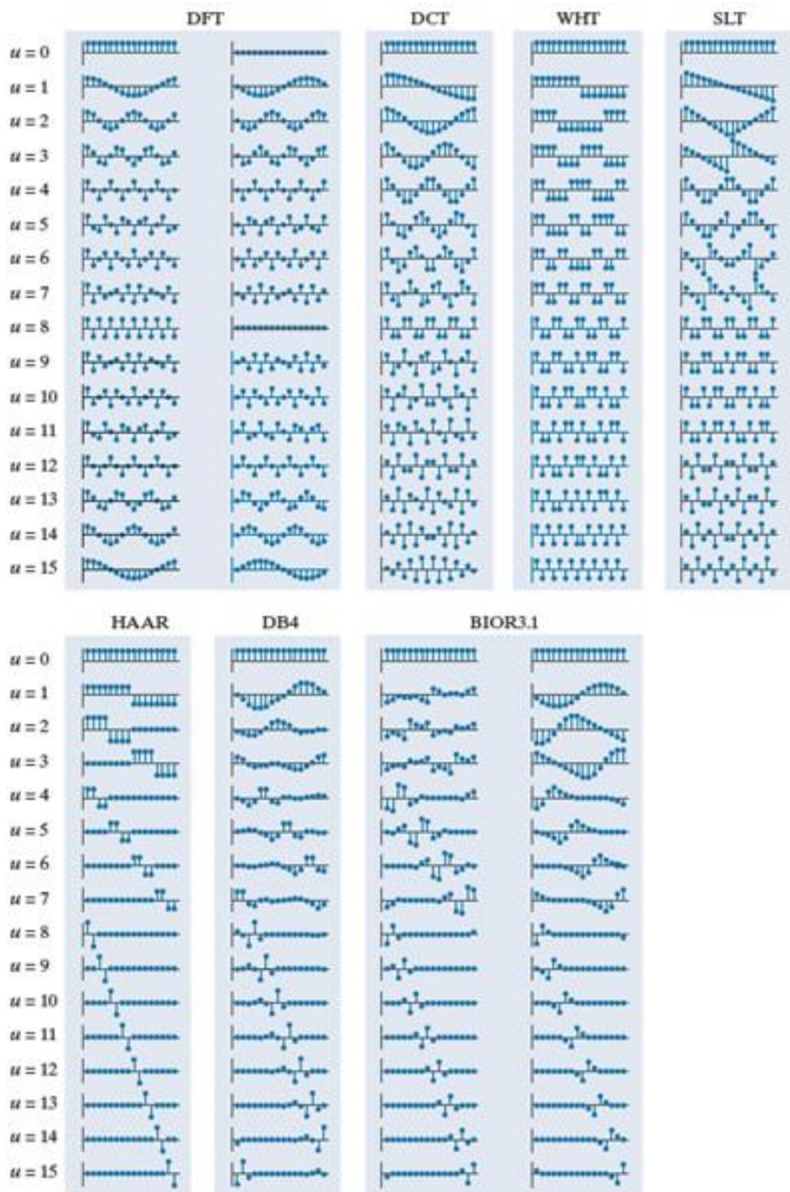
ECE 5256 Digital Image Processing
S. Kozaitis

Chapter 6.10 Wavelet Transforms

Basis Functions of Other Transforms

a b c d
e f g h

FIGURE 6.3 Basis vectors (for $N = 16$) of some commonly encountered transforms: (a) Fourier basis (real and imaginary parts), (b) discrete Cosine basis, (c) Walsh-Hadamard basis, (d) Slant basis, (e) Haar basis, (f) Daubechies basis, (g) Biorthogonal B-spline basis and its dual, and

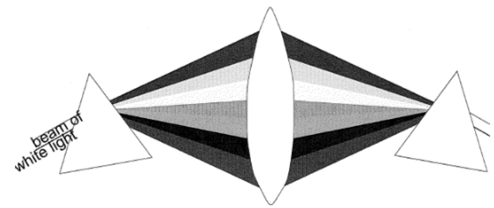


Bottom row are wavelet transforms

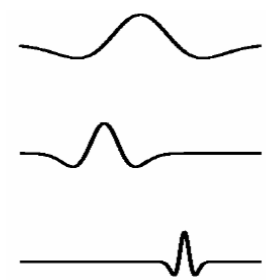
What is the main difference between the rows?

Wavelet Transforms are Similar to Fourier Transform But Often Better

- A wavelet transform is a tool for analyzing data of different frequency ranges, allowing one to study each component separately.

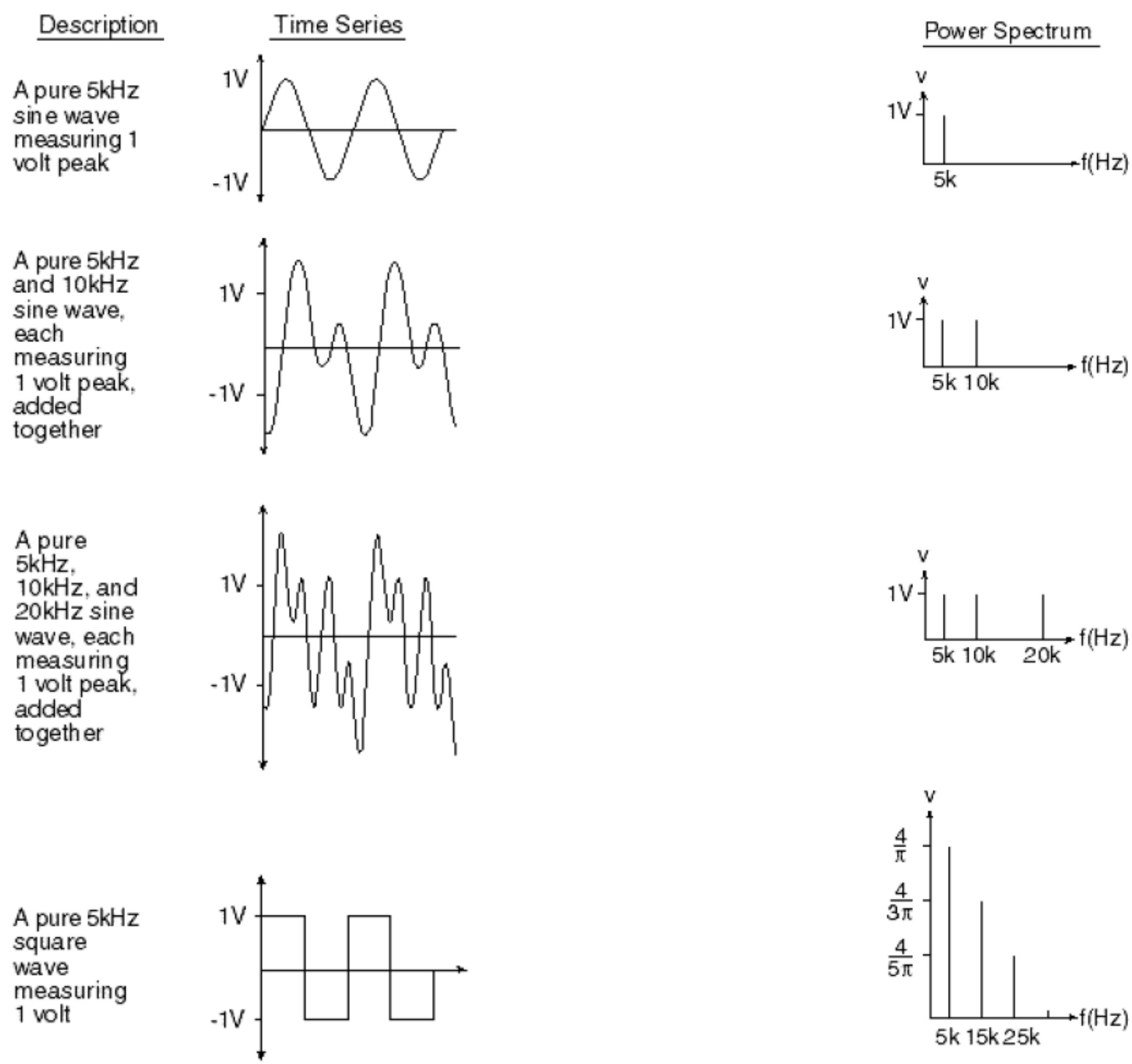


- The basic idea of the wavelet transform is to represent any arbitrary function as a superposition of a set of such wavelets (basis functions). It is similar in concept to the Fourier Transform.



- The basis functions in the wavelet transform are very different from the Fourier transform. They are obtained from a single **localized function** which is scaled and shifted.

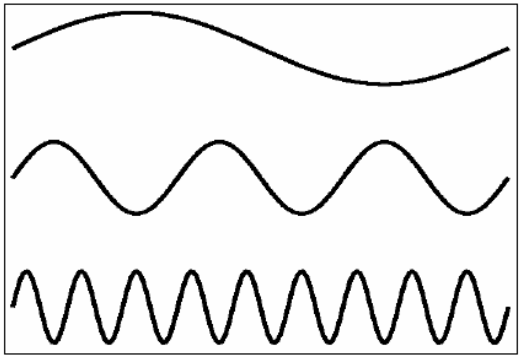
Fourier Transform Has Been Used for Frequency Analysis



Fourier Transform Basis Functions are Sines and Cosines

- The Fourier transform (FT) represents a signal in terms of different frequencies

- Sines and cosines are the basis function of the Fourier transform



FT Basis Functions

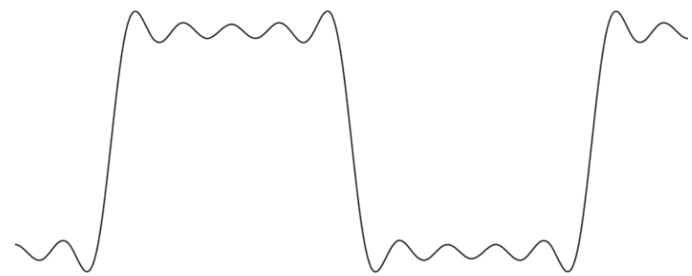
- The basis functions of the Fourier transform are not localized (they extend to infinity

Many Basis Functions Needed to Reconstruct Edges

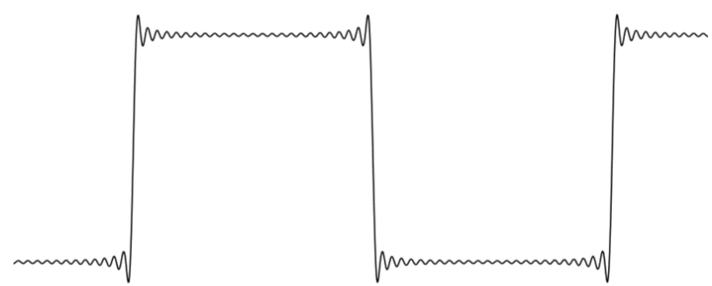
(not good)

Square wave reconstructed with

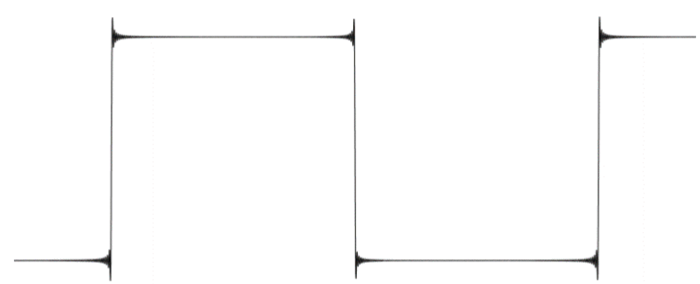
5 harmonics



25 harmonics



125 harmonics



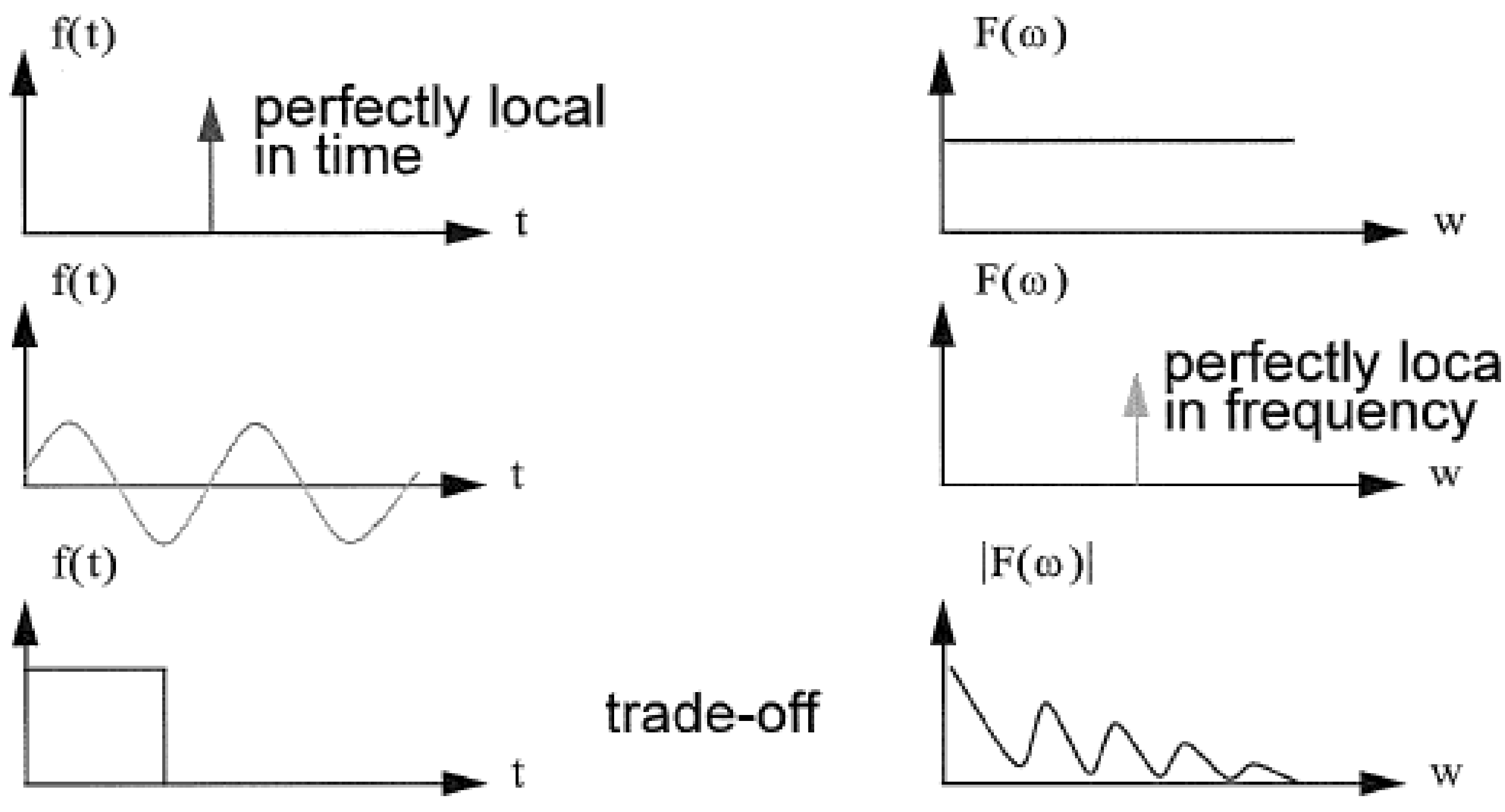
What is Wrong with the Fourier Transform?

- Short signals are described by many frequencies
- Edges, changes, beginning and ends of events are often difficult to isolate

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

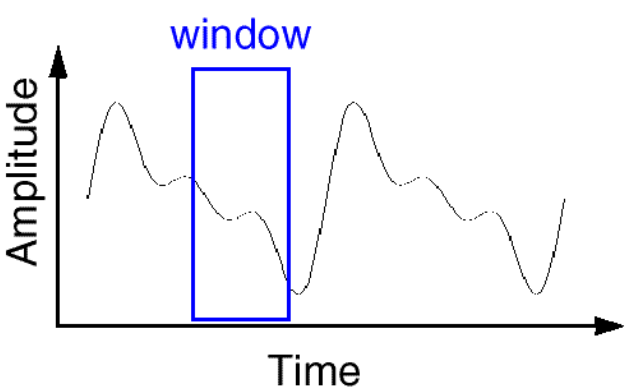
basis functions

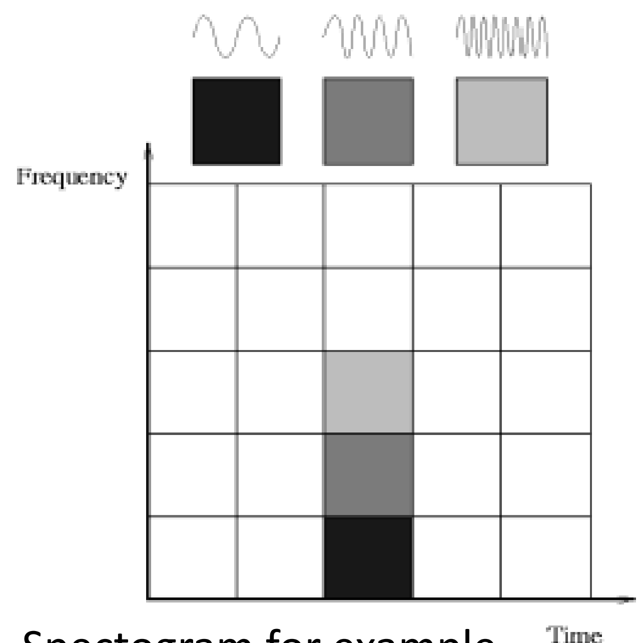
Trade-off between Time and Frequency Localization



Short Time Fourier Analysis Might Be Better

- Take the Fourier Transform of a "window" of a signal
- Both time and frequency are represented in limited precision





Spectogram for example

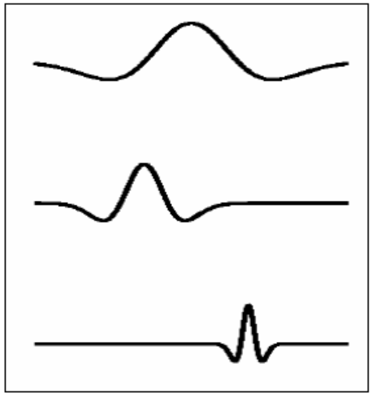
What is Wrong with the Short-Time Fourier Transform?

$$STFT_x(t, f) = \int [x(t^{'})g(t^{'} - t)]e^{-j2\pi f t^{'}} dt^{'}$$

- Requires *short time* window (large frequency window) to extract features of a signal.
- Requires *short frequency* window (long time window) to extract features of a signal.
- Many signals require a more flexible approach – a variable window size to determine either time of frequency feature more accurately

What Are Wavelets?

- Wavelets are *localized* waves that can be scaled and shifted
- Any signal can be decomposed in terms of wavelets.



WT Basis Functions

- Wavelets are the basis functions of a wavelet transforms
- There are many wavelet transforms

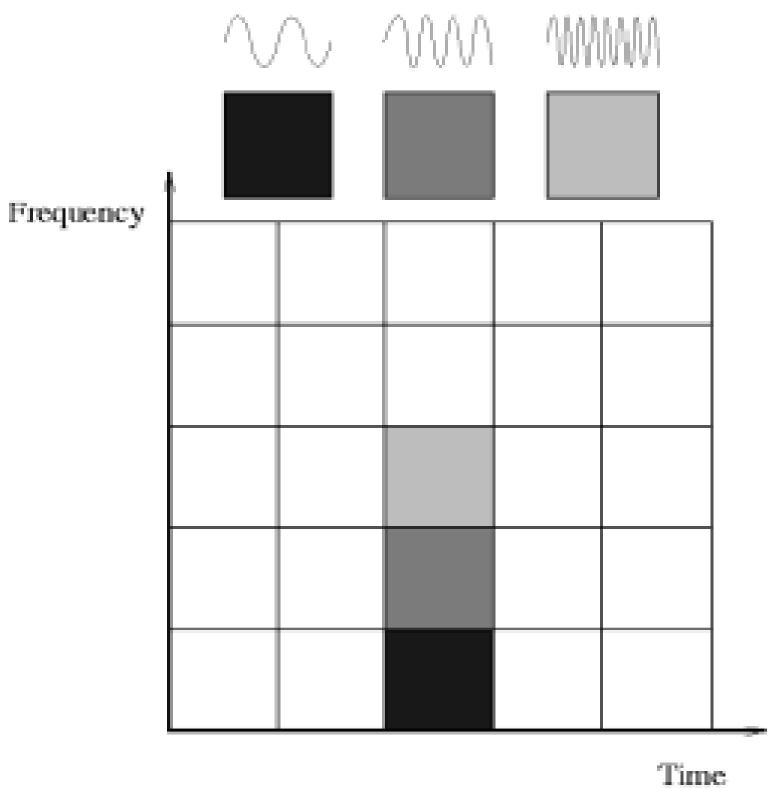
Wavelet History

- 1805 Fourier analysis developed
- 1910 Alfred Haar discovers the Haar transform
- 1980’s beginnings of wavelets in physics, vision, speech processing (ad hoc)
- … little theory … why/when do wavelets work?
- 1987 Mallat developed multiresolution theory, DWT, wavelet construction techniques (but still noncompact)
- 1988 Daubechies added theory: found compact, orthogonal wavelets with arbitrary number of vanishing moments!
- 1990’s: wavelets took off, attracting both theoreticians and engineers

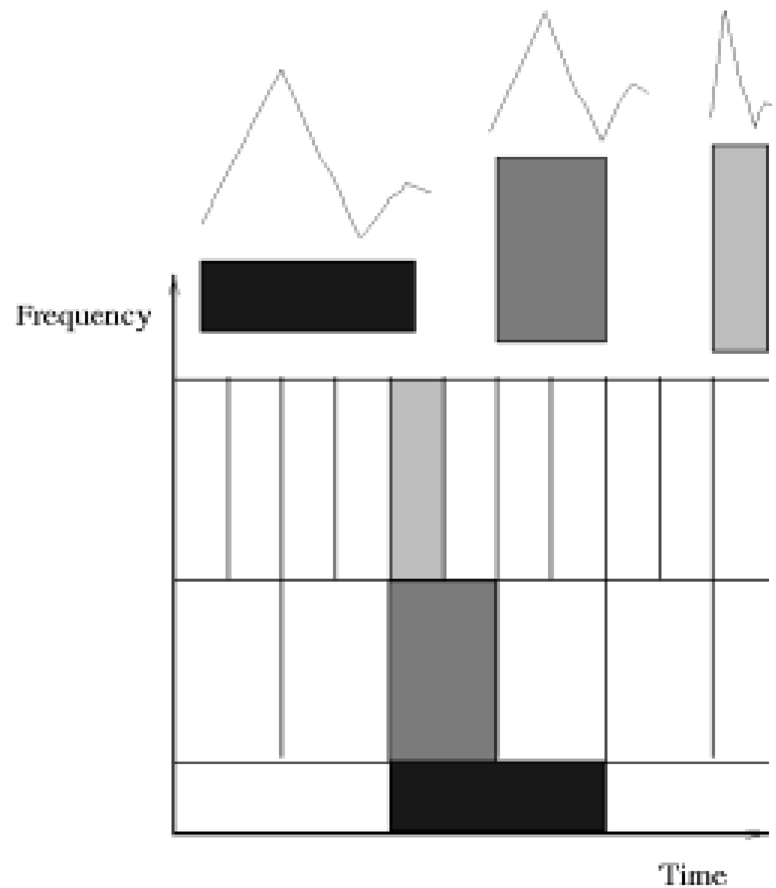
Comparison to Fourier Analysis

- Fourier analysis
  - Basis is global
  - Sinusoids with frequencies in arithmetic progression
- Short-time Fourier Transform
  - Basis is local
  - Fixed-width Gaussian "window"
- Wavelet
  - Basis is local
  - Frequencies in geometric progression
  - Basis has constant shape independent of scale

Wavelet Transform Automatically Produces a Multiresolution Analysis



Short-time Fourier Transform



Wavelet Transform

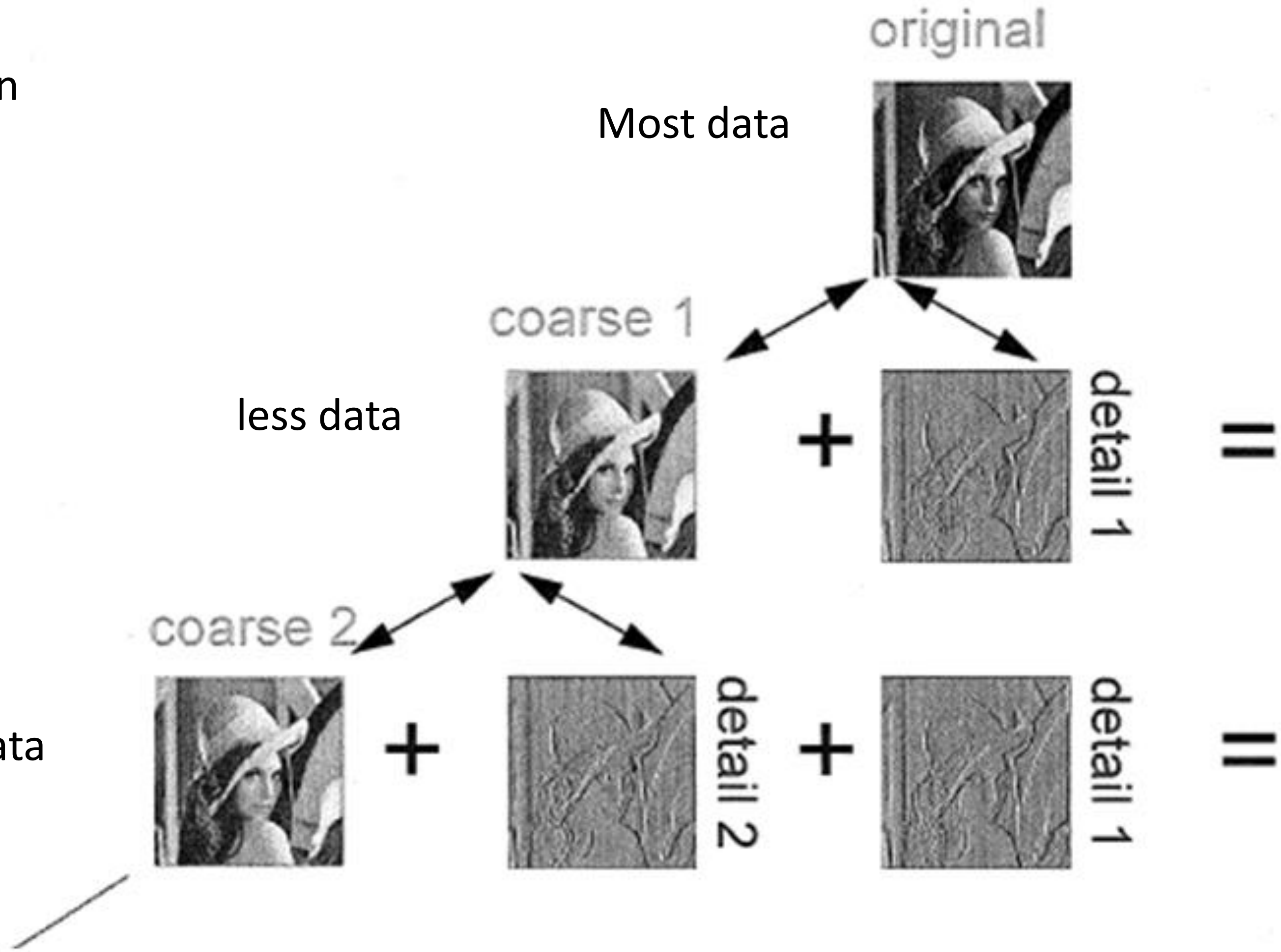
Multiresolution Analysis is Useful for Refinement of a Signal

Data Compression

Most data

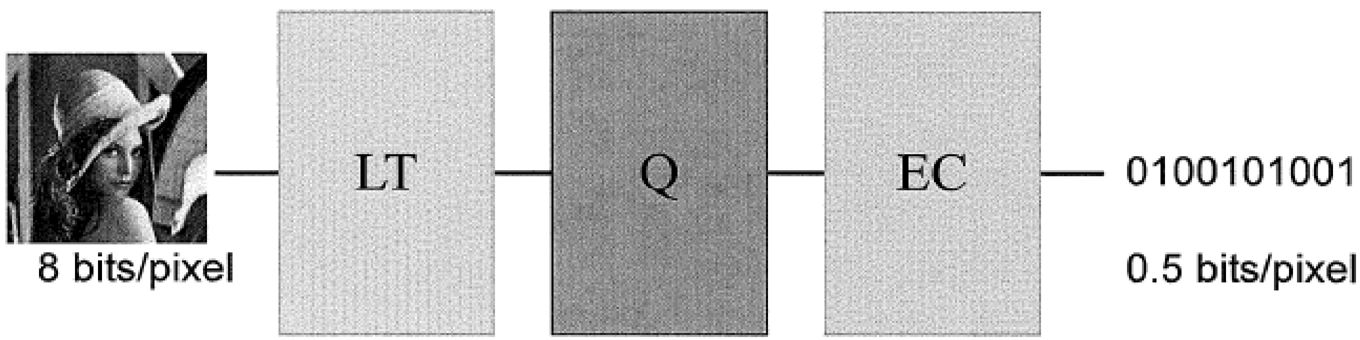
less data

even less data



Wavelet Transform Compacts Energy Better Than the Fourier Transform

Data compression



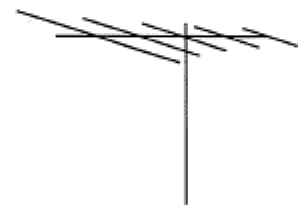
Linear transform

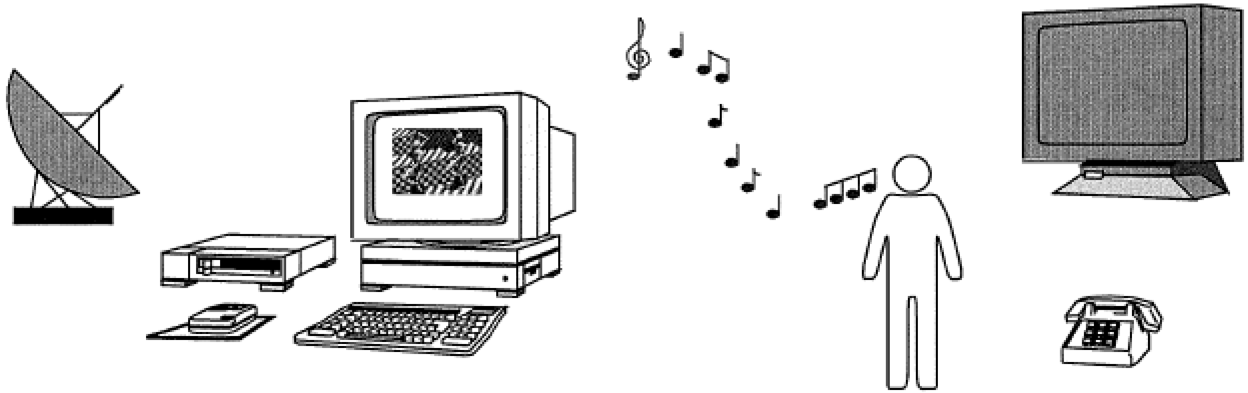
Quantization

Entropy Coder

Many Uses of Multiresolution

- digital audio and video coding
- conversions between TV standards
- digital HDTV and audio broadcast
- remote image databases with searching
- storage media with random access
- MR coding for multicast over the Internet
- MR graphics





Compression: still a key technique in communications

Only One General Type of Useful Wavelet Transform

- **Continuous** – useful for theoretical work

Input continuous, wavelet continuous, shift and scales continuous

- **Other versions**
  - Input continuous, wavelet continuous, shift and scales discrete
  - Input discrete, wavelet discrete, shift and scales discrete

- **Discrete Dyadic (scales are powers of 2)**
  - Input and wavelet are discrete and wavelets scale and shift by powers of two
  - *N* input samples and *N* output samples

Main Wavelet Transform Equation

$$b[j,k] = \sum_{n=0}^{n=N-1} f[n] 2^{j/2} \psi^{*}[2^{j} n - k]$$

wavelet coefficients

wavelet

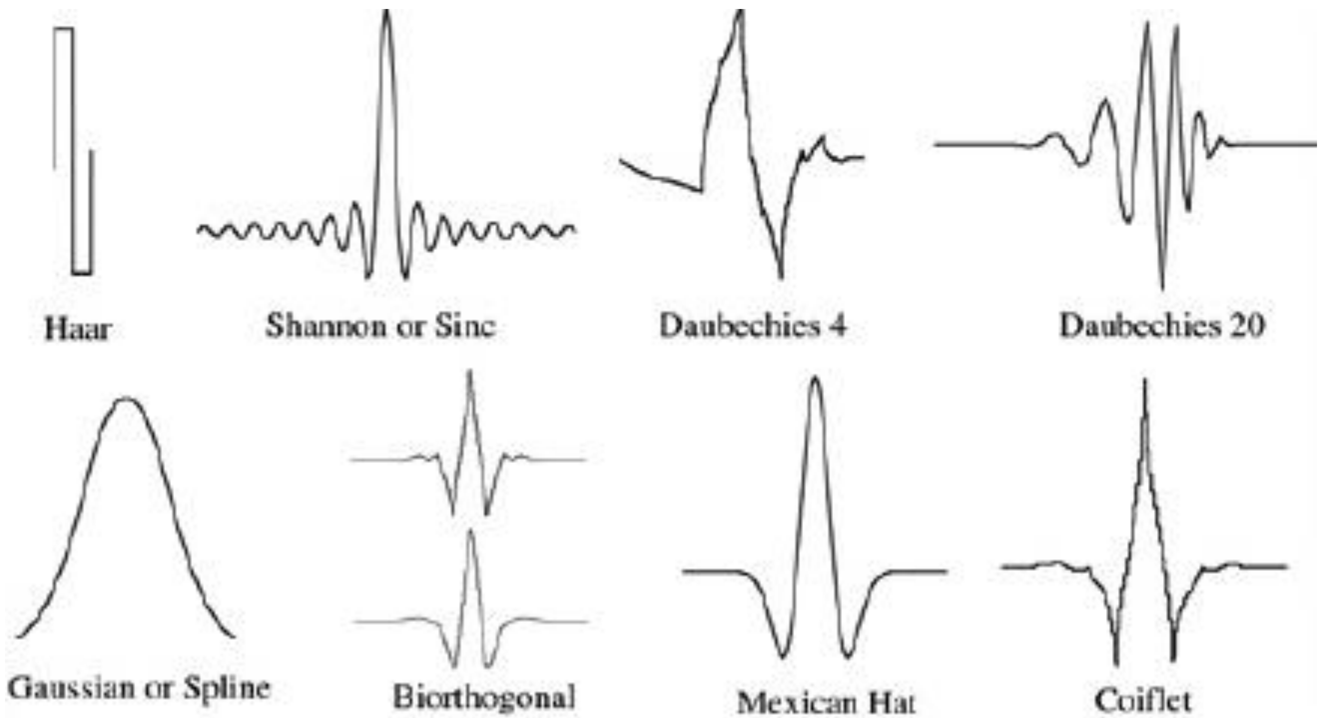
N = length of signal

$j = 0, 1, 2 \ldots.$ $(\log_2 N) - 1$ scales (0 scale is largest wavelet)

2^j shifts at each scale (Wavelet transform has j levels)

Looks complicated but easy implementation

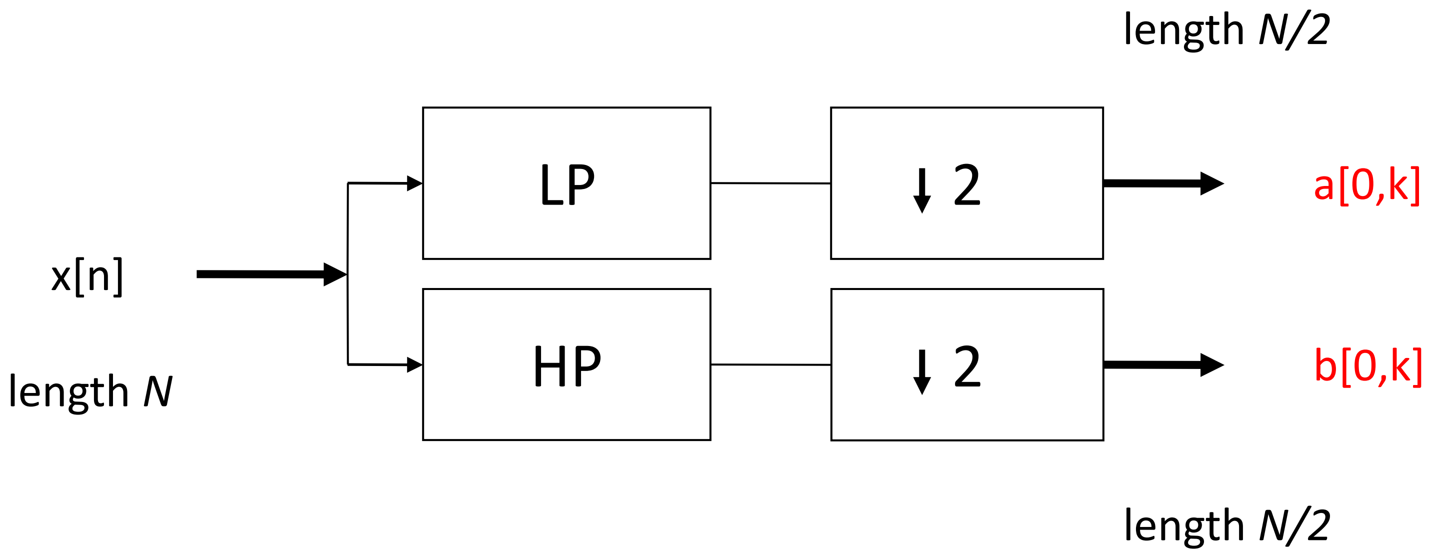
Infinite Number of Wavelets



Wavelets Implemented by Digital Filter Banks

(wavelet not obvious)

One level of the wavelet transform (level 0)

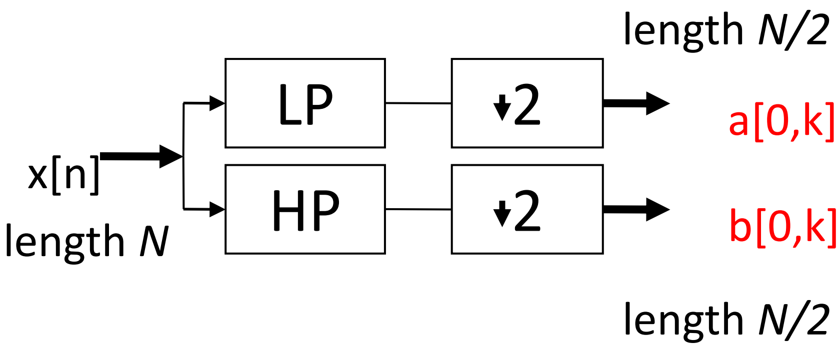


1-level of the WT is y[n] = [a[0,k] b[0,k]]

LP – low-pass filter
HP – high-pass filter

Wavelets Implemented by Digital Filter Banks

One level of the wavelet transform



Filter coefficients = [f[0], f[1], . . .]

Haar wavelet

Filter bank written in matrix form

$$[\mathbf{y}] = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} [\mathbf{x}]$$

(4 x 1 column vector)

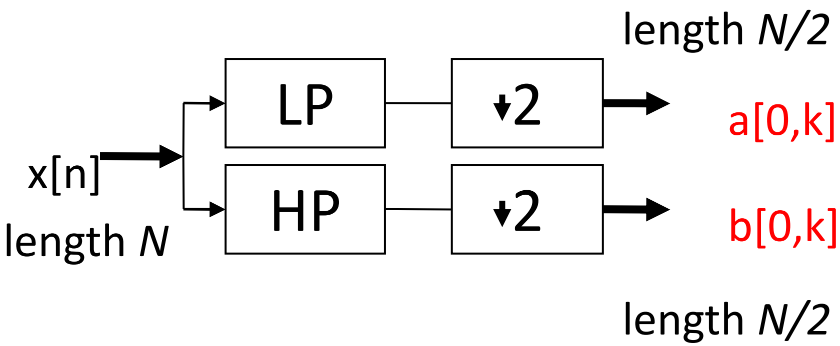
column vectors

- length 4 signal
- downsampling removes every other row

The Haar wavelet is often used in examples because it is the shortest

Wavelets Implemented by Digital Filter Banks

One level of the wavelet transform



Length 8 signal

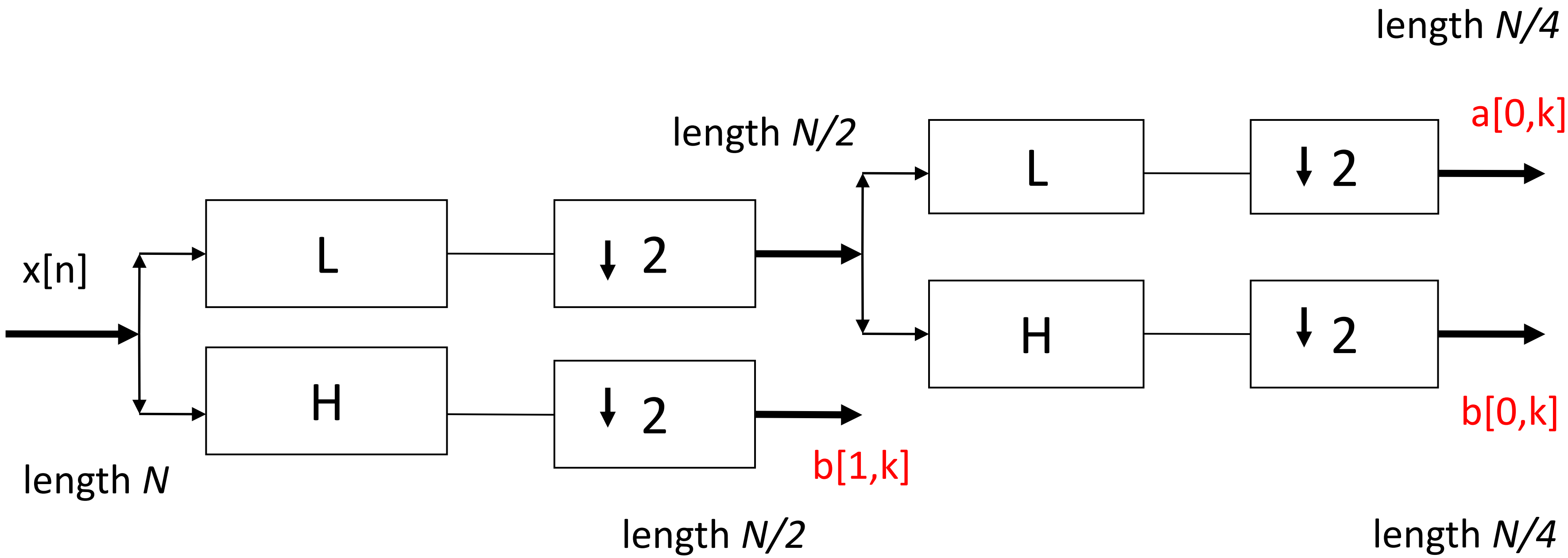
Analysis matrix is sparse!

$$[\mathbf{y}] = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} [\mathbf{x}]$$

[**x**] (8 x 1 vector)

Add a Filter Bank for Another Level

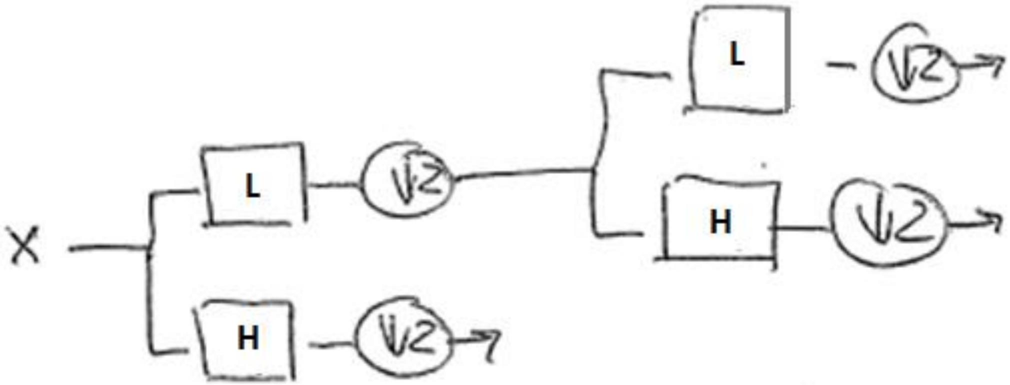
Add to the low-pass side for every level



wavelet transform is always length *N*

2-levels of the WT is y[n] = [a[0,k] b[0,k] b[1,k]]

Generate the Analysis Matrix for Another Level



$$\begin{bmatrix} L \\ H \end{bmatrix} \qquad \begin{bmatrix} \begin{matrix} L \\ H \end{matrix} & \\ & I \end{bmatrix}$$

Multiply in reverse order of the block diagram

$$A = \begin{bmatrix} \begin{matrix} L \\ H \end{matrix} & \\ & I \end{bmatrix} \begin{bmatrix} L \\ H \end{bmatrix}$$

Add a Filter Bank for Another Level

$$\mathbf{A} = \left[\begin{array}{c|c} \begin{matrix} L \\ H \end{matrix} & \\ \hline & I \end{array}\right] \begin{bmatrix} L \\ H \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{array}\right] \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$= I$

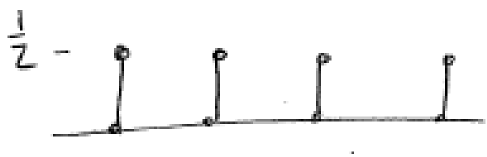
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ .707 & -.707 & 0 & 0 \\ 0 & 0 & .707 & -.707 \end{bmatrix}$$

The rows of the **A** matrix are the basis functions of the transform - wavelets

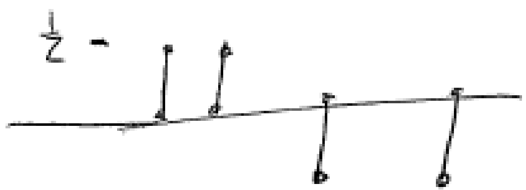
The first row is called a scaling function because it didn't use a high-pass filter, **a** coefficients are called scaling function coefficients

Multiplying a signal **x** by **A** is equivalent to correlating **x** with each basis function

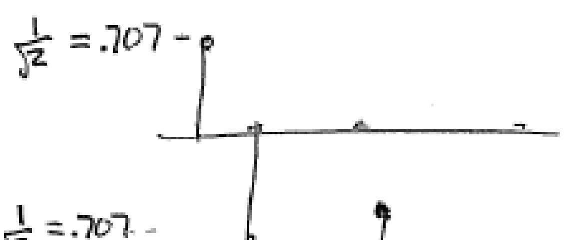
- The four basis function (wavelets) are the rows of the A matrix
- They all have the same energy



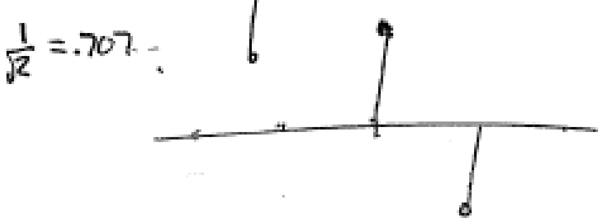
$$\frac{1}{2}^2+\frac{1}{2}^2+\frac{1}{2}^2+\frac{1}{2}^2=1$$



$$\frac{1}{2}^2+\frac{1}{2}^2+-\frac{1}{2}^2+-\frac{1}{2}^2=1$$

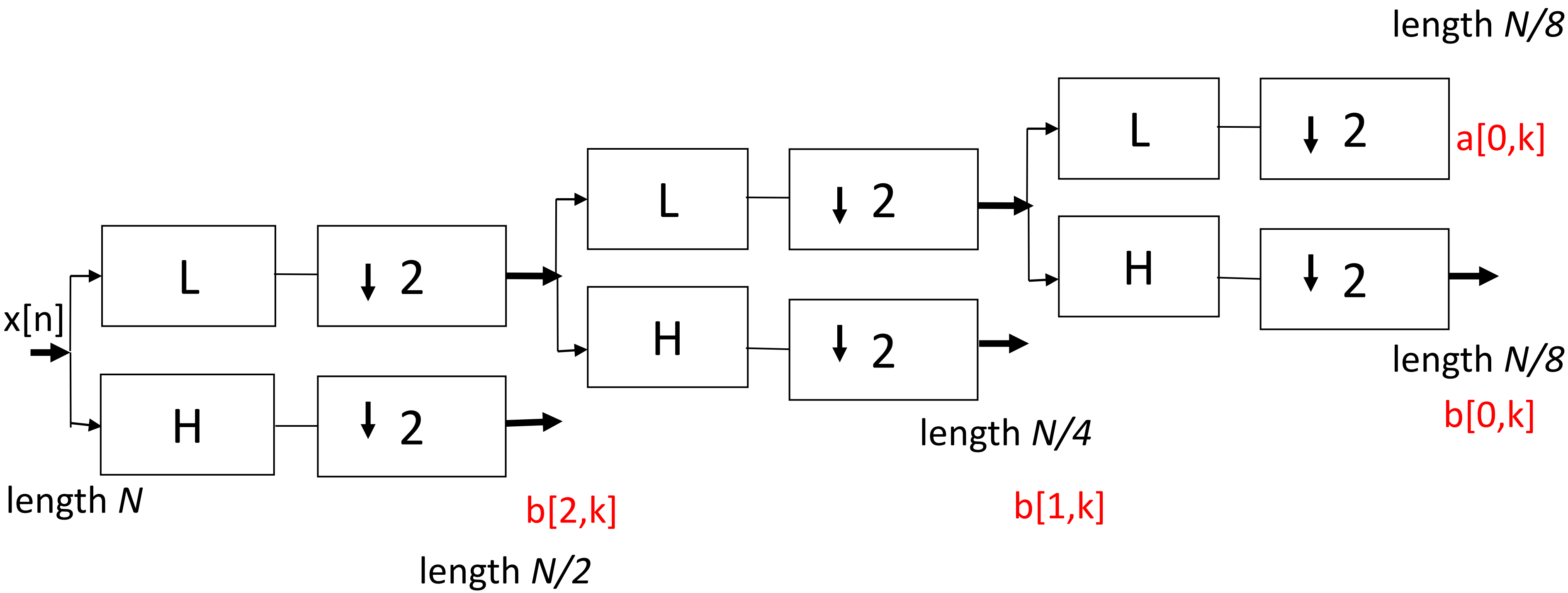


$$\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+0^2+0^2=1$$



$$0^2+0^2+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{-1}{\sqrt{2}}\right)^2=1$$

N Banks for *N* Levels



wavelet transform is length *N*

Analysis Matrix of 3-level Haar Wavelet Transform for Length 8 signal

Let $r = \frac{1}{\sqrt{2}}$

Easy to generate matrix from filter banks

$$\mathbf{A} = \begin{bmatrix} r & r & & & & & & \\ r & -r & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} r & r & & & & & & \\ & & r & r & & & & \\ r & -r & & & & & & \\ & & r & -r & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} r & r & & & & & & \\ & & r & r & & & & \\ & & & & r & r & & \\ & & & & & & r & r \\ r & -r & & & & & & \\ & & r & -r & & & & \\ & & & & r & -r & & \\ & & & & & & r & -r \end{bmatrix}$$

$$\mathbf{A} = \begin{array}{c} \begin{bmatrix} r^3 & r^3 & r^3 & r^3 & r^3 & r^3 & r^3 & r^3 \\ r^3 & r^3 & r^3 & r^3 & -r^3 & -r^3 & -r^3 & -r^3 \\ r^2 & r^2 & -r^2 & -r^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r^2 & r^2 & -r^2 & -r^2 \\ r & -r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & -r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & -r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & -r \end{bmatrix} \end{array} \begin{array}{l} j=0 \\ \\ j=1 \\ \\ \\ j=2 \\ \\ \\ \end{array}$$

Each matrix is 8 x 8

j is the level of the transform

The rows of the matrix are the wavelets!

Can Have Different Levels of a Wavelet Transform

These are all wavelet transforms of a length N signal

a[0,k]	b[0,k]		
Length [N/2 +	N/2]		
a[0,k]	b[0,k]	b[1,k]	
Length [N/4 +	N/4 +	N/2]	
a[0,k]	b[0,k]	b[1,k]	b[2,k]
Length [N/8 +	N/8 +	N/4 +	N/2]