\ \mathbf{F}(\uparrow 2)\mathbf{F}(\uparrow 2)\mathbf{x}$ (length(2G-1) + 4*length F-1)