

Homework 4

AMATH 481/581

Problem 1

Consider the initial-boundary value problem

$$\begin{aligned}u_t &= 5u_{xx} \text{ for all } -1 < x < 2 \text{ and } t > 0 \\u(t, -1) &= 0 \text{ and } u(t, 2) = 0 \text{ for all } t \geq 0 \\u(0, x) &= \sin\left(\frac{4\pi}{3}(x+1)\right)\end{aligned}$$

The true solution to this initial-boundary value problem is

$$u(t, x) = \exp\left(-\frac{5 \cdot 16\pi^2}{9}t\right) \sin\left(\frac{4\pi}{3}(x+1)\right)$$

In this problem, you will solve the PDE using the method of lines as discussed in class. For all parts, use a second order central difference scheme with $\Delta x = 0.125$ (i.e., 25 total grid points) to discretize u_{xx} . You should get a system of ODEs of the form

$$\mathbf{u}' = A\mathbf{u} \tag{1}$$

The different parts will explore different methods for solving this system. For each part, solve from time $t = 0$ until time $t = 0.25$.

- (a) Use the forward Euler method to solve (1) with $\Delta t = 2^{-10}$ (i.e., 257 total time points, including $t = 0$ and $t = 0.25$). Store your approximation of $u(0.25, 0)$ (i.e., the value of u at $t = 0.25$ and $x = 0$) in a variable named **A1**. Store the global error (that is, the maximum of the absolute values of the error at each grid point) in a variable named **A2**.
- (b) Use the forward Euler method to solve (1) with $\Delta t = 1/552$ (i.e., 139 total time points, including $t = 0$ and $t = 0.25$). Store your approximation of $u(0.25, 0)$ (i.e., the value of u at $t = 0.25$ and $x = 0$) in a variable named **A3**. Store the global error (that is, the maximum of the absolute values of the error at each grid point) in a variable named **A4**.

- (c) Use the backward Euler method to solve (1) with $\Delta t = 0.05$ (i.e., 6 total time points, including $t = 0$ and $t = 0.25$). Store your approximation of $u(0.25, 0)$ (i.e., the value of u at $t = 0.25$ and $x = 0$) in a variable named **A5**. Store the global error (that is, the maximum of the absolute values of the error at each grid point) in a variable named **A6**.
- (d) Use the backward Euler method to solve (1) with $\Delta t = 0.005$ (i.e., 51 total time points, including $t = 0$ and $t = 0.25$). Store your approximation of $u(0.25, 0)$ (i.e., the value of u at $t = 0.25$ and $x = 0$) in a variable named **A7**. Store the global error (that is, the maximum of the absolute values of the error at each grid point) in a variable named **A8**.

Problem 2

Consider the initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx} + 10x \text{ for all } 0 < x < 1 \text{ and } t > 0 \\ u(t, 0) &= 0 \text{ and } u(t, 1) = 10t \text{ for all } t \geq 0 \\ u(0, x) &= \sin(\pi x) - 0.8 \sin(3\pi x) \end{aligned}$$

The true solution to this initial-boundary value problem is

$$u(t, x) = 10tx + e^{-\pi^2 t} \sin(\pi x) - 0.8e^{-9\pi^2 t} \sin(3\pi x)$$

In this problem, you will solve the PDE using the method of lines as discussed in class. For all parts, use a second order central difference scheme with $\Delta x = 0.05$ (i.e., 21 total grid points) to discretize u_{xx} . You should get a system of ODEs of the form

$$\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{c}(t) \tag{2}$$

where $\mathbf{c}(t)$ is an 19×1 vector. The different parts will explore different methods for solving this system. For each part, solve from time $t = 0$ until time $t = 0.1$.

- (a) Use the forward Euler method to solve (2) with $\Delta t = 1/550$ (i.e., 56 total time points, including $t = 0$ and $t = 0.1$). Store your approximation of $u(0.1, 0.5)$ (i.e., the value of u at $t = 0.1$ and $x = 0.5$) in a variable named **A9**. Store the global error (that is, the maximum of the absolute values of the error at each grid point) in a variable named **A10**.

- (b) Use the forward Euler method to solve (2) with $\Delta t = 0.0005$ (i.e., 201 total time points, including $t = 0$ and $t = 0.1$). Store your approximation of $u(0.1, 0.5)$ (i.e., the value of u at $t = 0.1$ and $x = 0.5$) in a variable named **A11**. Store the global error (that is, the maximum of the absolute values of the error at each grid point) in a variable named **A12**.
- (c) Use the trapezoidal method to solve (2) with $\Delta t = 0.01$ (i.e., 11 total time points, including $t = 0$ and $t = 0.1$). Store your approximation of $u(0.1, 0.5)$ (i.e., the value of u at $t = 0.1$ and $x = 0.5$) in a variable named **A13**. Store the global error (that is, the maximum of the absolute values of the error at each grid point) in a variable named **A14**.
- (d) Use the trapezoidal method to solve (2) with $\Delta t = 0.001$ (i.e., 101 total time points, including $t = 0$ and $t = 0.1$). Store your approximation of $u(0.1, 0.5)$ (i.e., the value of u at $t = 0.1$ and $x = 0.5$) in a variable named **A15**. Store the global error (that is, the maximum of the absolute values of the error at each grid point) in a variable named **A16**.

Problem 3 - Relevant to the written report

Consider the initial-boundary value problem

$$\begin{aligned}
 u_t &= \kappa (u_{xx} + u_{yy}) \quad \text{for all } 0 < x, y < 1 \text{ and } t > 0 \\
 u(t, 0, y) &= u(t, 1, y) = 0 \quad \text{for all } 0 \leq y \leq 1 \text{ and } t > 0 \\
 u(t, x, 0) &= u(t, x, 1) = 0 \quad \text{for all } 0 \leq x \leq 1 \text{ and } t > 0 \\
 u(0, x, y) &= \begin{cases} e^{-10((x-0.5)^2 + (y-0.5)^2)} & 0 < x, y < 1 \\ 0 & x = 0, y = 0, x = 1 \text{ or } y = 1 \end{cases}
 \end{aligned}$$

where $\kappa = 0.1$ is the diffusion coefficient (just a constant).

Note: I updated the boundary conditions for clarification. The problem is discontinuous at the boundary at time $t = 0$, but this doesn't have a substantial effect on the solution (because the heat equation smooths out discontinuities).

In this problem, you will solve the (2 dimensional) PDE using the method of lines. For all parts, use a second order central difference scheme for both u_{xx} and u_{yy} with $\Delta x = \Delta y = 0.1$ (i.e., 11 total points in both the x and y directions). You should get a system of ODEs of the form

$$\mathbf{u}' = \mathbf{A}\mathbf{u} \tag{3}$$

where each component $u_i(t)$ is the value of u at one of the grid points at time t . Order grid points in the same way that we have when studying the Laplace equation. You should end up with the same (symmetric, negative definite) matrix A that we used throughout the last homework (with an extra factor of κ).

The different parts will explore different methods for solving this system. For each part, solve from time $t = 0$ until time $t = 1$.

- (a) Use the forward Euler method to solve (3) with $\Delta t = 1/3$ (i.e., using 4 total time points, including $t = 0$ and $t = 1$). Store your approximation of $u(1, 0.5, 0.5)$ (i.e., the value of u at $t = 1$, $x = 0.5$ and $y = 0.5$) in a variable named **A17**.

For the written report: Make a surface plot of the solution at times $t = 0$, $t = 1/3$, $t = 2/3$ and $t = 1$. Make sure that all four plots are visible at once and are all legible. (A good solution would be to use subplots to put all four in the same figure, but it's ok as long as all four are on the same page and a reasonable size.)

- (b) Use the forward Euler method to solve (3) with $\Delta t = 0.01$ (i.e., using 101 total time points, including $t = 0$ and $t = 1$). Store your approximation of $u(1, 0.5, 0.5)$ (i.e., the value of u at $t = 1$, $x = 0.5$ and $y = 0.5$) in a variable named **A18**.

For the written report: Make a surface plot of the solution at times $t = 0$, $t = 0.33$, $t = 0.67$ and $t = 1$. Make sure that all four plots are visible at once and are all legible. (A good solution would be to use subplots to put all four in the same figure, but it's ok as long as all four are on the same page and a reasonable size.)

- (c) Use the trapezoidal method to solve (3) with $\Delta t = 1/3$ (i.e., using 4 total time points, including $t = 0$ and $t = 1$). Store your approximation of $u(1, 0.5, 0.5)$ (i.e., the value of u at $t = 1$, $x = 0.5$ and $y = 0.5$) in a variable named **A19**.

For the written report: Make a surface plot of the solution at times $t = 0$, $t = 1/3$, $t = 2/3$ and $t = 1$. Make sure that all four plots are visible at once and are all legible. (A good solution would be to use subplots to put all four in the same figure, but it's ok as long as all four are on the same page and a reasonable size.)

- (d) Use the trapezoidal method to solve (3) with $\Delta t = 0.01$ (i.e., using 101 total time points, including $t = 0$ and $t = 1$). Store your approximation of $u(1, 0.5, 0.5)$ (i.e., the value of u at $t = 1$, $x = 0.5$ and $y = 0.5$) in a variable named **A20**.

For the written report: Make a surface plot of the solution at times $t = 0$, $t = 0.33$, $t = 0.67$ and $t = 1$. Make sure that all four plots are visible at once and

are all legible. (A good solution would be to use subplots to put all four in the same figure, but it's ok as long as all four are on the same page and a reasonable size.)

For the written report: Which of your solutions exhibit stability issues? (This should be obvious.) Suppose that your goal was just to get a rough picture of the solution at $t = 0$, $t \approx 1/3$, $t \approx 2/3$ and $t = 1$. Which method do you think would be better for this purpose: Forward Euler or the Trapezoidal method? Your answer should include a sentence or two about the pros and cons of each method.