

## Lec 9 复合 (隐) 函数微分法

### 9.1 复合函数 (composition) 微分法

#### 定理 9.1

设  $z = f(u, v)$  在区域  $D$  中可微, 且  $\begin{cases} u = g(x, y) \\ v = h(x, y) \end{cases}$  都在区域  $E$  中可微, 当复合  $f(g(x, y), h(x, y))$  有意义时,  $z$  通过中间变量  $u, v$  成为  $x, y$  的多元复合函数, 且有求偏导数的链式法则如下:

$$\begin{cases} \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \\ \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}; \end{cases} \quad (9.1)$$

同时,  $z$  作为  $x, y$  的多元复合函数可微, 且不论  $u, v$  是作为  $f(u, v)$  的自变量, 还是作为复合函数  $f(g(x, y), h(x, y))$  的中间变量, 总有:

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv. \quad (9.2)$$

?? 称为全微分的一阶形式不变性.



#### 证明

?? 固定  $y$ , 令  $x$  有增量  $\Delta x$ , 则

$$\begin{cases} \Delta u_x = g(x + \Delta x, y) - g(x, y), \\ \Delta v_x = h(x + \Delta x, y) - h(x, y), \\ \Delta z_x = f(u + \Delta u_x, v + \Delta v_x) - f(u, v) = \frac{\partial z}{\partial u} \Delta u_x + \frac{\partial z}{\partial v} \Delta v_x + o(\rho); \end{cases}$$

$$\text{其中 } \rho = \sqrt{(\Delta u_x)^2 + (\Delta v_x)^2}, \text{ 并有 } \Delta x \rightarrow 0 \Rightarrow \begin{cases} \Delta u_x \rightarrow 0, \\ \Delta v_x \rightarrow 0; \end{cases} \Rightarrow \rho \rightarrow 0.$$

利用

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{o(\rho)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{o(\rho)}{\rho} \frac{\rho}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{o(\rho)}{\rho} \lim_{\Delta x \rightarrow 0} \frac{\rho}{\Delta x} \\ &= \lim_{\rho \rightarrow 0} \frac{o(\rho)}{\rho} \lim_{\Delta x \rightarrow 0} \sqrt{\left(\frac{\Delta u_x}{\Delta x}\right)^2 + \left(\frac{\Delta v_x}{\Delta x}\right)^2} \\ &= 0 \cdot \sqrt{\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial v_x}{\partial x}\right)^2} \\ &= 0 \end{aligned}$$

以及

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}, \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} = \frac{\partial v}{\partial x};$$

因此有

$$\begin{aligned} \frac{\partial z}{\partial x} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta z_x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left( \frac{\partial z}{\partial u} \frac{\Delta u_x}{\Delta x} + \frac{\partial z}{\partial v} \frac{\Delta v_x}{\Delta x} + \frac{o(\rho)}{\Delta x} \right) \\ &= \frac{\partial z}{\partial u} \lim_{\Delta x \rightarrow 0} \frac{\Delta u_x}{\Delta x} + \frac{\partial z}{\partial v} \lim_{\Delta x \rightarrow 0} \frac{\Delta v_x}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{o(\rho)}{\Delta x} \\ &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \end{aligned}$$

同理, 对  $y$  有

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

可微性 记

$$\begin{cases} \Delta u = g(x + \Delta x, y + \Delta y) - g(x, y), \\ \Delta v = h(x + \Delta x, y + \Delta y) - h(x, y), \\ \Delta z = f(u + \Delta u, v + \Delta v) - f(u, v), \\ r = \sqrt{(\Delta x)^2 + (\Delta y)^2}, \\ \rho = \sqrt{(\Delta u)^2 + (\Delta v)^2}; \end{cases}$$

因此我们有

$$\begin{aligned} \Delta z &= \frac{\partial z}{\partial u} \Delta u + \frac{\partial z}{\partial v} \Delta v + o(\rho) \\ &= \frac{\partial z}{\partial u} \left( \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + o(r) \right) + \frac{\partial z}{\partial v} \left( \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + o(r) \right) + o(\rho) \\ &= \left( \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \right) \Delta x + \left( \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} \right) \Delta y + o(r) + o(\rho) \end{aligned}$$

同时, 当  $r \rightarrow 0$ , 有  $\rho \rightarrow 0$  与  $\frac{o(r)}{r}$  有界, 因此

$$\begin{aligned} \frac{\rho}{r} &= \frac{\sqrt{(\Delta u)^2 + (\Delta v)^2}}{r} \\ &= \sqrt{\left( \frac{\partial u}{\partial x} \frac{\Delta x}{r} + \frac{\partial u}{\partial y} \frac{\Delta y}{r} + \frac{o(r)}{r} \right)^2 + \left( \frac{\partial v}{\partial x} \frac{\Delta x}{r} + \frac{\partial v}{\partial y} \frac{\Delta y}{r} + \frac{o(r)}{r} \right)^2} \\ &\leq \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + M_0} \triangleq M, r \rightarrow 0 \end{aligned}$$

因此

$$\begin{aligned}\lim_{r \rightarrow 0} \left| \frac{o(\rho)}{r} \right| &= \lim_{r \rightarrow 0} \left| \frac{o(\rho)}{\rho} \right| \frac{\rho}{r} \\ &\leq M \lim_{r \rightarrow 0} \left| \frac{o(\rho)}{\rho} \right| \\ &= 0\end{aligned}$$

故

$$\Delta z = \left( \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \right) \Delta x + \left( \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} \right) \Delta y + o(r)$$

表明  $z$  作为  $x, y$  的多元复合函数可微.

?? (a). 当  $u, v$  作为  $f(u, v)$  的自变量时,  $z = f(u, v)$  可微, 自然有

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv.$$

(b). 当  $u, v$  作为复合函数  $f(g(x, y), h(x, y))$  的中间变量时,

$$\begin{aligned}dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= \left( \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \right) dx + \left( \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} \right) dy \\ &= \frac{\partial z}{\partial u} \left( \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + \frac{\partial z}{\partial v} \left( \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) \\ &= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv\end{aligned}$$

## 9.2 隐函数 (implicit function) 微分法

例 9.1 方程

$$3x + 4y - 5z + 7 = 0$$

可确定

$$\begin{cases} z = \frac{3}{5}x + \frac{4}{5}y + \frac{7}{5}, \\ \text{or } y = -\frac{3}{4}x + \frac{4}{5}z - \frac{7}{4}, \\ \text{or } x = -\frac{4}{3}y + \frac{5}{3}z - \frac{7}{3}; \end{cases}$$

三个函数, 分别可得

$$\begin{cases} \frac{\partial z}{\partial x} = \frac{3}{5}, \\ \frac{\partial z}{\partial y} = \frac{4}{5}; \end{cases} \quad \begin{cases} \frac{\partial y}{\partial z} = \frac{5}{4}, \\ \frac{\partial y}{\partial x} = -\frac{3}{4}; \end{cases} \quad \begin{cases} \frac{\partial x}{\partial y} = -\frac{4}{3}, \\ \frac{\partial x}{\partial z} = \frac{5}{3}; \end{cases}$$

可得

$$\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} = \frac{3}{5} \times \left(-\frac{4}{3}\right) \times \frac{5}{4} = -1, \quad \frac{\partial x}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x} = \frac{5}{3} \times \frac{4}{5} \times \left(-\frac{3}{4}\right) = -1.$$

上述的三个二元函数, 都是方程  $F(x, y, z) = 3x + 4y - 5z + 7 = 0$  所确定的隐函数.

## 定理 9.2

设方程  $F(x, y) = 0$  满足:

1.  $F(x, y) \in C^1(D)$ ,  $D$  为区域,
2.  $F(M_0) = F(x_0, y_0) = 0$ ,  $M_0 \in D$ ,
3.  $F'_y(M_0) = F'_y(x_0, y_0) \neq 0$ .

则方程  $F(x, y) = 0$  可在点  $M_0$  的某个  $\delta$  邻域  $\bar{U}(M_0, \delta)$  中确定唯一隐函数:  $y = \varphi(x)$  满足

$$\begin{cases} \varphi(x_0) = y_0, \\ \frac{dy}{dx} = \varphi'(x) = -\frac{F'_x(x, y)}{F'_y(x, y)} \in C \end{cases}$$



**证明** 不妨设  $F'_y(x_0, y_0) > 0$ , 则  $F(x_0, y)$  在  $y_0$  附近严格单调递增, 即在  $M(x_0, y_0)$  附近形成了一条唯一存在的严格单调递增平面曲线, 设此曲线的表达式为  $y = \varphi(x)$ ,  $(x, y) \in \bar{U}(M_0, \delta)$ , 则  $y = \varphi(x)$  即为所求的隐函数.

显然  $y = \varphi(x)$  穿过点  $M_0(x_0, y_0)$ , 即  $\varphi(x_0) = y_0$ , 且从  $F(x, \varphi(x)) \equiv 0$ , 两边对  $x$  求导, 有:  $F'_x \cdot 1 + F'_y \cdot \frac{d\varphi(x)}{dx} \equiv 0 \Rightarrow \frac{dy}{dx} = \varphi'(x) = -\frac{F'_x(x, y)}{F'_y(x, y)}$

从  $F \in C^1(D)$  知,  $\varphi'(x)$  是连续函数.

## 定理 9.3

设方程  $F(x, y, z) = 0$  满足:

1.  $F(x, y, z) \in C^1(D)$ ,  $D$  为区域,
2.  $F(M_0) = F(x_0, y_0, z_0) = 0$ ,  $M_0 \in D$ ,
3.  $F'_z(M_0) = F'_z(x_0, y_0, z_0) \neq 0$ .

则方程  $F(x, y, z) = 0$  可在点  $M_0$  的某个  $\delta$  邻域  $\bar{U}(M_0, \delta)$  中确定唯一隐函数:  $z = \varphi(x, y)$  满足

$$\begin{cases} \varphi(x_0, y_0) = z_0, \\ \frac{\partial z}{\partial x} = -\frac{F'_x(x, y, z)}{F'_z(x, y, z)}, \frac{\partial z}{\partial y} = -\frac{F'_y(x, y, z)}{F'_z(x, y, z)}. \end{cases}$$



**注** 值得注意的是, 上述隐函数  $y = \varphi(x)$  或者  $z = \varphi(x, y)$  只理论上存在, 实际问题中未必能求出来, 但隐函数的导数或偏导数是能够从已知方程  $F(x, y) = 0$  或  $F(x, y, z) = 0$  中求出来的.

例如, 已知  $z = \varphi(x, y)$  是方程  $F(x, y, z) = 0$  确定的隐函数, 则由  $F(x, y, \varphi(x, y)) \equiv 0$ , 两边对  $x, y$  分别求导, 有

$$\begin{cases} F'_x \cdot 1 + F'_z \cdot \varphi'_x(x, y) = 0 \\ F'_y \cdot 1 + F'_z \cdot \varphi'_y(x, y) = 0 \end{cases} \Rightarrow \begin{cases} \varphi'_x(x, y) = \frac{\partial z}{\partial x} = -\frac{F'_x(x, y, z)}{F'_z(x, y, z)} \\ \varphi'_y(x, y) = \frac{\partial z}{\partial y} = -\frac{F'_y(x, y, z)}{F'_z(x, y, z)}. \end{cases}$$

## 9.3 例题

例 9.2 证明:

$$u = \frac{1}{r}, r = \sqrt{x^2 + y^2 + z^2}$$

满足 Laplace 方程:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \equiv 0, \forall (x, y, z) \neq (0, 0, 0).$$

证明

$$\text{由于 } \frac{\partial u}{\partial x} = \frac{du}{dr} \frac{\partial r}{\partial x} = -\frac{1}{r^2} \frac{x}{r} = -\frac{x}{r^3},$$

$$\text{因此 } \frac{\partial^2 u}{\partial x^2} = -\left(\frac{x}{r^3}\right)'_x = -\frac{r^3 - 3r^2 \frac{x}{r}}{r^6} = -\frac{r^2 - 3x^2}{r^5},$$

$$\text{由 } u = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \text{ 的对称性知 } \begin{cases} \frac{\partial^2 u}{\partial y^2} = -\frac{r^2 - 3y^2}{r^5}, \\ \frac{\partial^2 u}{\partial z^2} = -\frac{r^2 - 3z^2}{r^5}; \end{cases}$$

$$\text{故 } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3r^2 - 3(x^2 + y^2 + z^2)}{r^5} = -\frac{3r^2 - 3r^2}{r^5} = 0.$$

例 9.3 证明:

$$u = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} (x > 0, t > 0, a > 0 \text{ 常数})$$

满足热传导方程:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

证明

$$\frac{\partial u}{\partial t} = \frac{1}{2a\sqrt{\pi}} (t^{-\frac{1}{2}})'_t e^{-\frac{x^2}{4a^2 t}} + \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} \left(-\frac{x^2}{4a^2 t}\right)'_t = \frac{1}{4a\sqrt{\pi t^3}} e^{-\frac{x^2}{4a^2 t}} \left(\frac{x^2}{2a^2 t} - 1\right).$$

$$\text{另外 } \frac{\partial u}{\partial x} = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} \left(-\frac{x^2}{4a^2 t}\right)'_x = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} \left(-\frac{x}{2a^2 t}\right),$$

$$\text{可得 } \frac{\partial^2 u}{\partial x^2} = \frac{1}{2a\sqrt{\pi t}} \left[ e^{-\frac{x^2}{4a^2 t}} \left(-\frac{x}{2a^2 t}\right)^2 + e^{-\frac{x^2}{4a^2 t}} \left(-\frac{1}{2a^2 t}\right) \right] = \frac{1}{4a\sqrt{\pi t^3}} e^{-\frac{x^2}{4a^2 t}} \left(\frac{x^2}{2a^4 t} - \frac{1}{a^2}\right),$$

$$\text{因此 } a^2 \frac{\partial^2 u}{\partial x^2} = \frac{1}{4a\sqrt{\pi t^3}} e^{-\frac{x^2}{4a^2 t}} \left(\frac{x^2}{2a^2 t} - 1\right) = \frac{\partial u}{\partial t}.$$

例 9.4 证明: 设

$$\varphi, \psi \in C^2(I), u = \varphi(x - at) + \psi(x + at), (x \in \mathbb{R}, t > 0, a > 0 \text{ 常数})$$

满足波动方程:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

证明

$$\text{令 } \begin{cases} v = x - at, \\ w = x + at; \end{cases} \text{ 则有 } u = \varphi(v) + \psi(w) \text{ 且 } \begin{cases} \frac{\partial v}{\partial x} = 1, \\ \frac{\partial w}{\partial x} = 1; \end{cases} \text{ 与 } \begin{cases} \frac{\partial v}{\partial t} = -a, \\ \frac{\partial w}{\partial t} = a; \end{cases}$$

$$\text{因此我们有 } \frac{\partial u}{\partial x} = \varphi'(v) \frac{\partial v}{\partial x} + \psi'(w) \frac{\partial w}{\partial x} = \varphi'(v) + \psi'(w),$$

$$\text{故 } \frac{\partial^2 u}{\partial x^2} = \varphi''(v) \frac{\partial v}{\partial x} + \psi''(w) \frac{\partial w}{\partial x} = \varphi''(v) + \psi''(w).$$

$$\text{同时 } \frac{\partial u}{\partial t} = \varphi'(v) \frac{\partial v}{\partial t} + \psi'(w) \frac{\partial w}{\partial t} = -a\varphi'(v) + a\psi'(w),$$

$$\text{故 } \frac{\partial^2 u}{\partial x^2} = -a\varphi''(v) \frac{\partial v}{\partial x} + a\psi''(w) \frac{\partial w}{\partial x} = a^2(\varphi''(v) + \psi''(w)) = a^2 \frac{\partial^2 u}{\partial x^2}.$$

**例 9.5** 球面方程  $x^2 + y^2 + z^2 = a^2$  ( $a > 0$  常数) 在第一卦限内可确定三个隐函数

$$x = \sqrt{a^2 - y^2 - z^2}, y = \sqrt{a^2 - x^2 - z^2}, z = \sqrt{a^2 - x^2 - y^2};$$

证明:

$$\frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} \equiv -1.$$

**证明**

$$\frac{\partial x}{\partial y} = -\frac{2y}{2\sqrt{a^2 - y^2 - z^2}} = -\frac{y}{x}, \frac{\partial y}{\partial z} = -\frac{2z}{2\sqrt{a^2 - x^2 - z^2}} = -\frac{z}{y},$$

$$\frac{\partial z}{\partial x} = -\frac{2x}{2\sqrt{a^2 - x^2 - y^2}} = -\frac{x}{z}; (x > 0, y > 0, z > 0),$$

$$\text{因此 } \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = \left(-\frac{y}{x}\right) \left(-\frac{z}{y}\right) \left(-\frac{x}{z}\right) \equiv -1, \forall x > 0, y > 0, z > 0, x^2 + y^2 + z^2 = a^2.$$

**例 9.6** 设  $F(x, y) \in C^2(D)$ ,  $D$  是区域, 函数  $y = \varphi(x)$  由方程  $F(x, y) = 0$  确定,


证明:

$$\varphi''(x) = \frac{d^2 y}{dx^2} = -\frac{\frac{\partial^2 F}{\partial x^2} \left(\frac{\partial F}{\partial y}\right)^2 - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial F}{\partial x} \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2} \left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}$$

**证明** 可知

$$\begin{aligned} \varphi''(x) &= (\varphi'(x))'_x = -\left(\frac{F'_x(x, y)}{F'_y(x, y)}\right)'_x \\ &= -\frac{(F'_x(x, y))'_x F'_y(x, y) - (F'_y(x, y))'_x F'_x(x, y)}{(F'_y(x, y))^2} \\ &= -\frac{(F''_{xx} \cdot 1 + F''_{xy} \cdot y'_x) F'_y - (F''_{yx} \cdot 1 + F''_{yy} \cdot y'_x) F'_x}{(F'_y)^2} \\ &= -\frac{\left(F''_{xx} + F''_{xy} \left(-\frac{F'_x}{F'_y}\right)\right) F'_y - \left(F''_{yx} + F''_{yy} \left(-\frac{F'_x}{F'_y}\right)\right) F'_x}{(F'_y)^2} \\ &= -\frac{F''_{xx} (F'_y)^2 - F''_{xy} F'_x F'_y - F''_{yx} F'_x F'_y + F''_{yy} (F'_x)^2}{(F'_y)^3} \\ &= -\frac{\frac{\partial^2 F}{\partial x^2} \left(\frac{\partial F}{\partial y}\right)^2 - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial F}{\partial x} \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2} \left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3} \end{aligned}$$

$$\text{其中 } y'_x = \frac{dy}{dx} = -\frac{F'_x(x, y)}{F'_y(x, y)}.$$

 **作业** ex9.2:20(2)(3)(4),25,28,32;ex9.3:1(1),2(2)(5),4(1).