Tut 3

3.1 求偏导

例 3.1 设函数 $z=(x^2-y^2)\mathrm{e}^{\frac{x}{y}}$, 求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$.

解 解法一

引入中间变量 $u=x^2-y^2, v=\dfrac{x}{y}$, 则 $z=u\mathrm{e}^v$.

$$\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\mathrm{e}^v\cdot 2x+u\mathrm{e}^v\cdot\frac{1}{y}=2x\mathrm{e}^{\frac{x}{y}}+\frac{x^2-y^2}{y}\mathrm{e}^{\frac{x}{y}},\\ \frac{\partial z}{\partial y}&=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\mathrm{e}^v\cdot(-2y)+u\mathrm{e}^v\cdot\left(-\frac{x}{y^2}\right)=-2y\mathrm{e}^{\frac{x}{y}}-\frac{x^2-y^2}{y^2}\mathrm{e}^{\frac{x}{y}}.\end{aligned}$$

解 解法二

固定 y, 将 z 视为只与 x 有关的函数, 则

$$\begin{aligned}\frac{\partial z}{\partial x}&=\left(2x+\frac{x^2-y^2}{y}\right)\mathrm{e}^{\frac{x}{y}}=2x\mathrm{e}^{\frac{x}{y}}+\left(2x+\frac{x^2-y^2}{y}\right)\mathrm{e}^{\frac{x}{y}}\cdot\frac{1}{y}=2x\mathrm{e}^{\frac{x}{y}}+\frac{x^2-y^2}{y}\mathrm{e}^{\frac{x}{y}},\\ \frac{\partial z}{\partial y}&=-\left(2y+\frac{(x^2-y^2)x}{y^2}\right)\mathrm{e}^{\frac{x}{y}}=-2y\mathrm{e}^{\frac{x}{y}}-\left(2y+\frac{(x^2-y^2)x}{y^2}\right)\mathrm{e}^{\frac{x}{y}}\cdot\frac{1}{y}=-2y\mathrm{e}^{\frac{x}{y}}-\frac{x^2-y^2}{y^2}\mathrm{e}^{\frac{x}{y}}.\end{aligned}$$

解 解法三 (这是我最推荐的解法)

对等式两边求微分, 得

$$\begin{aligned}\mathrm{d}z&=\mathrm{e}^{\frac{x}{y}}\,\mathrm{d}(x^2-y^2)+(x^2-y^2)\mathrm{e}^{\frac{x}{y}}\,\mathrm{d}\left(\frac{x}{y}\right)\\ &=\mathrm{e}^{\frac{x}{y}}\,(2x\,\mathrm{d}x-2y\,\mathrm{d}y)+(x^2-y^2)\mathrm{e}^{\frac{x}{y}}\left(\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{y^2}\right)\\ &=\left(2x\mathrm{e}^{\frac{x}{y}}+\frac{x^2-y^2}{y}\mathrm{e}^{\frac{x}{y}}\right)\mathrm{d}x+\left(-2y\mathrm{e}^{\frac{x}{y}}-\frac{x^2-y^2}{y^2}\mathrm{e}^{\frac{x}{y}}\right)\mathrm{d}y.\end{aligned}$$

将上式对照全微分 $\mathrm{d}z=\dfrac{\partial z}{\partial x}\mathrm{d}x+\dfrac{\partial z}{\partial y}\mathrm{d}y$ 比较, 得

$$\begin{aligned}\frac{\partial z}{\partial x}&=2x\mathrm{e}^{\frac{x}{y}}+\frac{x^2-y^2}{y}\mathrm{e}^{\frac{x}{y}},\\ \frac{\partial z}{\partial y}&=-2y\mathrm{e}^{\frac{x}{y}}-\frac{x^2-y^2}{y^2}\mathrm{e}^{\frac{x}{y}}.\end{aligned}$$

注 第三种方法的优势在于, 一次性写出了 $\mathrm{d}z$, 从而避免了中间变量的引入, 也避免了计算过程中的繁琐. 而且能够很好的处理各种中间变量以及隐函数带来的麻烦.

我们用一些例子说明用微分来求偏导的优势.

求高阶偏导

例 3.2 设函数 $z=f(x^2-y,g(xy))$, 求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y\partial x}$.

解 对等式两边求微分, 得

$$\begin{aligned}
\mathrm{d}z &= f_1'\,\mathrm{d}(x^2-y)+f_2'\,\mathrm{d}(g(xy))\\
&= f_1'\,(2x\,\mathrm{d}x-\mathrm{d}y)+f_2'g'\,(y\,\mathrm{d}x+x\,\mathrm{d}y)\\
&= (2xf_1'+yf_2'g')\,\mathrm{d}x+(-f_1'+xf_2'g')\,\mathrm{d}y.
\end{aligned}$$

将上式对照全微分 $\mathrm{d}z=\dfrac{\partial z}{\partial x}\mathrm{d}x+\dfrac{\partial z}{\partial y}\mathrm{d}y$ 比较, 得

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2xf_1'+yf_2'g',\\
\frac{\partial z}{\partial y} &= -f_1'+xf_2'g'.
\end{aligned}$$

这里要想清楚, 这里简写的 f_1' 是什么? 他在这里全部展开来写是

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2xf_1'+yf_2'g' = 2xf_1'(x^2-y,g(xy))+yf_2'(x^2-y,g(xy))\frac{\partial g}{\partial xy},\\
\frac{\partial z}{\partial y} &= -f_1'+xf_2'g' = -f_1'(x^2-y,g(xy))+xf_2'(x^2-y,g(xy))\frac{\partial g}{\partial xy}.
\end{aligned}$$

对 $\dfrac{\partial z}{\partial x}$ 再求 x 的偏导, 得

$$\begin{aligned}
\frac{\partial z}{\partial y\partial x} = \frac{\partial\frac{\partial z}{\partial x}}{\partial y} &= \frac{\partial}{\partial y}\,(2xf_1'+yf_2'g')\\
&= 2x\,(-f_{11}'+xg'f_{12}')+g'f_2'+y\,(-f_21''+xg'f_{22}'')\,g'+yf_2'g''x\\
&= (g'+xyg'')f_2'-2xf_{11}''+(2x^2-y)g'f_{12}''+xy(g')^2f_{22}''.
\end{aligned}$$

求变上限积分的偏导

例 3.3 设函数 $u(x,y)=\varphi(x,y)+\varphi(x-y)+\displaystyle\int_{x-y}^{x+y}\psi(t)\,\mathrm{d}t$, 其中函数 φ 具有二阶导数, 函数 ψ 具有一阶导数, 证明:

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}.$$

解 对 $u(x,y)$ 求微分, 得

$$\begin{aligned}
\mathrm{d}u &= (\mathrm{d}x+\mathrm{d}y)\varphi'(x+y)+(\mathrm{d}x-\mathrm{d}y)\varphi'(x-y)+\psi(x+y)\,\mathrm{d}(x+y)-\psi(x-y)\,\mathrm{d}(x-y)\\
&= (\varphi'(x+y)+\varphi'(x-y)+\psi(x+y)-\psi(x-y))\,\mathrm{d}x+(\varphi'(x+y)-\varphi'(x-y)+\psi(x+y)+\psi(x-y))\,\mathrm{d}
\end{aligned}$$

由此得

$$\begin{aligned}\frac{\partial u}{\partial x}&=\varphi'(x+y)+\varphi'(x-y)+\psi(x+y)-\psi(x-y),\\ \frac{\partial u}{\partial y}&=\varphi'(x+y)-\varphi'(x-y)+\psi(x+y)+\psi(x-y).\end{aligned}$$

从而

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial}{\partial x}\left(\varphi'(x+y)+\varphi'(x-y)+\psi(x+y)-\psi(x-y)\right)\\ &=\varphi''(x+y)+\varphi''(x-y)+\psi'(x+y)-\psi'(x-y),\\ \frac{\partial^2 u}{\partial y^2}&=\frac{\partial}{\partial y}\left(\varphi'(x+y)-\varphi'(x-y)+\psi(x+y)+\psi(x-y)\right)\\ &=\varphi''(x+y)-\varphi''(x-y)+\psi'(x+y)+\psi'(x-y).\end{aligned}$$

隐函数的偏导

例 3.4 设 $z=z(x,y)$ 是由方程 $2\sin(x+2y-3z)=x+2y-3z$ 确定的隐函数, 求

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1.$$

解 对方程两边求微分, 得

$$2\cos(x+2y-3z)\,(\mathrm{d}x+2\,\mathrm{d}y-3\,\mathrm{d}z)=\mathrm{d}x+2\,\mathrm{d}y-3\,\mathrm{d}z$$

整理得

$$(3-6\cos(x+2y-3z))\,\mathrm{d}z=(1-2\cos(x+2y-3z))\,\mathrm{d}x+2(1-2\cos(x+2y-3z))\,\mathrm{d}y$$

由此得

$$\begin{aligned}\frac{\partial z}{\partial x}&=\frac{1-2\cos(x+2y-3z)}{3-6\cos(x+2y-3z)},\\ \frac{\partial z}{\partial y}&=\frac{2(1-2\cos(x+2y-3z))}{3-6\cos(x+2y-3z)}\end{aligned}$$

于是

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{1-2\cos(x+2y-3z)+2(1-2\cos(x+2y-3z))}{3-6\cos(x+2y-3z)}=1.$$

求复杂变量代换的偏导

例 3.5 变量代换

$$u=\frac{x}{y},v=x,w=xz-y,$$

把函数 $z=z(x,y)$ 的方程

$$y\frac{\partial^2 z}{\partial y^2}+x\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial z}{\partial y}=\frac{1}{x}-\frac{x}{y}$$

化为 u,v,w 的方程 $w=w(u,v)$, 求 $w=w(u,v)$ 所满足的方程.

解 全部都求微分, 得

$$\begin{aligned}
\mathrm{d}u &= \frac{1}{y}\,\mathrm{d}x - \frac{x}{y^2}\,\mathrm{d}y \\
\mathrm{d}v &= \mathrm{d}x \\
\mathrm{d}w &= z\,\mathrm{d}x + x\,\mathrm{d}z - \mathrm{d}y
\end{aligned}$$

由 $\mathrm{d}w = w_1'\,\mathrm{d}u + w_2'\,\mathrm{d}v$ 得

$$z\,\mathrm{d}x + x\,\mathrm{d}z - \mathrm{d}y = w_1'\left(\frac{1}{y}\,\mathrm{d}x - \frac{x}{y^2}\,\mathrm{d}y\right) + w_2'\,\mathrm{d}x$$

由此得

$$\mathrm{d}z = \left(\frac{w_1'}{xy} + \frac{w_2'}{x} - \frac{z}{x}\right)\mathrm{d}x + \left(-\frac{w_1'}{y^2} + \frac{1}{x}\right)\mathrm{d}y$$

因此

$$\frac{\partial z}{\partial y} = -\frac{w_1'}{y^2} + \frac{1}{x}$$

对此作微分, 得

$$\mathrm{d}\left(\frac{\partial z}{\partial y}\right) = w_1'\left(-\frac{2}{y^3}\,\mathrm{d}y\right) + \frac{1}{y^2}(w_{11}''\,\mathrm{d}u + w_{12}''\,\mathrm{d}v) - \frac{1}{x^2}\,\mathrm{d}x$$

$$\begin{aligned}
\mathrm{d}\left(\frac{\partial z}{\partial y}\right) &= w_1'\left(-\frac{2}{y^3}\,\mathrm{d}y\right) + \frac{1}{y^2}(w_{11}''\,\mathrm{d}u + w_{12}''\,\mathrm{d}v) - \frac{1}{x^2}\,\mathrm{d}x \\
&= \left(\frac{xw_{11}''}{y^4} + \frac{2w_1'}{y^3}\right)\mathrm{d}y + \left(-\frac{w_{11}''}{y^3} - \frac{w_{12}''}{y^2} - \frac{1}{x^2}\right)\mathrm{d}x
\end{aligned}$$

由此得

$$\begin{aligned}
\frac{\partial^2 z}{\partial y^2} &= \frac{xw_{11}''}{y^4} + \frac{2w_1'}{y^3} \\
\frac{\partial^2 z}{\partial x\partial y} &= -\frac{w_{11}''}{y^3} - \frac{w_{12}''}{y^2} - \frac{1}{x^2}
\end{aligned}$$

代入 $y\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial^2 z}{\partial x\partial y} + 2\dfrac{\partial z}{\partial y} = \dfrac{1}{x} - \dfrac{x}{y}$ 得

$$y\left(\frac{xw_{11}''}{y^4} + \frac{2w_1'}{y^3}\right) + x\left(-\frac{w_{11}''}{y^3} - \frac{w_{12}''}{y^2} - \frac{1}{x^2}\right) + 2\left(\frac{w_1'}{y^2} + \frac{1}{x}\right) = \frac{1}{x} - \frac{x}{y}$$

化简得

$$w_{12}'' = y = \frac{x}{u} = \frac{v}{u}$$

或者写为

$$\frac{\partial^2 w}{\partial v\partial u} = \frac{v}{u}$$

注 一定不要写出 $\dfrac{\partial z}{\partial(x+y)}$, 尽管这个的意义是明确的, 但是大家要到第五周或者第六周学了 Jacobin 矩阵之后, 才能理解这个的意义.