Elastica determines the 2*n constants associated with the double integration of n piecewise functions and plots the resulting curve. Specifically Elastica solves boundary value problems of elastic curves.

Elastica is set up to solve and plot an elastic curve that has 1 to 6 intervals. However, the algorithm for determining the 2*n unknowns is general, and can be set to handle a greater number of intervals.

Elastica introduces and makes use of an end-conditions free-body. The free-body allows a graphical method of analyzing and specifying the end and continuity conditions needed to solve the curve. Though the method is described here, it can be more easily understood if it can be seen. A "PDF" file, if not available with this document, may be had by sending an e-mail with the words "Elastica method" in either the subject heading or in the first line of text to mcb4588@omega.uta.edu

Setting Up

The Elastica Group file has 3 items:
Elastica (program)
EUtility (program)
Elas (graph database)

Run EUtility to install the few resources Elastica needs to run (a few small lists). EUtility can also be used to clear all the resources and variables Elastica uses or creates from memory, if you have a TI-86. (If you have a TI-85, you can print a copy of EUtility and use that to delete the variables by hand.) EUtility also provides a way to set Elastica to solve a curve with more than 6 intervals (to solve for more than 12 unknowns).

Conventions Used

Questions asked by the program are answered one of four ways.
1. To answer "no," enter 0 (zero)(binary false).
2. To answer "yes," enter 1 (one)(binary true).
3. To answer a datum query, key in the datum.
4. To answer a question in the form "A or B," enter 0 for A, 1 for B.

Elastica needs the Elas graph database file to plot curves and to evaluate values and derivatives along the elastic curve. When you've finished using Elastica you can delete the Elas functions from the y-editor in the GRAPH application. Then, the next time you use Elastica, you'll have to recall the Elas graph database from the GRAPH menu before you can plot or evaluate the elastic curve. (Elastica does not need the Elas graph database to solve the elastic curve.)

Example and Step by Step

First sketch a free body of the beam. Let the beam be fix-connected on the left-hand side (label it A) and roller supported on the right-hand side (label it B). Let the length be 12 meters. At the center of the beam is one 50-Newton concentrated load acting downward. This is a first-degree indeterminate beam problem. The flexural rigidity of the beam, EI, for our purpose, can be unity, 1.

Our problem is to find the values of the 4 constants needed to describe the shape of the elastic curve.

First we need to find the number of intervals on the elastic curve. There are two moment equations needed to describe the beam. In general, with elastic curves, the number of intervals equals the number of moment equations plus the number of internal hinges (since internal hinges cause discontinuities on the elastic curve, but not on the moment diagram).

So we have two intervals, 0 to 6 and 6 to 12 meters. Now we can sketch a free body of the end-and-continuity conditions. Sketch the expected shape of the elastic curve for the beam. Let the curve be y(x). At A, x = 0. Draw two small squares one-on-top-of-the-other to represent the values of y'(0) and y(0) respectively. Then draw another pair of squares just to the left of y(6). Then another pair of squares just to the right of y(6); then one last pair of squares to the left of y(12).

The boxes representing y'(x) run along the top of the sketch of the elastic curve. The ones representing y(x) run along the bottom. These 8 boxes represent the "end conditions" of the curve. We can choose which ones of those to assign a value, and then give that to Elastica, which the program then stores in a matrix named "endCond," which it will use to solve the curve.

Because we have two intervals, we have also two equations that will be double integrated. So we have 4 constants to find, two for each interval. C_1 is associated with the derivative of y(x) over the first interval. C_2 is associated with y(x) over the first interval. C_3 is associated with the derivative of y(x) over the second interval, and C_4 is associated with y(x) over the second interval. If we let the intervals of y go from a to b, then we have along the curve a_1, b_1, a_2, b_2. Now each box is unique: C_1a_1, C_1b_1, C_2a_1, C_2b_1, C_3a_2, C_3b_2, C_4a_2, C_4b_2. Alternatively, the boxes can be referred to as y'_1a_1, y'_1b_1, y_1a_1, y_1b_1, y'_2a_2, y'_2b_2, y_2a_2, y_2b_2.

To solve for the 4 unknowns, we need 4 equations. We can specify four equations by using the end-condition values and relationships.

With our example we'll specify the value of y'_1a_1 as 0, and y_1a_1 as 0. That gives two equations: $y'_1(0)=0$, and $y_1(0)=0$. We need two more (linearly independent equations). We'll specify the relations, $y'_1b_1 = y'_2a_2$, and $y_1b_1 = y_2a_2$, which give the equations $y'_1(6) = y'_2(6)$, and $y_1(6) = y_2(6)$. That gives us a total of four linearly independent equations.

We want to be able to represent this on the end condition free body. To represent $y'_1(a_1) = 0$, darken that box, C_1a_1, and indicate the value by writing it just below and to the right of the box. To represent $y_1(a_1)=0$, darken that box, C_2a_1, and indicate its value by writing it just below and to the right of the box. To represent $y'_1(b_1) = y'_2(a_2)$, connect those two boxes, C_1b_1 and C_3a_2, by drawing an upside-down U above them. To represent $y_1(b_1) = y_2(a_2)$, connect those two boxes in a similar way, with a U-shape below the two boxes.

So we have four equations. They can be counted: two darkened squares and two U-connections. Provided that they are linearly independent and together involve the four constants, C_1 to C_4, we should have a solution to the curve.

Before we run Elastic we have two more steps. First we need to integrate the moment equations one time, and assign them to an equation. The moments are

$M_1(x) = 34.375\ x - 112.5$
$M_2(x) = -15.625\ x + 187.5.$

Integrating with respect to x we get

(34.375/2) x^2 - 112.5 x
(-15.625/2) x^2 + 187.5 x
(We don't need the constants of integration here.)

On the calculator we assign the integral associated with $M_1(x)$ to DP1, and assign the one associated with $M_2(x)$ to DP2:

DP1=(34.375/2) x^2 - 112.5 x
DP2=(-15.625/2) x^2 + 187.5 x

(Be sure and use the "negation function," (-), when the negative sign is out in front as in DP2.)

The last step before running Elastica is to set the x-limits in the GRAPH application with the "RANGE" or "WIND" menu. We can set xMin to 0, and xMax to 12. (We could guess at the y-range, but we won't.)

Now run Elastica.
To the prompt (comments are in parenthesis)

Length?	enter 12 (for 12 meters, the length of the beam)
Intervals?	enter 2
Limit list?	enter the list: {0,6,6,12} (This is a TI List, the intervals in the form (a,b))
endCond exists Use it?	enter 0 (for No. The endCond matrix, in our case, should not have the end conditions to our problem. This prompt allows us to bypass input-of-end-conditions into the endCond matrix.)
y'(x) , rhs of 0 is known? huh?	enter 1 (for Yes. From our free-body of end-conditions, we want to use the value of y'(x) at the right-hand-side of zero as an equation.)
y'(x) = ?	enter 0 (from our free-body we said that was equal to zero (the slope at A is zero, because that end is fixed).)
y(x), rhs of 0 is known huh?	enter 1 (from the free-body..., we blackened that box.)

$y(x) = ?$	enter 0 (we said it was zero. There is no displacement at A.)
$y'(x)$ lhs of 6 is known? huh?	enter 0 (From the free-body, that box is off, not blackened.)
$y(x)$ lhs of 6 is known? huh?	enter 0
$y'(x)$ rhs of 6 is known? huh?	enter 0
$y(x)$ rhs of 6 is known? huh?	enter 0
$y'(x)$ lhs of 12 is known? huh?	enter 0
$y(x)$ lhs of 12 is known? huh?	enter 0 (We know $y(x)$ is zero here, but we don't need another condition. So we don't enter it.)
$y_1(b_1) == y_2(a_2)$?	enter 1 (for Yes. From the free-body, we want to use this relation to determine the constants and solve the curve.)
Let f, g be y 2 1 f'(a) == g'(b)?	enter 1 (for Yes. We want to use the relation $y'_1(b_1) = y'_2(a_2)$. That's what this awkward question is asking. You can write it out: $f = y_2$; $g = y_1$; $f'(a) = y'_2(a)$; $g'(b) = y'_1(b)$...)

Is C4 equal to 0?	enter 0 (for No. If C_4 is zero, then $y_2(0) = 0$. That is not apparent from the sketch of the elastic curve, so we answer No. Elastica here has ended up with an equation $C_4 = 0$. To understand why, one can study the free-body of the end conditions with an understanding of how Elastica proceeds algorithmically. But never mind for now. By answering No Elastica sets the relation $C_4 = 0$ aside, and uses the four we specified.
Plot curve?	enter 1
Include prev. curve?	enter 0 (for No. In our case, we have no previous curve to include to plot over.)

Because we did not specify the y-range in the GRAPH application, we (probably) won't be able to see the elastic curve. So we can interrupt the graphing by pressing "ON." Then press "QUIT." (Don't press "GOTO" because you'll cause Elastica to be recompiled the next time you run it, which takes a long time.) We'll find the range and then re-plot the curve.

Now to find the lower limit of y, from the sketch of the curve we know it ought to be around x = 6 or 7. That's in the y_2 interval (a_2_b_2). Now go to the "CALC" menu and key in this:

evalF(y2, x, fMin(y2, x, 6, 8))

This does two things, fMin(y2, x, 6, 8) finds the x where y(x) is the minima in the interval of x = 6 to x = 8.

evalF(y2, x, fMin(y2, x, 6, 8)) then uses that x to find the value of y(x).

The value is -805.

We can now set the y-range in the GRAPH application using the "RANGE" or "WIND" menu. We'll set yMin = -900 and yMax = 100.

Exit to the home screen, then run Elastica again (we'll skip most of the input this time).

Length?	enter 12
Intervals?	enter 2
Limit list?	enter {0,6,6,12} (You can use the "LIST" menu to find "limLst" and answer this prompt with that, since it contains the limits keyed in last time, if you don't want to re-key in the list.)
endCond exists Use it?	enter 1 (for Yes. We don't want to re-key in the "end conditions.")
Calc. constants?	enter 0 (for No. We don't want to re-solve the curve.)
Plot curve?	enter 1 (Yes. Plot the curve.)
Include prev. curve?	enter 0 (No. We have no previous curve to include to be plotted over.)

Now the elastic curve plots. You can press "ON" to get out of plotting it. If you'll let it plot, you'll see the minima occurs between 6 and 7 as we anticipated.

Elastica's objects

Under the "MATRX" menu:

"AMat" is the "left-hand-side" of the equations we specified.

"BMat" is the "right-hand-side" of the equations we specified.

That is: AMat * CMat = BMat   (#1)

And from the operations [AMat]^(-1) * BMat = CMat   (#2)
we get the solution to the curve, the constants C_1 to C_4.

But CMat is a row matrix, and from the function Elastica uses to solve (#2) above, that row matrix is assigned to a vector. So CMat is a vector. Under the "VECTR" menu you'll find "CMat," which is [0, 0, -900, 1800]. And the solution to the curve is

For x = 0 to 6:
y(x) = f(x) + (0) x + 0

And for x = 6 to 12:
y(x) = g(x) -900 x + 1800

Back under the "MATRX" menu is the "ABMat." It is an augmented matrix (AMat and BMat) and contains all the relations Elastica generated. The rows contain the equations in the order they were evaluated by Elastica with respect to C_i. Equations below row n are extra relations that are added when Elastica does not find a relation specifically for C_i.

Elastica's Procedural Set Up

For example, we did not specify a relation specifically for C_4. Algorithmically, the relation we specified, $y_1(b_1) = y_2(a_2)$, provided an equation for C_2, not for C_4. The equation relates both C_2 and C_4, but procedurally, the equation $y_i(b_i) = y_{i+1}(a_{i+1})$ is associated with C_i(even), not C_{i+1}(even). Conversely, by algorithm, Elastica associates the equation $y_i'(b_i) = y_{i+1}'(a_{i+1})$ to C_{i+1}(odd) not C_i(odd).

In our case, $y'_1(0)=0$ was associated with row 1 of the ABMat, with C_1. $y_1(0)=0$ was associated with row 2 of the ABMat, with C_2. $y'_1(6) = y'_2(6)$ was associated with C_{i+1}(odd), (with row 3, with C_3). And since we had already associated an equation with C_2, the relation $y_1(6) = y_2(6)$ was an "extra" equation, and was put into row n+1 of the ABMat. When C_4 was evaluated by Elastica, there was no relation assigned to it. It kept its default relation $C_4 = 0$ associated with it. At the end of the algorithm Elastica asked us if we wanted to keep that equation: Is C_4=0? We said No. Elastica then substituted that equation with the extra one in row 5 into the AMat and BMat, and solved the curve.

With the ABMat matrix, you can elide rows "by hand", since it contains all the equations generated by Elastica, and then use "rref" (reduced row echelon form) under the "MATRX" "OPS" menu to find solutions.

End Condition Analogy

Finally, there is the "endCond" matrix. It contains the "end conditions" we specified (values of y'(x) and y(x)), not the relations. Now the transpose the endCond matrix is the free-body for the "end conditions."

endCond^T =
1 0 0 0
0 0 0 0
1 0 0 0
0 0 0 0

Row one represents $y'_1(a_1)$, $y'_1(b_1)$, $y'_2(a_2)$, $y'_2(b_2)$. Where the digit 1 represents "On" or the blackened box. The values in row two are the values we assigned. (They are zero by default, but that is superfluous except when, like in row-one-column-one, a box is turned on. In other words, all the values in row 2 are meaningless except for row-two-column-one, because its box (row-one-column-one) is turned on.)

Row three holds the analogous relations for y(x). That is row three represents $y_1(a_1)$, $y_1(b_1)$, $y_2(a_2)$, $y_2(b_2)$. And we specified a value for $y_1(a_1)$, we said it was zero. Hence $y_1(a_1)$ (row-three-column-one) is "On" and the value below it (row-four-column-one) has meaning.
Graphically, the transpose of the endCond matrix and the "end conditions" on the free-body for the end conditions are alike.

(The end condition values take on more prominence with some other boundary-continuity-value problems, i.e. elastic curves for frames and where the corollary to the conjugate beam theorem is used.)

Here are the main matrices from the example problem.

ABMat =
1 0 0 0 0
0 1 0 0 0
-1 0 1 0 -900
0 0 0 1 0
6 1 –6 –1 3600

CMat =
0 0 -900 1800

endCond =
1 0 1 0
0 0 0 0
0 0 0 0
0 0 0 0

Evaluating the Elastic Curve

Any point on the elastic curve can be evaluated (when the y-functions from the Elas graph database are in the y-editor of the GRAPH application.):

evalF(y1, x, 5) = -690.1
evalF(y2, x, 10) = -429.16

And any derivative on the curve can be evaluated:

nDer(y1, x, 2) = -156.25
nDer(y2, x, 12) = 225

And the area under any interval of y(x) can be found by using Simpson's rule, i.e.:

Simpson(y1, x, 0, 6) = - 2193.75
Simpson(y2, x, 6, 12) = -3206.25

The moments can be found by evaluating DPn, i.e.

der1(DP1, x, 6) = 93.75
der1(DP2, x, 6) = 93.75

And similarly, the shear at any point is

der2(DP1, x, 6) = 34.375
der2(DP2, x, 6) = -15.625

Note: Because the elastic curve, y(x), (in the y-editor) is itself an integrand (see the y-editor) it is "illegal" in the GRAPH application, in another integral (fnInt()) and in the "exact" derivative functions. This is of no consequence, as was demonstrated above.

More Examples

It is convenient and instructive to use Elastica to rework the example problem using its statically determinate components: That is to rework the problem as a both-ends-simply-supported problem, then as a right-end-cantilevered problem, and finally as a left-end-cantilevered problem. Both-ends-fixed is another variation.

The CMat matrices are:
Simply-supported: [-450, 0, -1350, 1800]
Right-end-cantilever (rhs overhanging) : [0, 0, -900, 1800]
Left-end-cantilever: [900, -900, 0, -7200]
Both-ends-fixed: [0, 0, -900, 1800]

Summary of Elastica Procedure

The procedure is always to
1. Determine the moment equations
2. Determine the number of intervals, n (equals the number of moment equations plus the number of internal hinges).
3. Integrate $M_i(x)$ with respect to x (ignoring the constants of integration) and assign that result to DPi, for i = 1 to n.
4. Draw a free-body of the end conditions and identify the 2*n equations to use to solve the curve on the free-body.
5. Run the program and answer its queries based on the free-body.

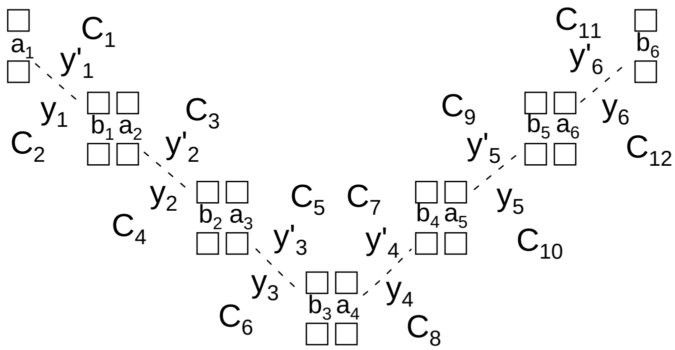
(To plot the curve, or to evaluate values along the curve, you need to have the Elas graph database installed in the GRAPH application, in the y-editor. This is not needed to solve for the constants, C_1 to C_{2n}. Elastica will either solve the curve based on the end and continuity conditions it was given, or, if too few equations and relations are given, it will tell you that AMat is not square. (In this case, see the ABMat matrix, and reanalyze the end and continuity conditions, redraw the free-body for them, and try again.) Or, if the end and continuity conditions Elastica uses (the AMat and BMat) to solve for the constants (C_1 to C_{2n}) are not linearly independent, the calculator will return the error message "Singularity." In this case, first, don't press the "GOTO" button because Elastica is working okay, and doing so would cause the program to be recompiled the next time you run it and that takes a long time. Second, do what is suggested for the case of the non-square matrix above (reanalyze the end and continuity conditions).)

For some worked examples (including one of a left-end-cantilever and one of a beam with internal hinges) and instructions for setting Elastica to solve curves with more than 12 unknowns, a "PDF" file may be had by sending an e-mail with the words "Elastica examples" in either the subject heading or in the first line of text to mcb4588@omega.uta.edu

Good luck.

Mark C. Bourland
The UTA graduate school
mcb4588@omega.uta.edu

End Condition Free-Body (General Case)



Each box is unique:

C_1a_1 ; C_1b_1 ; C_3a_2 ; C_3b_2 ; C_5a_3 ; C_5b_3 ; C_7a_4 ; C_7b_4 ; C_9a_5 ; C_9b_5 ; $C_{11}a_6$; $C_{11}b_6$
C_2a_1 ; C_2b_1 ; C_4a_2 ; C_4b_2 ; C_6a_3 ; C_6b_3 ; C_8a_4 ; C_8b_4 ; $C_{10}a_5$; $C_{10}b_5$; $C_{12}a_6$; $C_{12}b_6$

or,

y'_1a_1 ; y'_1b_1 ; y'_2a_2 ; y'_2b_2 ; y'_3a_3 ; y'_3b_3 ; y'_4a_4 ; y'_4b_4 ; y'_5a_5 ; y'_5b_5 ; y'_6a_6 ; y'_6b_6
y_1a_1 ; y_1b_1 ; y_2a_2 ; y_2b_2 ; y_3a_3 ; y_3b_3 ; y_4a_4 ; y_4b_4 ; y_5a_5 ; y_5b_5 ; y_6a_6 ; y_6b_6

The Elastica

$$\frac{\mathrm{M}(x)}{\mathrm{EI}} = \frac{\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}$$

Elastic Curve (for dy/dx ≈ 0 ; vertical displacements)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{M}(x)}{\mathrm{EI}} \qquad \left(\text{Or,} \quad \frac{\mathrm{d}^4\mathrm{y}}{\mathrm{d}x^4} = \frac{-w}{\mathrm{EI}}\right)$$

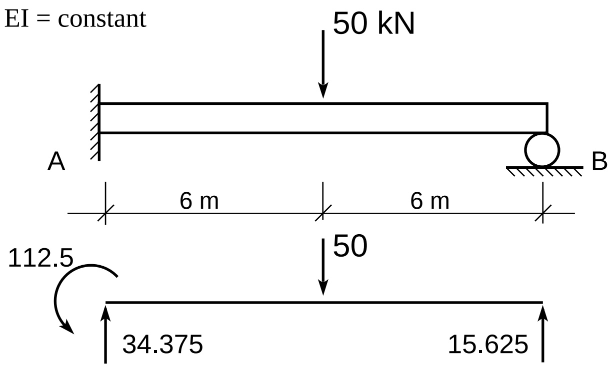
$$y(x) = \int\int \frac{\mathrm{M}(x)}{\mathrm{EI}}\,\mathrm{d}x\;\mathrm{d}x$$

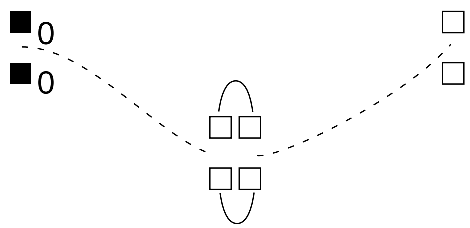
$$\mathrm{EI}\;y(x) = \mathrm{F}(x) + \mathrm{C}_1x + \mathrm{C}_2$$

For a six-interval Elastic Curve, with EI = 1:

$$\left.\begin{aligned}
y_1(x) &= \mathrm{F}_1(x) + \mathrm{C}_1\,x + \mathrm{C}_2\\
y_2(x) &= \mathrm{F}_2(x) + \mathrm{C}_3\,x + \mathrm{C}_4\\
y_3(x) &= \mathrm{F}_3(x) + \mathrm{C}_5\,x + \mathrm{C}_6\\
y_4(x) &= \mathrm{F}_4(x) + \mathrm{C}_7\,x + \mathrm{C}_8\\
y_5(x) &= \mathrm{F}_5(x) + \mathrm{C}_9\,x + \mathrm{C}_{10}\\
y_6(x) &= \mathrm{F}_6(x) + \mathrm{C}_{11}\,x + \mathrm{C}_{12}
\end{aligned}\right\} \begin{array}{l}\text{6 intervals}\\ \text{6 equations}\\ \text{12 unknowns}\end{array}$$

Twelve equations are needed to find the twelve unknowns.





End Condition Free-Body for Example Problem

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