Failure of Newton-Raphson Method

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The Newton-Raphson Method

The easiest case of the Newton-Raphson method is defined by the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

assuming that f is differentiable and its derivatives are non-zero. It is easy to remember, and is effective in solving real life problems. However, a well chosen point for the starting point x_0 is very important because Newton-Raphson method may fail to converge even for the simplest equations. This paper tries to show the failure of Newton-Raphson to locate the zeros of a function at several instances.

After implementing Newton-Raphson with the conditions: a) break if current iteration reached a very large number in absolute value, and b) break if iteration does not converge after a maximum number of iterates. We set the tolerance value, *tolerance* = 10^{-6}, and the maximum number of iterations, *maxIterations* = 100, and the very large number, *large* = 10^{200}.

For each of the following functions:

(i) $y = tan^{-1}(x)$; $x_0 = 1.45$

(ii) $y = xexp(-x)$; $x_0 = 2.0$

(iii) $y = x^3 - x - 3$; $x_0 = 0.0$

We will see how Newton-Raphson method fails to locate the root nearest to the starting point for these instances.

For (i) $y = tan^{-1}(x)$, when we chose x_0 = 1.45, the method indeed failed and the value of x diverges as shown in Figure 1.1

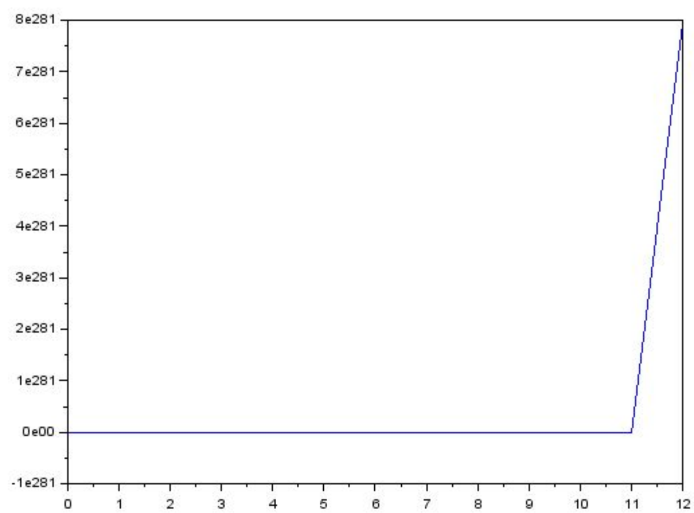

Figure 1.1

Had we chosen a different point, say $x_0 = 1.0$,

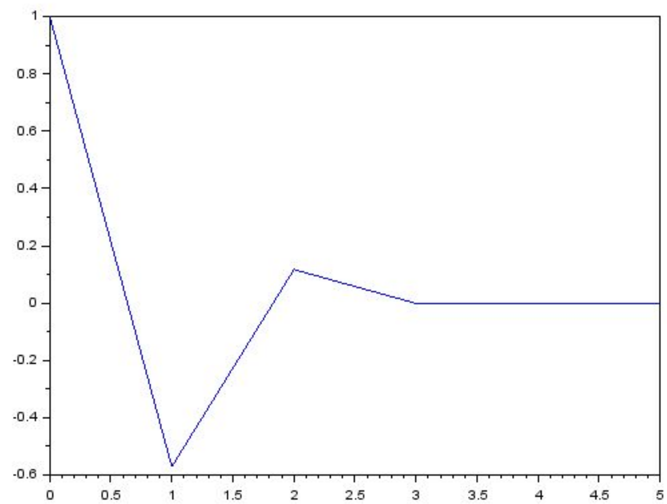

Figure 1.2

from Figure 1.2, we can see that the method eventually converges to 0.

For (ii) $y = x\exp(-x)$, when we chose $x_0 = 2.0$, the method has failed to converge as the value of x continually increases per iteration as we can see in Figure 1.3.

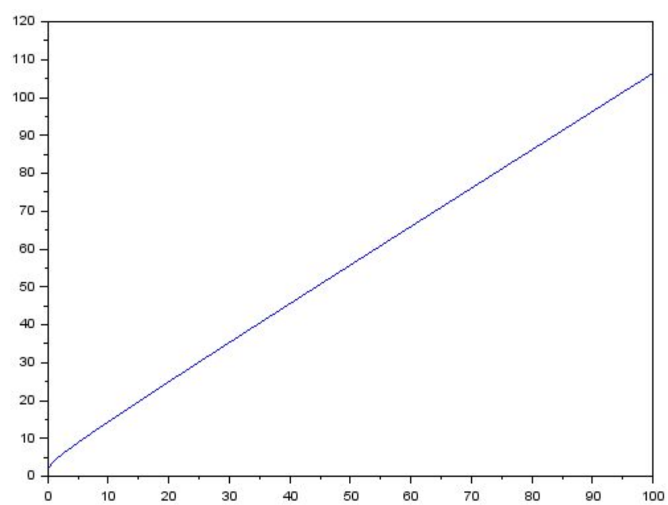

Figure 1.3

Choosing $x_0 = 0.5$ as the starting point, we can see a different behavior, and the method actually converges to 0.

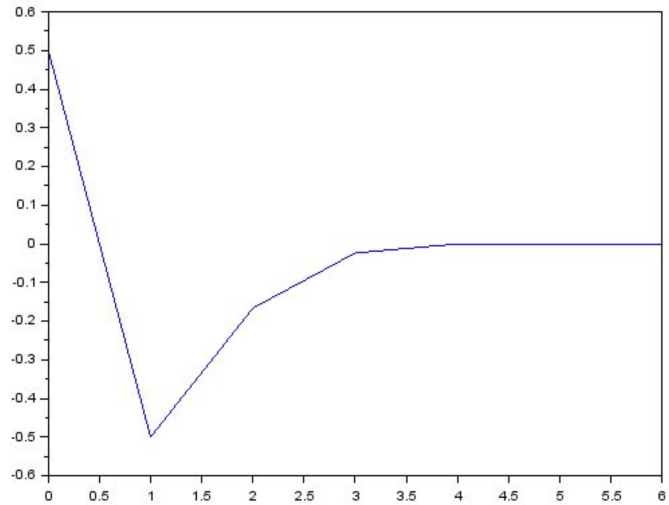

Figure 1.4

For (iii) $y = x^3 - x - 3$; $x_0 = 0.0$, we can see a fluctuating value of x, between the values -3.0004983, -1.9618967, -1.1475016, and -0.0074461. It has failed to converge, and continued to fluctuate until it has reached our set maximum number of iterations, 100.

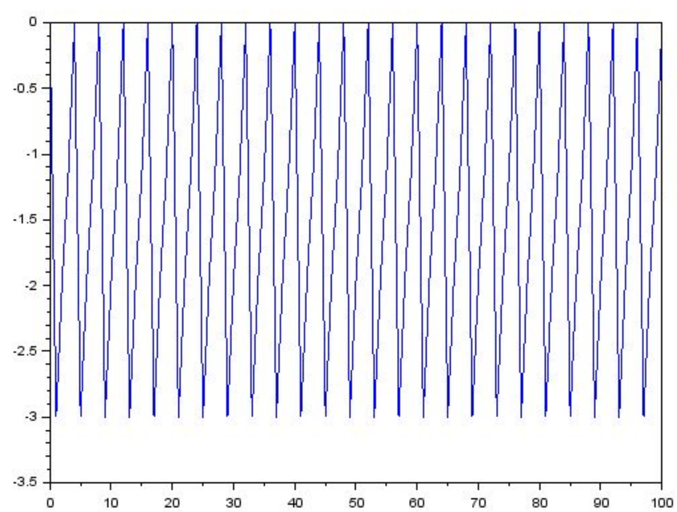

Figure 1.5

However, if we take $x_0 = 1.0$ instead, we can see in Figure 1.6 that the method converges to the value, 1.6716999 after 6 iterations.

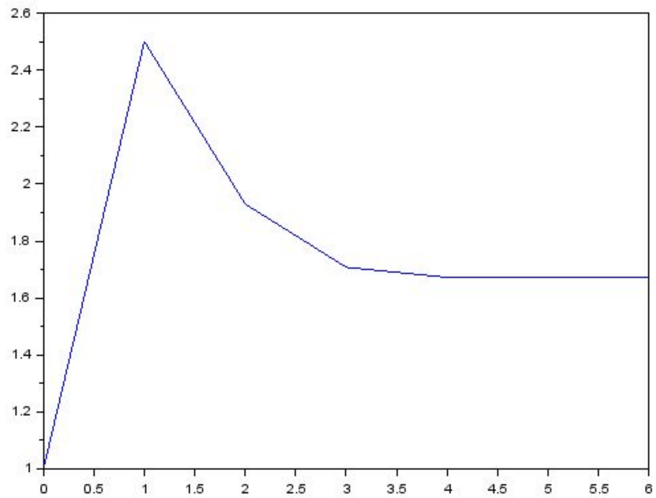

Figure 1.6

Failure of Newton-Raphson Method

If we do not know what the graph looks like, we might accidentally pick a point wherein the slope of the tangent line is equal to 0. In that case, Newton-Raphson diverges.

Let us take a look at the graph of the functions. In Figure 1.7 we can see that a root exists in x = 0. However, using $x_0 = 1.45$, observe in Figure 1.8.1 to Figure 1.8.5 that the x values go farther and farther from the root as the slope of the point approaches 0 or when the first derivative nearly vanishes.

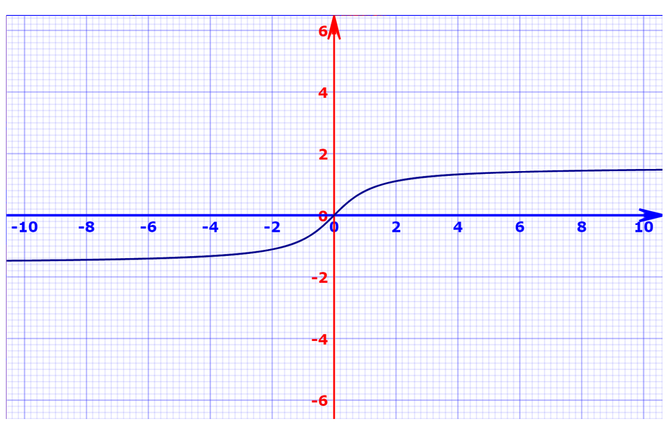

Figure 1.7 y = $tan^{-1}(x)$

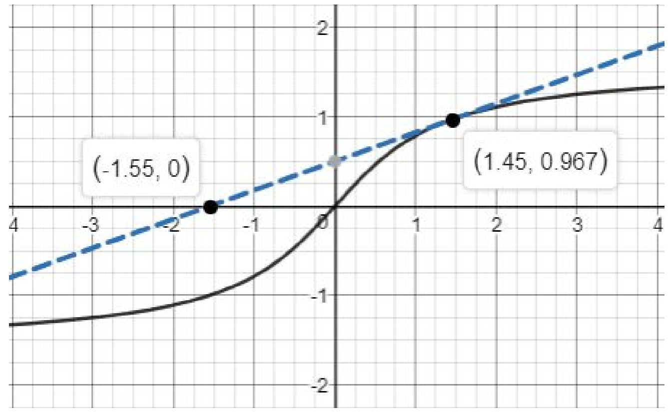

Figure 1.8.1 $x_0 = 1.45$

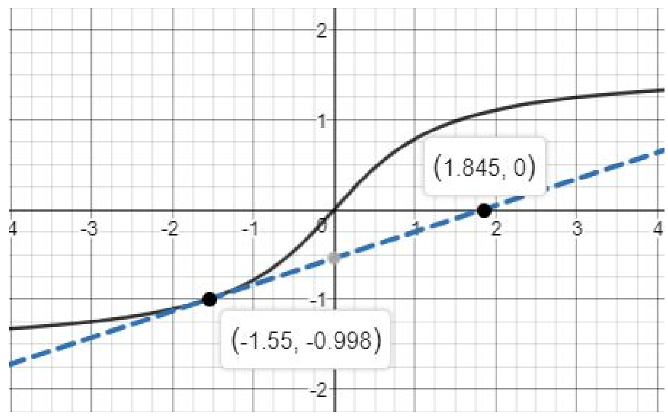

Figure 1.8.2 $x_1 = -1.55$

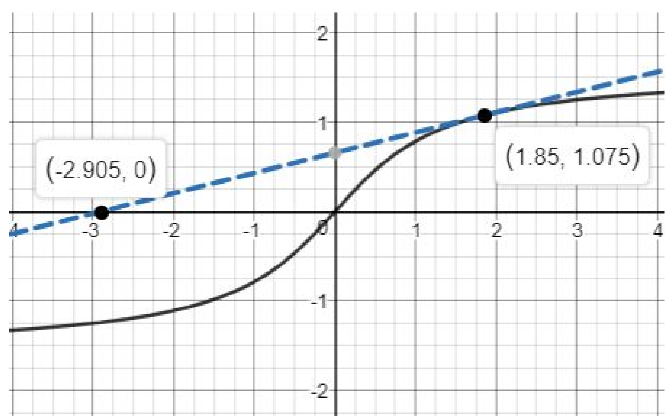

Figure 1.8.3 $x_2 = 1.85$

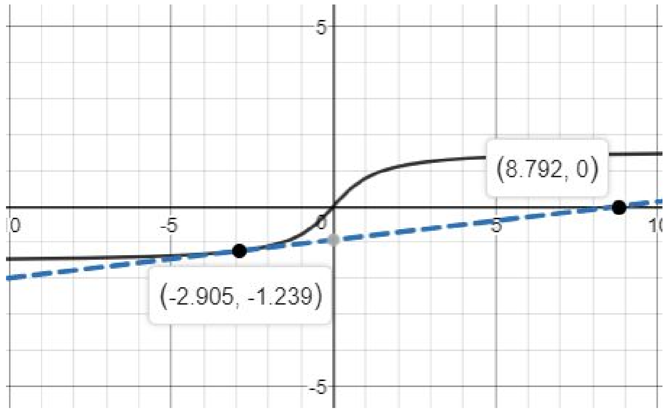

Figure 1.8.4 $x_3 = -2.905$

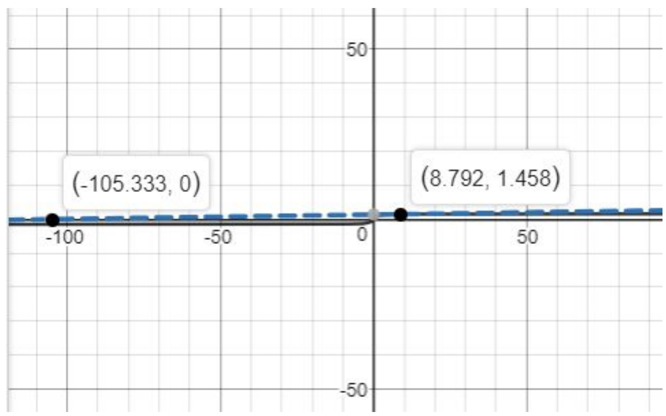

Figure 1.8.1 $x_4 = 8.792$ and $x_5 = -105.333$

Similar behavior could be observed for function (ii) as shown in Figures 1.9.1 to 1.9.5

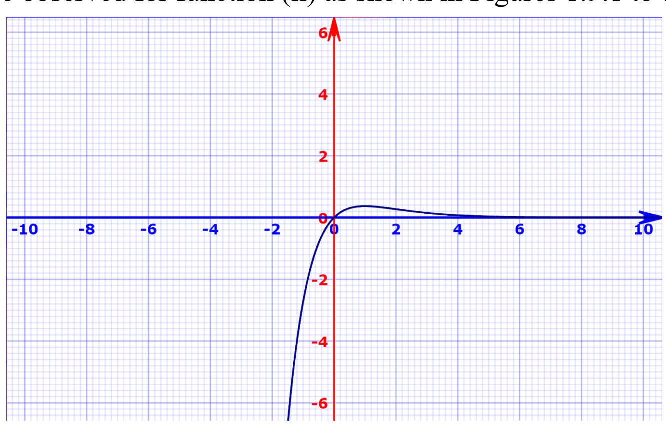

Figure 1.9 y = $xexp(-x)$

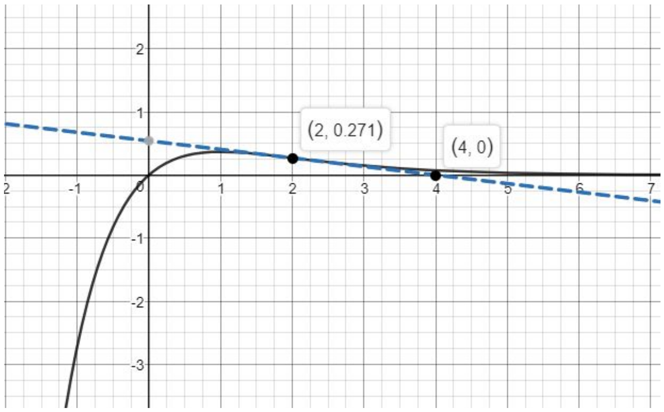

Figure 1.9.1 $x_0 = 2.0$

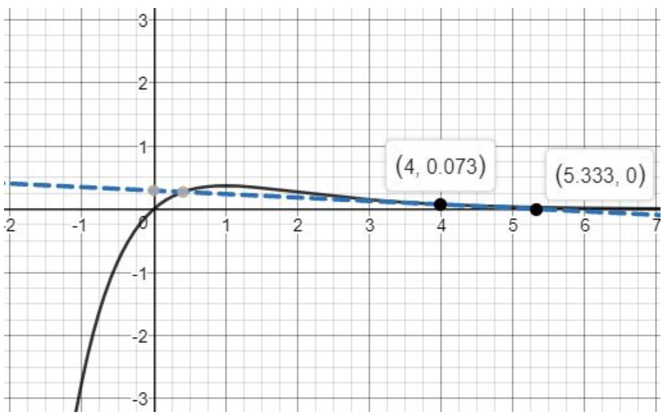

Figure 1.9.2 $x_1 = 4.0$

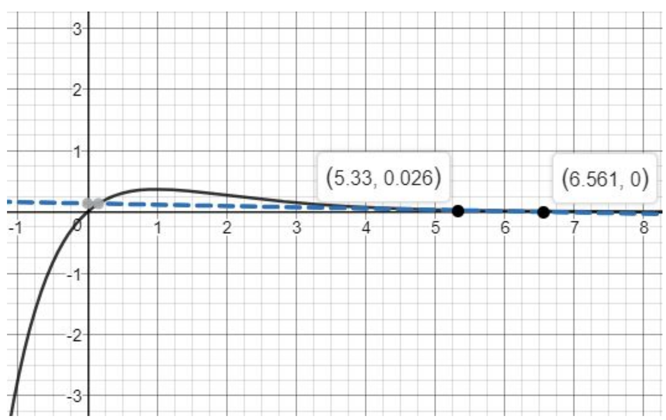

Figure 1.9.3 $x_2 = 5.33$

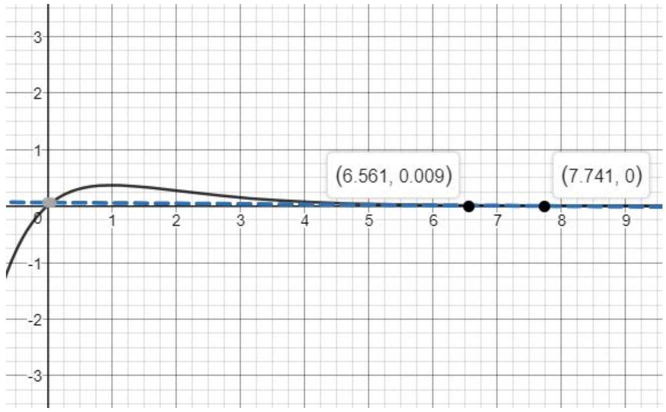

Figure 1.9.4 $x_3 = 6.561$

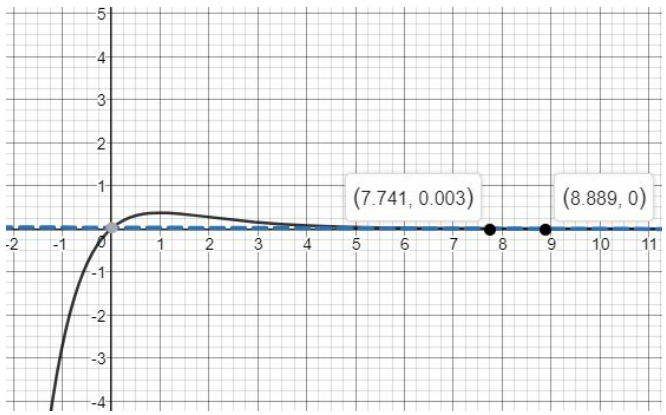

Figure 1.9.5 $x_4 = 7.741$ and $x_5 = 8.889$

Moreover, when we plot the points from Newton-Raphson Method evaluation, we can see that the method also fails for function (iii).

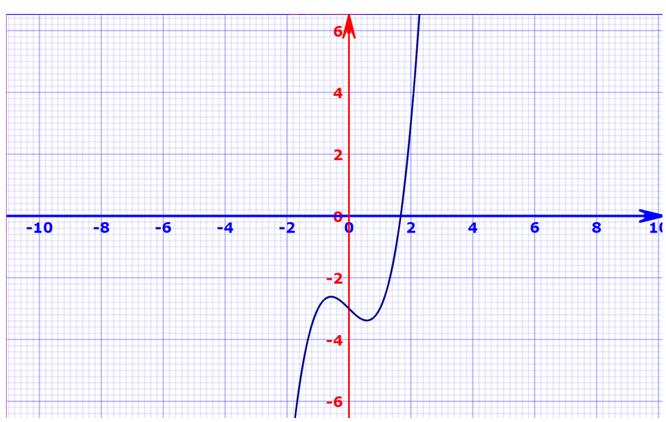

Figure 1.10 $y = x^3 - x - 3$

From these instances, we can see that the choice of x_0 can be very important in determining whether Newton-Raphson's method will converge. Unfortunately, there isn't a strategy that is always effective in choosing x_0. Choosing a bad starting point can give inaccurate, and meaningless values. For instance, the initial guess x_0 for the root might be outside the range of guaranteed convergence, and might include a local maximum or minimum of the function on the search interval. If an iteration places a trial guess near a local extremum, the Newton-Raphson method may fail.[2]

Modified Newton-Raphson Method

This part of the paper will show how knowing the multiplicity of roots may improve the Newton-Raphson Method rate of convergence, from quadratic to linear convergence.

Let's first take a look at how Newton-Raphson method fails to locate the root of

$$y = x^3 - 1.2x^2 - 8.19x + 13.23,\ \ with\ x_0 = 4$$

within just few iterations. Let's set the maximum number of iteration to 7 iterations only

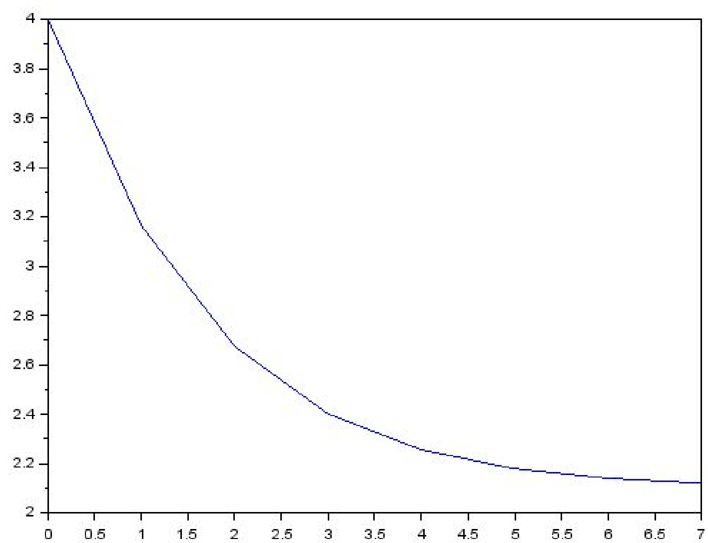

Figure 1.11 Failed to converge after 7 iterations

After 7 iterations, the original Newton-Raphson method hasn't converged yet. Although it is expected to converge after 22 iterations as shown in Figure 1.12, we can actually use the modified method to improve the rate of convergence.

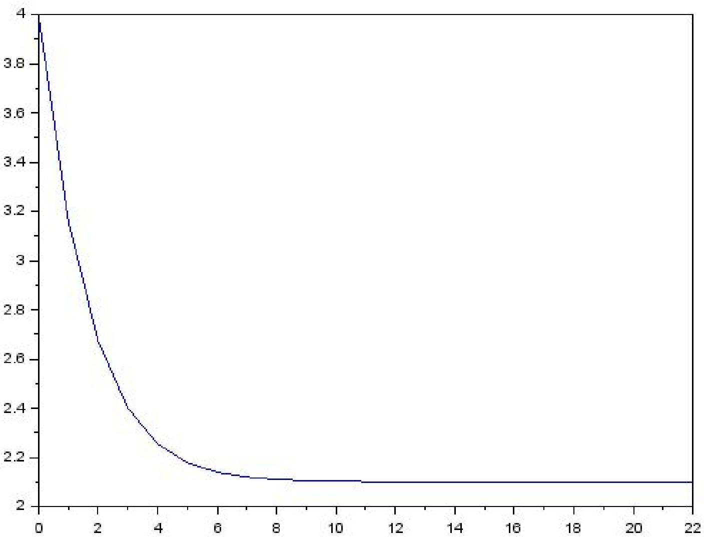

Figure 1.12 Converging after 22 iterations

Modifying the method as

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)},$$

gives us a faster convergence; converging at x = 2.1 after just 7 iterations as shown in Figure 1.13.

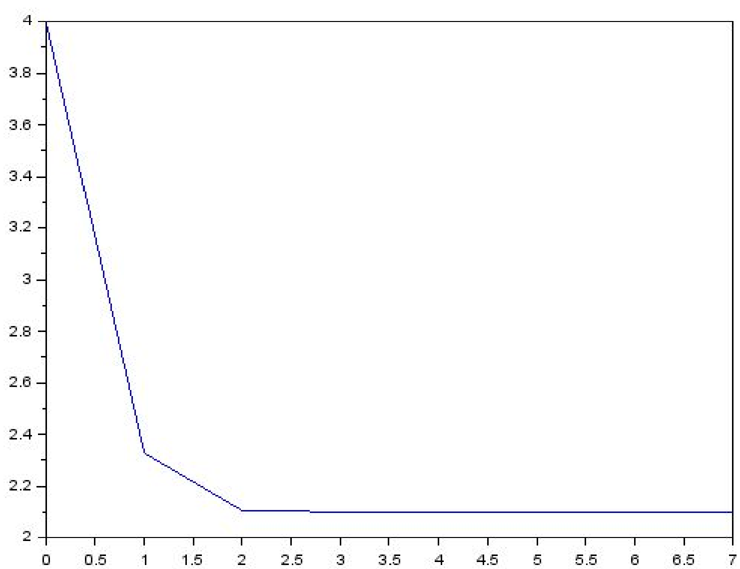

Figure 1.13 Modified Newton Raphson Method

The factor 2 in $x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$, is called the ***multiplicity*** of the root.[3] Generally, the Newton Raphson method has a quadratic rate of convergence (r = 2) especially when finding simple roots. However, if the root that we are trying to find has a multiplicity greater than one, (*m > 1)* then linear convergence is possible. [1]

The general formula for the Modified Newton-Raphson Method is of the form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

We have seen from the examples shown that choosing a starting point may be crucial in determining whether the Newton-Raphson Method will converge. Although the method may fail in some cases, especially if a bad starting point is chosen, it doesn't limit the method's usefulness, in practice, a particularly good guess is not required.

References:

[1]"Assessment of Newton`s Method", *Ens.utulsa.edu*, 2018. [Online]. Available: http://www.ens.utulsa.edu/~diaz/cs4533/newtcpp/node7.html. [Accessed: 10- Nov- 2018].

[2]"NUMERICAL RECIPES IN C: THE ART OF SCIENTIFIC COMPUTING", *Aip.de*, 1988. [Online]. Available: http://www.aip.de/groups/soe/local/numres/bookcpdf/c9-4.pdf. [Accessed: 10- Nov- 2018].

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[5]"Convergence of the Newton Method and Modified Newton Method", *Macs.citadel.edu*, 2010. [Online]. Available: http://macs.citadel.edu/chenm/343.dir/11.dir/lect2_4.pdf. [Accessed: 10- Nov- 2018].