Lec 10 多元函数微分法习题课（1）

10.1 例题

例 10.1 设方程:$u^3-3(x+y)u^2+z^3=0$ 确定了隐函数 $u=f(x,y,z)$, 求 $\mathrm{d}u$.

解 解法一:

原方程两边取全微分 d, 得

$$\begin{aligned}&\mathrm{d}(u^3-3(x+y)u^2+z^3)=\mathrm{d}(0)=0\\ \Rightarrow&\,\mathrm{d}(u^3)-3\,\mathrm{d}((x+y)u^2)+\mathrm{d}(z^3)=0\\ \Rightarrow&3u^2\,\mathrm{d}u-3u^2(\mathrm{d}x+\mathrm{d}y)-3(x+y)2u\,\mathrm{d}u+3z^2\,\mathrm{d}z=0\end{aligned}$$

整理得

$$\mathrm{d}u=\frac{3u^2(\mathrm{d}x+\mathrm{d}y)+3z^2\,\mathrm{d}z}{3u^2+6(x+y)u}=\frac{u^2\,\mathrm{d}x+u^2\,\mathrm{d}y-z^2\,\mathrm{d}z}{u^2-2(x+y)u}$$

解 解法二:

令 $F(x,y,z,u)=u^3-3(x+y)u^2+z^3$, 其中 x,y,z,u 地位相同, 则 $F_u'=3u^2-6(x+y)u$,$F_x'=-3u^2$,$F_y'=-3u^2$,$F_z'=3z^2$. 从而

$$\begin{aligned}\frac{\partial u}{\partial x}&=-\frac{F_x'}{F_u'}=\frac{u^2}{-2(x+y)u+u^2};\\ \frac{\partial u}{\partial y}&=-\frac{F_y'}{F_u'}=\frac{u^2}{-2(x+y)u+u^2};\\ \frac{\partial u}{\partial z}&=\frac{F_z'}{F_u'}=\frac{z^2}{-2(x+y)u+u^2}.\end{aligned}$$

因此

$$\mathrm{d}u=\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y+\frac{\partial u}{\partial z}\,\mathrm{d}z=\frac{u^2\,\mathrm{d}x+u^2\,\mathrm{d}y-z^2\,\mathrm{d}z}{u^2-2(x+y)u}.$$

解 解法三:

原方程两边分别对 x,y,z 求偏导数, 解出 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ 再代入 $\mathrm{d}u=\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y+\frac{\partial u}{\partial z}\,\mathrm{d}z$ 即可.

例 10.2 试证明方程

$$\frac{\partial^2u}{\partial x^2}+2\frac{\partial^2u}{\partial x\partial y}-3\frac{\partial^2u}{\partial y^2}+2\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=0$$

在线性变换

$$\begin{cases}\xi=x+y,\\ \eta=3x-y\end{cases}$$

下可以化简为

$$\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0$$

解 证法一:

从线性变换 $\begin{cases}\xi=x+y,\\ \eta=3x-y\end{cases}$ 可得 $x=\frac{1}{4}(\xi+\eta)$,$y=\frac{1}{4}(3\xi-\eta)$, 因此 $\frac{\partial x}{\partial\xi}=\frac{1}{4},\frac{\partial x}{\partial\eta}=\frac{1}{4},\frac{\partial y}{\partial\xi}=\frac{3}{4},\frac{\partial y}{\partial\eta}=-\frac{1}{4}$. 由此得

$$\begin{aligned}
\frac{\partial u}{\partial\xi}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\xi}=\frac{1}{4}\frac{\partial u}{\partial x}+\frac{3}{4}\frac{\partial u}{\partial y}\\
\Rightarrow\frac{\partial^2 u}{\partial\eta\partial\xi}&=\left(\frac{\partial u}{\partial\xi}\right)'_\eta=\left(\frac{1}{4}\frac{\partial u}{\partial x}\right)'_\eta+\left(\frac{3}{4}\frac{\partial u}{\partial y}\right)'_\eta\\
&=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial\eta}\right)+\frac{3}{4}\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial\eta}\right)\\
&=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\frac{1}{4}+\frac{\partial^2 u}{\partial x\partial y}\left(-\frac{1}{4}\right)\right)+\frac{3}{4}\left(\frac{\partial^2 u}{\partial x\partial y}\frac{1}{4}+\frac{\partial^2 u}{\partial y^2}\left(-\frac{1}{4}\right)\right)\\
&=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}\right)
\end{aligned}$$

即有

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}=16\frac{\partial^2 u}{\partial\eta\partial\xi}\\ \frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=4\frac{\partial u}{\partial\xi}\end{cases}$$

故原偏微分方程化简为

$$\begin{aligned}
16\frac{\partial^2 u}{\partial\eta\partial\xi}+2\left(4\frac{\partial u}{\partial\xi}\right)=0\\
\Rightarrow\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0
\end{aligned}$$

解 解法二:

从线性变换 $\begin{cases}\xi=x+y,\\ \eta=3x-y\end{cases}\Rightarrow\frac{\partial\xi}{\partial x}=1,\frac{\partial\xi}{\partial y}=1,\frac{\partial\eta}{\partial x}=3,\frac{\partial\eta}{\partial y}=-1$. 而 $u(x,y)$ 通过中间变量可视为 ξ,η 的函数, 从而

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x}=\frac{\partial u}{\partial\xi}+3\frac{\partial u}{\partial\eta},\\ \frac{\partial u}{\partial y}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial y}=\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}.\end{cases},$$

从而

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial u}{\partial\xi}\right)'_x+3\left(\frac{\partial u}{\partial\eta}\right)'_x=\left(\frac{\partial^2 u}{\partial\xi^2}+3\frac{\partial^2 u}{\partial\eta\partial\xi}\right)+3\left(\frac{\partial^2 u}{\partial\xi\partial\eta}+3\frac{\partial^2 u}{\partial\eta^2}\right),\\ \frac{\partial^2 u}{\partial x\partial y}=\left(\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}\right)'_x=\left(\frac{\partial^2 u}{\partial\xi^2}+3\frac{\partial^2 u}{\partial\xi\partial\eta}\right)-\left(\frac{\partial^2 u}{\partial\xi\partial\eta}+3\frac{\partial^2 u}{\partial\eta^2}\right),\\ \frac{\partial^2 u}{\partial y^2}=\left(\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}\right)'_y=\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\xi\partial\eta}\right)-\left(\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{\partial^2 u}{\partial\eta^2}\right).\end{cases}$$

即

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial \xi^2} + 6\dfrac{\partial^2 u}{\partial \xi \partial \eta} + 9\dfrac{\partial^2 u}{\partial \eta^2}, \\ \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial \xi^2} + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} - 3\dfrac{\partial^2 u}{\partial \eta^2}, \\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial \xi^2} - 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} + \dfrac{\partial^2 u}{\partial \eta^2}. \end{cases}$$

且

$$2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 2\left(\frac{\partial u}{\partial \xi} + 3\frac{\partial u}{\partial \eta}\right) + 6\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = 8\frac{\partial u}{\partial \xi},$$

从而原方程化为

$$\frac{\partial^2 u}{\partial \xi^2}(1+2-3) + \frac{\partial^2 u}{\partial \xi \partial \eta}(6+4+6) + \frac{\partial^2 u}{\partial \eta^2}(9-6-3) + 8\frac{\partial u}{\partial \xi} = 0,$$

即

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{2}\frac{\partial u}{\partial \xi} = 0.$$

例 10.3 设 $u = f(x,y,z), \varphi(x^2, \mathrm{e}^y, z) = 0, y = \sin x$, 且 $f, \varphi \in C^1$, $\dfrac{\partial \varphi}{\partial z} \neq 0$, 求 $\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

解 从 $\varphi(x^2, \mathrm{e}^{\sin x}, z) = 0$ 及 $\varphi'_z \neq 0$ 可知, 由方程 $\varphi(x^2, \mathrm{e}^{\sin x}, z) = 0$ 可确定 z 是 x, y 的隐函数, 从而 z 是 x 的复合函数. 故从 $u = f(x,y,z)$ 知,u 是 x 的一元函数.

注 助教注: 这个地方可以理解为由隐函数定理 $F(x,y,z) = \varphi(x^2, \mathrm{e}^y, z) = 0$, 确定了隐函数 $z = z(x,y)$, 从而 $u = f(x,y,z) = f(x,y,z(x,y)) = f(x, \sin x, z(x, \sin x))$ 确定了 u 是 x 的函数.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = f'_1 \cdot 1 + f'_2 \cdot y'_x + f'_3 \cdot z'_x = f'_1 + f'_2 \cdot \cos x \cdot 2x + f'_3 \cdot z'_x.$$

令 $F(x,y,z) = \varphi(x^2, \mathrm{e}^y, z)$, 则

$$\begin{cases} F'_x(x,y,z) = \varphi'_1 \cdot 2x + \varphi'_2 \mathrm{e}^{\sin x} \cos x, \\ F'_y(x,y,z) = \varphi'_3 \cdot 1 = \varphi'_3. \end{cases}$$

故 $z'_x = -\dfrac{F'_x}{F'_z} = -\dfrac{\varphi'_1 \cdot 2x + \varphi'_2 \mathrm{e}^{\sin x} \cos x}{\varphi'_3}$. 代入 $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ 即有

$$\frac{\mathrm{d}u}{\mathrm{d}x} = f'_1 + f'_2 \cdot y'_x + f'_3 \cdot \left(-\frac{\varphi'_1 \cdot 2x + \varphi'_2 \mathrm{e}^{\sin x} \cos x}{\varphi'_3}\right).$$

例 10.4 证明: 全微分也具有一阶微分形式不变性, 即, 若 $f(x,y)$ 可微, 则不论 x, y 是自变量还是中间变量, 则 $z = f(x,y)$, 总有

$$\mathrm{d}z = \mathrm{d}f(x,y) = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y = f'_x \,\mathrm{d}x + f'_y \,\mathrm{d}y.$$

证明

1. 当 x, y 是自变量时, 显然有 $\mathrm{d}z = f'_x \,\mathrm{d}x + f'_y \,\mathrm{d}y$.
2. 当 x, y 是中间变量时, 设 $\begin{cases} x = g(s,t), \\ y = h(s,t), \end{cases}$ 可微, 且 $f(g(s,t), h(s,t))$ 有意义时,z 通过中间

变量 x,y 成为 s,t 的复合函数, 且有求偏导数的链式法则如下:

$$\mathrm{d}x=\frac{\partial x}{\partial s}\,\mathrm{d}s+\frac{\partial x}{\partial t}\,\mathrm{d}t,$$
$$\mathrm{d}y=\frac{\partial y}{\partial s}\,\mathrm{d}s+\frac{\partial y}{\partial t}\,\mathrm{d}t.$$

且

$$\begin{aligned}\mathrm{d}z&=\frac{\partial z}{\partial s}\,\mathrm{d}s+\frac{\partial z}{\partial t}\,\mathrm{d}t=\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\right)\mathrm{d}s+\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right)\mathrm{d}t\\&=\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\,\mathrm{d}s+\frac{\partial x}{\partial t}\,\mathrm{d}t\right)+\frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\,\mathrm{d}s+\frac{\partial y}{\partial t}\,\mathrm{d}t\right)\\&=\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y.\end{aligned}$$

即 x,y 是中间变量时, 也有 $\mathrm{d}z=f'_x\,\mathrm{d}x+f'_y\,\mathrm{d}y$.

注 利用全微分的一阶微分形式不变性, 可导出多元可微函数的如下的微分四则运算法则:

1. $\mathrm{d}(u\pm v)=\mathrm{d}u\pm\mathrm{d}v$;
2. $\mathrm{d}(uv)=u\,\mathrm{d}v+v\,\mathrm{d}u$;
3. $\mathrm{d}\left(\frac{u}{v}\right)=\frac{v\,\mathrm{d}u-u\,\mathrm{d}v}{v^2}$, 其中 u,v 均可微, 且 $v\neq 0$;

证明

1. 令 $f(u,v)=u+v$, 则 $f(u,v)\in C^1$, 从而 $f(u,v)$ 可微, 无论 u,v 是自变量还是中间变量, 总有
$$\mathrm{d}(u+v)=\mathrm{d}f(u,v)=\frac{\partial f}{\partial u}\,\mathrm{d}u+\frac{\partial f}{\partial v}\,\mathrm{d}v=\mathrm{d}u+\mathrm{d}v$$
从而有 $\mathrm{d}(u\pm v)=\mathrm{d}u\pm\mathrm{d}v$. 这里 d 是全微分.
2. 令 $f(u,v)=uv$, 则 $f(u,v)\in C^1$, 从而 $f(u,v)$ 可微, 无论 u,v 是自变量还是中间变量, 总有
$$\mathrm{d}(uv)=\mathrm{d}f(u,v)=\frac{\partial f}{\partial u}\,\mathrm{d}u+\frac{\partial f}{\partial v}\,\mathrm{d}v=v\,\mathrm{d}u+u\,\mathrm{d}v$$
从而有 $\mathrm{d}(uv)=u\,\mathrm{d}v+v\,\mathrm{d}u$.
3. 令 $f(u,v)=\frac{u}{v}$, 则 $f(u,v)\in C^1$, 从而 $f(u,v)$ 可微, 无论 u,v 是自变量还是中间变量, 总有
$$\mathrm{d}\left(\frac{u}{v}\right)=\mathrm{d}f(u,v)=\frac{\partial f}{\partial u}\,\mathrm{d}u+\frac{\partial f}{\partial v}\,\mathrm{d}v=\frac{1}{v}\,\mathrm{d}u+\left(-\frac{u}{v^2}\right)\mathrm{d}v=\frac{v\,\mathrm{d}u-u\,\mathrm{d}v}{v^2}$$

注 二阶及以上的微分通常没有形式不变性, 具体而言, 设 $f(x,y)\in C^2$, 则 $z=f(x,y)\Rightarrow \mathrm{d}z=\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y$.

$$\begin{aligned}\mathrm{d}(\mathrm{d}z):=\mathrm{d}^2z&=\mathrm{d}\left(\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y\right)\\&=\left(\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y\right)'_x\mathrm{d}x+\left(\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y\right)'_y\mathrm{d}y\\&=\left(\frac{\partial^2 z}{\partial x^2}\,\mathrm{d}x+\frac{\partial^2 z}{\partial x\partial y}\,\mathrm{d}y\right)\mathrm{d}x+\left(\frac{\partial^2 z}{\partial x\partial y}\,\mathrm{d}x+\frac{\partial^2 z}{\partial y^2}\,\mathrm{d}y\right)\mathrm{d}y\\&=\frac{\partial^2 z}{\partial x^2}(\mathrm{d}x)^2+2\frac{\partial^2 z}{\partial x\partial y}\,\mathrm{d}x\,\mathrm{d}y+\frac{\partial^2 z}{\partial y^2}(\mathrm{d}y)^2.\end{aligned}$$

$\mathrm{d}^2 z$ 是 x, y 是自变量时的 $z = f(x,y)$ 的二阶微分, 而 $\mathrm{d}^2 z$ 是 x, y 是中间变量时的 $z = f(x,y)$ 的二阶微分, 二者通常不相等.

例 10.5 设 $u = u(x,y), v = v(x,y)$ 是由方程组

$$\begin{cases} u = f(ux, v+y), \\ v = g(u-x, v^2 y) \end{cases}$$

所确定的隐函数组, 求变换 $\begin{cases} u = u(x,y), \\ v = v(x,y) \end{cases}$ 的 Jacobi 行列式:

$$\begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} := \frac{\partial(u,v)}{\partial(x,y)} \quad f, g \in C^1.$$

解 令 $A = ux, B = v + y, E = u - x, F = v^2 y$, 则方程组可化为

$$\begin{cases} u = f(A,B), \\ v = g(E,F). \end{cases}$$

方程组两边关于 x 求偏导

$$\begin{cases} u'_x = f'_1 \cdot (u + xu'_x) + f'_2 \cdot (v'_x + 0), \\ v'_x = g'_1 \cdot (u'_x - 1) + g'_2 \cdot 2vv'_x y. \end{cases}$$

标准化为

$$\begin{cases} (xf'_1 - 1)u'_x + f'_2 v'_x = -f'_1 u, \\ g'_1 u'_x + (2vg'_2 y - 1)v'_x = g'_1. \end{cases}$$

令 $D = \begin{vmatrix} xf'_1 - 1 & f'_2 \\ g'_1 & 2vg'_2 y - 1 \end{vmatrix}$, 则 $D \neq 0$, 再令

$$D_1 = \begin{vmatrix} -f'_1 u & f'_2 \\ g'_1 & 2vg'_2 y - 1 \end{vmatrix}, \quad D_2 = \begin{vmatrix} xf'_1 - 1 & -f'_1 u \\ g'_1 & g'_1 \end{vmatrix},$$

由克莱姆法则可得

$$u'_x = \frac{D_1}{D}, \quad v'_x = \frac{D_2}{D}.$$

方程组 $\begin{cases} u = f(A,B), \\ v = g(E,F) \end{cases}$ 两边同时对 y 求偏导, 可得

$$\begin{cases} u'_y = f'_1 \cdot u'_y \cdot x + f'_2 \cdot (v'_y + 1), \\ v'_y = g'_1 \cdot u'_y + g'_2 \cdot (2vv'_y y + 2v^2). \end{cases}$$

标准化为

$$\begin{cases} (xf'_1 - 1)u'_y + f'_2 v'_y = -f'_2, \\ g'_1 u'_y + (2vg'_2 y - 1)v'_y = 2vg'_2. \end{cases}$$

令 $D=\begin{vmatrix} xf_1'-1 & f_2' \\ g_1' & 2vg_2'y-1 \end{vmatrix}$, 则 $D\neq 0$, 再令

$$\tilde{D}_1=\begin{vmatrix} -f_2' & f_2' \\ v^2g_2' & 2vg_2'y-1 \end{vmatrix},\quad \tilde{D}_2=\begin{vmatrix} xf_1'-1 & -f_2' \\ g_1' & g_1'v^2 \end{vmatrix},$$

由克莱姆法则可得

$$u_y'=\frac{\tilde{D}_1}{D},\quad v_y'=\frac{\tilde{D}_2}{D}.$$

从而

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x' & u_y' \\ v_x' & v_y' \end{vmatrix}=\frac{\begin{vmatrix} D_1 & \tilde{D}_1 \\ D_2 & \tilde{D}_2 \end{vmatrix}}{D}$$

例 10.6 设 $\begin{cases} u=u(x,y), \\ v=v(x,y) \end{cases}$ 是由方程组

$$\begin{cases} F(x,y,u,v)=0, \\ G(x,y,u,v)=0 \end{cases}$$

确定的隐函数组,$F,G\in C^1$, 且 $\dfrac{\partial(F,G)}{\partial(u,v)}\neq 0$, 求 $\mathrm{d}u,\mathrm{d}v$.

解 解法一:

$$\mathrm{d}u=\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y,\ \mathrm{d}v=\frac{\partial v}{\partial x}\,\mathrm{d}x+\frac{\partial v}{\partial y}\,\mathrm{d}y,\text{ 对 }\begin{cases} F(x,y,u,v)=0, \\ G(x,y,u,v)=0 \end{cases}$$ 两边关于 x 求偏导, 可得

$$\begin{cases} F_x'\cdot 1+F_u'\cdot\dfrac{\partial u}{\partial x}+F_v'\cdot\dfrac{\partial v}{\partial x}=0, \\ G_x'\cdot 1+G_u'\cdot\dfrac{\partial u}{\partial x}+G_v'\cdot\dfrac{\partial v}{\partial x}=0. \end{cases}$$

标准化为

$$\begin{cases} F_u'\dfrac{\partial u}{\partial x}+F_v'\dfrac{\partial v}{\partial x}=-F_x', \\ G_u'\dfrac{\partial u}{\partial x}+G_v'\dfrac{\partial v}{\partial x}=-G_x'. \end{cases}$$

令 $D=\begin{vmatrix} F_u' & F_v' \\ G_u' & G_v' \end{vmatrix}$, 则 $D\neq 0$, 再令

$$D_1=\begin{vmatrix} -F_x' & F_v' \\ -G_x' & G_v' \end{vmatrix},\quad D_2=\begin{vmatrix} F_u' & -F_x' \\ G_u' & -G_x' \end{vmatrix},$$

此时注意到 $D_1=\begin{vmatrix}F'_v & F'_x\\ G'_v & G'_x\end{vmatrix}=\dfrac{\partial(F,G)}{\partial(v,x)}$, $D_2=\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix}=\dfrac{\partial(F,G)}{\partial(u,v)}$, 由克莱姆法则可得

$$\frac{\partial u}{\partial x}=\frac{D_1}{D}=\frac{\frac{\partial(F,G)}{\partial(v,x)}}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$\frac{\partial v}{\partial x}=\frac{D_2}{D}=\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

对原方程组两边对 y 求偏导, 同样可得

$$\frac{\partial u}{\partial y}=\frac{\frac{\partial(F,G)}{\partial(v,y)}}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$\frac{\partial v}{\partial y}=\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

从而

$$\mathrm{d}u=\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y=\frac{\frac{\partial(F,G)}{\partial(v,x)}\,\mathrm{d}x+\frac{\partial(F,G)}{\partial(v,y)}\,\mathrm{d}y}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$\mathrm{d}v=\frac{\partial v}{\partial x}\,\mathrm{d}x+\frac{\partial v}{\partial y}\,\mathrm{d}y=\frac{\frac{\partial(F,G)}{\partial(u,x)}\,\mathrm{d}x+\frac{\partial(F,G)}{\partial(u,y)}\,\mathrm{d}y}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

解 解法二:

对原方程两边同时取全微分, 可得

$$\begin{cases}F'_x\,\mathrm{d}x+F'_y\,\mathrm{d}y+F'_u\,\mathrm{d}u+F'_v\,\mathrm{d}v=0,\\ G'_x\,\mathrm{d}x+G'_y\,\mathrm{d}y+G'_u\,\mathrm{d}u+G'_v\,\mathrm{d}v=0.\end{cases}$$

以 $\mathrm{d}u,\mathrm{d}v$ 为变量. 依 cramer 法则, 解得

$$\mathrm{d}u=\frac{D_1}{D}=\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y=\frac{\frac{\partial(F,G)}{\partial(v,x)}\,\mathrm{d}x+\frac{\partial(F,G)}{\partial(v,y)}\,\mathrm{d}y}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$\mathrm{d}v=\frac{D_2}{D}=\frac{\partial v}{\partial x}\,\mathrm{d}x+\frac{\partial v}{\partial y}\,\mathrm{d}y=\frac{\frac{\partial(F,G)}{\partial(u,x)}\,\mathrm{d}x+\frac{\partial(F,G)}{\partial(u,y)}\,\mathrm{d}y}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

其中,

$$D_1=\begin{vmatrix}-(F'_x\,\mathrm{d}x+F'_y\,\mathrm{d}y) & F'_v\\ -(G'_x\,\mathrm{d}x+G'_y\,\mathrm{d}y) & G'_v\end{vmatrix},\quad D_2=\begin{vmatrix}F'_u & -(F'_x\,\mathrm{d}x+F'_y\,\mathrm{d}y)\\ G'_u & -(G'_x\,\mathrm{d}x+G'_y\,\mathrm{d}y)\end{vmatrix},\quad D=\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix}=\frac{\partial(F,G)}{\partial(u,v)}.$$

✍ **作业** ex9.2:31;ex9.3:6,7,8,10,11(1),14.