Week 8

8.1 Apr 7 ex10.1:3,5,6,7; ex10.2:1(1),2(1)(2),3(1)

✍ **习题 10.1.3** 利用函数的奇偶性计算下列积分：

1. $\iint_D (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y, D: -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1;$
2. $\iint_D \sin x \sin y\,\mathrm{d}x\,\mathrm{d}y, D: x^2-y^2=1, x^2+y^2=9$ 围成含原点的部分;

解

1.
$$\begin{aligned}\iint_D (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}(x^2+y^2)\,\mathrm{d}x \\ &= 4\int_0^1\mathrm{d}y\int_0^1(x^2+y^2)\,\mathrm{d}x \\ &= 4\int_0^1\left(\frac{1}{3}+y^2\right)\mathrm{d}y \\ &= \frac{8}{3}\end{aligned}$$

2. 令
$$D_1=\{(x,y)\in D \mid y\geqslant 0\},\quad D_2=\{(x,y)\in D \mid y<0\}$$
则
$$\begin{aligned}\iint_D \sin x\sin y\,\mathrm{d}x\,\mathrm{d}y &= \iint_{D_1}\sin x\sin y\,\mathrm{d}x\,\mathrm{d}y+\iint_{D_2}\sin x\sin y\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{D_1}\sin x\sin y\,\mathrm{d}x\,\mathrm{d}y-\iint_{D_1}\sin x\sin y\,\mathrm{d}x\,\mathrm{d}y \\ &= 0\end{aligned}$$

✍ **习题 10.1.5**

设函数 $f(x)$ 在 $[0,a]$ 上连续, 证明
$$\int_0^a\mathrm{d}x\int_0^x f(x)f(y)\,\mathrm{d}y=\frac{1}{2}\left(\int_0^a f(x)\,\mathrm{d}x\right)^2,$$
$$\int_0^a\mathrm{d}x\int_0^x f(y)\,\mathrm{d}y=\int_0^a(a-x)f(x)\,\mathrm{d}x.$$

证明 由对称性
$$\int_0^a\mathrm{d}x\int_0^x f(x)f(y)\,\mathrm{d}y=\int_0^a\mathrm{d}y\int_0^y f(x)f(y)\,\mathrm{d}x$$

因此

$$\begin{aligned}\int_0^a \mathrm{d}x \int_0^x f(x)f(y)\,\mathrm{d}y &= \frac{1}{2}\int_0^a \mathrm{d}x \int_0^x f(x)f(y)\,\mathrm{d}y + \frac{1}{2}\int_0^a \mathrm{d}y \int_0^y f(x)f(y)\,\mathrm{d}x \\ &= \frac{1}{2}\iint_{D_1} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\iint_{[0,a]\times[0,a]} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x \int_0^a f(y)\,\mathrm{d}y \\ &= \frac{1}{2}\left(\int_0^a f(x)\,\mathrm{d}x\right)^2\end{aligned}$$

其中

$$D_1 = \{(x,y) \mid 0 \leqslant x, y \leqslant a,\ y \leqslant x\}, \quad D_2 = \{(x,y) \mid 0 \leqslant x, y \leqslant a,\ x < y\}$$

另一方面

$$\int_0^a \mathrm{d}x \int_0^x f(y)\,\mathrm{d}y = \int_0^a \mathrm{d}y \int_y^a f(y)\,\mathrm{d}x = \int_0^a (a-y)f(y)\,\mathrm{d}y = \int_0^a (a-x)f(x)\,\mathrm{d}x$$

习题 10.1.6

设函数 $f(x,y)$ 有连续的二阶偏导数, 在 $D=[a,b]\times[c,d]$ 上, 求积分

$$\iint_D \frac{\partial^2 f(x,y)}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y$$

解

$$\begin{aligned}\iint_D \frac{\partial^2 f}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y &= \int_c^d \mathrm{d}y \int_a^b \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\mathrm{d}x \\ &= \int_c^d \left(f_y(b,y) - f_y(a,y)\right)\mathrm{d}y \\ &= f(b,d) - f(a,d) - f(b,c) + f(a,c)\end{aligned}$$

习题 10.1.7

设函数 $f(x,y)$ 连续, 求极限

$$\lim_{r\to 0} \frac{1}{\pi r^2} \iint_{x^2+y^2\leqslant r^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

解 由题，$\forall \varepsilon > 0$，$\exists \delta > 0$，当 $\sqrt{x^2+y^2} < \delta$ 时，有

$$|f(x,y) - f(0,0)| < \varepsilon$$

因此，只要 $r<\delta$，就有

$$\begin{aligned}\left|\frac{1}{\pi r^2}\iint_{B(0,r)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y-f(0,0)\right| &= \left|\frac{1}{\pi r^2}\iint_{B(0,r)} (f(x,y)-f(0,0))\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\leqslant \frac{1}{\pi r^2}\iint_{B(0,r)} |f(x,y)-f(0,0)|\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \frac{1}{\pi r^2}\iint_{B(0,r)} \varepsilon\,\mathrm{d}x\,\mathrm{d}y \\ &= \varepsilon\end{aligned}$$

即

$$\lim_{r\to 0}\frac{1}{\pi r^2}\iint_{B(0,r)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y=f(0,0)$$

✍ 习题 10.2.1(1)

计算下列积分.

1. $\displaystyle\int_0^R \mathrm{d}x\int_0^{\sqrt{R^2-x^2}} \ln\left(1+x^2+y^2\right)\mathrm{d}y$;

解

1. 令 $t=r^2$，则

$$\begin{aligned}\int_0^R \mathrm{d}x\int_0^{\sqrt{R^2-x^2}} \ln\left(1+x^2+y^2\right)\mathrm{d}y &= \iint_D \ln\left(1+x^2+y^2\right)\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R r\ln\left(1+r^2\right)\mathrm{d}r \\ &= \frac{\pi}{4}\int_0^{R^2}\ln(1+t)\,\mathrm{d}t \\ &= \frac{\pi}{4}\left[\left(1+R^2\right)\ln\left(1+R^2\right)-R^2\right]\end{aligned}$$

✍ 习题 10.2.2(1)(2)

计算下面二重积分.

1. $\displaystyle\iint_D \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y, D: x^2+y^2\leqslant x+y$;
2. $\displaystyle\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,\mathrm{d}x\,\mathrm{d}y, D: x^2+y^2=4, y=0, y=x$ 所围成的第一象限部分;

解

1. 令 $x=r\cos\theta, y=r\sin\theta$，则

$$x^2+y^2\leqslant x+y\Longrightarrow r\leqslant\sin\theta+\cos\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$$

于是由于 $r \geqslant 0$，知 $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{3\pi}{4}$，于是

$$\begin{aligned}
\iint_D \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \mathrm{d}\theta \int_0^{\sin\theta+\cos\theta} r^2\,\mathrm{d}r \\
&= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin\theta+\cos\theta)^3\,\mathrm{d}\theta \\
&= \frac{1}{\sqrt{2}}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^3\left(\theta+\frac{\pi}{4}\right)\mathrm{d}\theta \\
&= \frac{1}{\sqrt{2}}\int_{-\frac{\pi}{4}}^{\pi} \sin^3\theta\,\mathrm{d}\theta \\
&= \frac{8\sqrt{2}}{9}
\end{aligned}$$

2. 令 $x = ar\cos\theta,\ y = br\sin\theta$，则

$$0 \leqslant y \leqslant x \Longrightarrow 0 \leqslant \tan\theta \leqslant \frac{b}{a} \Longrightarrow 0 \leqslant \theta \leqslant \arctan\frac{b}{a}$$

且

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

于是不难得到

$$\begin{aligned}
\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,\mathrm{d}x\,\mathrm{d}y &= ab\int_0^{\arctan\frac{b}{a}} \mathrm{d}\theta \int_0^1 r^2\,\mathrm{d}r \\
&= ab\cdot\arctan\frac{b}{a}\cdot\int_0^2 r^2\,\mathrm{d}r \\
&= ab\cdot\arctan\frac{b}{a}\cdot\frac{8}{3} \\
&= \frac{8}{3}ab\arctan\frac{b}{a}
\end{aligned}$$

✍ **习题 10.2.3(1)**

求下列曲线所围成的平面区域的面积:

1. $x^2+2y^2=3$ 和 $xy=1$(不含原点部分);

解

1. 设该图形第一象限的部分为 D。不难得到两曲线在第一象限交于 $(1,1)$ 和 $\left(\sqrt{2},\frac{\sqrt{2}}{2}\right)$，于是：

$$
\begin{aligned}
S = 2\iint_D \mathrm{d}x\,\mathrm{d}y &= \int_1^{\sqrt{2}} \mathrm{d}x \int_{1/x}^{\sqrt{\frac{3-x^2}{2}}} \mathrm{d}y \\
&= \int_1^{\sqrt{2}} \left(\sqrt{\frac{3-x^2}{2}} - \frac{1}{x}\right) \mathrm{d}x \\
&= \frac{3}{\sqrt{2}} \arcsin\frac{1}{3} - \ln 2
\end{aligned}
$$

8.2 Apr 9 ex10.1:1(4)(6),2(6)(8); ex10.2:2(3)(4)(7)(9),3(3),5

✍ **习题 10.1.1(4)(6)** 改变下列积分的顺序.

4. $\displaystyle\int_a^b \mathrm{d}y \int_y^b f(x,y)\,\mathrm{d}x$;
6. $\displaystyle\int_0^1 \mathrm{d}y \int_{\frac{1}{2}}^1 f(x,y)\,\mathrm{d}x + \int_1^2 \mathrm{d}y \int_{\frac{1}{2}}^{\frac{1}{y}} f(x,y)\,\mathrm{d}x$.

解

4.
$$\int_a^b \mathrm{d}y \int_y^b f(x,y)\,\mathrm{d}x = \int_a^b \mathrm{d}x \int_a^x f(x,y)\,\mathrm{d}y$$

6.
$$\int_0^1 \mathrm{d}y \int_{1/2}^1 f(x,y)\,\mathrm{d}x + \int_1^2 \mathrm{d}y \int_{1/2}^{1/y} f(x,y)\,\mathrm{d}x = \int_{1/2}^1 \mathrm{d}x \int_0^{1/x} f(x,y)\,\mathrm{d}y$$

✍ **习题 10.1.2(6)(8)**

计算下列积分.

6. $\displaystyle\iint_D \frac{\sin y}{y}\,\mathrm{d}x\,\mathrm{d}y, D:$ 由 $y=x$ 和 $x=y^2$ 围成;
8. $\displaystyle\iint_D |\cos(x+y)|\,\mathrm{d}x\,\mathrm{d}y$, 其中 D 是由直线 $y=x, y=0, x=\dfrac{\pi}{2}$ 所围成;

解

6.
$$\iint_D \frac{\sin y}{y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_{y^2}^y \frac{\sin y}{y}\,\mathrm{d}x = \int_0^1 (\sin y - y\sin y)\,\mathrm{d}y = 1 - \sin 1$$

8. 记
$$D_1 = \{(x,y) \in D : x+y \leqslant \frac{\pi}{2}\}$$
$$D_2 = \{(x,y) \in D : x+y > \frac{\pi}{2}\}$$

则

$$\begin{aligned}\iint_D |\cos(x+y)|\,\mathrm{d}x\,\mathrm{d}y &= \iint_{D_1} \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{D_2} \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\pi/4} \mathrm{d}y \int_y^{\pi/2-x} \cos(x+y)\,\mathrm{d}x - \int_{\pi/4}^{\pi/2} \mathrm{d}x \int_{\pi/2-x}^{x} \cos(x+y)\,\mathrm{d}y \\ &= \int_0^{\pi/4} (1-\sin 2y)\,\mathrm{d}y - \int_{\pi/4}^{\pi/2} (\sin 2x - 1)\,\mathrm{d}x \\ &= \int_0^{\pi/2} (1-\sin 2y)\,\mathrm{d}y \\ &= \frac{\pi}{2} - 1\end{aligned}$$

✍ **习题 10.2.2(3)(4)(7)(9)**

计算下面二重积分.

3. $\iint_D (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y, D$: 由 $xy=1, xy=2, y=x, y=2x$ 所围成的第一象限部分;
4. $\iint_D \mathrm{d}x\,\mathrm{d}y, D$: 由 $y^2=ax, y^2=bx, x^2=my, x^2=ny$ 所围成的区域 $(a>b>0, m>n>0)$;
7. $\iint_D \dfrac{x^2-y^2}{\sqrt{x+y+3}}\,\mathrm{d}x\,\mathrm{d}y, D: |x|+|y| \leqslant 1$;
9. $\iint_D |xy|\,\mathrm{d}x\,\mathrm{d}y, D = \{(x,y) : x^2+y^2 \leqslant a^2\}$

解

3. 令 $s=xy,\ t=\dfrac{y}{x}$, 则

$$\begin{cases} x = \sqrt{\dfrac{s}{t}} \\ y = \sqrt{st} \end{cases}$$

则

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \dfrac{1}{2\sqrt{st}} & -\dfrac{1}{2t}\sqrt{\dfrac{s}{t}} \\ \dfrac{1}{2}\sqrt{\dfrac{t}{s}} & \dfrac{1}{2}\sqrt{\dfrac{s}{t}} \end{vmatrix} = \frac{1}{2t}$$

注 如果你不想用 $s=s(x,y), t=t(x,y)$ 反解出 $x=x(s,t), y=y(s,t)$, 你也可以:

$$\frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y} \\ \dfrac{\partial t}{\partial x} & \dfrac{\partial t}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{vmatrix} = 2\frac{y}{x}$$

于是

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{1}{2\frac{y}{x}} = \frac{1}{2t}$$

于是不难得到

$$\begin{aligned}\iint_D (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y &= \int_1^2 \mathrm{d}s \int_1^2 \frac{1}{2t}\left(\frac{s}{t}+st\right)\mathrm{d}t \\ &= \int_1^2 \mathrm{d}s \int_1^2 \frac{1}{2t}\cdot\frac{s(1+t^2)}{t}\,\mathrm{d}t \\ &= \int_1^2 \mathrm{d}s \cdot s \cdot \frac{3}{4} \\ &= \frac{3}{4}\int_1^2 s\,\mathrm{d}s = \frac{3}{4}\cdot\frac{3}{2} = \frac{9}{8}\end{aligned}$$

4. 令 $s=\dfrac{x^2}{y}$, $t=\dfrac{y^2}{x}$，则

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{2}{3}\sqrt[3]{\frac{t}{s}} & \frac{1}{3}\sqrt[3]{\frac{s^2}{t^2}} \\ \frac{1}{3}\sqrt[3]{\frac{t^2}{s^2}} & \frac{2}{3}\sqrt[3]{\frac{s}{t}} \end{vmatrix} = \frac{1}{3}$$

或者

$$\frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 3 \Rightarrow \frac{\partial(x,y)}{\partial(s,t)} = \frac{1}{3}$$

于是

$$\iint_D \mathrm{d}x\,\mathrm{d}y = \int_n^m \mathrm{d}s \int_b^a \frac{1}{3}\,\mathrm{d}t = \frac{(a-b)(m-n)}{3}$$

7. 令 $s=x+y$, $t=x-y$，则

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

或者

$$\frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2 \Rightarrow \frac{\partial(x,y)}{\partial(s,t)} = \frac{1}{2}$$

于是

$$\begin{aligned}\iint_D \frac{x^2-y^2}{x+y+3}\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{2}\int_{-1}^1 \mathrm{d}s \int_{-1}^1 \frac{st}{\sqrt{s+3}}\,\mathrm{d}t \\ &= \int_{-1}^1 \frac{s}{\sqrt{s+3}}\,\mathrm{d}s \cdot \int_{-1}^1 t\,\mathrm{d}t = 0\end{aligned}$$

9. 令 $x=r\cos\theta$, $y=r\sin\theta$，并取

$$D_1 = \{(x,y)\in D \mid x,y \geqslant 0\}$$

则

$$\begin{aligned}\iint_D |xy|\,\mathrm{d}x\,\mathrm{d}y &= 4\iint_{D_1} |xy|\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^a r^3\,|\sin\theta\cos\theta|\,\mathrm{d}r \\ &= 4\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,\mathrm{d}\theta\cdot\int_0^a r^3\,\mathrm{d}r \\ &= 4\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2\theta)\,\mathrm{d}\theta\cdot\frac{a^4}{4} = 4\cdot\frac{1}{2}\cdot 1\cdot\frac{a^4}{4} = \frac{1}{2}a^4\end{aligned}$$

✍ 习题 10.2.3(3)

求下列曲线所围成的平面区域的面积:

3. 由直线 $x+y=a, x+y=b, y=kx, y=mx(0<a<b, 0<k<m)$ 所围成的的平面区域.

解

3. 令 $s=x+y,\ t=\dfrac{y}{x}$, 则

$$\frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -\dfrac{y}{x^2} & \dfrac{1}{x}\end{vmatrix} = \frac{1}{x}+\frac{y}{x^2}$$

于是

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{1}{\frac{1}{x}+\frac{y}{x^2}} = \frac{x^2}{x+y} = \frac{\left(\frac{s}{1+t}\right)^2}{s} = \frac{s}{(1+t)^2}$$

于是

$$\begin{aligned}S = \iint_D \mathrm{d}x\,\mathrm{d}y &= \int_a^b \mathrm{d}s\int_k^m \frac{s}{(1+t)^2}\,\mathrm{d}t = \int_a^b s\,\mathrm{d}s\int_k^m\frac{1}{(1+t)^2}\,\mathrm{d}t \\ &= \int_a^b s\,\mathrm{d}s\cdot\left[\frac{1}{1+k}-\frac{1}{1+m}\right] = \frac{b^2-a^2}{2}\left(\frac{1}{1+k}-\frac{1}{1+m}\right)\end{aligned}$$

✍ 习题 10.2.5

设 $f(x)$ 在 $[0,1]$ 上连续, 证明

$$\int_0^1 \mathrm{e}^{f(x)}\,\mathrm{d}x\cdot\int_0^1 \mathrm{e}^{-f(y)}\,\mathrm{d}y \geqslant 1.$$

证明 由对称性知

$$\int_0^1 \mathrm{e}^{f(x)}\,\mathrm{d}x\int_0^1 \mathrm{e}^{-f(y)}\,\mathrm{d}y = \iint_{[0,1]^2}\mathrm{e}^{f(x)-f(y)}\,\mathrm{d}x\,\mathrm{d}y = \iint_{[0,1]^2}\mathrm{e}^{f(y)-f(x)}\,\mathrm{d}x\,\mathrm{d}y$$

于是

$$\begin{aligned}\int_0^1 \mathrm{e}^{f(x)}\,\mathrm{d}x\int_0^1 \mathrm{e}^{-f(y)}\,\mathrm{d}y &= \frac{1}{2}\iint_{[0,1]^2}\mathrm{e}^{f(x)-f(y)}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\iint_{[0,1]^2}\mathrm{e}^{f(y)-f(x)}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\iint_{[0,1]^2}\left(\mathrm{e}^{f(x)-f(y)}+\mathrm{e}^{f(y)-f(x)}\right)\mathrm{d}x\,\mathrm{d}y \\ &\geqslant \frac{1}{2}\iint_{[0,1]^2} 2\,\mathrm{d}x\,\mathrm{d}y = 1\end{aligned}$$

8.3 Apr 11 ex10.3:1(1)(2),2(1)(2)(3),3(1)(3)(6),7,8

✍ **习题 10.3.1(1)(2)**

计算下列三重积分.

1. $\iiint_V xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, V: 1 \leqslant x \leqslant 2, -2 \leqslant y \leqslant 1, 0 \leqslant z \leqslant \frac{1}{2}$;
2. $\iiint_V xy^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, V: 由 $z = xy, y = x, x = 1, z = 0$ 围成;

解

1.
$$\iiint_V xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_1^2 x\,\mathrm{d}x \int_{-2}^1 y\,\mathrm{d}y \int_0^{\frac{1}{2}} \mathrm{d}z = \frac{3}{2} \cdot \frac{-3}{2} \cdot \frac{1}{2} = -\frac{9}{8}$$

2.
$$\begin{aligned}\iiint_V xy^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^{xy} x^2y^2z^3\,\mathrm{d}z \\ &= \int_0^1 \mathrm{d}x \int_0^x \frac{1}{4}x^5y^6\,\mathrm{d}y \\ &= \int_0^1 \frac{1}{4} \cdot \frac{1}{7}x^{12}\,\mathrm{d}x \\ &= \frac{1}{4} \cdot \frac{1}{7} \cdot \frac{1}{13} = \frac{1}{364}\end{aligned}$$

✍ **习题 10.3.2(1)(2)(3)**

计算下列积分值.

1. $\int_0^2 \mathrm{d}x \int_0^{\sqrt{2x-x^2}} \mathrm{d}y \int_0^a z\sqrt{x^2+y^2}\,\mathrm{d}z$;
2. $\int_{-R}^R \mathrm{d}x \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \mathrm{d}y \int_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\,\mathrm{d}z$;
3. $\int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,\mathrm{d}z$;

解

1.
$$D = \{(x,y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant \sqrt{2x-x^2}\} = \{(x,y) \mid (x-1)^2+y^2 \leqslant 1, 0 \leqslant y\}$$
$$\text{令 } x = r\cos\theta, y = r\sin\theta\text{，则 } y = \sqrt{2x-x^2} \Rightarrow x^2+y^2 \leqslant 0 \Rightarrow r \leqslant 2\cos\theta\text{, 结合图像可知}$$
$$D = \{(r,\theta) \mid 0 \leqslant r \leqslant 2\cos\theta, 0 \leqslant \theta \leqslant \frac{\pi}{2}\}$$

$$\begin{aligned}\int_0^2 \mathrm{d}x \int_0^{\sqrt{2x-x^2}} \mathrm{d}y \int_0^a z\sqrt{x^2+y^2}\,\mathrm{d}z &= \int_0^a z\,\mathrm{d}z \int_D \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^a z\,\mathrm{d}z \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{2\cos\theta} r\cdot\frac{\partial(x,y)}{\partial(r,\theta)}\,\mathrm{d}r \\ &= \int_0^a z\,\mathrm{d}z \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{2\cos\theta} r^2\,\mathrm{d}r \\ &= \int_0^a z\,\mathrm{d}z \int_0^{\frac{\pi}{2}} \mathrm{d}\theta\cdot\frac{8}{3}\cos^3\theta \\ &= \int_0^a z\,\mathrm{d}z\cdot\frac{8}{3}\cdot\frac{2}{3} = \frac{8}{9}a^2\end{aligned}$$

2. 区域

$$\begin{aligned}V &= \{(x,y,z)|-R\leqslant x\leqslant R, -\sqrt{R^2-x^2}\leqslant y\leqslant\sqrt{R^2-x^2}, 0\leqslant z\leqslant\sqrt{R^2-x^2-y^2}\} \\ &= \{(x,y,z)|x^2+y^2+z^2\leqslant R^2, 0\leqslant z\}\end{aligned}$$

令 $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$，则

$$V = \{(r,\theta,\varphi)|0\leqslant r\leqslant R, 0\leqslant\theta\leqslant\frac{\pi}{2}, 0\leqslant\varphi\leqslant 2\pi\}$$

于是

$$\iiint_V x^2+y^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R (r^2\sin^2\theta)\cdot(r^2\sin\theta)\,\mathrm{d}r = \frac{4\pi}{15}R^5$$

3. 区域

$$\begin{aligned}V &= \{(x,y,z)|0\leqslant x\leqslant 1, 0\leqslant y\leqslant\sqrt{1-x^2}, 0\leqslant z\leqslant\sqrt{1-x^2-y^2}\} \\ &= \{(x,y,z)|x^2+y^2+z^2\leqslant 1, x,y,z\geqslant 0\} \\ &= \{(r,\theta,\varphi)|0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant\frac{\pi}{2}, 0\leqslant\varphi\leqslant\frac{\pi}{2}\}\end{aligned}$$

令 $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$，则

$$\int_0^1\mathrm{d}x\int_0^{\sqrt{1-x^2}}\mathrm{d}y\int_0^{\sqrt{1-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,\mathrm{d}z = \int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 (r)\cdot(r^2\sin\theta)\,\mathrm{d}r = \frac{\pi}{8}$$

✍ **习题 10.3.3(1)(3)(6)**

计算下列三重积分.

1. $\iiint_V (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, V$: 由 $x^2+y^2=2z, z=2$ 围成;
3. $\iiint_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, V$: 由 $\sqrt{4-x^2-y^2}=z, x^2+y^2=3z$ 围成;
6. $\iiint_V |x^2+y^2+z^2-1|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, V : x^2+y^2+z^2\leqslant 4.$

解

1. 令 $x = r\cos\theta, y = r\sin\theta, z = z$，则

$$\begin{aligned}\iiint_V (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^2 \mathrm{d}z \int_{x^2+y^2\leqslant 2z} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{2z}} r^2\cdot r\,\mathrm{d}r \\ &= \int_0^2 \frac{1}{4}(\sqrt{2z})^4\,\mathrm{d}z\cdot 2\pi = 2\pi\int_0^2 z^2\,\mathrm{d}z \\ &= 2\pi\cdot\frac{8}{3} = \frac{16\pi}{3}\end{aligned}$$

3. 令 $x = r\cos\theta, y = r\sin\theta, z = z$，则

$$\begin{aligned}V &= \{(x,y,z)\mid \sqrt{4-x^2-y^2}\leqslant z\leqslant \frac{1}{3}(x^2+y^2), x^2+y^2\leqslant 3\} \\ &= \{(r,\theta,z)\mid 0\leqslant r\leqslant\sqrt{3}, \sqrt{4-r^2}\leqslant z\leqslant\frac{1}{3}r^2, 0\leqslant\theta\leqslant 2\pi\}\end{aligned}$$

于是

$$\begin{aligned}\iiint_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{3}}\mathrm{d}r\int_{\sqrt{4-r^2}}^{\frac{1}{3}r^2} z\cdot r\,\mathrm{d}z \\ &= \pi\int_0^{\sqrt{3}} r\left(\frac{r^4}{9}-4+r^2\right)\mathrm{d}r \\ &= \frac{13\pi}{4}\end{aligned}$$

6. 令 $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$，则

$$V = \{(x,y,z)\mid x^2+y^2+z^2\leqslant 4\} = \{(r,\theta,\varphi)\mid 0\leqslant r\leqslant 2, 0\leqslant\theta\leqslant\pi, 0\leqslant\varphi\leqslant 2\pi\}$$

于是

$$\begin{aligned}\iiint_V \left|x^2+y^2+z^2-1\right|\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^2 \left|r^2-1\right| r^2\sin\theta\,\mathrm{d}r \\ &= 4\pi\left(\int_0^1 (1-r^2)r^2\,\mathrm{d}r + \int_1^2 (r^2-1)r^2\,\mathrm{d}r\right) \\ &= 4\pi\left(\int_0^1 (r^2-r^4)\,\mathrm{d}r + \int_1^2 (r^4-r^2)\,\mathrm{d}r\right) \\ &= 16\pi\end{aligned}$$

✍ **习题 10.3.7**

设 $F(t) = \iiint_{x^2+y^2+z^2\leqslant t^2} f(x^2+y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, 其中 f 为可微函数, 求 $F'(t)$.

证明 令 $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$，则

$$V = \{(x,y,z)\mid x^2+y^2+z^2\leqslant t^2\} = \{(r,\theta,\varphi)\mid 0\leqslant r\leqslant t, 0\leqslant\theta\leqslant\pi, 0\leqslant\varphi\leqslant 2\pi\}$$

于是

$$\begin{aligned} F(t) &= \iiint_{x^2+y^2+z^2\leqslant t^2} f(x^2+y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^t f(r^2)r^2\sin\theta\,\mathrm{d}r \\ &= 4\pi\int_0^t f(r^2)r^2\,\mathrm{d}r \end{aligned}$$

因此

$$F'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left(4\pi\int_0^t f(r^2)r^2\,\mathrm{d}r\right)=4\pi t^2 f(t^2)$$

✍ **习题 10.3.8**

证明:

$$\iiint_{x^2+y^2+z^2\leqslant 1} f(z)\,\mathrm{d}V=\boldsymbol{\pi}\int_{-1}^{1} f(z)(1-z^2)\,\mathrm{d}z.$$

证明 令 $x=r\cos\theta, y=r\sin\theta, z=z$，则

$$V=\{(x,y,z)\mid x^2+y^2+z^2\leqslant 1\}=\{(r,\theta,z)\mid 0\leqslant r\leqslant\sqrt{1-z^2}, 0\leqslant\theta\leqslant 2\pi, -1\leqslant z\leqslant 1\}$$

于是

$$\begin{aligned} \iiint_{x^2+y^2+z^2\leqslant 1} f(z)\,\mathrm{d}V &= \iiint_{x^2+y^2+z^2\leqslant 1} f(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{1-z^2}}\mathrm{d}r\int_0^1 f(z)r\,\mathrm{d}z \\ &=\pi\int_{-1}^{1} f(z)(1-z^2)\,\mathrm{d}z \end{aligned}$$