Week 4

4.1 Mar 17 ex9.2:31,ex9.3:6,7,8,10,11(1),14.

✍ **习题 9.2.31** 试证: 方程 $\frac{\partial^2 u}{\partial x}+2\cos x\frac{\partial u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y}\sin^2 x-\frac{\partial u}{\partial y}\sin x=0$ 经变换 $\begin{cases}\xi=x-\sin x+y,\\ \eta=x+\sin x-y\end{cases}$ 后可化为 $\frac{\partial u}{\partial\xi\partial\eta}=0$.(其中二阶偏导数均连续)

解 由链式法则得

$$\begin{aligned}
\frac{\partial u}{\partial x}&=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}=(1-\cos x)\frac{\partial u}{\partial \xi}+(1+\cos x)\frac{\partial u}{\partial \eta}\\
\frac{\partial u}{\partial y}&=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}=\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}
\end{aligned}$$

因此

$$\begin{aligned}
\frac{\partial^2 u}{\partial x}&=\frac{\partial}{\partial x}\left((1-\cos x)\frac{\partial u}{\partial \xi}+(1+\cos x)\frac{\partial u}{\partial \eta}\right)\\
&=\left((1-\cos x)\frac{\partial}{\partial \xi}+(1+\cos x)\frac{\partial}{\partial \eta}\right)\left((1-\cos x)\frac{\partial u}{\partial \xi}+(1+\cos x)\frac{\partial u}{\partial \eta}\right)\\
&=(1-\cos x)^2\frac{\partial^2 u}{\partial \xi}+(1+\cos x)^2\frac{\partial^2 u}{\partial \eta}+2(1-\cos x)(1+\cos x)\frac{\partial u}{\partial \xi\partial \eta}\\
\frac{\partial^2 u}{\partial y}&=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)=\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)-\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)\\
&=\frac{\partial^2 u}{\partial \xi}-\frac{\partial^2 u}{\partial \eta}-2\frac{\partial u}{\partial \xi\partial \eta}\\
\frac{\partial u}{\partial x\partial y}&=\frac{\partial}{\partial y}\left((1-\cos x)\frac{\partial u}{\partial \xi}+(1+\cos x)\frac{\partial u}{\partial \eta}\right)\\
&=\left(\frac{\partial}{\partial \xi}-\frac{\partial}{\partial \eta}\right)\left((1-\cos x)\frac{\partial u}{\partial \xi}+(1+\cos x)\frac{\partial u}{\partial \eta}\right)\\
&=(1-\cos x)\frac{\partial^2 u}{\partial \xi}-(1+\cos x)\frac{\partial^2 u}{\partial \eta}+2\cos x\frac{\partial u}{\partial \xi\partial \eta}
\end{aligned}$$

代入得

$$\begin{aligned}
0 &= \frac{\partial^2 u}{\partial x} + 2\cos x \frac{\partial u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y}\sin^2 x - \frac{\partial u}{\partial y}\sin x \\
&= \left((1-\cos x)^2 + (1-\cos x)2\cos x - \sin^2 x\right)\frac{\partial^2 u}{\partial \xi} \\
&+ \left((1-\cos x)2(1-\cos^2 x) + 2\cos x 2\cos x - \sin^2 x(-2)\right)\frac{\partial u}{\partial \xi \partial \eta} \\
&+ \left((1-\cos x)(1+\cos x)^2 + 2\cos x(-(1+\cos x)) - \sin^2 x(1+\cos x)\right)\frac{\partial^2 u}{\partial \eta} \\
&= 2\frac{\partial u}{\partial \xi \partial \eta}
\end{aligned}$$

习题 9.3.6 设 $z = z(x,y)$ 是由方程 $2\sin(x+2y-3z) = x+2y-3z$ 确定的隐函数, 求

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1.$$

解 对方程两边求微分, 得

$$2\cos(x+2y-3z)\,(\mathrm{d}x + 2\,\mathrm{d}y - 3\,\mathrm{d}z) = \mathrm{d}x + 2\,\mathrm{d}y - 3\,\mathrm{d}z$$

整理得

$$(3 - 6\cos(x+2y-3z))\,\mathrm{d}z = (1 - 2\cos(x+2y-3z))\,\mathrm{d}x + 2(1-2\cos(x+2y-3z))\,\mathrm{d}y$$

由此得

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1-2\cos(x+2y-3z)}{3-6\cos(x+2y-3z)}, \\
\frac{\partial z}{\partial y} &= \frac{2(1-2\cos(x+2y-3z))}{3-6\cos(x+2y-3z)}
\end{aligned}$$

于是

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{1-2\cos(x+2y-3z) + 2(1-2\cos(x+2y-3z))}{3-6\cos(x+2y-3z)} = 1.$$

习题 9.3.7 设 $z = z(x,y)$ 是由方程 $\varphi(cx - az, cy - bz) = 0$ 确定的隐函数, 其中 φ 可微, 证明 $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c$.

解 对方程两边求微分, 得

$$\frac{\partial \varphi}{\partial x}(c\,\mathrm{d}x - a\,\mathrm{d}z) + \frac{\partial \varphi}{\partial y}(c\,\mathrm{d}y - b\,\mathrm{d}z) = 0$$

整理得

$$(c\frac{\partial \varphi}{\partial x} - a\frac{\partial \varphi}{\partial y})\,\mathrm{d}z = c\frac{\partial \varphi}{\partial x}\,\mathrm{d}x + c\frac{\partial \varphi}{\partial y}\,\mathrm{d}y$$

由此得

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{c\frac{\partial \varphi}{\partial x}}{c\frac{\partial \varphi}{\partial x} - a\frac{\partial \varphi}{\partial y}}, \\
\frac{\partial z}{\partial y} &= \frac{c\frac{\partial \varphi}{\partial y}}{c\frac{\partial \varphi}{\partial x} - a\frac{\partial \varphi}{\partial y}}
\end{aligned}$$

于是

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=\frac{ac\frac{\partial\varphi}{\partial x}+bc\frac{\partial\varphi}{\partial y}}{c\frac{\partial\varphi}{\partial x}-a\frac{\partial\varphi}{\partial y}}=c.$$

✍ **习题 9.3.8** 设 $z=x^2+y^2$ 其中 $y=y(x)$ 为由方程 $x^2-xy+y^2=1$ 所定义的函数, 求 $\frac{\mathrm{d}z}{\mathrm{d}x}, \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}$.

解 对方程两边求微分, 得

$$2x\,\mathrm{d}x+2y\,\mathrm{d}y=0$$

整理得

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2x-y}{x-2y}$$

由此得

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2}&=\frac{(x-2y)(2-\frac{\mathrm{d}y}{\mathrm{d}x})-(2x-y)(1-2\frac{\mathrm{d}y}{\mathrm{d}x})}{(x-2y)^2}\\&=-\frac{3y}{(x-2y)^2}+\frac{6x^2-3xy}{(x-2y)^3}=6\frac{(x-y)^2}{(x-2y)^3}\end{aligned}$$

故

$$\frac{\mathrm{d}z}{\mathrm{d}x}=2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=2x+2y\frac{2x-y}{x-2y}=\frac{2x^2-2y^2}{x-2y}$$

$$\begin{aligned}\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}&=\frac{\mathrm{d}}{\mathrm{d}x}\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\&=2+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\\&=\frac{10x^3-24x^2y+30xy^2-8y^3}{(x-2y)^3}\end{aligned}$$

✍ **习题 9.3.10** 设 $x=x(z), y=y(z)$ 是由方程组 $\begin{cases}x+y+z=0,\\x^2+y^2+z^2=1\end{cases}$ 所确定的隐函数组, 求 $\frac{\mathrm{d}x}{\mathrm{d}z}, \frac{\mathrm{d}y}{\mathrm{d}z}$.

解 对方程组两边求微分得到

$$\mathrm{d}x+\mathrm{d}y+\mathrm{d}z=0,$$

$$2x\,\mathrm{d}x+2y\,\mathrm{d}y+2z\,\mathrm{d}z=0$$

将其视为关于 $\mathrm{d}x, \mathrm{d}y$ 的方程组, 解得

$$\begin{aligned}\mathrm{d}x&=-\frac{y-z}{y-x}\mathrm{d}z,\\\mathrm{d}y&=-\frac{x-z}{y-x}\mathrm{d}z\end{aligned}$$

✍ **习题 9.3.11(1)** 设 $u=u(x,y), v=v(x,y)$ 是由下列方程组所确定的隐函数组, 求 $\frac{\partial(u,v)}{\partial(x\partial y)}$.

(1) $\begin{cases} u^2+v^2+x^2+y^2=1, \\ u+v+x+y=0. \end{cases}$

解 对方程组两边求微分得到

$$\begin{aligned} 2u\,\mathrm{d}u+2v\,\mathrm{d}v+2x\,\mathrm{d}x+2y\,\mathrm{d}y&=0, \\ \mathrm{d}u+\mathrm{d}v+\mathrm{d}x+\mathrm{d}y&=0 \end{aligned}$$

将其视为关于 $\mathrm{d}u,\mathrm{d}v$ 的方程组, 解得

$$\begin{aligned} \mathrm{d}u&=\frac{x-v}{u-v}\,\mathrm{d}x+\frac{y-v}{u-v}\,\mathrm{d}y, \\ \mathrm{d}v&=\frac{x-u}{v-u}\,\mathrm{d}x+\frac{y-u}{v-u}\,\mathrm{d}y \end{aligned}$$

于是

$$\frac{\partial(u,v)}{\partial(x\partial y)}=\begin{vmatrix} \dfrac{x-v}{u-v} & \dfrac{y-v}{u-v} \\ \dfrac{x-u}{v-u} & \dfrac{y-u}{v-u} \end{vmatrix}=\frac{x-y}{u-v}$$

✍ **习题 9.3.14** 设 $y=y(x), z=z(x)$ 是由方程 $z=xf(x+y)$ 和 $F(x,y,z)=0$ 所确定的函数, 其中 f 和 F 分别具有一阶连续导数和一阶连续偏导数, 求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

解 对方程组两边求微分得到

$$\begin{aligned} \mathrm{d}z&=f(x+y)\,\mathrm{d}x+xf'(x+y)\,\mathrm{d}y(\mathrm{d}x+\mathrm{d}y), \\ 0&=F'_x\,\mathrm{d}x+F'_y\,\mathrm{d}y+F'_z\,\mathrm{d}z \end{aligned}$$

将其视为关于 $\mathrm{d}z,\mathrm{d}y$ 的方程组, 解得

$$\mathrm{d}z=\frac{F'_zf+xF'_xf'-xF'_xf'}{F'_z(1+xf')}\,\mathrm{d}x$$

于是

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{F'_zf+xF'_xf'-xF'_xf'}{F'_z(1+xf')}$$

4.2 Mar 19 ex9.2:21,22,23,24,36(2)(5),38

✍ **习题 9.2.21** 求函数 $u=xyz$ 在点 $(1,2,-1)$ 沿方向 $\boldsymbol{l}=(3,-1,1)$ 的方向微商.

解

$$\nabla u=\left(\frac{\partial}{\partial x}u,\frac{\partial}{\partial y}u,\frac{\partial}{\partial z}u\right)=(yz,xz,xy)$$

于是所求方向微商为

$$\nabla u\cdot\frac{\boldsymbol{l}}{|\boldsymbol{l}|}=\left(\frac{\partial}{\partial x}u,\frac{\partial}{\partial y}u,\frac{\partial}{\partial z}u\right)\cdot\frac{\boldsymbol{l}}{|\boldsymbol{l}|}=(3,-1,1)\cdot\frac{(3,-1,1)}{\sqrt{11}}=\frac{3}{\sqrt{11}}$$

✍ **习题 9.2.22** 试求函数 $z=\arctan\dfrac{y}{x}$ 在圆 $x^2+y^2-2x=0$ 上一点 $P(\dfrac{1}{2},\dfrac{\sqrt{3}}{2})$ 处沿该圆周逆时针方向上的方向微商.

解 将圆参数化为 $\begin{cases} x = 1 + \cos\theta, \\ y = \sin\theta \end{cases}$, 则 $\theta = \dfrac{2\pi}{3}$ 时位于 P 点. 且方向向量为 $\boldsymbol{l} = (x'(\theta), y'(\theta)) = (-\sin\theta, \cos\theta) = (-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2})$, 于是所求方向微商为

$$\nabla z\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \cdot \boldsymbol{l} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)\bigg|_{(x,y)=(\frac{1}{2},\frac{\sqrt{3}}{2})} \cdot (-\frac{\sqrt{3}}{2}, -\frac{1}{2}) = \frac{1}{2}$$

习题 9.2.23 求函数 $u = x^2 + 2y^2 + 3z^2 + xy + 3x - 2y - 6z$ 在点 $(1, 1, -1)$ 的梯度和最大方向微商.

解

$$\nabla u = (2x + y + 3, 4y + x - 2, 6z - 6)$$

在点 $(1, 1, -1)$ 处

$$\nabla u = (6, 3, -12)$$

最大方向微商为沿着 ∇u 的方向, 即

$$\nabla u \cdot \frac{\nabla u}{|\nabla u|} = (6, 3, -12) \cdot \frac{(6, 3, -12)}{\sqrt{6^2 + 3^2 + 12^2}} = 3\sqrt{21}$$

习题 9.2.24 设 $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, r = |\boldsymbol{r}|$, 试求 (1) $\mathbf{grad}\,\dfrac{1}{r^2}$; (2) $\mathbf{grad}\ln r$.

解

$$\mathbf{grad}\,\frac{1}{r^2} = \mathbf{grad}\,\frac{1}{x^2+y^2+z^2} = -\frac{2x}{(x^2+y^2+z^2)^2}\boldsymbol{i} - \frac{2y}{(x^2+y^2+z^2)^2}\boldsymbol{j} - \frac{2z}{(x^2+y^2+z^2)^2}\boldsymbol{k} = -\frac{2\boldsymbol{r}}{r^4};$$

$$\mathbf{grad}\ln r = \mathbf{grad}\ln\sqrt{x^2+y^2+z^2} = \frac{x}{x^2+y^2+z^2}\boldsymbol{i} + \frac{y}{x^2+y^2+z^2}\boldsymbol{j} + \frac{z}{x^2+y^2+z^2}\boldsymbol{k} = \frac{\boldsymbol{r}}{r^2}.$$

习题 9.2.36(2)(5) 求下列复合函数的微分 $\mathrm{d}u$

(2) $u = f(\xi, \eta), \xi = xy, \eta = \dfrac{x}{y}$;

(5) $u = f(\xi, \eta, \zeta), \xi = x^2 + y^2, \eta = x^2 - y^2, \zeta = 2xy$.

解

(2)

$$\begin{aligned}
\mathrm{d}u &= f_1' \,\mathrm{d}\xi + f_2' \,\mathrm{d}\eta \\
&= f_1'(y\,\mathrm{d}x + x\,\mathrm{d}y) + f_2'\left(\frac{1}{y}\,\mathrm{d}x - \frac{x}{y^2}\,\mathrm{d}y\right) \\
&= (f_1'y + f_2'\frac{1}{y})\,\mathrm{d}x + (f_1'x - f_2'\frac{x}{y^2})\,\mathrm{d}y
\end{aligned}$$

(5)

$$\begin{aligned}
\mathrm{d}u &= f_1' \,\mathrm{d}\xi + f_2' \,\mathrm{d}\eta + f_3' \,\mathrm{d}\zeta \\
&= f_1'(2x\,\mathrm{d}x + 2y\,\mathrm{d}y) + f_2'(2x\,\mathrm{d}x - 2y\,\mathrm{d}y) + f_3'(2y\,\mathrm{d}x + 2x\,\mathrm{d}y) \\
&= (2xf_1' + 2xf_2' + 2yf_3')\,\mathrm{d}x + (2yf_1' - 2yf_2' + 2xf_3')\,\mathrm{d}y
\end{aligned}$$

✍ **习题 9.2.38** 求直角坐标和极坐标的坐标变换 $x = x(r,\theta) = r\cos\theta, y = y(r,\theta) = r\sin\theta$ 的 Jacobi 行列式.

解 坐标变换为

$$x = r\cos\theta,\quad y = r\sin\theta$$

偏导数为

$$\frac{\partial x}{\partial r} = \cos\theta,\quad \frac{\partial x}{\partial \theta} = -r$$
$$\frac{\partial y}{\partial r} = \sin\theta,\quad \frac{\partial y}{\partial \theta} = r$$

Jacobi 行列式为

$$\frac{\partial(x,y)}{\partial(r\partial\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

4.3 Mar 21 ex9.4:3,4,8(1)(4),9,11,16(1),17(2)

✍ **习题 9.4.3** 证明曲线 $x = a\cos t, y = a\sin t, z = bt$ 的切线与 Oz 轴成定角.

解 由 $\boldsymbol{r}(t) = (a\cos t, a\sin t, bt)$ 可知

$$\boldsymbol{r}'(t) = (-a\sin t, a\cos t, b)$$

切线方向向量为 $\boldsymbol{r}'(t)$, 与 Oz 轴的夹角 θ 有

$$\cos\theta = \frac{\boldsymbol{r}'(t)\cdot\boldsymbol{k}}{|\boldsymbol{r}'(t)|} = \frac{b}{\sqrt{a^2+b^2}}$$

为常数, 所以切线与 Oz 轴成定角.

✍ **习题 9.4.4** 设 $\boldsymbol{r} = \left(\frac{t}{1+t}, \frac{1+t}{t}, t^2\right)$ $(t > 0)$, 判断它是不是简单曲线, 是不是光滑曲线, 并求出它在 $t = 1$ 时的切线方程和法平面方程.

解 简单曲线: 无自交点, 即 $\boldsymbol{r}(t_1) = \boldsymbol{r}(t_2) \Rightarrow t_1 = t_2$.

设 $\boldsymbol{r}(t_1) = \boldsymbol{r}(t_2)$, 则 $t_1^2 = t_2^2 \Rightarrow t_1 = \pm t_2$, 但 $t > 0$, 所以 $t_1 = t_2$, 故是简单曲线.

光滑曲线:$\boldsymbol{r}$ 满足 $\boldsymbol{r}^{(n)}(t) = \left(\frac{(-1)^{n+1}n!}{(t+1)^{n+1}}, -\frac{(-1)^{n+1}n!}{t^{n+1}}, 0\right), n \geqslant 3$. 即 $\boldsymbol{r}$ 的各阶导数都存在且连续, 且 $\boldsymbol{r}'(t) \neq \boldsymbol{0}$, 所以是光滑曲线.

在 $t = 1$ 时, 切线方程为

$$\boldsymbol{r}'(1) = \left(\frac{1}{4}, -1, 2\right)$$

所以切线方程为

$$\frac{x-\frac{1}{2}}{\frac{1}{4}} = \frac{y-2}{-1} = \frac{z-1}{2}$$

设法平面方程为 $\frac{1}{4}x - y + 2z = d$, 代入 $\boldsymbol{r}(1) = \left(\frac{1}{2}, 2, 1\right)$ 得 $d = \frac{1}{8}$, 所以法平面方程为

$$\frac{x}{4} - y + 2z = \frac{1}{8}$$

✍ **习题 9.4.8(1)(4)** 求下列曲面在指定点的切平面和法线方程.

(1) $z=\sqrt{x^2+y^2}-xy$, 在点 $(3,4,-7)$;

(4) $4+\sqrt{x^2+y^2+z^2}=x+y+z$, 在点 $(2,3,6)$.

解

1. 曲面可参数化为

$$r(x,y)=\left(x,y,\sqrt{x^2+y^2}-xy\right)$$

于是

$$r_x=\left(1,0,\frac{x}{\sqrt{x^2+y^2}}-y\right)\quad\Rightarrow\quad r_x(3,4)=\left(1,0,-\frac{17}{5}\right)$$
$$r_y=\left(0,1,\frac{y}{\sqrt{x^2+y^2}}-x\right)\quad\Rightarrow\quad r_y(3,4)=\left(0,1,-\frac{11}{5}\right)$$

因此 $(3,4,-7)$ 处法向量为

$$\boldsymbol{n}=\left(1,0,\frac{-17}{5}\right)\times\left(0,1,\frac{-11}{5}\right)=\left(\frac{17}{5},\frac{11}{5},1\right)$$

法线为

$$\frac{x-3}{17}=\frac{y-4}{11}=\frac{z+7}{5}$$

设切平面为 $17x+11y+5z+d=0$，代入 $(3,4,-7)$ 得到 $d=-60$。于是切平面方程为

$$17x+11y+5z-60=0$$

2. 对于隐式曲面

$$F(x,y,z)=\sqrt{x^2+y^2+z^2}-(x+y+z)+4=0$$

于是

$$F_x=\frac{x}{\sqrt{x^2+y^2+z^2}}-1\quad\Rightarrow\quad F_x(2,3,6)=\frac{-5}{7}$$
$$F_y=\frac{y}{\sqrt{x^2+y^2+z^2}}-1\quad\Rightarrow\quad F_y(2,3,6)=\frac{-4}{7}\quad F_z=\frac{z}{\sqrt{x^2+y^2+z^2}}-1\quad\Rightarrow\quad F_z(2,3,6)=\frac{-1}{7}$$

因此 $(2,1,0)$ 处法向量为

$$\boldsymbol{n}=(5,4,1)$$

法线为

$$\frac{x-2}{5}=\frac{y-3}{4}=z-6$$

设切平面为 $5x+4y+z+d=0$，代入 $(2,3,6)$ 得到 $d=-28$。于是切平面方程为

$$5x+4y+z-28=0$$

✍ **习题 9.4.9** 求椭球面 $x^2+2y^2+z^2=1$ 上平行于平面 $x-y+2z=0$ 的切平面方程.

解 设切点为 (x_0,y_0,z_0), 对于椭圆而言, 在这一点的切平面的法向量为 $(2x_0,4y_0,2z_0)$, 而平面

$x-y+2z=0$ 的法向量为 $(1,-1,2)$, 切平面的法向量与平面 $x-y+2z=0$ 的法向量平行, 即

$$\begin{cases} 2x_0 = \lambda, \\ 4y_0 = -\lambda, \\ 2z_0 = 2\lambda. \end{cases}$$

又由椭球面方程得

$$x_0^2 + 2y_0^2 + z_0^2 = 1.$$

联立解得

$$(x_0, y_0, z_0) = \left(\frac{2}{\sqrt{22}}, -\frac{1}{\sqrt{22}}, \frac{4}{\sqrt{22}}\right) \quad \text{或} \quad (x_0, y_0, z_0) = \left(-\frac{2}{\sqrt{22}}, \frac{1}{\sqrt{22}}, -\frac{4}{\sqrt{22}}\right).$$

代入切平面方程 $2x_0(x-x_0) + 4y_0(y-y_0) + 2z_0(z-z_0) = 0$ 得

$$x + 2y + z = \frac{\sqrt{22}}{2} \quad \text{或} \quad x + 2y + z = -\frac{\sqrt{22}}{2}.$$

✍ **习题 9.4.11** 求椭球面 $x^2+2y^2+3z^2=21$ 在某点 M 处的切平面 π 的方程, 使 π 过已知直线 $L: \dfrac{x-6}{2} = \dfrac{y-3}{1} = \dfrac{2z-1}{-2}$.

解 椭球面可以写为隐式曲面 $F(x,y,z) = x^2+2y^2+3z^2-21 = 0$, 由此得

$$F_x = 2x, \quad F_y = 4y, \quad F_z = 6z$$

故 (x_0, y_0, z_0) 处的切平面方程为

$$x_0(x-x_0) + 2y_0(y-y_0) + 3z_0(z-z_0) = 0$$

展开得

$$x_0x + 2y_0y + 3z_0z = x_0^2 + 2y_0^2 + 3z_0^2 = 21$$

任取直线 L 上两点 $(6,3,\frac{1}{2})$ 和 $(0,0,\frac{7}{2})$, 代入切平面方程, 得到 $z_0 = \frac{7}{2}$, 进而切平面方程为

$$6x_0 + 6y_0 = 21 - \frac{3}{2}z_0 = \frac{63}{4}$$

再结合

$$x_0^2 + 2y_0^2 + z_0^2 = 21$$

解得

$$(x_0, y_0, z_0) = (3, 0, 2) \quad \text{或} \quad (x_0, y_0, z_0) = (1, 2, 2)$$

进而切平面方程为

$$x + 2z = 7 \quad \text{或} \quad x + 4y + 6z = 21$$

✍ **习题 9.4.16(1)** 求下列曲线在给定点的切线和法平面方程

(1) $x^3y + xy^3 = 3 - x^2y^2$ 在点 $(1,1)$.

解

(1) 对于隐式曲线

$$F(x,y) = x^3y + xy^3 + x^2 + y^2 - 3 = 0$$

有

$$F_x = 3x^2y + y^3 + 2xy^2 \quad \Rightarrow \quad F_x(1,1) = 6$$

$$F_y = x^3 + 3xy^2 + 2x^2y \quad \Rightarrow \quad F_y(1,1) = 6$$

故 $(1,1)$ 处法向量为 $n = (1,1)$，进一步切向量为 $\boldsymbol{t} = (1,-1)$。
进而切线和法线依次为

$$y = -x + 2 \quad 和 \quad y = x$$

✍ **习题 9.4.17(2)** 求下列曲线在给定点的切线和法平面方程

(2) $\begin{cases} 2x^2 + 3y^2 + z^2 = 47, \\ x^2 + 2y^2 = z \end{cases}$ 在点 $(-2,1,6)$.

解

(2) 考虑隐式曲面

$$F(x,y,z) = 2x^2 + 3y^2 + z^2 - 47 = 0, \quad G(x,y,z) = x^2 + 2y^2 - z = 0$$

不难得到它们在 $(-2,1,6)$ 处的法向量分别为

$$\boldsymbol{n}_1 = (-4,3,6), \quad \boldsymbol{n}_2 = (-4,4,-1)$$

于是曲线的切向量为

$$\boldsymbol{t} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = (-27,-28,-4)$$

因此 $(1,3,4)$ 处切线为

$$\frac{x+2}{27} = \frac{y-1}{28} = \frac{z-6}{4}$$

设切平面为 $27x + 28y + 4z + d = 0$，代入 $(-2,1,6)$ 得到 $d = 2$。于是切平面方程为

$$27x + 28y + 4z + 2 = 0$$