Lec 10 多元函数微分法习题课（1）

10.1 例题

例 10.1 设方程:$u^3-3(x+y)u^2+z^3=0$ 确定了隐函数 $u=f(x,y,z)$, 求 $\mathrm{d}u$.

解 解法一:

原方程两边取全微分 d, 得

$$\begin{aligned}&\mathrm{d}(u^3-3(x+y)u^2+z^3)=\mathrm{d}(0)=0\\ \Rightarrow&\,\mathrm{d}(u^3)-3\,\mathrm{d}((x+y)u^2)+\mathrm{d}(z^3)=0\\ \Rightarrow&3u^2\,\mathrm{d}u-3u^2(\mathrm{d}x+\mathrm{d}y)-3(x+y)2u\,\mathrm{d}u+3z^2\,\mathrm{d}z=0\end{aligned}$$

整理得

$$\mathrm{d}u=\frac{3u^2(\mathrm{d}x+\mathrm{d}y)+3z^2\,\mathrm{d}z}{3u^2+6(x+y)u}=\frac{u^2\,\mathrm{d}x+u^2\,\mathrm{d}y-z^2\,\mathrm{d}z}{u^2-2(x+y)u}$$

解 解法二:

令 $F(x,y,z,u)=u^3-3(x+y)u^2+z^3$, 其中 x,y,z,u 地位相同, 则 $F'_u=3u^2-6(x+y)u$,$F'_x=-3u^2$,$F'_y=-3u^2$,$F'_z=3z^2$. 从而

$$\begin{aligned}\frac{\partial u}{\partial x}&=-\frac{F'_x}{F'_u}=\frac{u^2}{-2(x+y)u+u^2};\\ \frac{\partial u}{\partial y}&=-\frac{F'_y}{F'_u}=\frac{u^2}{-2(x+y)u+u^2};\\ \frac{\partial u}{\partial z}&=\frac{F'_z}{F'_u}=\frac{z^2}{-2(x+y)u+u^2}.\end{aligned}$$

因此

$$\mathrm{d}u=\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y+\frac{\partial u}{\partial z}\,\mathrm{d}z=\frac{u^2\,\mathrm{d}x+u^2\,\mathrm{d}y-z^2\,\mathrm{d}z}{u^2-2(x+y)u}.$$

解 解法三:

原方程两边分别对 x,y,z 求偏导数, 解出 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ 再代入 $\mathrm{d}u=\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y+\frac{\partial u}{\partial z}\,\mathrm{d}z$ 即可.

例 10.2 试证明方程

$$\frac{\partial^2u}{\partial x^2}+2\frac{\partial^2u}{\partial x\partial y}-3\frac{\partial^2u}{\partial y^2}+2\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=0$$

在线性变换

$$\begin{cases}\xi=x+y,\\ \eta=3x-y\end{cases}$$

下可以化简为

$$\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0$$

解 证法一:

从线性变换 $\begin{cases}\xi=x+y,\\ \eta=3x-y\end{cases}$ 可得 $x=\frac{1}{4}(\xi+\eta)$,$y=\frac{1}{4}(3\xi-\eta)$, 因此 $\frac{\partial x}{\partial\xi}=\frac{1}{4},\frac{\partial x}{\partial\eta}=\frac{1}{4},\frac{\partial y}{\partial\xi}=\frac{3}{4},\frac{\partial y}{\partial\eta}=-\frac{1}{4}$. 由此得

$$\begin{aligned}\frac{\partial u}{\partial\xi}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\xi}=\frac{1}{4}\frac{\partial u}{\partial x}+\frac{3}{4}\frac{\partial u}{\partial y}\\ \Rightarrow\frac{\partial^2 u}{\partial\eta\partial\xi}&=\left(\frac{\partial u}{\partial\xi}\right)'_\eta=\left(\frac{1}{4}\frac{\partial u}{\partial x}\right)'_\eta+\left(\frac{3}{4}\frac{\partial u}{\partial y}\right)'_\eta\\ &=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial\eta}\right)+\frac{3}{4}\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial\eta}\right)\\ &=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\frac{1}{4}+\frac{\partial^2 u}{\partial x\partial y}\left(-\frac{1}{4}\right)\right)+\frac{3}{4}\left(\frac{\partial^2 u}{\partial x\partial y}\frac{1}{4}+\frac{\partial^2 u}{\partial y^2}\left(-\frac{1}{4}\right)\right)\\ &=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}\right)\end{aligned}$$

即有

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}=16\frac{\partial^2 u}{\partial\eta\partial\xi}\\ \frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=4\frac{\partial u}{\partial\xi}\end{cases}$$

故原偏微分方程化简为

$$\begin{aligned}16\frac{\partial^2 u}{\partial\eta\partial\xi}+2\left(4\frac{\partial u}{\partial\xi}\right)=0\\ \Rightarrow\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0\end{aligned}$$

解 解法二:

从线性变换 $\begin{cases}\xi=x+y,\\ \eta=3x-y\end{cases}\Rightarrow\frac{\partial\xi}{\partial x}=1,\frac{\partial\xi}{\partial y}=1,\frac{\partial\eta}{\partial x}=3,\frac{\partial\eta}{\partial y}=-1$. 而 $u(x,y)$ 通过中间变量可视为 ξ,η 的函数, 从而

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x}=\frac{\partial u}{\partial\xi}+3\frac{\partial u}{\partial\eta},\\ \frac{\partial u}{\partial y}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial y}=\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}.\end{cases},$$

从而

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial u}{\partial\xi}\right)'_x+3\left(\frac{\partial u}{\partial\eta}\right)'_x=\left(\frac{\partial^2 u}{\partial\xi^2}+3\frac{\partial^2 u}{\partial\eta\partial\xi}\right)+3\left(\frac{\partial^2 u}{\partial\xi\partial\eta}+3\frac{\partial^2 u}{\partial\eta^2}\right),\\ \frac{\partial^2 u}{\partial x\partial y}=\left(\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}\right)'_x=\left(\frac{\partial^2 u}{\partial\xi^2}+3\frac{\partial^2 u}{\partial\xi\partial\eta}\right)-\left(\frac{\partial^2 u}{\partial\xi\partial\eta}+3\frac{\partial^2 u}{\partial\eta^2}\right),\\ \frac{\partial^2 u}{\partial y^2}=\left(\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}\right)'_y=\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\xi\partial\eta}\right)-\left(\frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{\partial^2 u}{\partial\eta^2}\right).\end{cases}$$

即

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial \xi^2} + 6\dfrac{\partial^2 u}{\partial \xi \partial \eta} + 9\dfrac{\partial^2 u}{\partial \eta^2}, \\ \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial \xi^2} + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} - 3\dfrac{\partial^2 u}{\partial \eta^2}, \\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial \xi^2} - 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} + \dfrac{\partial^2 u}{\partial \eta^2}. \end{cases}$$

且

$$2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 2\left(\frac{\partial u}{\partial \xi} + 3\frac{\partial u}{\partial \eta}\right) + 6\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = 8\frac{\partial u}{\partial \xi},$$

从而原方程化为

$$\frac{\partial^2 u}{\partial \xi^2}(1+2-3) + \frac{\partial^2 u}{\partial \xi \partial \eta}(6+4+6) + \frac{\partial^2 u}{\partial \eta^2}(9-6-3) + 8\frac{\partial u}{\partial \xi} = 0,$$

即

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{2}\frac{\partial u}{\partial \xi} = 0.$$

例 10.3 设 $u = f(x,y,z), \varphi(x^2, \mathrm{e}^y, z) = 0, y = \sin x$, 且 $f, \varphi \in C^1$, $\dfrac{\partial \varphi}{\partial z} \neq 0$, 求 $\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

解 从 $\varphi(x^2, \mathrm{e}^{\sin x}, z) = 0$ 及 $\varphi_z' \neq 0$ 可知, 由方程 $\varphi(x^2, \mathrm{e}^{\sin x}, z) = 0$ 可确定 z 是 x, y 的隐函数, 从而 z 是 x 的复合函数. 故从 $u = f(x,y,z)$ 知,u 是 x 的一元函数.

注 助教注: 这个地方可以理解为由隐函数定理 $F(x,y,z) = \varphi(x^2, \mathrm{e}^y, z) = 0$, 确定了隐函数 $z = z(x,y)$, 从而 $u = f(x,y,z) = f(x,y,z(x,y)) = f(x, \sin x, z(x, \sin x))$ 确定了 u 是 x 的函数.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = f_1' \cdot 1 + f_2' \cdot y_x' + f_3' \cdot z_x' = f_1' + f_2' \cdot \cos x \cdot 2x + f_3' \cdot z_x'.$$

令 $F(x,y,z) = \varphi(x^2, \mathrm{e}^y, z)$, 则

$$\begin{cases} F_x'(x,y,z) = \varphi_1' \cdot 2x + \varphi_2' \mathrm{e}^{\sin x} \cos x, \\ F_z'(x,y,z) = \varphi_3' \cdot 1 = \varphi_3'. \end{cases}$$

故 $z_x' = -\dfrac{F_x'}{F_z'} = -\dfrac{\varphi_1' \cdot 2x + \varphi_2' \mathrm{e}^{\sin x} \cos x}{\varphi_3'}$. 代入 $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ 即有

$$\frac{\mathrm{d}u}{\mathrm{d}x} = f_1' + f_2' \cdot y_x' + f_3' \cdot \left(-\frac{\varphi_1' \cdot 2x + \varphi_2' \mathrm{e}^{\sin x} \cos x}{\varphi_3'}\right).$$

注 助教注: 这里老师写的确实很模糊. 我们要区分两个式子和他们分别的含义.

1. 令 $F(x,y,z) = \varphi(x^2, \mathrm{e}^y, z)$, 则 $F(x,y,z) = 0$ 确定了 $z = z(x,y)$, 其中 $z_x' = -\dfrac{F_x'}{F_z'}$. 这时候 z_x' 表示的是 $\dfrac{\partial z}{\partial x}$. $F_x'(x,y,z) = \varphi_1' \cdot 2x, F_z'(x,y,z) = \varphi_3'$, 从而 $z_x' = -\dfrac{F_x'}{F_z'} = -\dfrac{2x\varphi_1'}{\varphi_3'}$.
2. 令 $F(x,z) = \varphi(x^2, \mathrm{e}^{\sin x}, z)$, 则 $F(x,z) = 0$ 确定了 $z = z(x)$. 这时候 z_x' 表示的是 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$. $F_x'(x,z) = \varphi_1' \cdot 2x + \varphi_2' \mathrm{e}^{\sin x} \cos x, F_z'(x,z) = \varphi_3'$, 从而 $z_x' = -\dfrac{F_x'}{F_z'} = -\dfrac{2x\varphi_1' + \varphi_2' \mathrm{e}^{\sin x} \cos x}{\varphi_3'}$.

老师要表示的实际是第二种情况, 即 $z = z(x)$. 只不过写成的形式看起来像是第一种情况.

注 正是因为老师这里写模糊了, 所以会有疑问为什么当 $F(x,z)=\varphi(x^2,\mathrm{e}^{\sin x},z)$ 时,

$$F'_x=\varphi'_1\cdot 2x+\varphi'_2\mathrm{e}^{\sin x}\cos x$$

而不是

$$F'_x=\varphi'_1\cdot 2x+\varphi'_2\cdot\mathrm{e}^{\sin x}\cos x+\varphi'_3\cdot z'_x$$

后者是 $F(x)=\varphi(x^2,\mathrm{e}^{\sin x},z(x))$ 时的 $F'_x=F'(x)$. 而为了求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$, 我们需要对 $F(x,z)$ 这个函数利用隐函数定理, 此时 x,z 都是这个 $F(x,z)$ 的自变量, 因此 $F'_x=\varphi'_1\cdot 2x+\varphi'_2\mathrm{e}^{\sin x}\cos x$.

例 10.4 证明: 全微分也具有一阶微分形式不变性, 即, 若 $f(x,y)$ 可微, 则不论 x,y 是自变量还是中间变量, 则 $z=f(x,y)$, 总有

$$\mathrm{d}z=\mathrm{d}f(x,y)=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=f'_x\,\mathrm{d}x+f'_y\,\mathrm{d}y.$$

证明

1. 当 x,y 是自变量时, 显然有 $\mathrm{d}z=f'_x\,\mathrm{d}x+f'_y\,\mathrm{d}y$.
2. 当 x,y 是中间变量时, 设 $\begin{cases}x=g(s,t),\\ y=h(s,t),\end{cases}$ 可微, 且 $f(g(s,t),h(s,t))$ 有意义时,z 通过中间变量 x,y 成为 s,t 的复合函数, 且有求偏导数的链式法则如下:

$$\mathrm{d}x=\frac{\partial x}{\partial s}\mathrm{d}s+\frac{\partial x}{\partial t}\mathrm{d}t,$$
$$\mathrm{d}y=\frac{\partial y}{\partial s}\mathrm{d}s+\frac{\partial y}{\partial t}\mathrm{d}t.$$

且

$$\begin{aligned}\mathrm{d}z&=\frac{\partial z}{\partial s}\mathrm{d}s+\frac{\partial z}{\partial t}\mathrm{d}t=\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\right)\mathrm{d}s+\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right)\mathrm{d}t\\&=\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\mathrm{d}s+\frac{\partial x}{\partial t}\mathrm{d}t\right)+\frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\mathrm{d}s+\frac{\partial y}{\partial t}\mathrm{d}t\right)\\&=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y.\end{aligned}$$

即 x,y 是中间变量时, 也有 $\mathrm{d}z=f'_x\,\mathrm{d}x+f'_y\,\mathrm{d}y$.

注 利用全微分的一阶微分形式不变性, 可导出多元可微函数的如下的微分四则运算法则:

1. $\mathrm{d}(u\pm v)=\mathrm{d}u\pm\mathrm{d}v$;
2. $\mathrm{d}(uv)=u\,\mathrm{d}v+v\,\mathrm{d}u$;
3. $\mathrm{d}\left(\dfrac{u}{v}\right)=\dfrac{v\,\mathrm{d}u-u\,\mathrm{d}v}{v^2}$, 其中 u,v 均可微, 且 $v\neq 0$;

证明

1. 令 $f(u,v)=u+v$, 则 $f(u,v)\in C^1$, 从而 $f(u,v)$ 可微, 无论 u,v 是自变量还是中间变量, 总有

$$\mathrm{d}(u+v)=\mathrm{d}f(u,v)=\frac{\partial f}{\partial u}\mathrm{d}u+\frac{\partial f}{\partial v}\mathrm{d}v=\mathrm{d}u+\mathrm{d}v$$

从而有 $\mathrm{d}(u\pm v)=\mathrm{d}u\pm\mathrm{d}v$. 这里 d 是全微分.

2. 令 $f(u,v)=uv$, 则 $f(u,v)\in C^1$, 从而 $f(u,v)$ 可微, 无论 u,v 是自变量还是中间变量, 总

有
$$\mathrm{d}(uv)=\mathrm{d}f(u,v)=\frac{\partial f}{\partial u}\,\mathrm{d}u+\frac{\partial f}{\partial v}\,\mathrm{d}v=v\,\mathrm{d}u+u\,\mathrm{d}v$$
从而有 $\mathrm{d}(uv)=u\,\mathrm{d}v+v\,\mathrm{d}u$.

3. 令 $f(u,v)=\dfrac{u}{v}$, 则 $f(u,v)\in C^1$, 从而 $f(u,v)$ 可微, 无论 u,v 是自变量还是中间变量, 总有
$$\mathrm{d}\left(\frac{u}{v}\right)=\mathrm{d}f(u,v)=\frac{\partial f}{\partial u}\,\mathrm{d}u+\frac{\partial f}{\partial v}\,\mathrm{d}v=\frac{1}{v}\,\mathrm{d}u+\left(-\frac{u}{v^2}\right)\mathrm{d}v=\frac{v\,\mathrm{d}u-u\,\mathrm{d}v}{v^2}$$

注 二阶及以上的微分通常没有形式不变性, 具体而言, 设 $f(x,y)\in C^2$, 则 $z=f(x,y)\Rightarrow \mathrm{d}z=\dfrac{\partial z}{\partial x}\,\mathrm{d}x+\dfrac{\partial z}{\partial y}\,\mathrm{d}y$.
$$\begin{aligned}\mathrm{d}(\mathrm{d}z):=\mathrm{d}^2z&=\mathrm{d}\left(\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y\right)\\&=\left(\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y\right)'_x\mathrm{d}x+\left(\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y\right)'_y\mathrm{d}y\\&=\left(\frac{\partial^2 z}{\partial x^2}\,\mathrm{d}x+\frac{\partial^2 z}{\partial x\partial y}\,\mathrm{d}y\right)\mathrm{d}x+\left(\frac{\partial^2 z}{\partial x\partial y}\,\mathrm{d}x+\frac{\partial^2 z}{\partial y^2}\,\mathrm{d}y\right)\mathrm{d}y\\&=\frac{\partial^2 z}{\partial x^2}(\mathrm{d}x)^2+2\frac{\partial^2 z}{\partial x\partial y}\,\mathrm{d}x\,\mathrm{d}y+\frac{\partial^2 z}{\partial y^2}(\mathrm{d}y)^2.\end{aligned}$$
d^2z 是 x,y 是自变量时的 $z=f(x,y)$ 的二阶微分, 而 d^2z 是 x,y 是中间变量时的 $z=f(x,y)$ 的二阶微分, 二者通常不相等.

例 10.5 设 $u=u(x,y),v=v(x,y)$ 是由方程组
$$\begin{cases}u=f(ux,v+y),\\ v=g(u-x,v^2y)\end{cases}$$
所确定的隐函数组, 求变换 $\begin{cases}u=u(x,y),\\ v=v(x,y)\end{cases}$ 的 Jacobi 行列式:
$$\begin{vmatrix}u'_x & u'_y\\ v'_x & v'_y\end{vmatrix}:=\frac{\partial(u,v)}{\partial(x,y)}\qquad f,g\in C^1.$$

解 令 $A=ux,B=v+y,E=u-x,F=v^2y$, 则方程组可化为
$$\begin{cases}u=f(A,B),\\ v=g(E,F).\end{cases}$$
方程组两边关于 x 求偏导
$$\begin{cases}u'_x=f'_1\cdot(u+xu'_x)+f'_2\cdot(v'_x+0),\\ v'_x=g'_1\cdot(u'_x-1)+g'_2\cdot 2vv'_xy.\end{cases}$$
标准化为
$$\begin{cases}(xf'_1-1)u'_x+f'_2v'_x=-f'_1u,\\ g'_1u'_x+(2vg'_2y-1)v'_x=g'_1.\end{cases}$$

令 $D=\begin{vmatrix} xf_1'-1 & f_2' \\ g_1' & 2vg_2'y-1 \end{vmatrix}$, 则 $D\neq 0$, 再令

$$D_1=\begin{vmatrix} -f_1'u & f_2' \\ g_1' & 2vg_2'y-1 \end{vmatrix}, \quad D_2=\begin{vmatrix} xf_1'-1 & -f_1'u \\ g_1' & g_1' \end{vmatrix},$$

由克莱姆法则可得

$$u_x'=\frac{D_1}{D}, \quad v_x'=\frac{D_2}{D}.$$

方程组 $\begin{cases} u=f(A,B), \\ v=g(E,F) \end{cases}$ 两边同时对 y 求偏导, 可得

$$\begin{cases} u_y'=f_1'\cdot u_y'\cdot x+f_2'\cdot(v_y'+1), \\ v_y'=g_1'\cdot u_y'+g_2'\cdot(2vv_y'y+2v^2). \end{cases}$$

标准化为

$$\begin{cases} (xf_1'-1)u_y'+f_2'v_y'=-f_2', \\ g_1'u_y'+(2vg_2'y-1)v_y'=2vg_2'. \end{cases}$$

令 $D=\begin{vmatrix} xf_1'-1 & f_2' \\ g_1' & 2vg_2'y-1 \end{vmatrix}$, 则 $D\neq 0$, 再令

$$\tilde{D}_1=\begin{vmatrix} -f_2' & f_2' \\ v^2g_2' & 2vg_2'y-1 \end{vmatrix}, \quad \tilde{D}_2=\begin{vmatrix} xf_1'-1 & -f_2' \\ g_1' & g_1'v^2 \end{vmatrix},$$

由克莱姆法则可得

$$u_y'=\frac{\tilde{D}_1}{D}, \quad v_y'=\frac{\tilde{D}_2}{D}.$$

从而

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x' & u_y' \\ v_x' & v_y' \end{vmatrix}=\frac{\begin{vmatrix} D_1 & \tilde{D}_1 \\ D_2 & \tilde{D}_2 \end{vmatrix}}{D}$$

例 10.6 设 $\begin{cases} u=u(x,y), \\ v=v(x,y) \end{cases}$ 是由方程组

$$\begin{cases} F(x,y,u,v)=0, \\ G(x,y,u,v)=0 \end{cases}$$

确定的隐函数组,$F,G\in C^1$, 且 $\dfrac{\partial(F,G)}{\partial(u,v)}\neq 0$, 求 $\mathrm{d}u,\mathrm{d}v$.

解 解法一:

$$\mathrm{d}u=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y, \mathrm{d}v=\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y, \text{对} \begin{cases} F(x,y,u(x,y),v(x,y))=0, \\ G(x,y,u(x,y),v(x,y))=0 \end{cases} \text{两边关于 } x$$

求偏导, 可得

$$\begin{cases} F_x' \cdot 1 + F_u' \cdot \dfrac{\partial u}{\partial x} + F_v' \cdot \dfrac{\partial v}{\partial x} = 0, \\ G_x' \cdot 1 + G_u' \cdot \dfrac{\partial u}{\partial x} + G_v' \cdot \dfrac{\partial v}{\partial x} = 0. \end{cases}$$

注 助教注: 这里的对 $F = 0$ 两侧对 x 求偏导, 我们区分两个描述

1. $F(x, y, u, v) = 0$, 两侧对 x 求偏导, 也就是对第一个分量求偏导, 得到 $F_x' = 0$.
2. 我们令 $\tilde{F}(x, y) = F(x, y, u(x, y), v(x, y))$, 两侧对 x 求偏导, 求的是 $\dfrac{\partial \tilde{F}}{\partial x}$, 得到

$$\frac{\partial \tilde{F}}{\partial x} = F_x' + F_u' \cdot \frac{\partial u}{\partial x} + F_v' \cdot \frac{\partial v}{\partial x} = 0$$

后者才是隐函数定理的表述, 多元函数的隐函数定理是这样的

定理 10.1 (多元函数隐函数定理)

设 $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ 是一个连续可微函数, 设 $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{y} \in \mathbb{R}^m$. 若方程

$$\begin{cases} F^1(\boldsymbol{x}, \boldsymbol{y}) = 0, \\ F^2(\boldsymbol{x}, \boldsymbol{y}) = 0, \\ \cdots \\ F^m(\boldsymbol{x}, \boldsymbol{y}) = 0 \end{cases}$$

满足在 $(\boldsymbol{x_0}, \boldsymbol{y_0})$ 处, 有 $F(\boldsymbol{x_0}, \boldsymbol{y_0}) = 0$, 且 $\dfrac{\partial(F^1, F^2, \cdots, F^m)}{\partial(\boldsymbol{y})} \neq 0$, 则存在一个邻域 U 和一个函数

$$\boldsymbol{y} = \boldsymbol{\varphi}(\boldsymbol{x}) = (\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \cdots, \varphi_m(\boldsymbol{x}))$$

使得在 U 中, 有 $\boldsymbol{y} = \boldsymbol{\varphi}(\boldsymbol{x})$ 是 $\boldsymbol{x}$ 的函数, 且在 U 中有解集可以写为

$$F^1(\boldsymbol{x}, \boldsymbol{\varphi}(\boldsymbol{x})) = 0, F^2(\boldsymbol{x}, \boldsymbol{\varphi}(\boldsymbol{x})) = 0, \cdots, F^m(\boldsymbol{x}, \boldsymbol{\varphi}(\boldsymbol{x})) = 0 \tag{10.1}$$

我们希望求隐函数的偏导数, 是希望求 $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_m)$ 的偏导数. 也就是说

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial \varphi_1}{\partial x_1}$$

而 $\dfrac{\partial \varphi_1}{\partial x_1}$ 由 10.1 求出.

♡

因此我们在求 $\dfrac{\partial u}{\partial x}$ 时, 我们是对 $F(x, y, u(x, y), v(x, y))$ 两侧对 x 求偏导.

注 那有的同学会说, 这不对啊, 为什么例 10.3 中就是对 $F(x, z) = 0$ 对 x 求偏导时就是把 x, z 视作不相关的自变量呢?

这是因为, 如果我们已知 $z = z(x)$, 然后对 $F(x, z(x)) = 0$ 两侧对 x 求导, 得到

$$F_x' + F_z' \cdot z_x' = 0$$

还是能得到 $z_x' = -\dfrac{F_x'}{F_z'}$.

我们接下来继续原来的题目.

标准化为

$$\begin{cases} F'_u\dfrac{\partial u}{\partial x}+F'_v\dfrac{\partial v}{\partial x}=-F'_x, \\ G'_u\dfrac{\partial u}{\partial x}+G'_v\dfrac{\partial v}{\partial x}=-G'_x. \end{cases}$$

令 $D=\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}$, 则 $D\neq 0$, 再令

$$D_1=\begin{vmatrix} -F'_x & F'_v \\ -G'_x & G'_v \end{vmatrix},\quad D_2=\begin{vmatrix} F'_u & -F'_x \\ G'_u & -G'_x \end{vmatrix},$$

此时注意到 $D_1=\begin{vmatrix} F'_v & F'_x \\ G'_v & G'_x \end{vmatrix}=\dfrac{\partial(F,G)}{\partial(v,x)}$, $D_2=\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}=\dfrac{\partial(F,G)}{\partial(u,v)}$, 由克莱姆法则可得

$$\frac{\partial u}{\partial x}=\frac{D_1}{D}=\frac{\frac{\partial(F,G)}{\partial(v,x)}}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$\frac{\partial v}{\partial x}=\frac{D_2}{D}=\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

对原方程组两边对 y 求偏导, 同样可得

$$\frac{\partial u}{\partial y}=\frac{\frac{\partial(F,G)}{\partial(v,y)}}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$\frac{\partial v}{\partial y}=\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

从而

$$\mathrm{d}u=\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y=\frac{\frac{\partial(F,G)}{\partial(v,x)}\,\mathrm{d}x+\frac{\partial(F,G)}{\partial(v,y)}\,\mathrm{d}y}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$\mathrm{d}v=\frac{\partial v}{\partial x}\,\mathrm{d}x+\frac{\partial v}{\partial y}\,\mathrm{d}y=\frac{\frac{\partial(F,G)}{\partial(u,x)}\,\mathrm{d}x+\frac{\partial(F,G)}{\partial(u,y)}\,\mathrm{d}y}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

解 解法二:

对原方程两边同时取全微分, 可得

$$\begin{cases} F'_x\,\mathrm{d}x+F'_y\,\mathrm{d}y+F'_u\,\mathrm{d}u+F'_v\,\mathrm{d}v=0, \\ G'_x\,\mathrm{d}x+G'_y\,\mathrm{d}y+G'_u\,\mathrm{d}u+G'_v\,\mathrm{d}v=0. \end{cases}$$

以 $\mathrm{d}u,\mathrm{d}v$ 为变量. 依 cramer 法则, 解得

$$\mathrm{d}u=\frac{D_1}{D}=\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y=\frac{\frac{\partial(F,G)}{\partial(v,x)}\,\mathrm{d}x+\frac{\partial(F,G)}{\partial(v,y)}\,\mathrm{d}y}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$\mathrm{d}v=\frac{D_2}{D}=\frac{\partial v}{\partial x}\,\mathrm{d}x+\frac{\partial v}{\partial y}\,\mathrm{d}y=\frac{\frac{\partial(F,G)}{\partial(u,x)}\,\mathrm{d}x+\frac{\partial(F,G)}{\partial(u,y)}\,\mathrm{d}y}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

其中，

$$D_1 = \begin{vmatrix} -(F'_x\,\mathrm{d}x + F'_y\,\mathrm{d}y) & F'_v \\ -(G'_x\,\mathrm{d}x + G'_y\,\mathrm{d}y) & G'_v \end{vmatrix}, \quad D_2 = \begin{vmatrix} F'_u & -(F'_x\,\mathrm{d}x + F'_y\,\mathrm{d}y) \\ G'_u & -(G'_x\,\mathrm{d}x + G'_y\,\mathrm{d}y) \end{vmatrix}, \quad D = \begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix} = \frac{\partial(F,G)}{\partial(u,v)}.$$

✍ **作业** ex9.2:31;ex9.3:6,7,8,10,11(1),14.