3-D Pose and Reference Frames

Contents

1 Representing 3-D pose

We can describe the pose of an object in space by its rotation and translation with respect to a global coordinate frame or with respect to another frame of reference. Figure 1 shows the pose of a cylinder (i.e., Frame {1}) with respect to a frame of reference (i.e., Frame {0}). In this example, Frame {1} is the object's frame and Frame {0} is the global frame. The cylinder major axis is aligned with the z-axis of its coordinate frame.

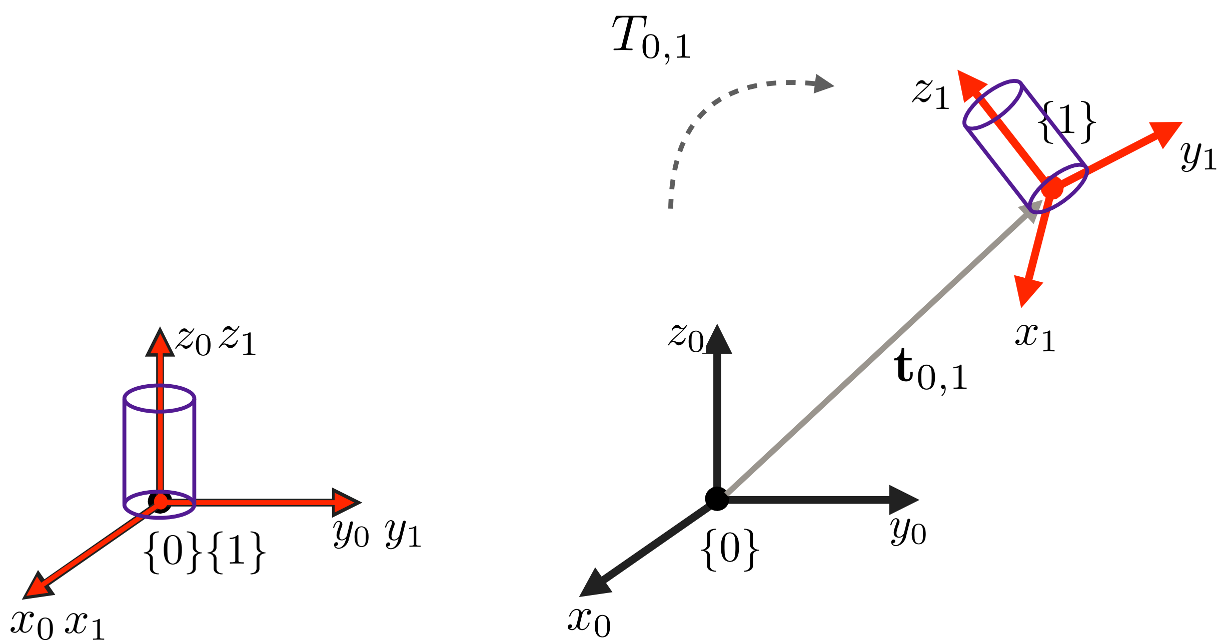

Figure 1: Object pose and coordinate-frame transformations. Left: We can assume that the two frames were initially aligned. Right: Frame {1} is transformed by $T_{0,1}$ which consists of a rotation followed by a translation with respect to Frame {0}.

The two coordinate frames in Figure 1 are related by a rigid-body transformation, which includes a rotation followed by a translation. Here, we assume that Frame {1} was initially aligned with frame Frame {0}, and it was then moved to a new position in space by transformation $T_{0,1}$. Transformation $T_{0,1}$ can be written in matrix form as:

$$T_{0,1} = \begin{bmatrix} R_{0,1} & \mathbf{t}_{0,1} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{1}$$

where $R_{0,1}$ is a rotation matrix and $\mathbf{t}_{0,1}$ is the translation (i.e., displacement) of frame 1's origin, all transformations are given in terms of frame 0's coordinates. Conversely, to obtain the pose of Frame {0} with respect to Frame {1}, we calculate the inverse transformation (i.e., the inverse of the matrix in Equation 1), which is given by:

$$T_{1,0} = T_{0,1}^{-1} = \begin{bmatrix} R_{0,1}^{\mathsf{T}} & -R^{\mathsf{T}}\,\mathbf{t}_{0,1} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{2}$$

These transformation equations are called *homogeneous transformations*. Homogeneous representation allows us to write affine transformations as single matrices, which simplifies calculations and shortens mathematical notation.

1.1 Numerical example

The following example shows the effect of transforming Frame {1} by rotating it by an angle of $\pi/4$ (i.e., 45 degrees) and then translated it by a vector $\mathbf{t} = (1/2, 1/2)^{\mathsf{T}}$ with respect to the Frame {0}. We show the transformation in 2-D for simplicity. Figure 2 shows Frame {0} and Frame {1} in their original aligned configuration.

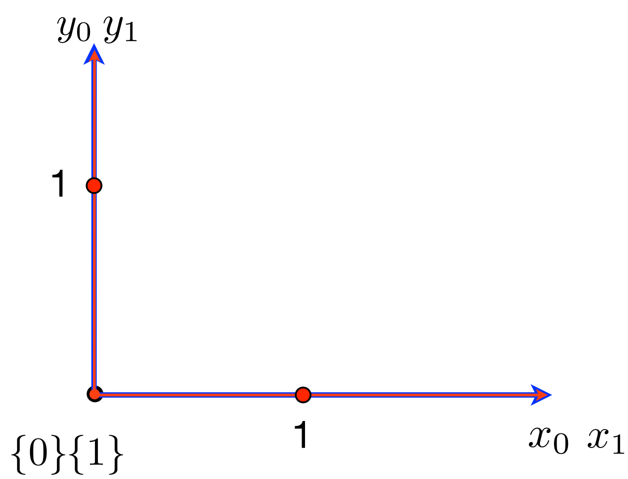

Figure 2: Coordinate frame transformations. Frame {0} (blue) and Frame {1} (red) are initially aligned to each other. Here, they frames are shown superimposed.

Numerically, we can represent Frame {0} (and Frame {1}) using the homogeneous matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

Here, the matrix's first column represents the unit vector $\mathbf{i} = (1, 0)^\top$, the second column represents the unit vector $\mathbf{j} = (0, 1)^\top$, which are the horizontal and vertical axes, respectively. The matrix's third column represents the frame's origin, which is currently located at $(0, 0)^\top$.

As pointed out earlier, the transformation that we want to apply in this example is:

$$T_{0,1} = \begin{bmatrix} R_{0,1} & \mathbf{t}_{0,1} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & \frac{1}{2} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Let us name this transformation $T_{0,1}$, and use the $_{0,1}$ label to indicate that the transformation "moves" the representation from Frame $\{0\}$ to Frame $\{1\}$. Conversely, a transformation that moves from Frame $\{1\}$ to Frame $\{0\}$ will be denoted $T_{1,0}$.

We use the transformation in Equation 4 to change the pose of Frame $\{1\}$ as shown in Figure 3. The pose of Frame $\{1\}$ with respect to Frame $\{0\}$ is obtained by multiplying the transformation matrix $T_{0,1}$ by the matrix of Frame $\{0\}$, i.e.:

$$\begin{aligned} \text{Frame}\,\{1\} &= T_{0,1}\, I \\ &= \begin{bmatrix} R_{0,1} & \mathbf{t}_{0,1} \\ \mathbf{0} & 1 \end{bmatrix} I \\ &= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= T_{0,1}. \end{aligned} \tag{5}$$

If we want to "move" back to the representation of Frame $\{0\}$, we need to calculate the inverse of $T_{0,1}$ which gives us $T_{1,0}$, and then multiply this inverse transformation by the matrix of

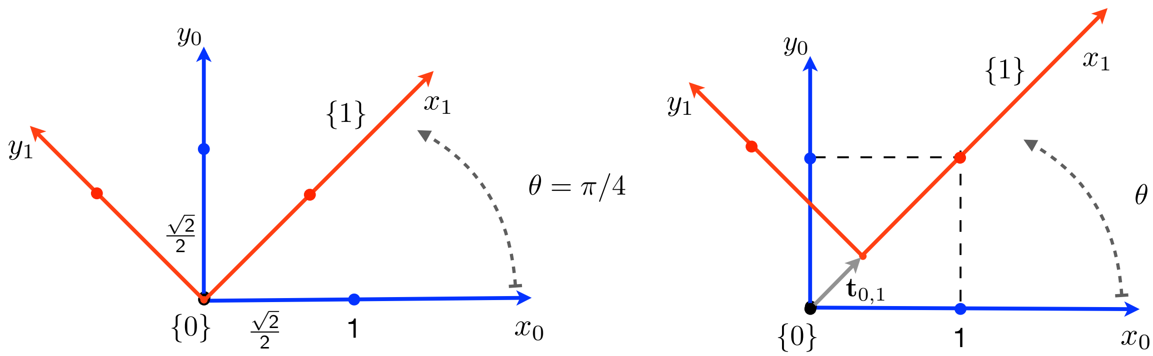

Figure 3: Coordinate frame transformations. The rotation transformation (left) is followed by a translation (right).

Frame $\{1\}$ as follows:

$$\begin{aligned}
\text{Frame}\,\{0\} &= T_{0,1}^{-1}\,T_{0,1} = T_{1,0}\,T_{0,1} \\
&= \begin{bmatrix} R_{0,1}^T & -R_{0,1}^T\mathbf{t}_{0,1} \\ \mathbf{0} & 1 \end{bmatrix} T_{0,1} \\
&= \begin{bmatrix} R_{1,0} & \mathbf{t}_{1,0} \\ \mathbf{0} & 1 \end{bmatrix} T_{0,1} \\
&= \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.
\end{aligned} \tag{6}$$

Note that the changes in reference indices and also that $T_{0,1}^{-1} = T_{1,0}$.

2 Changing reference frames

Often, it is useful to combine multiple transformations to form new transformations. We can also change reference frames to represent pose at multiple locations. For example, these calculations help us analyze the motion of an object that changes pose as it moves from one place to another or to animate a kinematic chain representing a robot arm.

To understand how we can change through multiple coordinate frames, we will upgrade the two-frame example shown in Figure 1 to a three-frame system. The three-frame system in shown in Figure 4. In the figure, transformation $T_{0,1}$ is the pose of Frame $\{1\}$ with respect to Frame $\{0\}$, and $T_{1,2}$ is the pose of Frame $\{2\}$ with respect to Frame $\{1\}$. Interestingly, we can combine the two consecutive transformations to obtain the pose of Frame $\{2\}$ with respect to Frame $\{0\}$. Thus, Frame $\{2\}$, expressed in terms of Frame $\{0\}$ is given by:

$$T_{0,2} = T_{0,1}T_{1,2} = \begin{bmatrix} R_{0,1} & \mathbf{t}_{0,1} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_{1,2} & \mathbf{t}_{1,2} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} R_{0,2} & \mathbf{t}_{0,2} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{7}$$

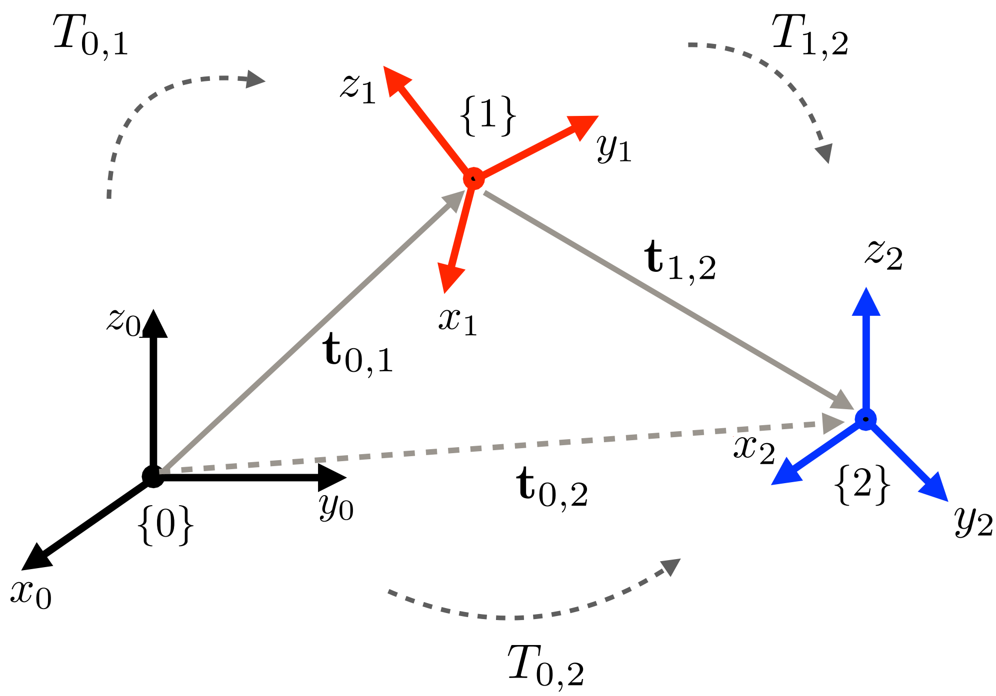

Figure 4: Coordinate frame 2 in world coordinates (frame 0) is given by a composed transformation formed by multiplying the matrices of the transformation chain, i.e., $T_{0,2} = T_{0,1}T_{1,2}$.

Note that we can also move in the reverse direction. For example, we can obtain the pose of frame 0 with respect to frame 2 by "moving" in the reverse direction, i.e.:

$$T_{2,0} = T_{0,2}^{-1} = T_{1,2}^{-1}T_{0,1}^{-1} = T_{2,1}T_{1,0}. \tag{8}$$

This recursive-composition approach works for any number of coordinate frames. Equation 9 shows an example of composing transformations for 6 coordinate frames, each one represented with respect to its predecessor, where frame 0 is the world reference frame.

$$
\begin{aligned}
T_{0,1} &= T_{0,1}. \\
T_{0,2} &= T_{0,1} T_{1,2}. \\
T_{0,3} &= T_{0,2} T_{2,3} = T_{0,1} T_{1,2} T_{2,3}. \\
T_{0,4} &= T_{0,3} T_{3,4} = T_{0,1} T_{1,2} T_{2,3} T_{3,4}. \\
T_{0,5} &= T_{0,4} T_{4,5} = T_{0,1} T_{1,2} T_{2,3} T_{3,4} T_{4,5}.
\end{aligned} \tag{9}
$$

3 Changing reference frames of a point in space

Given a point in space, we can change its reference frame by simply multiplying the point by the correct change-of-coordinates transformation. Because the transformations are homogeneous transformations, we must also represent the point in homogeneous coordinates.

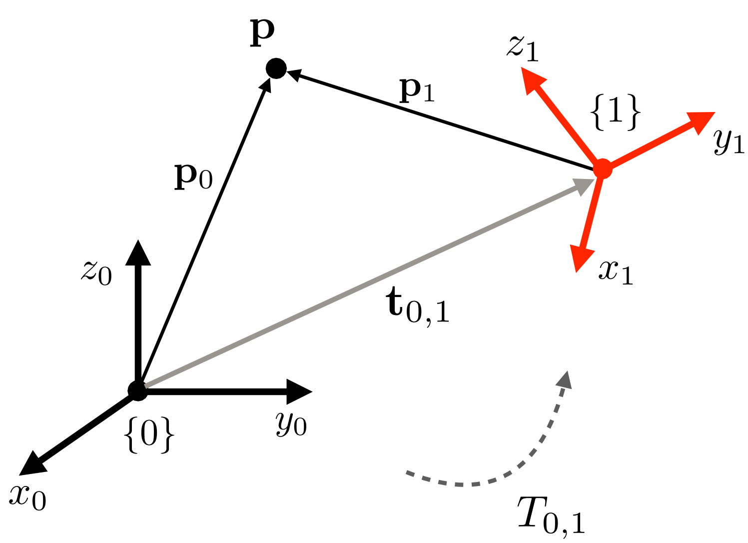

Figure 5: A 3-D point represented with respect to two different frames of reference. Each frame has its own representation of point **p**. It is represented by vector $\mathbf{p}_0$ with respect to frame 0 and by vector $\mathbf{p}_1$ with respect to frame 1.

Figure 5 shows an example of a point **p** represented with respect to frame 0 and frame 1. In each reference frame has its own representation of point **p**. To move from one frame representation to the other, we need to do a change-of-coordinates calculation. Here, to move from the representation of **p** in frame 1 to the representation in frame 0, we transform the point coordinates as follows:

$$\mathbf{p}_0 = T_{0,1}\,\mathbf{p}_1 = \begin{bmatrix} R_{0,1} & \mathbf{t}_{0,1} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}. \tag{10}$$

A numerical example of change of coordinate frames in 2-D is show in Figure 6. In this example, we will use homogeneous coordinates. Here, the representation of point $\mathbf{p}$ with respect to frame 0 is $\mathbf{p}_0 = (1,1,1)^\mathsf{T}$. The same point represented with respect to frame 0 is $\mathbf{p}_1 = \left(\sqrt{2}/2, 0, 1\right)^\mathsf{T}$. Frame 1 is rotated by an angle of $\pi/4$ (i.e., 45 degrees) and translated by a vector $\mathbf{t}$ with respect to the frame 0.

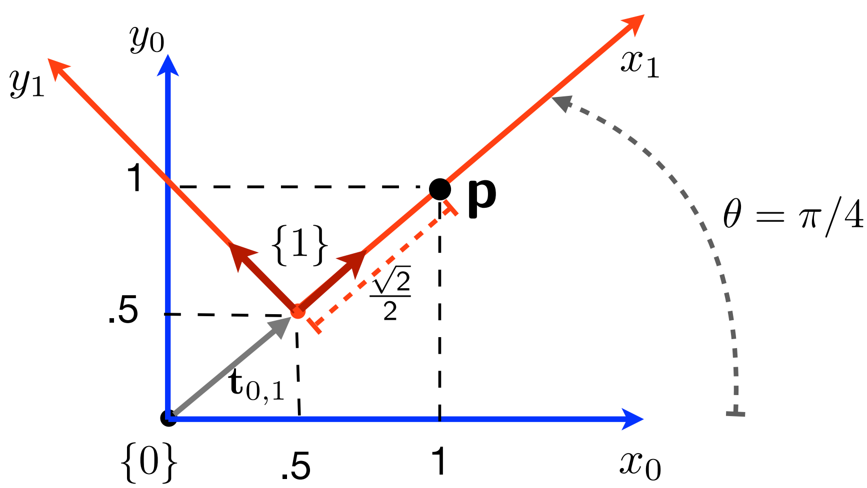

Figure 6: Change of coordinate frames of a point in 2-D.

To convert from $\mathbf{p}_1$ to $\mathbf{p}_0$, we use Equation 10 as follows:

$$\begin{aligned} \mathbf{p}_0 = T_{0,1}\,\mathbf{p}_1 &= \begin{bmatrix} R_{0,1} & \mathbf{t}_{0,1} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{11}$$

A Rigid-body transformation

Rotations about x-axis, y-axis, and z-axis:

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{12}$$

$$R_y(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \tag{13}$$

$$R_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Translation by a vector $\mathbf{t} = (t_x, t_y, t_z)^{\mathsf{T}}$. The rigid-body transformation in block-matrix notation:

$$T = \begin{bmatrix} r_1 & r_2 & r_3 & t_x \\ r_4 & r_5 & r_6 & t_y \\ r_7 & r_8 & r_9 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{3\times3} & \mathbf{t}_{3\times1} \\ \mathbf{0}_{1x3} & 1 \end{bmatrix} \tag{15}$$

Example transformation with rotation about the z-axis:

$$T = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & t_x \\ \sin\phi & \cos\phi & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

References