


## Week 4

### 4.1 Mar 17 ex9.2:31,ex9.3:6,7,8,10,11(1),14.

 **习题 9.2.31** 试证: 方程  $\frac{\partial^2 u}{\partial x^2} + 2 \cos x \frac{\partial u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} \sin^2 x - \frac{\partial u}{\partial y} \sin x = 0$  经变换  $\begin{cases} \xi = x - \sin x + y, \\ \eta = x + \sin x - y \end{cases}$

后可化为  $\frac{\partial u}{\partial \xi \partial \eta} = 0$ . (其中二阶偏导数均连续)

**解** 由链式法则得


$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = (1 - \cos x) \frac{\partial u}{\partial \xi} + (1 + \cos x) \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\end{aligned}$$

因此

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left( (1 - \cos x) \frac{\partial u}{\partial \xi} + (1 + \cos x) \frac{\partial u}{\partial \eta} \right) \\ &= \left( (1 - \cos x) \frac{\partial}{\partial \xi} + (1 + \cos x) \frac{\partial}{\partial \eta} \right) \left( (1 - \cos x) \frac{\partial u}{\partial \xi} + (1 + \cos x) \frac{\partial u}{\partial \eta} \right) \\ &= (1 - \cos x)^2 \frac{\partial^2 u}{\partial \xi^2} + (1 + \cos x)^2 \frac{\partial^2 u}{\partial \eta^2} + 2(1 - \cos x)(1 + \cos x) \frac{\partial u}{\partial \xi \partial \eta} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) = \frac{\partial}{\partial \xi} \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) - \frac{\partial}{\partial \eta} \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) \\ &= \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} - 2 \frac{\partial u}{\partial \xi \partial \eta} \\ \frac{\partial u}{\partial x \partial y} &= \frac{\partial}{\partial y} \left( (1 - \cos x) \frac{\partial u}{\partial \xi} + (1 + \cos x) \frac{\partial u}{\partial \eta} \right) \\ &= \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) \left( (1 - \cos x) \frac{\partial u}{\partial \xi} + (1 + \cos x) \frac{\partial u}{\partial \eta} \right) \\ &= (1 - \cos x) \frac{\partial^2 u}{\partial \xi^2} - (1 + \cos x) \frac{\partial^2 u}{\partial \eta^2} + 2 \cos x \frac{\partial u}{\partial \xi \partial \eta}\end{aligned}$$

代入得

$$\begin{aligned}
 0 &= \frac{\partial^2 u}{\partial x^2} + 2 \cos x \frac{\partial u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} \sin^2 x - \frac{\partial u}{\partial y} \sin x \\
 &= ((1 - \cos x)^2 + (1 - \cos x)2 \cos x - \sin^2 x) \frac{\partial^2 u}{\partial \xi^2} \\
 &\quad + ((1 - \cos x)2(1 - \cos^2 x) + 2 \cos x 2 \cos x - \sin^2 x(-2)) \frac{\partial u}{\partial \xi \partial \eta} \\
 &\quad + ((1 - \cos x)(1 + \cos x)^2 + 2 \cos x(-(1 + \cos x)) - \sin^2 x(1 + \cos x)) \frac{\partial^2 u}{\partial \eta^2} \\
 &= 2 \frac{\partial u}{\partial \xi \partial \eta}
 \end{aligned}$$

 **习题 9.3.6** 设  $z = z(x, y)$  是由方程  $2 \sin(x + 2y - 3z) = x + 2y - 3z$  确定的隐函数, 求

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1.$$

**解** 对方程两边求微分, 得

$$2 \cos(x + 2y - 3z) (dx + 2 dy - 3 dz) = dx + 2 dy - 3 dz$$

整理得


$$(3 - 6 \cos(x + 2y - 3z)) dz = (1 - 2 \cos(x + 2y - 3z)) dx + 2(1 - 2 \cos(x + 2y - 3z)) dy$$

由此得

$$\begin{aligned}
 \frac{\partial z}{\partial x} &= \frac{1 - 2 \cos(x + 2y - 3z)}{3 - 6 \cos(x + 2y - 3z)}, \\
 \frac{\partial z}{\partial y} &= \frac{2(1 - 2 \cos(x + 2y - 3z))}{3 - 6 \cos(x + 2y - 3z)}
 \end{aligned}$$

于是

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{1 - 2 \cos(x + 2y - 3z) + 2(1 - 2 \cos(x + 2y - 3z))}{3 - 6 \cos(x + 2y - 3z)} = 1.$$

 **习题 9.3.7** 设  $z = z(x, y)$  是由方程  $\varphi(cx - az, cy - bz) = 0$  确定的隐函数, 其中  $\varphi$  可微, 证明

$$a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = c.$$

**解** 对方程两边求微分, 得

$$\frac{\partial \varphi}{\partial x} (c dx - a dz) + \frac{\partial \varphi}{\partial y} (c dy - b dz) = 0$$

整理得

$$(c \frac{\partial \varphi}{\partial x} - a \frac{\partial \varphi}{\partial y}) dz = c \frac{\partial \varphi}{\partial x} dx + c \frac{\partial \varphi}{\partial y} dy$$

由此得

$$\begin{aligned}
 \frac{\partial z}{\partial x} &= \frac{c \frac{\partial \varphi}{\partial x}}{c \frac{\partial \varphi}{\partial x} - a \frac{\partial \varphi}{\partial y}}, \\
 \frac{\partial z}{\partial y} &= \frac{c \frac{\partial \varphi}{\partial y}}{c \frac{\partial \varphi}{\partial x} - a \frac{\partial \varphi}{\partial y}}
 \end{aligned}$$

于是

$$a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = \frac{ac \frac{\partial \varphi}{\partial x} + bc \frac{\partial \varphi}{\partial y}}{c \frac{\partial \varphi}{\partial x} - a \frac{\partial \varphi}{\partial y}} = c.$$

习题 9.3.8 设  $z = x^2 + y^2$  其中  $y = y(x)$  为由方程  $x^2 - xy + y^2 = 1$  所定义的函数, 求  $\frac{dz}{dx}, \frac{d^2 z}{dx^2}$ .

解 对方程两边求微分, 得

$$2x dx + 2y dy = 0$$

整理得

$$\frac{dy}{dx} = -\frac{2x - y}{x - 2y}$$

由此得

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{(x - 2y)(2 - \frac{dy}{dx}) - (2x - y)(1 - 2\frac{dy}{dx})}{(x - 2y)^2} \\ &= -\frac{3y}{(x - 2y)^2} + \frac{6x^2 - 3xy}{(x - 2y)^3} = 6 \frac{(x - y)^2}{(x - 2y)^3} \end{aligned}$$

故

$$\frac{dz}{dx} = 2x + 2y \frac{dy}{dx} = 2x + 2y \frac{2x - y}{x - 2y} = \frac{2x^2 - 2y^2}{x - 2y}$$

$$\begin{aligned} \frac{d^2 z}{dx^2} &= \frac{d}{dx} \left( 2x + 2y \frac{dy}{dx} \right) \\ &= 2 + 2 \left( \frac{dy}{dx} \right)^2 + 2y \frac{d^2 y}{dx^2} \\ &= \frac{10x^3 - 24x^2 y + 18xy^2 - 8y^3}{(x - 2y)^3} \end{aligned}$$

习题 9.3.10 设  $x = x(z), y = y(z)$  是由方程组  $\begin{cases} x + y + z = 0, \\ x^2 + y^2 + z^2 = 1 \end{cases}$  所确定的隐函数组, 求

$$\frac{dx}{dz}, \frac{dy}{dz}.$$

解 对方程组两边求微分得到

$$dx + dy + dz = 0,$$

$$2x dx + 2y dy + 2z dz = 0$$

将其视为关于  $dx, dy$  的方程组, 解得

$$\begin{aligned} dx &= -\frac{y - z}{y - x} dz, \\ dy &= -\frac{x - z}{y - x} dz \end{aligned}$$

习题 9.3.11(1) 设  $u = u(x, y), v = v(x, y)$  是由下列方程组所确定的隐函数组, 求  $\frac{\partial(u, v)}{\partial(x, y)}$ .

$$(1) \begin{cases} u^2 + v^2 + x^2 + y^2 = 1, \\ u + v + x + y = 0. \end{cases}$$

解 对方程组两边求微分得到

$$2u \, du + 2v \, dv + 2x \, dx + 2y \, dy = 0,$$

$$du + dv + dx + dy = 0$$

将其视为关于  $du, dv$  的方程组, 解得

$$du = \frac{x-v}{u-v} dx + \frac{y-v}{u-v} dy,$$

$$dv = \frac{x-u}{v-u} dx + \frac{y-u}{v-u} dy$$

于是

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{x-v}{u-v} & \frac{y-v}{u-v} \\ \frac{x-u}{v-u} & \frac{y-u}{v-u} \end{vmatrix} = \frac{x-y}{u-v}$$

习题 9.3.14 设  $y = y(x), z = z(x)$  是由方程  $z = xf(x+y)$  和  $F(x, y, z) = 0$  所确定的函数, 其中  $f$  和  $F$  分别具有一阶连续导数和一阶连续偏导数, 求  $\frac{dz}{dx}$ .

解 对方程组两边求微分得到

$$dz = f(x+y) dx + xf'(x+y) dy(dx + dy),$$

$$0 = F'_x dx + F'_y dy + F'_z dz$$

将其视为关于  $dz, dy$  的方程组, 解得

$$dz = \frac{F'_z f + xF'_x f' - xF'_x f'}{F'_z(1 + xf')} dx$$

于是

$$\frac{dz}{dx} = \frac{F'_z f + xF'_x f' - xF'_x f'}{F'_z(1 + xf')}$$

## 4.2 Mar 19 ex9.2:21,22,23,24,36(2)(5),38

习题 9.2.21 求函数  $u = xyz$  在点  $(1, 2, -1)$  沿方向  $\mathbf{l} = (3, -1, 1)$  的方向微商.

解

$$\nabla u = \left( \frac{\partial}{\partial x} u, \frac{\partial}{\partial y} u, \frac{\partial}{\partial z} u \right) = (yz, xz, xy)$$

于是所求方向微商为

$$\nabla u \cdot \frac{\mathbf{l}}{|\mathbf{l}|} = \left( \frac{\partial}{\partial x} u, \frac{\partial}{\partial y} u, \frac{\partial}{\partial z} u \right) \cdot \frac{\mathbf{l}}{|\mathbf{l}|} = (3, -1, 1) \cdot \frac{(3, -1, 1)}{\sqrt{11}} = \frac{3}{\sqrt{11}}$$

习题 9.2.22 试求函数  $z = \arctan \frac{y}{x}$  在圆  $x^2 + y^2 - 2x = 0$  上一点  $P(\frac{1}{2}, \frac{\sqrt{3}}{2})$  处沿该圆周逆时针方向上的方向微商.

**解** 将圆参数化为  $\begin{cases} x = 1 + \cos \theta, \\ y = \sin \theta \end{cases}$ , 则  $\theta = \frac{2\pi}{3}$  时位于  $P$  点. 且方向向量为  $\boldsymbol{l} = (x'(\theta), y'(\theta)) =$

$(-\sin \theta, \cos \theta) = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ , 于是所求方向微商为

$$\nabla z \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \cdot \boldsymbol{l} = \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \bigg|_{(x,y)=(\frac{1}{2}, \frac{\sqrt{3}}{2})} \cdot \left( -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right) = \frac{1}{2}$$

**习题 9.2.23** 求函数  $u = x^2 + 2y^2 + 3z^2 + xy + 3x - 2y - 6z$  在点  $(1, 1, -1)$  的梯度和最大方向微商.

**解**

$$\nabla u = (2x + y + 3, 4y + x - 2, 6z - 6)$$

在点  $(1, 1, -1)$  处

$$\nabla u = (6, 3, -12)$$

最大方向微商为沿着  $\nabla u$  的方向, 即

$$\nabla u \cdot \frac{\nabla u}{|\nabla u|} = (6, 3, -12) \cdot \frac{(6, 3, -12)}{\sqrt{6^2 + 3^2 + 12^2}} = 3\sqrt{21}$$

**习题 9.2.24** 设  $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ ,  $r = |\boldsymbol{r}|$ , 试求 (1)  $\text{grad } \frac{1}{r^2}$ ; (2)  $\text{grad } \ln r$ .

**解**

$$\text{grad } \frac{1}{r^2} = \text{grad } \frac{1}{x^2 + y^2 + z^2} = -\frac{2x}{(x^2 + y^2 + z^2)^2} \boldsymbol{i} - \frac{2y}{(x^2 + y^2 + z^2)^2} \boldsymbol{j} - \frac{2z}{(x^2 + y^2 + z^2)^2} \boldsymbol{k} = -\frac{2\boldsymbol{r}}{r^4};$$

$$\text{grad } \ln r = \text{grad } \ln \sqrt{x^2 + y^2 + z^2} = \frac{x}{x^2 + y^2 + z^2} \boldsymbol{i} + \frac{y}{x^2 + y^2 + z^2} \boldsymbol{j} + \frac{z}{x^2 + y^2 + z^2} \boldsymbol{k} = \frac{\boldsymbol{r}}{r^2}.$$

**习题 9.2.36(2)(5)** 求下列复合函数的微分  $du$

$$(2) \quad u = f(\xi, \eta), \xi = xy, \eta = \frac{x}{y};$$

$$(5) \quad u = f(\xi, \eta, \zeta), \xi = x^2 + y^2, \eta = x^2 - y^2, \zeta = 2xy.$$

**解**

(2)

$$\begin{aligned} du &= f'_1 d\xi + f'_2 d\eta \\ &= f'_1(y dx + x dy) + f'_2 \left( \frac{1}{y} dx - \frac{x}{y^2} dy \right) \\ &= (f'_1 y + f'_2 \frac{1}{y}) dx + (f'_1 x - f'_2 \frac{x}{y^2}) dy \end{aligned}$$

(5)

$$\begin{aligned} du &= f'_1 d\xi + f'_2 d\eta + f'_3 d\zeta \\ &= f'_1(2x dx + 2y dy) + f'_2(2x dx - 2y dy) + f'_3(2y dx + 2x dy) \\ &= (2x f'_1 + 2x f'_2 + 2y f'_3) dx + (2y f'_1 - 2y f'_2 + 2x f'_3) dy \end{aligned}$$

习题 9.2.38 求直角坐标和极坐标的坐标变换  $x = x(r, \theta) = r \cos \theta, y = y(r, \theta) = r \sin \theta$  的 Jacobi 行列式.

解 坐标变换为

$$x = r \cos \theta, \quad y = r \sin \theta$$

偏导数为

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos \theta, & \frac{\partial x}{\partial \theta} &= -r \\ \frac{\partial y}{\partial r} &= \sin \theta, & \frac{\partial y}{\partial \theta} &= r \end{aligned}$$

Jacobi 行列式为

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.$$

### 4.3 Mar 21 ex9.4:3,4,8(1)(4),9,11,16(1),17(2)

习题 9.4.3 证明曲线  $x = a \cos t, y = a \sin t, z = bt$  的切线与  $Oz$  轴成定角.

解 由  $\mathbf{r}(t) = (a \cos t, a \sin t, bt)$  可知

$$\mathbf{r}'(t) = (-a \sin t, a \cos t, b)$$

切线方向向量为  $\mathbf{r}'(t)$ , 与  $Oz$  轴的夹角  $\theta$  有

$$\cos \theta = \frac{\mathbf{r}'(t) \cdot \mathbf{k}}{|\mathbf{r}'(t)|} = \frac{b}{\sqrt{a^2 + b^2}}$$

为常数, 所以切线与  $Oz$  轴成定角.

习题 9.4.4 设  $\mathbf{r} = \left( \frac{t}{1+t}, \frac{1+t}{t}, t^2 \right) (t > 0)$ , 判断它是不是简单曲线, 是不是光滑曲线, 并求出它在  $t = 1$  时的切线方程和法平面方程.

解 简单曲线: 无自交点, 即  $\mathbf{r}(t_1) = \mathbf{r}(t_2) \Rightarrow t_1 = t_2$ .

设  $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ , 则  $t_1^2 = t_2^2 \Rightarrow t_1 = \pm t_2$ , 但  $t > 0$ , 所以  $t_1 = t_2$ , 故是简单曲线.

光滑曲线:  $\mathbf{r}$  满足  $\mathbf{r}^{(n)}(t) = \left( \frac{(-1)^{n+1}n!}{(t+1)^{n+1}}, -\frac{(-1)^{n+1}n!}{t^{n+1}}, 0 \right), n \geq 3$ . 即  $\mathbf{r}$  的各阶导数都存在且连续, 且  $\mathbf{r}'(t) \neq \mathbf{0}$ , 所以是光滑曲线.

在  $t = 1$  时, 切线方程为

$$\mathbf{r}'(1) = \left( \frac{1}{4}, -1, 2 \right)$$

所以切线方程为

$$\frac{x - \frac{1}{2}}{\frac{1}{4}} = \frac{y - 2}{-1} = \frac{z - 1}{2}$$

设法平面方程为  $\frac{1}{4}x - y + 2z = d$ , 代入  $\mathbf{r}(1) = \left( \frac{1}{2}, 2, 1 \right)$  得  $d = \frac{1}{8}$ , 所以法平面方程为

$$\frac{x}{4} - y + 2z = \frac{1}{8}$$

习题 9.4.8(1)(4) 求下列曲面在指定点的切平面和法线方程.

(1)  $z = \sqrt{x^2 + y^2} - xy$ , 在点  $(3, 4, -7)$ ;

(4)  $4 + \sqrt{x^2 + y^2 + z^2} = x + y + z$ , 在点  $(2, 3, 6)$ .

解

1. 曲面可参数化为

$$r(x, y) = \left( x, y, \sqrt{x^2 + y^2} - xy \right)$$

于是

$$r_x = \left( 1, 0, \frac{x}{\sqrt{x^2 + y^2}} - y \right) \Rightarrow r_x(3, 4) = \left( 1, 0, -\frac{17}{5} \right)$$

$$r_y = \left( 0, 1, \frac{y}{\sqrt{x^2 + y^2}} - x \right) \Rightarrow r_y(3, 4) = \left( 0, 1, -\frac{11}{5} \right)$$

因此  $(3, 4, -7)$  处法向量为

$$\mathbf{n} = \left( 1, 0, -\frac{17}{5} \right) \times \left( 0, 1, -\frac{11}{5} \right) = \left( \frac{17}{5}, \frac{11}{5}, 1 \right)$$

法线为

$$\frac{x-3}{17} = \frac{y-4}{11} = \frac{z+7}{5}$$

设切平面为  $17x + 11y + 5z + d = 0$ , 代入  $(3, 4, -7)$  得到  $d = -60$ 。于是切平面方程为

$$17x + 11y + 5z - 60 = 0$$

2. 对于隐式曲面

$$F(x, y, z) = \sqrt{x^2 + y^2 + z^2} - (x + y + z) + 4 = 0$$

于是

$$F_x = \frac{x}{\sqrt{x^2 + y^2 + z^2}} - 1 \Rightarrow F_x(2, 3, 6) = \frac{-5}{7}$$

$$F_y = \frac{y}{\sqrt{x^2 + y^2 + z^2}} - 1 \Rightarrow F_y(2, 3, 6) = \frac{-4}{7} \quad F_z = \frac{z}{\sqrt{x^2 + y^2 + z^2}} - 1 \Rightarrow F_z(2, 3, 6) = \frac{-1}{7}$$

因此  $(2, 1, 0)$  处法向量为

$$\mathbf{n} = (5, 4, 1)$$

法线为

$$\frac{x-2}{5} = \frac{y-3}{4} = z-6$$

设切平面为  $5x + 4y + z + d = 0$ , 代入  $(2, 3, 6)$  得到  $d = -28$ 。于是切平面方程为

$$5x + 4y + z - 28 = 0$$

习题 9.4.9 求椭球面  $x^2 + 2y^2 + z^2 = 1$  上平行于平面  $x - y + 2z = 0$  的切平面方程.

解 切点处的法向量也是平面  $x - y + 2z = 0$  的法向量  $\mathbf{n} = (1, -1, 2)$ , 于是切点处椭球面的一个法向量为  $\mathbf{n} = (1, -1, 2)$ .

设切点为  $(x_0, y_0, z_0)$ , 则切平面方程为

$$x_0(x - x_0) + 2y_0(y - y_0) + z_0(z - z_0) = 0$$

注意到  $(x_0, y_0, z_0) = \mathbf{n} = (1, -1, 2)$ , 所以设切点为  $(t, -t, 2t)$ , 代入椭圆方程得

$$t^2 + 2t^2 + 4t^2 = 1 \Rightarrow t = \pm \frac{1}{\sqrt{6}}$$

故切平面方程为


$$(x - \frac{1}{\sqrt{6}}) - (y + \frac{1}{\sqrt{6}}) + 2(z - \frac{2}{\sqrt{6}}) = 0$$

和

$$(x + \frac{1}{\sqrt{6}}) - (y - \frac{1}{\sqrt{6}}) + 2(z + \frac{2}{\sqrt{6}}) = 0$$

即

$$x - y + 2z = \pm \frac{4}{\sqrt{6}}$$

 **习题 9.4.11** 求椭球面  $x^2 + 2y^2 + 3z^2 = 21$  在某点  $M$  处的切平面  $\pi$  的方程, 使  $\pi$  过已知直线  $L: \frac{x-6}{2} = \frac{y-3}{1} = \frac{2z-1}{-2}$ .

**解** 椭球面可以写为隐式曲面  $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 21 = 0$ , 由此得

$$F_x = 2x, \quad F_y = 4y, \quad F_z = 6z$$

故  $(x_0, y_0, z_0)$  处的切平面方程为

$$x_0(x - x_0) + 2y_0(y - y_0) + 3z_0(z - z_0) = 0$$

展开得

$$x_0x + 2y_0y + 3z_0z = x_0^2 + 2y_0^2 + 3z_0^2 = 21$$

任取直线  $L$  上两点  $(6, 3, \frac{1}{2})$  和  $(0, 0, \frac{7}{2})$ , 代入切平面方程, 得到  $z_0 = \frac{7}{2}$ , 进而切平面方程为

$$6x_0 + 6y_0 = 21 - \frac{3}{2}z_0 = \frac{63}{4}$$

再结合


$$x_0^2 + 2y_0^2 + z_0^2 = 21$$

解得

$$(x_0, y_0, z_0) = (3, 0, 2) \quad \text{或} \quad (x_0, y_0, z_0) = (1, 2, 2)$$

进而切平面方程为

$$x + 2z = 7 \quad \text{或} \quad x + 4y + 6z = 21$$

 **习题 9.4.16(1)** 求下列曲线在给定点的切线和法平面方程

(1)  $x^3y + xy^3 = 3 - x^2y^2$  在点  $(1, 1)$ .

**解**

(1) 对于隐式曲线

$$F(x, y) = x^3y + xy^3 + x^2 + y^2 - 3 = 0$$



有


$$F_x = 3x^2y + y^3 + 2xy^2 \Rightarrow F_x(1, 1) = 6$$

$$F_y = x^3 + 3xy^2 + 2x^2y \Rightarrow F_y(1, 1) = 6$$

故  $(1, 1)$  处法向量为  $n = (1, 1)$ , 进一步切向量为  $t = (1, -1)$ 。

进而切线和法线依次为

$$y = -x + 2 \quad \text{和} \quad y = x$$

 **习题 9.4.17(2)** 求下列曲线在给定点的切线和法平面方程

$$(2) \quad \begin{cases} 2x^2 + 3y^2 + z^2 = 47, \\ x^2 + 2y^2 = z \end{cases} \quad \text{在点 } (-2, 1, 6).$$

解

(2) 考虑隐式曲面

$$F(x, y, z) = 2x^2 + 3y^2 + z^2 - 47 = 0, \quad G(x, y, z) = x^2 + 2y^2 - z = 0$$

不难得到它们在  $(-2, 1, 6)$  处的法向量分别为

$$\mathbf{n}_1 = (-4, 3, 6), \quad \mathbf{n}_2 = (-4, 4, -1)$$

于是曲线的切向量为

$$\mathbf{t} = \mathbf{n}_1 \times \mathbf{n}_2 = (-27, -28, -4)$$

因此  $(-2, 1, 6)$  处切线为

$$\frac{x + 2}{-27} = \frac{y - 1}{-28} = \frac{z - 6}{-4}$$

设切平面为  $27x + 28y + 4z + d = 0$ , 代入  $(-2, 1, 6)$  得到  $d = 2$ 。于是切平面方程为

$$27x + 28y + 4z + 2 = 0$$