Why is $\log n! = \theta(n \log n)$?

COMP 250 Review session

First recall that in order to prove that $f(n) = \theta(g(n))$, you need to prove that both $f(n) = O(g(n))$ and $g(n) = O(f(n))$, i.e. $f(n)$ is bounded above and below by $c \times g(n)$ $\forall n > n_0$.

1. Upper bound: $\log(n!) = O(n \log n)$

$\log(n!) = \log(n(n-1)\ldots 2 \times 1) = \sum_{i=1}^{n} \log(i)$

Now, $\sum_{i=1}^{n} \log(i) \leq \sum_{i=1}^{n} \log(n) = n \log n$. This is true because $\log n$ is the biggest term in the sum. Since we have n terms, then the sum has to be less or equal than $n \log n$.

$\implies \log(n!) \leq n \log n$

$\therefore \log(n!) = O(n \log n)$

2. Lower bound: $n \log n = O(\log n!)$

$\log n! = \sum_{i=1}^{n} \log(i) \geq \sum\limits_{i=\frac{n}{2}}^{n} \log(i) \geq \sum\limits_{i=\frac{n}{2}}^{n} \log(\frac{n}{2}) = \frac{n}{2} \log(\frac{n}{2}) = \frac{n}{2} \log n - \log 2 \times \frac{n}{2}$. Similarly as in (1), $\log \frac{n}{2}$ is the smallest term in the sum, therefore $\frac{n}{2} \log(\frac{n}{2})$ has to be less or equal than $\sum\limits_{i=\frac{n}{2}}^{n} \log(i)$.

For simplicity, let us assume that the base of the log is $2 \implies \frac{n}{2} \log 2 = \frac{n}{2}$.

How do we get rid of $\frac{n}{2}$?

We can prove that $\frac{n}{c} \log n \leq \frac{n}{2} \log n - \frac{n}{2}$ by choosing a good constant c. In our case, it suffices that $c \geq 4$.

<u>Lemma:</u> $\frac{n}{4} \log n \leq \frac{n}{2} \log n - \frac{n}{2}$

<u>Proof:</u> $\log n \geq 2 \quad \forall n \geq 4$

$\implies \frac{n}{4} \log n \geq \frac{n}{2}$

$\implies \frac{n}{2} \log n - \frac{n}{4} \log n \geq \frac{n}{2}$

$\implies \frac{n}{2} \log n - \frac{n}{2} \geq \frac{n}{4} \log n$, as required.

$\implies \frac{n}{4}\log n \leq n\log n - \frac{n}{2} \leq \log(n!) \implies n\log n \leq 4\log(n!) = O(\log n!)$
$\therefore n\log n = O(\log n!)$

Good luck with your midterm!!