Lec 11 方向导数与梯度

11.1 方向导数

定义 11.1

设函数 $u=f(x,y)$ 定义在 $\bar{U}(M_0,\delta)$ 中,$M_t=(x_0+t\cos\alpha,y_0+t\cos\beta)\in\bar{U}(M_0,\delta),\boldsymbol{l}=(\cos\alpha,\sin\alpha)=(\cos\alpha,\cos\beta),\alpha+\beta=\dfrac{\pi}{2}$.

如果极限

$$\lim_{t\to 0}\frac{f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)}{t}$$

存在, 那么称 $\left.\dfrac{\partial u}{\partial \boldsymbol{l}}\right|_{M_0}=\lim\limits_{t\to 0}\dfrac{f(x_0+t\cos\alpha,y_0+t\sin\beta)-f(x_0,y_0)}{t}$ 为函数 $u=f(x,y)$ 在点 M_0 处沿方向 $\boldsymbol{l}=(\cos\alpha,\cos\beta)$ 的方向导数 (directional derivative), 表示 u 关于 $\boldsymbol{l}$ 方向在 M_0 处的变化率. ♣

例 11.1 若 $f'_x(M_0)=f'_x(x_0,y_0)$ 存在, 则 $u=f(x,y)$ 在 M_0 处沿 x 轴方向 $\boldsymbol{i}=(1,0)$ 的方向导数为

$$\left.\frac{\partial u}{\partial \boldsymbol{i}}\right|_{M_0}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=f'_x(x_0,y_0);$$

而 $u=f(x,y)$ 在 M_0 处沿 x 轴负向 $\boldsymbol{l}=(-1,0)=-i$ 的方向导数为

$$\left.\frac{\partial u}{\partial(-\boldsymbol{i})}\right|_{M_0}=\lim_{-\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{-\Delta x}=-\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=-f'_x(M_0).$$

同理, 当 $f'_y(x_0,y_0)=B$(常数) 存在时, 则 $u=f(x,y)$ 在 M_0 处沿 y 轴正负方向都存在, 且

$$\left.\frac{\partial u}{\partial \boldsymbol{j}}\right|_{M_0}=f'_y(M_0),\ \left.\frac{\partial u}{\partial(-\boldsymbol{j})}\right|_{M_0}=-f'_y(M_0).$$

这里 $\boldsymbol{j}=(0,1),-\boldsymbol{j}=(0,-1)$ 分别为 y 轴的正负向.

例 11.2 $u=f(x,y)=\sqrt{x^2+y^2}$ 在 $(0,0)$ 处连续, 但 $u=f(x,y)$ 在 $(0,0)$ 处沿任何方向 $\boldsymbol{l}^0=(\cos\alpha,\sin\alpha)$ 的方向导数均不存在,

这是由于

$$\left.\frac{\partial u}{\partial \boldsymbol{l}^0}\right|=\lim_{t\to 0}\frac{f(0+t\cos\alpha,0+t\sin\alpha)-f(0,0)}{t}=\lim_{t\to 0}\frac{\sqrt{(t\cos\alpha)^2+(\boldsymbol{t}\sin\alpha)^2}-0}{t}=\lim_{t\to 0}\frac{|t|}{t}.$$

但 $\lim\limits_{t\to 0^+}\dfrac{|t|}{t}=1\neq\lim\limits_{t\to 0^-}\dfrac{|t|}{t}=-1$

例 11.3 设函数 $u=f(x,y,z)$ 定义在 $\bar{U}(M_0,\delta)$ 中,$M_0(x_0,y_0,z_0)$, $M_t(x_0+t\cos\alpha,y_0+t\cos\beta,z_0+\boldsymbol{t}\cos\gamma)=M_0+t\boldsymbol{l}\in\bar{U}(M_0,\delta),\boldsymbol{l}=(\cos\alpha,\cos\beta,\cos\gamma)$ 已知, 则定义

$$\left.\frac{\partial u}{\partial \boldsymbol{l}}\right|=\lim_{t\to 0}\frac{f(M_t)-f(M_0)}{t}=\lim_{t\to 0}\frac{f(x_0+t\cos\alpha,y_0+t\cos\beta,z_0+t\cos\gamma)-f(x_0,y_0,z_0)}{t}.$$

当偏导数 $f'_x(x_0,y_0,z_0)=A$(常数) 存在时,$u=f(x,y,z)$ 在 M_0 处沿 x 轴正向 $\boldsymbol{i}=(1,0,0)$,

负向 $-\boldsymbol{i}=(-1,0,0)$ 的方向导数都存在, 且

$$\left.\frac{\partial u}{\partial \boldsymbol{i}}\right|_{M_0}=A,\ \left.\frac{\partial u}{\partial(-\boldsymbol{i})}\right|_{M_0}=-A,$$

其余情况可类推.

定理 11.1

当 $u=f(x,y)$ 在点 $M_0(x_0,y_0)$ 处可微时,u 在点 M_0 处沿任何方向 $\boldsymbol{l}^0=(\cos\alpha,\sin\alpha)$ 的方向导数都存在, 且

$$\left.\frac{\partial u}{\partial \boldsymbol{l}^0}\right|_{M_0}=f_x'(M_0)\cos\alpha+f_y'(M_0)\sin\alpha=f_x'(M_0)\cos\alpha+f_y'(M_0)\cos\beta,(\alpha+\beta=\frac{\pi}{2}) \tag{11.1}$$

证明 设 $M_t(x_0+t\cos\alpha,y_0+t\sin\alpha)\in\bar{U}(M_0,\delta)$, 则

$$f(x_0+t\cos\alpha,y_0+t\sin\alpha)-f(x_0,y_0)=f_x'(M_0)t\cos\alpha+f_y'(M_0)t\cos\beta+o(\rho),$$

其中

$$\begin{aligned}\rho&=\rho(M_0,M_t)\\&=\sqrt{(x_0+t\cos\alpha-x_0)^2+(y_0+t\sin\alpha-y_0)^2}\\&=\sqrt{(t\cos\alpha)^2+(t\sin\alpha)^2}\\&=\sqrt{t^2\cos^2\alpha+t^2\sin^2\alpha}\\&=\sqrt{t^2}\\&=|t|\end{aligned}$$

而有 $\lim\limits_{t\to0}\dfrac{o(\rho)}{t}=\lim\limits_{t\to0}\dfrac{o(|t|)}{t}=\lim\limits_{t\to0}\dfrac{o(|t|)}{|t|}\dfrac{|t|}{t}$

$\left|\dfrac{|t|}{t}\right|=|\pm1|=1\leqslant1$ 有界,$\lim\limits_{t\to0}\dfrac{o(|t|)}{|t|}=0.$

因此 $\lim\limits_{t\to0}\dfrac{o(\rho)}{t}=0$

$$\begin{aligned}\left.\frac{\partial u}{\partial \boldsymbol{l}^0}\right|_{M_0}&=\lim_{t\to0}\frac{f(x_0+t\cos\alpha,y_0+t\sin\alpha)-f(x_0,y_0)}{t}\\&=\lim_{t\to0}\frac{f_x'(M_0)t\cos\alpha+f_y'(M_0)t\sin\alpha+o(\rho)}{t}\\&=\lim_{t\to0}\left(f_x'(M_0)\cos\alpha+f_y'(M_0)\sin\alpha+\frac{o(\rho)}{t}\right)\\&=f_x'(M_0)\cos\alpha+f_y'(M_0)\sin\alpha\end{aligned}$$

例 11.4 设 $z=f(x,y)=\begin{cases}\dfrac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}, & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$,$\boldsymbol{l}=(\cos\theta,\sin\theta),\theta\in[0,2\pi)$,

求 $f'_x(0,0), f'_y(0,0), \left.\dfrac{\partial z}{\partial \boldsymbol{l}}\right|_{O(0,0)}$, 并证明在 $O(0,0)$ 处,$z=f(x,y)$ 不可微.

解

1.

$$
\begin{aligned}
f'_x(0,0) &= \lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\frac{(\Delta x)^2 0^2}{((\Delta x)^2+0^2)^{\frac{3}{2}}}-0}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{0}{\Delta x}\\
&= 0
\end{aligned}
$$

由对称性可知,$f'_y(0,0)=f'_x(0,0)=0$

2.

$$
\begin{aligned}
\left.\frac{\partial z}{\partial \boldsymbol{l}}\right|_{O(0,0)} &= \lim_{t\to 0}\frac{f(0+t\cos\theta,0+t\sin\theta)-f(0,0)}{t}\\
&= \lim_{t\to 0}\frac{\frac{(t\cos\theta)^2(t\sin\theta)^2}{((t\cos\theta)^2+(t\sin\theta)^2)^{\frac{3}{2}}}}{t}\\
&= \lim_{t\to \mathbf{0}}\frac{t^3\cos^2\theta\sin^2\theta}{|t|^3}
\end{aligned}
$$

$\dfrac{t^3}{|t|^3}=\pm 1$ 有界, 但趋于零时极限不存在,

因此当且仅当 $\theta=0,\dfrac{\pi}{2},\pi,\dfrac{3\pi}{2}$ 时,$\cos^2\theta\sin^2\theta=0, \left.\dfrac{\partial z}{\partial \boldsymbol{l}}\right|_{O(0,0)}=0$

即只在 $\theta=0,\dfrac{\pi}{2},\pi,\dfrac{3\pi}{2}$ 四个方向上存在方向导数, 且方向导数为 0, 其他方向上无方向导数.

3.

$$
\begin{aligned}
\lim_{\rho\to 0}\frac{\Delta z-f'_x(0,0)\Delta x+f'_y(0,0)\Delta y}{\rho} &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(\Delta x)^2(\Delta y)^2}{((\Delta x)^2+(\Delta y)^2)^2}\\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y=k\Delta x}}\frac{k^2(\Delta x)^4}{((\Delta x)^2+k^2(\Delta x)^2)^2}\\
&= \frac{k^2}{(1+k^2)^2}\neq 0
\end{aligned}
$$

故不可微.

4. 不可微这一问依照定理**??**的结论直接可得:

若可微则应有 $\left.\dfrac{\partial z}{\partial \boldsymbol{l}}\right|_{O(0,0)}=f'_x(0,0)\cos\theta+f'_y(0,0)\sin\theta=0$, 即沿各个方向的方向导数都存在, 均为 0. 但这与第二问中我们求出来的结果矛盾, 故不可微.

同理, 当 $u=f(x,y,z)$ 在点 $M_0(x_0,y_0,z_0)$ 处可微时,u 在点 M_0 处沿任何方向

$\boldsymbol{l}^0=(\cos\alpha,\cos\beta,\cos\gamma)$ 的方向导数都存在, 且

$$\left.\frac{\partial u}{\partial \boldsymbol{l}^0}\right|_{M_0}=f_x'(M_0)\cos\alpha+f_y'(M_0)\cos\beta+f_z'(M_0)\cos\gamma \tag{11.2}$$

在**??**中称向量 $(f_x'(M_0),f_y'(M_0))$ 为函数 $u=f(x,y)$ 在点 $M_0(x_0,y_0)$ 处的梯度; 在**??**中称向量 $(f_x'(M_0),f_y'(M_0),f_z'(M_0))$ 为函数 $u=f(x,y,z)$ 在点 $M_0(x_0,y_0,z_0)$ 处的梯度, 记作

$$\mathbf{grad}\,f(x_0,y_0)=(f_x'(M_0),f_y'(M_0));\mathbf{grad}\,f(x_0,y_0,z_0)=(f_x'(M_0),f_y'(M_0),f_z'(M_0)).$$

或

$$\mathbf{grad}\,f(x_0,y_0)=\left.\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\right|_{M_0};\mathbf{grad}\,f(x_0,y_0,z_0)=\left.\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\right|_{M_0}.$$

$u=f(x,y,z)$ 在 $\bar{U}(M_0,\delta)$ 中任一点 M 的梯度记作

$$\mathbf{grad}\,f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)=\left(\frac{\partial }{\partial x},\frac{\partial }{\partial y},\frac{\partial }{\partial z}\right)u=\nabla u$$

其中 $\nabla=\left(\frac{\partial }{\partial x},\frac{\partial }{\partial y},\frac{\partial }{\partial z}\right)$ 称为微分向量算子, 也称为 Hamilton 算子, 此时**??**可改写为:

$$\begin{aligned}\left.\frac{\partial u}{\partial \boldsymbol{l}^0}\right|_{M_0}&=\left(f_x'(M_0),f_y'(M_0),f_z'(M_0)\right)\cdot(\cos\alpha,\cos\beta,\cos\gamma)\\&=\nabla u|\cdot\boldsymbol{l}^0\\&=\mathbf{grad}\,f(x_0,y_0,z_0)\cdot\boldsymbol{l}^0\\&=|\mathbf{grad}\,f(x_0,y_0,z_0)|\,\left|\boldsymbol{l}^0\right|\cos\left(\mathbf{grad}\,\widehat{f(M_0),\boldsymbol{l}^0}\right)\\&\leqslant|\mathbf{grad}\,f(x_0,y_0,z_0)|\\&=|\nabla u(M_0)|\end{aligned}$$

等号当且仅当 $\boldsymbol{l}^0$ 与 $\mathbf{grad}\,f(M_0)$ 一致时取到.

11.2 函数的梯度 (陡度, 倾斜度)

设 $u=f(x,y,z)$ 在点 $M_0(x_0,y_0,z_0)$ 处可微, 则 $f(x,y,z)$ 在 $M_0(x_0,y_0,z_0)$ 处的梯度

$$\mathbf{grad}\,f(x_0,y_0,z_0)=(f_x'(M_0),f_y'(M_0),f_z'(M_0))$$

是一个向量. 这个向量的模 $|\mathbf{grad}\,f(x_0,y_0,z_0)|$ 是 $f(x,y,z)$ 在点 M_0 处所有方向的方向导数中的最大值, 而梯度的方向即是 $f(x,y,z)$ 在点 M_0 处所有方向的方向导数中取最大值的方向.

$$\left.\frac{\partial u}{\partial \boldsymbol{l}^0}\right|_{M_0}=\mathbf{grad}\,f(x_0,y_0,z_0)\cdot\boldsymbol{l}^0=|\mathbf{grad}\,f(x_0,y_0,z_0)|\,\left|\boldsymbol{l}^0\right|\cos\left(\mathbf{grad}\,\widehat{f(M_0),\boldsymbol{l}^0}\right)\leqslant|\nabla u(M_0)|$$

可知, 当 $\boldsymbol{l}^0$ 与 $\mathbf{grad}\,f(M_0)$ 方向一致时, $\left.\frac{\partial u}{\partial \boldsymbol{l}^0}\right|_{M_0}$ 取最大值 $|\mathbf{grad}\,f(M_0)|$; 而当 $\boldsymbol{l}^0$ 与 $\mathbf{grad}\,f(M_0)$ 方向相反时, $\left.\frac{\partial u}{\partial \boldsymbol{l}^0}\right|_{M_0}$ 取最小值 $-|\mathbf{grad}\,f(M_0)|$;

即

$$\left(\left.\frac{\partial u}{\partial \boldsymbol{l}}\right|_{M_0}\right)_{\max} = |\mathbf{grad}\, f(M_0)|\,, \left(\left.\frac{\partial u}{\partial \boldsymbol{l}}\right|_{M_0}\right)_{\min} = -\,|\mathbf{grad}\, f(M_0)|$$

换言之, 在点 M_0 处沿梯度 $\mathbf{grad}\, f(M_0)$ 的方向,$f(x,y,z)$ 的变化率是最大的, 而沿着 $-\,\mathbf{grad}\, f(M_0)$ 的方向,$f(x,y,z)$ 的变化率最小:

$$-\,|\mathbf{grad}\, f(M_0)| \leqslant \left.\frac{\partial u}{\partial \boldsymbol{l}}\right|_{M_0} \leqslant |\mathbf{grad}\, f(M_0)|$$

并由 $\mathbf{grad}\, f(M_0) = \nabla u(M_0) = \left.\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)\right|_{M_0}$, 从而有

$$-\,|\nabla u(M_0)| \leqslant \left.\frac{\partial u}{\partial \boldsymbol{l}}\right|_{M_0} \leqslant |\nabla u(M_0)|$$

命题 11.1

求函数或数量场 u 的梯度是一种特定的微分运算, 设 $u_2 = f_1(x,y,z), u_2 = f_2(x,y,z)$ 均可微, 或 $f_1, f_2 \in C^1$, 则必有:

1. $\nabla(c_1u_1 + c_2u_2) = c_1\nabla u_1 + c_2\nabla u_2$,$c_1, c_2$ 为任意常数;
2. $\nabla(u_1u_2) = u_2\nabla u_1 + u_1\nabla u_2$;
3. $\nabla f(u_1) = f'(u)\nabla u, \forall f \in C^1$.

证明

1. c_1, c_2 是常数

$$\begin{aligned}
\nabla(c_1u_1 + c_2u_2) &= \left((c_1u_1 + c_2u_2)'_x, (c_1u_1 + c_2u_2)'_y, (c_1u_1 + c_2u_2)'_z\right) \\
&= \left(c_1(u_1)'_x + c_2(u_2)'_x, c_1(u_1)'_y + c_2(u_2)'_y, c_1(u_1)'_z + c_2(u_2)'_z\right) \\
&= c_1\left((u_1)'_x, (u_1)'_y, (u_1)'_z\right) + c_2\left((u_2)'_x, (u_2)'_y, (u_2)'_z\right) \\
&= c_1\nabla u_1 + c_2\nabla u_2
\end{aligned}$$

2.

$$\begin{aligned}
\nabla(u_1u_2) &= \left((u_1u_2)'_x, (u_1u_2)'_y, (u_1u_2)'_z\right) \\
&= \left(u_2(u_1)'_x + u_1(u_2)'_x, u_2(u_1)'_y + u_1(u_2)'_y, u_2(u_1)'_z + u_1(u_2)'_z\right) \\
&= u_2\left((u_1)'_x, (u_1)'_y, (u_1)'_z\right) + u_1\left((u_2)'_x, (u_2)'_y, (u_2)'_z\right) \\
&= u_2\nabla u_1 + u_1\nabla u_2
\end{aligned}$$

3.

$$\begin{aligned}
\nabla f(u) &= \left((f(u))'_x, (f(u))'_y, (f(u))'_z\right) \\
&= \left(f'(u)u'_x, f'(u)u'_y, f'(u)u'_z\right) \\
&= f'(u)\left(u'_x, u'_y, u'_z\right) \\
&= f'(u)\nabla u
\end{aligned}$$

从这三条性质可知, 哈密顿算子 ∇ 与微分算子 d非常类似.

例 11.5 求解下列各题:

1. 求 $z=x^2+y^2$ 在点 $M_0(1,2)$ 处, 沿着 (1,2) 到 $(2,2+\sqrt{3})$ 方向的方向导数, 并求 $\left.\frac{\partial z}{\partial \boldsymbol{l}}\right|_{M_0(1,2)}$ 的最大值和最小值.
2. 求 $z=1-(\frac{x^2}{a^2}+\frac{y^2}{b^2})$ 在点 $M_0(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}})$ 处, 沿曲线 $L:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 在这点的内法线方向的方向导数.
3. 求数量场 $\frac{m}{r}$ 所产生的梯度场 $\nabla\frac{m}{r}$, 其中 $m>0$ 为常数,$r=\sqrt{x^2+y^2+z^2}$ 是向径 (x,y,z) 的模.

解

1. $\boldsymbol{l}=(2-1,2+\sqrt{3}-2)=\left(1,\sqrt{3}\right)\Rightarrow\boldsymbol{l}^0=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)=(\cos\alpha,\cos\beta)$

   $\nabla z(M_0)=\left(z_x'(M_0),z_y'(M_0)\right)=(2x,2y)\Big|_{M_0(1,2)}=(2,4)$

   因此
   $$\left.\frac{\partial z}{\partial \boldsymbol{l}}\right|_{M_0(1,2)}=(2,4)\cdot(\frac{1}{2},\frac{\sqrt{3}}{2})=1+2\sqrt{3}$$
   而又有 $|\nabla z(M_0)|=|(2,4)|=2\sqrt{5}$

   因此
   $$\left(\left.\frac{\partial z}{\partial \boldsymbol{l}}\right|_{M_0(1,2)}\right)_{max}=2\sqrt{5},\left(\left.\frac{\partial z}{\partial \boldsymbol{l}}\right|_{M_0(1,2)}\right)_{min}=-2\sqrt{5}$$

2. L 有参数方程表示
   $$\mathbf{r}(t)=(x(t),y(t))=(a\cos t,b\sin t),t\in[0,2\pi]$$
   因此 $M_0=\mathbf{r}(t_0),t_0=\frac{\pi}{4}$.

   有 $r'(t)=(x'(t),y'(t))\Big|_{M_0}=(-a\sin t,b\cos t)\Big|_{t=\frac{\pi}{4}}=\left(-\frac{\sqrt{2}}{2}a,\frac{\sqrt{2}}{2}b\right)$

   可取切向量 $\boldsymbol{\tau}=(-a,b)$, 则过 M_0 的外法向量为 $\boldsymbol{n}=(b,a)$, 因此过 M_0 的内法向量为 $\boldsymbol{l}=-\boldsymbol{n}=(-b,-a)\Rightarrow\boldsymbol{l}^0=-\frac{1}{\sqrt{a^2+b^2}}(b,a)$,

   同时 $z_x'(M_0)=-\left.\frac{2x}{a^2}\right|_{M_0}=-\frac{2}{a^2}\frac{a}{\sqrt{2}}=-\frac{\sqrt{2}}{a},z_y'(M_0)=-\left.\frac{2y}{b^2}\right|_{M_0}=-\frac{2}{b^2}\frac{b}{\sqrt{2}}=-\frac{\sqrt{2}}{b}$

   故
   $$\left.\frac{\partial z}{\partial \boldsymbol{l}}\right|_{M_0}=-\frac{1}{\sqrt{a^2+b^2}}\left(-\frac{\sqrt{2}}{a},-\frac{\sqrt{2}}{b}\right)\cdot(b,a)=\frac{\sqrt{2}}{\sqrt{a^2+b^2}}\left(\frac{b}{a}+\frac{a}{b}\right)=\frac{\sqrt{2(a^2+b^2)}}{ab}$$

3. $\nabla\left(\frac{m}{r}\right)=\left(\left(\frac{m}{r}\right)_x',\left(\frac{m}{r}\right)_y',\left(\frac{m}{r}\right)_z'\right)$.

   而 $\left(\frac{m}{r}\right)_x'=m\left(\frac{1}{r}\right)_x'=-\frac{m}{r^2}r_x'=-\frac{m}{r^2}\frac{x}{r}=-\frac{mx}{r^3}$.

   由对称性知 $\left(\frac{m}{r}\right)_y'=-\frac{my}{r^3},\left(\frac{m}{r}\right)_z'=-\frac{mz}{r^3}$

因此

$$\nabla\left(\frac{m}{r}\right)=-\frac{m}{r^3}(x,y,z)$$

令 $\mathbf{r}^0=\frac{1}{r}(x,y,z)$, 则

$$\nabla\left(\frac{m}{r}\right)=-\frac{m}{r^2}\mathbf{r}^0=-\frac{m\cdot 1}{r^2}\mathbf{r}^0 \tag{11.3}$$

??右端的力学解释: 位于原点 $O(0,0,0)$ 的质量为 m 的顶点, 对位于点 $M(x,y,z)$ 且质量为 1 的单位质点的引力, 该引力大小与两质点的质量乘积成正比, 而与它们的距离的平方成反比, 并且和这个引力的方向由点 M 指向原点.

在物理中, 称 $\nabla\left(\frac{m}{r}\right)=\frac{m\cdot 1}{r^2}(-\mathbf{r}^0)$ 为引力场, 这是一个向量场, 而称 $\frac{m}{r}=\frac{m}{\sqrt{x^2+y^2+z^2}}$ 为对应的引力势函数, 简称势函数.

因为引力场 $\frac{m}{r^2}(-\mathbf{r}^0)=-\frac{m}{x^2+y^2+z^2}\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$ 是通过势函数 $\frac{m}{r}$ 取梯度得到的, 因此, 也常成这个引力场为梯度场.

注 设 $\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$, 则 $\nabla^2=\nabla\cdot\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\cdot(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})=\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2+\left(\frac{\partial}{\partial z}\right)^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\triangleq\Delta$——Laplace 算子.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u=0\Rightarrow\Delta u=0$$

✍ 作业 ex9.2:21,22,23,24,36(2)(5),38.