Convergence of Allen-Cahn equations to multi-phase mean curvature flow

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Contents

1 The Allen–Cahn equation

1.1 Structure of the equation

This chapter follows [LS16], but since the authors decided to only sketch some of the proofs, we want to go into more detail.

Let $\Lambda > 0$ and define the flat torus $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$, where we work with periodic boundary conditions and write $\int \mathrm{d}x$ instead of $\int_{\mathbb{T}} \mathrm{d}x$. Then for $u \colon [0, \infty) \times \mathbb{T} \to \mathbb{R}^N$ and some potential $W \colon \mathbb{R}^N \to [0, \infty)$, the *Allen–Cahn equation* with parameter $\varepsilon > 0$ is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \tag{1.1}$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to u for a fixed time the real number

$$\mathrm{E}_\varepsilon(u) \coloneqq \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 \, \mathrm{d}x \,. \tag{1.2}$$

If everything is nice and smooth, we can compute that under the assumption that u satisfies equation (1.1), we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u) \,,\, \partial_t u \rangle + \varepsilon \langle \nabla u \,,\, \nabla \partial_t u \rangle \, \mathrm{d}x \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u \,,\, \partial_t u \right\rangle \mathrm{d}x \\ &= \int -\varepsilon |\partial_t u|^2 \, \mathrm{d}x \,. \end{aligned} \tag{1.1}$$

This calculation suggests that equation (1.1) is the L^2 gradient flow (rescaled by $\sqrt{\varepsilon}$) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (1.1) via De Giorgis minimizing movements scheme, which we will do in Theorem 1.2.1.

But first we need to clarify what our potential W should look like. Classic examples in the scalar case are given by $W(u) = \big(u^2 - 1\big)^2$ or $W(u) = u^2(u-1)^2$, and we call functions like these *doublewell potentials*, see also Figure 1.1.

In higher dimensions, we want to accept the following potentials: $W \colon \mathbb{R}^N \to [0, \infty)$ has to be a a smooth multiwell potential with finitely many zeros at $u = \alpha_1, \dots, \alpha_P \in \mathbb{R}^N$. Furthermore we aks for polynomial growth in the sense that there exists some $p \geq 2$ such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \tag{1.3}$$

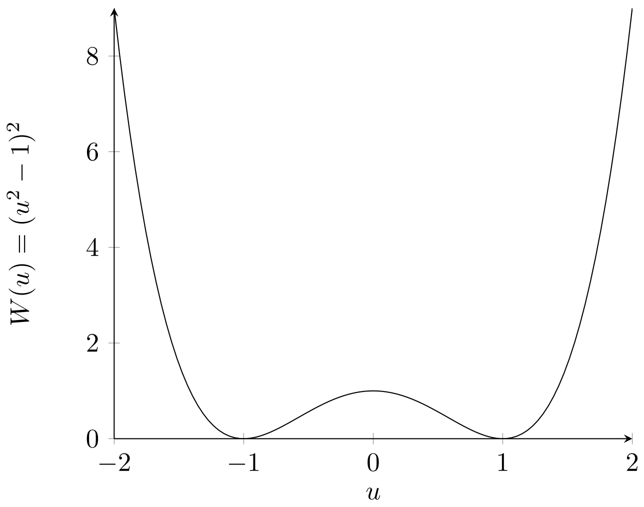

Figure 1.1: The graph of a doublewell potential

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \tag{1.4}$$

for all u sufficiently large. Lastly we want W to be convex up to a small perturbation in the sense that there exist smooth functions $W_{\text{conv}}, W_{\text{pert}} : \mathbb{R}^N \to [0, \infty)$ such that

$$W = W_{\text{conv}} + W_{\text{pert}}\,, \tag{1.5}$$

W_{conv} is convex and

$$\sup_{x \in \mathbb{R}^N} \left|\nabla^2 W_{\text{pert}}\right| < \infty. \tag{1.6}$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen-Cahn equation (1.1) as maps from $[0, T]$ into some suitable function space and thus use the following definition.

Definition 1.1.1. *We say that a function $u_\varepsilon \in \mathrm{C}\left([0,T]; \mathrm{L}^2\left(\mathbb{T}; \mathbb{R}^N\right)\right)$ which is also in $\mathrm{L}^\infty\left([0,T]; \mathrm{W}^{1,2}(\mathbb{T}; \mathbb{R}^N)\right)$ is a weak solution of the Allen–Cahn equation (1.1) with parameter $\varepsilon > 0$ and initial condition $u_\varepsilon^0 \in \mathrm{L}^2(\mathbb{T}; \mathbb{R}^N)$ if*

1. *the energy stays bounded, which means that*

$$\operatorname*{ess\,sup}_{0 \le t \le T} \mathrm{E}_\varepsilon(u_\varepsilon(t)) < \infty, \tag{1.7}$$

2. *its weak time derivative satisfies*

$$\partial_t u_\varepsilon \in \mathrm{L}^2\left([0,T] \times \mathbb{T}; \mathbb{R}^N\right), \tag{1.8}$$

3. *for almost every $t \in [0,T]$ and every $\xi \in \mathrm{L}^p([0,T]\times\mathbb{T};\mathbb{R}^N)\cap \mathrm{W}^{1,2}([0,T]\times\mathbb{T};\mathbb{R}^N)$, we have*

$$\int \left\langle \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon(t))\,,\,\xi \right\rangle + \langle \nabla u_\varepsilon(t)\,,\,\nabla\xi\rangle + \langle \partial_t u_\varepsilon(t)\,,\,\xi\rangle \,\mathrm{d}x = 0, \tag{1.9}$$

4. *the initial conditions are achieved in the sense that $u_\varepsilon(0) = u_\varepsilon^0$.*

Remark. Given our assumptions, we automatically have $\nabla W(u_\varepsilon)(t) \in \mathrm{L}^{p'}(\mathbb{T})$ since

$$|\nabla W(u_\varepsilon)|^{p/(p-1)} \lesssim 1 + |u_\varepsilon|^p \lesssim 1 + W(u), \tag{1.10}$$

which is integrable for almost every time t since we assume that the energy stays bounded, thus the integral in equation (1.9) is well defined.

Moreover we already obtain $1/2$ Hölder-continuity in time from the embedding

$$\mathrm{W}^{1,2}([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)) \hookrightarrow \mathrm{C}^{1/2}([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)), \tag{1.11}$$

which follows from a generalized version of the fundamental theorem of calculus and Hölder's inequality.

1.2 Existence of a solution

With Definition 1.1.1, we are able to state our existence results for solutions of the Allen–Cahn equation. The proof uses De Giorgis minimizing movements scheme and arguments from the theory of gradient flows and has been briefly sketched in [LS16].

Theorem 1.2.1. *Let $u_\varepsilon^0\colon \mathbb{T}\to\mathbb{R}^N$ be such that $\mathrm{E}_\varepsilon(u_\varepsilon^0) < \infty$. Then there exists a weak solution u_ε to the Allen–Cahn equation (1.1) in the sense of Definition 1.1.1 with initial data u_ε^0. Furthermore the solution satisfies the energy dissipation inequality*

$$\mathrm{E}_\varepsilon(u_\varepsilon(t)) + \int_0^t \int \varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s \le \mathrm{E}_\varepsilon(u_\varepsilon^0) \tag{1.12}$$

for every $t\in[0,T]$ and we additionally have $\partial^2_{i,j}u_\varepsilon, \nabla W(u_\varepsilon) \in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$ for all $1\le i,j\le d$. In particular we can test the weak form (1.9) with $\partial^2_{i,j}u_\varepsilon$.

Proof. **Step 1:** A minimization problem
Fix some $h>0$, $u_{n-1}\in \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$ and consider the functional

$$\begin{aligned}\mathcal{F}\colon\ \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N) &\to \mathbb{R} \\ u &\mapsto \mathrm{E}_\varepsilon(u) + \frac{1}{2h}\int |u-u_{n-1}|^2\,\mathrm{d}x\,.\end{aligned} \tag{1.13}$$

Then $\mathcal{F}$ is coercive with respect to $\|\cdot\|_{\mathrm{W}^{1,2}}$ and bounded from below by zero, thus we may take a $\mathrm{W}^{1,2}$-bounded sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$ such that $\mathcal{F}(v_k)\to\inf\mathcal{F}$

as $k \to \infty$. These v_k have a non-relabelled subsequence which converges weakly in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$ and strongly in $\mathrm{L}^2(\mathbb{T},\mathbb{R}^N)$ to some $u \in \mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$. Moreover v_k is bounded in $\mathrm{L}^p(\mathbb{T};\mathbb{R}^N$ by the lower growth assumption (1.3) on W and thus obtain by a duality argument that $u \in \mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$.

Lastly we have

$$\frac{1}{2h}\int |u-u_{n-1}|^2 \,\mathrm{d}x \le \liminf_{k\to\infty} \frac{1}{2h}\int |v_k-u_{n-1}|^2 \,\mathrm{d}x\,, \tag{1.14}$$

$$\int \frac{\varepsilon}{2}|\nabla u|^2 \,\mathrm{d}x \le \liminf_{k\to\infty} \int \frac{\varepsilon}{2}|\nabla v_k|^2 \,\mathrm{d}x \tag{1.15}$$

by the weak convergence in $\mathrm{W}^{1,2}(\mathbb{T},\mathbb{R}^N)$. By passing to another non-relabelled subsequence which converges pointwise almost everywhere, we moreover achieve by continuity of W that

$$\int \frac{1}{\varepsilon}W(u)\,\mathrm{d}x = \int \liminf_{k\to\infty} \frac{1}{\varepsilon}W(v_k)\,\mathrm{d}x \le \liminf_{k\to\infty}\int \frac{1}{\varepsilon}W(v_k)\,\mathrm{d}x\,,$$

which proves that u is a minimizer of $\mathcal{F}$.

Step 2: Minimizing movements scheme

By iteratively choosing minimizers u_n^h from Step 1, we obtain a sequence of functions $u_\varepsilon^0, u_1^h, \dots$. Thus we may define a function $u \in \mathrm{C}$ as the piecewise linear interpolation at the time-steps $0, h, 2h, \dots$ of these functions.

Step 3: Sharp energy dissipation inequality for u_n^h

We claim that there exists some constant $C > 0$ such that for all $h > 0$ and $n \in \mathbb{N}$, we have

$$\mathrm{E}_\varepsilon(u_n^h) + \left(\frac{1}{h} - \frac{C}{2\varepsilon}\right)\int \left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x \le \mathrm{E}_\varepsilon(u_{n-1}^h). \tag{1.16}$$

In order to prove this inequality, we notice that since $\left|\nabla^2 W_{\mathrm{pert}}\right| \le C$, the function $W_{\mathrm{pert}} + C|u|^2/2$ is convex for $C > 0$ sufficiently large, thus the functional

$$\tilde{\mathrm{E}}_\varepsilon(u) \coloneqq \int \frac{1}{\varepsilon}\left(W(u) + \frac{C}{2}|u|^2\right) + \frac{\varepsilon}{2}|\nabla u|^2 \,\mathrm{d}x$$

is convex on $\mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$. For a given $\xi \in \mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$, we thus have that the function $t \mapsto \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi)$ is convex and differentiable, which yields that

$$\tilde{\mathrm{E}}_\varepsilon(u_n^h + \xi) \ge \tilde{\mathrm{E}}_\varepsilon(u_n^h) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi). \tag{1.17}$$

But since u_n^h is a minimizer of the functional $\mathcal{F}$ defined by (1.13), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathcal{F}(u_n^h + t\xi) \\
&\quad + \frac{C}{2\varepsilon}\int \left|u_n^h + t\xi\right|^2 \mathrm{d}x - \frac{1}{2h}\int \left|u_n^h + t\xi - u_{n-1}^h\right|^2 \mathrm{d}x \\
&= \int \frac{C}{\varepsilon}\langle u_n^h\,,\,\xi\rangle - \frac{1}{h}\langle u_n^h - u_{n-1}^h\,,\,\xi\rangle\,\mathrm{d}x\,.
\end{aligned}$$

Plugging $\xi = u_{n-1}^h - u_n^h$ into this equation and using inequality (1.17) thus yields

$$\tilde{\mathrm{E}}_\varepsilon(u_{n-1}^h) \geq \tilde{\mathrm{E}}_\varepsilon(u_n^h) + \int \frac{C}{\varepsilon}\left(\langle u_{n-1}^h\,,\,u_n^h\rangle - \left|u_n^h\right|^2\right) + \frac{1}{h}\left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x\,,$$

which is equivalent to

$$\begin{aligned}
&\mathrm{E}_\varepsilon(u_{n-1}^h) \geq \mathrm{E}_\varepsilon(u_n^h) \\
&\qquad + \int \left(\frac{C}{2\varepsilon} - \frac{C}{\varepsilon}\right)\left|u_n^h\right|^2 - \frac{C}{2\varepsilon}\left|u_{n-1}^h\right|^2 + \frac{C}{\varepsilon}\langle u_{n-1}^h\,,\,u_n^h\rangle + \frac{1}{h}\left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x \\
&\quad = \mathrm{E}_\varepsilon(u_n^h) + \left(\frac{1}{h} - \frac{C}{2\varepsilon}\right)\int \left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x\,,
\end{aligned}$$

which is the claimed estimate (1.16).

Step 4: Hölder bounds for u_h

From the energy estimate (1.16) we deduce via an induction that

$$\begin{aligned}
&\mathrm{E}_\varepsilon(u_n^h) + \left(1 - \frac{Ch}{2\varepsilon}\right)\int_0^{nh}\int \left|\partial_t u^h\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&= \mathrm{E}_\varepsilon(u_n^h) + \left(h - \frac{Ch^2}{2\varepsilon}\right)\sum_{k=1}^{n}\int \left|\frac{u_n^h - u_{n-1}^h}{h}\right|^2 \mathrm{d}x \\
&\leq \mathrm{E}_\varepsilon(u_\varepsilon^0).
\end{aligned} \tag{1.18}$$

This gives use with the use of Jensen's inequality for $0 \leq s \leq t \leq T$ and $h > 0$ sufficiently small that

$$\begin{aligned}
\left\|u^h(t) - u^h(s)\right\|_{\mathrm{L}^2} &= \left\|\int_s^t \partial_t u_h(\tau)\,\mathrm{d}\tau\right\|_{\mathrm{L}^2} \\
&\leq \sqrt{t-s}\left(\int_0^T\int \left|\partial_t u^h(\tau,x)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
&\leq \sqrt{t-s}\left(1 - \frac{Ch}{2\varepsilon}\right)^{-1/2}\left(E_\varepsilon(u_\varepsilon^0)\right)^{1/2},
\end{aligned} \tag{1.19}$$

which gives us a uniform bound on the L^2-Hölder continuity of u^h in time as h tends to zero.

Step 5: Compactness

In order to apply Arzelà-Ascoli for the sequence (u_h) as h tends to zero, we need to check pointwise precompactness of the image and equicontinuity of the sequence. The equicontinuity follows from the previous estimate (1.19). In order to check the pointwise precompactness, we need to verify that for all $t \in [0,T]$, the set $\{u^h(t)\}_{\delta > h > 0}$ is a precompact subset of $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ (for $\delta > 0$ sufficiently small). But this follows from the energy bound $\mathrm{E}_\varepsilon(u_n^h) \leq \mathrm{E}_\varepsilon(u_0)$ (given by inequality (1.16)), which gives us a time-uniform bound on $\left\|\nabla u^h(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)}$ and on $\left\|u^h(t)\right\|_{\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)}$ and thus on $\left\|u^h(t)\right\|_{\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)}$. The compact embedding $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N) \hookrightarrow \mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ thus yields the desired pointwise precompactness.

Therefore we may apply Arzelà–Ascoli to obtain some $u \in \mathrm{C}^{1,2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N\right)$ and some sequence $h_n \to 0$ such that u^{h_n} converges uniformly to u on $[0,T]$ with respect to $\|\cdot\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)}$.

Step 6: Additional regularity 1

We first want to argue that from our construction, one already obtains that $u \in \mathrm{L}^\infty\left([0,T];\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\right)$ For this we first notice that for a fixed $t \in [0,T]$, the sequence $u^{h_n}(t)$ is by the energy bound (1.16) a bounded sequence in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$, thus we find some non-relabelled subsequence and $v \in \mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$ such that $u^{n_h}(t)$ converges weakly to v in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$. By uniqueness of the limit, we already have $u(t) = v$ almost everywhere, which yields $u(t) \in \mathrm{W}^{1,2}(\mathbb{T},\mathbb{R}^N)$, and by lower semicontinuity, we may also deduce that

$$\|u(t)\|_{\mathrm{W}^{1,2}} \leq \liminf_{n\to\infty} \left\|u^{h_n}(t)\right\|_{\mathrm{W}^{1,2}} \leq C\,\mathrm{E}_\varepsilon,$$

from which we deduce that $u \in \mathrm{L}^\infty\left([0,T];\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\right)$.

Secondly the boundedness of the energies

$$\sup_{0\leq t\leq T} \mathrm{E}_\varepsilon(u(t)) < \infty$$

follows from the lower semicontinuity of the energy and the pointwise L^2 convergence and pointwise weak convergence in $\mathrm{W}^{1,2}$ as described in step 1.

Lastly we want to argue that $\partial_t u \in \mathrm{L}^2\left([0,T]\times\mathbb{T};\mathbb{R}^N\right)$. From inequality (1.18) in step 4, we deduce that $\partial_t u^h$ is a bounded sequence in $\mathrm{L}^2[0,T]\times\mathbb{T};\mathbb{R}^N)$. Thus we find a non-relabelled subsequence of h_n and some $w \in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$ such that u^{h_n} converges weakly to w in $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. But then w is already the weak time derivative of u since for any testfunction ξ, we have

$$\begin{aligned}
\int_{[0,T]\times\mathbb{T}} \langle u\,,\,\partial_t\xi\rangle\,\mathrm{d}x\,\mathrm{d}t &= \lim_{n\to\infty}\int_{[0,T]\times\mathbb{T}} \left\langle u^{h_n}\,,\,\partial_t\xi\right\rangle\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{n\to\infty} -\int_{[0,T]\times\mathbb{T}} \left\langle \partial_t u^{h_n}\,,\,\xi\right\rangle\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{[0,T]\times\mathbb{T}} \langle w\,,\,\xi\rangle\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

Step 7: u is a weak solution

Going back to step 1, we see that u_n^h solves the Euler–Lagrange equation

$$\int \frac{1}{\varepsilon}\big\langle \nabla W(u_n^h)\,,\,\xi\big\rangle + \varepsilon\big\langle \nabla u_n^h\,,\,\nabla\xi\big\rangle + \left\langle \frac{u_n^h - u_{n-1}^h}{h}\,,\,\xi\right\rangle \mathrm{d}x = 0 \tag{1.20}$$

for any testfunction $\xi \in \mathrm{C}^\infty(\mathbb{T};\mathbb{R}^N)$. Let $t \in [0,T]$. Since u_h is defined as the pointwise linear interpolation of the functions u_n^h, we find for the sequence h_n corresponding sequences $\lambda_n \in [0,1]$ and $k_n \in \mathbb{N}$ such that

$$t = \lambda_n(k_n-1)h_n + (1-\lambda_n)k_n h_n$$

and therefore we can write

$$u_h(t) = \frac{k_n h_n - t}{h_n} u_{k_n-1}^h + \frac{t-(k_n-1)h_n}{h_n} u_{k_n}^{h_n}.$$

In order to pass to the limit in equation (1.20), we first note that $u_{k_n}^{h_n}$ converges to $u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ since

$$\begin{aligned}\left\|u(t) - u_{k_n}^{h_n}\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} &\le \left\|u(t) - u^{h_n}(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} + \left\|u^{h_n}(t) - u^{h_n}(k_n h_n)\right\|_{\mathrm{L}^2(\mathbb{T},\mathbb{R}^N)} \\ &\lesssim \left\|u(t) - u^{h_n}(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} + \sqrt{h_n}, \qquad (1.19)\end{aligned}$$

which goes to zero as n tends to infinity. This implies that

$$\begin{aligned}\int \varepsilon\langle \nabla u(t)\,,\,\nabla\xi\rangle\,\mathrm{d}x &= \int \varepsilon\langle u(t)\,,\,\operatorname{div}\nabla\xi\rangle\,\mathrm{d}x \\ &= \lim_{n\to\infty}\int \varepsilon\Big\langle u_{k_n}^{h_n}\,,\,\operatorname{div}\nabla\xi\Big\rangle\,\mathrm{d}x \\ &= \lim_{n\to\infty}\int \varepsilon\Big\langle \nabla u_{k_n}^{h_n}\,,\,\nabla\xi\Big\rangle\,\mathrm{d}x\,.\end{aligned}$$

In step 6, we moreover have shown that $\partial_t u^h$ converges weakly to $\partial_t u$ in $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. This yields, by choosing cylindrical testfunctions, that $\partial_t u^h(t)$ converges weakly to $\partial_t u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ for almost every $t \in [0,T]$. Thus we obtain for almost every $t \in [0,T]$ the convergence

$$\int \langle \partial_t u(t)\,,\,\xi\rangle\,\mathrm{d}x = \lim_{n\to\infty}\int \langle \partial_t u(t)\,,\,\xi\rangle\,\mathrm{d}x = \lim_{n\to\infty}\int \left\langle \frac{u_{k_n}^{h_n} - u_{k_n-1}^{h_n}}{h_n}\,,\,\xi\right\rangle \mathrm{d}x\,.$$

To obtain the weak equation, we still need to prove convergence of remaining term. For this we note that

$$\frac{1}{\varepsilon}\left|\Big\langle \nabla W(u_{k_n}^{h_n})\,,\,\xi\Big\rangle\right| \lesssim \left(1 + \left|u_{k_n}^{h_n}\right|^{p-1}\right)|\xi|,$$

which is a bounded sequence in $\mathrm{L}^{p'}(\mathbb{T};\mathbb{R}^N)$. Since we also may pass to a non-relabelled subsequence which converges almost everywhere, we thus obtain that

$$\int \langle \nabla W(u(t)), \xi \rangle \,\mathrm{d}x = \lim_{n\to\infty} \int \left\langle \nabla W(u_{k_n}^{h_n}), \xi \right\rangle \mathrm{d}x .$$

Therefore it follows from the Euler–Lagrange equation (1.20) that

$$\int \frac{1}{\varepsilon} \langle \nabla W(u(t)), \xi \rangle + \varepsilon \langle \nabla u, \nabla \xi \rangle + \langle \partial_t u, \xi \rangle \,\mathrm{d}x = 0$$

for all $\xi \in \mathrm{C}^\infty(\mathbb{T};\mathbb{R}^N)$. By continuity this extends to all $\xi \in \mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$.
Thus the last thing to check is that

$$\sup_{0\le t\le T} \mathrm{E}_\varepsilon(u(t)) < \infty,$$

which follows from the next step.

Step 8: Sharp energy dissipation inequality for u
Let $0 \le t \le T$ and define k_n as in step 7, where we also established that $u_{k_n}^{h_n}$ converges to $u(t)$ in $\mathrm{L}^2\left(\mathbb{T};\mathbb{R}^N\right)$, and thus we may pass to another non-relabelled subsequence to obtain pointwise convergence almost everywhere. By Fatou's Lemma, we thus obtain

$$\int \frac{1}{\varepsilon} W(u(t)) \,\mathrm{d}x \le \liminf_{n\to\infty} \int \frac{1}{\varepsilon} W(u_{k_n}^{h_n}) \,\mathrm{d}x .$$

Moreover we can deduce from the L^2-convergence that $\nabla u_{k_n}^{h_n}$ converges to $u(t)$ in the distributional sense. But $\nabla u_{k_n}^{h_n}$ is uniformly bounded in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ by the energy dissipation inequality (1.16), thus we already obtain that $\nabla u_{k_n}^{h_n}$ converges weakly to $\nabla u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ which yields

$$\int \frac{\varepsilon}{2} |\nabla u(t)|^2 \,\mathrm{d}x \le \liminf_{n\to\infty} \int \frac{\varepsilon}{2} \left|\nabla u_{k_n}^{h_n}\right|^2 \mathrm{d}x .$$

Lastly we have by the weak convergence of $\partial_t u^{h_n}$ to $\partial t u$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ proven in step 6 that

$$\begin{aligned} \int_0^t \int |\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{n\to\infty} \left(1 - \frac{Ch_n}{2\varepsilon}\right) \int_0^t \int \left|\partial_t u^{h_n}\right|^2 \mathrm{d}x\,\mathrm{d}t \\ &\le \liminf_{n\to\infty} \left(1 - \frac{Ch_n}{2\varepsilon}\right) \int_0^{k_n h_n} \int \left|\partial_t u^{h_n}\right|^2 \mathrm{d}x\,\mathrm{d}t . \end{aligned}$$

Summarizing these estimates, we obtain by the energy dissipation inequality for u^h (1.18) that for all $0 \le t \le T$ we have

$$\begin{aligned} \mathrm{E}_\varepsilon(u(t)) + \int_0^t \int |\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{n\to\infty} \mathrm{E}_\varepsilon(u_{k_n}) + \left(1 - \frac{Ch_n}{2\varepsilon}\right) \int_0^{k_n h_n} \int \left|\partial_t u^h\right|^2 \mathrm{d}x\,\mathrm{d}t \\ &\le \mathrm{E}_\varepsilon(u_\varepsilon^0). \end{aligned}$$

Step 8: Additional regularity 2

In order to complete the proof, we still have to show that $\partial^2_{i,j} u$ and$\nabla W(u)$ are elements of $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. In order to show that the second partial derivatives of u are square-integrable, we test the weak formulation (1.9) with the finite differences. To this end, define the finite differences as

$$\Delta_h^+ v(t,x) := \frac{v(t,x+hv)-v(t,x)}{h}, \quad \Delta_h^- v(t,x) := \frac{v(t,x-hv)-v(t,x)}{h}$$

for some $h>0$ and $v\in\mathbb{R}^d$. Thus plugging $\Delta_h^-\Delta_h^+ u$ into (1.9) yields by the transformation formula that

$$0 = \int_0^T \int \frac{1}{\varepsilon^2}\left\langle \Delta_h^+ \nabla W(u)\,,\, \Delta_h^+ u\right\rangle + \left\langle \nabla\Delta_h^+ u\,,\, \nabla\Delta_h^+ u\right\rangle + \left\langle \partial_t \Delta_h^+ u\,,\, \Delta_h^+ u\right\rangle \mathrm{d}x\,\mathrm{d}t\,,$$

which is equivalent to

$$\begin{aligned}
&\int_0^T \int \left|\Delta_h^+ \nabla u\right|^2 \mathrm{d}x\,\mathrm{d}t \\
=&\ -\int_0^T \int \partial_t \left(\frac{\left|\Delta_h^+ u\right|^2}{2}\right) + \frac{1}{\varepsilon^2}\left\langle \Delta_h^+ \nabla W(u)\,,\, \Delta_h^+ u\right\rangle \mathrm{d}x\,\mathrm{d}t \\
=&\ \int \frac{\left|\Delta_h^+ u(0)\right|^2 - \left|\Delta_h^+ u(T)\right|^2}{2}\,\mathrm{d}x \\
&\ -\int_0^1\int_0^T\int \left\langle \mathrm{D}^2\, W\left((1-s)u(t,x)+su(t,x+hv)\right)\right)\Delta_h^+ u\,,\, \Delta_h^+ u\right\rangle \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\,.
\end{aligned}$$

The first summand can be estimated by $\|u\|_{\mathrm{L}^\infty([0,T],\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N))}$. For the second summand, we partition W into the sum of W_{conv} and W_{pert}. The term involving the convex summand can then by estimated by

$$\int_0^1\int_0^T\int \left\langle \mathrm{D}^2\, W_{\mathrm{conv}}\left((1-s)u(t,x)+su(t,x+hv)\right)\Delta_h^+ u\,,\, \Delta_h^+ u\right\rangle \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \geq 0$$

and the for the pertubation term, we get via the bound on its second derivative that

$$\begin{aligned}
&\left|\int_0^1\int_0^T\int \left\langle \mathrm{D}^2\, W_{\mathrm{pert}}\left((1-s)u(t,x)+su(t,x+hv)\right)\Delta_h^+ u\,,\, \Delta_h^+ u\right\rangle \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\right| \\
\lesssim&\ \int_0^T\int |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}$$

which is also finite. Combining these estimates, we obtain that $\int_0^T\int\left|\Delta_h^+\nabla u\right|^2\mathrm{d}x\,\mathrm{d}t$ is uniformly bounded in h. Applying our calculation to all directions $v\in\mathbb{R}^d$, we get by the finite-differences theorem for all $1\leq i,j\leq d$ that $\partial^2_{i,j}u\in\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$.

In order to obtain $\nabla W(u) \in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$, we again consider the weak formulation (1.9) and notice that since we have already shown that both the time derivative and second space derivatives of u are square-integrable, our claim follows from a duality argument.

$\square$

Remark. The inequality (1.16) with the factor $1/2h$ instead of $1/h - C/2\varepsilon$ follows immediately from the definition of our optimization problem, but is not optimal for fixed ε if we want to study the behaviour as h tends to zero. Moreover this so called *sharp energy dissipation inequality* is important for later.

Remark. The energy dissipation inequality (1.12) can be deduced via the formal calculation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon}\langle \nabla W(u)\,,\,\partial_t u\rangle + \varepsilon\langle \nabla u\,,\,\nabla\partial_t u\rangle\,\mathrm{d}x\\ &= \int \left\langle \frac{1}{\varepsilon}\nabla W(u) - \varepsilon\Delta u\,,\,\partial_t u\right\rangle \mathrm{d}x\\ &= -\varepsilon\int |\partial_t u_\varepsilon|^2\,\mathrm{d}x\,.\end{aligned}$$

In order to make this calculation rigorous, we however need to show that $t \mapsto \mathrm{E}_\varepsilon(u(t))$ is absolutely continuous, which is non-trivial, but it would give us equality in energy dissipation inequality (1.12) Since we will only need the inequality, our proof will however suffice.

2 Convergence of the Allen–Cahn equations

The main content of this chapter is to look at the behaviour of solutions of the Allen–Cahn equation (1.1) as the parameter ε tends to zero. As it turns out the scalar case is significantly easier to handle than the vectorial case, thus we shall first focus on the case $N = 1$, also called the *twophase case*.

2.1 Convergence in the twophase case

When want to consider the behaviour of solutions $u_\varepsilon \colon [0,T] \times \mathbb{T} \to \mathbb{R}$ to (1.1) for $\varepsilon \to 0$. Let us for now assume that the energies of the initial functions $\mathrm{E}_\varepsilon(u_\varepsilon^0)$ stay uniformly bounded as ε tends to zero. Then due to the energy dissipation inequality (1.12), we already obtain that for all $0 \le t \le T$, we have that $\mathrm{E}_\varepsilon\left(u_\varepsilon(t)\right)$ stays uniformly bounded.

Another important observation for the convergence is the classic Modica Mortula trick (hier Referenz einfügen): let $\alpha < \beta$ be the two distinct zeros of the doublewell potential W. Then we define a primitive of $\sqrt{2W(u)}$ via

$$\phi(u) \coloneqq \int_\alpha^u \sqrt{2W(s)}\,\mathrm{d}s\,.$$

For $\psi_\varepsilon \coloneqq \phi \circ u_\varepsilon$, we can show that $\psi_\varepsilon \in \mathrm{W}^{1,1}((0,T) \times \mathbb{T})$ with weak derivatives $\nabla\psi_\varepsilon = \sqrt{2W(u_\varepsilon)}\nabla u_\varepsilon$ and $\partial_t\psi_\varepsilon = \sqrt{2W(u_\varepsilon)}\partial_t u_\varepsilon$. (we will later show this in more generality (Referenz einfügen)).

Thus via Young's inequality, we can compute

$$\begin{aligned} \mathrm{E}_\varepsilon(u_\varepsilon) &= \int \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\,\mathrm{d}x \\ &\ge \int \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x \\ &= \int |\nabla\psi_\varepsilon|\,\mathrm{d}x\,, \end{aligned} \tag{2.1}$$

which suggests that we might hope for good compactness properties of $\phi \circ u_\varepsilon$. We combine these two observations into the following Proposition.

Proposition 2.1.1. *Given initial data u_ε^0 whose energies stay uniformly bounded in the sense that*

$$\sup_{\varepsilon > 0} \mathrm{E}_\varepsilon(u_\varepsilon^0) < \infty, \tag{2.2}$$

there exists for any sequence $\varepsilon \to 0$ some non-relabelled subsequence such that the solutions of the Allen–Cahn equation (1.1) with initial condition u_ε^0 converge in $\mathrm{L}^1((0,T)\times\mathbb{T})$ to some $u = \alpha(1-\chi)+\beta\chi$ with $\chi \in \mathrm{BV}((0,T)\times\mathbb{T};\{0,1\})$. Moreover the compositions ψ_ε are uniformly bounded in $\mathrm{BV}((0,T)\times\mathbb{T})$ and converge to $\phi\circ u$ in $\mathrm{L}^1((0,T)\times\mathbb{T})$.

Proof. From the energy dissipation inequality (1.12) in Theorem 1.2.1 we infer that for all $\varepsilon > 0$, it holds that

$$\sup_{0\le t\le T} \mathrm{E}_\varepsilon(u_\varepsilon(t)) \le \mathrm{E}_\varepsilon(u_\varepsilon^0), \tag{2.3}$$

whose right hand side is by assumption uniformly bounded in ε. We want to use a similar calculation as (2.1) we thus infer that $\nabla\psi_\varepsilon$ is uniformly bounded in $\mathrm{L}^1((0,T)\times\mathbb{T})$. Moreover we may estimate

$$\begin{aligned}\int_0^T\int |\psi_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int\left|\int_\alpha^{u_\varepsilon}\sqrt{2W(s)}\right|\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_0^T\int |u_\varepsilon-\alpha| \sup_{s\in[\alpha,u_\varepsilon]}\sqrt{2W(s)}\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim 1+\int_0^T\int |u_\varepsilon|^{1+p/2}\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim 1+\int_0^T\int W(u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned} \tag{2.4}$$

which is uniformly bounded via the energy bound. Here the last two inequalities follow from the upper respectively lower growth assumptions (1.3) on W. Thus we have indeed that ψ_ε is a bounded sequence in $\mathrm{BV}((0,T)\times\mathbb{T})$ and therefore there exists some non-relabelled subsequence and some $\psi\in\mathrm{BV}((0,T)\times\mathbb{T})$ such that ψ_ε converges to ψ in $\mathrm{L}^1((0,T)\times\mathbb{T})$.

We notice that since W is non-negative and only has a discrete set of zeros, the function ϕ is strictly increasing and continuous on $\mathbb{R}$, and is thus invertible. Moreover we may pass to a further non-relabelled subsequence of ψ_ε which converges almost everywhere to ψ. Thus defining $u := \phi^{-1}(\psi)$, we obtain that

$$u_\varepsilon = \phi^{-1}(\psi_\varepsilon) \to \phi^{-1}(\psi) = u$$

converges pointwise almost everywhere.

Moreover we notice that by Fatou's Lemma and the boundedness of the energies that

$$\int W(u)\,\mathrm{d}x \le \liminf_{\varepsilon\to 0}\int W(u_\varepsilon)\,\mathrm{d}x \le \liminf_{\varepsilon\to0}\varepsilon\,\mathrm{E}_\varepsilon(u_\varepsilon) = 0.$$

Again using the non-negativity of W, this yields that $W(u)=0$ almost everywhere. Thus $u\in\{\alpha,\beta\}$ almost everywhere and we may write $u=\alpha(1-\chi)+\beta\chi$ for some $\chi\colon(0,T)\times\mathbb{T}\to\{0,1\}$. Looking again at the definition of u, we moreover obtain that

$$\psi = \phi(u) = \phi(\alpha)(1-\chi)+\phi(\beta)\chi = \int_\alpha^\beta\sqrt{2W(s)}\,\mathrm{d}s\,\chi =: \sigma\chi \tag{2.5}$$

and since ψ is a function of bounded variation, this implies that χ is of bounded variation as well.

Finally from the energy bound and the estimate $|u_\varepsilon| \lesssim 1 + W(u)$, we infer that u_ε is L^p-bounded, and since u_ε converges pointwise almost everywhere to u, we obtain the desired L^1-convergence. □

Nextup, we want to make sure that u respectively χ assume their initial data and on the way obtain a useful bound on the time derivative.

Lemma 2.1.2. *With the assumptions of Proposition 2.1.1, we have $\psi_\varepsilon \in \mathrm{W}^{1,2}([0,T];\mathrm{L}^1(\mathbb{T}))$ with the estimate*

$$\left(\int_0^T \left(\int |\partial_t \psi_\varepsilon| \,\mathrm{d}x\right)^2 \mathrm{d}t\right)^{1/2} \lesssim \mathrm{E}_\varepsilon(u_\varepsilon^0). \tag{2.6}$$

Furthermore the sequence u_ε is precompact in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$.

Proof. **Step 1:** $\psi_\varepsilon \in \mathrm{W}^{1,2}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$ and satisfies the inequality (2.6)

From estimate (2.4) we can infer that

$$\int_0^T \left(\int |\psi_\varepsilon| \,\mathrm{d}x\right)^2 \lesssim \int_0^T \left(1 + \int W(u_\varepsilon) \,\mathrm{d}x\right)^2 \mathrm{d}t < \infty$$

via the uniform boundedness of the energies (2.3). For the desired bound (2.6), we estimate via Hölder's inequality and the uniform boundedness of the energies that

$$\begin{aligned}
\int_0^T \left(\int |\partial_t \psi_\varepsilon| \,\mathrm{d}x\right)^2 \mathrm{d}t &\leq \int_0^T \left(\int \sqrt{2W(u_\varepsilon)}|\partial_t u_\varepsilon| \,\mathrm{d}x\right)^2 \mathrm{d}t \\
&= \int_0^T \left(\int \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\sqrt{\varepsilon}|\partial_t u_\varepsilon| \,\mathrm{d}x\right)^2 \mathrm{d}t \\
&\leq \int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon) \,\mathrm{d}x \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\leq 2\,\mathrm{E}_\varepsilon(u_\varepsilon^0) \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t && (2.3) \\
&\leq 2\left(\mathrm{E}_\varepsilon(u_\varepsilon^0)\right)^2, && (1.12)
\end{aligned}$$

which completes step 1.

Step 2: The sequence ψ_ε is precompact in $\mathrm{C}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$.

As noted in embedding (1.11), we have

$$\mathrm{W}^{1,2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right) \hookrightarrow \mathrm{C}^{1/2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)$$

from which the equicontinuity of the sequence follows with step 1. Moreover for fixed time t, estimate (2.1) yields that $\nabla\psi_\varepsilon(t)$ is bounded in $\mathrm{L}^1(\mathbb{T})$, and combined estimate (2.4) without integrating in time yields that $\psi_\varepsilon(t)$ is a bounded sequence in $\mathrm{W}^{1,1}(\mathbb{T})$ and thus precompact in $\mathrm{L}^1(\mathbb{T})$. Thus the Arzelà–Ascoli Theorem yields the desired precompactness.

Step 3: The sequence u_ε converges to u in measure uniformly in time.

From step 2, we infer that we already have

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int |\psi_\varepsilon(t,x) - \phi\circ u(t,x)| \,\mathrm{d}x = 0.$$

It especially follows in combination with equation (2.5) that

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int (1-\chi)|\psi_\varepsilon(t,x)| \,\mathrm{d}x = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int (1-\chi)|\psi_\varepsilon(t,x) - \phi\circ u(t,x)| \,\mathrm{d}x = 0$$

and

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int \chi|\psi_\varepsilon(t,x) - \sigma| \,\mathrm{d}x = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int \chi|\psi_\varepsilon(t,x) - \phi\circ u(t,x)| \,\mathrm{d}x = 0$$

Moreover we have by continuity and the p-growth of W (1.3) that for a given $\delta < \beta - \alpha$,

$$\min\{|\phi(t)| : \ |t-\alpha| > \delta/2 \text{ or } |t-\beta| > \delta/2\} = \rho > 0.$$

Therefore we may estimate

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\{x\in\mathbb{T} : \ |u_\varepsilon(t,x) - u(t,x)| > \delta\}\right)\\
\le\ &\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\{(1-\chi(t,x))|u_\varepsilon(t,x)-\alpha| > \delta/2\}\right) + \mathcal{L}^d\left(\{\chi(t,x)|u_\varepsilon(t,x)-\beta| > \delta/2\}\right)\\
\le\ &\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\{(1-\chi(t,x))|\psi_\varepsilon(t,x)| > \rho\}\right) + \mathcal{L}^d\left(\{\chi(t,x)|\psi_\varepsilon(t,x)-\sigma| > \rho\}\right)\\
\le\ &\operatorname*{ess\,sup}_{0\le t\le T} \frac{1}{\rho}\int (1-\chi(t,x))|\psi_\varepsilon(t,x)| \,\mathrm{d}x + \frac{1}{\rho}\int \chi(t,x)|\psi_\varepsilon(t,x)-\sigma| \,\mathrm{d}x\,,
\end{aligned}$$

which goes to zero as ε goes to zero, proving our claim.

Step 4: u_ε^2 is equiintegrable uniformly in time

We have

$$0 \le u_\varepsilon^2 \lesssim 1 + W(u_\varepsilon)$$

by the growth bounds (1.3) on W. Since $W(u_\varepsilon)$ converges to 0 in $\mathrm{L}^1(\mathbb{T})$ uniformly in time by the energy bound (2.3), it is equiintegrable uniformly in time. Thus u_ε^2 is equiintegrable uniformly in time as well.

Step 5: u_ε converges in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$.

We somewhat repeat the proof the convergence in measure and integrability imply L^1-convergence, while keeping the uniformity in time.

Take some $\delta > 0$. Then we decompose the integral

$$\int |u_\varepsilon - u|^2 \,\mathrm{d}x = \int_{\{|u_\varepsilon - u|\ge\delta\}} |u_\varepsilon - u|^2 \,\mathrm{d}x + \int_{\{|u_\varepsilon - u|<\delta\}} |u_\varepsilon - u|^2 \,\mathrm{d}x\,.$$

For the first summand, we notice that

$$\sup_{0\le t\le T}\int_{\{|u_\varepsilon-u|\ge\delta\}}|u_\varepsilon-u|^2\,\mathrm{d}x \lesssim \sup_{0\le t\le T}\int_{\{|u_\varepsilon-u|\ge\delta\}}1+u_\varepsilon^2\,\mathrm{d}x\to 0$$

as $\varepsilon\to 0$ since $\mathcal{L}^d\left(|u_\varepsilon-u|\ge\delta\right)\to 0$ uniformly in time by step 3 and u_ε^2+1 is equiintegrable uniformly in time by step 4.

For the second summand, we simply estimate

$$\sup_{0\le t\le T}\int_{\{|u_\varepsilon-u|<\delta\}}|u_\varepsilon-u|^2\,\mathrm{d}x=\delta^2\Lambda^d$$

Taking the limes superior as ε tends to zero of this inequality yields that the right hand side can be made arbitrarily small, which yields

$$\sup_{0\le t\le T}\int|u_\varepsilon-u|^2\,\mathrm{d}x\to 0$$

as ε tends to zero.

$\square$

Remark. From the previous Lemma, it follows that if the initial conditions u_ε^0 converge in L^1 or pointwise almost everywhere to the function $\alpha(1-\chi^0)+\beta\chi^0$ (and we know that the limit is of this form since the energies of the initial values stay bounded), then u also assumes this initial condition in $\mathrm{L}^2(\mathbb{T})$.

2.2 Convergence of the energies

As often in the Calculus of Variations, convergence of energies boost our modulus of convergence and gives our limit therefore additional regularity. Thus let us assume

$$\int_0^T \mathrm{E}_\varepsilon(u_\varepsilon)\,\mathrm{d}t\to\int_0^T\mathrm{E}(u)\,\mathrm{d}t\ \text{ as }\varepsilon\to 0, \tag{2.7}$$

where the surface tension energy is defined for $u=\alpha(1-\chi)+\beta\chi$ by

$$\mathrm{E}(u):=\sigma\int|\nabla\chi|. \tag{2.8}$$

Moreover we notice that the energy of u is exactly the total variation of $\psi=\sigma\chi$. Since for almost every time t, we have that $\psi_\varepsilon(t)$ converges to $\psi(t)$ in $\mathrm{L}^1(\mathbb{T})$ by Proposition 2.1.1, it follows from the lower semicontinuity of the variation measure and Young's inequality that

$$\mathrm{E}(u)\le\liminf_{\varepsilon\to 0}\int|\nabla\psi_\varepsilon|\,\mathrm{d}x\le\liminf_{\varepsilon\to 0}\mathrm{E}_\varepsilon(u_\varepsilon).$$

Moreover by the energy dissipation inequality (1.12), the energies stay uniformly bounded in time. Thus using the dominated convergence theorem, we see that our time integrated energy convergence assumption is equivalent to saying that for almost every time $t \in (0,T)$, we have convergence of the energies $\mathrm{E}_\varepsilon(u_\varepsilon(t)) \to \mathrm{E}(u(t))$.

Since the energies themselves can be interpreted as measures on the flat torus, we define for a continuous function $\varphi \in \mathrm{C}(\mathbb{T})$ the corresponding energy measures by

$$\begin{aligned} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) &:= \int \varphi \left(\frac{1}{\varepsilon} + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 \right) \mathrm{d}x \quad \text{and} \\ \mathrm{E}(u;\varphi) &:= \sigma \int \varphi |\nabla \chi|. \end{aligned}$$

Then from the energy convergence and lower semicontinuity just discussed, it follows that

$$\lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) = \mathrm{E}(u;\varphi). \tag{2.9}$$

A first additional regularity result under the energy convergence assumption is the following Proposition, which ensures that we have square-integrable normal velocities.

Proposition 2.2.1. *In the setting of Proposition 2.1.1 and given the energy convergence assumption (2.7), the measure $\partial_t\chi$ is absolutely continuous with respect to the measure $|\nabla\chi|\,\mathrm{d}t$ and the corresponding density V is square integrable with the estimate*

$$\int_0^T \int V^2 |\nabla \chi|\,\mathrm{d}t \lesssim \mathrm{E}^0,$$

where $\mathrm{E}^0 := \liminf_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon^0)$.

Proof. Take a smooth test function $\varphi \in \mathrm{C}_\mathrm{C}^\infty((0,T)\times\mathbb{T})$. Then via the L^1-convergence of $\psi_\varepsilon \to \psi$, we have

$$\begin{aligned} \partial_t\psi(\varphi) &= \liminf_{\varepsilon\to 0} \partial_t\psi_\varepsilon(\varphi) \\ &= \liminf_{\varepsilon\to 0} \int_0^T \int \sqrt{2W(u_\varepsilon)}\partial_t u_\varepsilon \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\le \liminf_{\varepsilon\to 0} \left(\int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon)\varphi^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \\ &\le \liminf_{\varepsilon\to 0} \left(2\int_0^T \mathrm{E}_\varepsilon\left(u_\varepsilon;\varphi^2\right)\mathrm{d}t \right)^{1/2} (\mathrm{E}_\varepsilon(u_\varepsilon))^{1/2} && (1.12) \\ &= \sqrt{2\sigma}\|\varphi\|_{\mathrm{L}^2(\mathbb{T},|\nabla\chi|\mathrm{d}t)} \sqrt{\mathrm{E}^0}. && (2.9) \end{aligned}$$

This proves both the absolute continuity and via a duality argument the desired bound since $\partial_t\psi = \sigma\partial_t\chi$ and $\sigma > 0$. □

We finish this section with a proof for the equipartition of the energy, which tells us that both the summand involving the potential $W(u_\varepsilon$ and the norm of the gradient contribute to the energy in similar parts.

Lemma 2.2.2. *Under the energy convergence assumption (2.7), we have for any continuous function $\varphi \in \mathrm{C}^{\infty}(\mathbb{T})$ that*

$$\begin{aligned} \mathrm{E}(u;\varphi) = \lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) &= \lim_{\varepsilon\to 0} \int \varphi\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| \,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int \varphi\varepsilon|\nabla u_\varepsilon|^2 \,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int \varphi\frac{1}{\varepsilon}2W(u_\varepsilon) \,\mathrm{d}x \end{aligned}$$

for almost every time $0 \le t \le T$.

Proof. We have already established the first equality before. For the second equality, we first assume that $\varphi \in \mathrm{C}(\mathbb{T};[0,\infty))$. By the lower semicontinuity of the variation measure, we immediately obtain

$$\liminf_{\varepsilon\to 0} \int \varphi\sqrt{2W(u_\varepsilon}|\nabla u_\varepsilon| \,\mathrm{d}x \ge \mathrm{E}(u;\varphi).$$

But by Young's inequality, we also have

$$\limsup_{\varepsilon\to 0} \int \varphi\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| \,\mathrm{d}x \le \limsup_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) = \mathrm{E}(u;\varphi).$$

For general $\varphi \in \mathrm{C}(\mathbb{T})$, we decompose φ into its positive and negative part and apply the previous argument to both in order to get the claim.

The third and fourth inequality follow for a given non-negative $\varphi \in \mathrm{C}(\mathbb{T};[0,\infty))$ by the L^2 estimate

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int \left|\sqrt{\varphi}\sqrt{\varepsilon}|\nabla u_\varepsilon| - \sqrt{\varphi}\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right|^2 \mathrm{d}x \\ =\ &\lim_{\varepsilon\to 0} 2\,\mathrm{E}_\varepsilon(u_\varepsilon;\varphi) - 2\int \varphi\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| \,\mathrm{d}x = 0 \end{aligned}$$

which implies

$$\lim_{\varepsilon\to 0} \int \varphi\varepsilon|\nabla u_\varepsilon|^2 \,\mathrm{d}x = \lim_{\varepsilon\to 0} \int \varphi\frac{1}{\varepsilon}2W(u_\varepsilon) \,\mathrm{d}x = \lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi),$$

finishing our proof. □

2.3 Convergence to twophase mean curvature flow

We start by definining a BV-formulation for motion by mean curvature.

Definition 2.3.1. *Fix some finite time horizon $T < \infty$ and initial data $\chi^0 \in \mathrm{BV}(\mathbb{T};\{0,1\})$, we say that*

$$\chi \in \mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T};\{0,1\})\right)$$

with $\operatorname{ess\,sup}_{0\le t\le T} \mathrm{E}(\chi)$ moves by mean curvature *if there is a normal velocity* $V \in \mathrm{L}^2(|\nabla\chi|\,\mathrm{d}t)$ *such that*

1. *For all* $\xi \in \mathrm{C}_\mathrm{C}^\infty\left((0,T)\times\mathbb{T};\mathbb{R}^d\right)$*, we have*
$$\int_0^T \int V\langle \xi\,,\,\nu\rangle - \langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu\otimes\nu\rangle|\nabla\chi|\,\mathrm{d}t = 0. \tag{2.10}$$
2. *The function* V *is the normal velocity of* χ *in the sense that*
$$\partial\chi = V|\nabla\chi|\,\mathrm{d}t$$
holds distributionally in $(0,T)\times\mathbb{T}$*.*
3. *The initial data* χ^0 *is achieved in* $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$*, which simply means that* $\chi(0)=\chi^0$ *as functions in* $\mathrm{L}^2(\mathbb{T})$*.*

Our main goal in this section is now to show that the function χ we have found in Proposition 2.1.1 moves my mean curvature. Looking at equation (1.9) which reads
$$\int \frac{1}{\varepsilon^2}W'(u_\varepsilon(t))\varphi + \nabla u_\varepsilon(t)\nabla\varphi + \partial_t u_\varepsilon(t)\varphi\,\mathrm{d}x = 0, \tag{1.9}$$
we expect that for a suitable choice of testfunctions φ_ε, the following two terms converge:
$$\lim_{\varepsilon\to 0}\int_0^T\int \partial_t u_\varepsilon\varphi_\varepsilon = \sigma\int_0^T\int V\langle\xi\,,\,\nu\rangle|\nabla\chi|\,\mathrm{d}t\,,$$
$$\lim_{\varepsilon\to 0}\int_0^T\int \frac{1}{\varepsilon^2}W'(u_\varepsilon)\varphi_\varepsilon + \langle\nabla u_\varepsilon\,,\,\nabla\varphi_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t = \sigma - \int_0^T\int\langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu\otimes\nu\rangle|\nabla\chi|\,\mathrm{d}t\,.$$
But how do we find these testfunctions? For this, we first note that the curvature term $\int\langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu\otimes\nu\rangle|\nabla\chi|\,\mathrm{d}t$ is by [Mag12, Thm. 17.5] the first inner variation with respect to ξ of the perimeter, which is just our energy E up to the surface tension constant $\sigma>0$. Thus it is plausible to compute the first inner variation $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}E(\rho_s)$ and then we can hopefully choose the testfunction φ_ε in such a way that it equals $\int\frac{1}{\varepsilon^2}W'(u_\varepsilon(t))\varphi + \nabla u_\varepsilon(t)\nabla\varphi\,\mathrm{d}x$.

Thus let $(\rho_s)_s$ be functions which solve the ODE
$$\begin{cases}\partial_t\rho_s + \langle\xi\,,\,\nabla\rho_s\rangle &= 0\\ \rho_0 &= u_\varepsilon.\end{cases}$$
Then we formally compute
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\int\frac{\varepsilon}{2}|\rho_s|^2 + \frac{1}{\varepsilon}W(\rho_s)\,\mathrm{d}x &= \int\varepsilon\langle\nabla u_\varepsilon\,,\,\nabla\left(-\langle\xi\,,\,\nabla u_\varepsilon\rangle\right)\rangle + \frac{1}{\varepsilon}W'(u_\varepsilon)(-\langle\xi\,,\,\nabla u_\varepsilon\rangle)\,\mathrm{d}x\\ &= \int\left(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle\xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x\,.\end{aligned}$$
We therefore test equation (1.9) against $\varphi_\varepsilon \coloneqq \langle\xi\,,\,\nabla u_\varepsilon\rangle$.

2.3.1 Convergence of the curvature term

The goal of this section is to prove the convergence

$$\lim_{\varepsilon\to 0}\int\left(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle\xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x=\sigma\int\langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu\otimes\nu\rangle|\nabla\chi|$$

for almost every time t.

3 i dont know yet

Baldo proved in his paper [Bal90] that the Cahn-Hilliard energies Γ-converge with respect to $\|\cdot\|_{\mathrm{L}^1}$ to an *optimal partition energy* given by

$$\mathrm{E}(\chi) := \frac{1}{2} \sum_{1\le i,j\le P} \sigma_{i,j} \int \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) \tag{3.1}$$

for a partition $\chi_1, \ldots, \chi_P \colon \mathbb{T} \to \{0,1\}$ satisfying $\chi_{1\le i\le P}\chi_i = 1$ almost everywhere. We may also define measurable sets Ω_i through the relation $\chi_i = \mathbb{1}_{\Omega_i}$. The link between a sequence u_ε and χ is given by $u_\varepsilon \to u := \sum_{1\le i\le P} \alpha_i \chi_i$ in L^1.

Moreover if we denote by $\partial_* \Omega_i$ the reduced boundary of Ω_i and by $\Sigma_{i,j} := \partial_* \Omega_i \cap \partial_* \Omega_j$ the interface between Ω_i and Ω_j, then we may rewrite equation (3.1) as

$$\mathrm{E}(\chi) = \frac{1}{2} \sum_{1\le i,j\le P} \sigma_{i,j}\, \mathcal{H}^{d-1}(\Sigma_{i,j}).$$

Here, the surface tensions $\sigma_{i,j}$ are the geodesic distances between the wells α_i of W with respect to the metric $2W(u)\langle \cdot\,,\,\cdot \rangle$, which can be written out as

$$\sigma_{i,j} = \mathrm{d}_W(\alpha_i, \alpha_j)$$

for the geodesic distance defined as

$$\mathrm{d}_W(u,v) := \inf \left\{ \int_0^1 \sqrt{2W(\gamma)} |\dot\gamma| \,\mathrm{d}t \;:\; \gamma \in \mathrm{C}^1\left([0,1], \mathbb{R}^N\right) \text{ with } \gamma(0) = u,\, \gamma(1) = v \right\}. \tag{3.2}$$

Geometrically speaking, the partition energy E measures the surface tensions between the sets and penalizes larges interfaces. Also observe that the factor 1/2 can be left out if we only count each interface once.

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