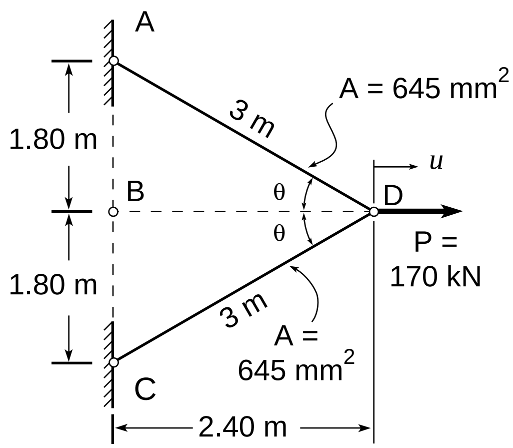
StaticsM Examples, Methods

August 25, 1998

Example: Axial loading and displacement relationships



$P =$ 170 kN

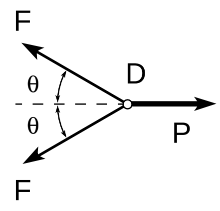
$u =$ horizontal displacement

$E =$ 211 GPa

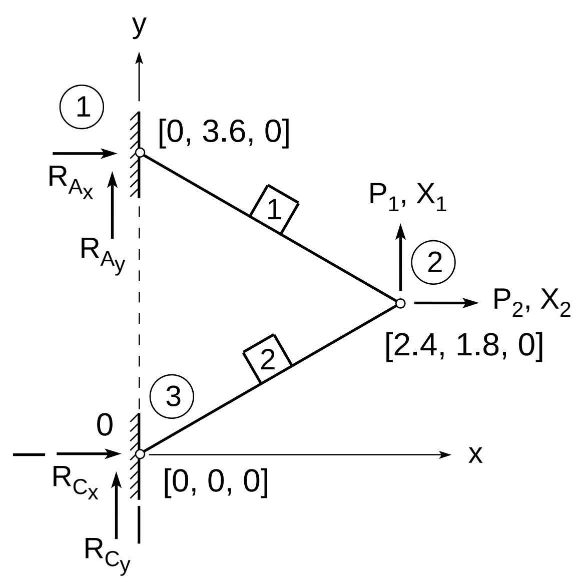
$A =$ 645 mm^2

$Y =$ 253 MPa

$$\in_y = \frac{Y}{E} = \frac{253 \text{ MPa}}{211 \text{ GPa}} = 0.0012$$



Free body



$$[A] = \begin{bmatrix} -.6 & .6 \\ .8 & .8 \end{bmatrix}$$

$$[S] = \begin{bmatrix} \frac{A_1 E}{L_1} & 0 \\ 0 & \frac{A_2 E}{L_2} \end{bmatrix} = \begin{bmatrix} 45365000 & 0 \\ 0 & 45365000 \end{bmatrix}$$

$$\{P\} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 170 \end{bmatrix} \text{ kN}$$

$$\left[[A]\,[S]\,[A]^T \right]^{-1} \{P\} = \{X\}$$

From Hooke's law

$F = k\,x$

$[K]\{X\} = \{P\}\ ;\ [K]^{-1}\{P\} = \{X\}$

$$\left[[S]\,[A]^T \right] \{X\} = \{F\}$$

From load-displacement relation.

$$e = \frac{PL}{AE} \quad ; [S]\{e\} = \{F\}$$

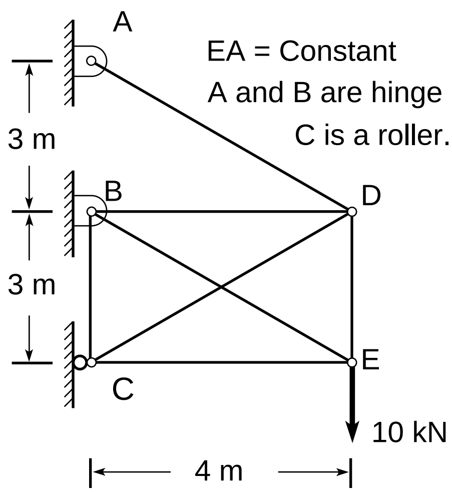
$$[A]^T \{X\} = \{e\}$$

$$\{X\} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.92774\text{E-}6 \end{bmatrix} \text{ m} \quad ; \quad u = X_2 = 2.92774\text{E-}6 \text{ m}$$

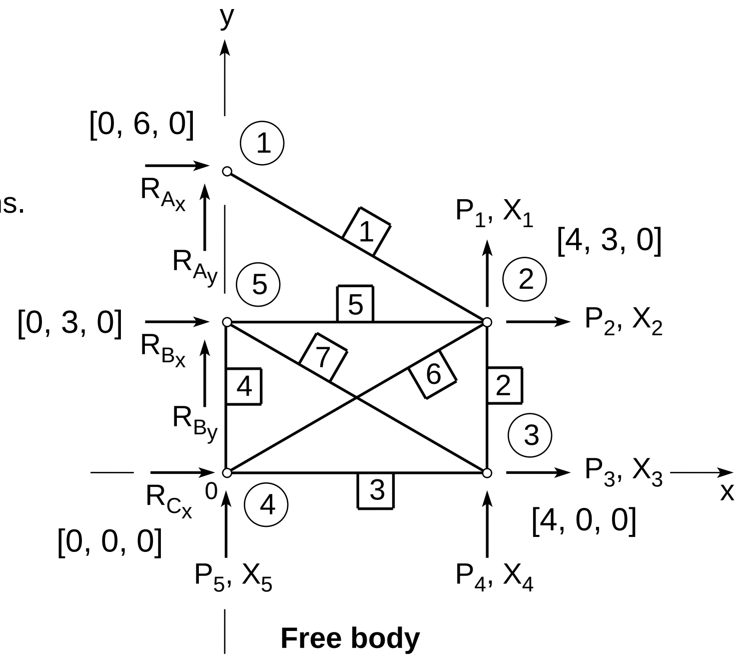
$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 106.25 \\ 106.25 \end{bmatrix} \text{ kN}$$

$$\{e\} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 2.342\text{E-}6 \\ 2.342\text{E-}6 \end{bmatrix} \text{ m} \quad ; \quad e = \text{elongation of bar}$$

Example: Indeterminate truss; load-displacement relationships. Matrix Displacement Method



EA = Constant
A and B are hinge connections.
C is a roller.



Free body

$$[A] = \begin{bmatrix} -.6 & 1 & 0 & 0 & 0 & .6 & 0 \\ .8 & 0 & 0 & 0 & 1 & .8 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & .8 \\ 0 & -1 & 0 & 0 & 0 & 0 & -.6 \\ 0 & 0 & 0 & -1 & 0 & -.6 & 0 \end{bmatrix} ; \quad \{P\} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -10 \\ 0 \end{bmatrix} \text{kN}$$

$$[S] = \begin{bmatrix} \frac{A_1E}{L_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{A_2E}{L_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{A_3E}{L_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{A_4E}{L_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{A_5E}{L_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{A_6E}{L_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{A_7E}{L_7} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

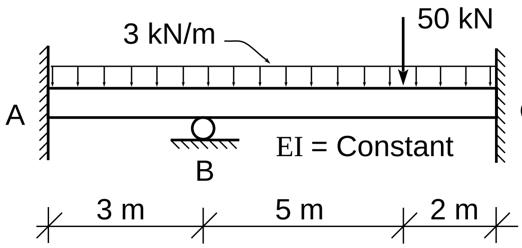
AE here is set equal to 1.
If interested in displacements and elongations, we can use a real set of AE factors.

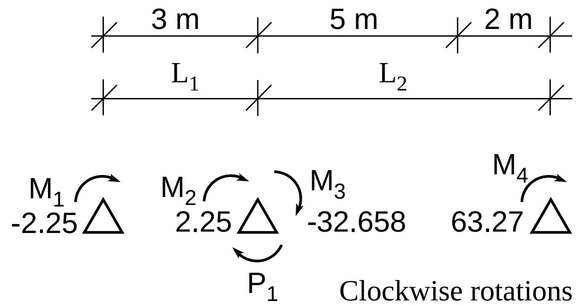
$$\left[[A]\,[S]\,[A]^T \right]^{-1} \{P\} = \{X\}$$

$$\left[[S]\,[A]^T \right] \{X\} = \{F\}$$

$$\{X\} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} -50.7907 \\ -1.7922 \\ -17.9534 \\ -70.6919 \\ -9.4465 \end{bmatrix} ; \quad \{F\} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} = \begin{bmatrix} 5.80813 \\ 6.6337 \\ -4.4883 \\ 3.1488 \\ -.44805 \\ -5.24806 \\ 5.61046 \end{bmatrix} \text{kN}$$

Example: Indeterminate beam; load-displacement relationships. Matrix Slope-Deflection Method





Clockwise rotations are postive.

Free body (of joints)

M_1, through M_4 are "fixed-end moments."
P_1 is the only "possible"; it's the only possible displacement (A and C are fixed).

(For reference, see "Slope-Deflection Equations.")

$$[A] = \begin{array}{c} \\ P_1 \end{array}\begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\{P\} = \begin{bmatrix} P_1 \end{bmatrix} = \begin{bmatrix} -(M_2 + M_3) \end{bmatrix} = \begin{bmatrix} +30.408 \end{bmatrix} \ \text{kN}\bullet\text{m}$$

$$[S] = \begin{bmatrix} \frac{4EI_1}{L_1} & \frac{2EI_1}{L_1} & 0 & 0 \\ \frac{2EI_1}{L_1} & \frac{4EI_1}{L_1} & 0 & 0 \\ 0 & 0 & \frac{4EI_2}{L_2} & \frac{2EI_2}{L_2} \\ 0 & 0 & \frac{2EI_2}{L_2} & \frac{4EI_2}{L_2} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{7} & \frac{2}{7} \\ 0 & 0 & \frac{2}{7} & \frac{4}{7} \end{bmatrix}$$

EI here is set equal to 1. If we are interested in displacements or rotations, we can use a real set of flexural rigidities

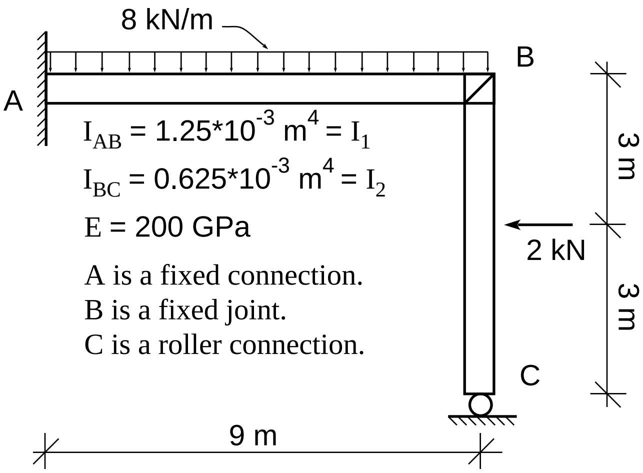
$$\left.\begin{array}{l} \left[[A]\,[S]\,[A]^T\right]^{-1}\{P\} = \{X\} \\ \left[[S]\,[A]^T\right]\{X\} = \{F\} \end{array}\right\}$$

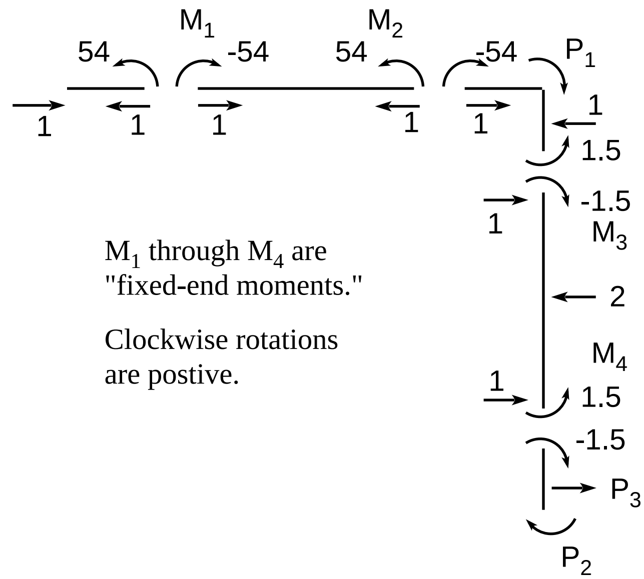
$$\{X\} = \begin{bmatrix} X_1 \end{bmatrix} = \begin{bmatrix} 15.9642 \end{bmatrix}$$

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 10.6428 \\ 21.2856 \\ 9.1224 \\ 4.5612 \end{bmatrix} \ \text{kN}\bullet\text{m}$$

$$\{M'\} = \begin{bmatrix} M'_1 \\ M'_2 \\ M'_3 \\ M'_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} 10.6428 \\ 21.2856 \\ 9.1224 \\ 4.5612 \end{bmatrix} + \begin{bmatrix} -2.25 \\ 2.25 \\ -32.658 \\ 63.27 \end{bmatrix} = \begin{bmatrix} 8.3928 \\ 23.5356 \\ -23.5356 \\ 67.8312 \end{bmatrix} \ \text{kN}\bullet\text{m}$$

Example: Indeterminate frame; load-displacement relationships. Matrix Slope-Deflection Method





Free body (of joints)

$$[A] = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/6 & -1/6 \end{bmatrix} \end{array} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/6 & -1/6 \end{bmatrix}$$

$$\{P\} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} -(M_2 + M_3) \\ -(M_4) \\ 1 \end{bmatrix} = \begin{bmatrix} -52.5 \\ -1.5 \\ 1 \end{bmatrix} \text{ kN•m}$$

$$[S] = \begin{bmatrix} \frac{4EI_1}{9} & \frac{2EI_1}{9} & 0 & 0 \\ \frac{2EI_1}{9} & \frac{4EI_1}{9} & 0 & 0 \\ 0 & 0 & \frac{4EI_2}{6} & \frac{2EI_2}{6} \\ 0 & 0 & \frac{2EI_2}{6} & \frac{4EI_2}{6} \end{bmatrix} = \begin{bmatrix} 111.111*10^6 & 55.555*10^6 & 0 & 0 \\ 55.555*10^6 & 111.111*10^6 & 0 & 0 \\ 0 & 0 & 83.333*10^6 & 41.666*10^6 \\ 0 & 0 & 41.666*10^6 & 83.333*10^6 \end{bmatrix}$$

$$\left[[A]\,[S]\,[A]^T \right]^{-1} \{P\} = \{X\}$$

$$\left[[S]\,[A]^T \right] \{X\} = \{F\}$$

$$\{X\} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -4.32\text{E-}7 \\ -3.6\text{E-}7 \\ -2.232\text{E-}6 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{radians (+, clockwise)} \\ \text{meters} \end{matrix}$$

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} -24 \\ -48 \\ -4.5 \\ -1.5 \end{bmatrix} \text{kN•m}$$

$$\{M'\} = \begin{bmatrix} M'_1 \\ M'_2 \\ M'_3 \\ M'_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} -24 \\ -48 \\ -4.5 \\ -1.5 \end{bmatrix} + \begin{bmatrix} -54 \\ 54 \\ -1.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} -78 \\ 6 \\ -6 \\ 0 \end{bmatrix} \text{kN•m}$$

Stiffness Matrix

For Trusses

$$[S] = \begin{array}{cc} & \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & \cdots & e_n \end{array} \\ \begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ \vdots \\ F_n \end{array} & \begin{bmatrix} \frac{A_1E}{L_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{A_2E}{L_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{A_3E}{L_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{A_4E}{L_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{A_5E}{L_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{A_nE}{L_n} \end{bmatrix} \end{array}$$

For Beams and Frames

$$[S] = \begin{array}{cc} & \begin{array}{cccccc} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \cdots & \cdots \end{array} \\ \begin{array}{c} M_1 \\ M_2 \\ M_3 \\ M_4 \\ \vdots \\ \vdots \end{array} & \begin{bmatrix} \frac{4EI_1}{L_1} & \frac{2EI_1}{L_1} & 0 & 0 & \cdots & \cdots \\ \frac{2EI_1}{L_1} & \frac{4EI_1}{L_1} & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \frac{4EI_2}{L_2} & \frac{2EI_2}{L_2} & \cdots & \cdots \\ 0 & 0 & \frac{2EI_2}{L_2} & \frac{4EI_2}{L_2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array}$$

Reference:

Wang, C. K., *Introductory Structural Analysis with Matrix Methods*, (1973)

McMinn, S. J., *Matrices for Structural Analysis*, (1962)

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