

## Week 8

### 8.1 Apr 14 补充题, CH10.5, 6, 8.

#### 习题 补充题 1

用五种方法计算  $\Omega: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$  的体积  $V(\Omega)$ .

解 先一后二, 先二后一, 球坐标换元, 柱坐标换元, 放缩

$$\begin{aligned} V &= \int_{-a}^a dx \iint_{\frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 - \frac{x^2}{a^2}} dy dz \\ &= \int_{-a}^a bc \left(1 - \frac{x^2}{a^2}\right) \pi dx = \frac{4}{3} \pi abc \end{aligned}$$

$$\begin{aligned} V &= \iint_{\frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1} \int_{-a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}}^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} dx dy dz \\ &= \int_{-b}^b dy \int_{-c}^c \sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}} dz \\ &= \frac{4}{3} \pi abc \end{aligned}$$

令  $(x, y, z) = (a \sin \theta \cos \varphi, b \sin \theta \sin \varphi, c \cos \theta)$ , 则

$$\begin{aligned} V &= \int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^1 r^2 \sin \theta dr \\ &= \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \int_0^1 r^2 dr = \frac{4}{3} \pi abc \end{aligned}$$


#### 习题 补充题 2

计算  $I_1 = \iiint_{x^2+y^2+z^2 \leq 1} \cos(ax+by+cz) dx dy dz$  与  $I_2 = \iiint_{x^2+y^2+z^2 \leq 1} (ax+by+cz)^m dx dy dz$  的值. 其中  $(a, b, c) \neq \theta$  为常向量,  $m \in N^+$ .

解

$$\begin{aligned} I_1 &= \iiint_{x^2+y^2+z^2 \leq 1} \cos(ax+by+cz) dV = \iiint_{x^2+y^2+z^2 \leq 1} \cos\left(\sqrt{a^2+b^2+c^2} x\right) dV \\ &= \int_{-1}^1 dx \iint_{y^2+z^2 \leq 1-x^2} \cos(rx) dS = \pi \int_{-1}^1 (1-x^2) \cos(rx) dx \\ &= -\frac{4\pi}{\sqrt{a^2+b^2+c^2}} \cos \sqrt{a^2+b^2+c^2} + \frac{4\pi}{(a^2+b^2+c^2)^{3/2}} \sin \sqrt{a^2+b^2+c^2} \end{aligned}$$

$$\begin{aligned}
I_2 &= \iiint_{x^2+y^2+z^2 \leq 1} (ax+by+cz)^m dV = r^m \iiint_{x^2+y^2+z^2 \leq 1} x^m dV \\
&= r^m \int_{-1}^1 (1-x^2)x^m dV = \pi \left( \frac{1-(-1)^{m+1}}{m+1} - \frac{1-(-1)^{m+3}}{m+3} \right) \\
&= \frac{2\pi}{(m+1)(m+3)} (1-(-1)^{m+1})
\end{aligned}$$

 **习题 CH10.5** 试求圆盘  $(x-a)^2 + (y-a)^2 \leq a^2$  与曲线  $(x^2+y^2)^2 = 8a^2xy$  的所围部分相交的区域  $D$  的面积.

**解** 由于所求区域为  $(x-a)^2 + (y-a)^2 \leq a^2$  与  $(x^2+y^2)^2 \leq 8a^2xy$  的交集,  $xy$  总是可以用  $u = \frac{x+y}{\sqrt{2}}, v = \frac{x-y}{\sqrt{2}}$  来变换成两个分开的  $u^2 - v^2$  的形式来简化计算.

令

$$u = \frac{x+y}{\sqrt{2}}, v = \frac{x-y}{\sqrt{2}}$$

则区域为

$$u^2 + v^2 - 2\sqrt{2}au + a^2 \leq 0, (u^2 + v^2)^2 \leq 4a^2(u^2 - v^2)$$

所求为

$$\frac{1}{2} \iint_{D'} du dv$$

此时可以反解计算, 但仍然比较麻烦, 需要对带累次根号的函数积分.

这时候再做换元, 令

$$u = r \cos \theta, v = r \sin \theta$$

则区域为

$$r^2 - 2\sqrt{2}ar \cos \theta + a^2 \leq 0, r^2 \leq 4a^2 \cos 2\theta$$

所求为

$$\frac{1}{2} \iint_{D''} r dr d\theta.$$

对于限制区域的两个方程, 可以分别解出  $r$  满足的范围为

$$[\sqrt{2}a \cos \theta - a\sqrt{\cos 2\theta}, \sqrt{2}a \cos \theta + a\sqrt{\cos 2\theta}], [0, 2a\sqrt{\cos 2\theta}]$$

由于  $\cos \theta = \sqrt{\frac{\cos 2\theta + 1}{2}}$ , 综合二者比较可得  $[a(\sqrt{1 + \cos 2\theta} - \sqrt{\cos 2\theta}), 2a\sqrt{\cos 2\theta}]$

因此得到

$$\cos 2\theta \geq \frac{1}{8}, \theta \in \left[-\frac{1}{2} \arccos \frac{1}{8}, \frac{1}{2} \arccos \frac{1}{8}\right]$$

因此此时得到

$$\begin{aligned}
 S &= \int_{-\frac{1}{2} \arccos \frac{1}{8}}^{\frac{1}{2} \arccos \frac{1}{8}} d\theta \int_{a(\sqrt{1+\cos 2\theta}-\sqrt{\cos 2\theta})}^{2a\sqrt{\cos 2\theta}} r dr \\
 &= \int_{-\frac{1}{2} \arccos \frac{1}{8}}^{\frac{1}{2} \arccos \frac{1}{8}} \frac{1}{2} \left( \left( 2a\sqrt{\cos 2\theta} \right)^2 - \left( a(\sqrt{1+\cos 2\theta}-\sqrt{\cos 2\theta}) \right)^2 \right) d\theta \\
 &= \frac{a^2}{2} \int_{-\frac{1}{2} \arccos \frac{1}{8}}^{\frac{1}{2} \arccos \frac{1}{8}} 2 \cos 2\theta - 1 + 2\sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta} d\theta \\
 &= a^2 \int_0^{\frac{1}{2} \arccos \frac{1}{8}} 2 \cos 2\theta - 1 + 2\sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta} d\theta \\
 &= a^2 (\sin 2\theta - \theta) \Big|_0^{\frac{1}{2} \arccos \frac{1}{8}} + 2a^2 \int_0^{\frac{1}{2} \arccos \frac{1}{8}} \sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta} d\theta \\
 &= a^2 \left( \frac{3\sqrt{7}}{8} - \frac{1}{2} \arccos \frac{1}{8} \right) + 2a^2 \int_0^{\frac{1}{2} \arccos \frac{1}{8}} \sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta} d\theta
 \end{aligned}$$

并计算

$$\begin{aligned}
 &\int_0^{\frac{1}{2} \arccos \frac{1}{8}} \sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta} d\theta \\
 &\stackrel{t=\cos 2\theta}{=} \int_{\frac{1}{8}}^1 \sqrt{1+t}\sqrt{t} \frac{1}{2\sqrt{1-t^2}} dt \\
 &= \frac{1}{2} \int_{\frac{1}{8}}^1 \sqrt{\frac{t}{1-t}} dt \\
 &= \frac{1}{2} \int_{\frac{1}{8}}^1 \frac{t}{\sqrt{\frac{1}{4} - (t - \frac{1}{2})^2}} dt \\
 &= \frac{1}{4} \int_{\frac{1}{8}}^1 \frac{2t-1}{\sqrt{\frac{1}{4} - (t - \frac{1}{2})^2}} + \frac{1}{\sqrt{\frac{1}{4} - (t - \frac{1}{2})^2}} dt \\
 &\stackrel{s=2t-1}{=} \frac{1}{4} \int_{-\frac{3}{4}}^1 \frac{s}{\sqrt{1-s^2}} + \frac{1}{\sqrt{1-s^2}} ds \\
 &= \frac{1}{4} \left( -\sqrt{1-s^2} + \arcsin s \right) \Big|_{-\frac{3}{4}}^1 \\
 &= \frac{\pi}{8} + \frac{1}{4} \arcsin \frac{3}{4} + \frac{\sqrt{7}}{16} \\
 &= \frac{1}{4} \arccos -\frac{3}{4} + \frac{\sqrt{7}}{16}
 \end{aligned}$$

带回得到

$$\begin{aligned} S &= a^2 \left( \frac{3\sqrt{7}}{8} - \frac{1}{2} \arccos \frac{1}{8} + 2 \left( \frac{1}{4} \arccos \frac{3}{4} + \frac{\sqrt{7}}{16} \right) \right) \\ &= a^2 \left( \frac{\sqrt{7}}{2} + \frac{1}{2} \left( \arccos \frac{3}{4} - \arccos \frac{1}{8} \right) \right) \\ &= a^2 \left( \frac{\sqrt{7}}{2} + \arccos \frac{5\sqrt{2}}{8} \right) \end{aligned}$$

其中  $\frac{1}{2} \left( \arccos \frac{3}{4} - \arccos \frac{1}{8} \right)$  可以考虑判断一下角的范围, 之后利用三角函数直接计算, 相当于求  $\frac{1}{2}(\theta_1 - \theta_2)$ , 由于  $0 < \theta_1 - \theta_2 < \frac{\pi}{2}$ , 因此  $\frac{1}{2}(\theta_1 - \theta_2) = \arccos \cos \frac{1}{2}(\theta_1 - \theta_2)$  因此

$$\begin{aligned} &\cos \frac{1}{2}(\theta_1 - \theta_2) \\ &= \sqrt{\frac{\cos(\theta_1 - \theta_2) + 1}{2}} \\ &= \sqrt{\frac{\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + 1}{2}} \\ &= \sqrt{\frac{-\frac{3}{4} \cdot \frac{1}{8} + \frac{\sqrt{7}}{4} \cdot \frac{3\sqrt{7}}{8} + 1}{2}} \\ &= \sqrt{\frac{\frac{9}{16} + 1}{2}} \\ &= \sqrt{\frac{25}{32}} \\ &= \frac{5\sqrt{2}}{8} \end{aligned}$$

即得.

### 习题 CH10.6 计算曲面

$$(x^2 + y^2)^2 + z^4 = y$$

所围成的体积  $V$ .

**解** 作球坐标换元

$$(x, y, z) = (r \sin \varphi \cos \theta, r \sin \varphi \sin \theta, r \cos \varphi)$$

原题中的积分区域  $\Omega$  关于  $xy$  平面和  $yz$  平面对称, 又  $y \geq 0$ , 由对称性可以只考虑

$$\Omega_1 = \{(x, y, z) \mid (x^2 + y^2)^2 + z^4 \leq y, x \geq 0, z \geq 0\}$$

的部分. 则

$$dx dy dz = r^2 \sin \varphi dr d\varphi d\theta$$

其中

$$(x^2 + y^2)^2 + z^4 \leq y \Rightarrow r^4 \sin^4 \varphi + r^4 \cos^4 \varphi \leq r \sin \varphi \sin \theta \Rightarrow 0 \leq r \leq \sqrt[3]{\frac{\sin \theta \sin \varphi}{\sin^4 \varphi + \cos^4 \varphi}}$$

$$x \geq 0, y \geq 0, z \geq 0 \Rightarrow \theta \in [0, \frac{\pi}{2}], \varphi \in [0, \frac{\pi}{2}]$$

故

$$\begin{aligned} V &= 4 \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\sqrt[3]{\frac{\sin \theta \sin \varphi}{\sin^4 \varphi + \cos^4 \varphi}}} r^2 \sin \varphi dr \\ &= 4 \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \sin \varphi \left( \frac{1}{3} \frac{\sin \theta \sin \varphi}{\sin^4 \varphi + \cos^4 \varphi} \right) \\ &= \frac{4}{3} \int_0^{\frac{\pi}{2}} \sin \theta d\theta \int_0^{\frac{\pi}{2}} \frac{\sin^2 \varphi}{\sin^4 \varphi + \cos^4 \varphi} d\varphi \\ &= \frac{4}{3} \cdot \int_0^{\frac{\pi}{2}} \frac{\sin^2 \varphi (\sin^2 \varphi + \cos^2 \varphi)}{\sin^4 \varphi + \cos^4 \varphi} d\varphi \\ &= \frac{4}{3} \cdot \int_0^{+\infty} \frac{t^2}{1+t^4} dt \\ &= \frac{\sqrt{2}}{3} \pi \end{aligned}$$

 习题 CH10.8 证明:

$$\iint_{x^2+y^2 \leq 1} f(ax+by+c) dx dy = 2 \int_{-1}^1 \sqrt{1-t^2} f(t\sqrt{a^2+b^2}+c) dt$$

解 令  $t = \frac{ax+by}{\sqrt{a^2+b^2}}, s = \frac{bx-ay}{\sqrt{a^2+b^2}}$ , 相当于正交变换,  $|J| = 1$ , 得到

$$\begin{aligned} I &= \iint_{t^2+s^2 \leq 1} f(t\sqrt{a^2+b^2}+c) dt ds \\ &= 2 \int_{-1}^1 \sqrt{1-t^2} f(t\sqrt{a^2+b^2}+c) dt \end{aligned}$$

## 8.2 Apr 16 ex10.3:5(8),6,12,14,16,19;CH10:4

 习题 ex10.3.5(8)

计算下列曲面围成的立体体积.

8.  $(x^2 + y^2 + z^2)^2 = a^3 x$ .

解 取球坐标换元,  $x = r \cos \theta, y = r \sin \theta \cos \varphi, z = r \sin \theta \sin \varphi$ , 则

$$dx dy dz = r^2 \sin \theta dr d\theta d\varphi$$

积分区域为

$$\Omega = \{(x, y, z) \mid (x^2 + y^2 + z^2)^2 = a^3 x\}$$

$$\Rightarrow (r^2)^2 = a^3 r \cos \theta \Rightarrow r^3 = a^3 \cos \theta$$

故  $\varphi \in [0, 2\pi], \theta \in [0, \frac{\pi}{2}], r \in [0, a\sqrt[3]{\cos \theta}]$

$$\begin{aligned} V &= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^{a\sqrt[3]{\cos \theta}} r^2 \sin \theta dr \\ &= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \sin \theta \left( \frac{1}{3} a^3 \cos \theta \right) \\ &= \frac{\pi}{3} a^3 \end{aligned}$$

### 习题 ex10.3.6

求函数  $f(x, y, z) = x^2 + y^2 + z^2$  在区域  $x^2 + y^2 + z^2 \leq x + y + z$  上的平均值.

解

$$x^2 + y^2 + z^2 \leq x + y + z \Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 \leq \frac{3}{4}$$

于是令

$$(x, y, z) = \left(\frac{1}{2} + \frac{1}{2}r \sin \theta \cos \varphi, \frac{1}{2} + \frac{1}{2}r \sin \theta \sin \varphi, \frac{1}{2} + \frac{1}{2}r \cos \theta\right)$$

区域的体积

$$V(\Omega) = \frac{\sqrt{3}}{2}\pi$$

$f$  在区域上的积分

$$I = \iiint_{\Omega} (x^2 + y^2 + z^2) dx dy dz$$

$$\begin{aligned} I &= \iiint_{\Omega} (x^2 + y^2 + z^2) dx dy dz \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 \left(\frac{1}{4} + \frac{1}{4}r^2 + r^2 \sin^2 \theta\right) r^2 \sin \theta dr \\ &= \frac{3\sqrt{3}}{5}\pi \end{aligned}$$

$$\text{故平均值 } \bar{f} = \frac{I}{V} = \frac{6}{5}.$$

### 习题 ex10.3.12

一个物体是由两个半径各为  $R$  和  $r (R \geq r)$  的同心球所围成, 已知材料的密度和到球心的距离成反比, 且在距离为 1 的球面出密度为  $k$ , 求该物体的质量.

解

$$\begin{aligned} I &= \iiint_{r^3 \leq x^2 + y^2 + z^2 \leq R^3} \frac{k}{\sqrt{x^2 + y^2 + z^2}} dx dy dz \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_r^R \frac{k}{r^2} \cdot r^2 \sin \theta dr \\ &= 2\pi k(R^2 - r^2) \end{aligned}$$

### 习题 ex10.3.14

有一个均匀质地的薄板, 它是由半径为  $a$  的半圆和一个长方形拼接而成, 为了使重心正好

在圆心上, 问长方形的宽  $b$  应为多少?

**解** 设圆心为原点, 薄板所在的区域为

$$\Omega = \{(x, y) \mid x^2 + y^2 \leq a^2, y \geq 0\} \cup \{(x, y) \mid -a \leq x \leq a, -b \leq y \leq 0\}$$

设重心坐标为  $(\bar{x}, \bar{y})$ , 则

$$\bar{x} = \frac{1}{I} \iint_{\Omega} x \, dx \, dy = 0$$

$$\bar{y} = 0 \Rightarrow \iint_{\Omega} y \, dx \, dy = 0 \Rightarrow \iint_{x^2+y^2 \leq a^2, y \geq 0} y \, dx \, dy + \iint_{-a \leq x \leq a, -b \leq y \leq 0} y \, dx \, dy = 0$$

其中

$$\iint_{x^2+y^2 \leq a^2, y \geq 0} y \, dx \, dy = \frac{1}{2} \int_0^{\pi} \int_0^a r \sin \theta \cdot r \, dr \, d\theta = \frac{1}{3} \pi a^3$$

$$\iint_{-a \leq x \leq a, -b \leq y \leq 0} y \, dx \, dy = \int_{-a}^a dx \int_{-b}^0 y \, dy = -\frac{1}{2} ab^2$$

因此

$$\frac{1}{3} \pi a^3 - \frac{1}{2} ab^2 = 0 \Rightarrow b = \frac{\sqrt{6}}{3} a$$

### 习题 ex10.3.16

设球体  $x^2 + y^2 + z^2 \leq 2az$  内各点密度与各点到原点的距离成反比, 求其重心坐标.

**解** 球体质量为

$$\begin{aligned} I &= \iiint_{x^2+y^2+z^2 \leq 2az} \frac{1}{\sqrt{x^2+y^2+z^2}} \, dx \, dy \, dz \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\sqrt{2a^2-2az}} r^2 \sin \theta \cdot \frac{1}{r} \, dr \\ &= 4\pi a^3 \end{aligned}$$

球体重心为

$$\begin{aligned} \bar{x} &= \frac{1}{I} \iiint_{x^2+y^2+z^2 \leq 2az} x \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} \, dx \, dy \, dz = 0 \\ \bar{y} &= \frac{1}{I} \iiint_{x^2+y^2+z^2 \leq 2az} y \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} \, dx \, dy \, dz = 0 \\ \bar{z} &= \frac{1}{I} \iiint_{x^2+y^2+z^2 \leq 2az} z \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} \, dx \, dy \, dz = \frac{4}{5} a \end{aligned}$$

故重心坐标为  $(0, 0, \frac{4}{5}a)$ .

### 习题 ex10.3.19

求密度为  $\rho$  的均匀球锥体对在其顶点为 1 单位质量的质点的引力, 设球的半径为  $R$ , 而轴截面的扇形的角度为  $2\alpha$ .

**解** 设球锥体的顶点为  $O$ , 球锥体的区域为

$$\Omega = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq R^2, z \geq 0, \frac{y}{x} \leq \tan \alpha\}$$

则  $(x, y, z)$  处  $dx dy dz$  的微元带来的引力为

$$d\mathbf{F} = -\frac{G\rho(x, y, z) dx dy dz}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z)$$

故总引力为


$$F_x = \iiint_{\Omega} -\frac{G\rho}{(x^2 + y^2 + z^2)^{3/2}} x dx dy dz = 0$$

$$F_y = \iiint_{\Omega} -\frac{G\rho}{(x^2 + y^2 + z^2)^{3/2}} y dx dy dz = 0$$

$$F_z = \iiint_{\Omega} -\frac{G\rho}{(x^2 + y^2 + z^2)^{3/2}} z dx dy dz = \pi GR\rho \sin^2 \alpha$$

因此

$$\|\mathbf{F}\| = \sqrt{F_x^2 + F_y^2 + F_z^2} = \pi GR\rho \sin^2 \alpha$$

 **习题 CH10.4** 设  $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ , 求  $I = \iint_D \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| dx dy$ .

**解** 不难得到

$$\begin{aligned} \iint_D \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| dx dy &= \iint_{B(0,1)} \left( x^2 + y^2 - \frac{x+y}{\sqrt{2}} \right) dx dy \\ &\quad + 2 \iint_{B(\frac{1}{2\sqrt{2}}, \frac{1}{2})} \left( \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right) dx dy \end{aligned}$$

一方面, 令  $x = r \cos \theta$ ,  $y = r \sin \theta$ , 则

$$\begin{aligned} I_1 &= \iint_{B(0,1)} \left( x^2 + y^2 - \frac{x+y}{\sqrt{2}} \right) dx dy \\ &= \int_0^{2\pi} d\theta \int_0^1 r \left( r^2 + \frac{r \cos \theta + r \sin \theta}{2} \right) dr \\ &= \int_0^{2\pi} \left( \frac{1}{4} + \frac{1}{3\sqrt{2}} (\cos \theta + \sin \theta) \right) d\theta \\ &= \frac{\pi}{2} \end{aligned}$$

另一方面, 令  $x = \frac{1}{2\sqrt{2}} + \frac{r}{2} \cos \theta$ ,  $y = \frac{1}{2\sqrt{2}} + \frac{r}{2} \sin \theta$ , 则

$$\begin{aligned} I_2 &= \iint_{B(\frac{1}{2\sqrt{2}}, \frac{1}{2})} \left( \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right) dx dy \\ &= \frac{1}{16} \int_0^{2\pi} d\theta \int_0^1 r(1 - r^2) dr \\ &= \frac{\pi}{32} \end{aligned}$$



因此

$$\iint_D \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| dx dy = I_1 - 2I_2 = \frac{9\pi}{16}$$

### 8.3 Apr 16 补充题,ex10.4:1;CH10:3

#### 习题 补充题 1

推导半径为  $R$  的  $n$  维球体的体积公式

$$V_n(R) = \frac{R^n}{n!} \int_0^{\frac{\pi}{2}} \sin^{n-1} \theta d\theta$$

#### 习题 化简

$$I = \int \cdots \int_{\Omega} f \left( \sum_{i=1}^6 a_i x_i \right) dx_1 \cdots dx_6$$

这里  $\Omega$  是  $\mathbb{R}^6$  的单位球.

**解** 对于  $\mathbf{a} = (a_1, \dots, a_6)$ , 记  $a = |\mathbf{a}|$ , 则旋转坐标系可得

$$I = \int \cdots \int_{\Omega} f(ax_1) dx_1 \cdots dx_6 = \int_{-1}^1 m(B(x_1)) f(ax_1) dx_1 = \frac{8\pi^2}{15} \int_{-1}^1 (1-x^2)^{\frac{5}{2}} f(ax) dx$$

这里

$$B(x_1) = \left\{ \mathbf{x}' = (\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6) \mid \|\mathbf{x}'\|^2 < 1 - x_1^2 \right\}$$

#### 习题 ex10.4.1

计算下列  $n$  重积分.

1.  $\int \cdots \int_{[0,1]^n} (x_1^2 + x_2^2 + \cdots + x_n^2) dx_1 \cdots dx_n;$
2.  $\int \cdots \int_{[0,1]^n} (x_1 + x_2 + \cdots + x_n)^2 dx_1 \cdots dx_n;$
3.  $\int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} x_1 \cdot x_2 \cdots x_n dx_n;$

**解**

1.

$$\begin{aligned} \int \cdots \int_{[0,1]^n} (x_1^2 + \cdots + x_n^2) dx_1 \cdots dx_n &= \sum_{i=1}^n \int \cdots \int_{[0,1]^n} x_i^2 dx_1 \cdots dx_n \\ &= \sum_{i=1}^n \int_0^1 x_i^2 dx_i = \frac{n}{3} \end{aligned}$$

2.

$$\begin{aligned}
& \int \cdots \int_{[0,1]^n} (x_1 + \cdots + x_n)^2 dx_1 \cdots dx_n \\
&= \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_n)^2 dx_n \\
&= \frac{1}{3} \int_0^1 \cdots \int_0^1 ((x_1 + \cdots + x_{n-1} + 1)^3 - (x_1 + \cdots + x_{n-1})^3) dx_{n-1} \\
&= \frac{1}{3} \int_0^1 \cdots \int_0^1 (3(x_1 + \cdots + x_{n-1})^2 + 3(x_1 + \cdots + x_{n-1}) + 1) dx_{n-1} \\
&= \frac{1}{3} + \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1})^2 dx_{n-1} + \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1}) dx_{n-1} \\
&= \frac{1}{3} + \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1})^2 dx_{n-1} + \sum_{i=1}^{n-1} \int_0^1 \cdots \int_0^1 x_i dx_{n-1} \\
&= \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1})^2 dx_{n-1} + \frac{n-1}{2} + \frac{1}{3} \\
&= \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_{n-2})^2 dx_{n-2} + \frac{n-1}{2} + \frac{n-2}{2} + \frac{2}{3} \\
&= \cdots \\
&= \frac{n(n-1)}{4} + \frac{n}{3} = \frac{n(3n+1)}{12}
\end{aligned}$$

3.

$$\begin{aligned}
\int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} x_1 x_2 \cdots x_n dx_n &= \frac{1}{2} \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} x_1 x_2 \cdots x_{n-2} x_{n-1}^3 dx_{n-1} \\
&= \frac{1}{8} \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-3}} x_1 x_2 \cdots x_{n-3} x_{n-2}^5 dx_{n-2} \\
&= \cdots \\
&= \frac{1}{(n-1)! 2^{n-1}} \int_0^1 x_1^{2^{n-1}} dx_1 \\
&= \frac{1}{n! 2^n}
\end{aligned}$$

 习题 CH10.3 计算

$$I_1 = \int_0^1 \sin\left(\ln \frac{1}{x}\right) \cdot \frac{x^b - x^a}{\ln x} dx, I_2 = \int_0^1 \cos\left(\ln \frac{1}{x}\right) \cdot \frac{x^b - x^a}{\ln x} dx$$

解

$$\begin{aligned}
I_1 &= \int_0^1 \sin\left(\ln \frac{1}{x}\right) \cdot \frac{x^b - x^a}{\ln x} dx \\
&= \int_0^1 \sin\left(\ln \frac{1}{x}\right) \left(\int_a^b x^y dy\right) dx \\
&= \int_a^b dy \int_0^1 \sin\left(\ln \frac{1}{x}\right) \cdot x^y dx \text{ (化为二重积分)} \\
&= \int_a^b \left(\int_0^1 \sin\left(\ln \frac{1}{x}\right) \cdot x^y dx\right) dy
\end{aligned}$$

因此问题转化为计算

$$g_1(y) = \int_0^1 \sin\left(\ln \frac{1}{x}\right) \cdot x^y dx, g_2(y) = \int_0^1 \cos\left(\ln \frac{1}{x}\right) \cdot x^y dx$$

分部一下, 得到

$$\begin{aligned}
g_1(y) &= \sin\left(\ln \frac{1}{x}\right) \cdot \frac{x^{y+1}}{y+1} \Big|_0^1 - \int_0^1 \cos\left(\ln \frac{1}{x}\right) \left(-\frac{1}{x}\right) \cdot \frac{x^{y+1}}{y+1} dx \\
&= \int_0^1 \cos\left(\ln \frac{1}{x}\right) \cdot \frac{x^y}{y+1} \\
&= \frac{1}{y+1} g_2(y) \\
g_2(y) &= \cos\left(\ln \frac{1}{x}\right) \cdot \frac{x^{y+1}}{y+1} \Big|_0^1 + \int_0^1 \sin\left(\ln \frac{1}{x}\right) \left(-\frac{1}{x}\right) \cdot \frac{x^{y+1}}{y+1} dx \\
&= \frac{1}{y+1} - \int_0^1 \sin\left(\ln \frac{1}{x}\right) \cdot \frac{x^y}{y+1} \\
&= \frac{1}{y+1} - \frac{1}{y+1} g_1(y)
\end{aligned}$$

因此  $g_1(y) = \frac{1}{1+(y+1)^2}, g_2(y) = \frac{y+1}{1+(y+1)^2}$

$$\begin{aligned}
I_1 &= \int_a^b \frac{1}{1+(y+1)^2} dy = \arctan(a+1) - \arctan(b+1) \\
I_2 &= \int_a^b \frac{y+1}{1+(y+1)^2} dy = \frac{1}{2} \ln(1+(a+1)^2) - \frac{1}{2} \ln(1+(b+1)^2)
\end{aligned}$$