Version: **PQR3468**

UNIVERSITY OF COLOMBO, SRI LANKA

UNIVERSITY OF COLOMBO SCHOOL OF COMPUTING

BACHELOR OF SCIENCE IN COMPUTER SCIENCE

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SCS 2107 – Mathematical Methods III

TWO (2) HOURS

Important Instructions to candidates:

1. The medium of instruction and questions is **English.**
2. If a page or a part of this question paper is not printed, please inform the supervisor immediately.
3. Note that questions appear on both sides of the paper. If a page is not printed, please inform the supervisor immediately.
4. Write your index number on each and every page of the answer paper.
5. This paper has **4** questions and **07** pages.
6. Answer **ALL** questions. All questions carry equal marks (**25** marks).
7. The **first two questions are MCQ** type. Mark your answers to these questions in the provided MCQ answer sheet.
8. **Clearly indicate the version of the exam on your MCQ answer sheet.**
9. Any electronic device capable of storing and retrieving text including electronic dictionaries and mobile phones are **not allowed.**
10. **Non-Programmable** calculators are **allowed.**

1. Mark your answers to these questions in the MCQ answer sheet provided.

(1) Which of the following statements is/are true?

(I) Every sequence converges.
(II) Every converging sequence is bounded.
(III) Every monotonic and bounded sequence converges.

(a) II only.
(b) III only.
(c) I and III only.
(d) I and II only.
(e) II and III only.

(2) Which of the following statements are true about the sequence given by

$$x_1 = 1, \qquad x_{n+1} = 3 - \frac{1}{x_n}.$$

(I) (x_n) is bounded.
(II) (x_n) is monotonic.
(III) (x_n) converges.

(a) I only.
(b) II only.
(c) I and III only.
(d) All three.
(e) None of the given.

Questions 3 - 5 are based on the following description.

Suppose that the Bisection Algorithm is applied to solve the equation $e^x + x = 0$, starting with the initial interval $[-1, 1]$ and its midpoint as the estimated solution. Take initial data as the 0-th iterate.

(3) The interval and the estimate of the solution in the second iterate are respectively,

(a) $[-1, -0.5]$ and -0.75.
(b) $[-0.5, 0]$ and -0.25.
(c) $[0, 0.5]$ and 0.25.
(d) $[0.5, 1]$ and 0.75.
(e) None of the given.

(4) The standard error bound for the approximate solution in the second iterate is

(a) 0.5
(b) 0.25
(c) 0.125
(d) 0.0625
(e) 0.03125

(5) If the error is required to be less than 10^{-3}, then determine the number of iterates required using the standard error estimate.

(a) 6
(b) 7
(c) 8
(d) 9
(e) 10

(6) Assuming the convergence, find the limit of the fixed point iteration given by $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(\frac{1}{ax_n} + x_n\right)$.

(a) $\frac{1}{\sqrt{a}}$

(b) $-\frac{1}{\sqrt{a}}$

(c) $-\sqrt{a}$

(d) $\sqrt{a}$

(e) None of the given.

(7) Consider the equation $x^5 - 2x - 2 = 0$. Which of the following is the function iterated by the Newton's method (Newton-Raphson method) to solve this equation? (You may assume that the initial value is $x_0 = 1$.)

(a) $\frac{x^6 - 5x^4 - 2x^2 - 2x + 2}{x^5 - 2x - 2}$

(b) $\frac{4x^5 + 2}{5x^4 - 2}$

(c) $\frac{5x^4 - 2}{x^5 - 2x - 2}$

(d) $\frac{x^5 - 2x - 2}{5x^4 - 2}$

(e) None of the given.

(8) Consider the fixed point iteration given by $x_{n+1} = x_n^2 - 1$. Which of the following statements are true?

(I) The initial value $x_0 = 1$ produces a periodic orbit.
(II) If x_0 is not any of the values ± 1 and 0, then the iteration converges.
(III) Iteration with $x_0 = 0.5$ converges.
(IV) Iteration diverge for all initial values.

(a) I, II and III only.
(b) I and III only.
(c) I and IV only.
(d) I only.
(e) III only.

(9) Which of the following is a polynomial passing through the points $(-1, 2)$, $(1, 2)$ and $(2, -1)$?

(a) $-\frac{1}{3}(x+1)(x-1) - (x+1)(x-2) + \frac{1}{3}(x-1)(x-2)$

(b) $-(x+1)(x-1) + 2(x+1)(x-2) + 2(x-1)(x-2)$

(c) $-(x+1)(x-1)\frac{1}{2}(x+1)(x-2) + \frac{1}{2}(x-1)(x-2)$

(d) $\frac{2}{9}(x-2)^2 - (x-2)(x+1)$

(e) None of the given.

(10) The interval $[a, b]$ was divided into certain number of sub intervals and the function values at the end points and midpoints of the sub-intervals were used to estimate $\int_a^b f(x)dx$ using midpoint and trapezoidal rules. The corresponding midpoint and trapezoidal sums are respectively $MSum = 5.32$ and $TSum = 4.36$. What is the approximation obtained by the Simpson's method for the data used by the above two methods?

(a) 4.84
(b) 4.68
(c) 5
(d) 4.82
(e) None of the given.

2. Mark your answers to these questions in the MCQ answer sheet provided.

<u>Questions 1 - 3</u> are based on the function $f(x) = \sqrt{x}$.

(1) Determine the degree 2 Taylor polynomial of the function $f(x) = \sqrt{x}$ around 4.

(a) $2 + \frac{1}{4}x - \frac{1}{32}x^2$

(b) $2 + \frac{1}{4}x - \frac{1}{64}x^2$

(c) $2 + \frac{1}{4}(x-4)^2 - \frac{1}{64}(x-4)^2$

(d) $2 + \frac{1}{4}(x-4)^2 - \frac{1}{32}(x-4)^2$

(e) None of the given.

(2) Approximate the value $\sqrt{4.1}$ using the degree 2 Taylor polynomial of the function $\sqrt{x}$ around 4.

(a) 2.02484375
(b) 2.0246875
(c) 2.0253125
(d) 2.02515625
(e) None of the given.

(3) Determine the standard error bound for the above approximation, using the bound $|f^{(3)}(x)| \leq 0.02$ in the interval $[4, 4.1]$.

(a) $\frac{1}{3} \times 10^{-4}$

(b) $\frac{1}{3} \times 10^{-5}$

(c) $\frac{1}{6} \times 10^{-4}$

(d) 2×10^{-4}

(e) None of the given.

<u>Questions 4 - 5</u> are based on the following description.

The velocity $v(t) = \dfrac{dx}{dt}$ of a moving particle at time t is given by $v = \dfrac{x^2 + x + 1}{x + 2}$, where $x = x(t)$ is the position at time t. The position of the particle at $t = 0$ is $x(0) = 0$.

(4) Use Euler's method with **two** steps to estimate $x(2)$.

(a) 0.85
(b) 0.5
(c) 1.2
(d) 0.25
(e) None of the given.

(5) The Runge-Kutta method (RK4) with step size h is given by
$x_{n+1} = x_n + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$ where,

$$\begin{aligned} m_1 &= f(t_n, x_n) \\ m_2 &= f(t_n + h/2, x_n + m_1 h/2) \\ m_3 &= f(t_n + h/2, x_n + m_2 h/2) \\ m_4 &= f(t_n + h, x_n + m_3 h). \end{aligned}$$

What is the estimate of $x(1)$ obtained by RK4 with one step?

(a) 0.5604
(b) 0.2704
(c) 0.6037
(d) 0.6432
(e) None of the given.

(6) Consider the binary operation $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, on $\mathbb{R}$, given by $a * b = a + b + ab$ for any $a, b \in \mathbb{R}$. Which of the following properties is/are true regarding $*$?

(I) $*$ is associative.
(II) There is an identity for $*$.
(III) Every element in $\mathbb{R}$ has an inverse with respect to $*$.

(a) All three.
(b) I and II only.
(c) II and III only.
(d) I only.
(e) None of the given.

(7) Which of the following is/are examples of groups? All the operations are standard ones.

(I) $(\{1, -1\}\ ,\ \times)$
(II) $(\mathbb{N}\ ,\ \times)$
(III) $(\mathbb{Z}\ ,\ \times)$
(IV) $(\mathbb{Z}\ ,\ +)$

(a) I only.
(b) IV only.
(c) I and II only.
(d) I and IV only.
(e) III and IV only.

<u>Questions 8 - 10</u> are based on modulo 159 multiplication, $\times_{159}$, defined on the set $Z_{159} = \{0, 1, \cdots, 157, 158\}$ of all non-negative remainders with respect to the division by 159. Hint: There are 104 multiplicatively invertible elements in Z_{159}.

(8) Find the multiplicative inverse of 11 in $(Z_{159}\ ,\ \times_{159})$.

(a) 35
(b) 11
(c) 17
(d) 14
(e) 29

(9) Determine the order of 16 in $(Z_{159}\ ,\ \times_{159})$.

(a) 13
(b) 26
(c) 159
(d) 158
(e) None of the given.

(10) Find 16^{-12} in $(Z_{159}\ ,\ \times_{159})$.

(a) 16
(b) 16^{12}
(c) 16^{-1}
(d) 1
(e) None of the given.

3. Use the booklet to write your answers to this question.

(a) The error bound, associated with the Simpson's method to estimate an integral $\int_a^b f(x)dx$, is such that

$$Error \leq \frac{h^4(b-a)M}{180}$$

where $|f^{(4)}(x)| \leq M$ for all $x \in [a,b]$ and $h = \dfrac{b-a}{n}$ and n is the (even) number of sub-intervals generated by all the points, including the midpoints, at which the function value is considered.

Consider the following values of the function f, given in the tabular form.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
f(x)	0.00	0.31	0.59	0.81	0.95	0.84	0.54	0.22	0.1	−0.1	−0.15

Suppose that for the function f, $\left|f^{(4)}(x)\right| \leq 10$ in the interval $[0,1]$.

Estimate the integral $\displaystyle\int_0^1 f(x)dx$ using the Simpson's method and give the error bound.

(b) Use Newton's Forward Difference method to find a polynomial passing through the points $(-h,a)$, $(0,b)$ and (h,c).

(c) Using a Taylor polynomial around 1 of suitable degree, determine the value $ln(2)$ with an error less than or equal to 0.25. Show your work to justify the answer.

4. Use the booklet to write your answers to this question.

(a) Find the remainder when $29^{41} + 41^{29}$ is divided by 35.

(b) Determine the modulo 5 addition table on $Z_5 = \{0, 1, 2, 3, 4\}$.

(c) Find the **additive** order of $3 \in Z_{115}$.

(d) Derive the Newton-Raphson (Newton's) fixed point iteration to find $\sqrt{N}$, where $N > 0$.

================ End of the Exam ================