# Week 3

## 3.1 Mar 10 ex9.2:2(2)(5)(8),3,4,6,13(4)(6),16.

✍ **习题 9.2.2** 求下列函数对于每个自变量的偏微商:

(2) $z = 3^{-y/x}$;

(5) $u = \arctan\left(\dfrac{x+y}{x-y}\right)$;

(8) $u = xe^{-z} + \ln(x + \ln y) + z$;

**解**

(2)

$$\frac{\partial z}{\partial x} = 3^{-y/x}\ln 3 \cdot \left(-\frac{y}{x}\right)' = 3^{-y/x}\ln 3 \cdot \frac{y}{x^2}$$
$$\frac{\partial z}{\partial y} = 3^{-y/x}\ln 3 \cdot \left(-\frac{y}{x}\right)' = 3^{-y/x}\ln 3 \cdot \left(-\frac{1}{x}\right)$$

(5)

$$\frac{\partial u}{\partial x} = \frac{(x-y)^2}{2x^2+2y^2}\frac{-2y}{(x-y)^2} = -\frac{y}{x^2+y^2}$$
$$\frac{\partial u}{\partial y} = \frac{(x-y)^2}{2x^2+2y^2}\frac{2x}{(x-y)^2} = \frac{x}{x^2+y^2}$$

(8)

$$\frac{\partial u}{\partial x} = \mathrm{e}^{-z} + \frac{1}{x+\ln y}$$
$$\frac{\partial u}{\partial y} = \frac{1}{y(x+\ln y)}$$
$$\frac{\partial u}{\partial z} = -x\mathrm{e}^{-z} + 1$$

✍ **习题 9.2.3** 设 $f(x,y) = \displaystyle\int_1^{x^2y} \frac{\sin t}{t}\,\mathrm{d}t$, 求 $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$.

**解**

$$\frac{\partial f}{\partial x} = \frac{\sin x^2y}{x^2y}\frac{\partial(x^2y)}{\partial x} = \frac{2\sin x^2y}{x}$$
$$\frac{\partial f}{\partial y} = \frac{\sin x^2y}{x^2y}\frac{\partial(x^2y)}{\partial y} = \frac{x^2\sin x^2y}{x^2y} = \frac{\sin x^2y}{y}$$

✍ **习题 9.2.4** 设 $f(x,y) = \begin{cases} y\sin\left(\dfrac{1}{x^2+y^2}\right), & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0. \end{cases}$ 考察函数 $f(x,y)$ 在原点 $(0,0)$ 的偏导数.

**解** $x$ 方向的偏导数为

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}0\cdot\sin\frac{1}{h^2+0^2}=0$$

$y$ 方向的偏导数为

$$\frac{\partial f}{\partial y}(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{h\sin\frac{1}{0^2+h^2}-0}{h}=\lim_{h\to 0}\sin\frac{1}{h^2}=\text{不存在}$$

**习题 9.2.6** 求曲面 $z=\dfrac{x^2+y^2}{4}$ 与平面 $y=4$ 的交线在点 $(2,4,5)$ 处的切线与 $Ox$ 轴正向所成的角度.

**解** 夹角 $\theta$ 的正弦为

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(2,4)}=\left(\frac{\partial}{\partial x}\frac{x^2+4^2}{4}\right)\Big|_{x=2}=1$$

因此夹角 $\theta=\dfrac{\pi}{4}$.

**习题 9.2.13** 求下列函数的微分, 或在给定点的微分

(4) $z=\arctan\dfrac{y}{x}$,

(6) $z=x^4+y^4-4x^2y^2$ 在点 $(0,0),(1,1)$.

**解** 解法一:

1.

$$\frac{\partial z}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}$$
$$\frac{\partial z}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$$

由此得

$$\mathrm{d}z=\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y=-\frac{y}{x^2+y^2}\,\mathrm{d}x+\frac{x}{x^2+y^2}\,\mathrm{d}y$$

2.

$$\frac{\partial z}{\partial x}=4x^3-8xy^2$$
$$\frac{\partial z}{\partial y}=4y^3-8x^2y$$

由此得

$$\mathrm{d}z=(4x^3-8xy^2)\,\mathrm{d}x+(4y^3-8x^2y)\,\mathrm{d}y$$

代入得

$$\mathrm{d}z|_{(0,0)}=0$$
$$\mathrm{d}z|_{(1,1)}=-4\,\mathrm{d}x-4\,\mathrm{d}y$$

**解** 解法二:

1.

$$\mathrm{d}z=\frac{1}{1+\left(\frac{y}{x}\right)^2}\mathrm{d}\left(\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\,\mathrm{d}x+\frac{1}{x}\,\mathrm{d}y\right)=\frac{-y\,\mathrm{d}x+x\,\mathrm{d}y}{x^2+y^2}$$

2.

$$\mathrm{d}z = 4x^3\,\mathrm{d}x + 4y^3\,\mathrm{d}y - 4(2xy^2\,\mathrm{d}x + 2x^2y\,\mathrm{d}y) = 4(x^3 - 2xy^2)\,\mathrm{d}x + 4(y^3 - 2x^2y)\,\mathrm{d}y$$

其余部分同解法一.

✍ **习题 9.2.16** 证明函数 $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0. \end{cases}$ 在点 $(0,0)$ 连续且偏导数存在, 但是在此点不可微.

**解** 由

$$0 \leqslant \lim_{x\to 0}\left|\frac{x^2y}{x^2+y^2}\right| \leqslant \lim_{x\to 0}\left|\frac{x}{2}\right| = 0$$

可得 $f(x,y)$ 在 $(0,0)$ 连续.$(0,0)$ 处的偏导数为

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{xy}{x^2+y^2}\bigg|_{y=0} = 0$$

$$\frac{\partial f}{\partial y} = \lim_{y\to 0}\frac{y^2}{x^2+y^2}\bigg|_{x=0} = 0$$

故 $f(x,y)$ 在 $(0,0)$ 处的偏导数存在.

下证不可微, 假设可微, 则极限

$$\lim_{x\to 0,y\to 0}\frac{f(x,y)}{\sqrt{x^2+y^2}} = \lim_{x\to 0,y\to 0}\frac{x^2y}{(x^2+y^2)^{3/2}}$$

存在, 但是

$$\lim_{x\to 0,y=0}\frac{x^2y}{(x^2+y^2)^{3/2}} = 0$$

$$\lim_{x\to 0,y=x}\frac{xy}{(x^2+y^2)^{3/2}} = \frac{1}{2\sqrt{2}}$$

矛盾, 故 $f(x,y)$ 在 $(0,0)$ 不可微.

## 3.2 Mar 12 ex9.2:2(7),8,11,15,26,27,28.

✍ **习题 9.2.2(7)** 求下列函数对于每个自变量的偏微商:

(7) $u = x^{y^z}$.

$$\frac{\partial u}{\partial x} = y^z x^{y^z-1}$$

$$\frac{\partial u}{\partial y} = zy^{z-1}x^{y^z}\ln x$$

$$\frac{\partial u}{\partial z} = y^z x^{y^z}\ln x\ln y$$

✍ **习题 9.2.8** 证明函数 $u = \dfrac{1}{\sqrt{t}}\mathrm{e}^{-\frac{x^2}{4t}}$ 满足热传导方程 $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$.

**解**

$$\frac{\partial u}{\partial t}=-\frac{1}{2}t^{-\frac{3}{2}}\mathrm{e}^{-\frac{x^2}{4t}}+\frac{1}{\sqrt{t}}\cdot\frac{x^2}{4t^2}\mathrm{e}^{-\frac{x^2}{4t}}$$
$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{t}}\mathrm{e}^{-\frac{x^2}{4t}}\cdot\left(-\frac{x}{2t}\right)$$
$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{\sqrt{t}}\mathrm{e}^{-\frac{x^2}{4t}}\cdot\left(-\frac{1}{2t}+\frac{x^2}{4t^2}\right)$$

**习题 9.2.11** 设 $r=\sqrt{x^2+y^2+z^2}$, 证明当 $r\neq 0$ 时, 有

(1) $\dfrac{\partial^2 r}{\partial x}+\dfrac{\partial^2 r}{\partial y}+\dfrac{\partial^2 r}{\partial z}=\dfrac{2}{r}$;

(2) $\dfrac{\partial^2 \ln r}{\partial x}+\dfrac{\partial^2 \ln r}{\partial y}+\dfrac{\partial^2 \ln r}{\partial z}=\dfrac{1}{r^2}$.

(3) $\dfrac{\partial^2}{\partial x}\dfrac{1}{r}+\dfrac{\partial^2}{\partial y}\dfrac{1}{r}+\dfrac{\partial^2}{\partial z}\dfrac{1}{r}=0$.

**证明** 证明一:

1.
$$\frac{\partial^2 r}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}\Rightarrow \Delta r=\frac{2}{r}$$

2.
$$\frac{\partial^2 \ln r}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)=\frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}\Rightarrow \Delta\ln r=\frac{1}{r^2}$$

3.
$$\frac{\partial^2}{\partial x}\frac{1}{r}=\frac{\partial}{\partial x}\left(-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)=\frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\Rightarrow \Delta\frac{1}{r}=0$$

**证明** 证明二:

1.
$$\nabla r=\left(\frac{\partial r}{\partial x},\frac{\partial r}{\partial y},\frac{\partial r}{\partial z}\right)=\left(\frac{x}{r},\frac{y}{r},\frac{z}{r}\right)=\frac{1}{r}(x,y,z):=\frac{1}{r}\boldsymbol{r}$$
$$\nabla\cdot\boldsymbol{r}=\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}z=3$$

由此得
$$\Delta r=\nabla^2 r=\nabla\cdot\nabla r=\nabla\cdot\frac{1}{r}\boldsymbol{r}=\frac{1}{r}\nabla\cdot\boldsymbol{r}+\boldsymbol{r}\cdot\nabla\left(\frac{1}{r}\right)=\frac{3-1}{r}=\frac{2}{r}$$

2.
$$\nabla\ln r=\left(\frac{\partial\ln r}{\partial x},\frac{\partial\ln r}{\partial y},\frac{\partial\ln r}{\partial z}\right)=\left(\frac{1}{r}\frac{\partial r}{\partial x},\frac{1}{r}\frac{\partial r}{\partial y},\frac{1}{r}\frac{\partial r}{\partial z}\right)=\frac{1}{r}\nabla r=\frac{1}{r^2}(x,y,z):=\frac{1}{r^2}\boldsymbol{r}$$

由此得
$$\Delta\ln r=\nabla^2\ln r=\nabla\cdot\nabla\ln r=\nabla\cdot\frac{1}{r^2}\boldsymbol{r}=\frac{1}{r^2}\nabla\cdot\boldsymbol{r}+\boldsymbol{r}\cdot\nabla\left(\frac{1}{r^2}\right)=\frac{3}{r^2}=\frac{1}{r^2}$$

3.
$$\nabla\frac{1}{r}=\left(\frac{\partial}{\partial x}\frac{1}{r},\frac{\partial}{\partial y}\frac{1}{r},\frac{\partial}{\partial z}\frac{1}{r}\right)=\left(-\frac{1}{r^2}\frac{\partial r}{\partial x},-\frac{1}{r^2}\frac{\partial r}{\partial y},-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)=-\frac{1}{r^2}\nabla r=-\frac{1}{r^3}(x,y,z):=-\frac{1}{r^3}\boldsymbol{r}$$

由此得

$$\Delta\frac{1}{r}=\nabla^2\frac{1}{r}=\nabla\cdot\nabla\frac{1}{r}=\nabla\cdot\left(-\frac{1}{r^3}\boldsymbol{r}\right)=-\frac{1}{r^3}\nabla\cdot\boldsymbol{r}-\boldsymbol{r}\cdot\nabla\left(\frac{1}{r^3}\right)=0$$

✍ **习题 9.2.15** 根据可微的定义证明, 函数 $f(x,y)=\sqrt{|xy|}$ 在原点不可微.

**证明** 假设 $f(x,y)=\sqrt{|xy|}$ 在原点可微, 则极限

$$f(h,k)-f(0,0)=\sqrt{|hk|}=ah+bk+o(\sqrt{h^2+k^2})$$

由对称性, 有 $a=b$, 然而取 $k=-h$ 时有

$$|h|=(a-b)h+o(h)=o(h)$$

矛盾, 故 $f(x,y)=\sqrt{|xy|}$ 在原点不可微.

✍ **习题 9.2.26** 设 $z=f(xy)$,$f$ 为可微函数. 证明 $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0$.

**证明** 由链式法则得

$$\frac{\partial z}{\partial x}=f'(xy)y,\quad \frac{\partial z}{\partial y}=f'(xy)x$$

代入得

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=f'(xy)xy-f'(xy)xy=0$$

✍ **习题 9.2.27** 设 $z=f(\ln x+\frac{1}{y})$, $f$ 为可微函数. 证明 $x\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=0$.

**证明** 由链式法则,

$$\frac{\partial z}{\partial x}=f'(\ln x+\frac{1}{y})\cdot\frac{1}{x}$$
$$\frac{\partial z}{\partial y}=f'(\ln x+\frac{1}{y})\cdot\left(-\frac{1}{y^2}\right)$$

代入得

$$x\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=f'(\ln x+\frac{1}{y})-f'(\ln x+\frac{1}{y})=0$$

✍ **习题 9.2.28** 证明函数 $u=\varphi(x-at)+\psi(x+at)$ 满足波动方程

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$

**证明**

令 $\begin{cases} v=x-at, \\ w=x+at; \end{cases}$ 则有 $u=\varphi(v)+\psi(w)$ 且 $\begin{cases} \frac{\partial v}{\partial x}=1, \\ \frac{\partial w}{\partial x}=1; \end{cases}$ 与 $\begin{cases} \frac{\partial v}{\partial t}=-a, \\ \frac{\partial w}{\partial t}=a; \end{cases}$

因此我们有 $\frac{\partial u}{\partial x}=\varphi'(v)\frac{\partial v}{\partial x}+\psi'(w)\frac{\partial w}{\partial x}=\varphi'(v)+\psi'(w)$,

故 $\frac{\partial^2 u}{\partial x^2}=\varphi''(v)\frac{\partial v}{\partial x}+\psi''(w)\frac{\partial w}{\partial x}=\varphi''(v)+\psi''(w)$.

同时 $\frac{\partial u}{\partial t}=\varphi'(v)\frac{\partial v}{\partial t}+\psi'(w)\frac{\partial w}{\partial t}=-a\varphi'(v)+a\psi'(w)$,

故 $\frac{\partial^2 u}{\partial x^2}=-a\varphi''(v)\frac{\partial v}{\partial x}+a\psi''(w)\frac{\partial w}{\partial x}=a^2\left(\varphi''(v)+\psi''(w)\right)=a^2\frac{\partial^2 u}{\partial x^2}$.

## 3.3 Mar 14 ex9.2:20(2)(3)(4),25,28,32; ex9.3:1(1),2(2)(5),4(1).

✍ **习题 9.2.20** 求下列复合函数的偏导数或倒数, 其中各题中的 $f$ 均有连续的二阶偏导数.

(2) 设 $u=f(x,y,z), x=\sin t, y=\cos t, z=\mathrm{e}^t$, 求 $\dfrac{\mathrm{d}u}{\mathrm{d}t}$;

(3) 设 $u=f(x^2-y^2,\mathrm{e}^{xy})$, 求 $\dfrac{\partial u}{\partial x}$ 和 $\dfrac{\partial u}{\partial x\partial y}$;

(4) 设 $u=f(x+y+z,x^2+y^2+z^2)$, 求 $\dfrac{\partial u}{\partial x},\dfrac{\partial^2 u}{\partial x},\dfrac{\partial u}{\partial x\partial y}$.

**解**

(2)

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t}&=\frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\\&=f_1'\cos t-f_2'\sin t+f_3'\mathrm{e}^t\end{aligned}$$

注 在 $u=f(x,y,z)$,$f$ 中的变量不复杂的时候可以使用 $f_x',f_y',f_z'$ 来表示 $f_1',f_2',f_3'$. 但是对后面两题使用 $f_x'$ 会有歧义.

(3)

$$\begin{aligned}\frac{\partial u}{\partial x}&=f_1'\cdot 2x-f_2'y\mathrm{e}^{xy}\\\frac{\partial u}{\partial x\partial y}&=\frac{\partial}{\partial x}\left(-2yf_1'+x\mathrm{e}^{xy}f_2'\right)\\&=-4xyf_{11}''+2(x^2-y^2)\mathrm{e}^{xy}f_{12}''+(x+y)\mathrm{e}^{xy}f_2'+xy\mathrm{e}^{xy}f_{22}''\end{aligned}$$

(4)

$$\begin{aligned}\frac{\partial u}{\partial x}&=f_1'+2xf_2'\\\frac{\partial^2 u}{\partial x}&=f_{11}''+4xf_{12}''+4x^2f_{22}''\\\frac{\partial u}{\partial x\partial y}&=\frac{\partial}{\partial x}\left(f_1'+2xf_2'\right)\\&=f_{11}''+2(x+y)f_{12}''+4xyf_{22}''\end{aligned}$$

✍ **习题 9.2.25** 设 $u=f(t), t=\varphi(xy,x+y)$, 其中 $f,\varphi$ 分别具有连续的二阶偏导数及偏导数, 求 $\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial x\partial y}$.

**解**

$$\begin{aligned}\frac{\partial u}{\partial x}&=y\varphi_1'f'(t)+\varphi_2'f'(t)\\\frac{\partial u}{\partial y}&=x\varphi_1'f'(t)+\varphi_2'f'(t)\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial x\partial y} &= \frac{\partial}{\partial x}(y\varphi_1' f'(t) + \varphi_2' f'(t)) \\
&= \varphi_1 f'(t) + xy\varphi_{11}'' f'(t) + x\varphi_{12}'' f'(t) + x(\varphi_1')^2 f''(t) + x\varphi_1'\varphi_2' f''(t) \\
&+ y\varphi_{21}'' f'(t) + \varphi_{22}'' f'(t) + y\varphi_1'\varphi_2' f''(t) + (\varphi_2)^2 f''(t) \\
&= f'(t)(\varphi_1' + xy\varphi_{11}'' + x\varphi_{12}'' + y\varphi_{21}'' + \varphi_{22}) + f''(t)(xy(\varphi_1')^2 + (x+y)\varphi_1'\varphi_2' + (\varphi_2)^2) \\
&= f'(t)\left(\varphi_1' + xy\varphi_{11}'' + (x+y)\varphi_{12}'' + \varphi_{22}''\right) + f''(t)\left(xy(\varphi_1')^2 + (x+y)\varphi_1'\varphi_2' + (\varphi_2)^2\right)
\end{aligned}$$

✍ **习题 9.2.28** 设 $u = f(x^2 - y^2, \mathrm{e}^{xy})$, 求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x\partial y}$.

**证明**

令 $\begin{cases} v = x - at, \\ w = x + at; \end{cases}$ 则有 $u = \varphi(v) + \psi(w)$ 且 $\begin{cases} \frac{\partial v}{\partial x} = 1, \\ \frac{\partial w}{\partial x} = 1; \end{cases}$ 与 $\begin{cases} \frac{\partial v}{\partial t} = -a, \\ \frac{\partial w}{\partial t} = a; \end{cases}$

因此我们有 $\frac{\partial u}{\partial x} = \varphi'(v)\frac{\partial v}{\partial x} + \psi'(w)\frac{\partial w}{\partial x} = \varphi'(v) + \psi'(w)$,

故 $\frac{\partial^2 u}{\partial x^2} = \varphi''(v)\frac{\partial v}{\partial x} + \psi''(w)\frac{\partial w}{\partial x} = \varphi''(v) + \psi''(w)$.

同时 $\frac{\partial u}{\partial t} = \varphi'(v)\frac{\partial v}{\partial t} + \psi'(w)\frac{\partial w}{\partial t} = -a\varphi'(v) + a\psi'(w)$,

故 $\frac{\partial^2 u}{\partial x^2} = -a\varphi''(v)\frac{\partial v}{\partial x} + a\psi''(w)\frac{\partial w}{\partial x} = a^2\left(\varphi''(v) + \psi''(w)\right) = a^2\frac{\partial^2 u}{\partial x^2}$.

✍ **习题 9.2.32** 设变换 $\begin{cases} u = x - 2y, \\ v = x + ay \end{cases}$ 可把方程 $6\frac{\partial^2 z}{\partial x} + \frac{\partial z}{\partial x\partial y} - \frac{\partial^2 z}{\partial y} = 0$ 化为 $\frac{\partial^2 w}{\partial u} = 0$, 求常数 $a$.

**解** 由链式法则得

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v} \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}
\end{aligned}$$

得

$$\begin{aligned}
\frac{\partial^2 z}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u} + 2\frac{\partial z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v} \\
\frac{\partial z}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(-2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}\right) = -2\frac{\partial^2 z}{\partial u} + (a-2)\frac{\partial z}{\partial u\partial v} + a\frac{\partial^2 z}{\partial v} \\
\frac{\partial^2 z}{\partial y} &= \frac{\partial}{\partial y}\left(-2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}\right) = \left(-2\frac{\partial}{\partial u} + a\frac{\partial}{\partial v}\right)\left(-2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}\right) = 4\frac{\partial z}{\partial u} - 4a\frac{\partial z}{\partial u\partial v} + a^2\frac{\partial z}{\partial v}
\end{aligned}$$

代入得

$$\begin{aligned}0 &= 6\frac{\partial^2 z}{\partial x} + \frac{\partial z}{\partial x\partial y} - \frac{\partial^2 z}{\partial y} \\ &= 6\frac{\partial^2 z}{\partial u} + 12\frac{\partial z}{\partial u\partial v} + 6\frac{\partial^2 z}{\partial v} + (a-2)\frac{\partial z}{\partial u\partial v} + a\frac{\partial^2 z}{\partial v} + 4\frac{\partial z}{\partial u} - 4a\frac{\partial z}{\partial u\partial v} + a^2\frac{\partial z}{\partial v} \\ &= (10+5a)\frac{\partial z}{\partial u\partial v} - (6+a-a^2)\frac{\partial^2 z}{\partial v}\end{aligned}$$

故 $a=3$ 或 $a=-2$, 但是当 $a=-2$ 时, 方程会变为 $0=0$, 无意义, 故 $a=3$.

**习题 9.3.1(1)** 证明下列方程在指定点的附近对 $y$ 有唯一解, 并求出 $y$ 对 $x$ 在该点处的一阶和二阶导数

(1) $x^2+xy+y^2=7$, 在 $(2,1)$ 处.

**解** 已知

$$F(x,y) = x^2+xy+y^2-7 \in C^1(\mathbb{R}^2)$$

且

$$F(2,1)=0, \quad F'_y(2,1)=4\neq 0$$

由隐函数定理, 知该方程在 $(2,1)$ 附近有唯一解 $y(x)$.

两边微分得到

$$\begin{aligned}2x\,\mathrm{d}x + y\,\mathrm{d}x + x\,\mathrm{d}y + 2y\,\mathrm{d}y = 0 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x+y}{x+2y} \\ \Rightarrow y'(2) = \frac{-5}{4}\end{aligned}$$

进一步

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= -\frac{(2\,\mathrm{d}x+\mathrm{d}y)(x+2y)-(2x+y)(\mathrm{d}x+2\,\mathrm{d}y)}{(x+2y)^2}\,\mathrm{d}x \\ &= \frac{(2x+y)-2(x+2y)}{(x+2y)^2} + \frac{2(2x+y)-(x+2y)}{(x+2y)^2}\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \frac{6(x^2+xy+y^2)}{(x+2y)^3} \\ \Rightarrow y''(2) &= -\frac{21}{32}\end{aligned}$$

**习题 9.3.2(2)** 求由下列方程所确定的隐函数的导数.

(2) $\ln\sqrt{x^2+y^2} = \arctan\frac{y}{x}$, 求 $\frac{\mathrm{d}y}{\mathrm{d}x}$ 和 $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$;

(5) $\frac{x}{z} = \ln\frac{z}{y}$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

**解**

1. 两边微分, 得到

$$\frac{x^2\,\mathrm{d}x + y\,\mathrm{d}y}{x^2+y^2} = \left(\frac{-y}{x^2}\,\mathrm{d}x + \frac{1}{x}\,\mathrm{d}y\right)$$

整理得

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}$$

故

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{1}{(x-y)^2}\left(((\mathrm{d}x+\mathrm{d}y)(x-y)-(x+y)(\mathrm{d}x+\mathrm{d}y))\right)$$

代入 $\mathrm{d}y = \frac{x+y}{x-y}$ 得

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{2y}{(x-y)^2} + \frac{2y(x+y)}{(x-y)^3} = \frac{4y^2}{(x-y)^3}$$

2. 两边微分, 得到

$$\frac{1}{z}\,\mathrm{d}x - \frac{x}{z^2}\,\mathrm{d}z$$

整理得

$$\frac{\partial z}{\partial x} = \frac{z}{x+z}$$

故

$$\frac{\partial z}{\partial y} = \frac{z^2}{y(x+z)}$$

✍ **习题 9.3.4(1)** 试求由下列方程所确定的隐函数的微分.

(1) $\cos^2 x + \cos^2 y + \cos^2 z = 1$, 求 $\mathrm{d}z$.

**解** 两边微分, 得到

$$-2\sin x\cos x\,\mathrm{d}x - 2\sin y\cos y\,\mathrm{d}y - 2\sin z\cos z\,\mathrm{d}z = 0$$

整理得

$$\mathrm{d}z = -\frac{\sin x\cos x\,\mathrm{d}x + \sin y\cos y\,\mathrm{d}y}{\sin z\cos z} = -\frac{\sin 2x\,\mathrm{d}x + \sin 2y\,\mathrm{d}y}{2\sin 2z}.$$