Lec 20 二重积分的一般变量代换

20.1 变量代换

我们考虑一下的变量的代换

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$$

其中通常有 $(u, v) \in D_{uv}$,$x(u, v), y(u, v) \in C^1(D_{uv})$. 且 Jacobian 行列式 $\dfrac{\partial(x, y)}{\partial(u, v)}$ 在 D_{uv} 中有界且不为零. 利用二元函数的全微分, 有:

$$\begin{cases} \mathrm{d}x = \mathrm{d}x(u, v) = x'_u \,\mathrm{d}u + x'_v \,\mathrm{d}v \\ \mathrm{d}y = \mathrm{d}y(u, v) = y'_u \,\mathrm{d}u + y'_v \,\mathrm{d}v \end{cases}$$

从而

$$\mathrm{d}x \,\mathrm{d}y = (x'_u \,\mathrm{d}u + x'_v \,\mathrm{d}v)(y'_u \,\mathrm{d}u + y'_v \,\mathrm{d}v) = (x'_u y'_v - x'_v y'_u) \,\mathrm{d}u \,\mathrm{d}v = \frac{\partial(x, y)}{\partial(u, v)} \,\mathrm{d}u \,\mathrm{d}v$$

注 汪老师 (也就是上面的讲义) 与课本讲的是不同的. 请注意区别, 汪老师上课讲的所有微是微分形式, 也就是说老师证明的是:

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{\partial(x, y)}{\partial(u, v)} \,\mathrm{d}u \wedge \mathrm{d}v$$

课本上讲的 $\mathrm{d}x \,\mathrm{d}y$ 是面积微元, 面积微元 $\mathrm{d}x \,\mathrm{d}y = \mathrm{d}y \,\mathrm{d}x$, 因此书上证明的是:

$$\mathrm{d}x \,\mathrm{d}y = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mathrm{d}u \,\mathrm{d}v$$

微分形式与面积微元的区别在于, 微分形式是有方向的, 而面积微元是无方向的. 因此在书上讲的 $\mathrm{d}x \,\mathrm{d}y$ 是无方向的, 而汪老师讲的 $\mathrm{d}x \wedge \mathrm{d}y$ 是有方向的.

助教推荐大家计算二重积分的时候, 用书上的换元公式, 也就是后者, 前者的积分的意义我们会在后面微分形式的积分中再提到.

例 20.1 作广义极坐标变换:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

则 $\dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$. 因此有

$$\mathrm{d}x \,\mathrm{d}y = r \,\mathrm{d}r \,\mathrm{d}\theta$$

20.2 例题

例 20.2 计算

$$I=\iint_D \frac{x^2}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$$

其中 $D:\{(x,y)|x^2+y^2\leqslant x\}$.

解 利用极坐标换元

$$(x,y)=(r\cos\theta,r\sin\theta)$$

则

$$D_{r,\theta}=\{(r,\theta)|r\geqslant 0,\theta\in[0,2\pi),(r\cos\theta)^2+(r\sin\theta)^2\leqslant r\cos\theta\}=\{(r,\theta)|\theta\in[-\frac{\pi}{2},\frac{\pi}{2}],r\leqslant\cos\theta\}$$

因此有

$$\begin{aligned}I&=\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\frac{(r\cos\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}r\,\mathrm{d}r\,\mathrm{d}\theta\\&=\int_0^{\frac{\pi}{2}}\cos^4\theta\,\mathrm{d}\theta=\frac{3}{16}\pi\end{aligned}$$

例 20.3 计算

$$I=\iint_D xy\,\mathrm{d}x\,\mathrm{d}y$$

其中 D 是第一象限中 $xy=a,xy=b,y^2=cx,y^2=dx$ 所围成的区域, 其中 $b>a>0,d>c>0$.

解 作代换 $u=xy,v=\dfrac{y^2}{x}$, 则

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}y & x\\ -\dfrac{y^2}{x^2} & \dfrac{2y}{x}\end{vmatrix}=\frac{3y^2}{x}$$

故

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{x}{3y^2}=\frac{1}{3v}$$

因此有

$$I=\int_a^b u\,\mathrm{d}u\int_c^d\frac{1}{3v}\,\mathrm{d}v=\frac{b^2-a^2}{6}\ln\frac{d}{c}$$

例 20.4 计算

$$I=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-(x^2+y^2)}\cos(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$$

解 作代换 $x=r\cos\theta,y=r\sin\theta$, 则

$$I=\int_0^{2\pi}\mathrm{d}\theta\int_0^{+\infty}\mathrm{e}^{-r^2}\cos(r^2)r\,\mathrm{d}r=2\pi\frac{1}{2}\int_0^{+\infty}\mathrm{e}^{t}\cos t\,\mathrm{d}t=\frac{\pi}{2}$$

例 20.5 设

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2}+\frac{(y-\mu_2)^2}{\sigma_2^2}-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right)}$$

其中 μ_1, μ_2 为常数, σ_1, σ_2 为正数, $\rho \in (-1, 1)$. 证明:

$$I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) \, \mathrm{d}x \, \mathrm{d}y = 1$$

解 作代换

$$\begin{cases} s = \dfrac{x - \mu_1}{\sigma_1} - \rho \dfrac{y - \mu_2}{\sigma_2} \\ t = \dfrac{y - \mu_2}{\sigma_2} \sqrt{1 - \rho^2} \end{cases}$$

则 Jacobian 行列式为

$$\frac{\partial(s, t)}{\partial(x, y)} = \begin{vmatrix} \dfrac{1}{\sigma_1} & -\dfrac{\rho}{\sigma_2} \\ 0 & \dfrac{1}{\sigma_2} \sqrt{1 - \rho^2} \end{vmatrix} = \frac{\sqrt{1 - \rho^2}}{\sigma_1 \sigma_2}$$

因此

$$I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\pi(1 - \rho^2)} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left(s^2 + t^2\right)} \, \mathrm{d}s \, \mathrm{d}t$$

再令 $u = \dfrac{s}{\sqrt{2(1 - \rho^2)}}, v = \dfrac{t}{\sqrt{2(1 - \rho^2)}}$, 则

$$I = \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{e}^{-u^2 - v^2} \, \mathrm{d}u \, \mathrm{d}v = \frac{1}{\pi} \left(\int_{-\infty}^{+\infty} \mathrm{e}^{-u^2} \, \mathrm{d}u \right)^2 = \frac{1}{\pi} \cdot \pi = 1$$

20.3 补充证明

下面我们将证明

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}}$$

当 $\dfrac{\partial(x, y)}{\partial(u, v)} \neq 0$ 时, 方程组 $\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$ 可唯一确定 $\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$,
方程中直接对 u 求导, 得到方程组

$$\frac{\partial u}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial u} = 1$$
$$\frac{\partial v}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial u} = 0$$

由此可以解出逆映射的偏微商

$$\frac{\partial x}{\partial u} = \frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y}}$$
$$\frac{\partial y}{\partial u} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y}}$$

同样的, 方程中对 v 求导, 得到方程组

$$\frac{\partial u}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial v} = 0$$

$$\frac{\partial v}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial v}=1$$

由此可以解出逆映射的偏微商

$$\frac{\partial x}{\partial v}=-\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}$$

$$\frac{\partial y}{\partial v}=\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}$$

因此

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}{\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)^2}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

✍ **作业** ex10.2:2(3)(4)(7)(9),3(3),5.