Week 8

8.1 Apr 14 补充题，CH10.5，6，8.

✍ **习题 补充题 1**

用五种方法计算 $\Omega: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\leqslant 1$ 的体积 $V(\Omega)$.

解 先一后二, 先二后一, 球坐标换元, 柱坐标换元, 放缩

$$\begin{aligned} V &= \int_{-a}^{a}\mathrm{d}x\iint_{\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1-\frac{x^2}{a^2}}\mathrm{d}y\,\mathrm{d}z \\ &= \int_{-a}^{a} bc\left(1-\frac{x^2}{a^2}\right)\pi\,\mathrm{d}x = \frac{4}{3}\pi abc \end{aligned}$$

$$\begin{aligned} V &= \iint_{\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1}\int_{-a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}}^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_{-b}^{b}\mathrm{d}y\int_{-c}^{c}\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,\mathrm{d}z \\ &= \frac{4}{3}\pi abc \end{aligned}$$

令 $(x,y,z)=(a\sin\theta\cos\varphi, b\sin\theta\sin\varphi, c\cos\theta)$，则

$$\begin{aligned} V &= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^1 r^2\sin\theta\,\mathrm{d}r \\ &= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^1 r^2\,\mathrm{d}r = \frac{4}{3}\pi abc \end{aligned}$$

✍ **习题 补充题 2**

计算 $I_1 = \iiint_{x^2+y^2+z^2\leqslant 1}\cos(ax+by+cz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ 与 $I_2 = \iiint_{x^2+y^2+z^2\leqslant 1}(ax+by+cz)^m\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ 的值. 其中 $(a,b,c)\neq\theta$ 为常向量,$m\in N^+$.

解

$$\begin{aligned} I_1 &= \iiint_{x^2+y^2+z^2\le 1}\cos(ax+by+cz)\ \mathrm{d}V = \iiint_{x^2+y^2+z^2\le 1}\cos\left(\sqrt{a^2+b^2+c^2}\,x\right)\mathrm{d}V \\ &= \int_{-1}^{1}\mathrm{d}x\iint_{y^2+z^2\le 1-x^2}\cos(rx)\ \mathrm{d}S = \pi\int_{-1}^{1}(1-x^2)\cos(rx)\ \mathrm{d}x \\ &= -\frac{4\pi}{\sqrt{a^2+b^2+c^2}}\cos\sqrt{a^2+b^2+c^2}+\frac{4\pi}{(a^2+b^2+c^2)^{3/2}}\sin\sqrt{a^2+b^2+c^2} \end{aligned}$$

$$
\begin{aligned}
I_2 &= \iiint_{x^2+y^2+z^2\le 1} (ax+by+cz)^m \, \mathrm{d}V = r^m \iiint_{x^2+y^2+z^2\le 1} x^m \, \mathrm{d}V \\
&= r^m \int_{-1}^{1} (1-x^2)x^m \, \mathrm{d}V = \pi \left(\frac{1-(-1)^{m+1}}{m+1} - \frac{1-(-1)^{m+3}}{m+3} \right) \\
&= \frac{2\pi}{(m+1)(m+3)} \left(1-(-1)^{m+1}\right)
\end{aligned}
$$

✍ **习题 CH10.5** 试求圆盘 $(x-a)^2+(y-a)^2 \leqslant a^2$ 与曲线 $(x^2+y^2)^2 = 8a^2xy$ 的所围部分相交的区域 D 的面积.

解 由于所求区域为 $(x-a)^2+(y-a)^2 \leqslant a^2$ 与 $(x^2+y^2)^2 \leqslant 8a^2xy$ 的交集,xy 总是可以用 $u=\dfrac{x+y}{\sqrt{2}}, v=\dfrac{x-y}{\sqrt{2}}$ 来变换成两个分开的 u^2-v^2 的形式来简化计算.

令

$$u=\frac{x+y}{\sqrt{2}}, v=\frac{x-y}{\sqrt{2}}$$

则区域为

$$u^2+v^2-2\sqrt{2}au+a^2 \leqslant 0, (u^2+v^2)^2 \leqslant 4a^2(u^2-v^2)$$

所求为

$$\frac{1}{2}\iint_{D'} \mathrm{d}u\,\mathrm{d}v$$

此时可以反解计算, 但仍然比较麻烦, 需要对带累次根号的函数积分.

这时候再做换元, 令

$$u=r\cos\theta, v=r\sin\theta$$

则区域为

$$r^2-2\sqrt{2}ar\cos\theta+a^2 \leqslant 0, r^2 \leqslant 4a^2\cos 2\theta$$

所求为

$$\frac{1}{2}\iint_{D''} r\,\mathrm{d}r\,\mathrm{d}\theta.$$

对于限制区域的两个方程, 可以分别解出 r 满足的范围为

$$[\sqrt{2}a\cos\theta - a\sqrt{\cos 2\theta}, \sqrt{2}a\cos\theta + a\sqrt{\cos 2\theta}], [0, 2a\sqrt{\cos 2\theta}]$$

由于 $\cos\theta = \sqrt{\dfrac{\cos 2\theta+1}{2}}$, 综合二者比较可得 $[a(\sqrt{1+\cos 2\theta}-\sqrt{\cos 2\theta}), 2a\sqrt{\cos 2\theta}]$

因此得到

$$\cos 2\theta \geqslant \frac{1}{8}, \theta \in [-\frac{1}{2}\arccos\frac{1}{8}, \frac{1}{2}\arccos\frac{1}{8}]$$

.

因此此时得到

$$
\begin{aligned}
S &= \int_{-\frac{1}{2}\arccos\frac{1}{8}}^{\frac{1}{2}\arccos\frac{1}{8}} \mathrm{d}\theta \int_{a(\sqrt{1+\cos 2\theta}-\sqrt{\cos 2\theta})}^{2a\sqrt{\cos 2\theta}} r\,\mathrm{d}r \\
&= \int_{-\frac{1}{2}\arccos\frac{1}{8}}^{\frac{1}{2}\arccos\frac{1}{8}} \frac{1}{2}\left(\left(2a\sqrt{\cos 2\theta}\right)^2 - \left(a(\sqrt{1+\cos 2\theta}-\sqrt{\cos 2\theta})\right)^2\right)\mathrm{d}\theta \\
&= \frac{a^2}{2}\int_{-\frac{1}{2}\arccos\frac{1}{8}}^{\frac{1}{2}\arccos\frac{1}{8}} 2\cos 2\theta - 1 + 2\sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta}\,\mathrm{d}\theta \\
&= a^2\int_{0}^{\frac{1}{2}\arccos\frac{1}{8}} 2\cos 2\theta - 1 + 2\sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta}\,\mathrm{d}\theta \\
&= a^2\left(\sin 2\theta - \theta\right)\Big|_0^{\arccos\frac{1}{8}} + 2a^2\int_{0}^{\frac{1}{2}\arccos\frac{1}{8}} \sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta}\,\mathrm{d}\theta \\
&= a^2\left(\frac{3\sqrt{7}}{8} - \frac{1}{2}\arccos\frac{1}{8}\right) + 2a^2\int_{0}^{\frac{1}{2}\arccos\frac{1}{8}} \sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta}\,\mathrm{d}\theta
\end{aligned}
$$

并计算

$$
\begin{aligned}
&\int_{0}^{\frac{1}{2}\arccos\frac{1}{8}} \sqrt{1+\cos 2\theta}\sqrt{\cos 2\theta}\,\mathrm{d}\theta \\
\overset{t=\cos 2\theta}{=\!=} &\int_{\frac{1}{8}}^{1} \sqrt{1+t}\sqrt{t}\frac{1}{2\sqrt{1-t^2}}\,\mathrm{d}t \\
= &\frac{1}{2}\int_{\frac{1}{8}}^{1} \sqrt{\frac{t}{1-t}}\,\mathrm{d}t \\
= &\frac{1}{2}\int_{\frac{1}{8}}^{1} \frac{t}{\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}}\,\mathrm{d}t \\
= &\frac{1}{4}\int_{\frac{1}{8}}^{1} \frac{2t-1}{\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}} + \frac{1}{\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}}\,\mathrm{d}t \\
\overset{s=2t-1}{=\!=} &\frac{1}{4}\int_{-\frac{3}{4}}^{1} \frac{s}{\sqrt{1-s^2}} + \frac{1}{\sqrt{1-s^2}}\,\mathrm{d}s \\
= &\frac{1}{4}\left(-\sqrt{1-s^2} + \arcsin s\right)\Big|_{-\frac{3}{4}}^{1} \\
= &\frac{\pi}{8} + \frac{1}{4}\arcsin\frac{3}{4} + \frac{\sqrt{7}}{16} \\
= &\frac{1}{4}\arccos -\frac{3}{4} + \frac{\sqrt{7}}{16}
\end{aligned}
$$

带回得到

$$
\begin{aligned}
S =& a^2\left(\frac{3\sqrt{7}}{8}-\frac{1}{2}\arccos\frac{1}{8}+2(\frac{1}{4}\arccos\frac{3}{4}+\frac{\sqrt{7}}{16})\right)\\
=& a^2\left(\frac{\sqrt{7}}{2}+\frac{1}{2}\left(\arccos-\frac{3}{4}-\arccos\frac{1}{8}\right)\right)\\
=& a^2\left(\frac{\sqrt{7}}{2}+\arccos\frac{5\sqrt{2}}{8}\right)
\end{aligned}
$$

其中 $\frac{1}{2}\left(\arccos\frac{3}{4}-\arccos\frac{1}{8}\right)$ 可以考虑判断一下角的范围，之后利用三角函数直接计算，相当于求 $\frac{1}{2}(\theta_1-\theta_2)$，由于 $0<\theta_1-\theta_2<\frac{\pi}{2}$，因此 $\frac{1}{2}(\theta_1-\theta_2)=\arccos\cos\frac{1}{2}(\theta_1-\theta_2)$ 因此

$$
\begin{aligned}
&\cos\frac{1}{2}(\theta_1-\theta_2)\\
=&\sqrt{\frac{\cos(\theta_1-\theta_2)+1}{2}}\\
=&\sqrt{\frac{\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2+1}{2}}\\
=&\sqrt{\frac{-\frac{3}{4}\cdot\frac{1}{8}+\frac{\sqrt{7}}{4}\cdot\frac{3\sqrt{7}}{8}+1}{2}}\\
=&\sqrt{\frac{\frac{9}{16}+1}{2}}\\
=&\sqrt{\frac{25}{32}}\\
=&\frac{5\sqrt{2}}{8}
\end{aligned}
$$

即得.

✍ **习题 CH10.6** 计算曲面

$$(x^2+y^2)^2+z^4=y$$

所围成的体积 V.

解 作球坐标换元

$$(x,y,z)=(r\sin\varphi\cos\theta,r\sin\varphi\sin\theta,r\cos\varphi)$$

原题中的积分区域 Ω 关于 xy 平面和 yz 平面对称，又 $y\geqslant 0$，由对称性可以只考虑

$$\Omega_1=\{(x,y,z)\mid (x^2+y^2)^2+z^4\leqslant y, x\geqslant 0, z\geqslant 0\}$$

的部分. 则

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=r^2\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta$$

其中

$$(x^2+y^2)^2+z^4 \leqslant y \Rightarrow r^4\sin^4\varphi+r^4\cos^4\varphi \leqslant r\sin\varphi\sin\theta \Rightarrow 0\leqslant r\leqslant \sqrt[3]{\frac{\sin\theta\sin\varphi}{\sin^4\varphi+\cos^4\varphi}}$$

$$x\geqslant 0, y\geqslant 0, z\geqslant 0 \Rightarrow \theta\in[0,\frac{\pi}{2}], \varphi\in[0,\frac{\pi}{2}]$$

故

$$\begin{aligned}
V &= 4\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{\sqrt[3]{\frac{\sin\theta\sin\varphi}{\sin^4\varphi+\cos^4\varphi}}} r^2\sin\varphi\,\mathrm{d}r \\
&= 4\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \mathrm{d}\varphi \sin\varphi\left(\frac{1}{3}\frac{\sin\theta\sin\varphi}{\sin^4\varphi+\cos^4\varphi}\right) \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta \int_0^{\frac{\pi}{2}} \frac{\sin^2\varphi}{\sin^4\varphi+\cos^4\varphi}\,\mathrm{d}\varphi \\
&= \frac{4}{3}\cdot\int_0^{\frac{\pi}{2}} \frac{\sin^2\varphi(\sin^2\varphi+\cos^2\varphi)}{\sin^4\varphi+\cos^4\varphi}\,\mathrm{d}\varphi \\
&= \frac{4}{3}\cdot\int_0^{+\infty} \frac{t^2}{1+t^4}\,\mathrm{d}t \\
&= \frac{\sqrt{2}}{3}\pi
\end{aligned}$$

✍ **习题 CH10.8** 证明:

$$\iint_{x^2+y^2\leqslant 1} f(ax+by+c)\,\mathrm{d}x\,\mathrm{d}y = 2\int_{-1}^{1}\sqrt{1-t^2}f(t\sqrt{a^2+b^2}+c)\,\mathrm{d}t$$

解 令 $t=\frac{ax+by}{\sqrt{a^2+b^2}}, s=\frac{bx-ay}{\sqrt{a^2+b^2}}$, 相当于正交变换,$|J|=1$, 得到

$$\begin{aligned}
I &= \iint_{t^2+s^2\leqslant 1} f(t\sqrt{a^2+b^2}+c)\,\mathrm{d}t\,\mathrm{d}s \\
&= 2\int_{-1}^{1}\sqrt{1-t^2}f(t\sqrt{a^2+b^2}+c)\,\mathrm{d}t
\end{aligned}$$

8.2 Apr 16 ex10.3:5(8),6,12,14,16,19;CH10:4

✍ **习题 ex10.3.5(8)**

计算下列曲面围成的立体体积.

8. $(x^2+y^2+z^2)^2=a^3x$.

解 取球坐标换元,$x=r\cos\theta, y=r\sin\theta\cos\varphi, z=r\sin\theta\sin\varphi$, 则

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

积分区域为

$$\Omega=\{(x,y,z)\mid (x^2+y^2+z^2)^2=a^3x\}$$

$$\Rightarrow (r^2)^2=a^3r\cos\theta \Rightarrow r^3=a^3\cos\theta$$

故 $\varphi \in [0, 2\pi], \theta \in [0, \frac{\pi}{2}], r \in [0, a\sqrt[3]{\cos\theta}]$

$$
\begin{aligned}
V &= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{a\sqrt[3]{\cos\theta}} r^2 \sin\theta \,\mathrm{d}r \\
&= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \sin\theta \left(\frac{1}{3}a^3\cos\theta\right) \\
&= \frac{\pi}{3}a^3
\end{aligned}
$$

✍ **习题 ex10.3.6**

求函数 $f(x,y,z) = x^2+y^2+z^2$ 在区域 $x^2+y^2+z^2 \leqslant x+y+z$ 上的平均值.

解

$$
x^2+y^2+z^2 \leqslant x+y+z \Rightarrow \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 + \left(z-\frac{1}{2}\right)^2 \leqslant \frac{3}{4}
$$

于是令

$$
(x,y,z) = \left(\frac{1}{2}+\frac{1}{2}r\sin\theta\cos\varphi, \frac{1}{2}+\frac{1}{2}r\sin\theta\sin\varphi, \frac{1}{2}+\frac{1}{2}r\cos\theta\right)
$$

区域的体积

$$
V(\Omega) = \frac{\sqrt{3}}{2}\pi
$$

f 在区域上的积分

$$
I = \iiiint_\Omega (x^2+y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
$$

$$
\begin{aligned}
I &= \iiiint_\Omega (x^2+y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^1 \left(\frac{1}{4}+\frac{1}{4}r^2+r^2\sin^2\theta\right) r^2\sin\theta\,\mathrm{d}r \\
&= \frac{3\sqrt{3}}{5}\pi
\end{aligned}
$$

故平均值 $\bar{f} = \frac{I}{V} = \frac{6}{5}$.

✍ **习题 ex10.3.12**

一个物体是由两个半径各为 R 和 $r(R \geqslant r)$ 的同心球所围成, 已知材料的密度和到球心的距离成反比, 且在距离为 1 的球面出密度为 k, 求该物体的质量.

解

$$
\begin{aligned}
I &= \iiint_{r^3 \leqslant x^2+y^2+z^2 \leqslant R^3} \frac{k}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_r^R \frac{k}{r^2} \cdot r^2\sin\theta\,\mathrm{d}r \\
&= 2\pi k(R^2-r^2)
\end{aligned}
$$

✍ **习题 ex10.3.14**

有一个均匀质地的薄板, 它是由半径为 a 的半圆和一个长方形拼接而成, 为了使重心正好

在圆心上, 问长方形的宽 b 应为多少?

解 设圆心为原点, 薄板所在的区域为

$$\Omega=\{(x,y)\mid x^2+y^2\leqslant a^2, y\geqslant 0\}\cup\{(x,y)\mid -a\leqslant x\leqslant a, -b\leqslant y\leqslant 0\}$$

设重心坐标为 $(\bar{x},\bar{y})$, 则

$$\bar{x}=\frac{1}{I}\iint_{\Omega}x\,\mathrm{d}x\,\mathrm{d}y=0$$

$$\bar{y}=0\Rightarrow\iint_{\Omega}y\,\mathrm{d}x\,\mathrm{d}y=0\Rightarrow\iint_{x^2+y^2\leqslant a^2,y\geqslant 0}y\,\mathrm{d}x\,\mathrm{d}y+\iint_{-a\leqslant x\leqslant a,-b\leqslant y\leqslant 0}y\,\mathrm{d}x\,\mathrm{d}y=0$$

其中

$$\iint_{x^2+y^2\leqslant a^2,y\geqslant 0}y\,\mathrm{d}x\,\mathrm{d}y=\frac{1}{2}\int_0^{\pi}\int_0^{a}r\sin\theta\cdot r\,\mathrm{d}r\,\mathrm{d}\theta=\frac{1}{3}\pi a^3$$

$$\iint_{-a\leqslant x\leqslant a,-b\leqslant y\leqslant 0}y\,\mathrm{d}x\,\mathrm{d}y=\int_{-a}^{a}\mathrm{d}x\int_{-b}^{0}y\,\mathrm{d}y=-\frac{1}{2}ab^2$$

因此

$$\frac{1}{3}\pi a^3-\frac{1}{2}ab^2=0\Rightarrow b=\frac{\sqrt{6}}{3}a$$

习题 ex10.3.16

设球体 $x^2+y^2+z^2\leqslant 2az$ 内各点密度与各点到原点的距离成反比, 求其重心坐标.

解 球体质量为

$$\begin{aligned}I&=\iiint_{x^2+y^2+z^2\leqslant 2az}\frac{1}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\&=\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^{\sqrt{2a^2-2az}}r^2\sin\theta\cdot\frac{1}{r}r^2\sin\theta\,\mathrm{d}r\\&=4\pi a^3\end{aligned}$$

球体重心为

$$\begin{aligned}\bar{x}&=\frac{1}{I}\iiint_{x^2+y^2+z^2\leqslant 2az}x\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0\\\bar{y}&=\frac{1}{I}\iiint_{x^2+y^2+z^2\leqslant 2az}y\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0\\\bar{z}&=\frac{1}{I}\iiint_{x^2+y^2+z^2\leqslant 2az}z\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\frac{4}{5}a\end{aligned}$$

故重心坐标为 $(0,0,\frac{4}{5}a)$.

习题 ex10.3.19

求密度为 ρ 的均匀球锥体对在其顶点为 1 单位质量的质点的引力, 设球的半径为 R, 而轴截面的扇形的角度为 2α.

解 设球锥体的顶点为 O, 球锥体的区域为

$$\Omega=\{(x,y,z)\mid x^2+y^2+z^2\leqslant R^2,z\geqslant 0,\frac{y}{x}\leqslant\tan\alpha\}$$

则 (x,y,z) 处 $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ 的微元带来的引力为

$$\mathrm{d}\boldsymbol{F}=-\frac{G\rho(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$$

故总引力为

$$F_x=\iiint_\Omega -\frac{G\rho}{(x^2+y^2+z^2)^{3/2}}x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0$$

$$F_y=\iiint_\Omega -\frac{G\rho}{(x^2+y^2+z^2)^{3/2}}y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0$$

$$F_z=\iiint_\Omega -\frac{G\rho}{(x^2+y^2+z^2)^{3/2}}z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\pi G R\rho\sin^2\alpha$$

因此

$$||\boldsymbol{F}||=\sqrt{F_x^2+F_y^2+F_z^2}=\pi G R\rho\sin^2\alpha$$

✍ **习题 CH10.4** 设 $D=\{(x,y)\mid x^2+y^2\leqslant 1\}$, 求 $I=\iint_D\left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|\mathrm{d}x\,\mathrm{d}y$.

解 不难得到

$$\begin{aligned}\iint_D\left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|\mathrm{d}x\,\mathrm{d}y&=\iint_{B(0,1)}\left(x^2+y^2-\frac{x+y}{\sqrt{2}}\right)\mathrm{d}x\,\mathrm{d}y\\&\quad+2\iint_{B\left(\frac{1}{2\sqrt{2}},\frac{1}{2}\right)}\left(\frac{x+y}{\sqrt{2}}-x^2-y^2\right)\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

一方面，令 $x=r\cos\theta$, $y=r\sin\theta$，则

$$\begin{aligned}I_1&=\iint_{B(0,1)}\left(x^2+y^2-\frac{x+y}{\sqrt{2}}\right)\mathrm{d}x\,\mathrm{d}y\\&=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\left(r^2+\frac{r\cos\theta+r\sin\theta}{2}\right)\mathrm{d}r\\&=\int_0^{2\pi}\left(\frac{1}{4}+\frac{1}{3\sqrt{2}}(\cos\theta+\sin\theta)\right)\mathrm{d}\theta\\&=\frac{\pi}{2}\end{aligned}$$

另一方面，令 $x=\frac{1}{2\sqrt{2}}+\frac{r}{2}\cos\theta$, $y=\frac{1}{2\sqrt{2}}+\frac{r}{2}\sin\theta$，则

$$\begin{aligned}I_2&=\iint_{B\left(\frac{1}{2\sqrt{2}},\frac{1}{2}\right)}\left(\frac{x+y}{\sqrt{2}}-x^2-y^2\right)\mathrm{d}x\,\mathrm{d}y\\&=\frac{1}{16}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r(1-r^2)\,\mathrm{d}r\\&=\frac{\pi}{32}\end{aligned}$$

因此

$$\iint_D \left|\frac{x+y}{\sqrt{2}} - x^2 - y^2\right| \mathrm{d}x\,\mathrm{d}y = I_1 - 2I_2 = \frac{9\pi}{16}$$

8.3 Apr 16 补充题,ex10.4:1;CH10:3

✍ **习题 补充题 1**

推导半径为 R 的 n 维球体的体积公式

$$V_n(R) = \frac{R^n}{n!}\int_0^{\frac{\pi}{2}} \sin^{n-1}\theta\,\mathrm{d}\theta$$

✍ **习题 化 简**

$$I = \int\cdots\int_\Omega f\left(\sum_{i=1}^6 a_i x_i\right) \mathrm{d}x_1\cdots\mathrm{d}x_6$$

这里 Ω 是 $\mathbb{R}^6$ 的单位球.

解 对于 $\boldsymbol{a} = (a_1,\cdots,a_6)$，记 $a = |\boldsymbol{a}|$，则旋转坐标系可得

$$I = \int\cdots\int_\Omega f(ax_1)\,\mathrm{d}x_1\cdots\mathrm{d}x_6 = \int_{-1}^1 m\left(B(x_1)\right) f(ax_1)\,\mathrm{d}x_1 = \frac{8\pi^2}{15}\int_{-1}^1 \left(1-x^2\right)^{\frac{5}{2}} f(ax)\,\mathrm{d}x$$

这里

$$B(x_1) = \left\{\mathbf{x}' = (\mathbf{x_2},\mathbf{x_3},\mathbf{x_4},\mathbf{x_5},\mathbf{x_6}) \,\middle|\, \|\mathbf{x}'\|^2 < 1 - x_1^2\right\}$$

✍ **习题 ex10.4.1**

计算下列 n 重积分.

1. $\displaystyle\int\cdots\int_{[0,1]^n} \left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \mathrm{d}x_1\cdots\mathrm{d}x_n$;
2. $\displaystyle\int\cdots\int_{[0,1]^n} \left(x_1 + x_2 + \cdots + x_n\right)^2 \mathrm{d}x_1\cdots\mathrm{d}x_n$;
3. $\displaystyle\int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-1}} x_1\cdot x_2\cdots x_n\,\mathrm{d}x_n$;

解

1.

$$\begin{aligned}\int\cdots\int_{[0,1]^n} \left(x_1^2 + \cdots + x_n^2\right) \mathrm{d}x_1\cdots\mathrm{d}x_n &= \sum_{i=1}^n \int\cdots\int_{[0,1]^n} x_i^2\,\mathrm{d}x_1\cdots\mathrm{d}x_n \\ &= \sum_{i=1}^n \int_0^1 x_i^2\,\mathrm{d}x_i = \frac{n}{3}\end{aligned}$$

2.

$$
\begin{aligned}
&\int\cdots\int_{[0,1]^n}(x_1+\cdots+x_n)^2\,\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=&\int_0^1\cdots\int_0^1(x_1+\cdots+x_n)^2\,\mathrm{d}x_n\\
=&\frac{1}{3}\int_0^1\cdots\int_0^1\left((x_1+\cdots+x_{n-1}+1)^3-(x_1+\cdots+x_{n-1})^3\right)\mathrm{d}x_{n-1}\\
=&\frac{1}{3}\int_0^1\cdots\int_0^1\left(3(x_1+\cdots+x_{n-1})^2+3(x_1+\cdots+x_{n-1})+1\right)\mathrm{d}x_{n-1}\\
=&\frac{1}{3}+\int_0^1\cdots\int_0^1(x_1+\cdots+x_{n-1})^2\,\mathrm{d}x_{n-1}+\int_0^1\cdots\int_0^1(x_1+\cdots+x_{n-1})\,\mathrm{d}x_{n-1}\\
=&\frac{1}{3}+\int_0^1\cdots\int_0^1(x_1+\cdots+x_{n-1})^2\,\mathrm{d}x_{n-1}+\sum_{i=1}^{n-1}\int_0^1\cdots\int_0^1 x_i\,\mathrm{d}x_{n-1}\\
=&\int_0^1\cdots\int_0^1(x_1+\cdots+x_{n-1})^2\,\mathrm{d}x_{n-1}+\frac{n-1}{2}+\frac{1}{3}\\
=&\int_0^1\cdots\int_0^1(x_1+\cdots+x_{n-2})^2\,\mathrm{d}x_{n-2}+\frac{n-1}{2}+\frac{n-2}{2}+\frac{2}{3}\\
=&\cdots\\
=&\frac{n(n-1)}{4}+\frac{n}{3}=\frac{n(3n+1)}{12}
\end{aligned}
$$

3.

$$
\begin{aligned}
\int_0^1\mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_{n-1}}x_1x_2\cdots x_n\,\mathrm{d}x_n&=\frac{1}{2}\int_0^1\mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_{n-2}}x_1x_2\cdots x_{n-2}x_{n-1}^3\,\mathrm{d}x_{n-1}\\
&=\frac{1}{8}\int_0^1\mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_{n-3}}x_1x_2\cdots x_{n-3}x_{n-2}^5\,\mathrm{d}x_{n-2}\\
&=\cdots\\
&=\frac{1}{(n-1)!\,2^{n-1}}\int_0^1 x_1^{2n-1}\,\mathrm{d}x_1\\
&=\frac{1}{n!\,2^n}
\end{aligned}
$$

✍ **习题 CH10.3** 计算

$$
I_1=\int_0^1\sin\left(\ln\frac{1}{x}\right)\cdot\frac{x^b-x^a}{\ln x}\,\mathrm{d}x,\ I_2=\int_0^1\cos\left(\ln\frac{1}{x}\right)\cdot\frac{x^b-x^a}{\ln x}\,\mathrm{d}x
$$

解

$$
\begin{aligned}
I_1 &= \int_0^1 \sin\left(\ln\frac{1}{x}\right)\cdot\frac{x^b-x^a}{\ln x}\,\mathrm{d}x \\
&= \int_0^1 \sin\left(\ln\frac{1}{x}\right)\left(\int_a^b x^y\,\mathrm{d}y\right)\mathrm{d}x \\
&= \int_a^b \mathrm{d}y\int_0^1 \sin\left(\ln\frac{1}{x}\right)\cdot x^y\,\mathrm{d}x(\text{化为二重积分}) \\
&= \int_a^b (\int_0^1 \sin\left(\ln\frac{1}{x}\right)\cdot x^y\,\mathrm{d}x)\,\mathrm{d}y
\end{aligned}
$$

因此问题转化为计算

$$
g_1(y) = \int_0^1 \sin\left(\ln\frac{1}{x}\right)\cdot x^y\,\mathrm{d}x, g_2(y) = \int_0^1 \cos\left(\ln\frac{1}{x}\right)\cdot x^y\,\mathrm{d}x
$$

分部一下, 得到

$$
\begin{aligned}
g_1(y) &= \sin\left(\ln\frac{1}{x}\right)\cdot\frac{x^{y+1}}{y+1}\Big|_0^1 - \int_0^1 \cos\left(\ln\frac{1}{x}\right)\left(-\frac{1}{x}\right)\cdot\frac{x^{y+1}}{y+1}\,\mathrm{d}x \\
&= \int_0^1 \cos\left(\ln\frac{1}{x}\right)\cdot\frac{x^y}{y+1} \\
&= \frac{1}{y+1}g_2(y) \\
g_2(y) &= \cos\left(\ln\frac{1}{x}\right)\cdot\frac{x^{y+1}}{y+1}\Big|_0^1 + \int_0^1 \sin\left(\ln\frac{1}{x}\right)\left(-\frac{1}{x}\right)\cdot\frac{x^{y+1}}{y+1}\,\mathrm{d}x \\
&= \frac{1}{y+1} - \int_0^1 \sin\left(\ln\frac{1}{x}\right)\cdot\frac{x^y}{y+1} \\
&= \frac{1}{y+1} - \frac{1}{y+1}g_1(y)
\end{aligned}
$$

因此 $g_1(y) = \dfrac{1}{1+(y+1)^2}, g_2(y) = \dfrac{y+1}{1+(y+1)^2}$

$$
I_1 = \int_a^b \frac{1}{1+(y+1)^2}\,\mathrm{d}y = \arctan(a+1) - \arctan(b+1)
$$

$$
I_2 = \int_a^b \frac{y+1}{1+(y+1)^2}\,\mathrm{d}y = \frac{1}{2}\ln\big(1+(a+1)^2\big) - \frac{1}{2}\ln\big(1+(b+1)^2\big)
$$