AMATH 581: Report 2

Minho Choi

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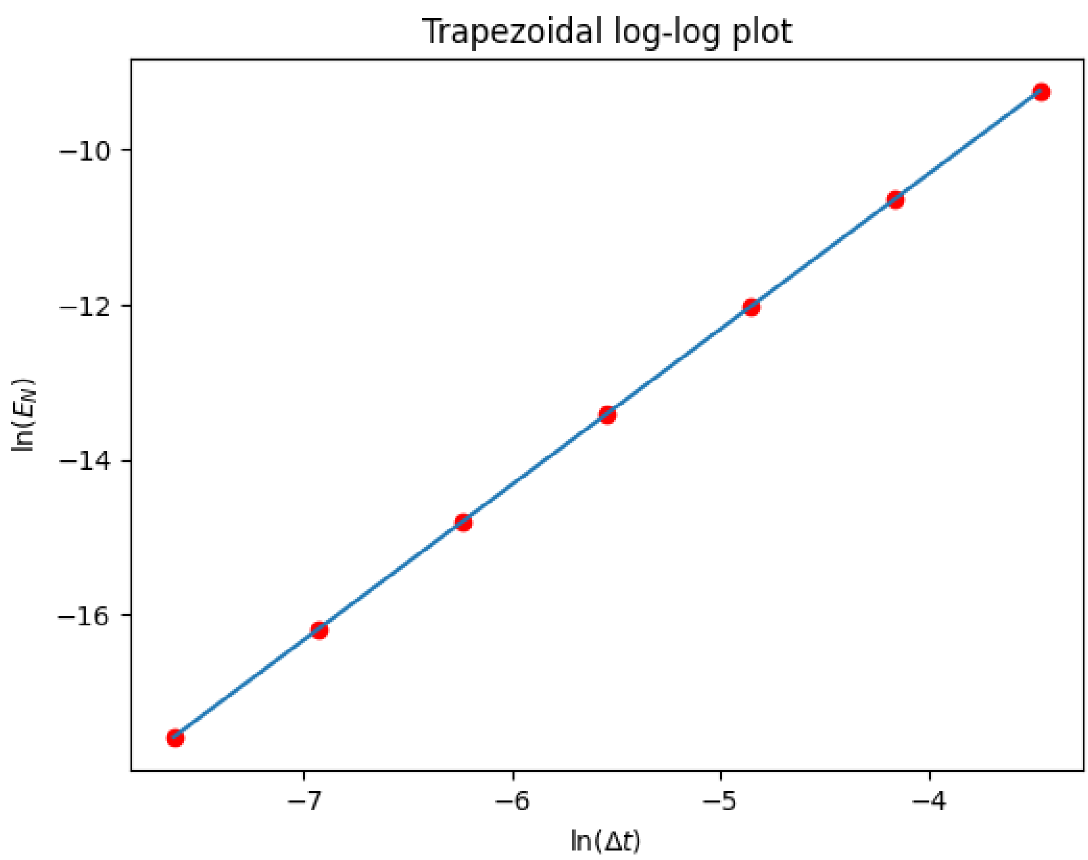
1 Order of Accuracy

To determine the order of accuracy for both the trapezoidal and midpoint methods, we use the numerical method of comparing the logarithms of global errors (E_N) and Δt (as we did in Report 1):

(i) **Trapezoidal Method**:
We obtain the following log-log plot:

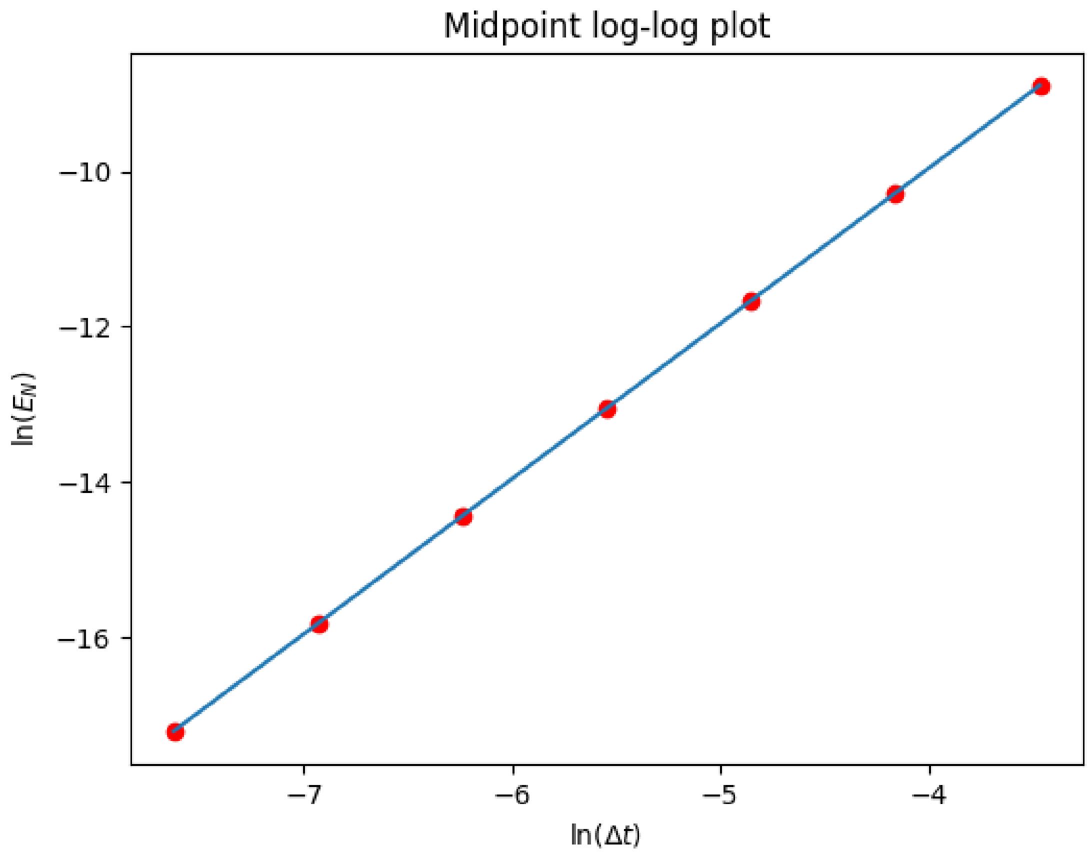


The equation of the best fit line is:

$$\ln(E_N) = 2.0000366392505855 \times \ln(\Delta t) - 2.3232260396583992$$

Therefore, the order of accuracy of trapezoidal method is $\mathcal{O}(\Delta t^2)$.

(ii) **Midpoint Method**:
We obtain the following log-log plot:



The equation of the best fit line is:

$$\ln(E_N) = 2.0000901145953613 \times \ln(\Delta t) - 1.963769831912312$$

Therefore, the order of accuracy of midpoint method is $\mathcal{O}(\Delta t^2)$.

2 Stability Region

To find the stability region for the trapezoidal method, we solve the following problem:

$$x' = \lambda x \implies f(t,x) = \lambda x$$

where $\lambda \in \mathbb{C}$. Based on the trapezoidal method, we have:

$$x_{k+1} = x_k + \frac{\Delta t}{2}(f(t_k, x_k) + f(t_{k+1}, x_{k+1})) = x_k + \frac{\Delta t}{2}(\lambda x_k + \lambda x_{k+1}) = x_k + \frac{\lambda \Delta t}{2} x_k + \frac{\lambda \Delta t}{2} x_{k+1}$$

$$\implies \left(1 - \frac{\lambda \Delta t}{2}\right) x_{k+1} = \left(1 + \frac{\lambda \Delta t}{2}\right) x_k \implies x_{k+1} = \frac{2 + \lambda \Delta t}{2 - \lambda \Delta t} x_k$$

By induction, we further have:

$$x_N = \left(\frac{2 + \lambda \Delta t}{2 - \lambda \Delta t}\right)^N x_0$$

Then, the approximation we got using trapezoidal method is stable if we have:

$$\left|\frac{2 + \lambda \Delta t}{2 - \lambda \Delta t}\right| < 1$$

Here note that:

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right| = \left|\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\right| = \left|\frac{r_1}{r_2}\right| \cdot \left|e^{i(\theta_1 - \theta_2)}\right| = \frac{|r_1|}{|r_2|} = \frac{|z_1|}{|z_2|}$$

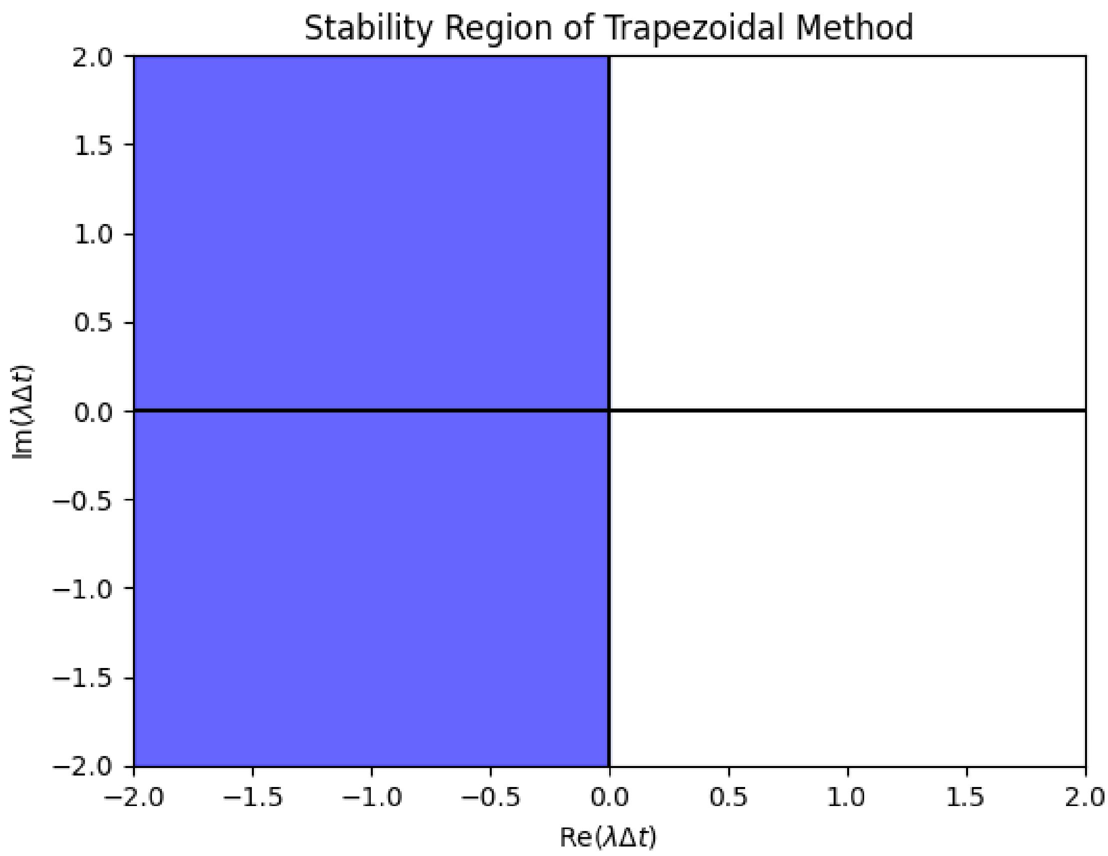
for any $z_1, z_2 \in \mathbb{C}$. Hence, we have:

$$\left|\frac{2 + \lambda \Delta t}{2 - \lambda \Delta t}\right| = \frac{|2 + \lambda \Delta t|}{|2 - \lambda \Delta t|} < 1$$

Let $\lambda\Delta t = a + bi$ for $a, b \in \mathbb{R}$. Then, we further have:

$$\frac{|2+a+bi|}{|2-a-bi|} < 1 \implies \frac{\sqrt{(2+a)^2+b^2}}{\sqrt{(2-a)^2+(-b)^2}} < 1 \implies \frac{(2+a)^2+b^2}{(2-a)^2+b^2} < 1$$
$$\implies 4+4a+a^2+b^2 < 4-4a+a^2+b^2 \implies 8a < 0 \implies a < 0$$

Therefore, the trapezoidal method is stable for $\mathrm{Re}(\lambda\Delta t) < 0$. The stability region looks like the following:



3 Stability in Problem 2(a) and Problem 2(b)

For **Problem 2**, we have:

$$\mathbf{x}' = A\mathbf{x} \text{ where } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence, the eigenvalues of A is:

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \implies \lambda_1 = 1, \lambda_2 = -1$$

Then, for **Problem 2(a)**, we have $\lambda_1\Delta t = 0.1 > 0$ and $\lambda_2\Delta t = -0.1 < 0$. Hence, since $\lambda\Delta t$ is not in the stability region when $\lambda = \lambda_1 = 1$, the trapezoidal method is not stable in **Problem 2(a)**. Similarly, for **Problem 2(b)**, we have $\lambda_1\Delta t = 0.01 > 0$ and $\lambda_2\Delta t = -0.01 < 0$. Hence, since $\lambda\Delta t$ is not in the stability region when $\lambda = \lambda_1 = 1$, the trapezoidal method is not stable in **Problem 2(b)**. However, since the final time is short, as $t = 1$, for **Problem 2(a), (b)**, the unstable property of trapezoidal method is not affecting the approximation too much.