Lec 8 可微条件与高阶偏导数

8.1 $z=f(x,y)$ 在 $M_0(x_0,y_0)$ 处可微的条件

定理 8.1

若 $z=f(x,y)$ 在 M_0 处可微, 则 $f'_x(M_0), f'_y(M_0)$ 存在. 反之未必.

证明 已知 $z=f(x,y)$ 在 $M_0(x_0,y_0)$ 处可微, 则

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = (A\Delta x + B\Delta y) + o(\rho),$$

令 $\Delta y = 0$, 则

$$\Delta z_x = f(x_0+\Delta x, y_0) - f(x_0,y_0) = A\Delta x + o(|\Delta x|),$$

由此得

$$f'_x(M_0) = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x} = A.$$

同理, 令 $\Delta x = 0$, 则 $f'_y(M_0) = B$.

即 $\mathrm{d}z|_{M_0} = A\Delta x + B\Delta y = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y \Rightarrow \mathrm{d}z = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y$. 将 f'_x 记为 $\dfrac{\partial f}{\partial x}$, 将 f'_y 记为 $\dfrac{\partial f}{\partial y}$, 则

$$\mathrm{d}z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y.$$

或者写成向量形式

$$\mathrm{d}z = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix}.$$

定理 8.2

若 $f(x,y)$ 在 M_0 处可微, 则 $z=f(x,y)$ 在 M_0 处必连续, 反之未必.

证明 已知 $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y + o(\rho)$, 且

$$\Delta x\to 0, \Delta y \to 0,$$

时, 有

$$f'_x(M_0)\Delta x + f'_y(M_0)\Delta y + o(\rho) \to 0, \quad \rho\to 0,$$

其中 $\rho = \sqrt{(\Delta x)^2+(\Delta y)^2}$, 因此 $\rho\to 0 \Leftrightarrow \Delta x\to 0, \Delta y\to 0$.

从而 $\lim\limits_{\Delta x\to 0,\Delta y\to 0}\Delta z = 0 \Leftrightarrow z=f(x,y)$ 在 M_0 处连续.

例 8.1 反例 1: $z=f(x,y)=\sqrt{x^2+y^2}$, 在 $M_0(0,0)$ 处连续. 但因 $f'_x(0,0)=f'_y(0,0)$ 都不存在, 所以 $f(x,y)$ 在 M_0 处不可微.

例 8.2 反例 2: $z=f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$ 在 $(0,0)$ 处有 $f'_x(0,0)=f'_y(0,0)=0$, 但 $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ 不存在, 所以 $f(x,y)$ 在 $(0,0)$ 处不连续. 由**??**可知 $f(x,y)$ 在 $(0,0)$ 处不可微.

定理 8.3

$z=f(x,y)$ 在 $M_0(x_0,y_0)$ 处可微的充分必要条件是

$$\lim_{\rho\to0}\frac{\Delta z-f'_x(M_0)\Delta x-f'_y(M_0)\Delta y}{\rho}=0.$$

证明 若 $z=f(x,y)$ 在 M_0 处可微, 则

$$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=f'_x(M_0)\Delta x+f'_y(M_0)\Delta y+o(\rho),$$

由此得

$$\lim_{\rho\to0}\frac{\Delta z-f'_x(M_0)\Delta x-f'_y(M_0)\Delta y}{\rho}=\lim_{\rho\to0}\frac{o(\rho)}{\rho}=0.$$

反之, 若

$$\lim_{\rho\to0}\frac{\Delta z-f'_x(M_0)\Delta x-f'_y(M_0)\Delta y}{\rho}=0,$$

则

$$\Delta-(f'_x(M_0)\Delta x+f'_y(M_0)\Delta y)=o(\rho)\Rightarrow\Delta z=f'_x(M_0)\Delta x+f'_y(M_0)\Delta y+o(\rho)=(A\Delta x+B\Delta y)+o(\rho),$$

从而 $f(x,y)$ 在 M_0 处可微.

定理 8.4

$z=f(x,y)$ 在 $M_0(x_0,y_0)$ 处可微的充分必要条件是 $f'_x(x_0,y_0),f'_y(x_0,y_0)$ 存在且连续.

证明 已知 $f'_x(x,y),f'_y(x,y)$ 在 M_0 处存在且连续, 则

$$\begin{aligned}\Delta z&=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=[f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0+\Delta y)]+[f(x_0,y_0+\Delta y)-f(x_0\\&=f'_x(x_0+\theta_1\Delta x,y_0+\Delta y)\Delta x+f'_y(x_0,y_0+\theta_2\Delta y)\Delta y,\end{aligned}$$

其中 $\theta_1,\theta_2\in(0,1)$. 利用 $f'_x(x,y),f'_y(x,y)$ 的连续性, 得

$$\lim_{\Delta x\to0,\Delta y\to0}f'_x(x_0+\theta_1\Delta x,y_0+\Delta y)=f'_x(x_0,y_0)$$
$$\lim_{\Delta x\to0,\Delta y\to0}f'_y(x_0,y_0+\theta_2\Delta y)=f'_y(x_0,y_0),$$

从而

$$f'_x(x_0+\theta_1\Delta x,y_0+\Delta y)=f'_x(x_0,y_0)+\alpha_1,\quad \alpha_1\to0,\ (\Delta x\to0,\Delta y\to0),$$
$$f'_y(x_0,y_0+\theta_2\Delta y)=f'_y(x_0,y_0)+\alpha_2,\quad \alpha_2\to0,\ (\Delta x\to0,\Delta y\to0),$$

即

$$\Delta z=(f'_x(M_0)+\alpha_1)\Delta x+(f'_y(M_0)+\alpha_2)\Delta y=f'_x(M_0)\Delta x+f'_y(M_0)\Delta y+\alpha_1\Delta x+\alpha_2\Delta y,$$

且 $\lim\limits_{\rho\to 0}\dfrac{\alpha_1\Delta x+\alpha_2\Delta y}{\rho}=\lim\limits_{\rho\to 0}(\alpha_1\cos\theta+\alpha_2\sin\theta)=0$, 从而 $\alpha_1\Delta x+\alpha_2\Delta y=o(\rho)$, 所以

$$\Delta z=f'_x(M_0)\Delta x+f'_y(M_0)\Delta y+o(\rho)=(A\Delta x+B\Delta y)+o(\rho),$$

从而 $f(x,y)$ 在 M_0 处可微.

例 8.3 反例 3: $z=f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$ 在 $(0,0)$ 处可微, 但 $f'_x(x,y), f'_y(x,y)$ 在 $(0,0)$ 处不连续.

8.2 高阶偏导数

设 $z=f(x,y)=x^2+xy+y^2+x^y+3x+4y$, 则

$$\frac{\partial z}{\partial x}=2x+y+yx^{y-1}+3,$$
$$\frac{\partial z}{\partial y}=x+2y+x^y\ln x+4.$$

由此得

$$\frac{\partial^2 z}{\partial y\partial x}=\left(\frac{\partial z}{\partial x}\right)'_y=(2x+y+yx^{y-1}+3)'_y=1+x^{y-1}+yx^{y-1}\ln x,$$
$$\frac{\partial^2 z}{\partial x\partial y}=\left(\frac{\partial z}{\partial y}\right)'_x=(x+2y+x^y\ln x+4)'_x=1+x^{y-1}+yx^{y-1}\ln x.$$

进一步得

$$\frac{\partial^3 z}{\partial x\partial y\partial x}=\left(\frac{\partial^2 z}{\partial y\partial x}\right)'_x=\left(1+x^{y-1}+yx^{y-1}\ln x\right)'_x=(y-1)x^{y-2}+y(y-1)x^{y-2}\ln x+yx^{y-2},$$
$$\frac{\partial^3 z}{\partial x^2\partial y}=\left(\frac{\partial^2 z}{\partial x\partial y}\right)'_x=(1+x^y\ln x+y)'_x=(y-1)x^{y-2}+y(y-1)x^{y-2}\ln x+yx^{y-2}.$$

对比得知, $\dfrac{\partial^2 z}{\partial y\partial x},\dfrac{\partial^2 z}{\partial x\partial y},\dfrac{\partial^2 z}{\partial x\partial y\partial x},\dfrac{\partial^2 z}{\partial y\partial x\partial y}$ 在区域 $D:x>0$ 中连续, 且

$$\frac{\partial^2 z}{\partial y\partial x}\equiv\frac{\partial^2 z}{\partial x\partial y},$$
$$\frac{\partial^3 z}{\partial x\partial y\partial x}\equiv\frac{\partial^3 z}{\partial x^2\partial y}.$$

对于 $(x,y)\in D$ 成立.

定理 8.5

若 $z=f(x,y)$ 在区域 D 中的高阶偏导数连续, 则高阶偏导数与求偏导的顺序无关.

证明 仅证 $\dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{\partial^2 z}{\partial x\partial y}$.

任取 $M_0=(x_0,y_0)\in D, B(M_0,r)\subset D$, 取 $h=\Delta x\neq 0, k=\Delta y\neq 0$, 使得 $(x_0+h,y_0+k)\in$

$B(M_0, r)$, 令

$$\varphi(x) = f(x, y_0 + k) - f(x, y),$$
$$\psi(y) = f(x_0 + h, y) - f(x_0, y).$$

是 $f(x, y)$ 分别对于 x 和 y 的偏差分。容易验证，如果 $\varphi(x)$ 和 $\psi(y)$ 分别对 x 和 y 再进行差分，那么差分的结果是都等于 $f(x, y)$ 的二阶混合差分（下列第二个等式的右端）

$$\begin{aligned}\varphi(x_0 + h) - \varphi(x_0) = \psi(y_0 + k) - \psi(y_0) \\ = f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) - f(x_0, y_0 + k) + f(x_0, y_0).\end{aligned}$$

由一元函数的微分公式可得

$$\begin{aligned}\varphi(x_0 + h) - \varphi(x_0) &= h\varphi'(x_0 + \theta_1 h) \\ &= h\left(f'(x_0 + \theta_1 h, y_0 + k) - f'(x_0 + \theta_1 h, y_0)\right) \\ &= hkf''_{xy}(x_0 + \theta_1 h, y_0 + \eta_1 k),\end{aligned}$$

其中 $0 < \theta_1, \eta_1 < 1$。类比存在 $0 < \theta_2, \eta_2 < 1$，使得

$$\psi(y + k) - \psi(y_0) = hkf''_{yx}(x_0 + \theta_2 h, y_0 + \eta_2 k).$$

故有

$$f''_{xy}(x_0 + \theta_1 h, y_0 + \eta_1 k) = f''_{yx}(x_0 + \theta_2 h, y_0 + \eta_2 k).$$

令 $(h, k) \to (0, 0)$，由混合偏导数的连续性即可证明定理。

8.3 例题

例 8.4 证明函数 $u = \dfrac{1}{r}, r = \sqrt{x^2 + y^2 + z^2} > 0$ 满足 Laplace 方程

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \equiv equiv0, \forall (x, y, z) \neq (0, 0, 0).$$

证明 $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}} \Rightarrow \dfrac{\partial u}{\partial x} = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \Rightarrow \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{(x^2 + y^2 + z^2) - 3x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$. 由于 $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ 是关于 x, y, z 的对称函数, 因此有

$$\frac{\partial^2 u}{\partial y^2} = -\frac{(x^2 + y^2 + z^2) - 3y^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}, \frac{\partial^2 u}{\partial z^2} = -\frac{(x^2 + y^2 + z^2) - 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}.$$

$$\frac{\partial^2 u}{\partial z^2} = -\frac{(x^2 + y^2 + z^2) - 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}.$$

从而

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3(x^2 + y^2 + z^2) - 3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0.$$

例 8.5 证明 $u=\dfrac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}, x>0, t>0, a>0$ 常数满足热传导方程
$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}.$$

证明
$$\begin{aligned}\frac{\partial u}{\partial t}&=\frac{(t^{-\frac{1}{2}})_t'}{2a\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{4a^2t}}+\frac{1}{2a\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(-\frac{x^2}{4a^2t}\right)_t'\\&=\frac{1}{2a\sqrt{\pi t}t}\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(-1+\frac{x^2}{2a^2t}\right).\end{aligned}$$

且有
$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(-\frac{x}{2a^2t}\right),\\\frac{\partial^2 u}{\partial x^2}&=\frac{1}{4a\sqrt{\pi t}t}\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(\frac{x^2}{2a^4t}-\frac{1}{a^2}\right).\end{aligned}$$

从而
$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\quad \forall t>0, x\in\mathbb{R}^+.$$

例 8.6 $\forall\phi,\psi\in C^2(I)$, $u=\phi(x-at)+\psi(x+at)$ 满足波动方程
$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$
其中 $a>0$ 为常数.

证明 令 $\begin{cases}v=x-at,\\w=x+at,\end{cases}$, 则 $u=\phi(v)+\psi(w)$, 且
$$\begin{aligned}\frac{\partial u}{\partial x}&=\phi'(v)\frac{\partial v}{\partial x}+\psi'(w)\frac{\partial w}{\partial x}=\phi'(v)+\psi'(w),\\\frac{\partial u}{\partial t}&=\phi'(v)\frac{\partial v}{\partial t}+\psi'(w)\frac{\partial w}{\partial t}=-a\phi'(v)+a\psi'(w).\end{aligned}$$

从而
$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\phi''(v)\frac{\partial v}{\partial x}+\psi''(w)\frac{\partial w}{\partial x}=\phi''(v)+\psi''(w),\\\frac{\partial^2 u}{\partial t^2}&=\phi''(v)\frac{\partial v}{\partial t}+\psi''(w)\frac{\partial w}{\partial t}=a^2\phi''(v)+a^2\psi''(w).\end{aligned}$$

因此有
$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2},\quad\forall t>0, x\in\mathbb{R}.$$

✍ **作业** ex9.2:2(7),8,11,15,26,27,28.