# Stability Properties of Predictor-Corrector Methods for Ordinary Differential Equations*

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## I. *Introduction*

Predictor-corrector methods furnish attractive algorithms for the numerical solution of ordinary differential equations because of the relatively small number of derivative evaluations required. For example, fourth degree predictor-corrector methods require two derivative evaluations per integration step while the corresponding Runge-Kutta fourth degree algorithm requires four derivative evaluations per integration step. In order to use these methods, however, an appropriate number of starting points must be provided in addition to the initial point and they must be obtained by another method.

One of the key factors to be considered in the selection of a particular predictor-corrector method is the stability of the numerical algorithm. This is particularly crucial when the differential equations being solved correspond to a system with a forcing function whose time duration or period is relatively long compared to the transient time constants of the system. Considerable effort has been directed toward the development of algorithms having improved stability characteristics [1, 2, 3, 4].

Much of the prior work in this field relates only to the limiting properties of algorithms as the interval of integration approaches zero. In many applications, such as the one described above, one needs additional information to infer the operating characteristics of a given algorithm. Dahlquist [2, 3] defines an algorithm as strongly unstable unless all of the characteristic roots are equal to or less than unity in absolute value as the integration interval approaches zero with the additional requirement that the roots of unit magnitude be simple. He obtains the important result that the degree of an algorithm cannot exceed its order by more than two without encountering strong instability. Both Dahlquist [3] and Henrici [5] study further the error propagation in the immediate vicinity of zero interval through the investigation of certain growth parameters. If an algorithm is strongly stable but exhibits undesirable error growth in the immediate vicinity of zero interval it is called weakly or conditionally unstable. Hamming [4], and Crane and Lambert [1] have synthesized corrector algorithms which are stable over an increased range of integration intervals. In this paper the stability properties of predictor-corrector algorithms is investigated for an increased range of integration intervals.

It is necessary to make a clear distinction between two modes of application

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of predictor-corrector algorithms. In one mode, a predictor formula is used to get a first estimate of the next value of the dependent variables and the corrector formula is applied iteratively until convergence is obtained. The number of derivative evaluations required is one greater than the number of iterations of the corrector and it is clear that this number may in fact exceed the number required by a Runge-Kutta algorithm. In this case, the stability properties of the algorithm are completely determined by the corrector equation alone and the predictor equation only influences the number of iterations required.

In a second much more commonly used mode, the values of the dependent variables obtained from one application of the corrector formula are regarded as the final values. The predicted and corrected values are compared to obtain an estimate of the truncation error associated with the integration step. The corrected values are accepted if this error estimate does not exceed a specified maximum value. Otherwise, the corrected values are rejected and the interval of integration is reduced starting from the last accepted point. Likewise, if the error estimate becomes unnecessarily small, the interval of integration may be increased. In this mode, only two derivative evaluations are required per integration step. However, the predictor formula is more influential in the stability properties of the predictor-corrector algorithm. Hamming [4] describes a modification of this mode of application in which the error estimate is incorporated in the final values of the dependent variables.

While it is quite evident that the predictor will influence the stability properties of the second type of predictor-corrector process it has often been at least tacitly assumed that the stability properties of the corrector equation alone adequately describe the characteristics of this second type of predictor-corrector process. Henrici [5, pp. 259–262], in fact, verifies that this is indeed the case in the immediate vicinity of zero integration interval so long as the degree (or order) of the predictor is at least as great as that of the corrector. In this paper it is shown that this is far from being the case for larger integration intervals. Detailed analyses for the methods of Milne [7, p. 134] and Hamming [4] are given. Conclusions of the study are verified through the experimental computation of the solution of the differential equation

$$y' = f(x, y) = -100y + 100, \qquad y(0) = 0,$$

which has the closed form solution

$$y = 1 - e^{-100x}.$$

Actual errors compare very closely with those expected from this theory.

To avoid confusion the following terminology pertaining to the various types of predictor-corrector processes is defined:

1. An iterative method refers to the use of the predictor equation once and then an iterative use of the corrector equation until convergence is obtained.
2. A predictor-corrector method refers to the use of the predictor equation with one subsequent application of the corrector equation.

3. A modified method, or modified predictor-corrector method, refers to the use of the predictor equation and one subsequent application of the corrector equation with incorporation of the error estimates as suggested in the paper by Hamming [4].

## II. *Analytical Results*

The first method to be analyzed was the predictor-corrector method of Milne, in which the predictor and corrector equations are respectively:

$$z_{n+1} = y_{n-3} + \frac{4h}{3}(2y_n' - y_{n-1}' + 2y_{n-2}'), \tag{1}$$

$$y_{n+1} = y_{n-1} + \frac{h}{3}(z_{n+1}' + 4y_n' + y_{n-1}'), \tag{2}$$

where

$$z_n' = f(x_n, z_n), \tag{3}$$

$$y_n' = f(x_n, y_n). \tag{4}$$

The difference equation and corresponding characteristic equation for the iterative Milne method are derived by Hamming [4] among others. The characteristic equation is

$$(\bar{h} - 3)\rho^2 + (4\bar{h})\rho + (\bar{h} + 3) = 0, \tag{5}$$

and the roots as a function of $\bar{h}$ are shown in Figure 1. For clarification all positive roots are designated by a circle ($\odot$), all negative roots by a square ($\boxdot$), and complex roots by an $\times$ ($\times$).

The same general approach used in getting the characteristic equation associated with the corrector equation will be used here in connection with the predictor-corrector equations. Let $\omega$ be an exact solution, that is:

$$\omega_n' = f(x_n, \omega_n), \tag{6}$$

where the $\omega_n$ approximately satisfy the relations (1) and (2) with an error term associated with each due to truncation. This error is assumed to be small at each point.

When the definitions

$$\epsilon_n = \omega_n - y_n, \qquad \epsilon_n' = \omega_n' - y_n', \qquad \nu_n = \omega_n - z_n, \qquad \nu_n' = \omega_n' - z_n'$$

are used, the following difference equations can be obtained:

$$\nu_{n+1} = \epsilon_{n-3} + \frac{4h}{3}(2\epsilon_n' - \epsilon_{n-1}' + 2\epsilon_{n-2}') + E_1, \tag{7}$$

$$\epsilon_{n+1} = \epsilon_{n-1} + \frac{h}{3}(\nu_{n+1}' + 4\epsilon_n' + \epsilon_{n-1}') + E_2. \tag{8}$$

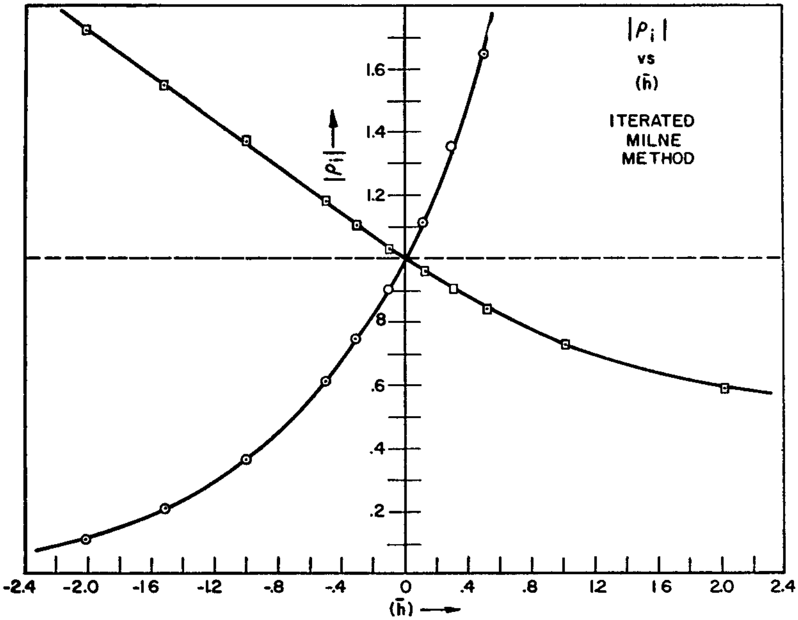



FIG. 1

Equation (4) is subtracted from (6) and the mean value theorem is applied to get

$$\epsilon_n' = \omega_n' - y_n' = f(x_n, \omega_n) - f(x_n, y_n) = \frac{\partial f}{\partial y} \epsilon_n \big|_{(x,y)=(x_n,\eta_1)} . \tag{9}$$

The same is done with (3) and (6) to get

$$\nu_n' = \frac{\partial f}{\partial y} \nu_n \big|_{(x,y)=(x_n,\eta_2)} . \tag{10}$$

At this point the usual assumptions are made, that $\partial f/\partial y$, $E_1$, and $E_2$ are all constants. For purposes of simplification let

$$\frac{\partial f}{\partial y} = A. \tag{11}$$

Now, from equation (7)

$$\nu_{n+1} = \epsilon_{n-3} + \frac{4Ah}{3} (2\epsilon_n - \epsilon_{n-1} + 2\epsilon_{n-2}) + E_1 , \tag{12}$$

and from (8)

$$\epsilon_{n+1} = \epsilon_{n-1} + \frac{Ah}{3} (\nu_{n+1} + 4\epsilon_n + \epsilon_{n-1}) + E_2 . \tag{13}$$

When (12) is substituted into (13) the following difference equation for $\epsilon_n$ is obtained:

$$\epsilon_{n+1} = \epsilon_{n-1} + \frac{Ah}{3}\left[\epsilon_{n-3} + \frac{4Ah}{3}(2\epsilon_n - \epsilon_{n-1} + 2\epsilon_{n-2}) + 4\epsilon_n + \epsilon_{n-1}\right] + E. \quad (14)$$

With the definition $\bar{h} = Ah$ and the substitution $\rho^n = \epsilon_n$, the following characteristic equation results:

$$\rho^4 - \rho^3\left(\frac{8\bar{h}^2}{9} + \frac{4\bar{h}}{3}\right) - \rho^2\left(1 + \frac{\bar{h}}{3} - \frac{4\bar{h}^2}{9}\right) - \rho\left(\frac{8\bar{h}^2}{9}\right) - \left(\frac{\bar{h}}{3}\right) = 0. \quad (15)$$

This gives a fourth degree equation as opposed to the quadratic obtained for the iterative Milne method, indicating that the predictor plays a considerable role in the overall solution. Figure 2 shows the magnitude of the roots of (15) as a function of $\bar{h}$.

The same type of analysis was applied to the method suggested by Hamming [4]. This method uses the same predictor equation as the Milne method and then uses the following corrector equation:

$$y_{n+1} = \tfrac{1}{8}[9y_n - y_{n-2} + 3h(z'_{n+1} + 2y_n' - y'_{n-1})], \quad (16)$$

where $z_{n+1}$ is again the predicted value (see equation (1)).

The characteristic equation corresponding to the iterated Hamming corrector is:

$$\rho^3\left(\frac{3\bar{h}}{8} - 1\right) + \rho^2\left(\frac{9}{8} + \frac{3\bar{h}}{4}\right) + \rho\left(-\frac{3\bar{h}}{8}\right) - \frac{1}{8} = 0. \quad (17)$$

The roots are shown in Figure 3 as a function of $\bar{h}$.

The modified Hamming method is specified by the following relations [4]:

$$p_{n+1} = y_{n-3} + \frac{4h}{3}(2y_n' - y'_{n-1} + 2y'_{n-2}), \quad (18)$$

$$m_{n+1} = p_{n+1} - \frac{112}{121}(p_n - c_n), \quad (19)$$

$$c_{n+1} = \tfrac{1}{8}[9y_n - y_{n-2} + 3h(m'_{n+1} + 2y_n' - y'_{n-1})], \quad (20)$$

$$y_{n+1} = c_{n+1} + \frac{9}{121}(p_{n+1} - c_{n+1}). \quad (21)$$

When the definitions

$$\nu_n = \omega_n - p_n, \qquad \gamma_n = \omega_n - m_n, \qquad \psi_n = \omega_n - c_n, \qquad \epsilon_n = \omega_n - y_n$$

are used the following difference equations are obtained:

$$\nu_{n+1} = \epsilon_{n-3} + \frac{4\bar{h}}{3}(2\epsilon_n - \epsilon_{n-1} + 2\epsilon_{n-2}), \quad (22)$$

$$\gamma_{n+1} = \nu_{n+1} - \frac{112}{121}(\nu_n - \psi_n), \quad (23)$$

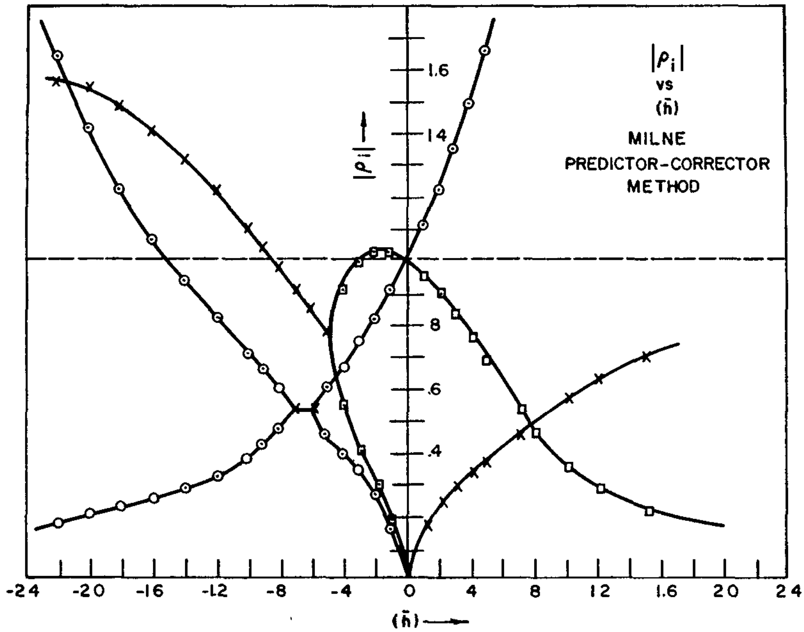

$|\rho_i|$
vs
$(\bar{h})$
MILNE
PREDICTOR-CORRECTOR
METHOD
$|\rho_i|$
1.6
1.4
.8
.6
.4
-2.4
-2.0
-1.6
-1.2
-.8
-.4
0
.4
.8
1.2
1.6
2.0
2.4
$(\bar{h})$

FIG. 2

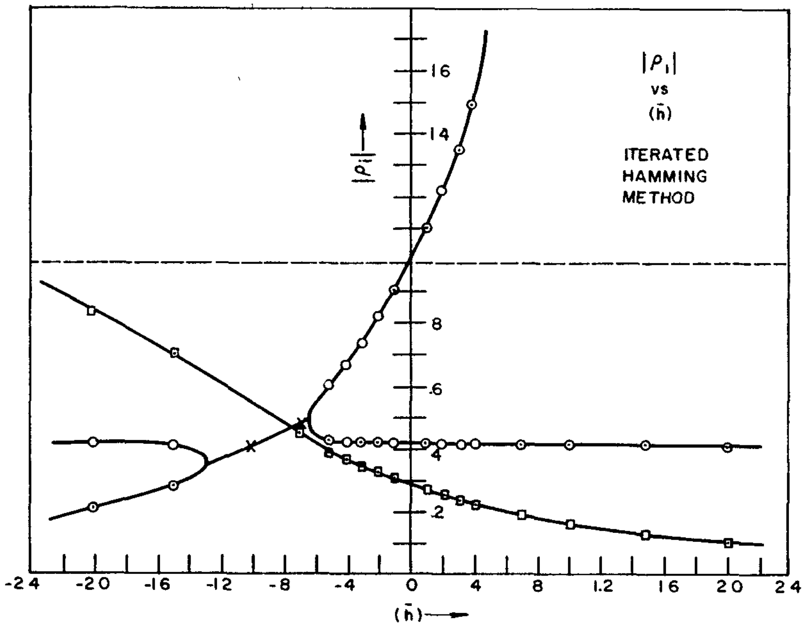

$|\rho_i|$
vs
$(\bar{h})$
ITERATED
HAMMING
METHOD
$|\rho_i|$
1.6
1.4
.8
.6
.4
.2
-2.4
-2.0
-1.6
-1.2
-.8
-.4
0
.4
.8
1.2
1.6
2.0
2.4
$(\bar{h})$

FIG. 3

$$\psi_{n+1} = \frac{9}{8}\epsilon_n - \frac{1}{8}\epsilon_{n-2} + \frac{3\bar{h}}{8}(\gamma_{n+1} + 2\epsilon_n - \epsilon_{n-1}), \tag{24}$$

$$\epsilon_{n+1} = \psi_{n+1} + \frac{9}{121}(\nu_{n+1} - \psi_{n+1}). \tag{25}$$

This system of simultaneous difference equations may be solved by assuming the solutions to be of the form

$$\nu_n = A\rho^n, \qquad \gamma_n = B\rho^n, \qquad \psi_n = C\rho^n, \qquad \epsilon_n = D\rho^n.$$

When these are inserted into the difference equations (22)–(25), one obtains a system of four simultaneous linear homogeneous equations in the constants $A$, $B$, $C$ and $D$. For this system to have a nonzero solution it is necessary and sufficient that the determinant of the coefficient matrix vanish. This leads to the characteristic equation

$$\begin{aligned}&\rho^5(121) + \rho^4(-126 - 150\bar{h} - 112\bar{h}^2) + \rho^3(54\bar{h} + 168\bar{h}^2)\\ &\quad + \rho^2(14 - 24\bar{h} - 168\bar{h}^2) + \rho(-9 - 42\bar{h} + 112\bar{h}^2) + (42\bar{h}) = 0,\end{aligned} \tag{26}$$

for the Hamming modified predictor-corrector method. The magnitude of its roots are shown as a function of $\bar{h}$ in Figure 4.

Comparison of the root loci of the various characteristic equations displays the relative merits of the different methods of solution. The roots of the characteristic equation for a given method indicate the behavior of the error in the solution when that method is used to solve an ordinary differential equation. This follows from the fact that the general solution of the difference equation for distinct roots different from unity is

$$\epsilon_n = C_1\rho_1^{\,n} + C_2\rho_2^{\,n} + \cdots + C_i\rho_i^{\,n} + C_{i+1}, \tag{27}$$

where $i$ is the number of roots in the characteristic equation. If the roots are not distinct, the form of the solution is modified. For instance, if $\rho_1 = \rho_2 = \rho$, then the combination $C_1\rho_1^{\,n} + C_2\rho_2^{\,n}$ is replaced by $(C_1 + C_2 n)\rho^n$. Also $C_{i+1}$ may be a function of $n$ rather than a constant if one or more roots are equal to plus one. In any case, the roots of the characteristic equation are important in the determination of the behavior of the error.

The two intervals of positive and negative $\bar{h}$ have been studied separately, since there are different requirements to be met in each case. In the interval of negative $\bar{h}$ where the true solution tends toward a constant, the criterion for stability will be that the magnitude of the roots be less than 1. In the interval of positive $\bar{h}$ the curve $e^{\bar{h}}$ gives more information as to stability. This is true since for positive $\bar{h}$ the solution is increasing exponentially and relative accuracy is maintained if the error does not grow more rapidly than the solution.

Figures 1, 2, 3 and 4 are curves of root magnitude for $\bar{h}$ for all the methods for which the analysis was presented. It is the region of negative $\bar{h}$ which is of maximum interest in connection with stability analysis of the algorithms. Figures 5 and 6 show the dominant roots for the various methods of Milne and

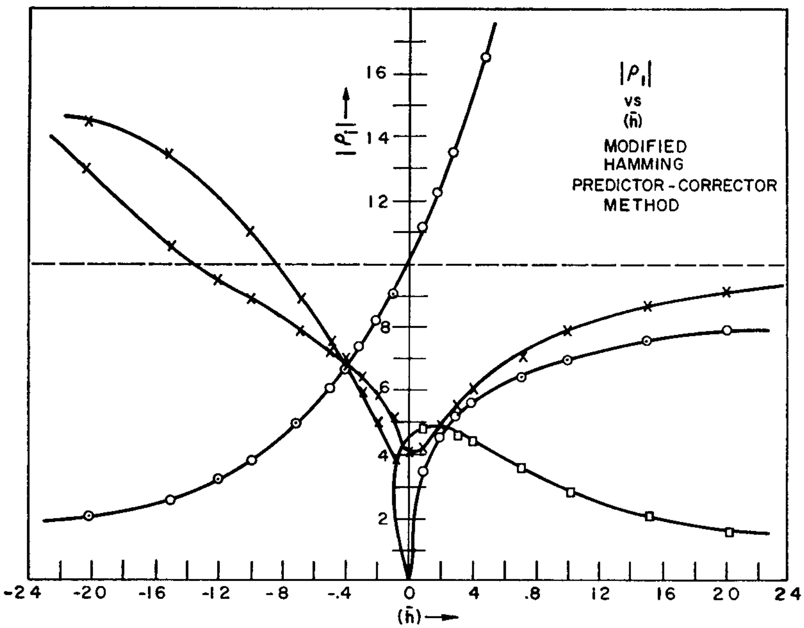

|ρ₁|
vs
(h̄)
MODIFIED
HAMMING
PREDICTOR - CORRECTOR
METHOD
16
14
12
8
6
4
2
-24
-20
-16
-12
-8
-.4
0
4
.8
12
16
20
24
(h̄)

Fig. 4

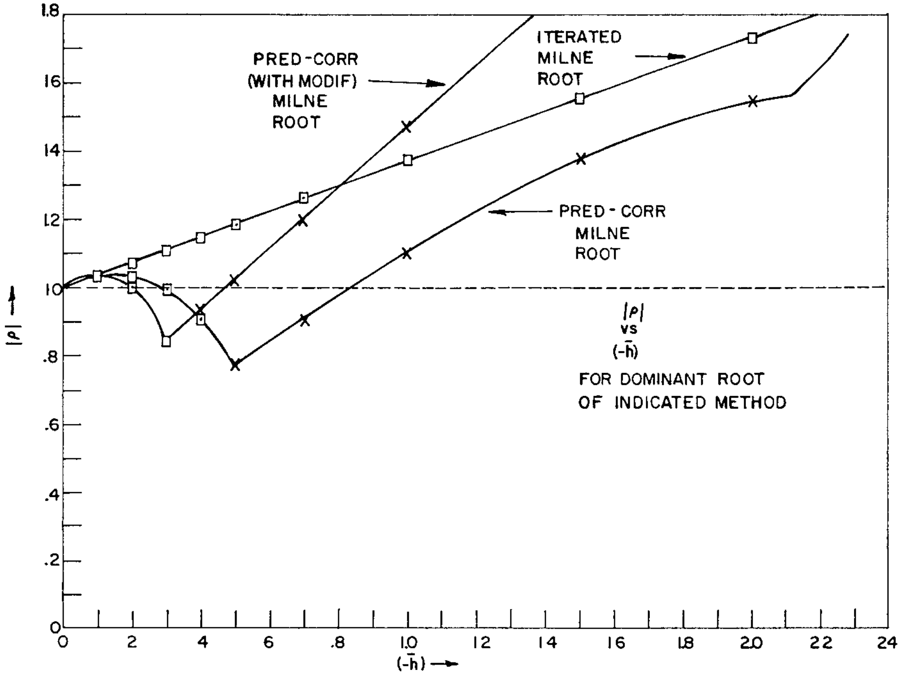

PRED-CORR
(WITH MODIF)
MILNE
ROOT
ITERATED
MILNE
ROOT
PRED - CORR
MILNE
ROOT
|ρ|
vs
(-h̄)
FOR DOMINANT ROOT
OF INDICATED METHOD
|ρ|
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
2
4
6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
(-h̄)

Fig. 5

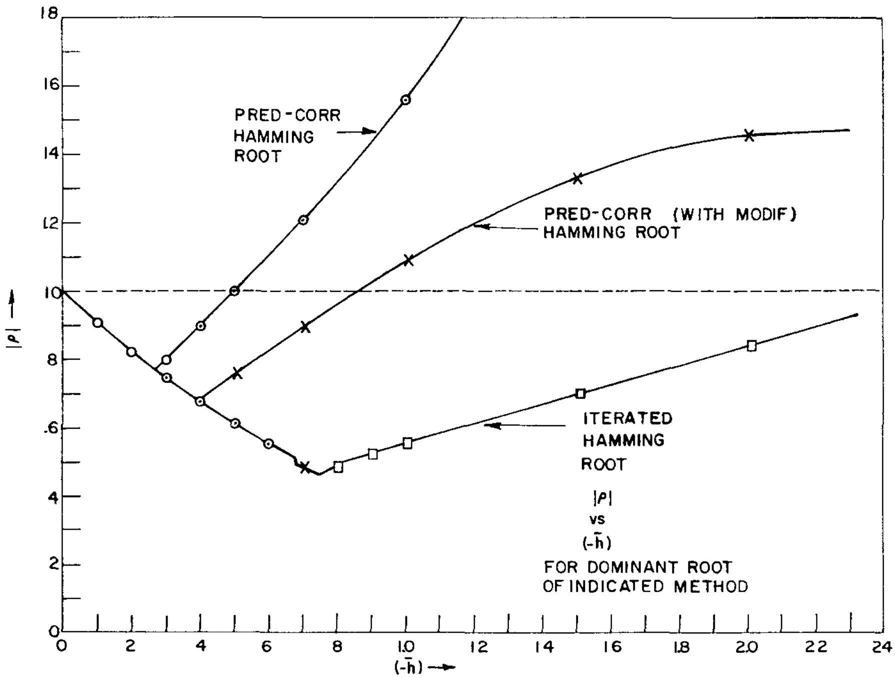



FIG. 6

Hamming, including two for which the specific analysis has not been included in this paper.

The most significant change in the loci of the roots is noticed in going from the iterated procedure to the predictor-corrector method. For example, the iterated Milne method shows no interval of stability for negative $\bar{h}$, yet the predictor-corrector Milne method is stable for $-0.8 < \bar{h} < -0.3$. The change from iterated Hamming to the predictor-corrector Hamming method has the effect of decreasing the interval of stability considerably.

No simple statement can be made about the effect of incorporating the error estimate in the solution for either of the methods. Use of the modification with the Milne predictor-corrector cuts the interval of stability almost in half (see Figure 5). Addition of the modification to the Hamming predictor-corrector, on the other hand, increases the interval of stability by a factor of nearly 2 (see Figure 6). In any case, the addition of the modification adds another root to the characteristic equations when compared with ordinary predictor-corrector methods.

In the region of positive $\bar{h}$, a significant change is again noted in going from iterated methods to the predictor-corrector methods. The iterated procedures for both Milne and Hamming have points at which a root becomes infinite. This in fact establishes the radius of convergence in $\bar{h}$ for the iterative procedure [6, p. 212]. For both the Milne and Hamming predictor-corrector methods, however, the dominant root lies below $e^{\bar{h}}$.

Of the predictor-corrector type methods analyzed here the modified Hamming method is probably the best, but the ordinary predictor-corrector Milne method is nearly as good. In both cases the cause of instability is a pair of complex roots whose magnitude becomes larger than unity. This occurs at $\bar{h} = -0.83$ for the Milne predictor-corrector method and at $\bar{h} = -0.85$ for the modified Hamming method. The Milne method has a root larger than unity in magnitude in the vicinity of $\bar{h} = 0$ but the root is only slightly larger than unity and corresponds to doubling the error in about 22 points.

### III. *Experimental Results*

The inadequacy of the theory incorporating the effect of the corrector equation only for predictor-corrector methods was first discovered through experimental computations on the prototype linear equation

$$y' = f(x, y) = -100y + 100, \qquad y(0) = 0,$$

which has the closed form solution

$$y = 1 - e^{-100x}.$$

Very poor correlation of actual errors with the errors expected on the basis of the properties of the corrector equation alone was obtained. This motivated the development of the theory described above.

With this theory excellent correlation between expected rates of error growth and actual rates of error growth is obtained. Figures 7–9 show actual error

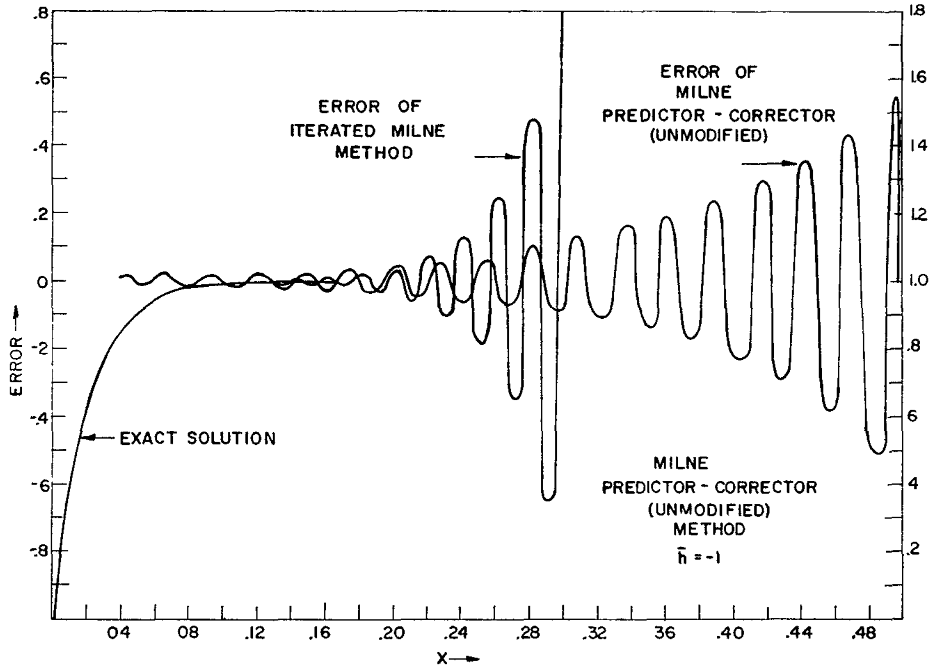
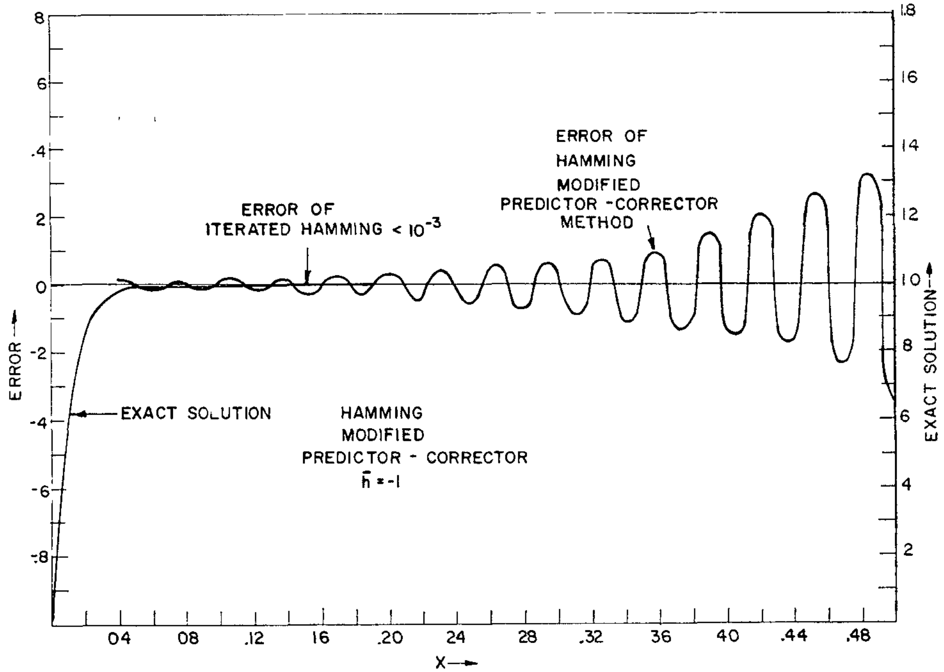
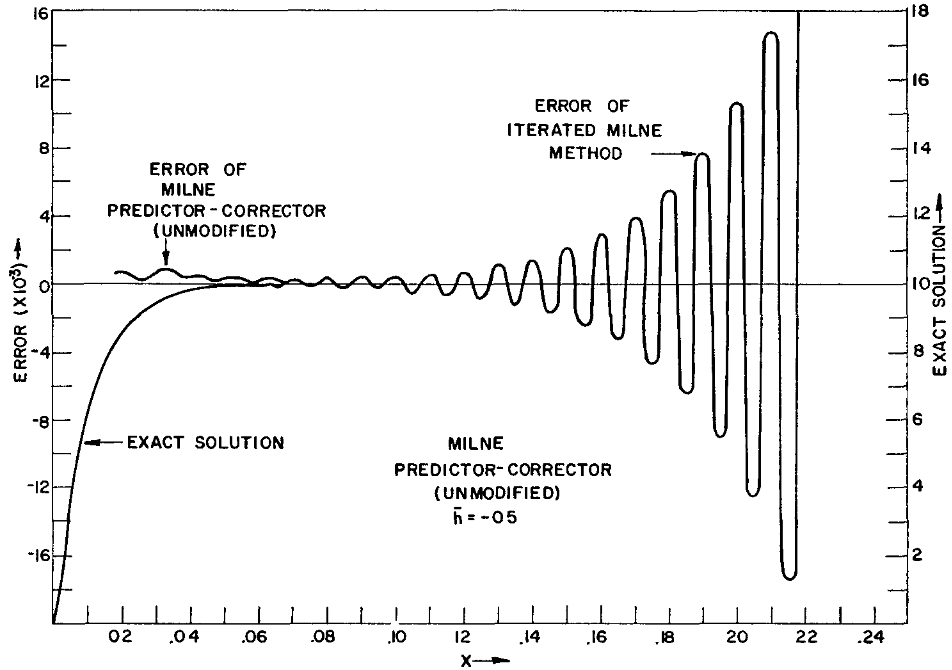



FIG. 7



ERROR OF HAMMING MODIFIED PREDICTOR-CORRECTOR METHOD
ERROR OF ITERATED HAMMING < $10^{-3}$
EXACT SOLUTION
HAMMING MODIFIED PREDICTOR - CORRECTOR
$\bar{h}$ = -1
ERROR
EXACT SOLUTION
X


FIG. 8



ERROR OF ITERATED MILNE METHOD
ERROR OF MILNE PREDICTOR-CORRECTOR (UNMODIFIED)
EXACT SOLUTION
MILNE PREDICTOR-CORRECTOR (UNMODIFIED)
$\bar{h}$ = -.05
ERROR (X$10^3$)
EXACT SOLUTION
X


FIG. 9

growth patterns for a number of algorithms and at two different intervals of integration. In Figure 7, for example, we see the envelope of the error curve growing from 0.10 to 0.59 as $x$ goes from 0.30 to 0.50 (corresponding to 20 integration steps). This implies a dominant characteristic root of magnitude 1.09 while the actual dominant characteristic root from Figure 2 has magnitude 1.095. It is particularly striking that the Milne predictor-corrector method provides a numerically stable solution for $\bar{h} = -0.5$ as illustrated in Figure 9. This is indeed what one would expect from roots of the characteristic equation at this interval (Figure 2).

## IV. *Conclusions*

This study has shown that a stability analysis of the corrector equation alone is not sufficient when the equation is to be used in a predictor-corrector combination. The analysis must include the predictor equation, the corrector equation and the manner in which they are to be used.

The use of a generalized predictor,

$$z_{n+1} = a_1 y_n + b_1 y_{n-1} + c_1 y_{n-2} + d_1 y_{n-3} + h(e_1 y_n' + f_1 y_{n-1}' + g_1 y_{n-2}' + k_1 y_{n-3}'), \tag{28}$$

and a generalized corrector,

$$y_{n+1} = a_2 y_n + b_2 y_{n-1} + c_2 y_{n-2} + h(d_2 z_{n+1}' + e_2 y_n' + f_2 y_{n-1}' + g_2 y_{n-2}'), \tag{29}$$

in the study of stability may lead to a better algorithm for the solution of ordinary differential equations.

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REFERENCES

1. Crane, R. L., and Lambert, R. J. Stability of a generalized corrector formula. *J. ACM 9* (Jan. 1962), 104–117.
2. Dahlquist, G. Convergence and stability in the numerical integration of ordinary differential equations. *Math. Scand. 4* (1956), 33–53.
3. Dahlquist, G. Stability and error bounds in the numerical integration of ordinary differential equations. *Trans. Roy. Inst. Technology*, No. 130, (1959), 1–87.
4. Hamming, R. W. Stable predictor-corrector methods for ordinary differential equations. *J. ACM 6* (Jan. 1959), 37–47.
5. Henrici, Peter. *Discrete Variable Methods in Ordinary Differential Equations*. Wiley & Sons, Inc., 1962.
6 Hildebrand, F. B. *Introduction to Numerical Analysis*. McGraw-Hill Book Co., Inc. 1956.
7. Milne, W. E. *Numerical Calculus*. Princeton University Press 1949