Mathematical Programming with Equilibrium Constraints: An Uncertainty Perspective

Master Thesis Presented to the

Department of Economics at the

Rheinische Friedrich-Wilhelms-Universität Bonn

in Partial Fulfillment of the Requirements for the Degree of

Master of Science (M.Sc.)

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Submitted in August 2020 by:

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Matriculation Number: 3220685

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Abbreviations

Term	Meaning
VI	Variational Inequality
NFXP	Nested Fixed Point Algorithm
MPEC	Mathematical Programming with Equilibrium Constraints
N – K	Newton-Kantorovich

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1 Mathematical Programming with Equilibrium Constraints

Mathematical Programming with Equilibrium Constraints can be traced back notation and concept wise to Game Theory. Specifically in the theory on Stackelberg games MPEC found its first development. It was due to Luo et al. (1996), though, that MPEC was set onto a mathematically rigorous foundation. They argue to have done so in order to present its manifold possibilities of application that had been overlooked previously.

Before we go into the details of MPEC regarding its mathematical formulation, applications and use in Economics, let us break down the lengthy term into its key components. The beginning "Mathematical Program" solely captures that we look at a mathematical optimization problem. The particularity of this problem comes in with the "Equilibrium Constraints". Mathematically this means that this optimization problem is subject to variational inequalities (VI) as constraints. Nagurney (1993) explains that VIs consist of but are not limited to nonlinear equations, optimization as well as fixed point problems. More broadly spoken VIs are able to harnesses our intuitive notion of economic equilibrium for which typically a functional or a system of equations must be solved for all possible values of a given input. This is tightly linked to what is looked for when solving a Stackelberg game. Essentially, an economic equilibrium has to be found. As a reminder in a Stackelberg game, there is one leader that moves first followed by the moves of some followers. Solving this problem involves the leader to solve an optimization problem that in turn is subject to an optimization procedure of the followers given every possible optimal value the leader might find. The variational inequality here is the problem of the followers which involves solving a decision problem for every possible move of the leader and which is cast into the optimization problem of the leader as a constraint. It can be seen from the fact that the leader moves first and followers move after, as noted by Luo et al. (1996), that the MPEC formulation is a hierarchical mathematical concept which captures multi-level optimization and hence can prove useful for the modeling of decision-making processes. They further explain that this feature can further be beneficial in other fields than just Economics. They showcase that a classification problem in machine learning can be formulated as an MPEC and they further describe some problems in robotics, chemical engineering and transportation networks in MPEC notation.

While this discussion shows that MPEC problems appear in theoretical Economics, Su and Judd (2012) enter with the novel idea to formulate an estimation procedure in structural econometrics as a MPEC. In the following I present their idea using the notation they originally suggested.

In order to estimate the structural parameters of an economic model using data, researchers commonly rely on the Generalized Method of Moments or maximum likelihood estimation. If the researchers opt for the most complex way of estimation (as opposed to using methods lowering the computational burden such as in Hotz and Miller (1993)) which involves solving the economic model at each guess of the structural parameters, they frequently employ the nested fixed point algorithm (NFXP) suggested by Rust (1987). In the case of maximum likelihood estimation, the approach works like the following: An unconstrained optimization algorithm guesses the structural parameters and for each of those guesses the underlying economic model is solved. The resulting outcome of the economic model allows to evaluate the maximum likelihood which then gives new information to the optimization algorithm to form a new guess of the structural parameters. This is repeated until some stopping criteria is met. To make it more explicit, let us introduce some mathematical notation. Let us assume that an economic model is described by some structural parameter vector θ and a state vector x as well as some endogenous vector σ. Assume we further observe some data consisting of $X = \{x_i, d_i\}_{i=1}^{M}$. Here, x_i is the observed state and d_i is the observed equilibrium outcome of the underlying economic decision model. M is the number of data points.

Let us further assume that generally σ depends on the parameters θ through a set of equilibrium conditions (or in the previous notation of variational inequalities), i.e. $\sigma(\theta)$. This includes e.g. Bellman equations. The consistency of σ with θ is expressed by the following condition:

$$h(\theta, \sigma) = 0.$$

For a given θ, let $\Sigma(\theta)$ denote the set of $\sigma(\theta)$ for which the equilibrium conditions hold, i.e. for which $h(\theta, \sigma) = 0$.

$$\Sigma(\theta) := \{\sigma : h(\theta, \sigma) = 0\}.$$

Let $\hat{\sigma}(\theta)$ denote an element of the above set. In the case of an infinite horizon dynamic discrete-choice model, this represents the expected value function evaluated at a specific parameter vector θ. In the case that a unique fixed point for the expected value function exists, $\hat{\sigma}(\theta)$ would be a single value but this does not have to hold in general. If the equilibrium condition involves solving a game for instance, one could easily imagine to find multiple equilibria which causes $\Sigma(\theta)$ to have multiple elements for a given θ.

For the case of multiple $\hat{\sigma}(\theta)$ the solution to the maximization of the log likelihood function $L(.)$ given the data X becomes:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \{ \max_{\hat{\sigma}(\theta) \in \Sigma(\theta)} L(\theta, \hat{\sigma}(\theta); X) \}. \tag{1}$$

This shows that the above problem boils down to finding the parameter vector θ that gives out possibly several $\hat{\sigma}(\theta)$ and which yields in combination with one of them the highest possible log likelihood of all combinations of θ and $\hat{\sigma}(\theta)$.

As already shortly described, the NFXP attempts to solve this problem in a nested loop. First, a guess for $\hat{\theta}$ is fixed for which the corresponding $\hat{\sigma}(\hat{\theta})$ (possibly multiple) are found. For those possibly multiple combinations of $\hat{\theta}$ and $\hat{\sigma}(\hat{\theta})$ the one that yields the highest log likelihood is chosen and this procedure is repeated until the $\hat{\theta}$ is found that solves equation (1). The NFXP therefore solves this problem by running an unconstrained optimization of the log likelihood function that involves solving the economic model at each parameter guess. For the simplified version of $\hat{\sigma}(\hat{\theta})$ being single-valued this idea is captured in the following pseudocode:

Algorithm 1: Nested Fixed Point Algorithm

Input: $\hat{\theta}_n$, $n = 0$, X;
while $f(||\hat{\theta}_{n+1} - \hat{\theta}_n||) \geq$ *stopping tolerance* **do**
  Calculate $\hat{\sigma}(\hat{\theta}_n)$ and evaluate $L(\hat{\theta}_n, \hat{\sigma}(\hat{\theta}_n); X)\}$;
  Based on that fix a new guess $\hat{\theta}_{n+1}$;
end

The above formulation clearly conveys two points already. The problem posed in equation (1) is essentially a hierarchical one. Additionally, we work with equilibrium conditions. This gives an indication that an MPEC formulation of the above problem might exist. Su and Judd (2012) formally prove exactly this. The difference to the NFXP way of writing the problem is that one know ensures differently that a guess of θ is consistent with the equilibrium condition $h(\theta, \sigma) = 0$. In the MPEC formulation σ is modeled explicitly as another parameter vector that can be chosen freely by an optimization algorithm instead of being derived from θ. This gives rise to a new log likelihood function $L(\theta, \sigma; X)$ for which they coin the term *augmented likelihood function.* Still they have to make sure, though, that the equilibrium condition holds meaning that the parameter guess for θ is consistent with the equilibrium σ. This is done by imposing it as a constraint to the augmented log likelihood function. The optimization problem now becomes a constrained optimization looking like the following:

$$
\begin{aligned}
&\max_{(\theta,\sigma)} L(\theta, \sigma; X) \\
&\text{subject to } h(\theta, \sigma) = 0.
\end{aligned}
\tag{2}
$$

Su and Judd (2012) provide a proof that the two formulations in the equations (1) and (11) are actually equivalent in the sense that they yield the same solution $\hat{\theta}$ for the structural parameters of the model. The general setup of the algorithm used for MPEC simplifies to the following:

Algorithm 2: Mathematical Programming with Equilibrium Constraints

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Input: $\hat{\theta}_n$, $\hat{\sigma}_n$, $n = 0$, X;
while $f(||(\hat{\theta}_{n+1}, \hat{\sigma}_{n+1}) - (\hat{\theta}_n, \hat{\sigma}_n)||) \geq$ *stopping tolerance* **do**
  Evaluate $L(\hat{\theta}_n, \hat{\sigma}_n; X)$;
  Based on that fix a new guess $(\hat{\theta}_{n+1}, \hat{\sigma}_{n+1})$;
end

Having established that the two algorithms or formulations theoretically yield the same solution for the structural parameters, Dong et al. (2017) note that the different way they achieve that can be characterized in the following way: The NFXP solves the problem with an unconstrained optimization algorithm by posing the problem as a low dimensional one. The MPEC formulation on the other hand is a high dimensional problem that needs to be solved using an optimizer that can handle constrained optimization problems involving nonlinear constraints. The difference in dimensionality stems from the fact that in the MPEC also the equilibrium variables need to be chosen. This observation automatically raises the question whether there is any advantage MPEC might have over NFXP as at first sight the problem seems to be more complicated. Su and Judd (2012) identify one major advantage which rests on the fact that the solving of the economic model does not have to be taken care of by the researcher but is cast to the optimization algorithm. The first immediate advantage comes from less coding effort. In the case of the problem of infinite-horizon dynamic discrete choice posed in Rust (1987) which Su and Judd base their comparison on, this makes a significant difference. Another advantage comes from the way modern solvers such as KNITRO (based on Byrd et al. (2006)) or IPOPT (see Pirnay et al. (2011)) handle constraints. Those constraints are not solved exactly until the last guess of the structural parameters which allows them to potentially perform faster than the NFXP in which at each guess of the structural parameters θ is calculated with high precision. This can especially be a factor when the underlying model is quite complicated such as for instance a game with multiple equilibria. Dubé et al. (2012), who look at the NFXP and MPEC for a BLP demand model, see another more practical factor that might make the case for MPEC. They report that practioners tend to loosen the convergence tolerance for solving the economic model when using the NFXP in order to speed up the process (especially when the model is computationally intensive). This leads to an increasing numerical error in the equilibrium outcome which might propagate into the guess of the structural parameters as the equilibrium outcome influences the likelihood function. This can result in wrong parameter estimates or even in failure of convergence. They further report that the existing literature on durable and semi-durable good markets might profit from MPEC as there are models that would need three nested loops when using the NFXP but two of them could be easily cast into the constraints of one major loop when opting for MPEC.

MPEC has one key limitation, though, that is mentioned by several different authors. Wright (2004) reports that the speed of modern solvers based on interior point algorithms (such as the before mentioned KNITRO and IPOPT) crucially depends on the sparsity of the Jacobian and the Hessian of the Langrangian. This highlights that the higher dimensionality (the size of the Jacobian and the Hessian) of MPEC problems does not need to generally cause a problem but if it comes with few zero elements in the before mentioned matrices it might, i.e. when those matrices are rather dense. This, in turn, depends on the economic model at hand and hence gives an indication that whether NFXP or MPEC should be prefered might depend on the specific context. This is confirmed by Dubé et al. who find that MPEC is faster and more reliable (looking at the convergence rate) than the NFXP but for problems that cause the constraint Jacobian and Hessian to be sparse but this advantage deteriorates when having dense matrices. Jørgensen (2013) confirms this for the case of estimating a continuous choice model. He states that MPEC needs too much memory when the state space is large and the before mentioned matrices are dense. In a more recent study Dong et al. (2017) compare MPEC and NFXP for an empirical matching model. They obtain a more fine-grained image of the previously noted tradeoff. In their estimation, they solve the same model with a more sparse and a more dense version of MPEC. They obtain this difference by first setting up MPEC with all equilibrium conditions as constraints (sparse version) and then again with a version where they substitute in some of the equilibrium conditions into the other ones (dense version). For the comparison of the two, they find that the sparse version has better convergence rates while the dense version has a speed advantage. The authors observe another interesting element when comparing NFXP and MPEC. In their application the inner loop can potentially fail depending on the structural parameter guess provided. This adds another problematic element to the use of the NFXP. This is due to the set up of separating the structural guess from the solving of the model. The optimizer cannot take into account whether a structural parameter guess might cause the inner loop (the solving of the economic model) to fail. This is different for the MPEC formulation in which the algorithm can jointly consider the structural parameter and the equilibrium outcome.

2 The Rust Model

This section follows up on the previous one in the sense that it introduces the model created by Rust (1987) and presents how NFXP and MPEC can be used to estimate its model parameters. My notation is mainly inspired by the one employed in Su and Judd (2012).

2.1 The Economic Model

Rust's model is based on the decision making process of Harold Zurcher who is in charge of a bus fleet and has to decide in each period $t = 0, 1, 2, ...$ whether to replace the engine ($d_t^i = 1$) of one or more buses $i = 1, 2, ..., M$ in his fleet or otherwise repair them in a less costly way ($d_t^i = 0$). The agent, hence, chooses from the action space $\mathcal{D} = \{0, 1\}$. He bases this choice on two state variables which are the observed cumulative mileage x_t^i of a bus since the last engine replacement and some unobserved (by the econometrician) factor ϵ_t^i. The state of a bus i in period t is therefore fully described by $(x_t^i, \epsilon_t^i) \in \mathcal{S}$. The agent receives an immediate reward in period t from the chosen replacement decision $d_t^i(x_t^i, \epsilon_t^i)$. The choice in turn affects the possible state space $\mathcal{S}'$ in the period $t + 1$ as the cumulative mileage after replacement x_{t+1}^i depends on the choice of d_t^i. As the agent is forward looking with a discount factor $\beta \in (0,1)$ he does not simply maximize the immediate reward but rather the expected discounted utility over an infinite horizon with a higher preference for reward occurring closer to the present. The immediate reward can be characterized in the following way:

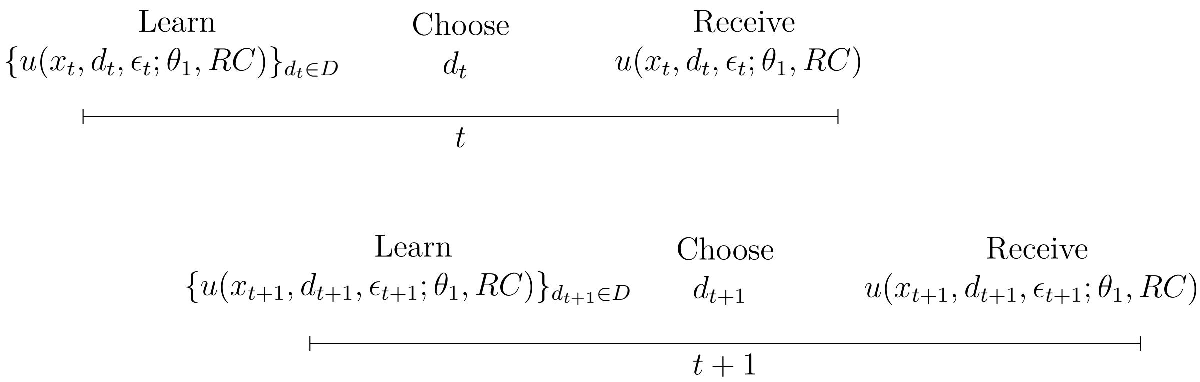
$$u(x_t^i, d_t^i, \epsilon_t^i; \theta_1, RC) = v(x_t^i, d_t^i; \theta_1, RC) + \epsilon_t^i \tag{3}$$

with

$$v(x_t^i, d_t^i; \theta_1, RC) = \begin{cases} -c(x; \theta_1) & \text{if } d_t^i = 0 \\ -RC - c(0; \theta_1) & \text{if } d_t^i = 1 \end{cases}$$

The immediate reward is hence determined by some operating cost $c(.)$ if regular maintenance as opposed to engine replacement is chosen. It consists of a replacement cost RC and the operating cost after resetting the cumulative mileage to zero if replacement is picked. This shows that The choice of the agent d_t^i depends crucially on the cost parameters θ_1 and RC. The timing of events for a single bus in the decision process of Harold Zurcher is depicted in Figure 1 below.

Figure 1. Timing of the Decision Model



The transition of the state vector $(x_t^i,\ \epsilon_t^i)$ is assumed to follow a Markov process, i.e. the current state only depends on the previous one and hence the utility maximization problem is time-invariant. Dropping the bus index i for convenience, the optimization problem of the agent gives rise to the following value function for a single bus:

$$V(x_t, \epsilon_t) = \max_{\{d_t, d_{t+1}, \dots\}} \mathbb{E}\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(x_\tau, d_\tau, \epsilon_\tau; \theta_1, RC)\right] \tag{4}$$

Solving this model leads to an optimal policy rule $\pi^* = (d_t^{\pi^*}(x_t, \epsilon_t))_t^\infty$.

2.2 The Model Solving

Given some simplifying assumptions such as conditional independence on the transition probabilities of the state vector, the assumption that the error term ϵ_t follows a multivariate extreme-value distribution and after discretizing the possible values of the state variable x, Rust derives from the original Bellman equation [1] the following contraction mapping needed to solve the economic model:

$$EV_f(\hat{x}_k, d) = \sum_{j=0}^{J} log \left\{ \sum_{d' \in \{0,1\}} exp[v(x', d'; \theta_1, RC) + \beta EV(x', d')] \right\} \times p_3(x'|\hat{x}_k, d; \theta_3). \quad (5)$$

with

$$p_3(x'|\hat{x}_k, d; \theta_3) = \begin{cases} Pr\{x' = \hat{x}_{k+j}|\theta_3\} & \text{if } d = 0 \\ Pr\{x' = \hat{x}_{1+j}|\theta_3\} & \text{if } d = 1 \end{cases}$$

for $j = 0, 1, ..., J$ indicating how many grid points the mileage state climbs up in the next period.

In the above equation, $EV_f(.)$ denotes the unique fixed point to a contraction mapping $T_f(EV_f, \theta)$ on the full state space $\Gamma_f = \{(\hat{x}_k, d)|\hat{x}_k \in \hat{\mathbf{x}}, d \in \mathcal{D}\}$. Here, $\hat{x}_k$ represents the grid point k of the state variable x while $\hat{x}_1 = 0$. The number of possible $\hat{x}_k$ depends on the choice of the grid size K set by the researcher. The set of possible grid points then is denoted as $\hat{\mathbf{x}} = \{\hat{x}_1, ..., \hat{x}_K\}$. All the variables v depict current period variables while variables v' display the possible value in the next period. The probability p_3 indicates how likely it is that a bus moves up a specific amount of grid points in the next period depending on the structural parameter θ_3. Imagine now we decide to set the grid size to $K = 90$ as done in Rust (1987), then we generally have to find the fixed point above which yields $EV = [EV(\hat{x}_1, 0), ..., EV(\hat{x}_{90}, 0), EV(\hat{x}_1, 1), ..., EV(\hat{x}_{90}, 1)]$. This simplifies, though, as all the expected values from $EV(\hat{x}_1, 1), ..., EV(\hat{x}_{90}, 1)$ are actually equivalent to $EV(\hat{x}_1, 0)$. This means that in our scenario we just have to find the fixed point for $d = 0$, i.e. in our example the vector EV for which we have to actually solve the contraction mapping has a dimension of 90. This observation will later be important for the difference between NFXP and MPEC. Su and Judd (2012), hence, denote the in dimension-reduced contraction mapping shorthand as:

$$EV_r = T_r(EV_r, \theta) \quad (6)$$

with $T_r(.)$ being a contraction mapping on the reduced state space $\Gamma_r = \{(\hat{x}_k, d = 0)|\hat{x}_k \in \hat{\mathbf{x}}\}$.

The unique fixed point can now be used to derive conditional choice probabilities of

[1] The Bellman equation can be found in formula 4.4 on page 1010 in Rust (1987)

the agent:

$$P(d|\hat{x};\theta) = \frac{exp[v(\hat{x},d;\theta_1,RC)+\beta EV(\hat{x},d)]}{\sum_{d'\in\{0,1\}} exp[v(\hat{x},d';\theta_1,RC)+\beta EV(\hat{x},d')]}. \tag{7}$$

The above equation describes the probability that the agent chooses d given that the observed mileage state is at a certain grid point $\hat{x}$. This derivation depends on both the cost parameters (θ_1, RC) directly and indirectly through $EV(.)$ on the transition parameter θ_3. These conditional choice probabilities together with the transition probabilities $p_3(.)$ become relevant in the next section when calibrating the model using maximum likelihood.

2.3 Calibration

In order to calibrate the parameter vector $\theta = (\theta_1, \theta_3, RC)$ using either NFXP or MPEC, let us assume that we observe some data set $X = (X^i)_{i=1}^M$ with $X^i = (x_t^i, d_t^i)_{t=1}^T$ for a single bus $i = 1, ..., M$. The data therefore consists of the engine replacement decision per bus and period as well as the cumulative mileage since the last engine replacement. The cumulative mileage x_t^i is assumed to already be discretized which means that that it takes values on the grid $\hat{\mathbf{x}} = \{\hat{x}_1, ..., \hat{x}_K\}$. The log likelihood of observing the data X now becomes:

$$L(\theta) = \sum_{i=1}^{M}\sum_{t=2}^{T} log[P(d_t^i|x_t^i;\theta)] + \sum_{i=1}^{M}\sum_{t=2}^{T} log[p_3(x_t^i|x_{t-1}^i, d_{t-1}^i;\theta_3)]. \tag{8}$$

To fulfill the aim of finding the optimal parameter vector θ one now has to solve the problem of maximizing the log likelihood:

$$\max_{\theta} L(\theta). \tag{9}$$

The path taken by the NFXP is now to hand this unconstrained optimization problem to an optimization algorithm which comes up with a guess for the optimal parameter $\hat{\theta}$ for which in a subroutine the expected values in equation 5 is calculated for. The expected value function is in turn needed to obtain the conditional choice probabilities in equation 7 which are then taken to evaluate the log likelihood $L(\theta)$. Based on this evaluation, the optimization algorithm comes up with a new guess for $\hat{\theta}$ and the above procedure is repeated until a certain convergence criteria of the algorithm is met. This procedure is again shown in pseudocode in the Algorithm 3 on the next page.

At every structural guess of the optimization algorithm the fixed point $EV(.)$ is calculate precisely as it is needed to evaluate the log likelihood $L(\theta)$. This is deemed inefficient by Su and Judd which gives rise to the augmented log likelihood mentioned before for which they insert the conditional choice probabilities $P(d_t^i|x_t^i;\theta)$ into $L(\theta)$ making the log likelihood explicitly depend on $EV(.)$. This results in the following log likelihood.

Algorithm 3: Nested Fixed Point Algorithm for the Rust Model

Input: $\hat{\theta}_n$, $n = 0$, X;

while $f(||(\hat{\theta}_{n+1}, E\hat{V}_{n+1}) - (\hat{\theta}_n, E\hat{V}_n)||) \geq$ *stopping tolerance* **do**

Solve fixed point

$$EV(\hat{x}_k, d) = \sum_{j=0}^{J} log\left\{\sum_{d' \in \{0,1\}} exp[v(x', d'; \hat{\theta}_{n,1}, \hat{RC}_n) + \beta EV(x', d')]\right\} \times p_3(x'|\hat{x}_k, d; \hat{\theta}_{n,3});$$

Given the solution to $EV(.)$ calculate

$$P(d|\hat{x}; \hat{\theta}_n) = \frac{exp[v(\hat{x}, d; \hat{\theta}_{n,1}, \hat{RC}_n) + \beta EV(\hat{x}, d)]}{\sum_{d' \in \{0,1\}} exp[v(\hat{x}, d'; \hat{\theta}_{n,1}, \hat{RC}_n)) + \beta EV(\hat{x}, d')]};$$

Evaluate the log likelihood

$$L(\hat{\theta}_n) = \sum_{i=1}^{M}\sum_{t=2}^{T} log[P(d_t^i|x_t^i; \hat{\theta}_n)] + \sum_{i=1}^{M}\sum_{t=2}^{T} log[p_3(x_t^i|x_{t-1}^i, d_{t-1}^i; \hat{\theta}_{n,3})];$$

Based on that fix a new guess $\hat{\theta}_{n+1}$;

end

$$\begin{aligned}\mathcal{L}(\theta, EV) = &\sum_{i=1}^{M}\sum_{t=2}^{T} log\left[\frac{exp[v(x_t^i, d_t^i, \theta) + \beta EV(x_t^i, d_t^i)]}{\sum_{d' \in \{0,1\}} exp[v(x_t^i, d', \theta) + \beta EV(x_t^i, d')]}\right] \\ &+ \sum_{i=1}^{M}\sum_{t=2}^{T} log[p_3(x_t^i|x_{t-1}^i, d_{t-1}^i; \theta_3)]\end{aligned} \tag{10}$$

In this function nothing guarantees that θ and EV are actually consistent. This is healed by imposing the fixed point equation 5 as constraints to the augmented likelihood function. The MPEC formulation of the calibration problem therefore looks like the following.

$$\begin{aligned}&\max_{(\theta, EV)} \mathcal{L}(\theta, EV) \\ &\text{subject to } EV = T(EV, \theta).\end{aligned} \tag{11}$$

For the MPEC formulation the problem is given to an optimization Algorithm that can handle nonlinear equality constraints. This algorithm then fixes a guess of (θ, EV) that satisfies the nonlinear constraints, i.e. that is consistent with the underlying economic model and for which the augmented log likelihood is evaluated. Based on this evaluation, the optimizer determines a new guess and the procedure starts over. Again this is done until a specific convergence criteria is met. This procedure is illustrated in algorithm 4 on the next page.

Algorithm 4: MPEC Algorithm for the Rust Model

Input: $\hat{\theta}_n$, $\hat{EV}_n$, $n = 0$, X;

while $f(||\hat{\theta}_{n+1} - \hat{\theta}_n||) \geq$ *stopping tolerance* **do**

Evaluate the augmented log likelihood

$$\mathcal{L}(\hat{\theta}_n, \hat{EV}_n) = \sum_{i=1}^{M} \sum_{t=2}^{T} log \left[\frac{exp[v(x_t^i, d_t^i, \hat{\theta}_n) + \beta \hat{EV}_n(x_t^i, d_t^i)]}{\sum_{d' \in \{0,1\}} exp[v(x_t^i, d', \hat{\theta}_n) + \beta \hat{EV}_n(x_t^i, d')]} \right]$$
$$+ \sum_{i=1}^{M} \sum_{t=2}^{T} log[p_3(x_t^i | x_{t-1}^i, d_{t-1}^i; \hat{\theta}_{n,3})]$$

Based on that fix a new guess $(\hat{\theta}_{n+1}, \hat{EV}_{n+1})$;

end

Having established both NFXP and MPEC for the Rust model, let us now turn to some particularities of the model that might be important for the performance of the two algorithms. First of all, Rust (1987) and Su and Judd (2012) show that both the likelihood for the NFXP and the MPEC are smooth and their first and second order derivatives exist. In the case of the NFXP this means that Newton's method can be used for the maximization problem. This further helps modern solvers such as KNITRO and IPOPT which are developed for smooth optimization problems (compare Byrd et al. (2006) and Wächter (2009)). Another special feature is that solving model involves finding a fixed point. In the case of the NFXP this is time consuming as it involves contraction iterations. This caused Rust to employ a polyalgorithm to find the fixed point using contraction steps at the beginning and switching to Newton-Kantorovich (N-K) steps as soon as a guess is already close to the unique fixed point. This practically speeds up the convergence when searching the fixed point as contraction iterations solely have a linear convergence rate while N-K iterations converge at a quadratic rate when being close to the fixed point (see Rust (1987, 2000)). As already noted before, MPEC on the other hand does not solve the fixed point but instead evaluates it once at every structural parameter guess without high precision until the last iteration of the optimization algorithm. Another factor explained in the general part on MPEC is the high dimensionality of it. Going back to our example when the grid size is set to 90, the MPEC formulation yields a problem consisting of 90 nonlinear constraints and $90 + |\theta|$ parameters to estimate. The NFXP has considerably less dimensions as only $|\theta|$ parameters have to be estimated and no constraints have to be considered by the optimizer. Su and Judd (2012) uncover the trade off of dimensionality and fixed point calculation to be the major one between MPEC and NFXP in the Rust model application. Following this line of arguments one might expect the chosen grid size to amplify this trade off into a certain direction as the grid size increases the dimensions of the MPEC problem while also make the fixed point calculation in the NFXP more computationally expensive.

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Affidavit

"I hereby confirm that the work presented has been performed and interpreted solely by myself except for where I explicitly identified the contrary. I assure that this work has not been presented in any other form for the fulfillment of any other degree or qualification. Ideas taken from other works in letter and in spirit are identified in every single case."

Place, Date

Signature