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Thomas W. Sederberg tom@cs.byu.edu

Takafumi Saito

**Guo-Jin Wang** 

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# Hodographs and Normals of Rational Curves and Surfaces

Takafumi Saito Nippon Telegraph and Telephone Co.

 $\begin{array}{c} \text{Guo-Jin Wang} \\ \textbf{Zhe jiang University} \end{array}$ 

Thomas W. Sederberg
Brigham Young University

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### Abstract

Derivatives and normals of rational Bézier curves and surface patches are discussed. A non-uniformly scaled hodograph of a degree  $m \times n$  tensor-product rational surface, which provides correct derivative direction but not magnitude, can be written as a degree  $(2m-2) \times 2n$  or  $2m \times (2n-2)$  vector function in polynomial Bézier form. Likewise, the scaled normal direction is degree  $(3m-2) \times (3n-2)$ . Efficient methods are developed for bounding these directions and the derivative magnitude.

### 1 Introduction

Derivatives and normal vectors of parametric curves and surfaces are important issues in computer graphics and geometric modeling. Although the derivative at any single point on a Bézier curve or surface can be calculated by subdivision, it is often useful to assess the possible range of derivatives or normal vectors for an entire curve segment of surface patch. For example, cusps and inflection points on a curve can be detected by this analysis. Bounds of derivative and normal directions also help in detecting intersections between two curves or surfaces [Sederberg '88][Hohmeyer '92]. In various other algorithms for curves and surfaces, efficiency can often be enhanced by a priori determination of a bound on derivative magnitude.

The first derivative of a curve is sometimes called its hodograph. It is well-known that the hodograph of a degree n polynomial Bézier curve can be expressed as a degree n-1 polynomial Bézier curve. For a tensor-product surface, the hodograph is defined as the partial derivative of the surface with respect to each parameter s or t. If the surface is a polynomial Bézier patch of degree m in s and n in t, the hodographs are also tensor-product, and the degree of the s-hodograph is m-1 in s and n in t, while the degree of the t-hodograph is  $m \times (n-1)$  [Boehm '84]. Since the normal direction can be obtained as the cross product of the s- and t-hodographs, its degree is  $(2m-1) \times (2n-1)$ .

These simple results, however, don't apply to rational curves and surfaces. The hodograph of a degree n rational curve is a rational curve of degree 2n. Since derivative direction is often more important than magnitude, [Sederberg '87] proposed a scaled hodograph, which expresses only the derivative direction, and is a degree 2n-2 polynomial Bézier curve. [Floater '92] provided

upper bounds of first and higher order derivative magnitudes for rational Bézier curves. However, derivatives and normals of rational surfaces have not been widely addressed.

We introduce a new approach for representing hodographs and normals. First, we apply it to rational curves and reconfirm the hodographs and the bounds of derivatives introduced by [Sederberg '87] and [Floater '92]. We then apply it to rational surfaces to get scaled hodographs and normal direction in polynomial Bézier form, from which we devise efficient methods for bounding their direction and magnitude.

In this paper, curve generally means rational Bézier curve, and surface means rational tensor-product Bézier patch. Also, note that scaled hodographs provide exact directions, but not magnitudes.

# 2 Rational Curves

In this section, we explain a new approach for bounding the derivative direction and magnitude of a curve. Although the final results are identical to those presented in [Sederberg '87] and [Floater '92], this new method extends directly to surfaces.

## 2.1 Direction Between Two Homogeneous Points

In projective geometry, points are usually specified in homogeneous form. A homogeneous point

$$\mathbf{P} = (X, Y, Z, W) \tag{1}$$

has the Cartesian coordinates

$$\tilde{\mathbf{P}} = (x, y, z) = \left(\frac{X}{W}, \frac{Y}{W}, \frac{Z}{W}\right). \tag{2}$$

In this paper, a single character in bold typeface signifies a homogeneous point, while one with a tilde denotes a Cartesian point or vector.

To aid in our discussion, we define:

$$Dir(\mathbf{P}_1, \mathbf{P}_2) \equiv (W_1 X_2 - W_2 X_1, W_1 Y_2 - W_2 Y_1, W_1 Z_2 - W_2 Z_1)$$
(3)

where  $\mathbf{P}_i = (X_i, Y_i, Z_i, W_i)$ . The 'Dir' function indicates the direction of the Cartesian vector between two points, because

$$Dir(\mathbf{P}_1, \mathbf{P}_2) = W_1 W_2 (\tilde{\mathbf{P}}_2 - \tilde{\mathbf{P}}_1)$$
(4)

if  $W_1W_2 \neq 0$ . Furthermore, one can easily verify that 'Dir' satisfies the following relations:

$$Dir(\mathbf{P}_{1}, \mathbf{P}_{1}) = \mathbf{0}$$

$$Dir(\mathbf{P}_{1}, \mathbf{P}_{2}) = -Dir(\mathbf{P}_{2}, \mathbf{P}_{1})$$

$$Dir(k\mathbf{P}_{1}, \mathbf{P}_{2}) = Dir(\mathbf{P}_{1}, k\mathbf{P}_{2}) = k Dir(\mathbf{P}_{1}, \mathbf{P}_{2})$$

$$Dir(\mathbf{P}_{1} + \mathbf{P}_{2}, \mathbf{P}_{3}) = Dir(\mathbf{P}_{1}, \mathbf{P}_{3}) + Dir(\mathbf{P}_{2}, \mathbf{P}_{3})$$

$$Dir(\mathbf{P}_{1}, \mathbf{P}_{2} + \mathbf{P}_{3}) = Dir(\mathbf{P}_{1}, \mathbf{P}_{2}) + Dir(\mathbf{P}_{1}, \mathbf{P}_{3}),$$
(5)

where k is a scalar value.

# 2.2 Hodograph of a Rational Curve

A rational Bézier curve  $\mathbf{P}[t]$  is defined with homogeneous control points  $\mathbf{P}_i = (X_i, Y_i, Z_i, W_i)$ :

$$\mathbf{P}[t] = (X[t], Y[t], Z[t], W[t])$$

$$= \sum_{i=0}^{n} B_i^n[t] \mathbf{P}_i, \qquad (6)$$

where

$$B_i^n[t] = \binom{n}{i} (1-t)^{n-i} t^i, \tag{7}$$

and n is the degree of the curve. In this paper, we assume that all  $W_i > 0$ . The derivative of the curve in homogeneous coordinates is

$$\mathbf{P}'[t] = (X'[t], Y'[t], Z'[t], W'[t])$$

$$= n \sum_{i=0}^{n-1} B_i^{n-1}[t] (\mathbf{P}_{i+1} - \mathbf{P}_i)$$
(8)

and the derivative in Cartesian coordinates is

$$\frac{d}{dt}\tilde{\mathbf{P}}[t] = \frac{d}{dt} \left( \frac{X[t]}{W[t]}, \frac{Y[t]}{W[t]}, \frac{Z[t]}{W[t]} \right) \\
= \frac{(W[t]X'[t] - W'[t]X[t], W[t]Y'[t] - W'[t]Y[t], W[t]Z'[t] - W'[t]Z[t])}{(W[t])^{2}} \\
= \frac{\text{Dir}(\mathbf{P}[t], \mathbf{P}'[t])}{(W[t])^{2}}.$$
(9)

Since  $(W[t])^2$  is a scalar function,  $Dir(\mathbf{P}[t], \mathbf{P}'[t])$  gives the derivative direction in Cartesian coordinate.

One might expect that the degree of  $Dir(\mathbf{P}[t], \mathbf{P}'[t])$  is 2n-1. However, the degree is at most 2n-2, which is shown as follows. Write

$$\mathbf{P}[t] = (1-t)\mathbf{F}_0[t] + t\,\mathbf{F}_1[t] \tag{10}$$

$$\mathbf{P}'[t] = n(\mathbf{F}_1[t] - \mathbf{F}_0[t]), \tag{11}$$

where

$$\mathbf{F}_{0}[t] = \sum_{i=0}^{n-1} B_{i}^{n-1}[t] \mathbf{P}_{i}$$
 (12)

$$\mathbf{F}_{1}[t] = \sum_{i=1}^{n} B_{i-1}^{n-1}[t] \mathbf{P}_{i}. \tag{13}$$

Therefore,

$$\operatorname{Dir}\left(\mathbf{P}[t], \mathbf{P}'[t]\right) = \operatorname{Dir}\left((1-t)\mathbf{F}_{0}[t], n\,\mathbf{F}_{1}[t]\right) - \operatorname{Dir}\left(t\,\mathbf{F}_{1}[t], n\,\mathbf{F}_{0}[t]\right)$$

$$= n\operatorname{Dir}\left(\mathbf{F}_{0}[t], \mathbf{F}_{1}[t]\right)$$

$$= n\sum_{k=0}^{2n-2} (1-t)^{2n-2-k} t^{k} \sum_{\substack{i+j=k+1\\0\leq i\leq n-1\\1\leq j\leq n}} \binom{n-1}{i} \binom{n-1}{j-1} \operatorname{Dir}\left(\mathbf{P}_{i}, \mathbf{P}_{j}\right). \tag{14}$$

For each combination of i and j which satisfies i + j = k + 1

$${\binom{n-1}{i} \binom{n-1}{j-1} \operatorname{Dir}(\mathbf{P}_{i}, \mathbf{P}_{j}) + \binom{n-1}{j} \binom{n-1}{i-1} \operatorname{Dir}(\mathbf{P}_{j}, \mathbf{P}_{i})}$$

$$= \left[ \binom{n-1}{i} \binom{n-1}{j-1} - \binom{n-1}{j} \binom{n-1}{i-1} \right] \operatorname{Dir}(\mathbf{P}_{i}, \mathbf{P}_{j})$$

$$= \binom{n}{i} \binom{n}{j} \frac{j-i}{n} \operatorname{Dir}(\mathbf{P}_{i}, \mathbf{P}_{j})$$

$$= \binom{n}{i} \binom{n}{k-i+1} \frac{k-2i+1}{n} \operatorname{Dir}(\mathbf{P}_{i}, \mathbf{P}_{k-i+1}).$$
(15)

Having eliminated j, we get

$$\operatorname{Dir}\left(\mathbf{P}[t], \mathbf{P}'[t]\right) \\
= \sum_{k=0}^{2n-2} (1-t)^{2n-2-k} t^k \sum_{i=\max(0,k-n+1)}^{\lfloor k/2 \rfloor} (k-2i+1) \binom{n}{i} \binom{n}{k-i+1} \operatorname{Dir}\left(\mathbf{P}_i, \mathbf{P}_{k-i+1}\right) \\
= \sum_{k=0}^{2n-2} B_k^{2n-2} [t] \tilde{\mathbf{H}}_k \tag{16}$$

and

$$\frac{d}{dt}\tilde{\mathbf{P}}[t] = \frac{1}{(W[t])^2} \sum_{k=0}^{2n-2} B_k^{2n-2}[t] \,\tilde{\mathbf{H}}_k \tag{17}$$

where

$$\tilde{\mathbf{H}}_k = \frac{1}{\binom{2n-2}{k}} \sum_{i=\max(0,k-n+1)}^{\lfloor k/2 \rfloor} (k-2i+1) \binom{n}{i} \binom{n}{k-i+1} \operatorname{Dir}(\mathbf{P}_i, \mathbf{P}_{k-i+1}). \tag{18}$$

Equation (16) gives a scaled hodograph, and its control points are  $\tilde{\mathbf{H}}_k$  ( $0 \le k \le 2n$ –2). Equation (17) gives the hodograph with the correct magnitude. These results are equivalent to the hodographs in [Sederberg '87]. For a rational cubic Bézier curve, the scaled hodograph is degree four with control

points:

$$\tilde{\mathbf{H}}_{0} = 3 \operatorname{Dir}(\mathbf{P}_{0}, \mathbf{P}_{1})$$

$$\tilde{\mathbf{H}}_{1} = \frac{3}{2} \operatorname{Dir}(\mathbf{P}_{0}, \mathbf{P}_{2})$$

$$\tilde{\mathbf{H}}_{2} = \frac{1}{2} \operatorname{Dir}(\mathbf{P}_{0}, \mathbf{P}_{3}) + \frac{3}{2} \operatorname{Dir}(\mathbf{P}_{1}, \mathbf{P}_{2})$$

$$\tilde{\mathbf{H}}_{3} = \frac{3}{2} \operatorname{Dir}(\mathbf{P}_{1}, \mathbf{P}_{3})$$

$$\tilde{\mathbf{H}}_{4} = 3 \operatorname{Dir}(\mathbf{P}_{2}, \mathbf{P}_{3}).$$
(19)

#### 2.3 **Bound of Derivative Direction**

In equation (18), each coefficient of  $Dir(\mathbf{P}_i, \mathbf{P}_{k-i+1})$  is positive. This guarantees that the derivative direction is bounded by the convex hull of the vectors  $Dir(\mathbf{P}_a, \mathbf{P}_b)$   $(0 \le a < b \le n)$ . For each  $Dir(\mathbf{P}_a, \mathbf{P}_b),$ 

$$\operatorname{Dir}(\mathbf{P}_{a}, \mathbf{P}_{b}) = W_{a}W_{b}\left(\tilde{\mathbf{P}}_{b} - \tilde{\mathbf{P}}_{a}\right)$$

$$= W_{a}W_{b}\sum_{i=a}^{b-1}\left(\tilde{\mathbf{P}}_{i+1} - \tilde{\mathbf{P}}_{i}\right), \tag{20}$$

which shows that  $Dir(\mathbf{P}_a, \mathbf{P}_b)$  is bounded by the convex hull of the b-a vectors  $\tilde{\mathbf{P}}_{i+1} - \tilde{\mathbf{P}}_i$  (i = a)  $a, a+1, \ldots, b-1$ ). As a result,  $Dir(\mathbf{P}[t], \mathbf{P}'[t])$   $(0 \le t \le 1)$  can be bounded by the convex hull of the n vectors  $\tilde{\mathbf{P}}_{i+1} - \tilde{\mathbf{P}}_i$   $(i = 0, 1, \dots, n-1)$ . This result is commonly used for polynomial curves, but here we note that the bound is not violated if arbitrary positive weights are assigned to the control points of a polynomial Bézier curve.

#### 2.4 Bound of Derivative Magnitude

We can also derive an upper bound of the derivative magnitude in the following way. Define

$$W_{\max} \equiv \max_{0 \le i \le n} W_i \tag{21}$$

$$W_{\min} \equiv \min_{0 \le i \le n} W_i \tag{22}$$

$$W_{\text{max}} \equiv \max_{0 \le i \le n} W_i$$

$$W_{\text{min}} \equiv \min_{0 \le i \le n} W_i$$

$$D_{\text{max}} \equiv \max_{0 \le i \le n-1} \left\| \tilde{\mathbf{P}}_{i+1} - \tilde{\mathbf{P}}_i \right\|.$$
(21)

It is obvious that

$$W_{\min} \le W[t] \le W_{\max} \qquad (0 \le t \le 1). \tag{24}$$

We also invoke the identity

$$\sum_{i=\max(0,k-n+1)}^{\lfloor k/2\rfloor} (k-2i+1)^2 \binom{n}{i} \binom{n}{k-i+1} = n \binom{2n-2}{k}$$
 (25)

which can be proven by applying equation (16) to the curve P(t) = (t, 0, 0, 1). In this case, Dir(P(t), P'(t)) = (1, 0, 0) and  $P_i = (\frac{i}{n}, 0, 0, 1)$ . From (16) it follows that  $\tilde{H}_k = (1, 0, 0)$  for  $k = 0, 1, \ldots, 2n - 2$ . Inserting this identity and  $Dir(P_i, P_j) = (\frac{j-i}{n}, 0, 0)$  in (18) one obtains (25).  $Dir(\mathbf{P}_a, \mathbf{P}_b)$  ( $0 \le a < b \le 1$ ) can be bounded as follows:

$$\|\operatorname{Dir}(\mathbf{P}_{a}, \mathbf{P}_{b})\| = W_{a}W_{b} \|\tilde{\mathbf{P}}_{b} - \tilde{\mathbf{P}}_{a}\|$$

$$\leq W_{a}W_{b} \sum_{i=a}^{b-1} \|\tilde{\mathbf{P}}_{i+1} - \tilde{\mathbf{P}}_{i}\|$$

$$\leq W_{\max}^{2}(b-a)D_{\max}.$$
(26)

From equations (18), (25), and (26),

$$\left\| \frac{d}{dt} \tilde{\mathbf{P}}[t] \right\| \leq \frac{nW_{\text{max}}^2 D_{\text{max}}}{W_{\text{min}}^2} \sum_{k=0}^{2n-2} B_k^{2n-2}[t] 
= \frac{nW_{\text{max}}^2 D_{\text{max}}}{W_{\text{min}}^2}.$$
(27)

This result is equivalent to [Floater '92].

# 3 Rational Surfaces

### 3.1 Hodograph for Each Parameter Direction

A tensor-product rational Bézier surface  $\mathbf{P}[s,t]$  is defined with homogeneous control points  $\mathbf{P}_{i,j} = (X_{i,j}, Y_{i,j}, Z_{i,j}, W_{i,j})$ :

$$\mathbf{P}[s,t] = (X[s,t], Y[s,t], Z[s,t], W[s,t])$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{n} B_{i}^{m}[s] B_{j}^{n}[t] \mathbf{P}_{i,j}.$$
(28)

For such a surface, the derivative direction is defined independently for each parameter s, t. The s-derivative in Cartesian coordinates is

where

$$\mathbf{P}_s[s,t] = (X_s[s,t], Y_s[s,t], Z_s[s,t], W_s[s,t]),$$

$$X_s[s,t] = \frac{\partial}{\partial s} X[s,t], \quad Y_s[s,t] = \frac{\partial}{\partial s} Y[s,t], \quad \dots$$

Thus, the s- derivative direction in Cartesian coordinates can be defined as  $Dir(\mathbf{P}[s,t],\mathbf{P}_s[s,t])$ . In this section, we only focus on the s-derivative direction; the t-derivative direction can be derived in like manner.

The surface P[s,t] can be expressed,

$$\mathbf{P}[s,t] = \sum_{i=0}^{m} B_i^m[s] \mathbf{Q}_i[t], \tag{30}$$

where

$$\mathbf{Q}_i[t] = \sum_{j=0}^n B_j^n[t] \, \mathbf{P}_{i,j}. \tag{31}$$

Notice that equation (30) can be interpreted as a curve with the parameter s, with control points  $\mathbf{Q}_i[t]$ . This means that the s-derivative of  $\mathbf{P}[s,t]$  can be viewed as simply the derivative of a curve. Thus, we can apply equation (16),

$$\operatorname{Dir}(\mathbf{P}[s,t],\mathbf{P}_{s}[s,t]) = \sum_{k=0}^{2m-2} (1-s)^{2m-2-k} s^{k} \sum_{i=\max(0,k-m+1)}^{\lfloor k/2 \rfloor} (k-2i+1) \binom{m}{i} \binom{m}{k-i+1} \operatorname{Dir}(\mathbf{Q}_{i}[t],\mathbf{Q}_{k-i+1}[t]) (32)$$

The next step is to compute  $Dir(\mathbf{Q}_a[t], \mathbf{Q}_b[t])$   $(0 \le a < b \le m)$ . In an expression reminiscent of the curve case, we have

$$\operatorname{Dir}\left(\mathbf{Q}_{a}[t], \mathbf{Q}_{b}[t]\right) = \sum_{l=0}^{2n} (1-t)^{2n-l} t^{l} \sum_{j=\max(0,l-n)}^{\min(l,n)} \binom{n}{j} \binom{n}{l-j} \operatorname{Dir}\left(\mathbf{P}_{a,j}, \mathbf{P}_{b,l-j}\right). \tag{33}$$

From equations (32) and (33),

$$Dir\left(\mathbf{P}[s,t],\mathbf{P}_{s}[s,t]\right) = \sum_{k=0}^{2m-2} \sum_{l=0}^{2n} B_{k}^{2m-2}[s] B_{l}^{2n}[t] \,\tilde{\mathbf{H}}_{k,l}$$
(34)

and

$$\frac{\partial}{\partial s}\tilde{\mathbf{P}}[s,t] = \frac{1}{(W[s,t])^2} \sum_{k=0}^{2m-2} \sum_{l=0}^{2n} B_k^{2m-2}[s] B_l^{2n}[t] \,\tilde{\mathbf{H}}_{k,l},\tag{35}$$

where

$$\tilde{\mathbf{H}}_{k,l} = \frac{\sum_{i=\max(0,k-m+1)}^{\lfloor k/2 \rfloor} \sum_{j=\max(0,l-n)}^{\min(l,n)} (k-2i+1) \binom{m}{i} \binom{m}{i} \binom{m}{k-i+1} \binom{n}{j} \binom{n}{l-j} \operatorname{Dir} \left(\mathbf{P}_{i,j}, \mathbf{P}_{k-i+1,l-j}\right)}{\binom{2m-2}{k} \binom{2n}{l}}. (36)$$

Equation (34) gives a scaled hodograph for s, and its control points are  $\tilde{\mathbf{H}}_{k,l}$ . Equation (35) is the hodograph for s with the correct magnitude.

The degree of the scaled hodograph is 2m-2 in s and 2n in t. For a rational bilinear surface, the degree of the hodograph is 0 in s and 2 in t, and its control points are

$$\tilde{\mathbf{H}}_{0,0} = \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{1,0}) 
\tilde{\mathbf{H}}_{0,1} = \frac{1}{2} \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{1,1}) + \frac{1}{2} \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{1,0}) 
\tilde{\mathbf{H}}_{0,2} = \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{1,1}).$$
(37)

For a rational biquadratic surface, the hodograph is degree  $2 \times 4$ , and its control points are

$$\begin{split} &\tilde{\mathbf{H}}_{0,0} &= 2 \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{1,0}) \\ &\tilde{\mathbf{H}}_{0,1} &= \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{1,1}) + \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{1,0}) \\ &\tilde{\mathbf{H}}_{0,2} &= \frac{1}{3} \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{1,2}) + \frac{4}{3} \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{1,1}) + \frac{1}{3} \operatorname{Dir}(\mathbf{P}_{0,2}, \mathbf{P}_{1,0}) \\ &\tilde{\mathbf{H}}_{0,3} &= \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{1,2}) + \operatorname{Dir}(\mathbf{P}_{0,2}, \mathbf{P}_{1,1}) \\ &\tilde{\mathbf{H}}_{0,4} &= 2 \operatorname{Dir}(\mathbf{P}_{0,2}, \mathbf{P}_{1,2}) \\ &\tilde{\mathbf{H}}_{1,0} &= \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{2,0}) \\ &\tilde{\mathbf{H}}_{1,1} &= \frac{1}{2} \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{2,1}) + \frac{1}{2} \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{2,0}) \\ &\tilde{\mathbf{H}}_{1,2} &= \frac{1}{6} \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{2,2}) + \frac{2}{3} \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{2,1}) + \frac{1}{6} \operatorname{Dir}(\mathbf{P}_{0,2}, \mathbf{P}_{2,0}) \\ &\tilde{\mathbf{H}}_{1,3} &= \frac{1}{2} \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{2,2}) + \frac{1}{2} \operatorname{Dir}(\mathbf{P}_{0,2}, \mathbf{P}_{2,1}) \\ &\tilde{\mathbf{H}}_{1,4} &= \operatorname{Dir}(\mathbf{P}_{0,2}, \mathbf{P}_{2,2}) \\ &\tilde{\mathbf{H}}_{2,0} &= 2 \operatorname{Dir}(\mathbf{P}_{1,0}, \mathbf{P}_{2,0}) \\ &\tilde{\mathbf{H}}_{2,1} &= \operatorname{Dir}(\mathbf{P}_{1,0}, \mathbf{P}_{2,1}) + \operatorname{Dir}(\mathbf{P}_{1,1}, \mathbf{P}_{2,0}) \\ &\tilde{\mathbf{H}}_{2,2} &= \frac{1}{3} \operatorname{Dir}(\mathbf{P}_{1,0}, \mathbf{P}_{2,2}) + \frac{4}{3} \operatorname{Dir}(\mathbf{P}_{1,1}, \mathbf{P}_{2,1}) + \frac{1}{3} \operatorname{Dir}(\mathbf{P}_{1,2}, \mathbf{P}_{2,0}) \\ &\tilde{\mathbf{H}}_{2,3} &= \operatorname{Dir}(\mathbf{P}_{1,1}, \mathbf{P}_{2,2}) + \operatorname{Dir}(\mathbf{P}_{1,2}, \mathbf{P}_{2,1}) \\ &\tilde{\mathbf{H}}_{2,4} &= 2 \operatorname{Dir}(\mathbf{P}_{1,2}, \mathbf{P}_{2,2}). \end{split}$$

Figure 1 (a) shows the structure of these control points.

## 3.2 Bound of Derivative Direction

Though (2m-1)(2n+1) control points are required for the scaled hodograph, the derivative direction can be bounded by smaller number of vectors. For equation (32), the method in section 2.3 can be applied, and thus,  $\operatorname{Dir}(\mathbf{P}[s,t],\mathbf{P}_s[s,t])$  can be bounded by the convex hull of the m vectors  $\operatorname{Dir}(\mathbf{Q}_i[t],\mathbf{Q}_{i+1}[t])$   $(i=0,1,\ldots,m-1)$ . For equation (33), however,  $\operatorname{Dir}(\mathbf{Q}_a[t],\mathbf{Q}_b[t])$  cannot be bounded in such a way. Therefore,  $\operatorname{Dir}(\mathbf{P}[s,t],\mathbf{P}_s[s,t])$  can be bounded by the convex hull of the m(2n+1) vectors

$$\sum_{j=\max(0,l-n)}^{\min(l,n)} \binom{n}{j} \binom{n}{l-j} \operatorname{Dir}(\mathbf{P}_{i,j}, \mathbf{P}_{i+1,l-j}) \qquad (i=0,1,\dots,m-1;\ l=0,1,\dots,2n).$$
 (39)

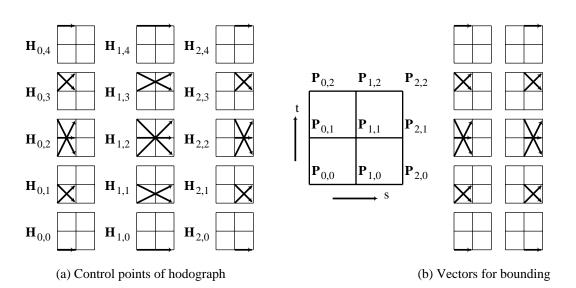


Figure 1: s-Derivative of Rational Biquadratic Surface

For example, the s-derivative direction of a biquadratic patch can be bounded by following 10 vectors:

$$\begin{aligned} & \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{1,0}) \\ & 2 \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{1,1}) + 2 \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{1,0}) \\ & \operatorname{Dir}(\mathbf{P}_{0,0}, \mathbf{P}_{1,2}) + 4 \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{1,1}) + \operatorname{Dir}(\mathbf{P}_{0,2}, \mathbf{P}_{1,0}) \\ & 2 \operatorname{Dir}(\mathbf{P}_{0,1}, \mathbf{P}_{1,2}) + 2 \operatorname{Dir}(\mathbf{P}_{0,2}, \mathbf{P}_{1,1}) \\ & \operatorname{Dir}(\mathbf{P}_{0,2}, \mathbf{P}_{1,2}) \\ & \operatorname{Dir}(\mathbf{P}_{1,0}, \mathbf{P}_{2,0}) \\ & 2 \operatorname{Dir}(\mathbf{P}_{1,0}, \mathbf{P}_{2,1}) + 2 \operatorname{Dir}(\mathbf{P}_{1,1}, \mathbf{P}_{2,0}) \\ & \operatorname{Dir}(\mathbf{P}_{1,0}, \mathbf{P}_{2,2}) + 4 \operatorname{Dir}(\mathbf{P}_{1,1}, \mathbf{P}_{2,1}) + \operatorname{Dir}(\mathbf{P}_{1,2}, \mathbf{P}_{2,0}) \\ & 2 \operatorname{Dir}(\mathbf{P}_{1,1}, \mathbf{P}_{2,2}) + 2 \operatorname{Dir}(\mathbf{P}_{1,2}, \mathbf{P}_{2,1}) \\ & \operatorname{Dir}(\mathbf{P}_{1,2}, \mathbf{P}_{2,2}). \end{aligned} \tag{40}$$

Figure 1 (b) shows the structure of these vectors.

## 3.3 Bound of Derivative Magnitude

An upper bound of the derivative magnitude for a rational surface can be obtained in the following way. From equations (29) and (32), by applying the method used in section 2.4, the derivative

magnitude can be bounded with the control points  $\mathbf{Q}_i[t]$  in equation (31):

$$\left\| \frac{\partial}{\partial s} \tilde{\mathbf{P}}[s,t] \right\| = \frac{m}{(W[s,t])^2} \left\| \sum_{k=0}^{2m-2} (1-t)^{2m-2-k} t^k \sum_{\substack{i+j=k+1\\0 \le i \le m-1\\1 \le j \le m}} {m-1 \choose i} \left( \frac{m-1}{j-1} \right) \operatorname{Dir} \left( \mathbf{Q}_i[t], \mathbf{Q}_j[t] \right) \right\|$$

$$\leq \frac{mW_{\max}^2}{W_{\min}^2} \max_{\substack{0 \le i \le m-1\\0 \le i \le m-1}} \left\| \tilde{\mathbf{Q}}_{i+1}[t] - \tilde{\mathbf{Q}}_i[t] \right\|, \tag{41}$$

where

$$W_{\max} \equiv \max_{0 \le i \le m} W_{i,j}, \tag{42}$$

$$W_{\min} \equiv \min_{\substack{0 \le i \le m \\ 0 \le i \le n}} W_{i,j}. \tag{43}$$

The distance between the control points  $\mathbf{Q}_i[t]$  can be bounded as follows:

$$\begin{split} \left\| \tilde{\mathbf{Q}}_{i+1}[t] - \tilde{\mathbf{Q}}_{i}[t] \right\| &= \frac{\left\| \operatorname{Dir}(\mathbf{Q}_{i}[t], \mathbf{Q}_{i+1}[t]) \right\|}{W_{i}[t]W_{i+1}[t]} \\ &\leq \frac{1}{W_{\min}^{2}} \left\| \sum_{l=0}^{2n} \frac{B_{l}^{2n}[t]}{\binom{2n}{l}} \sum_{j=\max(0,l-n)}^{\min(l,n)} \binom{n}{j} \binom{n}{l-j} \operatorname{Dir}(\mathbf{P}_{i,j}, \mathbf{P}_{i+1,l-j}) \right\| \\ &\leq \frac{S_{\max}}{W_{\min}^{2}}, \end{split}$$

$$\tag{44}$$

where  $W_i[t]$  denotes the weight of  $\mathbf{Q}_i[t]$ , and

$$S_{\max} \equiv \max_{\substack{0 \le i \le m-1 \\ 0 \le l \le 2n}} \left\| \frac{1}{\binom{2n}{l}} \sum_{j=\max(0,l-n)}^{\min(l,n)} \binom{n}{j} \binom{n}{l-j} \operatorname{Dir}\left(\mathbf{P}_{i,j}, \mathbf{P}_{i+1,l-j}\right) \right\|. \tag{45}$$

Therefore, we get the following upper bound:

$$\left\| \frac{\partial}{\partial s} \tilde{\mathbf{P}}[s, t] \right\| \le \frac{mW_{\text{max}}^2 S_{\text{max}}}{W_{\text{min}}^4}. \tag{46}$$

Notice that  $S_{\text{max}}$  is the maximum magnitude of the m(2n+1) vectors

$$\frac{1}{\binom{2n}{l}} \sum_{i=\max(0,l-n)}^{\min(l,n)} \binom{n}{j} \binom{n}{l-j} \operatorname{Dir}(\mathbf{P}_{i,j},\mathbf{P}_{i+1,l-j}) \qquad (i=0,1,\ldots,m-1;\ l=0,1,\ldots,2n), \quad (47)$$

each of which has the same direction as each vector in equation (39). This means that equation (47) forms a set of *bounding vectors* that can be used to bound both derivative direction and magnitude.

### 3.4 Normal Direction

The normal direction can be obtained simply by taking the cross product of the s- and t-scaled hodographs, where the degree of the normal direction is  $(4m-2) \times (4n-2)$ . [Hohmeyer '92] pointed out that the degree can be reduced to  $(3m-1) \times (3n-1)$ . We show here that it can actually be reduced to  $(3m-2) \times (3n-2)$ .

First of all, we define the notation:

$$Nrm(\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{3}) \equiv \begin{pmatrix} \begin{vmatrix} Y_{1} & Y_{2} & Y_{3} \\ Z_{1} & Z_{2} & Z_{3} \\ W_{1} & W_{2} & W_{3} \end{vmatrix}, \begin{vmatrix} Z_{1} & Z_{2} & Z_{3} \\ X_{1} & X_{2} & X_{3} \\ W_{1} & W_{2} & W_{3} \end{vmatrix}, \begin{vmatrix} X_{1} & X_{2} & X_{3} \\ Y_{1} & Y_{2} & Y_{3} \\ W_{1} & W_{2} & W_{3} \end{vmatrix}, (48)$$

which provides a normal direction of three homogeneous points. This notation 'Nrm' satisfies the following relations:

$$Nrm(\mathbf{P}_{1}, \mathbf{P}_{1}, \mathbf{P}_{1}) = Nrm(\mathbf{P}_{2}, \mathbf{P}_{1}, \mathbf{P}_{1}) = Nrm(\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{1}) = Nrm(\mathbf{P}_{1}, \mathbf{P}_{1}, \mathbf{P}_{2}) = \mathbf{0}$$

$$Nrm(\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{3}) = Nrm(\mathbf{P}_{2}, \mathbf{P}_{3}, \mathbf{P}_{1}) = Nrm(\mathbf{P}_{3}, \mathbf{P}_{1}, \mathbf{P}_{2})$$

$$= -Nrm(\mathbf{P}_{1}, \mathbf{P}_{3}, \mathbf{P}_{2}) = -Nrm(\mathbf{P}_{2}, \mathbf{P}_{1}, \mathbf{P}_{3}) = -Nrm(\mathbf{P}_{3}, \mathbf{P}_{2}, \mathbf{P}_{1})$$

$$Nrm(k\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{3}) = Nrm(\mathbf{P}_{1}, k\mathbf{P}_{2}, \mathbf{P}_{3}) = Nrm(\mathbf{P}_{1}, \mathbf{P}_{2}, k\mathbf{P}_{3}) = k Nrm(\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{3})$$

$$Nrm(\mathbf{P}_{1} + \mathbf{P}_{2}, \mathbf{P}_{3}, \mathbf{P}_{4}) = Nrm(\mathbf{P}_{1}, \mathbf{P}_{3}, \mathbf{P}_{4}) + Nrm(\mathbf{P}_{2}, \mathbf{P}_{3}, \mathbf{P}_{4})$$

$$Nrm(\mathbf{P}_{1}, \mathbf{P}_{2} + \mathbf{P}_{3}, \mathbf{P}_{4}) = Nrm(\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{4}) + Nrm(\mathbf{P}_{1}, \mathbf{P}_{3}, \mathbf{P}_{4})$$

$$Nrm(\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{3} + \mathbf{P}_{4}) = Nrm(\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{3}) + Nrm(\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{4}),$$

where k is a scalar value.

The normal direction at P[s,t] is given by

$$Nrm(\mathbf{P}[s,t], \mathbf{P}_s[s,t], \mathbf{P}_t[s,t]). \tag{50}$$

To calculate it efficiently, we define  $S_{ij}$  as follows:

$$\mathbf{S}_{00} \equiv \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} B_{i}^{m-1}[s] B_{j}^{n-1}[t] \mathbf{P}_{i,j}$$

$$\mathbf{S}_{10} \equiv \sum_{i=1}^{m} \sum_{j=0}^{n-1} B_{i-1}^{m-1}[s] B_{j}^{n-1}[t] \mathbf{P}_{i,j}$$

$$\mathbf{S}_{01} \equiv \sum_{i=0}^{m-1} \sum_{j=1}^{n} B_{i}^{m-1}[s] B_{j-1}^{n-1}[t] \mathbf{P}_{i,j}$$

$$\mathbf{S}_{11} \equiv \sum_{i=1}^{m} \sum_{j=1}^{n} B_{i-1}^{m-1}[s] B_{j-1}^{n-1}[t] \mathbf{P}_{i,j}.$$
(51)

patch	bilinear	biquadratic	bicubic	$m \times n$
s-hodograph	$0 \times 2$	$2 \times 4$	$4 \times 6$	$(2m-2)\times 2n$
t-hodograph	$2 \times 0$	$4 \times 2$	$6 \times 4$	$2m \times (2n-2)$
normal	1 × 1	$4 \times 4$	$7 \times 7$	$(3m-2) \times (3n-2)$

Table 1: Degree of Scaled Hodographs and Normal Directions

Then  $\mathbf{P}$ ,  $\mathbf{P}_s$ , and  $\mathbf{P}_t$  are simply expressed with  $\mathbf{S}_{ij}$ :

$$\mathbf{P}[s,t] = (1-s)(1-t)\mathbf{S}_{00} + s(1-t)\mathbf{S}_{10} + (1-s)t\mathbf{S}_{01} + st\mathbf{S}_{11}, 
\mathbf{P}_{s}[s,t] = m \sum_{i=0}^{m-1} \sum_{j=0}^{n} B_{i}^{m-1}[s]B_{j}^{n}[t](\mathbf{P}_{i+1,j} - \mathbf{P}_{i,j}) 
= m [(1-t)\mathbf{S}_{10} + t\mathbf{S}_{11} - (1-t)\mathbf{S}_{00} - t\mathbf{S}_{01}], 
\mathbf{P}_{t}[s,t] = n \sum_{i=0}^{m} \sum_{j=0}^{n-1} B_{i}^{m}[s]B_{j}^{n-1}[t](\mathbf{P}_{i,j+1} - \mathbf{P}_{i,j}) 
= n [(1-s)\mathbf{S}_{01} + s\mathbf{S}_{11} - (1-s)\mathbf{S}_{00} - s\mathbf{S}_{10}].$$
(52)

Substituting equations (52) into equation (50) and applying equation (49), we get

$$\operatorname{Nrm} (\mathbf{P}[s,t], \mathbf{P}_{s}[s,t], \mathbf{P}_{t}[s,t]) = mn \left[ (1-s)(1-t) \operatorname{Nrm}(\mathbf{S}_{00}, \mathbf{S}_{10}, \mathbf{S}_{01}) + s(1-t) \operatorname{Nrm}(\mathbf{S}_{00}, \mathbf{S}_{10}, \mathbf{S}_{11}) + (1-s)t \operatorname{Nrm}(\mathbf{S}_{00}, \mathbf{S}_{11}, \mathbf{S}_{01}) + st \operatorname{Nrm}(\mathbf{S}_{10}, \mathbf{S}_{11}, \mathbf{S}_{01}) \right].$$
(53)

Since the degree of  $\mathbf{S}_{ij}$  is  $(m-1) \times (n-1)$ , the degree of the normal direction is  $(3m-2) \times (3n-2)$ . For example, the normal direction of a bilinear surface is degree  $1 \times 1$ :

$$\operatorname{Nrm}(\mathbf{P}[s,t], \mathbf{P}_{s}[s,t], \mathbf{P}_{t}[s,t]) = (1-s)(1-t)\operatorname{Nrm}(\mathbf{P}_{0,0}, \mathbf{P}_{1,0}, \mathbf{P}_{0,1}) + s(1-t)\operatorname{Nrm}(\mathbf{P}_{0,0}, \mathbf{P}_{1,0}, \mathbf{P}_{1,1}) + (1-s)t\operatorname{Nrm}(\mathbf{P}_{0,0}, \mathbf{P}_{1,1}, \mathbf{P}_{0,1}) + st\operatorname{Nrm}(\mathbf{P}_{1,0}, \mathbf{P}_{1,1}, \mathbf{P}_{0,1}).$$
(54)

Table 1 summarizes the degree of scaled hodographs and normal direction for rational surfaces.

# 3.5 Bound of Normal Direction

There are two ways to create a bound of the normal direction. One is from the bounds of s-and t derivative directions, which is discussed in [Sederberg '88] for polynomial surfaces: create a bounding cone for each derivative direction from which a bounding cone for the normals can be calculated. This idea can be applied to a rational surface, where it is necessary to evaluate m(2n+1) vectors for s-derivative and (2m+1)n vectors for t-derivative. The other method is to calculate the normal direction by using the method in section 3.4. Here, the set of control points can bound the direction.

patch	bilinear	biquadratic	bicubic	$m \times n$
s-hodograph	3	10	21	m(2n+1)
t-hodograph	3	10	21	(2m+1)n
normal	4	25	64	(3m-1)(3n-1)

Table 2: Number of Vectors Required for Bounding

For polynomial surfaces, [Sederberg '88] claimed that the latter method generally gives a tighter bound, but requires much more computation. This is also true for rational surfaces. Table 2 shows the number of vectors required for the bounds. For both biquadratic and bicubic patches, the number of vectors required to bound normals is larger than total number of vectors for bounding s-and t-derivative directions. Furthermore, calculation of 'Nrm' is more expensive than that of 'Dir'. Thus, we recommend that normal bounds be computed from derivative bounds as in [Sederberg '88], except for bilinear surfaces.

# 4 Discussion

We have tried to find tighter or simpler bounds for derivative directions and normals of rational surfaces. The following hypotheses seem plausible, but have been proven false.

**Hypothesis 1.** For a degree  $m \times n$  rational surface  $\mathbf{P}[s,t]$ , the s-derivative direction is bounded by s-derivative directions of mn bilinear surfaces, each of which are defined by four control points:  $\mathbf{P}_{i,j}, \mathbf{P}_{i+1,j}, \mathbf{P}_{i,j+1}, \mathbf{P}_{i+1,j+1}$   $(i = 0, 1, \dots, m-1; j = 0, 1, \dots, m-1)$ .

**Hypothesis 2.** On a degree  $m \times n$  rational surface  $\mathbf{P}[s,t]$ , the surface normal is bounded by the normals of 4mn triangles:  $\triangle \mathbf{P}_{i,j} \mathbf{P}_{i+1,j} \mathbf{P}_{i,j+1}$ ,  $\triangle \mathbf{P}_{i,j} \mathbf{P}_{i+1,j+1}$ ,  $\triangle \mathbf{P}_{i,j} \mathbf{P}_{i+1,j+1} \mathbf{P}_{i,j+1}$ ,  $\triangle \mathbf{P}_{i+1,j} \mathbf{P}_{i+1,j+1} \mathbf{P}_{i,j+1}$ ,  $(i = 0, 1, \dots, m-1; j = 0, 1, \dots, n-1)$ .

Hypothesis 1 is true for any degree  $m \times 1$  surface, and Hypothesis 2 is true for any bilinear surface (see equation (54)). However, they are not true in general. Figure 2 shows a counterexample. For this degree  $1 \times 2$  patch, the s-derivative direction and the surface normal at  $\mathbf{P}[1, 0.5]$  are (1.00, -1.29, 0.66) and (-0.81, -0.41, 0.42), respectively. These directions defy the hypothesised bounds. If Hypothesis 1 were true, the projection of the s-derivative onto xy-plane, i.e. (1.00, -1.29, 0), should be inside of (1, 1, 0) and (1, -1, 0). If Hypothesis 2 were true, the normal direction should be bounded by (0, -1, 1), (0, 1, 1), (-1, -1, 1), and (-1, 1, 1).

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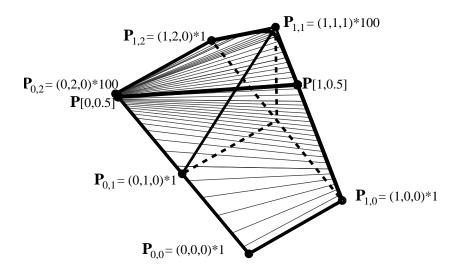


Figure 2: A Counterexample of the Hypotheses

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