第8讲：可微分条件与高阶偏导数

(一) $z=f(x,y)$ 在 $M_0(x_0,y_0)$ 处可微的条件：

Th1：若 $z=f(x,y)$ 在 M_0 处可微，则 $f'_x(M_0)$，$f'_y(M_0)$ 必存在，反之未必。

Th2：若 $f(x,y)$ 在 M_0 处可微，则 $z=f(x,y)$ 在 M_0 处必连续，反之未必。

Th3：$z=f(x,y)$ 在 $M_0(x_0,y_0)$ 处可微的充要条件：

$$\lim_{\rho\to 0}\frac{\Delta z-(f'_x(M_0)\Delta x+f'_y(M_0)\Delta y)}{\rho}=0.$$

Th4：$z=f(x,y)$ 在 $M_0(x_0,y_0)$ 处可微的充分条件：

$f'_x(x,y)$，$f'_y(x,y)$ 在 $M_0(x_0,y_0)$ 处存在且连续.

证Th1：已知 $z=f(x,y)$ 在 $M_0(x_0,y_0)$ 处可微 $\Rightarrow \Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$
$=(A\Delta x+B\Delta y)+o(\rho)$，令 $\Delta y=0$，则 $\Delta z_x=f(x_0+\Delta x,y_0)-f(x_0,y_0)=$

$A\Delta x+o(|\Delta x|)\Rightarrow f'_x(M_0)=\lim\limits_{\Delta x\to 0}\frac{\Delta z_x}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{A\Delta x+o(|\Delta x|)}{\Delta x}$

$=\lim\limits_{\Delta x\to 0}\left(A+\frac{o(|\Delta x|)}{|\Delta x|}\frac{|\Delta x|}{\Delta x}\right)=A+0=A$，同理 $f'_y(M_0)=B$.

即 $dz|_{M_0}=A\Delta x+B\Delta y=f'_x(M_0)\Delta x+f'_y(M_0)\Delta y\Rightarrow dz=f'_x\Delta x+f'_y\Delta y$

$$=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)\begin{pmatrix}dx\\dy\end{pmatrix}\quad(1).$$

证Th2：已知 $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y + o(\rho)$

且 $\begin{cases} \Delta x \to 0 \\ \Delta y \to 0 \end{cases}$ 时，$\begin{cases} f'_x(M_0)\Delta x + f'_y(M_0)\Delta y \longrightarrow 0 \\ o(\rho) \longrightarrow 0 \quad (\rho = \sqrt{(\Delta x)^2+(\Delta y)^2} \to 0 \text{时}) \end{cases}$，从而

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z = 0 \iff z = f(x,y)$ 在 $M_0(x_0,y_0)$ 处连续.

反例1：$z = f(x,y) = \sqrt{x^2+y^2}$ 在 $O(0,0)$ 处连续，但因 $f'_x(0,0)$，$f'_y(0,0)$

均不存在，由Th1，$f(x,y) = \sqrt{x^2+y^2}$ 在 $O(0,0)$ 处不可微.

反例2：$z = f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & x^2+y^2>0 \\ 0, & x^2+y^2=0 \end{cases}$ 在 $O(0,0)$ 处有

$f'_x(0,0) = 0 = f'_y(0,0)$，但因 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) \neq f(0,0) = 0$. 因此，

$f(x,y)$ 在 $O(0,0)$ 处不连续，由Th2，$f(x,y)$ 在 $O(0,0)$ 处不可微。

证Th3：若 $z = f(x,y)$ 在 $M_0(x_0,y_0)$ 处可微，则

$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y + o(\rho)$，$\Rightarrow$

$\lim\limits_{\rho\to 0} \dfrac{\Delta z - (f'_x(M_0)\Delta x + f'_y(M_0)\Delta y)}{\rho} = \lim\limits_{\rho\to 0} \dfrac{o(\rho)}{\rho} = 0$

反之，若 $\lim\limits_{\rho\to 0} \dfrac{\Delta z - (f'_x(M_0)\Delta x + f'_y(M_0)\Delta y)}{\rho} = 0$ 则

$\Delta z - (f'_x(M_0)\Delta x + f'_y(M_0)\Delta y) = o(\rho) \Rightarrow \Delta z = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y + o(\rho)$

(z).

$=(A\Delta x+B\Delta y)+o(\rho)$. 从而 $z=f(x,y)$ 在 M_0 处可微.

证明4: 已知 $f'_x(x,y), f'_y(x,y)$ 在 $M_0(x_0,y_0)$ 处存在且连续.

从而 $\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=[f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0+\Delta y)]+$

$[f(x_0,y_0+\Delta y)-f(x_0,y_0)] \xrightarrow[\text{中值定理}]{\text{Lagrange微分}} f'_x(x_0+\theta_1\Delta x,y_0+\Delta y)\Delta x+$

$f'_y(x_0,y_0+\theta_2\Delta y)\Delta y$, 其中 $\theta_1,\theta_2\in(0,1)$. 利用 $f'_x(x,y), f'_y(x,y)$ 在

$M_0(x_0,y_0)$ 处连续性可知, $\begin{cases}\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f'_x(x_0+\theta_1\Delta x,y_0+\Delta y)=f'_x(x_0,y_0)\\ \lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f'_y(x_0,y_0+\theta_2\Delta y)=f'_y(x_0,y_0)\end{cases}$

从而 $\begin{cases}f'_x(x_0+\theta_1\Delta x,y_0+\Delta y)=f'_x(x_0,y_0)+\alpha_1, \ \alpha_1\to 0 \ (\Delta x\to 0,\Delta y\to 0\text{时})\\ f'_y(x_0,y_0+\theta_2\Delta y)=f'_y(x_0,y_0)+\alpha_2, \ \alpha_2\to 0, \ (\Delta x\to 0,\Delta y\to 0\text{时}).\end{cases}$

即 $\Delta z=(f'_x(M_0)+\alpha_1)\Delta x+(f'_y(M_0)+\alpha_2)\Delta y=f'_x(M_0)\Delta x+f'_y(M_0)\Delta y+\alpha_1\Delta x+\alpha_2\Delta y$

且 $\lim\limits_{\rho\to 0}\dfrac{\alpha_1\Delta x+\alpha_2\Delta y}{\rho}=\lim\limits_{\rho\to 0}(\alpha_1\cos\theta+\alpha_2\sin\theta)=0+0=0$

$\therefore \alpha_1\Delta x+\alpha_2\Delta y=o(\rho)$. 故有:

$\Delta z=f'_x(M_0)\Delta x+f'_y(M_0)\Delta y+o(\rho)=(A\Delta x+B\Delta y)+o(\rho)$. 即

$z=f(x,y)$ 在 M_0 处可微.

(3).

反例3. $z=f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2>0,\\ 0, & x^2+y^2=0,\end{cases}$

在$O(0,0)$处可微. 但$f'_x(x,y)$, $f'_y(x,y)$在$O(0,0)$处不连续。

(三) 高阶偏导数 (higher order partial derivative)

设 $z=f(x,y)=x^2+xy+y^2+x^y+3x+4y$, $(x>0, y\in R)$.

则 $\begin{cases}\dfrac{\partial z}{\partial x}=2x+y+yx^{y-1}+3\\ \dfrac{\partial z}{\partial y}=x+2y+x^y\ln x+4.\end{cases}\Longrightarrow$

$$\begin{cases}\dfrac{\partial^2 z}{\partial y\partial x}=\left(\dfrac{\partial z}{\partial x}\right)'_y=(2x+y+yx^{y-1}+3)'_y=0+1+1\cdot x^{y-1}+yx^{y-1}\ln x+0\\ \dfrac{\partial^2 z}{\partial x\partial y}=\left(\dfrac{\partial z}{\partial y}\right)'_x=(x+2y+x^y\ln x+4)'_x=1+0+yx^{y-1}\ln x+x^{y-1}+0\end{cases}$$

$$\begin{cases}\dfrac{\partial^3 z}{\partial x\partial y\partial x}=\left(\dfrac{\partial^2 z}{\partial y\partial x}\right)'_x=(1+x^{y-1}+yx^{y-1}\ln x)'_x=(y-1)x^{y-2}+y(y-1)x^{y-2}\ln x+yx^{y-2}\\ \dfrac{\partial^3 z}{\partial x^2\partial y}=\left(\dfrac{\partial^2 z}{\partial x\partial y}\right)'_x=(1+yx^{y-1}\ln x+x^{y-1})'_x=y(y-1)x^{y-2}\ln x+yx^{y-2}+(y-1)x^{y-2}\end{cases}$$

显然, $\dfrac{\partial^2 z}{\partial y\partial x}$, $\dfrac{\partial^2 z}{\partial x\partial y}$, $\dfrac{\partial^3 z}{\partial x\partial y\partial x}$, $\dfrac{\partial^3 z}{\partial x^2\partial y}$ 均在区域 $D: x>0$ 中

连续且 $\dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{\partial^2 z}{\partial x\partial y}$, $\dfrac{\partial^3 z}{\partial x\partial y\partial x}=\dfrac{\partial^3 z}{\partial x^2\partial y}$, $\forall (x,y)\in D$.

Th5: 若 $z=f(x,y)$ 在区域D中混高阶偏导数连续, 则

高阶偏导数与求偏导的顺序无关。

证：设 $f(x,y)\in C^2(D)$，且 $M_0(x_0,y_0)$ 为 D 中任一点，设

$M_1(x_0+h,y_0+k)$，$M_2(x_0+h,y_0)$，$M_3(x_0,y_0+k)\in D$，$h\cdot k\neq 0$.

则只须：$f''_{xy}(M_0)=f''_{yx}(M_0)$，再由 M_0 在 D 中的任意性，

即可证得：$f''_{xy}(x,y)\equiv f''_{yx}(x,y)$ 或 $\frac{\partial^2 f}{\partial y\partial x}\equiv\frac{\partial^2 f}{\partial x\partial y}$，$\forall(x,y)\in D$.

令 $\begin{cases} m(x)=f(x,y_0+k)-f(x,y_0) \\ n(y)=f(x_0+h,y)-f(x_0,y)\end{cases}$ 则有

$$\begin{cases} m(x_0+h)-m(x_0)=f(x_0+h,y_0+k)-f(x_0+h,y_0)-f(x_0,y_0+k)+f(x_0,y_0) \\ n(y_0+k)-n(y_0)=f(x_0+h,y_0+k)-f(x_0,y_0+k)-f(x_0+h,y_0)+f(x_0,y_0)\end{cases}$$

即对 $\forall h\cdot k\neq 0$，有 $m(x_0+h)-m(x_0)\equiv n(y_0+k)-n(y_0)$. 利用

Lagrange 微分中值定理知：$m'_x(x_0+\theta_1 h)h\equiv n'_y(y_0+\theta_2 k)k$

且 $\theta_1,\theta_2\in(0,1)$. 即有：

$$(f'_x(x_0+\theta_1 h,y_0+k)-f'_x(x_0+\theta_1 h,y_0))h\equiv(f'_y(x_0+h,y_0+\theta_2 k)-f'_y(x_0,y_0+\theta_2 k))k$$

对 f'_x，f'_y 这两个函数再次使用 Lagrange 微分中值定理：

$$f''_{xy}(x_0+\theta_1 h,y_0+\theta_3 k)kh\equiv f''_{yx}(x_0+\theta_4 h,y_0+\theta_2 k)hk,\ \theta_3,\theta_4\in(0,1).$$

∵ $f\in C^2(D)$，∴ 在上式中令 $h\to0, k\to0$，则有

$f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0)$. 由 $M(x_0,y_0)$ 在 D 中的任意性.

即有 $f''_{xy}(x,y)=f''_{yx}(x,y) \Leftrightarrow \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$，$\forall(x,y)\in D$.

同理可证，若 $z=f(x,y)\in C^3(D)$，则必有

$\frac{\partial^3 z}{\partial x^2\partial y}=\frac{\partial^3 z}{\partial x\partial y\partial x}=\frac{\partial^3 z}{\partial y\partial x^2}$，证法同上。

(三) 例题：

例1. 证明函数 $u=\frac{1}{r}$，$r=\sqrt{x^2+y^2+z^2}>0$ 满足 Laplace

方程（即调和方程）：$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$，$\forall(x,y,z)\neq(0,0,0)$.

例2. 证明 $u=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$，$(x>0, t>0, a>0$ 常数$)$

满足热传导方程：$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$.

例3. 对 $\forall\varphi,\psi\in C^2(I)$，$u=\varphi(x-at)+\psi(x+at)$ 满足

波动方程：$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ $(a>0$，常数$)$，$t>0$，$x\in(-\infty,+\infty)$.

证例1：$u=(x^2+y^2+z^2)^{-\frac{1}{2}} \Rightarrow \frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x$

$$\Rightarrow \frac{\partial u}{\partial x} = -x(x^2+y^2+z^2)^{-\frac{3}{2}} \Rightarrow$$

$$\frac{\partial^2 u}{\partial x^2} = -1(x^2+y^2+z^2)^{-\frac{3}{2}} + (-x)(-\tfrac{3}{2})(x^2+y^2+z^2)^{-\frac{5}{2}} \cdot 2x$$

$$= -(x^2+y^2+z^2)^{-\frac{5}{2}}(x^2+y^2+z^2) + 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} = -\frac{(x^2+y^2+z^2)-3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

因为 $u=(x^2+y^2+z^2)^{-\frac{1}{2}}$ 是 x,y,z 的对称函数，因此，有：

$$\frac{\partial^2 u}{\partial y^2} = -\frac{(x^2+y^2+z^2)-3y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \quad \frac{\partial^2 u}{\partial z^2} = -\frac{(x^2+y^2+z^2)-3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

故 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3(x^2+y^2+z^2)-3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} \equiv 0$，$\forall (x,y,z) \neq (0,0,0)$，

证例2：$\because \frac{\partial u}{\partial t} = \frac{(t^{-\frac{1}{2}})'_t}{2a\sqrt{\pi}} e^{-\frac{x^2}{4a^2t}} + \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} \left(-\frac{x^2}{4a^2t}\right)'_t$

$$= \frac{1}{4a\sqrt{\pi t}\, t} e^{-\frac{x^2}{4a^2t}} \left(-1 + \frac{x^2}{2a^2t}\right)，且$$

$$\frac{\partial u}{\partial x} = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} \left(-\frac{x^2}{4a^2t}\right)'_x = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} \left(-\frac{x}{2a^2t}\right) \Rightarrow$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2a\sqrt{\pi t}} \left[e^{-\frac{x^2}{4a^2t}} \left(-\frac{x}{2a^2t}\right)^2 + e^{-\frac{x^2}{4a^2t}} \left(-\frac{1}{2a^2t}\right)\right]$$

$$= \frac{1}{4a\sqrt{\pi t}\, t} e^{-\frac{x^2}{4a^2t}} \left(\frac{x^2}{2a^4t} - \frac{1}{a^2}\right) \Rightarrow$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{1}{4a\sqrt{\pi t}\, t} e^{-\frac{x^2}{4a^2t}} \left(\frac{x^2}{2a^2t} - 1\right) \equiv \frac{\partial u}{\partial t}，\forall t>0, x \in R^+;$$

证例3：令 $\begin{cases} v = x-at \\ w = x+at \end{cases}$，则 $u = \varphi(v) + \psi(w)$

(1).

$$\Rightarrow \frac{\partial u}{\partial x}=\varphi'(v)\frac{\partial v}{\partial x}+\psi'(w)\frac{\partial w}{\partial x}=\varphi'(v)\cdot 1+\psi'(w)\cdot 1\Rightarrow$$

$$\frac{\partial^2 u}{\partial x^2}=\varphi''(v)\frac{\partial v}{\partial x}+\psi''(w)\frac{\partial w}{\partial x}=\varphi''(v)\cdot 1+\psi''(w)\cdot 1$$ 且

$$\frac{\partial u}{\partial t}=\varphi'(v)\frac{\partial v}{\partial t}+\psi'(w)\frac{\partial w}{\partial t}=\varphi'(v)(-a)+\psi'(w)a\Rightarrow$$

$$\frac{\partial^2 u}{\partial t^2}=\varphi''(v)\frac{\partial v}{\partial t}(-a)+\psi''(w)\frac{\partial w}{\partial t}(a)=\varphi''(v)(-a)^2+\psi''(w)a^2$$

$$=a^2(\varphi''(v)+\psi''(w))\equiv a^2\frac{\partial^2 u}{\partial x^2},\ \forall t>0,x\in R.$$

(四) 作业：EX9.2

2(7)；8；11；15；26；27；28.

附梯微分向量算子：$\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$，$u=f(x,y,z)\in C^2(D)$，

$\vec{A}(x,y,z)=(P(x,y,z),Q(x,y,z),R(x,y,z))\in C^2(D)$. 则

(1). $\nabla u=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})u=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})$ 为 $u=f(x,y,z)$ 的梯度；

(2). $\nabla\cdot\vec{A}=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\cdot(P,Q,R)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$ 为 $\vec{A}(x,y,z)$ 的散度

(3). $\nabla\times\vec{A}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R\end{vmatrix}=(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})i+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})j+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})k$

是 $\vec{A}$ 的旋度。