第10讲：多元函数微分法习题课(I)

例1. 设方程：$u^3-3(x+y)u^2+z^3=0$ 确定了隐函数 $u=f(x,y,z)$，求 du.

解法(一)：对方程两边取全微分 d：

$$d(u^3-3(x+y)u^2+z^3)=d(0)=0 \Rightarrow d(u^3)-3d((x+y)u^2)+d(z^3)=0$$

$$\Rightarrow 3u^2du-3(dx+dy)u^2-3(x+y)2udu+3z^2dz=0$$，解得

$$du=\frac{3u^2dx+3u^2dy-3z^2dz}{3u^2-6u(x+y)}=\frac{u^2dx+u^2dy-z^2dz}{u^2-2u(x+y)}$$

解法(二)：令 $F(x,y,z,u)=u^3-3(x+y)u^2+z^3$，其中 x,y,z,u 地位相同，则 $F'_u=3u^2-6(x+y)u$；$F'_x=-3u^2$，$F'_y=-3u^2$，$F'_z=3z^2$.

且 $F'_u=3u^2-6(x+y)u\neq0$. 从而 $\frac{\partial u}{\partial x}=-F'_x/F'_u=+3u^2/(3u^2-6(x+y)u)$

$=\frac{u^2}{u^2-2(x+y)u}$；$\frac{\partial u}{\partial y}=-F'_y/F'_u=3u^2/(3u^2-6(x+y)u)=\frac{u^2}{u^2-2(x+y)u}$，$\frac{\partial u}{\partial z}=-\frac{F'_z}{F'_u}$

$=-\frac{3z^2}{3u^2-6(x+y)u}=\frac{-z^2}{u^2-2(x+y)u}$，于是

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=\frac{u^2dx+u^2dy-z^2dz}{u^2-2(x+y)u}$$

解法(三)：对方程两边分别对 x,y,z 求偏导，解出 $\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial y}$，$\frac{\partial u}{\partial z}$，再代入 $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$ 即可。

例2.(ex9.2/30，$u=u(x,y)\in C^2$)

试证方程：$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}\cdot 2+\frac{\partial u}{\partial y}\cdot 6=0$ 在

线性变换 $\begin{cases}\xi=x+y\\ \eta=3x-y\end{cases}$ 下，可化简为：$\frac{\partial^2 u}{\partial \eta\partial \xi}+\frac{1}{2}\frac{\partial u}{\partial \xi}=0$.

证法(一)：从线性变换 $\begin{cases}\xi=x+y\\ \eta=3x-y\end{cases}$ 可得线性逆变换：

$$\begin{cases}x=\frac{1}{4}(\xi+\eta)\\ y=\frac{1}{4}(3\xi-\eta)\end{cases}\Rightarrow \frac{\partial x}{\partial \xi}=\frac{1}{4}=\frac{\partial x}{\partial \eta};\ \frac{\partial y}{\partial \xi}=\frac{3}{4},\ \frac{\partial y}{\partial \eta}=-\frac{1}{4}.$$

从 $\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}=\frac{\partial u}{\partial x}\frac{1}{4}+\frac{\partial u}{\partial y}\frac{3}{4}\Rightarrow$

$$\frac{\partial^2 u}{\partial \eta\partial \xi}=\left(\frac{\partial u}{\partial \xi}\right)'_\eta=\left(\frac{1}{4}\frac{\partial u}{\partial x}\right)'_\eta+\left(\frac{3}{4}\frac{\partial u}{\partial y}\right)'_\eta=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \eta}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial \eta}\right)+$$

$$\frac{3}{4}\left(\frac{\partial^2 u}{\partial y\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \eta}\right)=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\frac{1}{4}+\frac{\partial^2 u}{\partial x\partial y}\left(-\frac{1}{4}\right)\right)+\frac{3}{4}\left(\frac{\partial^2 u}{\partial y\partial x}\frac{1}{4}+\frac{\partial^2 u}{\partial y^2}\left(-\frac{1}{4}\right)\right)$$

$$=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}\right)$$

即 $\begin{cases}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}=16\frac{\partial^2 u}{\partial \eta\partial \xi}\\ \frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=4\frac{\partial u}{\partial \xi}\end{cases}$

根据原偏微分方程(PDE)化简为：$16\frac{\partial^2 u}{\partial \eta\partial \xi}+2\left(4\frac{\partial u}{\partial \xi}\right)=0$

即 $\frac{\partial^2 u}{\partial \eta\partial \xi}+\frac{1}{2}\frac{\partial u}{\partial \xi}=0$.

证法(2)：从线性变换 $\begin{cases}\xi = x+y\\ \eta = 3x-y\end{cases} \Rightarrow \dfrac{\partial \xi}{\partial x}=1, \dfrac{\partial \xi}{\partial y}=1,$

$\dfrac{\partial \eta}{\partial x}=3, \dfrac{\partial \eta}{\partial y}=-1$. 而 $u(x,y)$ 通过中间变量可视为 ξ,η 的函数.

从而 $\begin{cases}\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x} = \dfrac{\partial u}{\partial \xi}\cdot 1 + \dfrac{\partial u}{\partial \eta}\cdot 3 \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial y} = \dfrac{\partial u}{\partial \xi}\cdot 1 + \dfrac{\partial u}{\partial \eta}(-1)\end{cases}$，从而

$$\begin{cases}\dfrac{\partial^2 u}{\partial x^2} = \left(\dfrac{\partial u}{\partial \xi}\right)'_x + 3\left(\dfrac{\partial u}{\partial \eta}\right)'_x = \dfrac{\partial^2 u}{\partial \xi^2}\cdot 1 + \dfrac{\partial^2 u}{\partial \xi \partial \eta}\cdot 3 + 3\left(\dfrac{\partial^2 u}{\partial \eta \partial \xi}\cdot 1 + \dfrac{\partial^2 u}{\partial \eta^2}\cdot 3\right) \\ \dfrac{\partial^2 u}{\partial x \partial y} = \left(\dfrac{\partial u}{\partial \xi} - \dfrac{\partial u}{\partial \eta}\right)'_x = \dfrac{\partial^2 u}{\partial \xi^2}\cdot 1 + \dfrac{\partial^2 u}{\partial \xi \partial \eta}\cdot 3 - \left(\dfrac{\partial^2 u}{\partial \eta \partial \xi}\cdot 1 + \dfrac{\partial^2 u}{\partial \eta^2}\cdot 3\right) \\ \dfrac{\partial^2 u}{\partial y^2} = \left(\dfrac{\partial u}{\partial \xi} - \dfrac{\partial u}{\partial \eta}\right)'_y = \dfrac{\partial^2 u}{\partial \xi^2}\cdot 1 + \dfrac{\partial^2 u}{\partial \xi \partial \eta}(-1) - \left(\dfrac{\partial^2 u}{\partial \eta \partial \xi}\cdot 1 + \dfrac{\partial^2 u}{\partial \eta^2}(-1)\right)\end{cases}$$

即 $$\begin{cases}\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial \xi^2} + 6\dfrac{\partial^2 u}{\partial \eta \partial \xi} + 9\dfrac{\partial^2 u}{\partial \eta^2} \\ \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial \xi^2} + 2\dfrac{\partial^2 u}{\partial \eta \partial \xi} - 3\dfrac{\partial^2 u}{\partial \eta^2} \\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial \xi^2} - 2\dfrac{\partial^2 u}{\partial \eta \partial \xi} + \dfrac{\partial^2 u}{\partial \eta^2}\end{cases}$$

且 $2\dfrac{\partial u}{\partial x} + 6\dfrac{\partial u}{\partial y} = 2\left(\dfrac{\partial u}{\partial \xi} + 3\dfrac{\partial u}{\partial \eta}\right) + 6\left(\dfrac{\partial u}{\partial \xi} - \dfrac{\partial u}{\partial \eta}\right) = 8\dfrac{\partial u}{\partial \xi}$，从而

$\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} - 3\dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} + 6\dfrac{\partial u}{\partial y} = 0$，化为

$\dfrac{\partial^2 u}{\partial \xi^2}\cdot(1+2-3) + \dfrac{\partial^2 u}{\partial \xi \partial \eta}\cdot(6+4+6) + \dfrac{\partial^2 u}{\partial \eta^2}(9-6-3) + 8\dfrac{\partial u}{\partial \xi} = 0$

即 $16\dfrac{\partial^2 u}{\partial \xi \partial \eta} + 8\dfrac{\partial u}{\partial \xi} = 0 \Rightarrow \dfrac{\partial^2 u}{\partial \eta \partial \xi} + \dfrac{1}{2}\dfrac{\partial u}{\partial \xi} = 0$. (3).

例3.(ex9.3/13):

设 $u=f(x,y,z)$, $g(x^2,e^y,z)=0$, $y=\sin x$, 且 $f,g\in C^1$, $\frac{\partial g}{\partial z}\neq 0$. 求 $\frac{du}{dx}$.

解:(1). 从 $g(x^2,e^{\sin x},z)=0$ 及 $g_z'=\frac{\partial g}{\partial z}\neq 0$, 知. ~~故某~~

由方程 $g(x^2,e^{\sin x},z)=0$ 可確定 z 是 x,y 的函数, 从而

z 是 x 的复合函数. 故从 $u=f(x,y,z)$ 知. u 是 x 的一元函数

$$\frac{du}{dx}=f_1'\cdot 1+f_2'\cdot y_x'+f_3'\cdot z_x'=f_1'+f_2'\cdot\cos x+f_3'\cdot z_x'.$$

令 $F(x,y,z)=g(x^2,e^y,z)$, 则 $\begin{cases} F_x'(x,y,z)=g_1'\cdot 2x+g_2' e^{\sin x}\cos x \\ F_z'(x,y,z)=g_3'\cdot 1=g_3'. \end{cases}$

$$z_x'=-\frac{F_x'(x,y,z)}{F_z'(x,y,z)}=-\frac{2xg_1'+e^{\sin x}\cos x\, g_2'}{g_3'}$$

故 $\frac{du}{dx}=f_1'\cdot 1+f_2'\cdot y_x'+f_3'\cdot\left(-\frac{2xg_1'+e^{\sin x}\cos x\, g_2'}{g_3'}\right)$.

例4. 证明: 全微分也具有一阶微分形式不变性.

即, 若 $f(x,y)$ 可微, 则不论 x,y 是自变量, 还是中间

变量, 则 $z=f(x,y)$ 总有:

$dz = df(x,y) = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f'_x \cdot dx + f'_y \cdot dy$ （※1）

证(1)，当x, y是自变量且$z=f(x,y)$可微时，有

$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$，即（※1）成立.

(2)当$z=f(x,y)$可微，$\begin{cases} x=g(s,t) \\ y=h(s,t) \end{cases}$可微，且$f(g(s,t),h(s,t))$

有意义时，z通过中间变量x, y成为变量s, t的函数，此时，

有：$dx = \frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt$，$dy = \frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt$. 且

$dz = \frac{\partial z}{\partial s}ds + \frac{\partial z}{\partial t}dt = \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)ds + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)dt$

$= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt\right)$

$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$，即$x, y$是中间变量时，（※1）仍成立。

利用全微分的一阶微分形式不变性，可导出多元可微函数的如下的微分四则运算法则：

(1). $d(u \pm v) = du \pm dv$；(2). $d(u \cdot v) = vdu + udv$

(3) $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$，其中，$u, v$均可微，且$v \neq 0$.

证(1). 令 $f(u,v)=u+v$，则 $f(u,v)\in C^1$，从而 $f(u,v)$ 可微分.

无论 u,v 是自变量，还是中间变量，总有：

$$d(u+v)=df(u,v)=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv=1\cdot du+1\cdot dv=du+dv$$

从而有：$d(u\pm v)=du\pm dv$. 这里 d 是全微分。

证(3). 令 $f(u,v)=\frac{u}{v}$，$(v\neq 0)$. 则 $f\in C^1\Rightarrow d\left(\frac{u}{v}\right)=df(u,v)$

$$=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv=\frac{1}{v}du+\left(-\frac{u}{v^2}\right)dv=\frac{v\,du-u\,dv}{v^2}.$$

设 $f(x,y)\in C^2$，则 $z=f(x,y)\Rightarrow dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$$d(dz)\triangleq d^2z=d\left(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\right)=\left(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\right)'_x dx+\left(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\right)'_y dy$$

$$\xlongequal{dx,dy\text{视为常量}}\left(\frac{\partial^2 z}{\partial x^2}dx+\frac{\partial^2 z}{\partial x\partial y}dy\right)dx+\left(\frac{\partial^2 z}{\partial y\partial x}dx+\frac{\partial^2 z}{\partial y^2}dy\right)dy$$

$$=\frac{\partial^2 z}{\partial x^2}dx^2+\frac{\partial^2 z}{\partial x\partial y}dydx+\frac{\partial^2 z}{\partial y\partial x}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2 \quad (*)$$

$(*)$ 是 x,y 是自变量时的 $z=f(x,y)$ 的二阶全微分. 二阶以上的全微分通常不再具有形式不变性。例. 设 $z=x^2+y^2$，x,y 是自变量

则 $dz=2xdx+2ydy$，$d^2z=2(dx)^2+2(dy)^2$. 设 $\begin{cases}z=x^2+y^2\\ x=s^2\\ y=t^2\end{cases}$，则 $z=s^4+t^4$.

则 $\begin{cases}dx=2sds\\ dy=2tdt\end{cases}$

$dz=4s^3ds+4t^3dt$，$d^2z=12s^2(ds)^2+12t^2(dt)^2=3[4s^2(ds)^2+4t^2(dt)^2]=3((dx)^2+(dy)^2)$
$\neq 2((dx)^2+(dy)^2)$

例5.（ex9.3/11/(3)）. 设 $u=u(x,y)$，$v=v(x,y)$ 是由方程组：

$$\begin{cases} u=f(ux, v+y) \\ v=g(u-x, v^2y) \end{cases}$$ 所确定的隐函数组，求变换 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$

的 Jacobi（雅可比）行列式 $\begin{vmatrix} u'_x, & u'_y \\ v'_x & v'_y \end{vmatrix} \triangleq \frac{\partial(u,v)}{\partial(x,y)}$，$f,g \in C^1$.

解：令 $A=ux$，$B=v+y$，$E=u-x$，$F=v^2y$，则 $\begin{cases} u=f(A,B) \\ v=g(E,F) \end{cases}$.

方程组两边关于 x 求偏导：$\begin{cases} u'_x=f'_1\cdot(u+xu'_x)+f'_2\cdot(v'_x+0) \\ v'_x=g'_1\cdot(u'_x-1)+g'_2\cdot 2vv'_xy \end{cases}$

标准化：$\begin{cases} (xf'_1-1)u'_x+f'_2\cdot v'_x=-uf'_1 \\ g'_1u'_x+(2vyg'_2-1)v'_x=g'_1 \end{cases}$

令 $D=\begin{vmatrix} xf'_1-1, & f'_2 \\ g'_1 & 2vyg'_2-1 \end{vmatrix}$. 则 $D\neq 0$，且 $D_1=\begin{vmatrix} -uf'_1 & f'_2 \\ g'_1 & 2vyg'_2-1 \end{vmatrix}$

$D_2=\begin{vmatrix} xf'_1-1 & -uf'_1 \\ g'_1 & g'_1 \end{vmatrix}$，由克莱姆（Cramer）法则

$u'_x=\frac{D_1}{D}$，$v'_x=\frac{D_2}{D}$.

方程组 $\begin{cases} u=f(A,B) \\ v=g(E,F) \end{cases}$ 两边对 y 求偏导：

$$\begin{cases} u'_y=f'_1\cdot xu'_y+f'_2\cdot(v'_y+1) \\ v'_y=g'_1\cdot u'_y+g'_2\cdot(2vv'_yy+v^2) \end{cases} \Leftrightarrow \begin{cases} (xf'_1-1)u'_y+f'_2v'_y=-f'_2 \\ g'_1u'_y+(2vg'_2y-1)v'_y=-g'_2v^2. \end{cases}$$

(I).

$D=\begin{vmatrix} xf_1'-1 & f_2' \\ g_1' & 2vyg_2'-1 \end{vmatrix}\neq 0$, $\tilde{D}_1=\begin{vmatrix} -f_2' & f_2' \\ g_2'v^2, & 2vyg_2'-1 \end{vmatrix}$, $\tilde{D}_2=\begin{vmatrix} xf_1'-1 & -f_2' \\ g_1' & g_2'v^2 \end{vmatrix}$

$u_y'=\frac{\tilde{D}_1}{D}$, $v_y'=\frac{\tilde{D}_2}{D}$, $\Rightarrow$

$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x' & u_y' \\ v_x' & v_y' \end{vmatrix}=\begin{vmatrix} D_1 & \tilde{D}_1 \\ D_2 & \tilde{D}_2 \end{vmatrix}/D.$

例6 (ex9.3/15).

设 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$ 是由方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 确定的隐函数组, $F,G\in C^1$ 且 $\frac{\partial(F,G)}{\partial(u,v)}\neq 0$. 求 du, dv.

解: $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$; $dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$

在 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 两边关于 x 求偏导得

$\begin{cases} F_x'\cdot 1+F_u'\cdot\frac{\partial u}{\partial x}+F_v'\frac{\partial v}{\partial x}=0 \\ G_x'\cdot 1+G_u'\frac{\partial u}{\partial x}+G_v'\frac{\partial v}{\partial x}=0 \end{cases}$, 移项化: $\begin{cases} F_u'\frac{\partial u}{\partial x}+F_v'\frac{\partial v}{\partial x}=-F_x' \\ G_u'\frac{\partial u}{\partial x}+G_v'\frac{\partial v}{\partial x}=-G_x' \end{cases}$

令 $D=\begin{vmatrix} F_u' & F_v' \\ G_u' & G_v' \end{vmatrix}$. 则 $D=\frac{\partial(F,G)}{\partial(u,v)}\neq 0$.

令 $D_1=\begin{vmatrix} -F_x' & F_v' \\ -G_x' & G_v' \end{vmatrix}$, $D_2=\begin{vmatrix} F_u' & -F_x' \\ G_u' & -G_x' \end{vmatrix}$, 则 $D_1=\begin{vmatrix} F_v' & F_x' \\ G_v' & G_x' \end{vmatrix}=\frac{\partial(F,G)}{\partial(v,x)}$

$D_2 = \begin{vmatrix} F'_u & -F'_x \\ G'_u & -G'_x \end{vmatrix} = \begin{vmatrix} F'_x & F'_u \\ G'_x & G'_u \end{vmatrix} = \frac{\partial(F,G)}{\partial(x,u)}$，依 cramer 法则

$\frac{\partial u}{\partial x} = D_1/D = \frac{\partial(F,G)}{\partial(v,x)} / \frac{\partial(F,G)}{\partial(u,v)}$；$\frac{\partial v}{\partial x} = D_2/D = \frac{\partial(F,G)}{\partial(x,u)} / \frac{\partial(F,G)}{\partial(u,v)}$

对原方程组两边关于 y 求偏导. 由 cramer 法则，同样

可得：$\frac{\partial u}{\partial y} = \frac{\partial(F,G)}{\partial(v,y)} / \frac{\partial(F,G)}{\partial(u,v)}$；$\frac{\partial v}{\partial y} = \frac{\partial(F,G)}{\partial(y,u)} / \frac{\partial(F,G)}{\partial(u,v)}$.

于是，
$$\begin{cases} du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \left(\frac{\partial(F,G)}{\partial(v,x)}dx + \frac{\partial(F,G)}{\partial(v,y)}dy\right) / \frac{\partial(F,G)}{\partial(u,v)}; \\ dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \left(\frac{\partial(F,G)}{\partial(x,u)}dx + \frac{\partial(F,G)}{\partial(y,u)}dy\right) / \frac{\partial(F,G)}{\partial(u,v)}. \end{cases}$$

例6 解法2：对原方程组两边同时取全微分：

$$\begin{cases} F'_x dx + F'_y dy + F'_u du + F'_v dv = d(0) = 0 \\ G'_x dx + G'_y dy + G'_u du + G'_v dv = d(0) = 0 \end{cases}$$，视 du, dv 为未知量.

依 cramer 法则，解得：$du = D_1/D = \left(\frac{\partial(F,G)}{\partial(v,x)}dx + \frac{\partial(F,G)}{\partial(v,y)}dy\right) / \frac{\partial(F,G)}{\partial(u,v)}$；

$dv = D_2/D = \left(\frac{\partial(F,G)}{\partial(x,u)}dx + \frac{\partial(F,G)}{\partial(y,u)}dy\right) / \frac{\partial(F,G)}{\partial(u,v)}$. 其中，

$D_1 = \begin{vmatrix} -(F'_x dx + F'_y dy) & F'_v \\ -(G'_x dx + G'_y dy) & G'_v \end{vmatrix}$，$D_2 = \begin{vmatrix} F'_u & -(F'_x dx + F'_y dy) \\ G'_u & -(G'_x dx + G'_y dy) \end{vmatrix}$，$D = \frac{\partial(F,G)}{\partial(u,v)}$.

作业：ex9.2/31；ex9.3/6；7；8；10；11(1)；14.

附录：

例7. 设D是区域，$f \in C^2(D)$，$u=f(x+y+z, x^2+y^2+z^2)$，求 $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$ 及 du.

解(1). 令 $v=x+y+z$, $w=x^2+y^2+z^2$, 则 $u=f(v,w)$. 设 $f_1'=\frac{\partial f}{\partial v}$, $f_2'=\frac{\partial f}{\partial w}$.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial f}{\partial v}\cdot 1+\frac{\partial f}{\partial w}\cdot 2x=f_1'+f_2'\cdot 2x;$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial y}=\frac{\partial f}{\partial v}\cdot 1+\frac{\partial f}{\partial w}\cdot 2y=f_1'+f_2'\cdot 2y;$$

(2). $$\frac{\partial^2 u}{\partial x^2}=(f_1'+2xf_2')_x'=(f_1')_x'+(2xf_2')_x'=f_{11}''\cdot 1+f_{12}''\cdot 2x+$$

$$(2x)_x'\cdot f_2'+2x(f_{21}''\cdot 1+f_{22}''\cdot 2x)=f_{11}''+4xf_{12}''+4x^2f_{22}''+2f_2'.$$

$$\frac{\partial^2 u}{\partial x \partial y}=(f_1'+f_2'\cdot 2y)_x'=f_{11}''\cdot 1+f_{12}''\cdot 2x+2y(f_{21}''\cdot 1+f_{22}''\cdot 2x)$$

$$=f_{11}''+2f_{12}''\cdot(x+y)+4xyf_{22}''.$$

(3). $$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

$$=(f_1'+2xf_2')dx+(f_1'\cdot 1+2yf_2')dy+(f_1'\cdot 1+2zf_2')dz.$$

注：$\because f \in C^2(D)$，$\therefore f_{12}''=f_{21}''$，即 $\frac{\partial^2 f}{\partial w \partial v}=\frac{\partial^2 f}{\partial v \partial w}$.