

# Homework 5

AMATH 481/581

## Problem 1

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Consider the initial-boundary value problem

$$\begin{aligned}u_t &= -3u_x \text{ for all } -10 < x < 10 \text{ and } t > 0 \\u(t, -10) &= u(t, 10) \text{ for all } t \geq 0 \\u(0, x) &= f(x) \equiv \begin{cases} x+1 & -1 \leq x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

In this problem, you will solve the PDE using the method of lines as discussed in class. For all parts, use a second order central difference scheme with  $\Delta x = 0.25$  (i.e., 81 total grid points) to discretize  $u_x$ . You should get a system of ODEs of the form

$$\mathbf{u}' = \mathbf{A}\mathbf{u} \tag{1}$$

The different parts will explore different methods for solving this system. For each part, solve from time  $t = 0$  until time  $t = 3$ . Each question asks you to approximate  $u(3, 9)$ . This means your approximation for  $u$  at  $t = 3$  and  $x = 9$ .

- (a) Use the forward Euler method to solve (1) with  $\Delta t = 0.1$ . Store your approximation of  $u(3, 9)$  in a variable named **A1**.
- (b) Use the forward Euler method to solve (1) with  $\Delta t = 0.01$ . Store your approximation of  $u(3, 9)$  in a variable named **A2**.
- (c) Use the trapezoidal method to solve (1) with  $\Delta t = 0.1$ . Store your approximation of  $u(3, 9)$  in a variable named **A3**.
- (d) Use the trapezoidal method to solve (1) with  $\Delta t = 0.01$ . Store your approximation of  $u(3, 9)$  in a variable named **A4**.
- (e) Use the midpoint method to solve (1) with  $\Delta t = 0.1$ . Since the midpoint method is a multi-step method, you will need to use a different method for the first time step. Use the trapezoidal method for this purpose. Store your approximation of  $u(3, 9)$  in a variable named **A5**.

- (f) Use the midpoint method to solve (1) with  $\Delta t = 0.01$ . Since the midpoint method is a multi-step method, you will need to use a different method for the first time step. Use the trapezoidal method for this purpose. Store your approximation of  $u(3, 9)$  in a variable named **A6**.
- (g) Use the Lax-Friedrichs method to solve (1) with  $\Delta t = 0.1$ . Store your approximation of  $u(3, 9)$  in a variable named **A7**.
- (h) Use the Lax-Friedrichs method to solve (1) with  $\Delta t = 0.01$ . Store your approximation of  $u(3, 9)$  in a variable named **A8**.

## Problem 2

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Consider the initial-boundary value problem

$$\begin{aligned} u_t &= -(0.2 + \sin^2(x - 1)) u_x \text{ for all } 0 < x < 2\pi \text{ and } t > 0 \\ u(t, 0) &= u(t, 2\pi) \text{ for all } t \geq 0 \\ u(0, x) &= \cos(x) \end{aligned}$$

In this problem, you will solve the PDE using the method of lines as discussed in class. For all parts, use a second order central difference scheme with  $\Delta x = 2\pi/100$  (i.e., 101 total grid points) to discretize  $u_x$ . You should get a system of ODEs of the form

$$\mathbf{u}' = A(x)\mathbf{u} \tag{2}$$

The different parts will explore different methods for solving this system. For each part, solve from time  $t = 0$  until time  $t = 8$ . Each question asks you to approximate  $u(4, \pi)$ . This means your approximation for  $u$  at  $t = 4$  and  $x = \pi$ .

- (a) Use the backward Euler method to solve (2) with  $\Delta t = 1$ . Store your approximation of  $u(4, \pi)$  in a variable named **A9**.
- (b) Use the backward Euler method to solve (2) with  $\Delta t = 0.01$ . Store your approximation of  $u(4, \pi)$  in a variable named **A10**.
- (c) Use the trapezoidal method to solve (2) with  $\Delta t = 1$ . Store your approximation of  $u(4, \pi)$  in a variable named **A11**.
- (d) Use the trapezoidal method to solve (2) with  $\Delta t = 0.01$ . Store your approximation of  $u(4, \pi)$  in a variable named **A12**.

- (e) Use the midpoint method to solve (2) with  $\Delta t = 0.05$ . Since the midpoint method is a multi-step method, you will need to use a different method for the first time step. Use the trapezoidal method for this purpose. Store your approximation of  $u(4, \pi)$  in a variable named **A13**.
- (f) Use the midpoint method to solve (2) with  $\Delta t = 0.01$ . Since the midpoint method is a multi-step method, you will need to use a different method for the first time step. Use the trapezoidal method for this purpose. Store your approximation of  $u(4, \pi)$  in a variable named **A14**.
- (g) Use the Lax-Friedrichs method to solve (2) with  $\Delta t = 0.05$ . Store your approximation of  $u(4, \pi)$  in a variable named **A15**.
- (h) Use the Lax-Friedrichs method to solve (2) with  $\Delta t = 0.01$ . Store your approximation of  $u(4, \pi)$  in a variable named **A16**.

### Problem 3 - Relevant to the written report

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Consider the initial-boundary value problem

$$\begin{aligned}
 u_t &= -u_x \text{ for all } 0 < x < 25 \text{ and } t > 0 \\
 u(t, 0) &= u(t, 25) \text{ for all } t \geq 0 \\
 u(0, x) &= f(x) \equiv e^{-20(x-2)^2} + e^{-(x-5)^2} \text{ for all } 0 \leq x < 25
 \end{aligned}$$

(Note that the initial condition isn't actually a periodic function, but  $u(0, 0)$  and  $u(0, 25)$  are so close to zero that it doesn't really matter.)

In this problem, you will solve the PDE using the method of lines as discussed in class. For all parts, use a second order central difference scheme with  $\Delta x = 0.05$  (i.e., 501 total grid points) to discretize  $u_x$ . You should get a system of ODEs of the form

$$\mathbf{u}' = A\mathbf{u} \tag{3}$$

For each part, solve from  $t = 0$  until  $t = 17$  using  $\Delta t = 1/22$  (i.e., 375 total time points). At time  $t \leq 17$ , the true solution is

$$u(t, x) = f((x - t)(\text{mod}25)) \approx f(x - t)$$

(The exact formula is true for all time, but the approximation is only valid until one of the wave peaks reaches the boundary, which happens at some point after  $t = 17$ .)

Note that the true solution should have a peak at  $x = 2+t$ . Each part of this problem asks you to approximate  $u(17, 19)$ . This means your approximation at  $t = 17$  and  $x = 19$ . The true solution at this point is supposed to be 1.

- (a) Use the Lax-Friedrichs method to solve (3). Store your approximation of  $u(17, 19)$  in a variable named **A17**.

**For the written report:** Plot your approximation of  $u(17, x)$  alongside the true solution  $f(x - 17)$ . Does your solution appear to be dispersive? Does it appear to be dissipative? How can you tell? How could you reduce the dispersion and/or dissipation by modifying  $\Delta t$  and/or  $\Delta x$ ?

- (b) Use the midpoint method to solve (3). Since the midpoint method is a multi-step method, you will need to use a different method for the first time step. Use the trapezoidal method for this purpose. Store your approximation of  $u(17, 19)$  in a variable named **A18**.

**For the written report:** Plot your approximation of  $u(17, x)$  alongside the true solution  $f(x - 17)$ . Does your solution appear to be dispersive? Does it appear to be dissipative? How can you tell? How could you reduce the dispersion and/or dissipation by modifying  $\Delta t$  and/or  $\Delta x$ ?