Lec 9 复合 (隐) 函数微分法

9.1 复合函数 (composition) 微分法

定理 9.1

设 $z = f(u,v)$ 在区域 D 中可微, 且 $\begin{cases} u = g(x,y) \\ v = h(x,y) \end{cases}$ 都在区域 E 中可微, 当复合 $f(g(x,y),h(x,y))$ 有意义时,z 通过中间变量 u,v 成为 x,y 的多元复合函数, 且有求偏导数的链式法则如下:

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}, \\ \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}; \end{cases} \tag{9.1}$$

同时,z 作为 x,y 的多元复合函数可微, 且不论 u,v 是作为 $f(u,v)$ 的自变量, 还是作为复合函数 $f(g(x,y),h(x,y))$ 的中间变量, 总有:

$$\mathrm{d}z = \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v. \tag{9.2}$$

9.2称为全微分的一阶形式不变性.

♡

证明

9.1 固定 y, 令 x 有增量 Δx, 则

$$\begin{cases} \Delta u_x = g(x+\Delta x,y) - g(x,y), \\ \Delta v_x = h(x+\Delta x,y) - h(x,y), \\ \Delta z_x = f(u+\Delta u_x, v+\Delta v_x) - f(u,v) = \dfrac{\partial z}{\partial u}\Delta u_x + \dfrac{\partial z}{\partial v}\Delta v_x + o(\rho); \end{cases}$$

其中 $\rho = \sqrt{(\Delta u_x)^2 + (\Delta v_x)^2}$, 并有 $\Delta x \to 0 \Rightarrow \begin{cases} \Delta u_x \to 0, \\ \Delta v_x \to 0; \end{cases} \Rightarrow \rho \to 0.$

利用

$$\begin{aligned} \lim_{\Delta x\to 0}\frac{o(\rho)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{o(\rho)}{\rho}\frac{\rho}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{o(\rho)}{\rho}\lim_{\Delta x\to 0}\frac{\rho}{\Delta x} \\ &= \lim_{\rho\to 0}\frac{o(\rho)}{\rho}\lim_{\Delta x\to 0}\sqrt{\left(\frac{\Delta u_x}{\Delta x}\right)^2 + \left(\frac{\Delta v_x}{\Delta x}\right)^2} \\ &= 0\cdot\sqrt{\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial v_x}{\partial x}\right)^2} \\ &= 0 \end{aligned}$$

以及

$$\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\frac{\partial u}{\partial x},\ \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}=\frac{\partial v}{\partial x};$$

因此有

$$\begin{aligned}\frac{\partial z}{\partial x}&=\lim_{\Delta x\to 0}\frac{\Delta z_x}{\Delta x}\\&=\lim_{\Delta x\to 0}\left(\frac{\partial z}{\partial u}\frac{\Delta u_x}{\Delta x}+\frac{\partial z}{\partial v}\frac{\Delta v_x}{\Delta x}+\frac{o(\rho)}{\Delta x}\right)\\&=\frac{\partial z}{\partial u}\lim_{\Delta x\to 0}\frac{\Delta u_x}{\Delta x}+\frac{\partial z}{\partial v}\lim_{\Delta x\to 0}\frac{\Delta v_x}{\Delta x}+\lim_{\Delta x\to 0}\frac{o(\rho)}{\Delta x}\\&=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\end{aligned}$$

同理, 对 y 有

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

可微性 记

$$\begin{cases}\Delta u=g(x+\Delta x,y+\Delta y)-g(x,y),\\ \Delta v=h(x+\Delta x,y+\Delta y)-h(x,y),\\ \Delta z=f(u+\Delta u,v+\Delta v)-f(u,v),\\ r=\sqrt{(\Delta x)^2+(\Delta y)^2},\\ \rho=\sqrt{(\Delta u)^2+(\Delta v)^2};\end{cases}$$

因此我们有

$$\begin{aligned}\Delta z&=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho)\\&=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+o(r)\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y+o(r)\right)+o(\rho)\\&=\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\Delta y+o(r)+o(\rho)\end{aligned}$$

同时, 当 $r\to 0$, 有 $\rho\to 0$ 与 $\dfrac{o(r)}{r}$ 有界, 因此

$$\begin{aligned}\frac{\rho}{r}&=\frac{\sqrt{(\Delta u)^2+(\Delta v)^2}}{r}\\&=\sqrt{\left(\frac{\partial u}{\partial x}\frac{\Delta x}{r}+\frac{\partial u}{\partial y}\frac{\Delta y}{r}+\frac{o(r)}{r}\right)^2+\left(\frac{\partial v}{\partial x}\frac{\Delta x}{r}+\frac{\partial v}{\partial y}\frac{\Delta y}{r}+\frac{o(r)}{r}\right)^2}\\&\leqslant\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}+M_0\triangleq M,r\to 0\end{aligned}$$

因此

$$\begin{aligned}\lim_{r\to 0}\left|\frac{o(\rho)}{r}\right| &= \lim_{r\to 0}\left|\frac{o(\rho)}{\rho}\right|\left|\frac{\rho}{r}\right| \\ &\leqslant M\lim_{r\to 0}\left|\frac{o(\rho)}{\rho}\right| \\ &= 0\end{aligned}$$

故

$$\Delta z = \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\Delta y + o(r)$$

表明 z 作为 x, y 的多元复合函数可微.

9.2 (a). 当 u, v 作为 $f(u, v)$ 的自变量时,$z = f(u, v)$ 可微, 自然有

$$\mathrm{d}z = \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v.$$

(b). 当 u, v 作为复合函数 $f(g(x, y), h(x, y))$ 的中间变量时,

$$\begin{aligned}\mathrm{d}z &= \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y \\ &= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\mathrm{d}y \\ &= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right) \\ &= \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v\end{aligned}$$

9.2 隐函数 (implicit function) 微分法

例 9.1 方程

$$3x + 4y - 5z + 7 = 0$$

可确定

$$\begin{cases} \quad z = \frac{3}{5}x + \frac{4}{5}y + \frac{7}{5}, \\ \text{or} \quad y = -\frac{3}{4}x + \frac{5}{4}z - \frac{7}{4}, \\ \text{or} \quad x = -\frac{4}{3}y + \frac{5}{3}z - \frac{7}{3}; \end{cases}$$

三个函数, 分别可得

$$\begin{cases} \frac{\partial z}{\partial x} = \frac{3}{5}, \\ \frac{\partial z}{\partial y} = \frac{4}{5}; \end{cases} \quad \begin{cases} \frac{\partial y}{\partial z} = \frac{5}{4}, \\ \frac{\partial y}{\partial x} = -\frac{3}{4}; \end{cases} \quad \begin{cases} \frac{\partial x}{\partial y} = -\frac{4}{3}, \\ \frac{\partial x}{\partial z} = \frac{5}{3}; \end{cases}$$

可得

$$\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{3}{5}\times\left(-\frac{4}{3}\right)\times\frac{5}{4} = -1, \frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial x} = \frac{5}{3}\times\frac{4}{5}\times\left(-\frac{3}{4}\right) = -1.$$

上述的三个二元函数, 都是方程 $F(x, y, z) = 3x + 4y - 5z + 7 = 0$ 所确定的隐函数.

定理 9.2

设方程 $F(x,y)=0$ 满足:

1. $F(x,y)\in C^1(D)$,D 为区域,
2. $F(M_0)=F(x_0,y_0)=0, M_0\in D$,
3. $F'_y(M_0)=F'_y(x_0,y_0)\neq 0$.

则方程 $F(x,y)=0$ 可在点 M_0 的某个 δ 邻域 $\bar{U}(M_0,\delta)$ 中确定唯一隐函数:$y=\varphi(x)$ 满足

$$\begin{cases}\varphi(x_0)=y_0,\\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=\varphi'(x)=-\dfrac{F'_x(x,y)}{F'_y(x,y)}\in C\end{cases}$$

证明 不妨设 $F'_y(x_0,y_0)>0$, 则 $F(x_0,y)$ 在 y_0 附近严格单调递增, 即在 $M(x_0,y_0)$ 附近形成了一条唯一存在的严格单调递增平面曲线, 设此曲线的表达式为 $y=\varphi(x),(x,y)\in\bar{U}(M_0,\delta)$, 则 $y=\varphi(x)$ 即为所求的隐函数.

显然 $y=\varphi(x)$ 穿过点 $M_0(x_0,y_0)$, 即 $\varphi(x_0)=y_0$, 且从 $F(x,\varphi(x))\equiv 0$, 两边对 x 求导, 有:$F'_x\cdot 1+F'_y\cdot\dfrac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\equiv 0\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\varphi'(x)=-\dfrac{F'_x(x,y)}{F'_y(x,y)}$

从 $F\in C^1(D)$ 知,$\varphi'(x)$ 是连续函数.

定理 9.3

设方程 $F(x,y,z)=0$ 满足:

1. $F(x,y,z)\in C^1(D)$,D 为区域,
2. $F(M_0)=F(x_0,y_0,z_0)=0, M_0\in D$,
3. $F'_z(M_0)=F'_y(x_0,y_0,z_0)\neq 0$.

则方程 $F(x,y,z)=0$ 可在点 M_0 的某个 δ 邻域 $\bar{U}(M_0,\delta)$ 中确定唯一隐函数:$z=\varphi(x,y)$ 满足

$$\begin{cases}\varphi(x_0,y_0)=z_0,\\ \dfrac{\partial z}{\partial x}=-\dfrac{F'_x(x,y,z)}{F'_z(x,y,z)},\dfrac{\partial z}{\partial y}=-\dfrac{F'_y(x,y,z)}{F'_z(x,y,z)}.\end{cases}$$

注 值得注意的是, 上述隐函数 $y=\varphi(x)$ 或者 $z=\varphi(x,y)$ 只理论上存在, 实际问题中未必能求出来, 但隐函数的导数或偏导数是能够从已知方程 $F(x,y)=0$ 或 $F(x,y,z)=0$ 中求出来的.

例如, 已知 $z=\varphi(x,y)$ 是方程 $F(x,y,z)=0$ 确定的隐函数, 则由 $F(x,y,\varphi(x,y))\equiv 0$, 两边对 x,y 分别求导, 有

$$\begin{cases}F'_x\cdot 1+F'_z\cdot\varphi'_x(x,y)=0\\ F'_y\cdot 1+F'_z\cdot\varphi'_y(x,y)=0\end{cases}\Rightarrow\begin{cases}\varphi'_x(x,y)=\dfrac{\partial z}{\partial x}=-\dfrac{F'_x(x,y,z)}{F'_z(x,y,z)}\\ \varphi'_y(x,y)=\dfrac{\partial z}{\partial y}=-\dfrac{F'_y(x,y,z)}{F'_z(x,y,z)}.\end{cases}$$

9.3 例题

例 9.2 证明:

$$u=\frac{1}{r}, r=\sqrt{x^2+y^2+z^2}$$

满足 Laplace 方程:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\equiv 0, \forall(x,y,z)\neq(0,0,0).$$

证明

$$\text{由于 } \frac{\partial u}{\partial x}=\frac{\mathrm{d}u}{\mathrm{d}r}\frac{\partial r}{\partial x}=-\frac{1}{r^2}\frac{x}{r}=-\frac{x}{r^3},$$

$$\text{因此 } \frac{\partial^2 u}{\partial x^2}=-\left(\frac{x}{r^3}\right)'_x=-\frac{r^3-3r^2\frac{x}{r}x}{r^6}=-\frac{r^2-3x^2}{r^5},$$

$$\text{由 } u=\frac{1}{\sqrt{x^2+y^2+z^2}} \text{ 的对称性知 } \begin{cases}\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{r^2-3y^2}{r^5},\\ \dfrac{\partial^2 u}{\partial z^2}=-\dfrac{r^2-3z^2}{r^5};\end{cases}$$

$$\text{故 } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3r^2-3(x^2+y^2+z^2)}{r^5}=-\frac{3r^2-3r^2}{r^5}=0.$$

例 9.3 证明:

$$u=\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}(x>0,t>0,a>0\text{常数})$$

满足热传导方程:

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}.$$

证明

$$\frac{\partial u}{\partial t}=\frac{1}{2a\sqrt{\pi}}(t^{-\frac{1}{22}})'_t\mathrm{e}^{-\frac{x^2}{4a^2t}}+\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}(-\frac{x^2}{4a^2t})'_t=\frac{1}{4a\sqrt{\pi t^3}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(\frac{x^2}{2a^2t}-1\right).$$

$$\text{另外 } \frac{\partial u}{\partial x}=\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(-\frac{x^2}{4a^2t}\right)'_x=\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(-\frac{x}{2a^2t}\right),$$

$$\text{可得 } \frac{\partial^2 u}{\partial x^2}=\frac{1}{2a\sqrt{\pi t}}\left[\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(-\frac{x}{2a^2t}\right)^2+\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(-\frac{1}{2a^2t}\right)\right]=\frac{1}{4a\sqrt{\pi t^3}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(\frac{x^2}{2a^4t}-\frac{1}{a^2}\right),$$

$$\text{因此 } a^2\frac{\partial^2 u}{\partial x^2}=\frac{1}{4a\sqrt{\pi t^3}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\left(\frac{x^2}{2a^2t}-1\right)=\frac{\partial u}{\partial t}.$$

例 9.4 证明: 设

$$\varphi,\psi\in C^2(I), u=\varphi(x-at)+\psi(x+at),(x\in\mathbb{R},t>0,a>0\text{常数})$$

满足波动方程:

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$

证明

$$\text{令 } \begin{cases}v=x-at,\\ w=x+at;\end{cases} \quad \text{则有 } u=\varphi(v)+\psi(w) \text{ 且 } \begin{cases}\dfrac{\partial v}{\partial x}=1,\\ \dfrac{\partial w}{\partial x}=1;\end{cases} \quad \text{与 } \begin{cases}\dfrac{\partial v}{\partial t}=-a,\\ \dfrac{\partial w}{\partial t}=a;\end{cases}$$

$$\text{因此我们有 } \frac{\partial u}{\partial x}=\varphi'(v)\frac{\partial v}{\partial x}+\psi'(w)\frac{\partial w}{\partial x}=\varphi'(v)+\psi'(w),$$

故 $\dfrac{\partial^2 u}{\partial x^2}=\varphi''(v)\dfrac{\partial v}{\partial x}+\psi''(w)\dfrac{\partial w}{\partial x}=\varphi''(v)+\psi''(w)$.

同时 $\dfrac{\partial u}{\partial t}=\varphi'(v)\dfrac{\partial v}{\partial t}+\psi'(w)\dfrac{\partial w}{\partial t}=-a\varphi'(v)+a\psi'(w)$,

故 $\dfrac{\partial^2 u}{\partial x^2}=-a\varphi''(v)\dfrac{\partial v}{\partial x}+a\psi''(w)\dfrac{\partial w}{\partial x}=a^2\left(\varphi''(v)+\psi''(w)\right)=a^2\dfrac{\partial^2 u}{\partial x^2}$.

例 9.5 球面方程 $x^2+y^2+z^2=a^2(a>0$常数) 在第一卦限内可确定三个隐函数

$$x=\sqrt{a^2-y^2-z^2}, y=\sqrt{a^2-x^2-z^2}, z=\sqrt{a^2-x^2-y^2};$$

证明:

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}\equiv -1.$$

证明

$\dfrac{\partial x}{\partial y}=-\dfrac{2y}{2\sqrt{a^2-y^2-z^2}}=-\dfrac{y}{x}, \dfrac{\partial y}{\partial z}=-\dfrac{2z}{2\sqrt{a^2-x^2-z^2}}=-\dfrac{z}{y}$,

$\dfrac{\partial z}{\partial x}=-\dfrac{2x}{2\sqrt{a^2-x^2-y^2}}=-\dfrac{x}{z}; (x>0,y>0,z>0)$,

因此 $\dfrac{\partial x}{\partial y}\dfrac{\partial y}{\partial z}\dfrac{\partial z}{\partial x}=\left(-\dfrac{y}{x}\right)\left(-\dfrac{z}{y}\right)\left(-\dfrac{x}{z}\right)\equiv -1, \forall x>0,y>0,z>0,x^2+y^2+z^2=a^2$.

例 9.6 设 $F(x,y)\in C^2(D)$,D 是区域, 函数 $y=\varphi(x)$ 由方程 $F(x,y)=0$ 确定,

证明:

$$\varphi''(x)=\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=-\frac{\frac{\partial^2 F}{\partial x^2}\left(\frac{\partial F}{\partial y}\right)^2-2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}+\frac{\partial^2 F}{\partial y^2}\left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}$$

证明 可知

$$\begin{aligned}
\varphi''(x)=\left(\varphi'(x)\right)'_x&=-\left(\frac{F'_x(x,y)}{F'_y(x,y)}\right)'_x\\
&=-\frac{\left(F'_x(x,y)\right)'_x F'_y(x,y)-\left(F'_y(x,y)\right)'_x F'_x(x,y)}{\left(F'_y(x,y)\right)^2}\\
&=-\frac{\left(F''_{xx}\cdot 1+F''_{xy}\cdot y'_x\right)F'_y-\left(F''_{yx}\cdot 1+F''_{yy}\cdot y'_x\right)F'_x}{\left(F'_y\right)^2}\\
&=-\frac{\left(F''_{xx}+F''_{xy}\left(-\frac{F'_x}{F'_y}\right)\right)F'_y-\left(F''_{yx}+F''_{yy}\left(-\frac{F'_x}{F'_y}\right)\right)F'_x}{\left(F'_y\right)^2}\\
&=-\frac{F''_{xx}\left(F'_y\right)^2-F''_{xy}F'_xF'_y-F''_{xy}F'_xF'_y+F''_{yy}\left(F'_x\right)^2}{\left(F'_y\right)^3}\\
&=-\frac{\frac{\partial^2 F}{\partial x^2}\left(\frac{\partial F}{\partial y}\right)^2-2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}+\frac{\partial^2 F}{\partial y^2}\left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}
\end{aligned}$$

其中 $y'_x=\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{F'_x(x,y)}{F'_y(x,y)}$.

✍ **作业** ex9.2:20(2)(3)(4),25,28,32;ex9.3:1(1),2(2)(5),4(1).