Elastica Examples, Methods and Modifications.

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Example
Beam with Varied Stiffness

Intervals

n	a_n	b_n
1	0	10
2	10	20
3	20	30

$$M_1(x) = \frac{20}{EI}$$

$$M_2(x) = \frac{20}{2EI} = \frac{10}{EI}$$

$$M_3(x) = \frac{20}{EI}$$

EI $M_n(x)$

n	$M_n(x)$
1	= 20
2	= 10
3	= 20

$\Longrightarrow$

EI $DP_n(x) = \int M_n(x)\, dx$

n	$DP_n(x)$
1	= 20 x
2	= 10 x
3	= 20 x

$\Longrightarrow$ *Elastica*

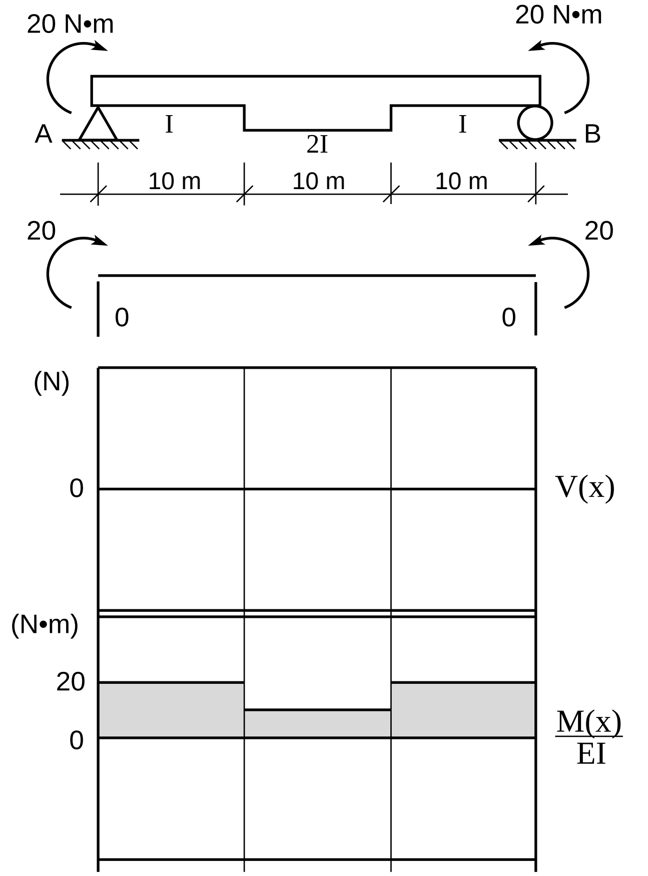
3 intervals
6 unknowns
6 equations

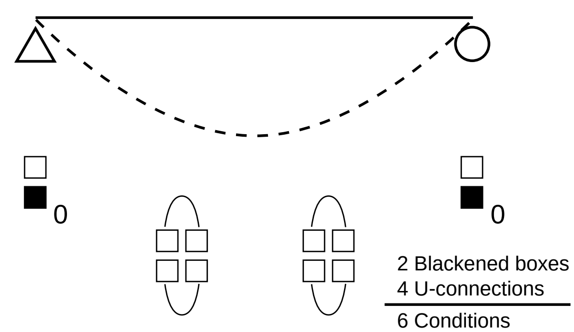
$\left.\begin{array}{c}\text{End and}\\ \text{continuity}\\ \text{conditions}\end{array}\right\} \Longrightarrow$ *Elastica*

$$\begin{bmatrix} 10 & 1 & -10 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 20 & 1 & -20 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 30 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} = \begin{bmatrix} -500 \\ 0 \\ 100 \\ 2000 \\ -200 \\ -9000 \end{bmatrix}$$

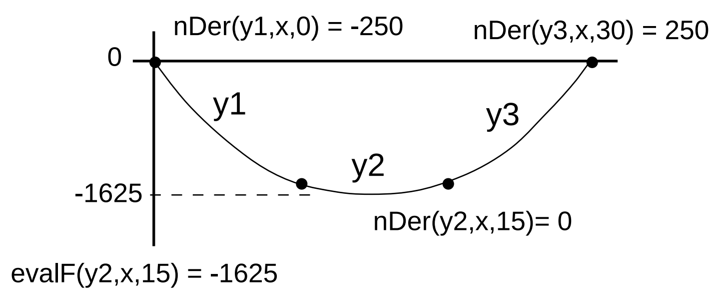
$$\begin{bmatrix} 10 & 1 & -10 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 20 & 1 & -20 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 30 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -500 \\ 0 \\ 100 \\ 2000 \\ -200 \\ -9000 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix}$$

$$\textit{Elastica} \Longrightarrow \text{CMat} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} = \begin{bmatrix} -250 \\ 0 \\ -150 \\ -500 \\ -350 \\ 1500 \end{bmatrix} \Longrightarrow$$





End and Continuity Conditions



Elastic Curve

Example
Beam with Internal Hinge

Intervals

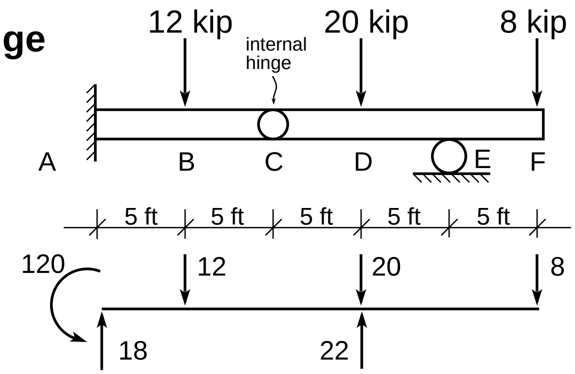
n	a_n	b_n
1	0	5
2	5	10
3	10	15
4	15	20
5	20	25

$M_1(x) = 18x - 120$

$M_2(x) = 6x - 60$

$M_3(x) = -14x + 240$

$M_4(x) = 8x - 200$



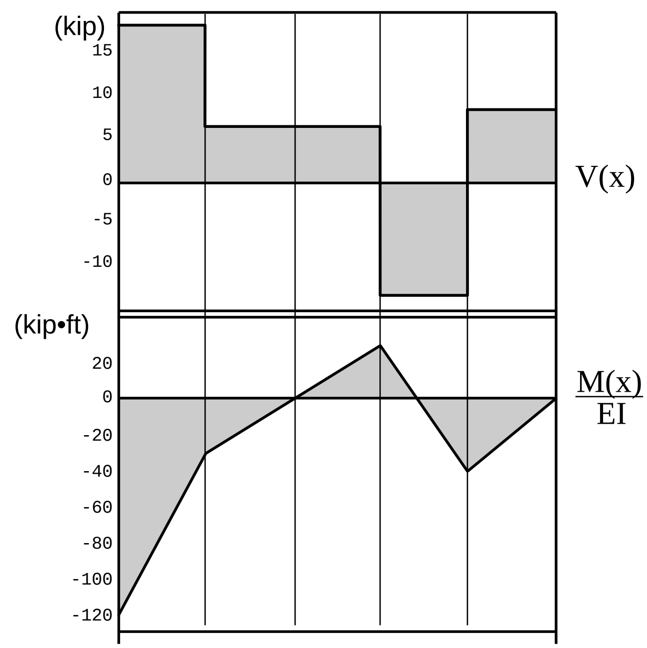
EI $M_n(x)$ EI $DP_n(x) = \int M_n(x)\, dx$

n	$M_n(x)$
1	$= 18x - 120$
2	$= 6x - 60$
3	$= 6x - 60$
4	$= -14x + 240$
5	$= 8x - 200$

$\Longrightarrow$

n	$DP_n(x)$
1	$= 9x^2 - 120x$
2	$= 3x^2 - 60x$
3	$= 3x^2 - 60x$
4	$= -7x^2 + 240x$
5	$= 4x^2 - 200x$

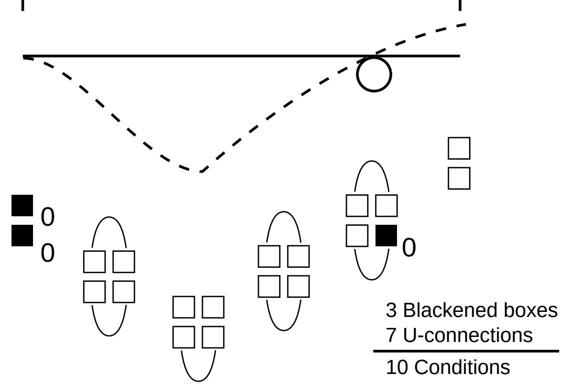
$\Longrightarrow$ *Elastica*



5 intervals
10 unknowns
10 equations

End and continuity conditions $\Longrightarrow$ *Elastica*

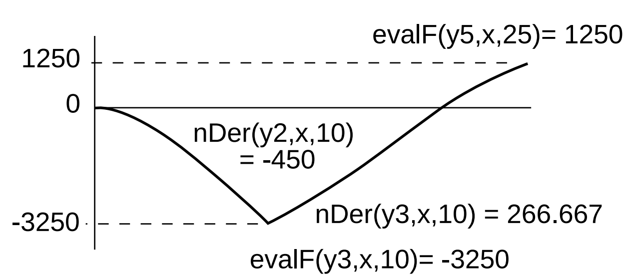
$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & 1 & -10 & -1 & 0 & 0 & 0 & 0 \\
5 & 1 & -5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 15 & 1 & -15 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 20 & 1 & -20 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 1
\end{bmatrix}
\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ -150 \\ 0 \\ 500 \\ 22500 \\ -2250 \\ -5866.667 \\ 4400 \\ 29333.33 \end{bmatrix}
$$



End and Continuity Conditions

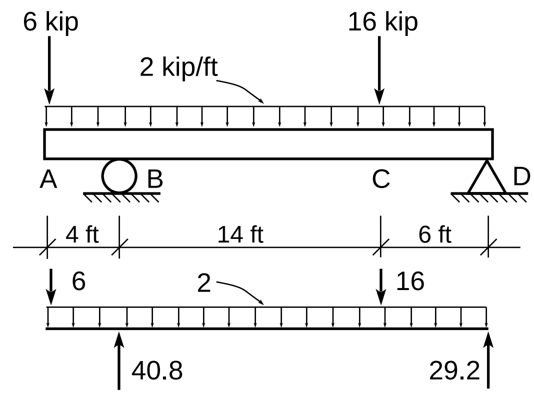
$$\text{AMat}^{-1} * \text{BMat} = \text{CMat}$$

$$
\textit{Elastica} \Longrightarrow \text{CMat} =
\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ -150 \\ 250 \\ 566.667 \\ -6916.667 \\ -1683.333 \\ 433.333 \\ 2716.666 \\ -25000 \end{bmatrix}
\Longrightarrow
$$



Elastic Curve

Example
Cantilever Beam with Distributed Load



Intervals

n	a_n	b_n
1	0	4
2	4	18
3	18	24

$M_1(x) = -6x - x^2$

$M_2(x) = -163.2 + 34.8x - x^2$

$M_3(x) = 124.8 + 18.8x - x^2$

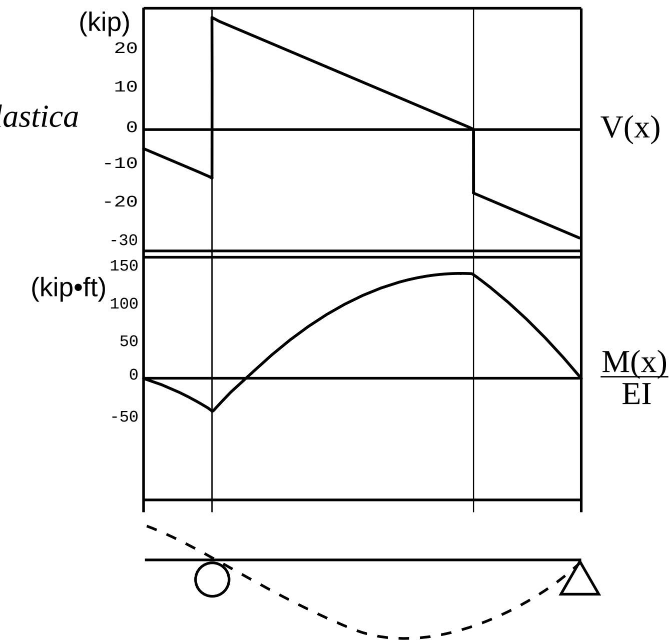
EI $M_n(x)$ EI $DP_n(x) = \int M_n(x)\,dx$

n	$M_n(x)$
1	$= M_1(x)$
2	$= M_2(x)$
3	$= M_3(x)$

⇒

n	$DP_n(x)$
1	$= -3x^2 - (1/3)x^3$
2	$= -163.2x + 17.4x^2 - (1/3)x^3$
3	$= 124.8x + 9.4x^2 - (1/3)x^3$

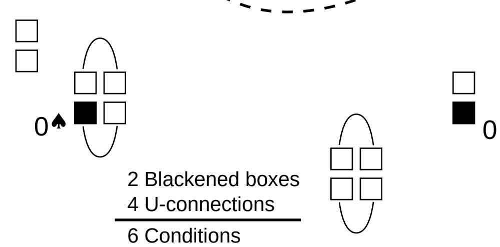
} ⇒ *Elastica*



3 intervals
6 unknowns
6 equations

End and continuity conditions } ⇒ *Elastica*

$$\text{endCond} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

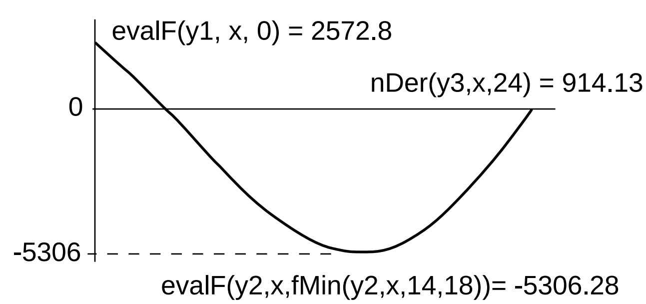


End and Continuity Conditions

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & -4 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 18 & 1 & -18 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 24 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} = \begin{bmatrix} 256/3 \\ -4352/5 \\ 1632/5 \\ 31104 \\ -2592 \\ -51609.6 \end{bmatrix}$$

$$\text{AMat}^{-1} * \text{BMat} = \text{CMat}$$

$$\textit{Elastica} \Rightarrow \text{CMat} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} \text{"▸Frac"} = \begin{bmatrix} -9328/15 \\ 12864/5 \\ -4432/15 \\ 10688/5 \\ -43312/15 \\ 17689.6 \end{bmatrix} \Rightarrow$$

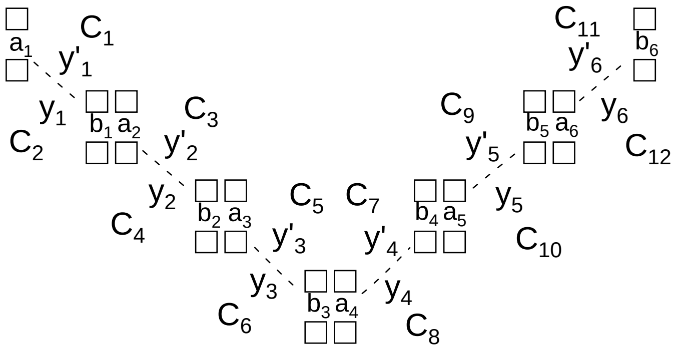


Elastic Curve

♠ This condition is a special case. Elastica gives the option of using $C_2b_1 = n$ when the boxes C_1a_1 and C_2a_1 are both off. $C_2b_1 = 0$ is not added to the endCond matrix because it would prevent $C_2b_1 == C_4a_2$ from being used as a condition. Both $C_2b_1 = n$, and $C_2b_1 == C_4a_2$ are needed in this case for a solution.

General Relationships

End Condition Free-Body (General Case)



Each box is unique:

C_1a_1 ; C_1b_1 ; C_3a_2 ; C_3b_2 ; C_5a_3 ; C_5b_3 ; C_7a_4 ; C_7b_4 ; C_9a_5 ; C_9b_5 ; $C_{11}a_6$; $C_{11}b_6$
C_2a_1 ; C_2b_1 ; C_4a_2 ; C_4b_2 ; C_6a_3 ; C_6b_3 ; C_8a_4 ; C_8b_4 ; $C_{10}a_5$; $C_{10}b_5$; $C_{12}a_6$; $C_{12}b_6$

or,

y'_1a_1 ; y'_1b_1 ; y'_2a_2 ; y'_2b_2 ; y'_3a_3 ; y'_3b_3 ; y'_4a_4 ; y'_4b_4 ; y'_5a_5 ; y'_5b_5 ; y'_6a_6 ; y'_6b_6
y_1a_1 ; y_1b_1 ; y_2a_2 ; y_2b_2 ; y_3a_3 ; y_3b_3 ; y_4a_4 ; y_4b_4 ; y_5a_5 ; y_5b_5 ; y_6a_6 ; y_6b_6

The Elastica

$$\frac{M(x)}{EI} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

Elastic Curve (for dy/dx ≈ 0 ; vertical displacements)

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \qquad \left(\text{Or,} \quad \frac{d^4y}{dx^4} = \frac{-w}{EI}\right)$$

$$y(x) = \int\int \frac{M(x)}{EI}\, dx\, dx$$

$$EI\, y(x) = F(x) + C_1x + C_2$$

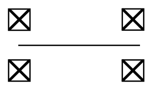
For a six-interval Elastic Curve, with EI = 1:

$$\left.\begin{aligned}
y_1(x) &= F_1(x) + C_1x + C_2\\
y_2(x) &= F_2(x) + C_3x + C_4\\
y_3(x) &= F_3(x) + C_5x + C_6\\
y_4(x) &= F_4(x) + C_7x + C_8\\
y_5(x) &= F_5(x) + C_9x + C_{10}\\
y_6(x) &= F_6(x) + C_{11}x + C_{12}
\end{aligned}\right\}\ \begin{array}{l}\text{6 intervals}\\ \text{6 equations}\\ \text{12 unknowns}\end{array}$$

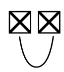
Twelve equations are needed to find the twelve unknowns.

“Bad” Patterns of End and Continuity Conditions on Free-Body

(In general, specify ends first.)

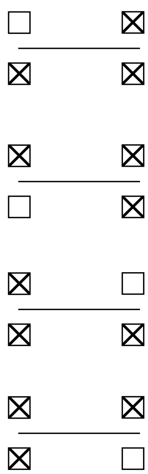


quadrangle

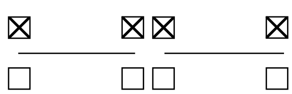


drunk

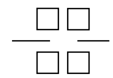
hanging drunk



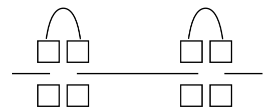
triangles



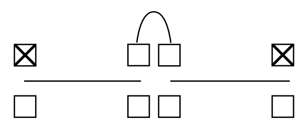
surface



ancloverleaf



roller skates



airplane

Trouble Shooting

Check DPi for i = 1 to n. All terms should involve x.

Check DPi for i = 1 to n. Check for correct sign. Use the negation function, (-), where it is appropriate.

Check the list of intervals, (a,b), in the TI list, limLst. There should be 2n items in the list.

Make sure the Elas graph database is "recalled" into the y-editor. Make sure y1 to y6 are deselected in the y-editor.

Use EUtility to reinstall the few resources Elastica uses. Once installed, Elastica should run repeatedly.

If Elastica is interrupted, don't use "GOTO" because it takes a long time to re-compile.

Solving an Elastic Curve with More Than 6 Intervals

Elastica's algorithm can solve for a greater number of intervals than the default n of six (where n is the number of intervals along the curve).

Set Elastica to solve for n intervals by using EUtility.

You will not be able to plot the curve or use the y-editor to evaluate the curve. Elastica will simply build the ABMat matrix and with good conditions, solve the curve (give the CMat matrix).

Elastica will offer to plot the curve, but the functions it uses and that algorithm are specifically for 6 or fewer intervals. They will not work with an n > 6.

To set Elastica to solve for n intervals (n > 6), use EUtility and the program editor and change the lines of code on the left, below, to the n intervals, as patterned on the right.

Make the changes to EUtility, run EUtility to install the new resources, then use Elastica as before, but don't use the plot function, and don't rely on the functions in the y-editor to evaluate the curve, unless, of course, your particular problem has 6 or fewer intervals. (i.e. You can solve for n intervals, and for 6 or fewer intervals, you can also plot the curve and use the functions in the y-editor to evaluate the curve.)

Change these lines in EUtility

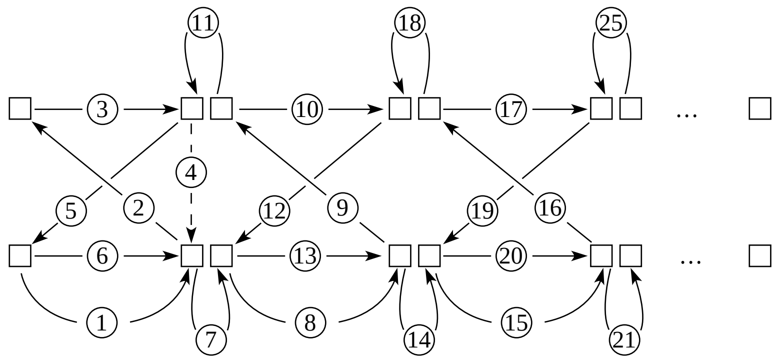
```
: {1,3,5,7,9,11} -> lwrLst
: {2,4,6,8,10,12} -> uprLst
: "DP1DP2DP3DP4DP5DP6DP7" -> elaStr
: {1,4,7,10,13,16,19} -> strLst
```

Use this pattern.

```
: {1,3,5,7,9,11,…,(2*n)–1} -> lwrLst
: {2,4,6,8,10,12,…,(2*n)} -> uprLst
: "DP1DP2DP3DP4DP5DP6DP7…DP(n+1)" -> elaStr
: {1,4,7,10,13,16,19,…,(n*3)+1} -> strLst
```

Where n is the number of intervals (n > 6).

Elastica's Procedural Method
(marching methods)



Steps 1, 8, 15… look for known "end-points-on-an-interval." For example, "$y_1(a_1) = q$ *and* $y_1(b_1) = r$"

Steps 11, 18, 25… look for y'(x)-relationships.

Steps 7, 14, 21… look for y(x)-relationships.

All other steps look for "end-condition values".

See also the Cantilever-beam-with-distributed-loading example about Step 4 above; and see the discussion on procedure in the "Elastica guide," (main example).

Evaluating the Elastic Curve

Any point on the elastic curve can be evaluated (when the y-functions from the Elas graph database are in the y-editor of the GRAPH application.):

evalF(y1, x, 5) = y1(5)
evalF(y2, x, 5) = y2(5)

And any derivative on the curve can be evaluated:

nDer(y1, x, 5) = y'1(5)
nDer(y2, x, 5) = y'2(5)

And the area under any interval of y(x) can be found by using Simpson's rule, i.e.:

Simpson(y1, x, 0, 6) = y1(6) - y1(0)
Simpson(y2, x, 6, 12) = y2(12) - y2(6)

The moments can be found by evaluating DPn, i.e.

der1(DP1, x, 6) = y"1(6)
der1(DP2, x, 6) = y"2(6)

And similarly, the shear at any point is

der2(DP1, x, 6) = y'"1(6)
der2(DP2, x, 6) = y'"2(6)

Note: Because the elastic curve, y(x), (in the y-editor) is itself an integrand (see the y-editor) it is "illegal" in the GRAPH application, in another integral (fnInt()) and in the "exact" derivative functions. This is of no consequence, as is demonstrated above.

Summary of Elastica Procedure

The procedure is always to
1. Determine the moment equations
2. Determine the number of intervals, n (equals the number of moment equations plus the number of internal hinges).
3. Integrate $M_i(x)$ with respect to x (ignoring the constants of integration) and assign that result to DPi, for i = 1 to n.
4. Draw a free-body of the end conditions and identify the 2*n equations to use to solve the curve on the free-body.
5. Run the program and answer its queries based on the free-body.

Elastica's objects

"AMat" is the "left-hand-side" of the equations specified.

"BMat" is the "right-hand-side" of the equations specified.

That is: AMat * CMat = BMat (#1)

And from the operations [AMat]^(-1) * BMat = CMat (#2)
we get the solution to the curve, the constants C_1 to C_n.

But CMat is a row matrix, and from the function Elastica uses to solve (#2) above, that row matrix is assigned to a vector. Under the "VECTR" menu you'll find "CMat."

Under the "MATRX" menu is the "ABMat." It is an augmented matrix (AMat and BMat) and contains all the relations Elastica generates.

With the ABMat matrix, you can elide rows "by hand", and then use "rref" (reduced row echelon form) under the "MATRX" "OPS" menu to find solutions.

The "endCond" matrix contains the "end conditions" specified (values of y'(x) and y(x)), not the relations. The transpose the endCond matrix is the free-body for the "end conditions."

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