AMATH 583: Exam 1

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Problem 1

(a) The given functions are linearly dependent, since:

$$1 \cdot f_1(x) + 2 \cdot f_2(x) - 1 \cdot f_3(x) = x^2 - 3 + 4 - 2x - x^2 + 2x - 1 = 0$$

(b) The given functions are linearly dependent, since:

$$e \cdot f_1(x) - 1 \cdot f_2(x) = e \cdot e^x - e^{x+1} = e^{x+1} - e^{x+1} = 0$$

(c) The given functions are linearly independent. Suppose c_1, c_2, c_3 are constants that satisfy:

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$$

If $x = 0$, we have:

$$c_1 + c_2 + c_3 = 0$$

If $x = 1$, we have:

$$c_1 e + c_2 e^2 + c_3 e^3 = 0$$

If $x = 2$, we have:

$$c_1 e^2 + c_2 e^4 + c_3 e^6 = 0$$

Using matrix form to solve for the coefficients c_1, c_2, c_3, we have:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ e & e^2 & e^3 & 0 \\ e^2 & e^4 & e^6 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & e(e-1) & e(e^2-1) & 0 \\ 0 & e^2(e^2-1) & e^2(e^4-1) & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & e(e-1) & e(e^2-1) & 0 \\ 0 & 0 & e^3(e^4-1)(e+1) & 0 \end{array}\right)$$

Using back substitution, we get:

$$c_1 = c_2 = c_3 = 0$$

Thus, the given functions are linearly independent.

Problem 2

(a) First, consider:

$$c_1 + c_2(1-x) + c_3(1-x)^2 = 0$$

where c_1, c_2, c_3 are constants. If $x = 0$, we have:

$$c_1 + c_2 + c_3 = 0$$

If $x = 1$, we have:

$$c_1 = 0$$

If $x = 2$, we have:

$$c_1 - c_2 + c_3 = 0$$

Based on the three equations above, we get:

$$c_1 = c_2 = c_3 = 0$$

Hence, the polynomials $\{1, 1-x, (1-x)^2\}$ are linearly independent.
Next, let $c_1x^2 + c_2x + c_3$ be an arbitrary polynomial in the space of quadratic polynomials, where c_1, c_2, c_3 are constants. Then, we have:

$$\begin{aligned} c_1+c_2+c_3+(-2c_1-c_2)(1-x)+c_1(1-x)^2 &= c_1+c_2+c_3-2c_1-c_2+2c_1x+c_2x+c_1x^2-2c_1x+c_1 \\ &= c_1x^2 + c_2x + c_3 \end{aligned}$$

Hence, the polynomials $\{1, 1-x, (1-x)^2\}$ span the space of quadratic polynomials.
Thus, the polynomials $\{1, 1-x, (1-x)^2\}$ form a basis for the space of quadratic polynomials.

(b) Based on part **(a)**, we have:

$$c_1x^2 + c_2x + c_3 = 1 + x^2$$

when $c_1 = 1, c_2 = 0, c_3 = 1$. Then, we get:

$$\begin{aligned} a &= c_1 + c_2 + c_3 = 1 + 0 + 1 = 2 \\ b &= -2c_1 - c_2 = -2 \\ c &= c_1 = 1 \end{aligned}$$

Therefore, we have:

$$1 + x^2 = 2 \cdot (1) - 2 \cdot (1-x) + 1 \cdot (1-x)^2$$

Problem 3

(a) We prove the 4 axioms for the norm:

(1) Note that:

$$\begin{aligned} &|f(1)| \geq 0 \quad \text{for all } f \in X \\ &|f'(x)| \geq 0 \quad \text{for all } f \in X \text{ and } 0 \leq x \leq 1 \end{aligned}$$

Hence, it is clear that:

$$N(f) = |f(1)| + \max_{0 \leq x \leq 1} |f'(x)| \geq 0$$

for all $f \in X$.

(2) Let $N(f) = 0$ for some $f \in X$. Then, we have:

$$|f(1)| = 0 \quad \text{and} \quad \max_{0 \leq x \leq 1} |f'(x)| = 0$$

Then, we know that $f'(x) = 0$, which means that $f(x) = c$ where c is a constant. In addition, since $|f(1)| = 0$, it means that $f(1) = 0 = c$. Hence, $f(x) = 0$.

(3) Let α be a constant and $f \in X$. Then, we have:

$$\begin{aligned} N(\alpha f) &= |\alpha f(1)| + \max_{0 \leq x \leq 1} |\alpha f'(x)| = |\alpha| \cdot |f(1)| + \max_{0 \leq x \leq 1} |\alpha| \cdot |f'(x)| \\ &= |\alpha| \left(|f(1)| + \max_{0 \leq x \leq 1} |f'(x)| \right) \\ &= |\alpha| \cdot N(f) \end{aligned}$$

(4) Let $f, g \in X$. Then, we have:

$$N(f+g) = |f(1) + g(1)| + \max_{0 \leq x \leq 1} |f'(x) + g'(x)|$$

By the triangle inequality of absolute value, we further have:

$$\begin{aligned} N(f+g) &= |f(1)| + |g(1)| + \max_{0 \leq x \leq 1} (|f'(x)| + |g'(x)|) \\ &= |f(1)| + \max_{0 \leq x \leq 1} |f'(x)| + |g(1)| + \max_{0 \leq x \leq 1} |g'(x)| \\ &= N(f) + N(g) \end{aligned}$$

Thus, N is a norm of X.

Problem 4

(a) We compute:

- $(x_1, x_1) = \int_{-1}^{1} t^4 \, dt = \left(\frac{t^5}{5} \Big|_{-1}^{1} \right) = \frac{2}{5}$
- $q_1(t) = \frac{x_1}{\sqrt{(x_1, x_1)}} = \sqrt{\frac{5}{2}} t^2$
- $\widetilde{q_2}(t) = x_2 - (x_2, q_1)q_1 = t - \sqrt{\frac{5}{2}} t^2 \int_{-1}^{1} \sqrt{\frac{5}{2}} t^3 \, dt = t - \frac{5}{2} t^2 \left(\frac{t^4}{4} \Big|_{-1}^{1} \right) = t$

  $(\widetilde{q_2}(t), \widetilde{q_2}(t)) = \int_{-1}^{1} t^2 \, dt = \left(\frac{t^3}{3} \Big|_{-1}^{1} \right) = \frac{2}{3}$

  $q_2(t) = \frac{\widetilde{q_2}(t)}{(\widetilde{q_2}(t), \widetilde{q_2}(t))} = \sqrt{\frac{3}{2}} t$
- $\widetilde{q_3}(t) = x_3 - (x_3, q_1)q_1 - (x_3, q_2)q_2 = 1 - \sqrt{\frac{5}{2}} t^2 \int_{-1}^{1} \sqrt{\frac{5}{2}} t^2 \, dt - \sqrt{\frac{3}{2}} t \int_{-1}^{1} \sqrt{\frac{3}{2}} t \, dt$

  $= 1 - \frac{5}{2} t^2 \left(\frac{t^3}{3} \Big|_{-1}^{1} \right) - \frac{3}{2} t \left(\frac{t^2}{2} \Big|_{-1}^{1} \right) = 1 - \frac{5}{3} t^2$

  $(\widetilde{q_3}(t), \widetilde{q_3}(t)) = \int_{-1}^{1} \left(1 - \frac{5}{3} t^2 \right)^2 dt = \int_{-1}^{1} 1 - \frac{10}{3} t^2 + \frac{25}{9} t^4 \, dt = \left(t - \frac{10}{3} t^3 + \frac{25}{9} t^5 \Big|_{-1}^{1} \right)$

  $= 2 \left(1 - \frac{10}{3} + \frac{25}{9} \right) = 2 \left(\frac{9}{9} - \frac{30}{9} + \frac{25}{9} \right) = \frac{8}{9}$

  $q_3(t) = \frac{\widetilde{q_3}(t)}{(\widetilde{q_3}(t), \widetilde{q_3}(t))} = \frac{3}{2\sqrt{2}} \left(1 - \frac{5}{3} t^2 \right) = \frac{3\sqrt{2}}{4} - \frac{5\sqrt{2}}{4} t^2$

Hence, we get:

$$\left\{ \sqrt{\frac{5}{2}} t^2, \ \sqrt{\frac{3}{2}} t, \ \frac{3\sqrt{2}}{4} - \frac{5\sqrt{2}}{4} t^2 \right\}$$

(b) Normality check:

(i) $(q_1, q_1) = \int_{-1}^{1} \frac{5}{2} t^4 \, dt = \left(\frac{t^5}{2} \Big|_{-1}^{1} = 1 \right)$

(ii) $(q_2, q_2) = \int_{-1}^{1} \frac{3}{2} t^2 \, dt = \left(\frac{t^3}{2} \Big|_{-1}^{1} = 1 \right)$

(iii) $(q_3, q_3) = \int_{-1}^{1} \left(\frac{3\sqrt{2}}{4} - \frac{5\sqrt{2}}{4} t^2 \right)^2 dt = \int_{-1}^{1} \frac{9}{8} - \frac{15}{4} t^2 + \frac{25}{8} t^4 \, dt$

$= \left(\frac{9}{8} t - \frac{5}{4} t^3 + \frac{5}{8} t^5 \Big|_{-1}^{1} \right) = 2 \left(\frac{9}{8} - \frac{5}{4} - \frac{5}{8} \right) = 2 \left(\frac{9}{8} - \frac{10}{8} - \frac{5}{8} \right) = 2 \cdot \frac{1}{2} = 1$

Orthogonality check:

(i) $(q_1, q_2) = \int_{-1}^{1} \frac{\sqrt{15}}{2} t^3 \, dt = \frac{\sqrt{15}}{2} \left(\frac{t^4}{4} \Big|_{-1}^{1} \right) = 0$

(ii) $(q_1, q_3) = \int_{-1}^{1} \sqrt{\frac{5}{2}} t^2 \left(\frac{3\sqrt{2}}{4} - \frac{5\sqrt{2}}{4} t^2 \right) dt = \int_{-1}^{1} \frac{3\sqrt{5}}{4} t^2 - \frac{5\sqrt{5}}{4} t^4 \, dt$

$= \left(\frac{\sqrt{5}}{4} t^3 - \frac{\sqrt{5}}{4} t^5 \Big|_{-1}^{1} \right) = \left(\frac{\sqrt{5}}{4} - \frac{\sqrt{5}}{4} + \frac{\sqrt{5}}{4} - \frac{\sqrt{5}}{4} \right) = 0$

$$\text{(iii)}\ \ (q_2, q_3) = \int_{-1}^{1} \sqrt{\frac{3}{2}} t \left(\frac{3\sqrt{2}}{4} - \frac{5\sqrt{2}}{4} t^2 \right) \, dt = \int_{-1}^{1} \frac{3\sqrt{3}}{4} t - \frac{5\sqrt{3}}{4} t^3 \, dt$$
$$= \left(\frac{3\sqrt{3}}{8} t^2 - \frac{5\sqrt{3}}{16} t^4 \Big|_{-1}^{1} \right) = 0$$

Problem 5

We compute:

$$A = \begin{pmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{4}{3} & 0 & 1 \end{pmatrix}$$
$$M_1 A = \begin{pmatrix} 3 & -6 & -3 \\ 0 & 4 & 8 \\ 0 & -1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{4} & 1 \end{pmatrix}$$
$$M_2 M_1 A = \begin{pmatrix} 3 & -6 & -3 \\ 0 & 4 & 8 \\ 0 & 0 & 2 \end{pmatrix}$$

Therefore, we get:

$$L = L_1 L_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{4} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{1}{4} & 1 \end{pmatrix}, \ U = \begin{pmatrix} 3 & -6 & -3 \\ 0 & 4 & 8 \\ 0 & 0 & 2 \end{pmatrix}$$
$$\Longrightarrow A = \begin{pmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} 3 & -6 & -3 \\ 0 & 4 & 8 \\ 0 & 0 & 2 \end{pmatrix} = LU$$