Convergence of Allen-Cahn equations to multiphase mean curvature flow

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Introduction

Allgemeine Notizen: Etwas zu Eindeutigkeit sagen; wir gehen sehr oft zu Teilfolgen über. Hilft uns hierbei die Weak strong uniqueness?

1. Abstract

This thesis presents a conditional convergence result of solutions to the Allen–Cahn equation with arbitrary potentials to a De Giorgi type BV-solution for multiphase mean curvature flow. For this we will recall the proof for the conditional convergence to BV-solutions in the sense of Laux and Simon. Lastly we show that De Giorgi type BV-solutions are De Giorgi type varifold solutions, and thus our solution concept exhibits a weak-strong uniqueness.

2. History and main goals

Multiphase mean curvature is an important geometric evolution equation which has been studied for a long time, bearing not only mathematical importance, but also for the applied sciences. Originally it was proposed to study the evolution of grain boundaries in annealed recrystallized metal, as described by Mullins in [Mul56], who cites Beck in [Bec52] as already having observed such a behaviour in 1952.

Over the years a number of different solution concepts for multiphase mean curvature flow have been proposed. Classically we have smooth solutions, where we require the evolution of sets to be smooth, for example described by Huisken in [Hui90]. Another description of smoothly evolving mean curvature flow can be found in the work of Gage and Hamilton [GH86], who proved the "shrinking conjecture" for convex planar curves. Brakke describes in his book [Bra78] the motion by mean curvature using varifolds, which yields a quite abstract and general notion for mean curvature flow and is based on the gradient flow structure of mean curvature flow. Luckhaus and Sturzenhecker introduced a distributional solution concept for mean curvature flow in their work [LS95]. Another solution concept is the viscosity solution concept, for example presented in ([CGG91], [ES91]), where it is shown that solutions of a certain parabolic equation have the property that if they are smooth, the corresponding level sets move by mean curvature.

The Allen–Cahn equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} \nabla W(u_\varepsilon) \tag{1}$$

is commonly used as a phase-field approximation for mean curvature flow and was first proposed by Allen and Cahn in their paper [AC79]. Here $W\colon \mathbb{R}^N \to [0,\infty)$ is some smooth potential with finitely many zeros $\alpha_1, \dots, \alpha_P$. Since then it has been a topic of long study, both from a numerical and analytical standpoint.

If we consider the two-phase case, which corresponds to setting $N = 1$ and $P = 2$, then the behaviour of the solutions to (1) are thoroughly researched. Chen has proven in [Che92] that as long as the interface of the limit evolves smoothly, we have convergence to mean curvature flow. The authors Bronsard and Kohn showed by utilization of the gradient flow structure of (1), more precisely by mainly utilizing the corresponding energy dissipation inequality, in [BK91] the compactness of solutions and regularity for the limit. Moreover they studied the behaviour for radially symmetric initial data and showed motion by mean curvature in this case. Ilmanen made fundamental contributions in [Ilm93] by proving the convergence to Brakke's mean curvature flow as long as the initial conditions are well-prepared by exploiting the equipartition of energies. However his methods seem to only work in the two-phase case since he uses a comparison principle whose generalization to the vectorial case is not clear.

The multiphase case for both the convergence of the Allen–Cahn equation and mean curvature flow is much more involved and a topic of current research. For the convergence of the Allen–Cahn equation, asymptotic expansions with Neumann boundary conditions have been studied by Keller, Rubinstein and Sternberg in [RSK89] (NOCHMAL MIT TIM BESPREHCEN). Moreover it has by shown by Bronsard and Reitich in [BR93] that for the three-phase case, we have short-time existence of a solution (in what sense????).

The analysis of multiphase mean curvature flow has been studied for example by Mantegazza, Novaga and Tortorelli in [MNT04], where they considered the planar case when only a single triple junction appears, and their work has been extended to several triple junctions in [Man+16]. Ilmanen, Neves and Schulze furthermore proved in [INS19] that even for non-regular initial data in the sense that not all angles at triple junctions are equal, we have short time existence by approximation through regular networks. For long time existence, it has been shown by Kim and Tonegawa in [KT17] through a modification of Brakke's approximation scheme that one can obtain non-trivial mean curvature flow even with singular initial data.

The first main goal of this thesis is to prove a conditional convergence result of solutions to the vectorial Allen–Cahn equation (1) to a De Giorgi type BV-solution of multiphase mean curvature flow in the sense of Definition 5.1.1. The proof is based mainly on a duality argument and the results of Laux and Simon in [LS16]. However the strong assumption (MIT TIM ÜBER SIEN PAPER DISKUTIEREN; WANN MAN DIES ZEIGEN KANN) we make here is that the Cahn-Hilliard energies (2.2) of the solutions to (1) converge to the perimeter functional (1.7) of the limit. This prevents that as ε tends to zero, approximate interfaces collapse, which would mean that energy dissipates, see also the discussion in Section 5.2. The energy convergence assumption provides us with the important equipartition of energies, whose proof under milder assumption was the main obstacle of Ilmanen in [Ilm93]. Moreover it is the key for our localization estimates and lets us localize on the different phases of mean curvature flow. Lastly it assures that the differential ∇u_ε can locally up to an error be written as a rank-one matrix which is the tensor of the approximate frozen unit normal and the gradient of the geodesic distance function associated to the majority phase evaluated at u_ε, see the proof of Proposition 4.2.7.

The second main goal is to compare De Giorgi type BV-solutions to De Giorgi type varifold solutions, which were proposed by Hensel and Laux in [HL21]. We will show that our solution concept is stronger in the sense that every BV-solution is also a varifold solution. Since Hensel and Laux have shown weak-strong uniqueness for their varifold solution concept, it follows that we also obtain weak-strong uniqueness for our De Giorgi type BV-solution concept, which is the best we can expect. In fact it has been shown on a numerical basis by Angenent, Chopp and Ilmanen in [AIC95] (AUCH MIT TIM BESPRECHEN) that even in three dimensions, there exists a hypersurface whose evolution by mean curvature flow admits a singularity at a certain time after which we have nonuniqueness.

Let us also mention some of the closely related unanswered questions. For one Hensel and Laux have shown in [HL21] that in the two-phase case and under well prepared initial conditions, then solutions of the Allen–Cahn equation (1) converge to a De Giorgi type varifold solution. However their methods have no obvious generalization to the multiphase case without the energy convergence assumption, and one even struggles to find an approximate sequence which constructs the desired varifolds. And even then one would have to find suitable substitutions for the localization estimates explained in Section 4.2.2, which are based on De Giorgi's structure theorem and thus only work for BV-functions. One other possible question would be how to generalize the results to the case of arbitrary mobilities: Throughout the thesis, and also the main background papers ([LS16], [HL21]), it is always assumed that the mobilities are fixed through the relation $\mu_{ij} = 1/\sigma_{ij}$, where σ_{ij} denotes the surface tension of the (i,j)-th interface, and μ_{ij} its mobility. As proposed by Bretin, Danescu, Penuelas, and Masnou in [Bre+18], passing to arbitrary mobilities should amount to multiplying an appropriate "mobility matrix" M onto the right-hand side of (1) and changing the metric of the underlying space accordingly to $\langle u, v\rangle = \int \langle Mu\,,\, v\rangle \,\mathrm{d}x$. The difference in their approach is that first uncouple their system so that they arrive at the scalar Allen–Cahn equation and then couple they components through a Lagrange-multiplier, which assures that the limit is a partition.

3. Structure of the thesis

In Chapter 1 we will give a soft mathematical introduction into the topic of gradient flows. We will derive De Giorgi's optimal energy dissipation inequality in a simple example and discuss its usefulness for reformulating the gradient flow equation. Afterwards we apply our observations to (multiphase) mean curvature flow.

Afterwards in Chapter 2, we consider the Allen–Cahn equation (1) already mentioned above. Here we will focus again on its gradient flow structure. We then propose a suitable solution concept and prove the existence of a solution through De Giorgi's minimizing movements scheme.

Continuing with Chapter 3, we are going to take a look at the behaviour of solutions to the Allen–Cahn equation as ε tends to zero. First we study the simple two-phase case in order to get a better feeling for the equation and highlight important techniques like the Modica–Mortola trick. We will show that we have convergence to an evolving set of finite perimeter in space in time. Afterwards we also prove precompactness of the sequence in $\mathrm{C}([0,T];\mathrm{L}^2)$ which lets us show that the initial data as attained. Next up is the multiphase case, which requires more finesse, but we are able to show similar results as in the two-phase case. This will require a generalized chain rule for distributional derivatives and careful analysis of the geodesic distance functions with respect to the potential W from equation (1).

Chapter 4 is concerned with the conditional convergence result proven by Laux and Simon in [LS16]. We show that the in Chapter 3 observed limit is a BV-solution to mean curvature flow under the crucial assumption of energy convergence. Again we first consider the simpler two-phase case in order to simplify some of the arguments. Here one has to show the existence of normal velocities followed by the equipartition of energies. Then we are going to separately show the convergence of the velocity term and the curvature term of the Allen–Cahn equation to the corresponding terms for mean curvature flow. For the multiphase equivalent our core strategy is to reduce to the two-phase case through a localization argument, which we detail in Section 4.2.2. Afterwards we show the same results as in the two-phase case.

Lastly in Chapter 5 we are going to present new results building on the previous insights. We start off by presenting a De Giorgi type BV-solution concept for multiphase mean curvature flow. This is followed by proving a similar conditional convergence result as in the previous chapter, but now to this new solution. In the final Section 5.2, we are going to discuss the assumption of energy convergence. Moreover we show that every De Giorgi type BV-solution is a De Giorgi type varifold solution in the sense of Hensel and Laux in [HL21], whose solution concept does not rely on the assumption of energy convergence.

1. Gradient flows and mean curvature flow

1.1. Gradient flows

In the simplest case, a gradient flow of a given energy $\mathrm{E}\colon \mathbb{R}^N \to \mathbb{R}$ (with respect to the euclidean inner product) is a solution to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\nabla \mathrm{E}(x(t)), \tag{1.1}$$

where we usually prescribe some initial value $x(0) = x_0 \in \mathbb{R}^N$. The central structure here is that on the right hand side of the equation, we do not have any function, but the gradient of some continuously differentiable function. What a solution x does is that it moves in the direction of the steepest descent of the energy E. Moreover this allows for the following computation given a continuously differentiable solution x of (1.1):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}(x(t)) &= \langle \nabla \mathrm{E}(x(t))\,,\, x'(t) \rangle = -|\nabla \mathrm{E}(x(t))|^2 = -|x'(t)|^2 \\ &= -\frac{1}{2}\left(|x'(t)|^2 + |\nabla \mathrm{E}(x(t))|^2 \right).\end{aligned}$$

We especially obtain that the function $\mathrm{E}(x(t))$ is non-increasing, which coincides with our intuition that x moves along the steepest descent of the energy. But more precisely, we obtain from the fundamental theorem of calculus the *energy dissipation identity*

$$\mathrm{E}(x(T)) + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla \mathrm{E}(x(t))|^2 \,\mathrm{d}t = \mathrm{E}(x(0)). \tag{1.2}$$

One could now raise the question if this is identity already characterizes equation (1.1). But as it turns out, we can go even one step further, namely we only ask for the *optimal energy dissipation inequality* given by

$$\mathrm{E}(x(T)) + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla \mathrm{E}(x(t))|^2 \,\mathrm{d}t \leq \mathrm{E}(x(0)). \tag{1.3}$$

We call this inequality optimal since as demonstrated before, we usually expect an equality to hold. Now let us assume that x satisfies (1.3) and has sufficient regularity.

Then we can estimate again by the fundamental theorem that

$$\begin{aligned}&\frac{1}{2}\int_0^T |\nabla \mathrm{E}(x(t)) + x'(t)|^2 \,\mathrm{d}t \\ =& \int_0^T \langle \nabla \mathrm{E}(x(t)), x'(t)\rangle \,\mathrm{d}t + \frac{1}{2}\int_0^T |x'(t)|^2 + |\nabla \mathrm{E}(x(t))|^2 \,\mathrm{d}t \\ =& \mathrm{E}(x(T)) - \mathrm{E}(x(0)) + \frac{1}{2}\int_0^T |x'(t)|^2 + |\nabla \mathrm{E}(x(t))|^2 \,\mathrm{d}t \le 0.\end{aligned}$$

Since we started with an integral over a non-negative function, this implies that for almost every time t, we have that $x'(t) = -\nabla \mathrm{E}(x(t))$, which proves that x (again under sufficient regularity assumptions this then must already hold for every time) is a gradient flow of the energy E.

The real strength of formulating the differential equation (1.1) via inequality (1.3) becomes clear if we want to consider gradient flows in a more complicated setting. In order to formulate equation (1.1), we need to have a notion of differentiation and a gradient in the target of x. Therefore we may use pre-Hilbert spaces or smooth Riemannian manifolds as a suitable substitutes for $\mathbb{R}^N$.

Examples for such more complicated gradient flows include the heat equation, which can be written as the L^2-gradient flow of the Dirichlet energy, the Fokker-Planck equation, the Allen–Cahn equation and mean curvature flow. The two latter of course play a key role for us.

These observations have been around for a long time, and are credited to De Giorgi and his paper [De 93]. Sandier and Serfaty have also written an excellent paper on this topic, as well as Ambrosio, Gigli and Savaré [AGS05], to which we refer the interested reader.

1.2. Mean curvature flow

Mean curvature flow describes the geometric evolution of a set $(\Omega(t))_{t\geq 0}$ respectively the evolution of its boundary $\Sigma(t) := \partial\Omega(t)$. It is formulated through the equation

$$\frac{1}{\mu} V = -\sigma H \quad \text{on } \Sigma. \tag{1.4}$$

Here $\mu > 0$ is a positive constant which is called the *mobility* and $\sigma > 0$ is a positive constant as well which we define as the *surface tension*. By V we denote the normal velocity of the set and by H its mean curvature. which is defined as the sum of the principle curvatures at a given point. There are a lot of different notions of solutions to this equation as already mentioned in the Introduction. For us the central structure of this equation is its gradient flow structure. Formally we want to consider the space

$$\mathcal{M} := \{(d-1)\text{-dimensional surfaces on } \mathbb{T}\},$$

where the tangent space at a given $\Sigma \in \mathcal{M}$ consists of the normal velocities on Σ. The metric tensor at a surface Σ of two normal velocities is then given by the rescaled L^2

inner product on Σ
$$\langle V\,,\,W\rangle_\Sigma \coloneqq \frac{1}{\mu}\int_\Sigma VW\,\mathrm{d}\,\mathcal{H}^{d-1}$$
and our energy will simply be the rescaled perimeter functional
$$\mathrm{E}(\Sigma) \coloneqq \sigma\,\mathcal{H}^{d-1}(\Sigma).$$
Since the first inner variation of the perimeter functional is given by the mean curvature vector (see [Mag12, Thm. 17.5]), the gradient of the energy should simply be the mean curvature vector multiplied by $\sigma\mu$. Thus the mean curvature flow equation (1.4) corresponds exactly to the gradient flow equation (1.1). Note however that this metric tensor induces a degenerate metric in the sense that the distance between any two hypersurfaces is zero, which has been shown by Michor and Mumford in[MM06]. To give some intuition for this result, we can roughen up a hypersurface, then move it, and afterwards flatten it back, while only producing an arbitrarily small distance due to a scaling invariance.

In Section 1.1 we highlighted the importance of De Giorgi's optimal energy dissipation inequality (1.3). In our setting this translates to the inequality
$$\begin{aligned}&\mathrm{E}(\Sigma(T)) + \frac{1}{2}\int_0^T \langle V(t)\,,\,V(t)\rangle_{\Sigma(t)} + \left\langle \nabla_{\Sigma(t)}E\,,\,\nabla_{\Sigma(t)}E\right\rangle_{\Sigma(t)}\,\mathrm{d}t\\ =\;&\mathrm{E}(\Sigma(T)) + \frac{1}{2}\int_0^T\int_{\Sigma(t)}\frac{1}{\mu}V(t)^2 + \sigma^2\mu H(t)^2\,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t \le \mathrm{E}(\Sigma(0)).\end{aligned}$$
This is the main motivation for the definition of De Giorgi type solutions to mean curvature flow in the two-phase case, see Definition 5.1.1 and Definition 5.2.2.

Far more important for us shall however be multiphase mean curvature flow, which is of high importance in the applied sciences and mathematically quite complex. Essentially instead of just considering the evolution of one single set, we look at the evolution of a partition of the flat torus and require that at each interface of the sets, the mean curvature flow equation (1.4) is satisfied.

We say that a partition $(\Omega_i(t))_{i=1,\ldots,P}$ with $\Sigma_{ij} \coloneqq \partial\Omega_i\cap\partial\Omega_j$ satisfies multiphase mean curvature with mobilities μ_{ij} and surface tensions σ_{ij} if
$$\frac{1}{\mu_{ij}}V_{ij} = -\sigma_{ij}H_{ij} \qquad \text{on } \Sigma_{ij} \text{ for all } i\neq j \text{ and} \tag{1.5}$$
$$\sigma_{ij}\nu_{ij} + \sigma_{jk}\nu_{jk} + \sigma_{ki}\nu_{ki} = 0 \qquad \text{at triple junctions.} \tag{1.6}$$
The second equation is a stability condition when more than two sets meet and is called *Herring's angle condition.* Here ν_{ij} is the outer unit normal on $\Sigma_{i,j}$ pointing from Ω_i to Ω_j. One could argue that we would also want stability conditions at for example quadruple junctions. In two dimensions though, such quadruple junctions are expected to immediatly dissipate. In higher dimensions, quadruple junctions might become stable, but only on lower dimensional sets, and shall thus not be relevant for us.

As our space, we shall thus now consider tuples of $(d-1)$-dimensional surfaces on $\mathbb{T}$. The energy and metric tensor are given by

$$\mathrm{E}(\Sigma) \coloneqq \sum_{i<j} \sigma_{ij}\, \mathcal{H}^{d-1}(\Sigma_{ij}) \tag{1.7}$$

and

$$\langle V\,,\, W\rangle_\Sigma \coloneqq \sum_{i<j} \frac{1}{\mu_{ij}} \int_{\Sigma_{ij}} V_{ij} W_{ij}\, \mathrm{d}\,\mathcal{H}^{d-1}\,.$$

Of course this again produced a degenerate metric. Using a variant of the divergence theorem on surfaces ([Mag12, Thm. 11.8]) and again the computation for the first variation of the perimeter ([Mag12, Thm. 17.5]), we see that multiphase mean curvature has the desired gradient flow structure and that De Giorgi's inequality (1.3) translates to

$$\mathrm{E}(\Sigma(T)) + \frac{1}{2}\sum_{i<j}\int_0^T \int_{\Sigma_{ij}} \frac{1}{\mu_{ij}} V_{ij}^2 + \sigma_{ij}^2 \mu_{ij} H_{ij}^2\, \mathrm{d}\,\mathcal{H}^{d-1}\, \mathrm{d}t \le \mathrm{E}(\Sigma(0)).$$

If we make the simplifying assumption that the mobilities are already determined through the surface tensions by the relation $\mu_{ij} = 1/\sigma_{ij}$, then this inequality becomes

$$\mathrm{E}(\Sigma(T)) + \frac{1}{2}\sum_{i<j}\sigma_{ij}\int_0^T \int_{\Sigma_{ij}} V_{ij}^2 + H_{ij}^2\, \mathrm{d}\,\mathcal{H}^{d-1}\, \mathrm{d}t \le \mathrm{E}(\Sigma(0)),$$

which motivated the multiphase phase of Definition 5.1.1 and Definition 5.2.2.

2. The Allen–Cahn equation

2.1. Structure of the equation

This chapter follows [LS16], but since the authors decided to only sketch some of the proofs, we want to go into more detail.

Let $\Lambda > 0$ and define the flat torus $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$, where we work with periodic boundary conditions and write $\int \mathrm{d}x$ instead of $\int_{\mathbb{T}} \mathrm{d}x$. Then for $u \colon [0, \infty) \times \mathbb{T} \to \mathbb{R}^N$ and some potential $W \colon \mathbb{R}^N \to [0, \infty)$, the *Allen–Cahn equation* with parameter $\varepsilon > 0$ is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \tag{2.1}$$

To understand this equation better, we consider the *Cahn–Hilliard energy*, which assigns to u for a fixed time the real number

$$\mathrm{E}_\varepsilon(u) := \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 \,\mathrm{d}x \,. \tag{2.2}$$

If everything is nice and smooth, we can compute that under the assumption that u satisfies equation (2.1), we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u) \,,\, \partial_t u \rangle + \varepsilon \langle \nabla u \,,\, \nabla \partial_t u \rangle \,\mathrm{d}x \\
&= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u \,,\, \partial_t u \right\rangle \mathrm{d}x \\
&= \int -\varepsilon |\partial_t u|^2 \,\mathrm{d}x \,. && (2.1)
\end{aligned}$$

This calculation suggests that equation (2.1) is the L^2 gradient flow (rescaled by $\sqrt{\varepsilon}$) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (2.1) via De Giorgis minimizing movements scheme ([De 93]), which we will do in Theorem 2.2.1.

But first we need to clarify what our potential W should look like. Classic examples in the scalar case are given by $W(u) = \left(u^2 - 1\right)^2$ or $W(u) = u^2(u-1)^2$, and we call functions like these *doublewell potentials*, see also Figure 2.1.

In higher dimensions, we want to accept the following potentials: $W \colon \mathbb{R}^N \to [0, \infty)$ has to be a a smooth multiwell potential, meaning that it has finitely many zeros at $u = \alpha_1, \ldots, \alpha_P \in \mathbb{R}^N$. Furthermore we aks for polynomial growth in the sense that there exists some $p \geq 2$ such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \tag{2.3}$$

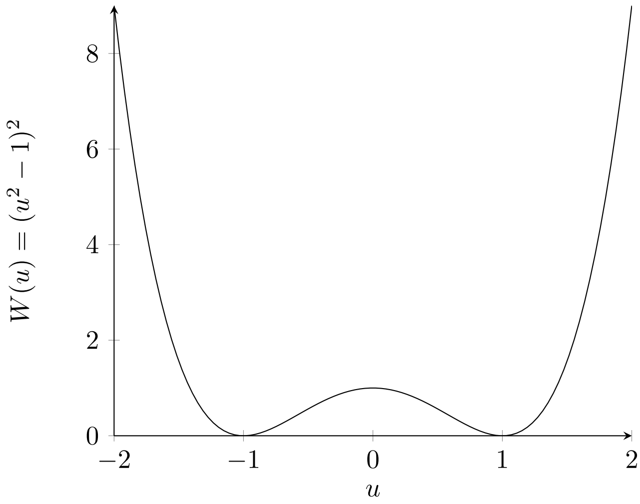


Figure 2.1.: The graph of a doublewell potential

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \tag{2.4}$$

for all u sufficiently large. Lastly we want W to be convex up to a small perturbation in the sense that there exist smooth functions $W_{\text{conv}}, W_{\text{pert}} : \mathbb{R}^N \to [0, \infty)$ such that

$$W = W_{\text{conv}} + W_{\text{pert}}\,, \tag{2.5}$$

W_{conv} is convex and

$$\sup_{x\in\mathbb{R}^N} \left|\nabla^2 W_{\text{pert}}\right| < \infty. \tag{2.6}$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen-Cahn equation (2.1) as maps from $[0,T]$ into some suitable function space and thus use the following definition.

Definition 2.1.1. *We say that a function $u_\varepsilon \in \mathrm{C}\left([0,T]; \mathrm{L}^2\left(\mathbb{T};\mathbb{R}^N\right)\right)$ which is also in $\mathrm{L}^\infty\left([0,T]; \mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\right)$ is a weak solution of the Allen–Cahn equation (2.1) with parameter $\varepsilon > 0$ and initial condition $u_\varepsilon^0 \in \mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ if*

1. *the energy stays bounded, which means that*

$$\operatorname*{ess\,sup}_{0\le t\le T} \mathrm{E}_\varepsilon(u_\varepsilon(t)) < \infty, \tag{2.7}$$

2. *its weak time derivative satisfies*

$$\partial_t u_\varepsilon \in \mathrm{L}^2\left([0,T]\times\mathbb{T};\mathbb{R}^N\right), \tag{2.8}$$

3. *for almost every $t \in [0,T]$ and every $\xi \in \mathrm{L}^p([0,T]\times\mathbb{T};\mathbb{R}^N)\cap \mathrm{W}^{1,2}([0,T]\times\mathbb{T};\mathbb{R}^N)$, we have*

$$\int \left\langle \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon(t))\,,\,\xi\right\rangle + \langle \nabla u_\varepsilon(t)\,,\,\nabla\xi\rangle + \langle \partial_t u_\varepsilon(t)\,,\,\xi\rangle \,\mathrm{d}x = 0, \tag{2.9}$$

4. *the initial conditions are achieved in the sense that $u_\varepsilon(0) = u_\varepsilon^0$.*

Remark 2.1.2. Given our assumptions, we automatically have $\nabla W(u_\varepsilon)(t) \in \mathrm{L}^{p'}(\mathbb{T})$ since

$$|\nabla W(u_\varepsilon)|^{p/(p-1)} \lesssim 1 + |u_\varepsilon|^p \lesssim 1 + W(u), \tag{2.10}$$

which is integrable for almost every time t since we assume that the energy stays bounded, thus the integral in equation (2.9) is well defined.

Moreover we already obtain 1/2 Hölder-continuity in time from the embedding

$$\mathrm{W}^{1,2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right) \hookrightarrow \mathrm{C}^{1/2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right), \tag{2.11}$$

which follows from a generalized version of the fundamental theorem of calculus and Hölder's inequality.

2.2. Existence of a solution

With Definition 2.1.1, we are able to state our existence results for solutions of the Allen–Cahn equation. The proof uses De Giorgis minimizing movements scheme and arguments from the theory of gradient flows and has been very briefly sketched in [LS16].

Theorem 2.2.1. *Let $u^0\colon \mathbb{T}\to\mathbb{R}^N$ be such that $\mathrm{E}_\varepsilon(u^0) < \infty$. Then there exists a weak solution u_ε to the Allen–Cahn equation (2.1) in the sense of Definition 2.1.1 with initial data u^0. Furthermore the solution satisfies the energy dissipation inequality*

$$\mathrm{E}_\varepsilon(u_\varepsilon(t)) + \int_0^t \int \varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s \le \mathrm{E}_\varepsilon(u^0) \tag{2.12}$$

for every $t\in[0,T]$ and we additionally have $\partial^2_{i,j}u_\varepsilon, \nabla W(u_\varepsilon) \in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$ for all $1\le i,j\le d$. In particular we can test the weak form (2.9) with $\partial^2_{i,j}u_\varepsilon$.

Proof. **Step 1:** A minimization problem.
Fix some $h>0$, $u_{n-1}\in \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$ and consider the functional

$$\begin{aligned}\mathcal{F}\colon\ \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N) &\to \mathbb{R} \\ u &\mapsto \mathrm{E}_\varepsilon(u) + \frac{1}{2h}\int \varepsilon|u-u_{n-1}|^2\,\mathrm{d}x\,.\end{aligned} \tag{2.13}$$

Then $\mathcal{F}$ is coercive with respect to $\|\cdot\|_{\mathrm{W}^{1,2}}$ and bounded from below by zero, thus we may take a $\mathrm{W}^{1,2}$-bounded sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$ such that $\mathcal{F}(v_k)\to\inf\mathcal{F}$

as $k \to \infty$. These v_k have a non-relabelled subsequence which converges weakly in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$ and strongly in $\mathrm{L}^2(\mathbb{T},\mathbb{R}^N)$ to some $u \in \mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$. Moreover v_k is bounded in $\mathrm{L}^p\left(\mathbb{T};\mathbb{R}^N\right)$ by the lower growth assumption (2.3) on W and thus we obtain by a duality argument that $u \in \mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$. Lastly we have

$$\frac{1}{2h}\int \varepsilon|u-u_{n-1}|^2\,\mathrm{d}x \le \liminf_{k\to\infty}\frac{1}{2h}\int \varepsilon|v_k-u_{n-1}|^2\,\mathrm{d}x\,,$$
$$\int \frac{\varepsilon}{2}|\nabla u|^2\,\mathrm{d}x \le \liminf_{k\to\infty}\int \frac{\varepsilon}{2}|\nabla v_k|^2\,\mathrm{d}x$$

by the weak convergence in $\mathrm{W}^{1,2}(\mathbb{T},\mathbb{R}^N)$. By passing to another non-relabelled subsequence which converges pointwise almost everywhere, we moreover achieve by the continuity of W that

$$\int \frac{1}{\varepsilon}W(u)\,\mathrm{d}x = \int \liminf_{k\to\infty}\frac{1}{\varepsilon}W(v_k)\,\mathrm{d}x \le \liminf_{k\to\infty}\int\frac{1}{\varepsilon}W(v_k)\,\mathrm{d}x\,,$$

which proves that u is a minimizer of $\mathcal{F}$.

Step 2: Minimizing movements scheme.

By iteratively choosing minimizers u_n^h from Step 1, we obtain a sequence of functions $u^0, u_1^h, \ldots$. Thus we may define a function u as the piecewise linear interpolation at the time-steps $0, h, 2h, \ldots$ of these functions.

Step 3: Sharp energy dissipation inequality for u_n^h.

We claim that there exists some constant $C > 0$ such that for all $h > 0$ and $n \in \mathbb{N}$, we have

$$\mathrm{E}_\varepsilon(u_n^h) + \left(\frac{1}{h} - \frac{C}{2\varepsilon^2}\right)\int \varepsilon\left|u_n^h - u_{n-1}^h\right|^2\mathrm{d}x \le \mathrm{E}_\varepsilon(u_{n-1}^h). \tag{2.14}$$

In order to prove this inequality, we notice that since $\left|\nabla^2 W_{\text{pert}}\right| \le C$, the function $W_{\text{pert}} + C|u|^2/2$ is convex for $C > 0$ sufficiently large, thus the functional

$$\tilde{\mathrm{E}}_\varepsilon(u) := \int \frac{1}{\varepsilon}\left(W(u) + \frac{C}{2}|u|^2\right) + \frac{\varepsilon}{2}|\nabla u|^2\,\mathrm{d}x$$

is convex on $\mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$. For a given $\xi \in \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$, we thus have that the function $t \mapsto \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi)$ is convex and differentiable, which yields that

$$\tilde{\mathrm{E}}_\varepsilon(u_n^h + \xi) \ge \tilde{\mathrm{E}}_\varepsilon(u_n^h) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi). \tag{2.15}$$

But since u_n^h is a minimizer of the functional $\mathcal{F}$ defined by (2.13), we have

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi)\\
=\;&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathcal{F}(u_n^h + t\xi) + \frac{C}{2\varepsilon}\int\left|u_n^h + t\xi\right|^2\mathrm{d}x - \frac{1}{2h}\int\varepsilon\left|u_n^h + t\xi - u_{n-1}^h\right|^2\mathrm{d}x\right)\\
=\;&\int \frac{C}{\varepsilon}\langle u_n^h\,,\,\xi\rangle - \frac{1}{h}\varepsilon\langle u_n^h - u_{n-1}^h\,,\,\xi\rangle\,\mathrm{d}x\,.
\end{aligned}$$

Plugging $\xi = u_{n-1}^h - u_n^h$ into this equation and using inequality (2.15) thus yields

$$\tilde{\mathrm{E}}_\varepsilon(u_{n-1}^h) \geq \tilde{\mathrm{E}}_\varepsilon(u_n^h) + \int \frac{C}{\varepsilon}\left(\langle u_n^h\,,\,u_{n-1}^h\rangle - \left|u_n^h\right|^2\right) + \frac{1}{h}\varepsilon\left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x\,,$$

which is equivalent to

$$\begin{aligned}
&\mathrm{E}_\varepsilon(u_{n-1}^h)\\
&\geq \mathrm{E}_\varepsilon(u_n^h) + \int \left(\frac{C}{2\varepsilon} - \frac{C}{\varepsilon}\right)\left|u_n^h\right|^2 - \frac{C}{2\varepsilon}\left|u_{n-1}^h\right|^2 + \frac{C}{\varepsilon}\langle u_n^h\,,\,u_{n-1}^h\rangle + \frac{1}{h}\varepsilon\left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x\\
&= \mathrm{E}_\varepsilon(u_n^h) + \left(\frac{1}{h} - \frac{C}{2\varepsilon^2}\right)\int \varepsilon\left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x\,.
\end{aligned}$$

This is the claimed estimate (2.14).

Step 4: Hölder bounds for u_h.

From the energy estimate (2.14) we deduce via an induction that for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathrm{E}_\varepsilon(u_n^h) + \left(1 - \frac{Ch}{2\varepsilon^2}\right)\int_0^{nh}\int \varepsilon\left|\partial_t u^h\right|^2 \mathrm{d}x\,\mathrm{d}t\\
&= \mathrm{E}_\varepsilon(u_n^h) + \left(h - \frac{Ch^2}{2\varepsilon^2}\right)\sum_{k=1}^{n}\int \varepsilon\left|\frac{u_k^h - u_{k-1}^h}{h}\right|^2 \mathrm{d}x\\
&\leq \mathrm{E}_\varepsilon(u^0).
\end{aligned} \tag{2.16}$$

This gives us with the use of Jensen's inequality for $0 \leq s \leq t \leq T$ and $h > 0$ sufficiently small that

$$\begin{aligned}
\left\|u^h(t) - u^h(s)\right\|_{\mathrm{L}^2} &= \left\|\int_s^t \partial_t u_h(\tau)\,\mathrm{d}\tau\right\|_{\mathrm{L}^2}\\
&\leq \sqrt{t-s}\left(\int_0^T\int \left|\partial_t u^h(\tau,x)\right|^2 \mathrm{d}x\,\mathrm{d}\tau\right)^{1/2}\\
&\leq \sqrt{t-s}\left(\varepsilon - \frac{Ch}{2\varepsilon}\right)^{-1/2}\left(E_\varepsilon(u_\varepsilon^0)\right)^{1/2},
\end{aligned} \tag{2.17}$$

which gives us a uniform bound on the L^2-Hölder continuity of u^h in time as h tends to zero since ε is fixed.

Step 5: Compactness.

In order to apply the Arzelà–Ascoli theorem for the sequence (u_h) in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)$ as h tends to zero, we need to check pointwise precompactness of the image and equicontinuity of the sequence. The equicontinuity follows from the previous estimate (2.17). In order to check the pointwise precompactness, we need to verify that for all $t \in [0,T]$, the set $\{u^h(t)\}_{\delta>h>0}$ is a precompact subset of $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ (for $\delta>0$ sufficiently small). But this follows from the energy bound $\mathrm{E}_\varepsilon(u_n^h)\le \mathrm{E}_\varepsilon(u_0)$ (given by inequality (2.14)), which gives us a time-uniform bound on $\left\|\nabla u^h(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)}$ and on $\left\|u^h(t)\right\|_{\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)}$ and thus on $\left\|u^h(t)\right\|_{\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)}$. The compact embedding $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\hookrightarrow \mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ thus yields the desired pointwise precompactness.

Therefore we may apply Arzelà–Ascoli to obtain some $u\in \mathrm{C}^{1,2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N\right)$ and some sequence $h_n\to 0$ such that u^{h_n} converges uniformly to u on $[0,T]$ with respect to $\|\cdot\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)}$.

Step 6: Additional regularity 1.

We first want to argue that from our construction, one already obtains that $u\in \mathrm{L}^\infty\left([0,T];\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\right)$. For this we first notice that for a fixed $t\in[0,T]$, we find by the pointwise precompactness some non-relabelled subsequence and $v\in \mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$ such that $u^{h_n}(t)$ converges weakly to v in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$. By uniqueness of the limit, we already have $u(t)=v$ almost everywhere, which yields $u(t)\in \mathrm{W}^{1,2}(\mathbb{T},\mathbb{R}^N)$. Applying the lower semicontinuity of the norm, we may also deduce that

$$\|u(t)\|_{\mathrm{W}^{1,2}}\le \liminf_{n\to\infty}\left\|u^{h_n}(t)\right\|_{\mathrm{W}^{1,2}}\le C\,\mathrm{E}_\varepsilon(u^0),$$

from which we deduce that $u\in \mathrm{L}^\infty\left([0,T];\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\right)$. Secondly the boundedness of the energies

$$\sup_{0\le t\le T}\mathrm{E}_\varepsilon(u(t))<\infty$$

follows from the lower semicontinuity of the energy and the pointwise L^2 convergence and pointwise weak convergence in $\mathrm{W}^{1,2}$ as described in Step 1. Lastly we want to argue that $\partial_t u\in \mathrm{L}^2\left([0,T]\times\mathbb{T};\mathbb{R}^N\right)$. From inequality (2.16) in step 4, we deduce that $\partial_t u^h$ is a bounded sequence in $\mathrm{L}^2\left([0,T]\times\mathbb{T};\mathbb{R}^N\right)$. Thus we find a non-relabelled subsequence of h_n and some $w\in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$ such that u^{h_n} converges weakly to w in $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. But then w is already the weak time derivative of u since for any testfunction ξ, we have

$$\begin{aligned}\int_{[0,T]\times\mathbb{T}}\langle u\,,\,\partial_t\xi\rangle\,\mathrm{d}x\,\mathrm{d}t &= \lim_{n\to\infty}\int_{[0,T]\times\mathbb{T}}\left\langle u^{h_n}\,,\,\partial_t\xi\right\rangle\mathrm{d}x\,\mathrm{d}t\\ &= \lim_{n\to\infty}-\int_{[0,T]\times\mathbb{T}}\left\langle \partial_t u^{h_n}\,,\,\xi\right\rangle\mathrm{d}x\,\mathrm{d}t\\ &= -\int_{[0,T]\times\mathbb{T}}\langle w\,,\,\xi\rangle\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Step 7: u is a weak solution.

Going back to step 1, we see that u_n^h solves the Euler–Lagrange equation

$$\int \frac{1}{\varepsilon^2}\left\langle\nabla W(u_n^h)\,,\,\xi\right\rangle+\left\langle\nabla u_n^h\,,\,\nabla\xi\right\rangle+\left\langle\frac{u_n^h-u_{n-1}^h}{h}\,,\,\xi\right\rangle\mathrm{d}x=0 \tag{2.18}$$

for any testfunction $\xi \in \mathrm{C}^\infty(\mathbb{T};\mathbb{R}^N)$. Let $t \in [0,T]$. Since u_h is defined as the pointwise linear interpolation of the functions u_n^h, we find for the sequence h_n corresponding sequences $\lambda_n \in [0,1]$ and $k_n \in \mathbb{N}$ such that

$$t = \lambda_n(k_n-1)h_n + (1-\lambda_n)k_n h_n$$

and therefore we can write

$$u_h(t) = \frac{k_n h_n - t}{h_n} u^h_{k_n-1} + \frac{t-(k_n-1)h_n}{h_n} u^{h_n}_{k_n}.$$

In order to pass to the limit in equation (2.18), we first note that $u^{h_n}_{k_n}$ converges to $u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ since

$$\begin{aligned} \left\| u(t) - u^{h_n}_{k_n} \right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} &\le \left\| u(t) - u^{h_n}(t) \right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} + \left\| u^{h_n}(t) - u^{h_n}(k_n h_n) \right\|_{\mathrm{L}^2(\mathbb{T},\mathbb{R}^N)} \\ &\lesssim \left\| u(t) - u^{h_n}(t) \right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} + \sqrt{h_n}, \end{aligned}$$

which goes to zero as n tends to infinity. Here we used (2.17) for the second inequality. This implies that

$$\begin{aligned} \int \langle \nabla u(t)\,,\, \nabla\xi \rangle \,\mathrm{d}x &= -\int \langle u(t)\,,\, \operatorname{div} \nabla \xi \rangle \,\mathrm{d}x \\ &= -\lim_{n\to\infty} \int \left\langle u^{h_n}_{k_n}\,,\, \operatorname{div} \nabla\xi \right\rangle \mathrm{d}x \\ &= \lim_{n\to\infty} \int \left\langle \nabla u^{h_n}_{k_n}\,,\, \nabla\xi \right\rangle \mathrm{d}x\,. \end{aligned}$$

In Step 6, we moreover have shown that $\partial_t u^{h_n}$ converges weakly to $\partial_t u$ in $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. This yields, by choosing cylindrical testfunctions, that $\partial_t u^{h_n}(t)$ converges weakly to $\partial_t u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ for almost every $t \in [0,T]$. Thus we obtain for almost every $t \in [0,T]$ the convergence

$$\int \langle \partial_t u(t)\,,\, \xi \rangle \,\mathrm{d}x = \lim_{n\to\infty} \int \left\langle \partial_t u^{h_n}(t)\,,\, \xi \right\rangle \mathrm{d}x = \lim_{n\to\infty} \int \left\langle \frac{u^{h_n}_{k_n} - u^{h_n}_{k_n-1}}{h_n}\,,\, \xi \right\rangle \mathrm{d}x\,.$$

To obtain the weak equation, we still need to prove convergence of remaining term. For this we note that

$$\left| \left\langle \nabla W(u^{h_n}_{k_n})\,,\, \xi \right\rangle \right| \lesssim \left(1 + \left| u^{h_n}_{k_n} \right|^{p-1} \right) |\xi|,$$

which is a bounded sequence in $\mathrm{L}^{p'}(\mathbb{T};\mathbb{R}^N)$. Since we also may pass to a non-relabelled subsequence which converges almost everywhere, we thus obtain by the continuity of ∇W that

$$\int \langle \nabla W(u(t))\,,\, \xi \rangle \,\mathrm{d}x = \lim_{n\to\infty} \int \left\langle \nabla W(u^{h_n}_{k_n})\,,\, \xi \right\rangle \mathrm{d}x\,.$$

Therefore it follows from the Euler–Lagrange equation (2.18) that

$$\int \frac{1}{\varepsilon^2}\langle \nabla W(u(t)), \xi\rangle + \langle \nabla u(t), \nabla \xi\rangle + \langle \partial_t u(t), \xi\rangle \,\mathrm{d}x = 0$$

for almost every t and all $\xi \in \mathrm{C}^\infty(\mathbb{T};\mathbb{R}^N)$. By continuity this extends to all $\xi \in \mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$.

Step 8: Sharp energy dissipation inequality for u.

Let $0 \le t \le T$ and define k_n as in Step 7, where we also established that $u_{k_n}^{h_n}$ converges to $u(t)$ in $\mathrm{L}^2\left(\mathbb{T};\mathbb{R}^N\right)$, and thus we may pass to another non-relabelled subsequence to obtain pointwise convergence almost everywhere. By Fatou's Lemma, we thus obtain

$$\int \frac{1}{\varepsilon} W(u(t))\,\mathrm{d}x \le \liminf_{n\to\infty} \int \frac{1}{\varepsilon} W(u_{k_n}^{h_n})\,\mathrm{d}x\,.$$

Moreover we can deduce from the L^2-convergence that $\nabla u_{k_n}^{h_n}$ converges to $\nabla u(t)$ in the distributional sense. But $\nabla u_{k_n}^{h_n}$ is uniformly bounded in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ by the energy dissipation inequality (2.14), thus we already obtain that $\nabla u_{k_n}^{h_n}$ converges weakly to $\nabla u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ which yields

$$\int \frac{\varepsilon}{2}|\nabla u(t)|^2\,\mathrm{d}x \le \liminf_{n\to\infty} \int \frac{\varepsilon}{2}\left|\nabla u_{k_n}^{h_n}\right|^2\mathrm{d}x\,.$$

Lastly we have by the weak convergence of $\partial_t u^{h_n}$ to $\partial_t u$ in $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$ proven in Step 6 that

$$\begin{aligned}\int_0^t \int \varepsilon|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{n\to\infty}\left(1-\frac{Ch_n}{2\varepsilon^2}\right)\int_0^t\int \varepsilon\left|\partial_t u^{h_n}\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\le \liminf_{n\to\infty}\left(1-\frac{Ch_n}{2\varepsilon^2}\right)\int_0^{k_n h_n}\int \varepsilon\left|\partial_t u^{h_n}\right|^2\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Summarizing these estimates, we obtain by the energy dissipation inequality for u^h (2.16) that for all $0 \le t \le T$ we have

$$\begin{aligned}\mathrm{E}_\varepsilon(u(t)) + \int_0^t\int \varepsilon|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{n\to\infty} \mathrm{E}_\varepsilon(u_{k_n}) + \left(1-\frac{Ch_n}{2\varepsilon^2}\right)\int_0^{k_n h_n}\int \varepsilon\left|\partial_t u^h\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\le \mathrm{E}_\varepsilon(u_\varepsilon^0).\end{aligned}$$

Step 8: Additional regularity 2.

In order to complete the proof, we still have to show that $\partial_{i,j}^2 u$ and $\nabla W(u)$ are elements of $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. In order to show that the second partial derivatives of u are square-integrable, we test the weak formulation (2.9) with the finite differences. To this end, define the finite differences as

$$\Delta_h^+ v(t,x) := \frac{v(t,x+hv)-v(t,x)}{h} \quad \text{and} \quad \Delta_h^- v(t,x) := \frac{v(t,x-hv)-v(t,x)}{h}$$

for some $h > 0$ and $v \in \mathbb{R}^d$. Thus plugging $\Delta_h^- \Delta_h^+ u$ into (2.9) yields by the transformation formula that

$$0 = \int_0^T \int \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u) \, , \, \Delta_h^+ u \rangle + \langle \nabla \Delta_h^+ u \, , \, \nabla \Delta_h^+ u \rangle + \langle \partial_t \Delta_h^+ u \, , \, \Delta_h^+ u \rangle \, \mathrm{d}x \, \mathrm{d}t \, ,$$

which is equivalent to

$$\begin{aligned}
&\int_0^T \int \left| \Delta_h^+ \nabla u \right|^2 \mathrm{d}x \, \mathrm{d}t \\
&= - \int_0^T \int \partial_t \left(\frac{\left|\Delta_h^+ u\right|^2}{2} \right) + \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u) \, , \, \Delta_h^+ u \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&= \int \frac{\left|\Delta_h^+ u(0)\right|^2 - \left|\Delta_h^+ u(T)\right|^2}{2} \, \mathrm{d}x \\
&\quad - \int_0^1 \int_0^T \int \langle \mathrm{D}^2 \, W \left((1-s) u(t,x) + s u(t, x + hv) \right)) \, \Delta_h^+ u \, , \, \Delta_h^+ u \rangle \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}s \, .
\end{aligned}$$

The first summand can be estimated by $\|u\|_{\mathrm{L}^\infty([0,T], \mathrm{W}^{1,2}(\mathbb{T}; \mathbb{R}^N))}$ (we are working with the representative chosen before). For the second summand, we partition W into the sum of W_{conv} and W_{pert}. The term involving the convex summand can then by estimated by

$$\int_0^1 \int_0^T \int \langle \mathrm{D}^2 \, W_{\mathrm{conv}} \left((1-s) u(t,x) + s u(t, x + hv) \right) \Delta_h^+ u \, , \, \Delta_h^+ u \rangle \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}s \geq 0$$

and the for the pertubation term, we get via the bound on its second derivative that

$$\begin{aligned}
&\left| \int_0^1 \int_0^T \int \langle \mathrm{D}^2 \, W_{\mathrm{pert}} \left((1-s) u(t,x) + s u(t, x + hv) \right) \Delta_h^+ u \, , \, \Delta_h^+ u \rangle \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}s \right| \\
&\lesssim \int_0^T \int |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \, ,
\end{aligned}$$

which is also finite. Combining these estimates, we obtain that $\int_0^T \int \left|\Delta_h^+ \nabla u\right|^2 \mathrm{d}x \, \mathrm{d}t$ is uniformly bounded in h. Applying our calculation to all directions $v \in \mathbb{R}^d$, we get by the finite-differences theorem for all $1 \leq i, j \leq d$ that $\partial_{i,j}^2 u \in \mathrm{L}^2([0,T] \times \mathbb{T}; \mathbb{R}^N)$.

In order to obtain $\nabla W(u) \in \mathrm{L}^2([0,T] \times \mathbb{T}; \mathbb{R}^N)$, we again consider the weak formulation (2.9) and notice that since we have already shown that both the time derivative and second space derivatives of u are square-integrable, our claim follows from a duality argument.

This completes the proof. □

Remark 2.2.2. The inequality (2.14) with the factor $1/2h$ instead of $1/h - C/2\varepsilon^2$ follows immediately from the definition of our optimization problem, but is not optimal for fixed ε if we want to study the behaviour as h tends to zero.

Remark 2.2.3. The energy dissipation inequality (2.12) can be deduced via the formal calculation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon}\langle \nabla W(u)\,,\, \partial_t u\rangle + \varepsilon \langle \nabla u\,,\, \nabla \partial_t u\rangle \,\mathrm{d}x \\ &= \int \left\langle \frac{1}{\varepsilon}\nabla W(u) - \varepsilon \Delta u\,,\, \partial_t u \right\rangle \mathrm{d}x \\ &= -\varepsilon \int |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,.\end{aligned}$$

In order to make this calculation rigorous, we however need to show that $t \mapsto \mathrm{E}_\varepsilon(u(t))$ is absolutely continuous, which is non-trivial, but it would give us equality in energy dissipation inequality (2.12). Since we will only need the inequality, our proof will however suffice.

3. Convergence to an evolving partition

The main content of this chapter is to look at the behaviour of solutions of the Allen–Cahn equation (2.1) as the parameter ε tends to zero with minimal assumptions. As it turns out the scalar case is significantly easier to handle than the vectorial case, thus we shall first focus on the case $N = 1$ and $P = 2$, also called the *two-phase case*. We will show that as ε tends to zero, u_ε converges to an evolving partition of the flat torus which only takes values in the zeros of the potential W.

3.1. Convergence in the two-phase case

This section is again based on [LS16], but the proofs are done more thoroughly, for example the existence The proof in this section is loosely based on its multiphase version presented in [LS16], but the proofs simplify in the two-phase case and thus gives an easier introduction into the topic.

The only assumption for now that we make is that the energies of the initial functions $\mathrm{E}_\varepsilon(u_\varepsilon^0)$ stay uniformly bounded as ε tends to zero. Then due to the energy dissipation inequality (2.12), we already obtain that for all $0 \leq t \leq T$, we have that $\mathrm{E}_\varepsilon\left(u_\varepsilon(t)\right)$ stays uniformly bounded.

Another important observation for the convergence is the classic Modica Mortola trick ([MM77]) : Let $\alpha < \beta$ be the two distinct zeros of the doublewell potential W. Then we define a primitive of $\sqrt{2W(u)}$ via

$$\phi(u) := \int_\alpha^u \sqrt{2W(s)}\,\mathrm{d}s\,.$$

For $\psi_\varepsilon := \phi \circ u_\varepsilon$, we can show that $\psi_\varepsilon \in \mathrm{W}^{1,1}((0,T)\times\mathbb{T})$ with weak derivatives $\nabla\psi_\varepsilon = \sqrt{2W(u_\varepsilon)}\nabla u_\varepsilon$ and $\partial_t\psi_\varepsilon = \sqrt{2W(u_\varepsilon)}\partial_t u_\varepsilon$.

Thus via Young's inequality, we can compute

$$\begin{aligned}
\mathrm{E}_\varepsilon(u_\varepsilon) &= \int \frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\,\mathrm{d}x \\
&\geq \int \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x \\
&= \int |\nabla\psi_\varepsilon|\,\mathrm{d}x\,, \qquad (3.1)
\end{aligned}$$

which suggests that we might hope for good compactness properties of $\phi\circ u_\varepsilon$. Moreover since the energies stay bounded and the summand $1/\varepsilon W(u)$ penalizes any mass outside of the wells, we expect that a limit function is concentrated in α and β. Indeed this is given by the following Proposition.

Proposition 3.1.1. *Given well prepared initial data u_ε^0 in the sense that*

$$\lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon^0) = \mathrm{E}(u^0) =: \mathrm{E}_0 < \infty, \tag{3.2}$$

there exists for any sequence $\varepsilon \to 0$ some non-relabelled subsequence such that the solutions of the Allen–Cahn equation (2.1) with initial condition u_ε^0 converge in $\mathrm{L}^1((0,T)\times\mathbb{T})$ to some $u = \alpha(1-\chi)+\beta\chi$ with $\chi \in \mathrm{BV}((0,T)\times\mathbb{T};\{0,1\})$. Moreover the compositions ψ_ε are uniformly bounded in $\mathrm{BV}((0,T)\times\mathbb{T})$ and converge to $\phi\circ u$ in $\mathrm{L}^1((0,T)\times\mathbb{T})$.

Proof. From the energy dissipation inequality (2.12) in Theorem 2.2.1 we infer that for all $\varepsilon > 0$, it holds that

$$\sup_{0\le t\le T} \mathrm{E}_\varepsilon(u_\varepsilon(t)) \le \mathrm{E}_\varepsilon(u_\varepsilon^0), \tag{3.3}$$

whose right hand side is by assumption uniformly bounded in ε. By the calculation (3.1) we thus infer that $\nabla\psi_\varepsilon$ is uniformly bounded in $\mathrm{L}^1((0,T)\times\mathbb{T})$. Moreover we may estimate by the upper and lower growth bound on W (2.3)

$$\begin{aligned}
\int_0^T\int |\psi_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int \left|\int_\alpha^{u_\varepsilon}\sqrt{2W(s)}\right|\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^T\int |u_\varepsilon-\alpha| \sup_{s\in[\alpha,u_\varepsilon]}\sqrt{2W(s)}\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim 1+\int_0^T\int |u_\varepsilon|^{1+p/2}\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim 1+\int_0^T\int W(u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}\tag{3.4}$$

which is uniformly bounded via the energy bound. Thus we have indeed that ψ_ε is a bounded sequence in $\mathrm{BV}((0,T)\times\mathbb{T})$ and therefore there exists some non-relabelled subsequence and some $\psi\in\mathrm{BV}((0,T)\times\mathbb{T})$ such that ψ_ε converges to ψ in $\mathrm{L}^1((0,T)\times\mathbb{T})$.

We notice that since W is non-negative and only has a discrete set of zeros, the function ϕ is strictly increasing and continuous on $\mathbb{R}$, and is thus invertible. Moreover we may pass to a further non-relabelled subsequence of ψ_ε which converges almost everywhere to ψ. Thus defining $u := \phi^{-1}(\psi)$, we obtain that

$$u_\varepsilon = \phi^{-1}(\psi_\varepsilon) \to \phi^{-1}(\psi) = u$$

converges pointwise almost everywhere.

Moreover we notice that by Fatou's Lemma and the boundedness of the energies that

$$\int W(u)\,\mathrm{d}x \le \liminf_{\varepsilon\to0}\int W(u_\varepsilon)\,\mathrm{d}x \le \liminf_{\varepsilon\to0}\varepsilon\,\mathrm{E}_\varepsilon(u_\varepsilon) = 0.$$

Again using the non-negativity of W, this yields that $W(u) = 0$ almost everywhere. Thus $u \in \{\alpha, \beta\}$ almost everywhere and we may write $u = \alpha(1-\chi) + \beta\chi$ for some $\chi \colon (0,T) \times \mathbb{T} \to \{0,1\}$. Looking again at the definition of u, we moreover obtain that

$$\psi = \phi(u) = \phi(\alpha)(1-\chi) + \phi(\beta)\chi = \int_\alpha^\beta \sqrt{2W(s)}\,\mathrm{d}s\,\chi =: \sigma\chi \tag{3.5}$$

and since ψ is a function of bounded variation, this implies that χ is of bounded variation as well.

Finally from the energy bound and the estimate $|u_\varepsilon|^p \lesssim 1 + W(u_\varepsilon)$, we infer that u_ε is L^p-bounded, and since u_ε converges pointwise almost everywhere to u, we obtain the desired L^1-convergence. □

Notice that in the prof, we defined the constant $\sigma := \int_\alpha^\beta \sqrt{2W(s)}\,\mathrm{d}s$, which will later be the surface tension for the mean curvature flow. Nextup, we want to make sure that u respectively χ assume their initial data and on the way obtain a useful bound on the time derivative.

Lemma 3.1.2. *With the assumptions of Proposition 3.1.1, we have $\psi_\varepsilon \in \mathrm{W}^{1,2}([0,T];\mathrm{L}^1(\mathbb{T}))$ with the estimate*

$$\left(\int_0^T \left(\int |\partial_t \psi_\varepsilon|\,\mathrm{d}x\right)^2 \mathrm{d}t\right)^{1/2} \lesssim \mathrm{E}_\varepsilon(u_\varepsilon^0). \tag{3.6}$$

Furthermore the sequence u_ε is precompact in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$.

Proof. The desired regularity will follow quite directly from Hölder's inequality. For the desired precompactness, we will first show precompactness of ψ_ε by Arzelà–Ascoli, which implies that u_ε converges in measure. Using the equiintegrability of u_ε uniformly in time, we may then deduce our claim.

Step 1: $\psi_\varepsilon \in \mathrm{W}^{1,2}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$ and satisfies the inequality (3.6).

Using the same estimates as in (3.4) we can infer that

$$\int_0^T \left(\int |\psi_\varepsilon|\,\mathrm{d}x\right)^2 \mathrm{d}t \lesssim \int_0^T \left(1 + \int W(u_\varepsilon)\,\mathrm{d}x\right)^2 \mathrm{d}t < \infty$$

via the uniform boundedness of the energies (3.3). For the desired bound (3.6), we estimate via Hölder's inequality and the uniform boundedness of the energies that

$$\begin{aligned}\int_0^T \left(\int |\partial_t \psi_\varepsilon| \,\mathrm{d}x\right)^2 \mathrm{d}t &= \int_0^T \left(\int \sqrt{2W(u_\varepsilon)}|\partial_t u_\varepsilon| \,\mathrm{d}x\right)^2 \mathrm{d}t \\ &= \int_0^T \left(\int \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\sqrt{\varepsilon}|\partial_t u_\varepsilon| \,\mathrm{d}x\right)^2 \mathrm{d}t \\ &\le \int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon)\,\mathrm{d}x \int \varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le 2\,\mathrm{E}_\varepsilon(u_\varepsilon^0) \int_0^T \int \varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le 2\left(\mathrm{E}_\varepsilon(u_\varepsilon^0)\right)^2,\end{aligned}$$

where the last two inequalities follow from the energy dissipation estimate (2.12).

Step 2: The sequence ψ_ε is precompact in $\mathrm{C}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$.

As noted in embedding (2.11), we have

$$\mathrm{W}^{1,2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right) \hookrightarrow \mathrm{C}^{1/2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)$$

from which the equicontinuity of the sequence follows with Step 1. Moreover for fixed time t, the arguments from Proposition 3.1.1 yield that $\psi_\varepsilon(t)$ is a bounded sequence in $\mathrm{W}^{1,1}(\mathbb{T})$ and is thus precompact in $\mathrm{L}^1(\mathbb{T})$. Thus the Arzelà–Ascoli Theorem yields the desired precompactness.

Step 3: The sequence u_ε converges to u in measure uniformly in time.

From Step 2, we infer that

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int |\psi_\varepsilon(t,x) - \phi\circ u(t,x)| \,\mathrm{d}x = 0.$$

It especially follows in combination with equation (3.5) that

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int (1-\chi)|\psi_\varepsilon(t,x)| \,\mathrm{d}x = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int (1-\chi)|\psi_\varepsilon(t,x) - \phi\circ u(t,x)| \,\mathrm{d}x = 0$$

and

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int \chi|\psi_\varepsilon(t,x) - \sigma| \,\mathrm{d}x = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int \chi|\psi_\varepsilon(t,x) - \phi\circ u(t,x)| \,\mathrm{d}x = 0$$

Moreover since ϕ is strictly increasing we have that for a given $\delta < \beta - \alpha$,

$$\min\left(\inf\{|\phi(s)| : \ |s-\alpha| > \delta/2\}, \inf\{|\phi(s)-\sigma| : \ |s-\beta| > \delta/2\}\right) = \rho > 0.$$

Therefore we may estimate

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\{x\in\mathbb{T}:\ |u_\varepsilon(t,x)-u(t,x)|>\delta\}\right)\\
\le\ &\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\{(1-\chi(t,x))|u_\varepsilon(t,x)-\alpha|>\delta/2\}\right)+\mathcal{L}^d\left(\{\chi(t,x)|u_\varepsilon(t,x)-\beta|>\delta/2\}\right)\\
\le\ &\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\{(1-\chi(t,x))|\psi_\varepsilon(t,x)|>\rho\}\right)+\mathcal{L}^d\left(\{\chi(t,x)|\psi_\varepsilon(t,x)-\sigma|>\rho\}\right)\\
\le\ &\operatorname*{ess\,sup}_{0\le t\le T} \frac{1}{\rho}\int(1-\chi(t,x))|\psi_\varepsilon(t,x)|\,\mathrm{d}x+\frac{1}{\rho}\int\chi(t,x)|\psi_\varepsilon(t,x)-\sigma|\,\mathrm{d}x\,,
\end{aligned}$$

which vanishes as ε goes to zero, proving our claim.

Step 4: u_ε^2 is equiintegrable uniformly in time.

We have

$$0\le u_\varepsilon^2\lesssim 1+W(u_\varepsilon)$$

by the growth bounds (2.3) on W. Since $W(u_\varepsilon)$ converges to 0 in $\mathrm{L}^1(\mathbb{T})$ uniformly in time by the energy bound (3.3), it is equiintegrable uniformly in time. Thus u_ε^2 is equiintegrable uniformly in time as well.

Step 5: u_ε converges in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$.

We somewhat repeat the proof that convergence in measure and equiintegrability imply L^1-convergence, while keeping the uniformity in time.

Take some $\delta>0$. Then we decompose the integral

$$\int|u_\varepsilon-u|^2\,\mathrm{d}x=\int_{\{|u_\varepsilon-u|\ge\delta\}}|u_\varepsilon-u|^2\,\mathrm{d}x+\int_{\{|u_\varepsilon-u|<\delta\}}|u_\varepsilon-u|^2\,\mathrm{d}x\,.$$

For the first summand, we notice that

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_{\{|u_\varepsilon-u|\ge\delta\}}|u_\varepsilon-u|^2\,\mathrm{d}x\lesssim\operatorname*{ess\,sup}_{0\le t\le T}\int_{\{|u_\varepsilon-u|\ge\delta\}}1+u_\varepsilon^2\,\mathrm{d}x\to 0$$

as ε goes to zero since $\mathcal{L}^d\left(|u_\varepsilon-u|\ge\delta\right)$ converges to zero uniformly in time by Step 3 and u_ε^2+1 is equiintegrable uniformly in time by Step 4. For the second summand, we simply estimate

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_{\{|u_\varepsilon-u|<\delta\}}|u_\varepsilon-u|^2\,\mathrm{d}x=\delta^2\Lambda^d$$

Taking the limes superior as ε tends to zero of this inequality yields that the right hand side can be made arbitrarily small, which yields

$$\operatorname*{ess\,sup}_{0\le t\le T}\int|u_\varepsilon-u|^2\,\mathrm{d}x\to 0$$

as ε tends to zero.

This concludes the proof. □

Remark 3.1.3. From the previous Lemma, it follows that if the initial conditions u_ε^0 converge in L^1 or pointwise almost everywhere to the function $\alpha(1-\chi^0)+\beta\chi^0$ (and we know that the limit is of this form since the energies of the initial values stay bounded), then u also assumes this initial data in $\mathrm{L}^2(\mathbb{T})$.

3.2. Convergence in the multiphase case

Let us now turn to the much more interesting and more challenging case where we consider systems of the Allen–Cahn equation (2.1), or in other words, we want to consider the case where u_ε maps to $\mathbb{R}^N$ and therefore our potential W is a map from $\mathbb{R}^N$ to $[0,\infty)$. The for us most relevant case is when W has exactly $P = N+1$ zeros given by $\alpha_1,\dots,\alpha_P$, but it is no limitation for us to allow more general amounts of zeros.

Let us also fix some notation for this section. For a function $u = \sum_{i=1}^P \mathbb{1}_{\Omega_i}\alpha_i$ with $\Omega_i \in \mathrm{BV}$ and the corresponding interfaces $\Sigma_{ij} \coloneqq \partial_*\Omega_i \cap \partial_*\Omega_j$, we define for a given testfunction $\varphi \in \mathrm{C}^\infty(\mathbb{T})$ the localized energies by

$$\mathrm{E}_\varepsilon(u_\varepsilon;\varphi) \coloneqq \int \varphi\left(\frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\right)\mathrm{d}x \tag{3.7}$$

and

$$\mathrm{E}(u;\varphi) \coloneqq \sum_{i<j}\sigma_{ij}\int_{\Sigma_{ij}}\varphi\,\mathrm{d}\,\mathcal{H}^{d-1}. \tag{3.8}$$

The latter is the localized version of the weighted perimeter functional introduced in Section 1.2.

One of the many difficulties in the vectorial case is that that there is no easy choice of a primitive for $\sqrt{2W(u)}$ compared to the scalar case. We there saw for example through the Modica–Mortola trick (3.1) that this provided a very powerful tool for us, and comparing the composition $\phi\circ u_\varepsilon$ to u_ε was quite simple since the map ϕ was invertible as a consequence of the non-negativity of W.

As a suitable replacement, we shall consider the *geodesic distance* defined as

$$\mathrm{d_W}(u,v) \coloneqq \inf\left\{\int_0^1 \sqrt{2W(\gamma)}|\dot\gamma|\,\mathrm{d}t \,:\, \gamma\in\mathrm{C}^1\left([0,1];\mathbb{R}^N\right) \text{ with } \gamma(0)=u,\, \gamma(1)=v\right\}.$$

This indeed defines a metric on $\mathbb{R}^N$: If $\mathrm{d_W}(v,w)=0$, then by the continuity of W and since it only has a discrete set of zeros, we may deduce that $v=w$. Symmetry can be seen by reversing a given path between two points and the triangle inequality follows from concatenation of two paths and rescaling. Note moreover that by an approximation argument (for example through splines), we may also take paths γ which are piecewise continuously differentiable which makes constructions of paths easier.

The *geodesic distances* generated by W are defined as

$$\sigma_{ij} \coloneqq \mathrm{d_W}(\alpha_i,\alpha_j)$$

and as a consequence of $\mathrm{d_W}$ being a metric satisfy

$$\sigma_{ik} \leq \sigma_{ij} + \sigma_{jk},$$

$\sigma_{ij} = 0$ if and only if i is equal to j and $\sigma_{ij} = \sigma_{ji}$.

Our replacement for the primitive ϕ is now given for $1 \leq i \leq P$ by the geodesic distance function

$$\phi_i(u) \coloneqq \mathrm{d_W}(\alpha_i, u).$$

Our first obstacle is the regularity of the function $\psi_{\varepsilon,i} \coloneqq \phi_i \circ u_\varepsilon$. A priori we only know that ϕ_i is locally Lipschitz continuous on $\mathbb{R}^N$ and thus differentiable almost everywhere. If $N = 1$, then this would already suffice to deduce that $\psi_{\varepsilon,i}$ is weakly differentiable, but in higher dimensions, u could for example move along a hypersurface where ϕ_i could in theory be nowhere differentiable since the hypersurface is a Lebesgue nullset. This can however be salvaged through the following chain rule for distributional derivatives by Ambrosio and Maso [AM90, Cor. 3.2].

Theorem 3.2.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set, $p \in [1,\infty]$, $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ and let $f\colon \mathbb{R}^N \to \mathbb{R}^k$ be a Lipschitz continuous function such that $f(0) = 0$. Then $v \coloneqq f \circ u \in \mathrm{W}^{1,p}\left(\Omega, \mathbb{R}^k\right)$. Furthermore for almost every $x \in \Omega$ the restriction of f to the affine space*

$$T_x^u \coloneqq \left\{ y \in \mathbb{R}^N : \ y = u(x) + \mathrm{D}\, u(x) z \text{ for some } z \in \mathbb{R}^d \right\} = u(x) + \dot{T}_x^u$$

is differentiable at $u(x)$ and

$$\mathrm{D}\, v = \mathrm{D}\left(f|_{T_x^u}\right)(u)\, \mathrm{D}\, u$$

holds almost everywhere in Ω.

Remark 3.2.2. The matrix $\mathrm{D}\left(f|_{T_x^u}\right)(u)$ can be interpreted as some matrix in $\mathbb{R}^{k\times n}$ which acts on $v \in \dot{T}_x^u$ by

$$\mathrm{D}\left(f|_{T_x^u}\right)(u(x))[v] = \lim_{h\to 0} \frac{f(u(x)+hv) - f(u(x))}{h}.$$

Thus we may choose a suitable representative since the product $\mathrm{D}\left(f|_{T_x^u}\right)(u)\,\mathrm{D}\, u$ will not change by definition of $\dot{T}_x^u$.

Moreover the assumption $f(0) = 0$ can be left out on bounded domains by simply subtracting the constant $f(0)$.

We are now in the position to prove the following regularity result.

Lemma 3.2.3. *Let $u \in \mathrm{C}\left([0,T]; \mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)$ with*

$$\operatorname*{ess\,sup}_{0\leq t\leq T} \mathrm{E}_\varepsilon(u) + \int_0^T \int \varepsilon |\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty$$

for some $\varepsilon > 0$. Then for all $1 \leq i \leq P$ there exists a map

$$\mathrm{D}\,\phi_i(u)\colon [0,T]\times\mathbb{T} \to \mathrm{Lin}\left(\mathbb{R}^N, \mathbb{R}\right)$$

such that the chain rule is valid with the pair $\mathrm{D}\,\phi_i(u)$ *and* $(\partial_t, \mathrm{D})u$*:*
For almost every $(t,x) \in [0,T] \times \mathbb{T}$ *we have*

$$\mathrm{D}(\phi_i \circ u) = \mathrm{D}\,\phi_i(u)\,\mathrm{D}\,u \quad \textit{and} \quad \partial_t(\phi_i \circ u) = \mathrm{D}\,\phi_i(u)\partial_t u.$$

Furthermore we can control the modulus of $\mathrm{D}\,\phi_i(u)$ *almost everywhere in time and space via the estimate*

$$|\mathrm{D}\,\phi_i(u)| \leq \sqrt{2W(u)}. \tag{3.9}$$

Additionally we have $\phi_i \circ u \in \mathrm{L}^\infty\left([0,T]; \mathrm{W}^{1,1}(\mathbb{T})\right) \cap \mathrm{W}^{1,1}([0,T] \times \mathbb{T})$ *with the estimates*

$$\operatorname*{ess\,sup}_{0 \leq t \leq T} \int |\phi_i \circ u| \,\mathrm{d}x \lesssim 1 + \operatorname*{ess\,sup}_{0 \leq t \leq T} \varepsilon\,\mathrm{E}_\varepsilon(u), \tag{3.10}$$

$$\operatorname*{ess\,sup}_{0 \leq t \leq T} \int |\nabla(\phi_i \circ u)| \,\mathrm{d}x \leq \operatorname*{ess\,sup}_{0 \leq t \leq T} \mathrm{E}_\varepsilon(u) \tag{3.11}$$

and

$$\int_0^T \int |\partial_t(\phi_i \circ u)| \,\mathrm{d}x\,\mathrm{d}t \leq T \operatorname*{ess\,sup}_{0 \leq t \leq T} \mathrm{E}_\varepsilon(u) + \int_0^T \int \varepsilon |\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t\,. \tag{3.12}$$

Proof. Since Theorem 3.2.1 requires Lipschitz continuity of ϕ_i, but we only have local Lipschitz continuity, let us first assume that u is bounded in space and time. Then we may modify ϕ_i outside of a compact set such that it is (globally) Lipschitz continuous and does not change on the image of u.

Since we have via the energy estimate that $u \in \mathrm{W}^{1,2}([0,T] \times \mathbb{T}; \mathbb{R}^N)$, we obtain by the distributional chain rule (Theorem 3.2.1) that $\psi_i = \phi_i \circ u \in \mathrm{W}^{1,2}([0,T] \times \mathbb{T})$.

Let $\Pi(t,x)$ denote the orthogonal projection of $\mathbb{R}^N$ onto $\dot{T}^u_{t,x}$ and define

$$\mathrm{D}\,\phi_i(u)(t,x)[v] \coloneqq \mathrm{D}(\phi_i|_{T^u_{t,x}})(u(t,x))[\Pi(t,x)v].$$

This defines a unique row vector and thus we can now proceed to prove inequality (3.9). Let $v \in \dot{T}^u_{t,x}$, let $h \in \mathbb{R} \setminus \{0\}$ and let $\gamma\colon [0,1] \to \mathbb{R}^N$ be a path connecting α_i and u. Then we define the new path $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^N$ by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(2t) & , t \leq \frac{1}{2} \\ u + \left(t - \frac{1}{2}\right) 2hv & , t \geq \frac{1}{2}. \end{cases}$$

We observe that $\tilde{\gamma}$ is a piecewise continuously differentiable path connecting α_i and $u + hv$, thus we can estimate by a substitution that

$$\begin{aligned} &\mathrm{d_W}(\alpha_i, u + hv) - \int_0^1 \sqrt{2W(\gamma(t))}|\gamma'(t)| \,\mathrm{d}t \\ \leq &\int_0^1 \sqrt{2W(\tilde{\gamma}(t))}|\tilde{\gamma}'(t)| \,\mathrm{d}t - \int_0^1 \sqrt{2W(\gamma(t))}|\gamma'(t)| \,\mathrm{d}t \\ = &\int_0^1 \sqrt{2W(u + thv)}|hv| \,\mathrm{d}t\,. \end{aligned}$$

Taking the infimum over all C^1-paths connecting α_i and u yields

$$\mathrm{d_W}(\alpha_i, u+hv) - \mathrm{d_W}(\alpha_i, u) \le \int_0^1 \sqrt{2W(u+thv)}|hv| \,\mathrm{d}t\,.$$

Using a similar strategy but with a reversed path moreover yields the inequality

$$\mathrm{d_W}(\alpha_i, u) - \mathrm{d_W}(\alpha_i, u+hv) \le \int_0^1 \sqrt{2W(u+thv)}|hv| \,\mathrm{d}t\,.$$

Thus we obtain by the dominated convergence theorem

$$\limsup_{h\to 0}\left|\frac{\phi(u+hv)-\phi(u)}{h}\right| \le \limsup_{h\to 0}\int_0^1 \sqrt{2W(u+thv)}|v| \,\mathrm{d}t = \sqrt{2W(u)}|v|,$$

which yields

$$\left|\mathrm{D}(\phi_i|_{T^u_{t,x}})(u)[v]\right| \le \sqrt{2W(u)}|v|$$

and thus gives us the desired inequality (3.9) since $|\Pi(v)| \le |v|$.

Now let us consider the general case and denote by u_M the truncation of u defined by

$$u^j_M := \begin{cases} u & , \text{ if} \left|u^j\right| \le M \\ M\frac{u^j}{|u^j|} & , \text{ else.} \end{cases}$$

Then we still have $u_M \in \mathrm{W}^{1,2}([0,T]\times\mathbb{T})$ and obtain by the previous step that $\phi_i \circ u_M \in \mathrm{W}^{1,2}([0,T]\times\mathbb{T})$ and that for almost every $(t,x)\in[0,T]\times\mathbb{T}$, the function ϕ_i is differentiable on $T^{u_M}_{t,x}$. Moreover if $(t,x)\in u^{-1}([-M,M]^N)$, then we obtain $T^{u_M}_{t,x} = T^u_{t,x}$. Next we we want to show that $\phi_i\circ u_M$ converges to $\phi_i \circ u$ in a suitable sense. First we recognize that $\phi_i\circ u_M$ converges to $\phi_i\circ u$ pointwise almost everywhere. Moreover we find a majorant since

$$\begin{aligned}\phi_i(v) &\le \int_0^1 \sqrt{2W(\alpha_i + s(v-\alpha_i))}|v-\alpha_i| \,\mathrm{d}s \le \left\|\sqrt{2W}\right\|_{\mathrm{L}^\infty[v,\alpha_i]}|v-\alpha_i| \\ &\lesssim 1+|v|^{1+p/2},\end{aligned} \tag{3.13}$$

thus we have $\phi_i\circ u_M \lesssim 1+|u|^p$, which is in integrable majorant. Therefore the dominated convergence theorem yields that $\phi_i\circ u_M$ converges to $\phi_i\circ u$ in $\mathrm{L}^1([0,T]\times\mathbb{T})$. Moreover estimate (3.13) together with the p-growth of W (2.3) already yield the desired L^1-estimate (3.10).

Furthermore we recognize that on the set $\{u_M = u\}$, we already have

$$(\partial_t, \mathrm{D})u_M = (\partial_t,\mathrm{D})u$$

and the sets $\{u_M = u\}$ are non-decreasing, thus

$$\lim_{M\to\infty} |\{u_M \ne u, (\partial_t,\mathrm{D})u_M \ne (\partial_t,\mathrm{D})u\}| = 0.$$

Moreover we have that for almost every (t,x), the derivative $\mathrm{D}\,\phi_i(u_M(t,x))$ eventually becomes stationary. We denote its almost everywhere pointwise limit by $\mathrm{D}\,\phi_i(u)$ and it satisfies almost everywhere

$$|\mathrm{D}\,\phi_i(u)| \leq \sqrt{2W(u)}.$$

In order to show that $\phi_i \circ u$ is weakly differentiable with derivative

$$(\partial_t, \mathrm{D})\phi_i \circ u = \mathrm{D}\,\phi_i(u)(\partial_t u, \mathrm{D}\,u),$$

we compute for a testfunction φ that by the L^1-convergence, we have

$$\begin{aligned}
\int_0^T \int \phi_i \circ u(\partial_t, \mathrm{D})\varphi \, \mathrm{d}x\, \mathrm{d}t &= \lim_{M\to\infty} \int_0^T \int \phi_i \circ u_M(\partial_t, \mathrm{D})\varphi \, \mathrm{d}x\, \mathrm{d}t \\
&= -\lim_{M\to\infty} \int \mathrm{D}\,\phi_i(u_M)(\partial_t, \mathrm{D})u_M \varphi \, \mathrm{d}x\, \mathrm{d}t \\
&= -\lim_{M\to\infty} \int_{|u|\leq M} \mathrm{D}\,\phi_i(u)(\partial_t, \mathrm{D})u\varphi \, \mathrm{d}(t,x) \\
&= \int_0^T \int \mathrm{D}\,\phi_i(u)(\partial_t, \mathrm{D})u\varphi \, \mathrm{d}x\, \mathrm{d}t \,.
\end{aligned}$$

The last inequality is due to the dominated convergence theorem with majorant $\sqrt{2W(u)}|(\partial_t, \mathrm{D})u||\varphi|$. It remains to prove the estimates (3.11) and (3.12). These follow from applications of Young's inequality

$$\begin{aligned}
\operatorname*{ess\,sup}_{0\leq t\leq T} \int |\mathrm{D}(\phi_i \circ u)| &= \operatorname*{ess\,sup}_{0\leq t\leq T} \int |\mathrm{D}\,\phi_i(u)\,\mathrm{D}\,u| \,\mathrm{d}x \\
&\leq \operatorname*{ess\,sup}_{0\leq t\leq T} \int \sqrt{2W(u)}|\nabla u| \,\mathrm{d}x \\
&\leq \sup_{0\leq t\leq T} \mathrm{E}_\varepsilon(u_\varepsilon)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \int |\partial_t(\phi_i \circ u)| \,\mathrm{d}x\,\mathrm{d}t &\leq \int_0^T \int \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u)}\sqrt{\varepsilon}|\partial_t u| \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^T \int \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2}|\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t \,,
\end{aligned}$$

which finishes our proof. □

With this powerful tool on our hands, we are in the position to prove a similar result to Proposition 3.1.1 for the multiphase case.

Proposition 3.2.4. *Given well prepared initial data $u_\varepsilon^0 = \sum_{i=1}^P \chi_i^0 \alpha_i$ in the sense that*

$$\mathrm{E}_\varepsilon(u_\varepsilon^0) \to \mathrm{E}(u^0) =: \mathrm{E}_0 < \infty,$$

there exists for any sequence $\varepsilon \to 0$ some non-relabelled subsequence such that the solutions of the Allen–Cahn equation (2.1) with initial conditions u_ε^0 converge in

$\mathrm{L}^1\left((0,T)\times\mathbb{T};\mathbb{R}^N\right)$ *to some* $u=\sum_{i=1}^P \chi_i\alpha_i$ *with a partition* $\chi\in\mathrm{BV}\left((0,T)\times\mathbb{T};\{0,1\}^P\right)$. *Furthermore we have*

$$\operatorname*{ess\,sup}_{0\le t\le T}\mathrm{E}(u)\le\mathrm{E}_0 \tag{3.14}$$

and for all $1\le i\le P$*, the compositions* $\phi_i\circ u_\varepsilon$ *are uniformly bounded in* $\mathrm{BV}((0,T)\times\mathbb{T})$ *and converge to* $\phi_i\circ u$ *in* $\mathrm{L}^1\left((0,T)\times\mathbb{T}\right)$.

Remark 3.2.5. The proof is quite similar to the proof of Proposition 3.1.1, but we have to work more in order to obtain the existence and desired convergence to u.

Proof. By Theorem 2.2.1 the solution u_ε of the Allen–Cahn equation (2.1) satisfies the assumptions of Lemma 3.2.3. Thus $\psi_{\varepsilon,i}:=\varphi_i\circ u_\varepsilon$ is uniformly bounded in $\mathrm{BV}\left((0,T)\times\mathbb{T}\right)$ as ε tends to zero for all $1\le i\le P$ and thus we find a non-relabelled subsequence and $v_i\in\mathrm{BV}\left((0,T)\times\mathbb{T}\right)$ such that $\psi_{\varepsilon,i}$ converges to v_i in $\mathrm{L}^1\left((0,T)\times\mathbb{T}\right)$ and pointwise almost everywhere for all $1\le i\le P$.

We now want to show that we can write $v_i=\phi_i\circ u$ for $u=\sum\chi_j\alpha_j$. An elegant proof of this presented in [FT89, Thm. 4.1] can be done through the fundamental theorem of Young measures ([Mül99, Thm. 3.1]). By passing to another non-relabelled subsequence of u_ε, we obtain that u_ε generates a Young measure ν. Since u is L^p-bounded, we have that for almost every (t,x), the measure $\nu_{(t,x)}$ is a probability measure. Moreover the sequence $W(u_\varepsilon)$ is uniformly integrable since

$$0\le W(u_\varepsilon)\le\frac{1}{\varepsilon}W(u_\varepsilon)\to 0$$

in L^1. Thus we obtain that for all $\varphi\in\mathrm{L}^\infty([0,T]\times\mathbb{T})$

$$0=\lim_{\varepsilon\to0}\int_0^T\int\varphi(t,x)W(u_\varepsilon(t,x))\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int\varphi(t,x)\int_{\mathbb{R}^N}W(y)\,\mathrm{d}\nu_{(t,x)}(y)\,\mathrm{d}x\,\mathrm{d}t\,,$$

which implies that for almost every (t,x), the probability measure $\nu_{(t,x)}$ is supported on the set $\{\alpha_1,\dots,\alpha_P\}$. Therefore we can write $\nu_{(t,x)}=\sum_{j=1}^P\lambda_j\delta_{\alpha_j}$ for non-negative numbers $\lambda_j\in[0,1]$ with $\sum\lambda_j=1$.

Now let $1\le i\le P$. For every $f\in\mathrm{C}_0(\mathbb{R})$, we have that $f\circ\phi_i\in\mathrm{C}_0(\mathbb{R}^N)$ by the p-growth of W. Thus we can compute that $\phi_i\circ u_\varepsilon$ generates the Young measure given at almost every point (t,x) by $\nu_{(t,x)}=\sum\lambda_j\delta_{\phi_i(\alpha_j)}$. But $\phi_i\circ u_\varepsilon$ converges to v_i in L^1, thus especially in measure, which implies by [Mül99, Cor. 3.2] that our convex combination already has to be trivial, or in other words, we can write

$$\nu_{(t,x)}=\sum_{j=1}^P\mathbb{1}_{\Omega_j(t)}(x)\delta_{\alpha_j}$$

for a time-dependent partition $(\Omega_j(t))_{j=1,\dots,P}$ of $\mathbb{T}$. On a technical side note, we already know that the sets $\Omega_j\subseteq[0,T]\times\mathbb{T}$ are measurable: Take a festfunction f such that $f(\alpha_j)=1$ and $f(\alpha_k)=0$ for $k\neq j$. Then Ω_j is exactly the preimage of $\{1\}$ under the measurable map

$$(t,x)\mapsto\int f(y)\,\mathrm{d}\nu_{t,x}(y)\,.$$

Proceeding with the proof we can write

$$\delta_{v_i(t,x)} = \sum_{j=1}^{P} \mathbb{1}_{\Omega_j(t)}(x)\delta_{\phi_i(\alpha_j)},$$

and by defining $u \coloneqq \sum_{j=1}^{P} \mathbb{1}_{\Omega_j(t)}(x)\alpha_j$, we obtain that

$$v_i = \phi_i \circ u = \sum_{j=1}^{P} \mathbb{1}_{\Omega_j(t)}(x)\sigma_{ij}.$$

What is left to show is the energy estimate (3.14) and that the partition function $\chi(t,x) \coloneqq \left(\mathbb{1}_{\Omega_1(t)}(x), \dots, \mathbb{1}_{\Omega_P(t)}(x)\right)$ is of bounded variation. The latter follows from the Fleming–Rishel co-area formula [FR60] which yields since $\phi_i \circ u \in \mathrm{BV}\left((0,T)\times\mathbb{T}\right)$ that

$$\begin{aligned}
\infty &> |(\partial_t, \mathrm{D})\phi_i \circ u| \left((0,T)\times\mathbb{T}\right) \\
&= \int_0^\infty \mathcal{H}^d \left(\partial_* \left(\{(t,x) :\ \phi_i(u(t,x)) \le s\}\right)\right) \mathrm{d}s \\
&\ge \int_0^{\min_{i\neq j}\sigma_{ij}} \mathcal{H}^d \left(\partial_* \{(t,x) \in (0,T)\times\mathbb{T} :\ \chi_i(t,x) = 1\}\right) \mathrm{d}s \\
&= \min_{i\neq j} \sigma_{ij} |(\partial_t, \mathrm{D})\chi_i| \left((0,T)\times\mathbb{T}\right).
\end{aligned}$$

Here we denote by $\partial_* A$ the measure theoretic boundary of a given measurable set A, as defined for example in [EG15, Def. 5.7]. Since we have $\sigma_{ij} > 0$ for $i \neq j$, this proves that $\chi_i \in \mathrm{BV}\left((0,T)\times\mathbb{T}\right)$.

The energy estimate is a non-trivial consequence of the lower semicontinuity of the variation measure and has been proven by Baldo in [Bal90]. We will recap the proof since it provides some insight into the geometry of the phases. □

Definition 3.2.6. *Given a partition $(\Omega_i)_{i=1,\dots,P}$ of the flat torus $\mathbb{T}$, where all sets Ω_i are of finite perimeter, we define the (i,j)-th interface as $\Sigma_{ij} \coloneqq \partial_*\Omega_i \cap \partial_*\Omega_j$.*

We already mentioned that we want to apply the lower semicontinuity of the variation measure. Since we can think of ϕ_i as being locally the correct choice for suitable i, we introduce the following notion.

Definition 3.2.7. *Let μ, ν be regular positive Borel measures on $\mathbb{T}$. Define the supremum $\mu \vee \nu$ of μ and ν as the smallest regular positive measure which is greater or equal than μ and ν on every Borel subset of $\mathbb{T}$. By the regularity of Radon measures, we have*

$$\mu\vee\nu(U) = \sup\left\{\mu(V) + \mu(W) :\ V \cap W = \emptyset, V \cup W \subseteq U, V \text{ and } W \text{ are open subsets of } \mathbb{T}\right\}$$

for any open subset $U \subseteq \mathbb{T}$.

It is natural to ask how we can characterize the total variation of $\psi_i = \phi_i \circ u = \sum \mathbb{1}_{\Omega_j} \sigma_{ij}$. For this we note that when going from the set Ω_j to the set Ω_k, our function jumps from σ_{ij} to σ_{ik}. Thus we obtain the following intuitive result.

Lemma 3.2.8. *In the setting of Proposition 3.2.4 we can write*

$$|\nabla \psi_i| = \sum_{1 \le j < k \le P} |\sigma_{ij} - \sigma_{ik}| \, \mathrm{d}\, \mathcal{H}^{d-1} \lfloor_{\Sigma_{jk}} .$$

The proof can be found in the Appendix. If we now consider the supremum of the measures $(|\nabla \psi_i|)_{i=1,\dots,P}$, we notice that all of them are supported on the interfaces Σ_{jk}, and that by the triangle inequality for the surface tensions, we have

$$\max_{1 \le i \le P} |\sigma_{ij} - \sigma_{ik}| = \sigma_{jk}. \tag{3.15}$$

The maximum is obtained by either plugging in $i = j$ or $i = k$. Thus we obtain the following.

Proposition 3.2.9. *In the situation of Proposition 3.2.4, we have*

$$\left(\bigvee_{i=1}^{P} |\nabla \psi_i| \right) = \sum_{1 \le i < j \le P} \sigma_{ij} \, \mathrm{d}\, \mathcal{H}^{d-1} \lfloor_{\Sigma_{ij}} .$$

Proof. This follows by combining Lemma 3.2.8 and Lemma A.0.3 with the above equation (3.15). □

Collecting our observations, we can finish our arguments.

Continuation of the proof for Proposition 3.2.4. We are left with proving the estimate (3.14). We first note that as for the proof of (3.11), it holds that for any open subset U of the flat torus, we have for almost every time t the estimate

$$\liminf_{\varepsilon \to 0} \int_U \frac{1}{\varepsilon} W(u_\varepsilon(t)) + \frac{\varepsilon}{2} |\nabla u_\varepsilon(t)|^2 \, \mathrm{d}x \ge |\nabla \psi_i(t)|(U).$$

Since

$$\left(\bigvee_{i=1}^{P} |\nabla \psi_i| \right)(\mathbb{T}) = \sup \left\{ \sum_{i=1}^{P} |\nabla \psi_i|(U_i) : \ (U_i)_i \text{ are disjoint open subsets of } \mathbb{T} \right\},$$

we finally deduce that for almost every $0 \le t \le T$, we have

$$\begin{aligned} \mathrm{E}(u(t)) = \sum_{1 \le i < j \le P} \sigma_{ij} \, \mathcal{H}^{d-1}(\Sigma_{ij}(t)) &= \left(\bigvee_{i=1}^{P} |\nabla \psi_i(t)| \right)(\mathbb{T}) \\ &\le \liminf_{\varepsilon \to 0} \mathrm{E}_\varepsilon(u_\varepsilon(t)) \\ &\le \liminf_{\varepsilon \to 0} \mathrm{E}_\varepsilon(u_\varepsilon^0) \\ &= \mathrm{E}_0, \end{aligned}$$

which completes the proof. □

Nextup, we want to prove a stronger convergence of u_ε to u. This will let us prove that u achieves its initial data continuously in L^2.

Lemma 3.2.10. *In the situation of Proposition 3.2.4 we have for all $1 \leq i \leq P$ that $\psi_{\varepsilon,i} \in \mathrm{W}^{1,2}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$ with the corresponding estimate*

$$\left(\int_0^T \left(\int |\partial_t \psi_{\varepsilon,i}| \,\mathrm{d}x\right)^2 \mathrm{d}t\right)^{1/2} \lesssim \mathrm{E}_\varepsilon(u_\varepsilon(0)). \tag{3.16}$$

Furthermore the sequence u_ε is precompact in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)$. In particular we get that u achieves the initial data in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)$.

Remark 3.2.11. This statement is almost equal to its two-phase equivalent Lemma 3.1.2 and the proof uses a similar strategy.

Proof. **Step 1:** Estimate (3.16) holds

We compute that by Hölder's inequality and the energy dissipation inequality (2.12)

$$\begin{aligned}
\int_0^T \left(\int |\partial_t \psi_{\varepsilon,i}| \,\mathrm{d}x\right)^2 \mathrm{d}t &\leq \int_0^T \left(\int \sqrt{2W(u_\varepsilon)}|\partial_t u_\varepsilon| \,\mathrm{d}x\right)^2 \mathrm{d}t \\
&= \int_0^T \left(\int \frac{2}{\varepsilon} W(u_\varepsilon) \,\mathrm{d}x \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\right) \mathrm{d}t \\
&\leq 2\,\mathrm{E}_\varepsilon(u_\varepsilon(0)) \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\leq 2\,\mathrm{E}_\varepsilon(u_\varepsilon(0))^2.
\end{aligned}$$

By estimate (3.11), we especially obtain $\psi_{\varepsilon,i} \in \mathrm{L}^2\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$.

Step 2: The sequence $\psi_{\varepsilon,i}$ is precompact in $\mathrm{C}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$.

This follows exactly as in the twophase case from the embedding of $\mathrm{W}^{1,2}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$ into $\mathrm{C}^{1/2}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$.

Step 3: The sequence u_ε converges to $\sum_i \chi_i \alpha_i$ in measure uniformly in time.

Let $\rho > 0$. Then by definition of ϕ_i, there exists some $\delta > 0$ such that for all $1 \leq i \leq P$, we have: If $|v - \alpha_i| > \rho$ holds , then we already have $\phi_i(v) > \delta$. Therefore we can estimate

$$\begin{aligned}
\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\left\{x:\ \left|u_\varepsilon - \sum_i \mathbb{1}_{\Omega_i\alpha_i}\right| > \rho\right\}\right) &= \operatorname*{ess\,sup}_{0\le t\le T} \sum_{i=1}^P \mathcal{L}^d\left(\{x\in\Omega_i:\ |u_\varepsilon-\alpha_i|>\rho\}\right)\\
&\le \operatorname*{ess\,sup}_{0\le t\le T} \sum_{i=1}^P \mathcal{L}^d\left(\{x\in\Omega_i:\ \phi_i(u_\varepsilon)>\delta\}\right)\\
&\le \operatorname*{ess\,sup}_{0\le t\le T} \frac{1}{\delta}\sum_{i=1}^P \int_{\Omega_i} |\psi_{\varepsilon,i}|\,\mathrm{d}x\\
&\le \operatorname*{ess\,sup}_{0\le t\le T} \frac{1}{\delta}\sum_{i=1}^P \int |\psi_{\varepsilon,i}-\psi_i|\,\mathrm{d}x\,,
\end{aligned}$$

which converges to 0 by Step 2.

Step 4: The sequence u_ε^2 is equiintegrable uniformly in time

This follows as in the twophase case as well.

Step 5: The sequence u_ε converges in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)$.

Here the proof does not change as well.

This completes our arguments. □

4. Conditional convergence of the Allen–Cahn equation

This chapter is dedicated to the main conditional convergence results by Laux and Simon [LS16]. The goal is to show that under the assumption that for almost every time, the Cahn–Hilliard energies of the solutions of the Allen–Cahn equation converge to the surface tension energy of the limit, we have that the limit is a BV-solution to mean curvature flow in the sense of Definition 4.1.1 for the two-phase case and Definition 4.2.1 for its multiphase equivalent.

The proofs we present are taken from [LS16], or from other authors, in which case we explicitly credit them. At some points however, some arguments in the original proofs are missing, which we then complete and also point this out to the reader.

Since the arguments are again quite involved, we shall focus on the easier two-phase case first and then turn towards the multiphase case.

4.1. Conditional convergence in the two-phase case

4.1.1. Convergence to a BV-solution to two-phase mean curvature flow

Convergence of energies often boosts our modulus of convergence and gives our limit therefore additional regularity. Thus let us assume

$$\int_0^T \mathrm{E}_\varepsilon(u_\varepsilon)\,\mathrm{d}t \to \int_0^T \mathrm{E}(u)\,\mathrm{d}t \;\text{ as } \varepsilon \to 0, \tag{4.1}$$

where in the two-phase case, the surface tension energy is defined for $u = \alpha(1-\chi) + \beta\chi$ by

$$\mathrm{E}(u) \coloneqq \mathrm{E}(\chi) \coloneqq \sigma \int |\nabla \chi| \tag{4.2}$$

already motivated in Section 1.2. We will go into more depth about why this assumption is important in Section 5.2. Moreover we notice that the energy of u is exactly the total variation of $\psi = \sigma\chi$. Since for almost every time t, we have that $\psi_\varepsilon(t)$ converges to $\psi(t)$ in $\mathrm{L}^1(\mathbb{T})$ by Proposition 3.1.1, it follows from the lower semicontinuity of the variation measure and Young's inequality that

$$\mathrm{E}(u) \le \liminf_{\varepsilon\to 0} \int |\nabla \psi_\varepsilon|\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon).$$

Moreover by the energy dissipation inequality (2.12), the energies stay uniformly bounded in time. Thus using the dominated convergence theorem, we see that our time integrated energy convergence assumption is equivalent to saying that for almost every time $t \in (0,T)$, we have convergence of the energies $\mathrm{E}_\varepsilon(u_\varepsilon(t)) \to \mathrm{E}(u(t))$.

Since the energies themselves can be interpreted as measures on the flat torus, we define for a continuous function $\varphi \in \mathrm{C}(\mathbb{T})$ the corresponding energy measures by

$$\begin{aligned} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) &:= \int \varphi \left(\frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \right) \mathrm{d}x \quad \text{and} \\ \mathrm{E}(u;\varphi) &:= \sigma \int \varphi |\nabla \chi| = \int \varphi |\nabla \psi|. \end{aligned}$$

Then from the energy convergence and lower semicontinuity just discussed, it follows that

$$\lim_{\varepsilon \to 0} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) = \mathrm{E}(u;\varphi). \tag{4.3}$$

We start by defining a BV-formulation for motion by mean curvature in the spirit of Luckhaus und Sturzenhecker [LS95].

Definition 4.1.1 (BV-solution to two-phase mean curvature flow)**.** *Fix some finite time horizon $T < \infty$ and initial data $\chi^0 \in \mathrm{BV}(\mathbb{T};\{0,1\})$. We say that*

$$\chi \in \mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T};\{0,1\})\right)$$

with $\operatorname{ess\,sup}_{0 \le t \le T} \mathrm{E}(\chi) < \infty$ moves by mean curvature with initial data χ^0 *if there is a normal velocity $V \in \mathrm{L}^2(|\nabla \chi|\,\mathrm{d}t)$ such that*

1. *For all $\xi \in \mathrm{C}_\mathrm{c}^\infty\left((0,T)\times\mathbb{T};\mathbb{R}^d\right)$, we have*

$$\int_0^T \int V\langle \xi\,,\,\nu\rangle - \langle \mathrm{D}\,\xi\,,\,\mathrm{Id} - \nu\otimes\nu\rangle |\nabla\chi|\,\mathrm{d}t = 0, \tag{4.4}$$

   where $\nu := \nabla\chi/|\nabla\chi|$ is the inner unit normal.

2. *The function V is the normal velocity of χ in the sense that*

$$\partial_t \chi = V|\nabla\chi|\,\mathrm{d}t$$

   holds distributionally in $(0,T)\times\mathbb{T}$.

3. *The initial data χ^0 is achieved in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$, which means that $\chi(0) = \chi^0$ as functions in $\mathrm{L}^2(\mathbb{T})$.*

Let us give a brief motivation for this definition. First we note that the boundedness of energies already implies that χ is of bounded variation in space for almost every time. Moreover equation (4.4) hinges on the distributional formulation for the mean curvature vector as stated in [Mag12] and yields that

$$\int \langle \mathrm{D}\,\xi\,,\,\mathrm{Id} - \nu\otimes\nu\rangle|\nabla\chi| = -\int H\langle\xi\,,\,\nu\rangle|\nabla\chi|.$$

Thus the equation is equivalent to $H = -V$ on Σ.

Our main goal in this section is now to show that the function χ we have found in Proposition 3.1.1 moves my mean curvature, assuming the time integrated energies converge (4.1). Thus our goal is to prove the following Theorem.

Theorem 4.1.2. *Let a smooth double-well potential $W\colon \mathbb{R} \to [0,\infty)$ satisfy the assumptions (2.3)-(2.6). Let $T < \infty$ be an arbitrary finite time horizon. Given a sequence of initial data $u_\varepsilon^0\colon \mathbb{T} \to \mathbb{R}$ with $\chi^0 \in \mathrm{BV}(\mathbb{T};\{0,1\})$ such that u_ε^0 converges to $u^0 = (1-\chi^0)\alpha + \chi^0\beta$ pointwise almost everyhwere and*

$$\mathrm{E}_0 \coloneqq \mathrm{E}(\chi^0) = \lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon^0) < \infty,$$

we have that that for some subsequence of solutions u_ε to (2.1), there exists a pointwise almost everywhere limit $u = (1-\chi)\alpha + \chi\beta$ with $\chi \in \mathrm{BV}((0,T)\times\mathbb{T};\{0,1\})$ and $\chi \in \mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T};\{0,1\})\right)$ which assumes the initial data in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$. If we additionally assume that the time-integrated energies converge (4.1), then χ moves by mean curvature in the sense of Definition 4.1.1.

Looking at the distributional form of the Allen–Cahn equation (2.9), which reads

$$\int \frac{1}{\varepsilon^2} W'(u_\varepsilon(t))\varphi + \langle \nabla u_\varepsilon(t)\,,\, \nabla\varphi\rangle + \partial_t u_\varepsilon(t)\varphi \,\mathrm{d}x = 0,$$

we expect that for a suitable choice of testfunctions φ_ε, the following two terms converge:

$$\lim_{\varepsilon\to 0}\int_0^T\int \partial_t u_\varepsilon \varphi_\varepsilon = \sigma\int_0^T\int V\langle \xi\,,\,\nu\rangle |\nabla\chi|\,\mathrm{d}t\,,$$
$$\lim_{\varepsilon\to 0}\int_0^T\int \frac{1}{\varepsilon^2}W'(u_\varepsilon)\varphi_\varepsilon + \langle\nabla u_\varepsilon\,,\,\nabla\varphi_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t = -\sigma\int_0^T\int\langle \mathrm{D}\,\xi\,,\,\mathrm{Id} - \nu\otimes\nu\rangle|\nabla\chi|\,\mathrm{d}t\,.$$

But how do we find these testfunctions? For this, we first note that the curvature term $\int\langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu\otimes\nu\rangle|\nabla\chi|\,\mathrm{d}t$ is by [Mag12, Thm. 17.5] the first inner variation with respect to ξ of the perimeter functional, which is just our energy E up to the surface tension constant $\sigma > 0$. Since the energy E_ε converges to E, we may also hope that their first variations converge to each other. Thus it is plausible to compute the first inner variation $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\mathrm{E}_\varepsilon(\rho_s)$ and then we can hopefully choose the testfunction φ_ε in such a way that it equals $\int 1/\varepsilon^2 W'(u_\varepsilon)\varphi + \langle\nabla u_\varepsilon\,,\,\nabla\varphi\rangle\,\mathrm{d}x$.

Thus let $(\rho_s)_s$ be functions which solve the ordinary differential equation

$$\begin{cases}\partial_s\rho_s + \langle\xi\,,\,\nabla\rho_s\rangle & = 0,\\ \rho_0 & = u_\varepsilon.\end{cases}$$

Then we formally compute that the first inner variation of the energy E_ε with respect to ξ is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\int\frac{\varepsilon}{2}|\rho_s|^2 + \frac{1}{\varepsilon}W(\rho_s)\,\mathrm{d}x &= \int \varepsilon\langle\nabla u_\varepsilon\,,\,\nabla\left(-\langle\xi\,,\,\nabla u_\varepsilon\rangle\right)\rangle + \frac{1}{\varepsilon}W'(u_\varepsilon)(-\langle\xi\,,\,\nabla u_\varepsilon\rangle)\,\mathrm{d}x\\ &= \int\left(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle\xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x\,.\end{aligned}$$

We therefore test equation (2.9) with $\varphi_\varepsilon \coloneqq \langle \xi \,,\, \nabla u_\varepsilon \rangle$.

A first additional regularity result under the energy convergence assumption is the following Proposition, which ensures that we have a square-integrable normal velocity.

Proposition 4.1.3. *In the setting of Proposition 3.1.1 and given the energy convergence assumption (4.1), the measure $\partial_t \chi$ is absolutely continuous with respect to the measure $|\nabla\chi|\,\mathrm{d}t$ and the corresponding density V is square integrable with the estimate*

$$\int_0^T \int V^2 |\nabla \chi| \,\mathrm{d}t \lesssim \mathrm{E}_0 \,.$$

Remark 4.1.4. By [AFP00, Thm. 3.103], we can disintegrate the measure $|\nabla\chi|_{d+1}$, which is the variation in space in time and space, into the measure $|\nabla\chi|_d \,\mathrm{d}t$, which we will use by a slight abuse of notation and drop the subindex depending on the situation at hand. Here $|\nabla\chi|_d$ simply denotes the variation in space for a fixed time.

Proof. Take a smooth test function $\varphi \in \mathrm{C}_c^\infty((0,T)\times\mathbb{T})$. Then via the L^1-convergence of ψ_ε to ψ, we have

$$\begin{aligned}
\partial_t \psi(\varphi) &= \liminf_{\varepsilon\to 0} \partial_t \psi_\varepsilon(\varphi) \\
&= \liminf_{\varepsilon\to 0} \int_0^T \int \sqrt{2W(u_\varepsilon)}\partial_t u_\varepsilon \varphi \,\mathrm{d}x \,\mathrm{d}t \\
&\le \liminf_{\varepsilon\to 0} \left(\int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon)\varphi^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \\
&\le \liminf_{\varepsilon\to 0} \left(2\int_0^T \mathrm{E}_\varepsilon\left(u_\varepsilon;\varphi^2\right) \mathrm{d}t \right)^{1/2} \left(\mathrm{E}_\varepsilon(u_\varepsilon)\right)^{1/2} \\
&= \sqrt{2\sigma}\|\varphi\|_{\mathrm{L}^2((0,T)\times\mathbb{T},|\nabla\chi|\mathrm{d}t)}\sqrt{\mathrm{E}_0}.
\end{aligned}$$

The second inequality is due to the energy dissipation inequality (2.12) and the last equality due to the convergence of the localized energies (4.3). This proves both the absolute continuity and via a duality argument the desired bound since $\partial_t\psi = \sigma\partial_t\chi$ and $\sigma > 0$. □

We finish this section with a proof for the *equipartition of the energyies*, which tells us that both the summand involving the potential $W(u_\varepsilon)$ and the norm of the gradient contribute to the energy in equal parts. Or in other words

$$\frac{1}{\varepsilon}W(u_\varepsilon) - \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 \rightharpoonup^* 0$$

holds in the distributional sense. Notice that under the energy convergence assumption (4.1), the proof is quite simple. Ilmanen actually showed in [Ilm93] that in the two-phase case this is true for a big class of well-prepared initial conditions, but the proof relies on a comparison principle, which has no obvious substitute in the multiphase case and the proof is much more involved.

Lemma 4.1.5. *In the situation of Proposition 3.1.1 and under the energy convergence assumption (4.1), we have for any continuous function $\varphi \in \mathrm{C}(\mathbb{T})$ that*

$$\begin{aligned}\mathrm{E}(u;\varphi) = \lim_{\varepsilon\to 0}\mathrm{E}_\varepsilon(u_\varepsilon;\varphi) &= \lim_{\varepsilon\to 0}\int \varphi\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int \varphi\varepsilon|\nabla u_\varepsilon|^2\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int \varphi\frac{1}{\varepsilon}2W(u_\varepsilon)\,\mathrm{d}x\end{aligned}$$

for almost every time $0 \le t \le T$.

Proof. We have already established the first equality before. For the second equality, we first assume that $\varphi \in \mathrm{C}(\mathbb{T})$ is non-negative. By the lower semicontinuity of the variation measure, we immediately obtain

$$\liminf_{\varepsilon\to 0}\int \varphi\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x = \liminf_{\varepsilon\to 0}\int \varphi|\nabla\psi_\varepsilon|\,\mathrm{d}x \ge \mathrm{E}(u;\varphi).$$

But by Young's inequality, we also have

$$\limsup_{\varepsilon\to 0}\int \varphi\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x \le \limsup_{\varepsilon\to 0}\mathrm{E}_\varepsilon(u_\varepsilon;\varphi) = \mathrm{E}(u;\varphi).$$

For general $\varphi \in \mathrm{C}(\mathbb{T})$, we decompose φ into its positive and negative part and apply the previous argument to both in order to get the claim.

The third and fourth equality follow for a given non-negative $\varphi \in \mathrm{C}(\mathbb{T};[0,\infty))$ by the L^2-estimate

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int\left|\sqrt{\varphi}\sqrt{\varepsilon}|\nabla u_\varepsilon| - \sqrt{\varphi}\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right|^2\mathrm{d}x \\ =\ &\lim_{\varepsilon\to 0} 2\,\mathrm{E}_\varepsilon(u_\varepsilon;\varphi) - 2\int \varphi\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x = 0\end{aligned}$$

which implies that

$$\lim_{\varepsilon\to 0}\int \varphi\varepsilon|\nabla u_\varepsilon|^2\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int \varphi\frac{1}{\varepsilon}2W(u_\varepsilon)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\mathrm{E}_\varepsilon(u_\varepsilon;\varphi),$$

since the integral of $\varphi 1/\varepsilon 2W(u_\varepsilon) + \varphi\varepsilon|\nabla u_\varepsilon|^2$ converges . □

4.1.2. Convergence of the curvature term

The goal of this section is to prove the convergence

$$\lim_{\varepsilon\to 0}\int\left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle\xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x = \sigma\int\langle \mathrm{D}\,\xi\,,\,\mathrm{Id} - \nu\otimes\nu\rangle|\nabla\chi|$$

for almost every time t. We directly follow the proof from Luckhaus and Modica in [LM89] with some extra steps.

Through an integration by parts, we obtain

$$\int \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle \xi\,,\,\nabla u_\varepsilon\rangle = \int -\varepsilon\sum_{i,j=1}^d \partial_{x_i}u_\varepsilon\left(\partial_{x_i}\xi^j\partial_{x_j}u_\varepsilon + \xi^j\partial^2_{x_ix_j}u_\varepsilon\right) + \frac{1}{\varepsilon}W(u_\varepsilon)\operatorname{div}\xi\,\mathrm{d}x\,. \tag{4.5}$$

Moreover by another integration by parts, we have

$$\begin{aligned}\int \sum_{i,j=1}^d \partial_{x_i}u_\varepsilon\xi^j\partial^2_{x_ix_j}u_\varepsilon\,\mathrm{d}x &= \int -\sum_{i,j=1}^d \partial_{x_i}u_\varepsilon\left(\partial^2_{x_ix_j}u_\varepsilon\xi^j + \partial_{x_j}\xi^j\partial_{x_i}u_\varepsilon\right)\mathrm{d}x\\ &= \int -|\nabla u_\varepsilon|^2\operatorname{div}\xi - \sum_{i,j=1}^d \partial_{x_i}u_\varepsilon\partial^2_{x_ix_j}u_\varepsilon\xi^j\,\mathrm{d}x\end{aligned}$$

which is equivalent to

$$\int \sum_{i,j=1}^d \partial_{x_i}u_\varepsilon\xi^j\partial^2_{x_ix_j}u_\varepsilon\,\mathrm{d}x = -\frac{1}{2}\int |\nabla u_\varepsilon|^2\operatorname{div}\xi\,\mathrm{d}x\,.$$

Plugging this equation into the first equation (4.5), we obtain

$$\begin{aligned}&\int \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle \xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x\\ &= \int -\varepsilon\sum_{i,j=1}^d \partial_{x_i}u_\varepsilon\partial_{x_j}u_\varepsilon\partial_{x_i}\xi^j + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\operatorname{div}\xi + \frac{1}{\varepsilon}W(u_\varepsilon)\operatorname{div}\xi\,\mathrm{d}x \qquad (4.6)\\ &= \varepsilon\int |\nabla u_\varepsilon|^2\operatorname{div}\xi - \sum_{i,j=1}^d \partial_{x_i}u_\varepsilon\partial_{x_j}u_\varepsilon\partial_{x_i}\xi^j\,\mathrm{d}x\\ &\quad + \int \frac{1}{\varepsilon}W(u_\varepsilon)\operatorname{div}\xi - \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\operatorname{div}\xi\,\mathrm{d}x\,.\end{aligned}$$

The last integral vanishes by the equipartition of the energies (Lemma 4.1.5). Since $\partial_{x_i}u_\varepsilon/|\nabla u_\varepsilon| = \partial_{x_i}\psi_\varepsilon/|\nabla\psi_\varepsilon|$ by the chain rule and non-negativity of W, the former integral can be written as

$$\int_{\mathbb{T}_\varepsilon} g(x,\nabla\psi_\varepsilon)\varepsilon|\nabla u_\varepsilon|^2\,\mathrm{d}x\,. \tag{4.7}$$

Here the function g is defined by

$$g(x,p) = \begin{cases}\sum_{i,j=1}^d -\frac{p_i}{|p|}\partial_{x_i}\xi^j\frac{p_j}{|p|} + \operatorname{div}\xi & \text{if } p\neq 0,\\ 0 & \text{else,}\end{cases}$$

and the set $\mathbb{T}_\varepsilon$ is defined as

$$\mathbb{T}_\varepsilon := \{x \in \mathbb{T} :\ \nabla\psi_\varepsilon(x) \neq 0\} = \{x \in \mathbb{T} :\ \nabla u_\varepsilon(x) \neq 0\} \cap \{x \in \mathbb{T} :\ u_\varepsilon(x) \notin \{\alpha, \beta\}\}\,.$$

For the representation (4.7), we also have to use that

$$\mathcal{L}^d\left(\{x \in \mathbb{T} :\ \nabla u_\varepsilon(x) \neq 0 \text{ and } u_\varepsilon(x) \in \{\alpha, \beta\}\}\right) = 0,$$

which is a known result for Sobolev functions.

Again by the equipartition of energies (Lemma 4.1.5) and the boundedness of g, we can replace $\varepsilon|\nabla u_\varepsilon|^2$ by $\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|$ in the integral (4.7) via the estimate

$$\begin{aligned}
&\int_{\mathbb{T}_\varepsilon} \left|\varepsilon|\nabla u_\varepsilon|^2 - \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\right| \mathrm{d}x \\
&\leq \left(\int_{\mathbb{T}_\varepsilon} \left|\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right|^2 \mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{T}_\varepsilon} \varepsilon|\nabla u_\varepsilon|^2 \,\mathrm{d}x\right)^{1/2} \\
&\leq \left(\int \varepsilon|\nabla u_\varepsilon|^2 - 2\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| + \frac{1}{\varepsilon}2W(u_\varepsilon)\,\mathrm{d}x\right) \sqrt{2\,\mathrm{E}_\varepsilon(u_\varepsilon)}
\end{aligned}$$

which vanishes as ε tends to zero. Thus

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\mathbb{T}_\varepsilon} g(x, \nabla\psi_\varepsilon)\varepsilon|\nabla u_\varepsilon|^2 \,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_{\mathbb{T}_\varepsilon} g(x, \nabla\psi_\varepsilon)\sqrt{2W(u_\varepsilon)}|u_\varepsilon| \,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{T}_\varepsilon} g(x, \nabla\psi_\varepsilon)|\nabla\psi_\varepsilon| \,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int F(x, \nabla\psi_\varepsilon)\,\mathrm{d}x\,,
\end{aligned}$$

where $F(x, p)$ is defined as $g(x, p)|p|$ at points where p is not equal to 0, and defined as 0 elsewhere. Since $F(x, \lambda p) = \lambda F(x, p)$ for positive λ and since F satisfies the periodic boundary condition in x, we are in the position to apply a Theorem proven by Reshetnyak in [Res68] and again by Luckhaus and Modica in [LM89]. We will later see a quantitative version of this in the proof of Proposition 4.2.7. Here it yields that since $|\nabla\psi_\varepsilon|(\mathbb{T}) \to |\nabla\psi|(\mathbb{T})$ by the equipartition of energies (Lemma 4.1.5), we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int F(x, \nabla\psi_\varepsilon)\,\mathrm{d}x &= \sigma\int F(x, \nu)|\nabla\chi| \\
&= \sigma\int\left(\sum_{i,j=1}^{d} -\frac{\nu_i}{|\nu|}\partial_{x_i}\xi^j\frac{\nu_j}{|\nu|} + \operatorname{div}\xi\right)|\nu||\nabla\chi| \\
&= \sigma\int \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - \nu\otimes\nu\rangle|\nabla\chi|,
\end{aligned}$$

which finishes the proof. The time integrated version given by

$$\lim_{\varepsilon\to 0}\int_0^T\int\left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle \xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t = \sigma\int_0^T\int \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - \nu\otimes\nu\rangle|\nabla\chi|\,\mathrm{d}t$$

follows from the generalized dominated convergence theorem via the equality (4.6) which yields

$$\left|\int \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle \xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x\right| \lesssim \mathrm{E}_\varepsilon(u_\varepsilon).$$

4.1.3. Convergence of the velocity term

We now want to prove the convergence of the velocity term given by

$$\lim_{\varepsilon\to 0}\int_0^T\int \partial_t u_\varepsilon\langle \xi\,,\,\varepsilon\nabla u_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t = \sigma\int_0^T\int V\langle \xi\,,\,\nu\rangle|\nabla\chi|\,\mathrm{d}t\,.$$

The difficulty here is that products of weakly converging sequences will in general not weakly converge. To be more precise, we only have $\partial_t u_\varepsilon \rightharpoonup V|\nabla\chi|\,\mathrm{d}t$ and $\varepsilon\nabla u_\varepsilon \approx \sigma\nu$ in a weak sense.

Therefore we try to freeze the normal in a fixed direction, apply the weak convergence of $\partial_t u_\varepsilon$ and then unfreeze the normal. Freezing the approximate normal $\varepsilon\nabla u_\varepsilon$ amounts to replacing $\varepsilon\nabla u_\varepsilon$ by $\varepsilon|\nabla u_\varepsilon|\nu^*$ for a suitably chosen $\nu^*\in\mathbb{S}^{d-1}$. Let η be a cutoff on the support of ξ and denote by $\nu_\varepsilon = \nabla u_\varepsilon/|\nabla u_\varepsilon|$ the approximate unit normal. Then the error we make can be estimated for all $\alpha > 0$ via Young's inequality by

$$\begin{aligned}
&\left|\int_0^T\int \partial_t u_\varepsilon\langle \xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int \partial_t u_\varepsilon\langle \xi\,,\,\varepsilon|\nabla u_\varepsilon|\nu^*\rangle\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\le \|\xi\|_{\mathrm{L}^\infty}\int_0^T\int \eta\sqrt{\varepsilon}|\partial_t u_\varepsilon|\sqrt{\varepsilon}|\nabla u_\varepsilon - |\nabla u_\varepsilon|\nu^*|\,\mathrm{d}x\,\mathrm{d}t \\
&\le \|\xi\|_{\mathrm{L}^\infty}\left(\alpha\int_0^T\int \eta\varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\alpha}\int_0^T\int \eta\varepsilon|\nabla u_\varepsilon|^2|\nu_\varepsilon-\nu^*|^2\,\mathrm{d}x\,\mathrm{d}t\right) \\
&\le \|\xi\|_{\mathrm{L}^\infty}\left(\alpha\int_0^T\int \eta\varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\alpha}\,\mathcal{E}_\varepsilon(\nu^*;\eta)\right).
\end{aligned}\tag{4.8}$$

Here the approximate tilt-excess in direction ν^* is given by

$$\mathcal{E}_\varepsilon(\nu^*;\eta) := \int_0^T\int \eta\varepsilon|\nabla u_\varepsilon|^2|\nu_\varepsilon-\nu^*|^2\,\mathrm{d}x\,\mathrm{d}t\,.$$

So let us accept the error term 4.8 for now. With our frozen normal, we now notice that via the equipartition of energies, we may replace $\varepsilon|\nabla u_\varepsilon|$ by $\sqrt{2W(u_\varepsilon)}$ via the estimate

$$\begin{aligned}
&\int_0^T\int |\partial_t u_\varepsilon|\eta\left|\varepsilon|\nabla u_\varepsilon| - \sqrt{2W(u_\varepsilon)}\right|\,\mathrm{d}x\,\mathrm{d}t \\
&\le \left(\int_0^T\int \varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int \eta^2\left(\varepsilon|\nabla u_\varepsilon|^2 - 2|\nabla u_\varepsilon|\sqrt{2W(u_\varepsilon)} + \frac{1}{\varepsilon}2W(u_\varepsilon)\right)\mathrm{d}x\,\mathrm{d}t\right)^{1/2}
\end{aligned}$$

The first factor is uniformly bounded by the energy dissipation inequality (2.12) and the second term vanishes as ε tends to zero by the equipartition of energies (Lemma 4.1.5). But now we recognize the identity

$$\int_0^T \int \partial_t u_\varepsilon \sqrt{2W(u_\varepsilon)} \langle \xi \,,\, \nu^* \rangle \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int \partial_t \psi_\varepsilon \langle \xi \,,\, \nu^* \rangle \,\mathrm{d}x\,\mathrm{d}t\,,$$

which converges as ε approaches zero to

$$\int_0^T \int \langle \xi \,,\, \nu^* \rangle \partial_t \psi = \sigma \int_0^T \int V \langle \xi \,,\, \nu^* \rangle |\nabla \chi| \,\mathrm{d}t\,. \tag{4.9}$$

Finally we want to unfreeze the normal, which means that we want to replace ν^* by ν on the right hand side of equation (4.9). This can be estimated again by Young's inequality via the error

$$\|\xi\|_{\mathrm{L}^\infty} \left(\alpha \int_0^T \int \eta V^2 |\nabla \chi| \,\mathrm{d}t + \frac{1}{\alpha}\, \mathcal{E}(\nu^*;\eta) \right),$$

where the tilt-excess is given by

$$\mathcal{E}(\nu^*;\eta) \coloneqq \sigma \int_0^T \int \eta |\nu - \nu^*|^2 |\nabla \chi| \,\mathrm{d}t\,.$$

This finishes the proof for the convergence of the velocity term up to arguing that the errors can be made arbitrarily small.

First we study the behaviour of the approximate tilt-excess $\mathcal{E}_\varepsilon$ by connecting it to $\mathcal{E}$. We notice that by expansion, we have

$$\begin{aligned}
\mathcal{E}_\varepsilon(\nu^*;\eta) &= 2\int_0^T \int \eta\varepsilon |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t - 2\left\langle \int_0^T \int \eta\varepsilon |\nabla u_\varepsilon| \nabla u_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \;,\; \nu^* \right\rangle \\
\mathcal{E}(\nu^*;\eta) &= 2\,\mathrm{E}(u;\eta) - 2\sigma \left\langle \int_0^T \int \eta \nu |\nabla \chi| \,\mathrm{d}t \;,\; \nu^* \right\rangle.
\end{aligned}$$

But by the equipartition of energies Lemma 4.1.5, we have that

$$2\int_0^T \int \eta\varepsilon |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \to 2\,\mathrm{E}(u;\eta)$$

and recognizing that $\sigma \int_0^T \int \eta\nu|\nabla\chi|\,\mathrm{d}t = \int_0^T \int \eta\nabla\psi$, we also obtain

$$\begin{aligned}
&\left|\int_0^T \int \eta\varepsilon|\nabla u_\varepsilon|\nabla u_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^T \int \eta\nu|\nabla\chi|\,\mathrm{d}t\right| \\
\leq &\left|\int_0^T \int \eta\sqrt{2W(u_\varepsilon)}\nabla u_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int \eta\nabla\psi\right| + \int_0^T \int \varepsilon|\nabla u_\varepsilon|\left|\nabla u_\varepsilon - \frac{1}{\varepsilon}\sqrt{2W(u_\varepsilon)}\right| \mathrm{d}x\,\mathrm{d}t \\
\leq &\left|\int_0^T \int \eta\nabla\psi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int \eta\nabla\psi\right| \\
&+ \left(\int_0^T \int \varepsilon|\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \left(\int_0^T \int \left(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2},
\end{aligned}$$

which vanishes as ε tends to zero by the weak convergence of $\nabla\psi_\varepsilon \rightharpoonup^* \nabla\psi$ and the equipartition of the energies. Since we are taking ε to zero, we thus only have to make sure that the tilt-excess is sufficiently small since the approximate tilt excess converges to the tilt excess.

We are now in the position to argue why the error can be made arbitrarily small. Let $\delta > 0$. Then we first choose our $\alpha > 0$ so small that

$$\limsup_{\varepsilon\to 0} \alpha\|\xi\|_{\mathrm{L}^\infty} \left(\int_0^T \int \varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int V^2|\nabla\chi|\,\mathrm{d}t\right) < \frac{\delta}{2},$$

which is possible by the energy dissipation inequality (2.12) and the square-integrability of the normal velocity (Proposition 4.1.3). Then we choose a partition of unity $(\eta_i)_{i=1,\dots,n}$ and approximate unit normals $(\nu_i^*)_{i=1,\dots,n}$ such that

$$\frac{2}{\alpha}\|\xi\|_{\mathrm{L}^\infty}\sigma\sum_{i=1}^n \int_0^T \int \eta_i|\nu - \nu_i^*|^2|\nabla\chi|\,\mathrm{d}t < \frac{\delta}{2}.$$

The existence of these can be seen by taking a smooth approximation of ν with respect to the measure $|\nabla\chi|\,\mathrm{d}t$. Collecting all of our errors, we obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left|\int_0^T \int \partial_t u_\varepsilon\langle\xi\,,\,\varepsilon\nabla u_\varepsilon\rangle \,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^T \int V\langle\xi\,,\,\nu\rangle|\nabla\chi|\,\mathrm{d}t\right| \\
= &\limsup_{\varepsilon\to 0}\left|\sum_{i=1}^n \int_0^T \int \eta_i\partial_t u_\varepsilon\langle\xi\,,\,\varepsilon\nabla u_\varepsilon\rangle \,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^T \int \eta_i V\langle\xi\,,\,\nu\rangle|\nabla\chi|\,\mathrm{d}t\right| \\
\leq &\limsup_{\varepsilon\to 0}\|\xi\|_{\mathrm{L}^\infty}\left(\sum_{i=1}^n \alpha\int_0^T \left(\int \eta_i\varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x + \int \eta_i V^2|\nabla\chi|\right)\mathrm{d}t + \frac{2}{\alpha}\,\mathcal{E}(\nu_i^*;\eta_i)\right) < \delta,
\end{aligned}$$

which finishes the proof.

4.2. Conditional convergence in the multiphase case

4.2.1. Convergence to a BV-solution to multiphase mean curvature flow

We start by defining a BV-formulation for motion by multiphase mean curvature flow under the simplifying assumption that the mobilities are already fixed by the surface tensions through the equation

$$\mu_{ij} = \frac{1}{\sigma_{ij}}. \tag{4.10}$$

Note moreover that in this section, all expressions refer to their multiphase equivalent.

Definition 4.2.1 (BV-solution to multiphase mean curvature flow)**.** *Fix some finite time horizon $T < \infty$, a $P \times P$ matrix of surface tensions σ and initial data $\chi^0\colon\ \mathbb{T} \to \{0,1\}^P$ with $\mathrm{E}_0 := \mathrm{E}(\chi^0) < \infty$ and $\sum_{i=1}^P \chi_i^0 = 1$. We say that*

$$\chi \in \mathrm{C}\left([0,T]; \mathrm{L}^2\left(\mathbb{T}; \{0,1\}^P\right)\right)$$

with $\operatorname{ess\,sup}_{0\le t\le T} \mathrm{E}(\chi)$ *and* $\sum_{i=1}^P \chi_i = 1$ moves by mean curvature with initial data χ^0 and surface tensions σ *if there exist normal velocities $V_i \in \mathrm{L}^2(|\nabla\chi_i|\,\mathrm{d}t)$ with*

$$\int_0^T \int V_i^2 |\nabla \chi_i| \,\mathrm{d}t < \infty$$

such that

1. *For all $\xi \in \mathrm{C}_\mathrm{c}^\infty\left((0,T)\times\mathbb{T};\mathbb{R}^d\right)$, we have*

$$\sum_{1\le i<j\le P} \sigma_{ij} \int_0^T \int \left(V_i \langle \xi\,,\, \nu_i\rangle - \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - \nu_i \otimes \nu_i\rangle\right) \times \tag{4.11}$$
$$\frac{1}{2}\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right)\mathrm{d}t = 0,$$

   where ν_i is the inner unit normal of χ_i.

2. *The functions V_i are the normal velocities of the interfaces in the sense that*

$$\partial_t \chi_i = V_i |\nabla\chi_i| \,\mathrm{d}t$$

   holds distributionally on $(0,T)\times\mathbb{T}$.

3. *The initial data is achieved in the space $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$.*

Since (4.11) is quite the long equation, we want to motivate it as a suitable weak formulation for multiphase mean curvature flow. Assuming that everything is nice and smooth, we have by the Gauss-Green theorem on surfaces [Mag12, Thm. 11.8] that

$$\int_{\Sigma_{ij}} \int \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - \nu\otimes\nu\rangle \,\mathrm{d}\,\mathcal{H}^{d-1} = -\int_{\Sigma_{ij}} \int H\langle \xi\,,\,\nu\rangle \,\mathrm{d}\,\mathcal{H}^{d-1} + \int_{\Gamma_{ij}} \int \langle \xi\,,\, \nu_\Gamma\rangle \,\mathrm{d}\,\mathcal{H}^{d-2}\,.$$

Here Γ_{ij} denotes the boundary of the surface Σ_{ij} and ν_Γ the corresponding unit normal. But then Herring's angle condition (1.6) tells us that the boundary terms cancel out. Therefore equation (4.11) is now the sum of integral over each interface of

$$(V_i + H)\langle \xi \,,\, \nu_i \rangle,$$

which is zero by equation (1.5) combined with the assumption for the mobilities (4.10).

Our main goal will be to arrive at a conditional convergence result to multiphase mean curvature flow in the sense of the above definition. This is captured by the following.

Theorem 4.2.2. *Let a smooth multiwell potential $W\colon \mathbb{R}^N \to [0,\infty)$ satisfy the assumptions (2.3)-(2.6). Let $T < \infty$ be an arbitrary finite time horizon. Given a sequence of initial data $u_\varepsilon^0\colon\, \mathbb{T} \to \mathbb{R}^N$ approximating a partition $\chi^0 \in \mathrm{BV}\left(\mathbb{T}; \{0,1\}^P\right)$ in the sense that $u_\varepsilon^0 \to u^0 = \sum_{1\le i\le P} \chi_i^0 \alpha_i$ holds pointwise almost everywhere and*

$$\mathrm{E}_0 \coloneqq \mathrm{E}(\chi^0) = \lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon^0) < \infty,$$

we have that for some subsequence of solutions u_ε to the Allen–Cahn equation (2.1) with initial datum u_ε^0, there exists a time-dependent partition χ with $\chi \in \mathrm{BV}\left((0,T)\times\mathbb{T}; \{0,1\}^P\right)$ and $\chi \in \mathrm{C}\left([0,T]; \mathrm{L}^2\left(\mathbb{T}; \{0,1\}^P\right)\right)$ such that u_ε converges to $u \coloneqq \sum_{1\le i\le P} \chi_i \alpha_i$ almost everywhere. Moreover u assumes the initial data u^0 in $\mathrm{C}\left([0,T]; \mathrm{L}^2(\mathbb{T})\right)$. If we additionally assume that the time-integrated energies converge (4.1), then χ moves by mean curvature in the sense of Definition 4.2.1.

First we are going to prove the existence of normal velocities V_i for the evolving sets Ω_i under the assumption of the energy convergence (4.1).

Proposition 4.2.3. *In the situation of Proposition 3.2.4 and under the energy convergence assumption (4.1), we have that for every $1 \le i \le P$, the signed measure $\partial_t \chi_i$ is absolutely continuous with respect to the measure $|\nabla \chi_i|\,\mathrm{d}t$ and the corresponding density V_i is square integrable with the corresponding estimate*

$$\int_0^T \int V_i^2 |\nabla \chi_i| \,\mathrm{d}t \lesssim \mathrm{E}_0\,. \tag{4.12}$$

Proof. The proof proceeds in four steps. First we show that $\partial_t \psi_i$ is absolutely continuous with respect to the energy measure. Afterwards, we show that we can already estimate $|\partial_t \chi_i|$ by $|\partial_t \psi_i|$. To finish the argument, we then we have to argue that $\partial_t \chi_i$ is already singular with respect to every part of the energy which is not contained in $|\nabla \chi_i|\,\mathrm{d}t$.

Step 1: The signed measure $\partial_t \psi_i$ is absolutely continuous with respect to the energy measure $\mathrm{E}(\cdot, u)\,\mathrm{d}t$ with square-integrable density.

Let φ be a testfunction. Then we can estimate by Hölder's inequality

$$\begin{aligned}
\partial_t\psi_i(\varphi) &= \liminf_{\varepsilon\to 0}\int_0^T\int \varphi\partial_t\psi_{\varepsilon,i}\,\mathrm{d}x\,\mathrm{d}t\\
&\le \liminf_{\varepsilon\to 0}\left(\int_0^T\int \varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int \varphi^2\frac{1}{\varepsilon}2W(u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\
&\le \sqrt{\mathrm{E}_0}\left(\int_0^T \mathrm{E}(u;\varphi)\,\mathrm{d}t\right)^{1/2}.
\end{aligned}$$

Here the last inequality is due to the energy dissipation (2.12) and the proof of our claim is therefore complete.

Step 2: For $d_i := \min_{j\neq i}\sigma_{ij}$, we have that $d_i|\partial_t\chi_i| \le |\partial_t\psi_i|$.

Using the disintegration Theorem [AFP00, Thm. 3.103] and the Fleming–Rishel co-area formula, we have for any open set $U\subseteq(0,T)\times\mathbb{T}$ that

$$\begin{aligned}
|\partial_t\psi_i|(U) &= \int |\partial_t\psi_i(\cdot,x)|(\pi_x(U))\,\mathrm{d}x\\
&= \int\int_0^\infty \mathcal{H}^0\left(\left\{t:\ \sum_j \chi_j\sigma_{ij} > s, (t,x)\in U\right\}\right)\mathrm{d}s\,\mathrm{d}x\\
&\ge \int\int_0^{d_i} \mathcal{H}^0\left(\{t:\ \chi_i(t,x)=1, (t,x)\in U\}\right)\mathrm{d}s\,\mathrm{d}x\\
&= d_i\int\int_0^\infty \mathcal{H}^0\left(\{t:\ \chi_i(t,x)>s, (t,x)\in U\}\right)\mathrm{d}s\,\mathrm{d}x\\
&= d_i|\partial_t\chi_i|(U).
\end{aligned}$$

Here the set $\pi_x(U)$ denotes the set of all t such that (t,x) is an element of U. This completes the proof.

Step 3: The signed measure $\partial_t\chi_i$ is singular to the wrong parts of $\mathrm{E}(u;\cdot)$. More precisely we claim that if $1\le j<k\le P$ and $i\notin\{j,k\}$, then the measures $\mathcal{H}^{d-1}|_{\Sigma_{jk}}\,\mathrm{d}t$ and $|\partial_t\chi_i|$ are mutually singular.

We again write $|\nabla\chi|_{d+1}$ for the space derivative in time and space and $|\nabla\chi|_d$ for the space derivative for some fixed time t, which is defined for almost every time. Moreover, we denote by $\tilde\Omega_i$ the set in time and space such that $\chi_i(t,x)=\mathbb{1}_{\tilde\Omega_i}(t,x)$ and by $\tilde\Sigma_{ij}$ the corresponding interfaces. By Lemma A.0.1 we can then write

$$|\partial_t\chi_i| \le |(\partial_t,\nabla)\chi_i| = \sum_{l\neq i}\mathcal{H}^d|_{\partial_*\tilde\Omega_i\cap\partial_*\tilde\Omega_l}.$$

But for all $l\neq i$, we have again by Lemma A.0.1 that either

$$|(\partial_t,\nabla)\chi_j|(\tilde\Sigma_{il}) = 0 \quad\text{or}\quad |(\partial_t,\nabla)\chi_k|(\tilde\Sigma_{il}) = 0.$$

Without loss of generality we assume the former. Then it follows that $|\nabla\chi_j|_{d+1}\left(\tilde{\Sigma}_{il}\right)=0$ and therefore we have

$$\begin{aligned}\mathcal{H}^{d-1}\lfloor_{\Sigma_{jk}}\,\mathrm{d}t\,(\tilde{\Sigma}_{il}) &= \frac{1}{2}\left(|\nabla\chi_j|_d+|\nabla\chi_k|_d-|\nabla(\chi_j+\chi_k)|_d\right)\mathrm{d}t\left(\tilde{\Sigma}_{il}\right)\\ &= \frac{1}{2}\left(|\nabla\chi_j|_{d+1}+|\nabla\chi_k|_{d+1}-|\nabla(\chi_j+\chi_k)|_{d+1}\right)\left(\tilde{\Sigma}_{il}\right)\\ &= \frac{1}{2}\left(|\nabla\chi_k|_{d+1}\left(\tilde{\Sigma}_{i,l}\right)-|\nabla\chi_k|_{d+1}\left(\tilde{\Sigma}_{il}\right)\right)=0.\end{aligned}$$

This proves the desired singularity.

Step 4: We have $|\partial_t\chi_i|\leq|\nabla\chi_i|\,\mathrm{d}t$ and estimate (4.12) holds.

Combining Step 1 and Step 2, we obtain that $|\partial_t\chi_i|$ is absolutely continuous with respect to the energy measure $\mathrm{E}(u;\cdot)$. Moreover we can write

$$\mathrm{E}(u;\cdot)\,\mathrm{d}t=\sum_{j\neq i}\sigma_{ij}\,\mathcal{H}^{d-1}\lfloor_{\Sigma_{ij}}\,\mathrm{d}t+\sum_{1\leq j<k\leq P,j,k\neq i}\sigma_{jk}\,\mathcal{H}^{d-1}\lfloor_{\Sigma_{jk}}\,\mathrm{d}t\,.$$

The second sum is by Step 3 singular to $|\partial_t\chi_i|$ and the first sum is bounded by $|\nabla\chi_i|\,\mathrm{d}t$, which proves that $\partial_t\chi_i$ is absolutely continuous with respect to $|\nabla\chi_i|\,\mathrm{d}t$. Lastly the desired estimate (4.12) is now a consequence of our previous arguments combined with a duality estimate. By Step 3 we find a Borel-measurable set S such that for all $j,k\neq i$, we have $\mathcal{H}^{d-1}\lfloor_{\Sigma_{jk}}\,\mathrm{d}t\,(S)=0$ and $|\partial_t\chi_i|((0,T)\times\mathbb{T}\setminus S)=0$. Fix some $K>0$ and let φ_n be a sequence of testfunctions which converges in $\mathrm{L}^2\left(|\nabla\chi_i|\,\mathrm{d}t\right)$ to the function $V_i\mathbb{1}_{|V_i|\leq K}\mathbb{1}_S$. Then combining the previous steps, we can estimate

$$\begin{aligned}\int_0^T\int V_i^2\mathbb{1}_{|V_i|\leq K}|\nabla\chi_i|\,\mathrm{d}t &= \liminf_{n\to\infty}\left|\int_0^T\int V_i\varphi_n|\nabla\chi_i|\,\mathrm{d}t\right|\\ &= \liminf_{n\to\infty}\left|\int_0^T\int\varphi_n\partial_t\chi_i\right|\\ &\lesssim \liminf_{n\to\infty}\int_0^T\int\varphi_n^2|\partial_t\psi_i|\\ &\leq \sqrt{\mathrm{E}_0}\liminf_{n\to\infty}\left(\int_0^T\mathrm{E}(u;\varphi_n^2)\,\mathrm{d}t\right)^{1/2}\\ &\lesssim \sqrt{\mathrm{E}_0}\left(\int_0^T\int V_i^2\mathbb{1}_{|V_i|\leq K}|\nabla\chi_i|\,\mathrm{d}t\right)^{1/2}.\end{aligned}$$

This proves by the monotone convergence theorem the desired claim.

Therefore our proof is complete. □

As in the twophase case Lemma 4.1.5, we have an equipartition of energies. The proof is similar and therefore left out.

Lemma 4.2.4. *In the situation of Proposition 3.2.4 and under the energy convergence assumption (4.1), we have for any continuous $\varphi \in \mathrm{C}(\mathbb{T})$ that*

$$\begin{aligned}\mathrm{E}(u;\varphi) = \lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) &= \lim_{\varepsilon\to 0} \int \varphi\varepsilon |\nabla u_\varepsilon|^2 \,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int \varphi \frac{1}{\varepsilon} 2W(u_\varepsilon)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon|\,\mathrm{d}x\end{aligned}$$

for almost every time $0 \le t \le T$.

4.2.2. Localization estimates

In order to prove convergence of the curvature and velocity term, we want to reduce the multiphase case to the two-phase case. The central idea here is to cover the flat torus with a suitable covering of balls and argue that, up to a small error, we can choose for each ball majority phases i, j such that the partition looks like a two-phase mean curvature flow on the ball, see Figure 4.1.

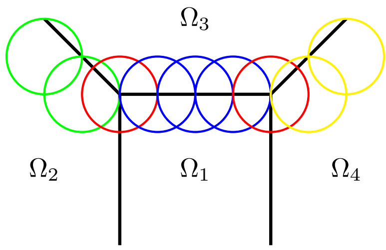


Figure 4.1.: Localization of multiphase mean curvature flow. For the green balls, we choose the majority phase (2,3), for the blue ones (1,3) and for the yellow balls (3,4). The red balls however give us an error.

To formulate this rigorously let $r > 0$ and define the covering $\mathcal{B}_r$ of the flat torus by

$$\mathcal{B}_r := \{B_r(c) : c \in \mathcal{L}_r\},$$

where the set of centers $\mathcal{L}_r$ is given by $\mathcal{L}_r := \mathbb{T} \cap (r/\sqrt{d})\mathbb{Z}^d$. Moreover let ρ_B be a smooth cutoff for the ball B with support in the ball with the same center, but double the radius.

Additionally to choosing a majority phase, we can also argue that along the chosen majority phase, we have a local flatness. Thus we may approximate the outer unit normal up to an arbitrarily small error by a constant unit vector. This is captured by the following Lemma found in [LO16]. The proof is based on De Giorgi's structure theorem for sets of finite perimeter.

Lemma 4.2.5. *For every $\delta > 0$ and every partition $\chi\colon \mathbb{T} \to \{0,1\}^P$ such that $\chi_i \in \mathrm{BV}(\mathbb{T})$ holds for all $1 \le i \le P$, there exist some $r_0 > 0$ such that for all $0 < r < r_0$, we find for every ball $B \in \mathcal{B}_r$ some unit vector ν_B such that*

$$\sum_{B\in\mathcal{B}_r} \min_{i\neq j} \int \rho_B |\nu_i - \nu_B|^2 |\nabla\chi_i| + \int \rho_B |\nu_j + \nu_B|^2 |\nabla\chi_j| + \sum_{k\notin\{i,j\}} \int \rho_B |\nabla\chi_k| \lesssim \delta\, \mathrm{E}(\chi).$$

Note that by Lemma A.0.1, integrating $|\nabla\chi_k|$ over the sum of all $k \notin \{i,j\}$ equates to summing over all interfaces which are not the (i,j)-th interface. Moreover the second summand is in some sense unnecessary, since the last summand provides us with a localization on the (i,j)-th interface, on which we have $\nu_i = -\nu_j$. However it is convenient to keep it so that we do not have to repeat this argument.

Remark 4.2.6. This localization estimate also implies the smallness of

$$\sum_{B\in\mathcal{B}_r} \min_i \left| \mathrm{E}(\chi;\rho_B) - \int \rho_B |\nabla\psi_i| \right| = \sum_{B\in\mathcal{B}_r} \mathrm{E}(\chi;\rho_B) - \max_i \int \rho_B |\nabla\psi_i| \tag{4.13}$$

in the same sense as in the above Lemma 4.2.5. Notice that the equality (4.13) follows for example from Proposition 3.2.9, which yields that $|\nabla\psi_i| \le \mathrm{E}(\chi;\cdot)$. The smallness follows since by Lemma 3.2.8, we have for every $i \neq j$ that

$$\begin{aligned}
&\mathrm{E}(\chi;\rho_B) - \int \rho_B |\nabla\psi_i| \\
&= \sum_{1\le k<l\le P} \sigma_{kl} \int_{\Sigma_{kl}} \rho_B \,\mathrm{d}\mathcal{H}^{d-1} - \sum_{1\le k<l\le P} |\sigma_{ik} - \sigma_{il}| \int_{\Sigma_{kl}} \rho_B \,\mathrm{d}\mathcal{H}^{d-1} \\
&= \sum_{\substack{1\le k<l\le P\\ \{k,l\}\neq\{i,j\}}} (\sigma_{kl} - |\sigma_{ik} - \sigma_{il}|) \int_{\Sigma_{kl}} \rho_B \,\mathrm{d}\mathcal{H}^{d-1} \\
&\lesssim \sum_{k\notin\{i,j\}} \int \rho_B |\nabla\chi_k|.
\end{aligned}$$

Thus we can estimate the error by the last summand of the error in Lemma 4.2.5.

4.2.3. Convergence of the curvature term

Proposition 4.2.7. *In the situation of Proposition 3.2.4 and under the energy convergence assumption (4.1), we have that the first variations converge in the sense that for almost every time $0 \le t \le T$, we have*

$$\lim_{\varepsilon\to 0} \int \left\langle \varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla W(u_\varepsilon)\,,\, \mathrm{D}\, u_\varepsilon \xi \right\rangle \mathrm{d}x = \sum_{1\le i<j\le P} \sigma_{ij} \int_{\Sigma_{ij}} \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - \nu_i \otimes \nu_i \rangle \,\mathrm{d}\mathcal{H}^{d-1}.$$

Additionally we have the estimate

$$\left| \int \left\langle \varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla W(u_\varepsilon)\,,\, \mathrm{D}\, u_\varepsilon \xi \right\rangle \mathrm{d}x \right| \lesssim \|\nabla\xi\|_{\sup} \mathrm{E}_\varepsilon(u_\varepsilon). \tag{4.14}$$

Remark 4.2.8. We can not proceed exactly as in the twophase case. There the key argument was that from the energy convergence, we could already infer that $|\nabla \psi_\varepsilon|(\mathbb{T})$ converges to $|\nabla \psi|(\mathbb{T})$, and thus enabled us to apply the Theorem by Reshetnyak. Firstly we do not have one primitive ϕ of $\sqrt{2W}$ in the multiphase case and more importantly, we can not expect the functions $\psi_{\varepsilon,i}$ to satisfy

$$|\psi_{\varepsilon,i}|(\mathbb{T}) \to |\psi_i|(\mathbb{T}), \tag{4.15}$$

so the original theorem from Reshetnyak will not work here. However we have by the lower semicontinuity of the variation measure under the assumption of energy convergence (4.1) that

$$\begin{aligned} |\nabla \psi_i|(\mathbb{T}) &\le \liminf_{\varepsilon \to 0} |\nabla \psi_{\varepsilon,i}|(\mathbb{T}) \\ &\le \liminf_{\varepsilon \to 0} \mathrm{E}_\varepsilon(u_\varepsilon) \\ &= \mathrm{E}(u) \\ &= \sum_{1 \le j < k \le P} \sigma_{jk}\, \mathcal{H}^{d-1}(\Sigma_{jk}). \end{aligned}$$

But by Lemma 3.2.8, we know that

$$|\nabla \psi_i| = \sum_{1 \le j < k \le P} |\sigma_{ij} - \sigma_{ik}|\, \mathcal{H}^{d-1} \lfloor_{\Sigma_{jk}}.$$

Since $|\sigma_{ii} - \sigma_{ik}| = \sigma_{ik}$ for all $k \neq i$, this yields that localized on $\partial_* \Omega_i$, we have the necessary convergence $|\nabla \psi_{\varepsilon,i}| \to |\nabla \psi_i|$. Thus we will develop a quantitative version of Reshetnyaks theorem and apply it to our setting.

Proof of Proposition 4.2.7. Using the same calculations as in the two-phase case, we obtain for a given test vector field ξ that

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int \left\langle \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon)\,,\, \mathrm{D}\, u_\varepsilon \xi \right\rangle \mathrm{d}x \\ = &\lim_{\varepsilon \to 0} \int \varepsilon \left(|\nabla u_\varepsilon|^2 \operatorname{div} \xi - \sum_{i=1}^{N} \sum_{j,k=1}^{d} \partial_{x_j} u_\varepsilon^i \partial_{x_j} \xi^k \partial_{x_k} u_\varepsilon^i \right) \mathrm{d}x =: \lim_{\varepsilon \to 0} I_\varepsilon \end{aligned}$$

Now however, we can not proceed as in the twophase case as explained in Remark 4.2.8. Instead we rewrite

$$I_\varepsilon = \int \varepsilon \langle \mathrm{D}\, \xi\,,\, \mathrm{Id} - N_\varepsilon^\top N_\varepsilon \rangle |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,, \tag{4.16}$$

where we define $N_\varepsilon := \mathrm{D}\, u_\varepsilon / |\mathrm{D}\, u_\varepsilon|$ wherever it is defined, and zero else. By the equipartition of energies (Lemma 4.2.4) and using that $|N_\varepsilon| \le 1$, we can replace $\varepsilon |\nabla u_\varepsilon|^2$ by $\sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon|$.

We can take care of the term involving the divergence again with the equipartition of energies. Thus summarizing our results, it suffices by rescaling to show for all $A \in \mathrm{C}^\infty(\mathbb{T};\mathbb{R}^{d\times d})$ with $|A| \le 1$ that

$$\lim_{\varepsilon\to 0} \int \langle A\,,\, N_\varepsilon^\top N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \,\mathrm{d}x = \sum_{1\le i<j\le P} \sigma_{i,j} \int_{\Sigma_{i,j}} \langle A\,,\, \nu_i \otimes \nu_i \rangle \,\mathrm{d}\,\mathcal{H}^{d-1}\,. \tag{4.17}$$

To this end, we will show the following three claims for a partition of unity function η.

Claim 1: We choose a majority phase by introducing the function ϕ_i on the right hand side of equation (4.17). The corresponding error is given by

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \left| \int \eta \langle A\,,\, N_\varepsilon^\top N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \,\mathrm{d}x - \int \eta \langle A\,,\, \theta_{\varepsilon,i} \otimes \theta_{\varepsilon,i} \rangle |\nabla \psi_{\varepsilon,i}| \,\mathrm{d}x \right| \\ &\lesssim \mathrm{E}(u;\eta) - \int \eta |\nabla \psi_i|,\end{aligned}$$

where the approximate normal of the i-th phase is defined by $\theta_i^\varepsilon := \nabla\psi_i^\varepsilon / |\nabla \psi_i^\varepsilon|$ wherever the gradient does not vanish, and zero else.

Claim 2: By a quantitative Reshetnyak type argument, we have

$$\limsup_{\varepsilon\to 0} \left| \int \eta \langle A\,,\, \theta_{\varepsilon,i} \otimes \theta_{\varepsilon,i} \rangle |\nabla \psi_{\varepsilon,i}| \,\mathrm{d}x - \int \eta \langle A\,,\, \theta_i \otimes \theta_i \rangle |\nabla \psi_i| \right| \lesssim \mathrm{E}(u;\eta) - \int \eta |\nabla \psi_i|,$$

where $\theta_i := \nabla \psi_i / |\nabla \psi_i|$.

Claim 3: We can undo the localization onto the majority phase. The corresponding error is given by

$$\left| \int \eta \langle A\,,\, \theta_i \otimes \theta_i \rangle |\nabla \psi_i| - \sum_{1\le j<k\le P} \sigma_{jk} \int_{\Sigma_{jk}} \eta \langle A\,,\, \nu_j \otimes \nu_j \rangle \,\mathrm{d}\,\mathcal{H}^{d-1} \right| \le \mathrm{E}(u;\eta) - \int \eta |\nabla \psi_i|,$$

where we remind that ν_j is the inner unit normal of Ω_j.

Let us first assume that we have proven those three claims and show how the desired convergence (4.17) follows from them. We take a partition of unity η_B of the flat torus with respect to the covering $\mathcal{B}_r$ introduced in Section 4.2.2. Then

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left|\int \left\langle A\,,\, N_\varepsilon^\top N_\varepsilon\right\rangle \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x - \sum_{1\le j<k\le P}\sigma_{jk}\int_{\Sigma_{jk}}\langle A\,,\,\nu_j\otimes\nu_j\rangle\,\mathrm{d}\mathcal{H}^{d-1}\right| \\
&= \left|\sum_{B\in\mathcal{B}_r}\int \left\langle A\eta_B\,,\, N_\varepsilon^\top N_\varepsilon\right\rangle \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x - \sum_{1\le j<k\le P}\sigma_{jk}\int_{\Sigma_{jk}}\langle A\eta_B\,,\,\nu_j\otimes\nu_j\rangle\,\mathrm{d}\mathcal{H}^{d-1}\right| \\
&\le \sum_{B\in\mathcal{B}_r}\min_{1\le i\le P}\left|\int \langle A\eta_B\,,\, N_\varepsilon^\top N_\varepsilon\rangle\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x - \int\langle A\eta_B\,,\,\theta_{\varepsilon,i}\otimes\theta_{\varepsilon,i}\rangle|\nabla\psi_{\varepsilon,i}|\,\mathrm{d}x\right| \\
&\qquad\qquad + \left|\int\langle A\eta_B\,,\,\theta_{\varepsilon,i}\otimes\theta_{\varepsilon,i}\rangle|\nabla\psi_{\varepsilon,i}|\,\mathrm{d}x - \int\langle A\eta_B\,,\,\theta_i\otimes\theta_i\rangle|\nabla\psi_i|\right| \\
&\qquad\qquad + \left|\int\langle A\eta_B\,,\,\theta_i\otimes\theta_i\rangle|\nabla\psi_i| - \sum_{1\le j<k\le P}\sigma_{jk}\int_{\Sigma_{jk}}\langle A\eta_B\,,\,\nu_j\otimes\nu_j\rangle\,\mathrm{d}\mathcal{H}^{d-1}\right| \\
&\lesssim \sum_{B\in\mathcal{B}_r}\min_{1\le i\le P}\mathrm{E}(u;\eta_B) - \int\eta_B|\nabla\psi_i|,
\end{aligned}$$

which vanishes as r tends to zero by Remark 4.2.6. Thus let us now prove the three claims.

Proof of Claim 1. For simplicity, we drop the index i and for now also ε. First, we replace the matrix N by the matrix πN, where we define the rank-one matrix π by

$$\pi := \frac{\nabla\phi}{|\nabla\phi|}\otimes\frac{\nabla\phi}{|\nabla\phi|}\in\mathbb{R}^N,$$

which is motivated by the chain rule. The multiplication with π is an orthogonal projection $\pi\colon\mathbb{R}^{N\times d}\to\mathbb{R}^{N\times d}$. Moreover, we can compute that

$$\begin{aligned}
((\pi N)^\top\pi N)_{ij} &= \sum_{k=1}^N(\pi N)_{ki}(\pi N)_{kj} = \sum_{k=1}^N\left(\sum_{l=1}^N\pi_{kl}N_{li}\right)\left(\sum_{r=1}^N\pi_{kr}N_{rj}\right) \\
&= \sum_{l,r=1}^N N_{li}N_{rj}\sum_{k=1}^N\pi_{kl}\pi_{kr} = \sum_{l,r=1}^N N_{li}N_{rj}\sum_{k=1}^N\frac{\partial_k\phi\partial_l\phi\partial_k\phi\partial_r\phi}{|\nabla\phi|^4} \\
&= \sum_{l,r=1}^N N_{li}\frac{\partial_l\phi}{|\nabla\phi|}N_{rj}\frac{\partial_r\phi}{|\nabla\phi|} = \frac{\partial_i\psi}{|\nabla u||\nabla\phi|}\,\frac{\partial_j\psi}{|\nabla u||\nabla\phi|}
\end{aligned}$$

Thus we infer that

$$\begin{aligned}\left\langle A\,,\,(\pi N_\varepsilon)^\top \pi N_\varepsilon\right\rangle &= \sum_{i,j=1}^{d} A_{ij}\frac{\partial_i\psi_\varepsilon}{|\nabla\phi||\nabla u_\varepsilon|}\frac{\partial_j\psi_\varepsilon}{|\nabla\phi||\nabla u_\varepsilon|}\\ &= \frac{|\nabla\psi_\varepsilon|^2}{|\nabla\phi|^2|\nabla u_\varepsilon|^2}\left\langle A\,,\,\frac{\nabla\psi_\varepsilon}{|\nabla\psi_\varepsilon|}\otimes\frac{\nabla\psi_\varepsilon}{|\nabla\psi_\varepsilon|}\right\rangle\\ &= |\pi N_\varepsilon|^2\langle A\,,\,\theta_\varepsilon\otimes\theta_\varepsilon\rangle, \end{aligned} \tag{4.18}$$

where we used that

$$|\pi N_\varepsilon|^2 = \frac{|\nabla\psi_\varepsilon|^2}{|\nabla\phi|^2|\nabla u_\varepsilon|^2}.$$

Moreover we recognize that since multiplication with π is an orthogonal projection, we have the Pythagorean Theorem

$$\begin{aligned}N^\top N &= (\pi N + N - \pi N)^\top(\pi N + N - \pi N)\\ &= (\pi N)^\top \pi N + (N-\pi N)^\top(N-\pi N) + (\pi N)^\top(N-\pi N) + (N-\pi N)^\top \pi N\\ &= (\pi N)^\top \pi N + (N-\pi N)^\top(N-\pi N).\end{aligned}$$

In order to prove the claim, we first get an error when replacing $N_\varepsilon^\top N_\varepsilon$ with $\theta_\varepsilon\otimes\theta_\varepsilon$ and the second error when replacing $\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|$ by $|\nabla\psi_\varepsilon|$. For the first error we can estimate by the Pythagorean Theorem and equation (4.18) that

$$\begin{aligned}&\left|\int \eta\left\langle A\,,\,N_\varepsilon^\top N_\varepsilon\right\rangle\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x - \int \eta\langle A\,,\,\theta_\varepsilon\otimes\theta_\varepsilon\rangle\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x\right|\\ \le&\left|\int \eta\left\langle A\,,\,N_\varepsilon^\top N_\varepsilon\right\rangle\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x - \int \eta\left\langle A\,,\,\pi N_\varepsilon^\top \pi N_\varepsilon\right\rangle\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x\right|\\ &+\left|\int \eta\left\langle A\,,\,\pi N_\varepsilon^\top \pi N_\varepsilon\right\rangle\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x - \int \eta\langle A\,,\,\theta_\varepsilon\otimes\theta_\varepsilon\rangle\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x\right|\\ \le&\int \eta|N_\varepsilon - \pi N_\varepsilon|^2\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x + \int \eta\,(1-|\pi N_\varepsilon|)^2\,\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x =: I_\varepsilon.\end{aligned}$$

Using again that multiplication with π is an orthogonal projection, we have

$$|(\mathrm{Id}-\pi)N_\varepsilon|^2 = |N_\varepsilon|^2 - |\pi N_\varepsilon|^2 = 1 - |\pi N_\varepsilon|^2 \lesssim 1 - |\pi N_\varepsilon| = 1 - \left|\frac{\mathrm{D}\,\phi}{|\nabla\phi|}N_\varepsilon\right|,$$

where for the last identity, we used that $\left|v^\top B\right| = |v\otimes vB|$ for all unit vectors v and matrices B. Therefore we can estimate

$$\begin{aligned}I_\varepsilon &\lesssim \int \eta\left(1-\left|\frac{\mathrm{D}\,\phi(u_\varepsilon)}{|\nabla\phi(u_\varepsilon)|}N_\varepsilon\right|\right)\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x\\ &= \int \eta\left(\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| - |\nabla\psi^\varepsilon|\frac{\sqrt{2W(u_\varepsilon)}}{|\nabla\phi(u_\varepsilon)|}\right)\mathrm{d}x\\ &\le \mathrm{E}_\varepsilon(u_\varepsilon;\eta) - \int \eta|\nabla\psi_\varepsilon|\,\mathrm{d}x\,.\end{aligned}$$

The last inequality is due to Young's inequality and $|\nabla\phi| \le \sqrt{2W(u_\varepsilon)}$. By the convergence of energies for almost every time and the lower semicontinuity of the variation measure, we thus have

$$\limsup_{\varepsilon\to 0} I_\varepsilon \lesssim \mathrm{E}(u;\eta) - \int \eta|\nabla\psi|.$$

The second error is given by

$$\begin{aligned}
&\left|\int \eta\langle A\,,\,\theta_\varepsilon\otimes\theta_\varepsilon\rangle\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x - \int \eta\langle A\,,\,\theta_\varepsilon\otimes\theta_\varepsilon\rangle|\nabla\psi_\varepsilon|\,\mathrm{d}x\right|\\
=&\left|\int \eta\langle A\,,\,\theta_\varepsilon\otimes\theta_\varepsilon\rangle\left(\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| - |\nabla\psi_\varepsilon|\right)\mathrm{d}x\right|\\
\le&\int \eta\left(\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| - |\nabla\psi_\varepsilon|\right)\mathrm{d}x\\
\le&\ \mathrm{E}_\varepsilon(u_\varepsilon;\eta) - \int \eta|\nabla\psi_\varepsilon|\,\mathrm{d}x\,,
\end{aligned}$$

which in the limes superior can be controlled by the desired term as above, finishing the proof for the first claim. □

Proof of Claim 2. The strategy here is to first pass to the space of measures in order to obtain some limit, and then sandwich the limit between the measure $|\nabla\psi_i|$ and the energy measure $\mathrm{E}(u;\cdot)$. Thus consider the sequence $(\mu_\varepsilon)_\varepsilon$ of Radon measures on $\mathbb{T}\times\mathbb{S}^{d-1}$ defined by

$$\mu_\varepsilon := |\nabla\psi_\varepsilon|\,\mathrm{d}x\otimes\left(\delta_{u_\varepsilon(x)}\right)_{x\in\mathbb{T}}$$

which acts on continuous functions φ through

$$\int_{\mathbb{T}\times\mathbb{S}^{d-1}} \varphi(x,\nu)\,\mathrm{d}\mu_\varepsilon(x,\nu) = \int_{\mathbb{T}} \varphi\left(x,\nu_\varepsilon(x)\right)|\nabla\psi_\varepsilon|\,\mathrm{d}x\,.$$

By Young's inequality and the boundedness of energies, μ_ε is a bounded sequence of Radon measures, thus we find a Radon measure $\tilde\mu$ on $\mathbb{T}\times\mathbb{S}^{d-1}$ and some non-relabelled subsequence such that $\mu_\varepsilon \rightharpoonup^* \tilde\mu$ as Radon measures on $\mathbb{T}\times\mathbb{S}^{d-1}$. By [AFP00, Thm. 2.28] we can disintegrate the measure $\tilde\mu$, which means that we find probability measures $(p_x)_{x\in\mathbb{T}}$ and a Radon measure μ on $\mathbb{T}$ such that $x\mapsto p_x$ is μ-measurable and we have the identity

$$\int_{\mathbb{T}\times\mathbb{S}^{d-1}} f(x,\tilde\nu)\,\mathrm{d}\tilde\mu(x,\tilde\nu) = \int_{\mathbb{T}}\int_{\mathbb{S}^{d-1}} f(x,\tilde\nu)\,\mathrm{d}p_x(\tilde\nu)\,\mathrm{d}\mu(x)$$

for all $f\in \mathrm{L}^1(\mathbb{T}\times\mathbb{S}^{d-1},\tilde\mu)$. Thus we have

$$\lim_{\varepsilon\to 0}\int \varphi(x,\nu_\varepsilon)|\nabla\psi^\varepsilon|\,\mathrm{d}x = \int_{\mathbb{T}}\int_{\mathbb{S}^{d-1}} \varphi(x,\tilde\nu)\,\mathrm{d}p_x(\tilde\nu)\,\mathrm{d}\mu(x)$$

for all continuous φ and plugging in $\varphi(x,\tilde{\nu}) := \langle A(x)\,,\,\tilde{\nu}\otimes\tilde{\nu}\rangle$ thus yields

$$\lim_{\varepsilon\to 0}\int \eta\langle A\,,\,\nu_\varepsilon\otimes\nu_\varepsilon\rangle|\nabla\psi_\varepsilon|\,\mathrm{d}x = \int \eta\Big\langle A(x)\,,\int \tilde{\nu}\otimes\tilde{\nu}\,\mathrm{d}p_x(\tilde{\nu})\Big\rangle\,\mathrm{d}\mu(x)\,.$$

We now would like to prove that up to an error bounded by $\mathrm{E}(u;\eta)-\int\eta|\nabla\psi|$, the right-hand side is equal to $\int\langle A\,,\,\theta\otimes\theta\rangle|\nabla\psi|$.

On the one hand, we have by the lower semicontinuity of the variation measure that

$$\int \eta|\nabla\psi| \le \liminf_{\varepsilon\to 0}\int \eta|\nabla\psi_\varepsilon|\,\mathrm{d}x = \int \eta(x)\int 1\,\mathrm{d}p_x(\tilde{\nu})\,\mathrm{d}\mu(x) = \int \eta(x)\,\mathrm{d}\mu(x)\,, \tag{4.19}$$

or in other words, we have $|\nabla\psi|\le\mu$.

On the other hand we have by the convergence of the energies and Young's inequality that we may estimate μ via

$$\int \eta\,\mathrm{d}\mu = \lim_{\varepsilon\to 0}\int \eta|\nabla\psi_\varepsilon|\,\mathrm{d}x = \liminf_{\varepsilon\to 0}\mathrm{E}_\varepsilon(u_\varepsilon;\eta) = \mathrm{E}(u;\eta). \tag{4.20}$$

Using that

$$|\tilde{\nu}\otimes\tilde{\nu}-\theta\otimes\theta| = |\tilde{\nu}\otimes(\tilde{\nu}-\theta)+(\tilde{\nu}-\theta)\otimes\theta| \le 2|\tilde{\nu}-\theta|$$

and inequality (4.19), we can therefore estimate

$$\begin{aligned}
&\left|\int\int_{\mathbb{S}^{d-1}} \eta\langle A\,,\,\tilde{\nu}\otimes\tilde{\nu}\rangle\,\mathrm{d}p_x(\tilde{\nu})\,\mathrm{d}\mu - \int \eta\langle A\,,\,\theta\otimes\theta\rangle|\nabla\psi|\right| \\
&\le \left|\int \eta\Big\langle A\,,\int_{\mathbb{S}^{d-1}}\tilde{\nu}\otimes\tilde{\nu}\,\mathrm{d}p_x(\tilde{\nu})\Big\rangle(\mathrm{d}\mu-|\nabla\psi|)\right| + \left|\int \eta\Big\langle A\,,\int_{\mathbb{S}^{d-1}}\tilde{\nu}\otimes\tilde{\nu}-\theta\otimes\theta\,\mathrm{d}p_x(\tilde{\nu})\Big\rangle|\nabla\psi|\right| \\
&\le \int \eta\,(\mathrm{d}\mu-|\nabla\psi|) + 2\int\eta\int_{\mathbb{S}^{d-1}}|\tilde{\nu}-\theta|\,\mathrm{d}p_x(\tilde{\nu})\,|\nabla\psi|.
\end{aligned}$$

The first summand can by inequality (4.20) be estimated by $\mathrm{E}(u;\eta)-\int\eta|\nabla\psi|$. For the second summand, we use a duality argument: Given a smooth vector field ξ, we have by the weak convergence of $\nabla\psi_\varepsilon$ to $\nabla\psi$ that

$$\int\langle\xi\,,\,\theta\rangle|\nabla\psi| = \lim_{\varepsilon\to 0}\int\langle\xi\,,\,\theta_\varepsilon\rangle|\nabla\psi_\varepsilon|\,\mathrm{d}x = \int\Big\langle\xi\,,\int\tilde{\nu}\,\mathrm{d}p_x(\tilde{\nu})\Big\rangle\,\mathrm{d}\mu\,,$$

from which we deduce that

$$\begin{aligned}
\int\Big\langle\xi\,,\int\theta-\tilde{\nu}\,\mathrm{d}p_x(\tilde{\nu})\Big\rangle|\nabla\psi| &= \int\Big\langle\xi\,,\int\tilde{\nu}\,\mathrm{d}p_x(\tilde{\nu})\Big\rangle\,(\mathrm{d}\mu-|\nabla\psi|) \\
&\le \int|\xi|\,(\mathrm{d}\mu-|\nabla\psi|) \\
&\le \mathrm{E}(u;|\xi|)-\int|\xi||\nabla\psi|,
\end{aligned}$$

which finishes the proof. □

Proof of Claim 3. First we notice that since $\psi_i = \sum_j \sigma_{i,j} \mathbb{1}_{\Omega_j}$, we have $\theta_i = \pm\nu_j$ on $\Sigma_{j,k}$ $|\nabla\psi_i|$ almost everywhere. Additionally using the representation of $|\nabla\psi_i|$ given in Lemma 3.2.8, we thus have

$$\begin{aligned}
&\left| \int \eta \langle A \,,\, \theta_i \otimes \theta_i \rangle |\nabla\psi_i| - \sum_{1\le j<k\le P} \sigma_{jk} \int_{\Sigma_{jk}} \eta \langle A \,,\, \nu_j \otimes \nu_j \rangle \,\mathrm{d}\,\mathcal{H}^{d-1} \right| \\
&= \left| \sum_{1\le j<k\le P} (|\sigma_{ij} - \sigma_{ik}| - \sigma_{jk}) \int_{\Sigma_{jk}} \eta \langle A \,,\, \nu_j \otimes \nu_j \rangle \,\mathrm{d}\,\mathcal{H}^{d-1} \right| \\
&\le \sum_{1\le j<k\le P} \int_{\Sigma_{jk}} \eta \left(\sigma_{jk} - |\sigma_{ij} - \sigma_{ik}|\right) \mathrm{d}\,\mathcal{H}^{d-1} \\
&= \mathrm{E}(u;\eta) - \int \eta |\nabla\psi_i|,
\end{aligned}$$

where for the inequality, we used that by the triangle inequality for the surface tensions, we have $|\sigma_{ij} - \sigma_{ik}| \le \sigma_{jk}$. □

This finishes the proofs for the three claims. Lastly we notice that we obtain the pointwise in time bound (4.14) from the rewritten term (4.16). □

We want to point out that Claim 3 and the corresponding proof do not appear in [LS16], but are necessary in order to finish the proof.

4.2.4. Convergence of the velocity term

As in the previous proof for the convergence of the curvature term we want to localize and argue as in the twophase case. Moreover we remember from the twophase case that we had to freeze the unit normal. However we now have the important difference that ∇u_ε describes both the change in physical space (domain) and the state space (codomain). Since the supposed limit ν_i only describes the change in physical space, we should only freeze the normal in this direction. We thus come up with the following definition.

Definition 4.2.9. *Let u_ε and χ be as in Proposition 3.2.4. Let $\nu^* \in \mathbb{S}^{d-1}$ and $\eta \in \mathrm{C}^\infty([0,T]\times\mathbb{T};[0,1])$. For $\varepsilon > 0$ the approximative localized tilt-excess of the i-th phase is given by*

$$\mathcal{E}_i^\varepsilon(\nu^*;\eta) := \int_0^T \int \eta \frac{1}{\varepsilon} |\varepsilon \,\mathrm{D}\, u_\varepsilon + \nabla\phi_i(u_\varepsilon) \otimes \nu^*|^2 \,\mathrm{d}x \,\mathrm{d}t\,.$$

In the limit $\varepsilon = 0$, we define the tilt-excess for $1 \le i,j \le P$, $i \ne j$ to be

$$\mathcal{E}_{ij}(\nu^*;\eta) := \int_0^T \int \eta|\nu_i - \nu^*|^2 |\nabla\chi_i| \,\mathrm{d}t + \int_0^T \int \eta|\nu_j + \nu^*|^2 |\nabla\chi_j| \,\mathrm{d}t + \sum_{k\in\{i,j\}} \int_0^T \int \eta|\nabla\chi_k| \,\mathrm{d}t\,.$$

Notice that we use $+\nabla\phi_i(u_\varepsilon)\otimes\nu^*$ in the definition of the approximate tilt excess instead of its negative since the normal of χ_i points inwards with respect to Ω_i, but $\nabla\phi_i/|\nabla\phi_i|$ points away from Ω_i.

Moreover we want to point out that the first two summands of the tilt-excess $\mathcal{E}_{ij}$ measure the local flatness of the boundary, and the last summand measure if mostly the (i,j)-th phase is present, see also the discussion in Section 4.2.2.

As in the twophase case, we first want to argue that when ε approaches zero, the approximate tilt-excess can be bounded by the tilt-excess of the limit.

Lemma 4.2.10. *Assume that we are in the situation of Proposition 3.2.4 and that the time-integrated energies converge (4.1). Then for every $1\le i,j\le P$ with $i\neq j$, every unit vector $\nu^*\in\mathbb{S}^{d-1}$ and $\eta\in C^\infty([0,T]\times\mathbb{T};[0,1])$, we have*

$$\limsup_{\varepsilon\to0}\mathcal{E}_i^\varepsilon(\nu^*;\eta)\lesssim\mathcal{E}_{ij}(\nu^*;\eta).$$

Proof. By expanding the square, we see that we can rewrite the approximate tilt-excess as

$$\begin{aligned}&\mathcal{E}_i^\varepsilon(\nu^*;\eta)\\ &=\int_0^T\int\eta\frac{1}{\varepsilon}\left(\varepsilon^2|\mathrm{D}\,u_\varepsilon|^2+2\varepsilon\langle\mathrm{D}\,u_\varepsilon\,,\,\nabla\phi_i(u_\varepsilon)\otimes\nu^*\rangle+|\nabla\phi_i(u_\varepsilon)\otimes\nu^*|^2\right)\mathrm{d}x\,\mathrm{d}t\\ &=:A_\varepsilon+B_\varepsilon+C_\varepsilon.\end{aligned}$$

Using the equipartition of energies (Lemma 4.2.4), we immediately get that

$$\limsup_{\varepsilon\to0}A_\varepsilon=\int_0^T\mathrm{E}(u;\eta)\,\mathrm{d}t$$

and

$$\begin{aligned}\limsup_{\varepsilon\to0}C_\varepsilon&\le\limsup_{\varepsilon\to0}\int_0^T\int\eta\frac{1}{\varepsilon}|\nabla\phi_i(u_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le\limsup_{\varepsilon\to0}\int_0^T\int\eta\frac{1}{\varepsilon}2W(u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\mathrm{E}(u;\eta)\,\mathrm{d}t\,.\end{aligned}$$

For the remaining summand, we note that

$$\limsup_{\varepsilon\to0}B_\varepsilon=\limsup_{\varepsilon\to0}\int_0^T\int2\eta\langle\nu^*\,,\,\nabla\psi_{\varepsilon,i}\rangle\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int2\eta\langle\nu^*\,,\,\nabla\psi_i\rangle\,\mathrm{d}t\,.$$

Summarizing these estimates, we have

$$\limsup_{\varepsilon\to0}\mathcal{E}_i^\varepsilon(\nu^*;\eta,u_\varepsilon)\le2\int_0^T\mathrm{E}(u;\eta)\,\mathrm{d}t+2\int_0^T\int\eta\langle\nu^*\,,\,\nabla\psi_i\rangle\,\mathrm{d}t\,.$$

By focusing on the (i,j)-th interface, we estimate the energy term by

$$\int_0^T \mathrm{E}(u;\eta)\,\mathrm{d}t \le \int_0^T \sigma_{ij} \int \eta|\nabla\chi_j|\,\mathrm{d}t + C\sum_{k\notin\{i,j\}}\int_0^T\int \eta|\nabla\chi_k|\,\mathrm{d}t$$

and the other term via

$$\begin{aligned}\int_0^T \left\langle \nu^*\,,\, \int \eta\nabla\psi_i \right\rangle \mathrm{d}t &= \sum_{1\le k\le P}\sigma_{ik}\int_0^T \left\langle \nu^*\,,\, \int \eta\nabla\chi_k\right\rangle \mathrm{d}t\\ &\le \int_0^T \sigma_{ij}\int \eta\langle \nu^*\,,\,\nu\rangle|\nabla\chi_j| + C\sum_{k\notin\{i,j\}}\int_0^T\int\eta|\nabla\chi_k|\,\mathrm{d}t\,,\end{aligned}$$

which yields that

$$\limsup_{\varepsilon\to 0}\mathcal{E}_i^\varepsilon(\nu^*;\eta) \le 2\sigma_{ij}\int_0^T\int \eta\left(1+\langle\nu^*\,,\,\nu_j\rangle\right)|\nabla\chi_j|\,\mathrm{d}t + C\sum_{k\notin\{i,j\}}\int_0^T\int\eta|\nabla\chi_k|\,\mathrm{d}t\,.$$

Since $1+\langle\nu^*\,,\,\nu_j\rangle = \frac{1}{2}|\nu^*+\nu_j|^2$, this finishes the proof. □

Note that we have actually proven a seemingly stronger estimate, since the summand $|\nu_i-\nu^*|$ is not included on the right-hand side of the last estimate. However it serves to symmetrize the multiphase excess.

Using Lemma 4.2.10, we can now prove a localized version of the convergence of the velocity term.

Proposition 4.2.11. *In the situation of Proposition 3.2.4 and under the assumption that the time-integrated energies converge (4.1), there exists a finite Radon measure μ on $[0,T]\times\mathbb{T}$ such that for every $1\le i,j\le P$ with $i\ne j$, every parameter $\alpha\in(0,1)$, every unit vector $\nu^*\in\mathbb{S}^{d-1}$ and every test vector field $\xi\in \mathrm{C}_\mathrm{c}^\infty\left((0,T)\times\mathbb{T};\mathbb{R}^d\right)$, we have*

$$\begin{aligned}&\limsup_{\varepsilon\to0}\left|\int_0^T\int \varepsilon\langle\partial_t u_\varepsilon\,,\,\mathrm{D}\,u_\varepsilon\eta\xi\rangle\,\mathrm{d}x\,\mathrm{d}t - \sigma_{ij}\int_0^T\int_{\Sigma_{ij}}\langle\eta\xi\,,\,\nu_i\rangle V_i\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\right|\\ &\lesssim \|\xi\|_{\sup}\left(\frac{1}{\alpha}\,\mathcal{E}_{ij}\left(\nu^*;\eta\right)+\alpha\mu(\eta)\right).\end{aligned}$$

Here $\eta\in \mathrm{C}^\infty\left([0,T]\times\mathbb{T};[0,1]\right)$ is some partition of unity function.

Proof. On a technical note, we first pass to a non-relabelled subsequence such that the limes superior becomes the regular limit. As in the twophase case, we first choose a majority phase and then freeze the approximate normal $\varepsilon\nabla u_\varepsilon$ by replacing it with

$-\nabla\phi_i(u_\varepsilon)\otimes\nu^*$. The error we can be estimated by Young's inequality through

$$\begin{aligned}
&\left|\int_0^T\int \varepsilon\langle\partial_t u_\varepsilon\,,\,\mathrm{D}\,u_\varepsilon\eta\xi\rangle\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int\langle\partial_t u_\varepsilon\,,\,\nabla\phi_i(u_\varepsilon)\otimes\nu^*\eta\xi\rangle\,\mathrm{d}x\,\mathrm{d}t\right|\\
&=\left|\int_0^T\int\left\langle\sqrt{\varepsilon}\partial_t u_\varepsilon\,,\,\left(\sqrt{\varepsilon}\,\mathrm{D}\,u_\varepsilon+\frac{1}{\sqrt{\varepsilon}}\nabla\phi_i\otimes\nu^*\right)\eta\xi\right\rangle\mathrm{d}x\,\mathrm{d}t\right|\\
&\le\frac12\|\xi\|_{\sup}\left(\alpha\int_0^T\int\eta\varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\alpha}\,\mathcal{E}_i^\varepsilon\left(\nu^*;\eta\right)\right).
\end{aligned}\tag{4.21}$$

We thus have arrived at the term

$$-\int_0^T\int\langle\partial_t u_\varepsilon\,,\,\nabla\phi_i(u_\varepsilon)\otimes\nu^*\eta\xi\rangle\,\mathrm{d}x\,\mathrm{d}t=-\int_0^T\int\partial_t\psi_{\varepsilon,i}\langle\eta\xi\,,\,\nu^*\rangle\,\mathrm{d}x\,\mathrm{d}t\,,$$

which by the weak convergence of $\partial_t\psi_{\varepsilon,i}$ to $\partial_t\psi_i$ converges to

$$-\sum_{1\le j\le P}\sigma_{ij}\int_0^T\int\langle\eta\xi\,,\,\nu^*\rangle V_j|\nabla\chi_j|\,\mathrm{d}t\,.$$

Thus

$$\begin{aligned}
&\limsup_{\varepsilon\to0}\left|-\int_0^T\int\langle\partial_t u_\varepsilon\,,\,\nabla\phi_i(u_\varepsilon)\otimes\nu^*\eta\xi\rangle\,\mathrm{d}x\,\mathrm{d}t-\sigma_{ij}\int_0^T\int_{\Sigma_{ij}}\langle\eta\xi\,,\,\nu_i\rangle V_i\,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t\right|\\
&=\left|\sum_{1\le k\le P}\sigma_{ik}\int_0^T\int\langle\eta\xi\,,\,\nu^*\rangle V_k|\nabla\chi_k|\,\mathrm{d}t+\sigma_{ij}\int_0^T\int_{\Sigma_{ij}}\langle\eta\xi\,,\,\nu_i\rangle V_i\,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t\right|\\
&\lesssim\|\xi\|_{\sup}\sum_{k\notin\{i,j\}}\int_0^T\int\eta|V_k||\nabla\chi_k|\,\mathrm{d}t+\left|\int_0^T\int_{\Sigma_{ij}}\left(-\langle\eta\xi\,,\,\nu^*\rangle+\langle\eta\xi\,,\,\nu_i\rangle\right)V_i\,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t\right|.
\end{aligned}$$

By applying Young's inequality twice and using again that on Σ_{ij}, we have $V_i=-V_j$, we furthermore estimate this term by

$$\begin{aligned}
&\|\xi\|_{\sup}\left(\sum_{k\notin\{i,j\}}\alpha\int_0^T\int\eta|V_k|^2|\nabla\chi_k|\,\mathrm{d}t+\frac{1}{\alpha}\int_0^T\int\eta|\nabla\chi_k|\,\mathrm{d}t+\int_0^T\int_{\Sigma_{ij}}\eta|V_i||\nu^*-\nu_i|\,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t\right)\\
&\le\|\xi\|_{\sup}\left(\alpha\sum_{1\le k\le P}\int_0^T\int\eta|V_k|^2|\nabla\chi_k|\,\mathrm{d}t+\frac{1}{\alpha}\,\mathcal{E}_{ij}(\nu^*;\eta)\right).
\end{aligned}$$

By looking again at the first error we made in inequality (4.21), we therefore define the measure μ through

$$\mu:=\sum_{1\le k\le P}|V_k|^2|\nabla\chi_k|\,\mathrm{d}t+\mu_2,$$

where μ_2 is the weak-star limit of some non-relabelled subsequence of the Radon measures $\varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t$, which stay bounded due to the energy dissipation inequality (2.12). This defines a finite Radon measure on $[0,T]\times\mathbb{T}$ by the square-integrability of the velocities observed in Proposition 4.2.3 and thus, the proof is complete. □

We are now in the position to prove the main result Theorem 4.2.2. This still requires some effort and the argument for the localization in time is missing in [LS16].

Proof of Theorem 4.2.2. Let us first collect all previous results. Proposition 3.2.4 guarantees that a limit χ as described in Theorem 4.2.2 exists. After that Lemma 3.2.10 ensures that χ is continuous in time with respect to the L^2-norm and therefore assumes the initial data. The existence of square-integrable normal velocities is proven in Proposition 4.2.3. For the distributional equation, the convergence of the curvature term has been proven in Proposition 4.2.7. Thus we still need the full convergence of the velocity term, and the bulk of the work has already been done in Proposition 4.2.11. We see that the only thing left to show is the full convergence of the velocity term, which states that

$$\lim_{\varepsilon\to 0}\int_0^T\int \varepsilon\langle \partial_t u_\varepsilon\,,\,\mathrm{D}\,u_\varepsilon\xi\rangle\,\mathrm{d}x\,\mathrm{d}t = \sum_{1\le i<j\le P}\sigma_{ij}\int_0^T\int_{\Sigma_{ij}}\langle\xi\,,\,\nu_i\rangle V_i\,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t\,.$$

The key problem we are facing is that the localization estimate Lemma 4.2.5 states that if we can choose the majority phase for a fixed time, then we expect smallness. However we see that the definition of the tilt-excess yields only a time-integrated error. Thus we want to take a partition of unity also in time and control the error.

To this end, let $0=T_0<\ldots<T_k=T$ be a partition of $[0,T]$ and for a given $\delta>0$, let $(g_k)_{k=1,\ldots,K}$ be a partition of unity with respect to the intervals $((T_{k-1}-\delta,T_k+\delta))$. As before let η_B be a partition of unity in space with respect to the covering $\mathcal{B}_r$. Then for any parameter $\alpha\in(0,1)$, we compute that by Proposition 4.2.11, we have

$$\begin{aligned}
A &:= \limsup_{\varepsilon\to 0}\left|\int_0^T\int \varepsilon\langle \partial_t u_\varepsilon\,,\,\mathrm{D}\,u_\varepsilon\xi\rangle\,\mathrm{d}x\,\mathrm{d}t - \sum_{1\le i<j\le P}\sigma_{ij}\int_0^T\int_{\Sigma_{ij}}\langle\xi\,,\,\nu_i\rangle V_i\,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t\right| \\
&\le \sum_{k=1}^K\sum_{B\in\mathcal{B}_r}\limsup_{\varepsilon\to 0}\left|\int_0^T\int \varepsilon g_k\eta_B\langle \partial_t u_\varepsilon\,,\,\mathrm{D}\,u_\varepsilon\xi\rangle\,\mathrm{d}x\,\mathrm{d}t\right. \\
&\qquad\qquad\left. - \sum_{1\le i<j\le P}\sigma_{ij}\int_0^T\int_{\Sigma_{ij}} g_k\eta_B\langle\xi\,,\,\nu_i\rangle V_i\,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t\right| \\
&\lesssim \|\xi\|_{\sup}\sum_{k=1}^K\sum_{B\in\mathcal{B}_r}\min_{i\neq j}\min_{\nu^*\in\mathbb{S}^{d-1}}\frac{1}{\alpha}\,\mathcal{E}_{ij}(\nu^*;g_k\eta_B)+\alpha\mu(g_k\eta_B)
\end{aligned}$$

Since no derivative falls on g_k, this term converges as δ tends to zero to

$$\|\xi\|_{\sup}\alpha\mu\left([0,T]\times\mathbb{T}\right)+\|\xi\|_{\sup}\frac{1}{\alpha}\sum_{k=1}^{K}\sum_{B\in\mathcal{B}_r}\min_{i\neq j}\min_{\nu^*\in\mathbb{S}^{d-1}}\int_{T_{k-1}}^{T_k}\int\eta_B|\nu_i-\nu^*|^2|\nabla\chi_i| \\ +\int\eta_B|\nu_j+\nu^*|^2|\nabla\chi_j|+\sum_{k\notin\{i,j\}}\int\eta_B|\nabla\chi_k|\,\mathrm{d}t\,. \tag{4.22}$$

Our problem is now that we would like to take the majority phase approximate inner normal dependent on time, which equates to pulling both minima inside the time integral. If our partition χ is smooth, then by choosing partitions $(T_k)_{k=1,\dots,K}$ whose width goes to zero, this would be possible. Since our partition will in general not be smooth, we instead choose a smooth approximation χ^n which converges to χ with respect to the strict metric as n tends to infinity. We estimate that for every $1\le i\le P$, every unit vector $\nu^*\in\mathbb{S}^{d-1}$, fixed index k and ball B, we have that when replacing χ with χ^n in a summand of equation (4.22), we get an error of size at most

$$\begin{aligned}&\int_{T_{k-1}}^{T_k}\left|\int\eta_B|\nu_i-\nu^*|^2|\nabla\chi_i|-\int\eta_B|\nu_i^n-\nu^*|^2|\nabla\chi_i^n|\right|\\&\quad+\left|\int\eta_B|\nu_j+\nu^*|^2|\nabla\chi_j|-\int\eta_B\left|\nu_j^n+\nu^*\right|^2\left|\nabla\chi_j^n\right|\right|+\sum_{l\notin\{i,j\}}\left|\int\eta_B|\nabla\chi_l|-\int\eta_B|\nabla\chi_l^n|\right|\mathrm{d}t\\&\lesssim\sum_{l=1}^{P}\int_{T_{k-1}}^{T_k}\left|\int\eta_B|\nabla\chi_l|-\int\eta_B|\nabla\chi_l^n|\right|+\left|\int\eta_B\nu_l|\nabla\chi_l|-\int\eta_B\nu_l^n|\nabla\chi_l^n|\right|\mathrm{d}t\,.\end{aligned}$$

Here ν_i^n is defined by $\nabla\chi_i^n/|\nabla\chi_i^n|$, and for the inequality we used that $|\vartheta-\gamma|^2=2(1-\langle\vartheta\,,\,\gamma\rangle)$ for unit vectors ϑ and γ. Thus when replacing χ by χ^n in (4.22), we get an error of size at most

$$\frac{1}{\alpha}\sum_{B\in\mathcal{B}_r}\int_0^T\sum_{1\le l\le P}\left|\int\eta_B|\nabla\chi_l|-\int\eta_B|\nabla\chi_l^n|\right|+\left|\int\eta_B\nu_l|\nabla\chi_l|-\int\eta_B\nu_l^n|\nabla\chi_l^n|\right|\mathrm{d}t\,,$$

which can be made arbitrarily small independent of the partition of $[0,T]$ we have chosen when we let n tend to infinity. Since χ^n is smooth, we can now make the width of our partition arbitrarily small and therefore obtain that

$$\begin{aligned}A\lesssim\ &\|\xi\|_{\sup}\alpha\mu\left([0,T]\times\mathbb{T}\right)+\delta(n)\\&+\|\xi\|_{\sup}\frac{1}{\alpha}\int_0^T\sum_{B\in\mathcal{B}_r}\min_{i\neq j}\min_{\nu^*\in\mathbb{S}^{d-1}}\int\rho_B|\nu_i^n-\nu^*|^2|\nabla\chi_i^n|+\int\rho_B\left|\nu_j^n+\nu^*\right|^2\left|\nabla\chi_j^n\right|\\&\qquad+\sum_{k\notin\{i,j\}}\int\rho_B|\nabla\chi_k^n|\,\mathrm{d}t\,,\end{aligned}$$

where $\delta(n)$ is some positive number which tends to zero as n approaches infinity. Now letting n tend to infinity yields by the generalized dominated convergence theorem with majorant $C\sum_i \int \eta_B|\nabla\chi_i^n|$ (whose time integral converges due to the strict convergence) that

$$
\begin{aligned}
A \lesssim &\|\xi\|_{\sup}\alpha\mu\left([0,T]\times\mathbb{T}\right) \\
&+ \|\xi\|_{\sup}\frac{1}{\alpha}\int_0^T \sum_{B\in\mathcal{B}_r} \min_{i\neq j}\min_{\nu^*\in\mathbb{S}^{d-1}} \int \rho_B|\nu_i-\nu^*|^2|\nabla\chi_i| + \int \rho_B|\nu_j+\nu^*|^2|\nabla\chi_j| \\
&\qquad + \sum_{k\notin\{i,j\}}\int \rho_B|\nabla\chi_k|\,\mathrm{d}t\,.
\end{aligned}
$$

Using the localization result Lemma 4.2.5 and the energy dissipation inequality (2.12) together with the dominated convergence theorem, the second summand vanishes as r tends to zero. Then sending α to zero completes the proof. □

5. De Giorgi's mean curvature flow

In this chapter, we build on the results of the previous chapters and introduce a different solution concept for mean curvature flow, namely a De Giorgi type BV-solution to mean curvature. A similar solution concept has been introduced in [LL21, Def. 1], but in the context of convergence of the thresholding scheme to mean curvature flow. Moreover, we will also compare it to the solution concept [HL21, Def. 1], which also permits oriented varifolds to be solutions to mean curvature flow. This provides a more general notion of solution, but is of use when the assumption of energy convergence falls away, see the later discussion.

5.1. Conditional convergence to De Giorgi's mean curvature flow

In this section, we shall state our solution concept and prove convergence to the aforementioned.

Definition 5.1.1 (De Giorgi type BV-solution to multiphase mean curvature flow)**.** *Fix some finite time horizon $T < \infty$, a $P \times P$ matrix of surface tensions σ and initial data $\chi^0 \colon \mathbb{T} \to \{0,1\}^P$ with $\mathrm{E}_0 \coloneqq \mathrm{E}(\chi^0) < \infty$ and $\sum_{i=1}^P \chi_i^0 = 1$. We say that*

$$\chi \in \mathrm{C}\left([0,T]; \mathrm{L}^2\left(\mathbb{T}; \{0,1\}^P\right)\right)$$

with $\operatorname{ess\,sup}_{0\le t\le T} \mathrm{E}(\chi) < \infty$ *and* $\sum_{i=1}^P \chi_i = \sum_{i=1}^P \mathbb{1}_{\Omega_i} = 1$ *is a* De Giorgi type BV-solution to multiphase mean curvature flow with initial data χ^0 and surface tensions σ *if the following holds.*

1. *For all $1 \le i \le P$, there exist normal velocities $V_i \in \mathrm{L}^2(|\nabla \chi_i| \,\mathrm{d}t)$ such that*

$$\partial_t \chi_i = V_i |\nabla \chi_i| \,\mathrm{d}t$$

   holds in the distributional sense on $(0,T) \times \mathbb{T}$.

2. *There exist a mean curvature vector $H \in \mathrm{L}^2(\mathrm{E}(u;\cdot)\,\mathrm{d}t\,; \mathbb{R}^d)$ which satisfies*

$$\begin{aligned} &\sum_{1\le i<j\le P} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle H\,,\, \xi\rangle \,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t \\ &= -\sum_{1\le i<j\le P} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - \nu_i \otimes \nu_i\rangle \,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t \end{aligned} \tag{5.1}$$

   for all test vector fields $\xi \in \mathrm{C}_\mathrm{c}^\infty\left((0,T)\times\mathbb{T}; \mathbb{R}^d\right)$, where $\nu_i \coloneqq \nabla\chi_i / |\nabla \chi_i|$ are the inner unit normals and $\Sigma_{ij} \coloneqq \partial_*\Omega_i \cap \partial_*\Omega_j$ is the (i,j)-th interface.

3. *The partition χ satisfies a De Giorgi type optimal energy dissipation inequality in the sense that for almost every time $0 < T' < T$, we have*

$$\mathrm{E}(\chi(T')) + \frac{1}{2} \sum_{1\le i<j\le P} \sigma_{ij} \int_0^{T'} \int_{\Sigma_{ij}} V_i^2 + |H|^2 \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \le \mathrm{E}_0\,. \tag{5.2}$$

4. *The initial data is achieved in the space $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$.*

We can now immediately prove a similar convergence result as in Theorem 4.2.2.

Theorem 5.1.2. *Let a smooth multiwell potential $W\colon \mathbb{R}^N \to [0,\infty)$ satisfy the assumptions (2.3)-(2.6). Let $T < \infty$ be an arbitrary finite time horizon. Given a sequence of initial data $u_\varepsilon^0\colon \mathbb{T} \to \mathbb{R}^N$ approximating a partition $\chi^0 \in \mathrm{BV}\left(\mathbb{T};\{0,1\}^P\right)$ in the sense that $u_\varepsilon^0 \to u^0 = \sum_{1\le i\le P} \chi_i^0 \alpha_i$ holds pointwise almost everywhere and*

$$\mathrm{E}_0 \coloneqq \mathrm{E}(\chi^0) = \lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon^0) < \infty,$$

we have that for some subsequence of solutions u_ε to the Allen–Cahn equation (2.1) with initial datum u_ε^0, there exists a time-dependent partition χ with $\chi \in \mathrm{BV}\left((0,T)\times\mathbb{T};\{0,1\}^P\right)$ and $\chi \in \mathrm{C}\left([0,T];\mathrm{L}^2\left(\mathbb{T};\{0,1\}^P\right)\right)$ such that u_ε converges to $u \coloneqq \sum_{1\le i\le P} \chi_i \alpha_i$ almost everyhwere. Moreover u assumes the initial data u^0 in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$. If we additionally assume that the time-integrated energies converge (4.1), then χ is a De Giorgi type BV*-solution to multiphase mean curvature flow in the sense of Definition 5.1.1.*

This result is similar to Theorem 4.2.2 and we only need to prove that χ is a BV-solution to De Giorgis mean curvature flow.

Proof. The idea of the proof is that we already have an optimal energy dissipation inequality for the Allen–Cahn equation given by (2.12). If we additionally use the Allen–Cahn equation once, we arrive at the reformulated version of the optimal energy dissipation inequality given by

$$\mathrm{E}_\varepsilon(u_\varepsilon(T')) + \frac{1}{2}\int_0^{T'}\int \varepsilon|\partial_t u_\varepsilon|^2 + \frac{1}{\varepsilon}\left|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla W(u_\varepsilon)\right|^2 \mathrm{d}x\,\mathrm{d}t \le \mathrm{E}_\varepsilon(u_\varepsilon^0).$$

Our hope is that as ε tends to zero, we can pass to the optimal energy dissipation inequality for χ (5.2) through lower semicontinuity. Since we assume the convergence of the initial energies and energy convergence for almost every time, the only terms we have to care about are the lower semicontinuity of the velocity term, which reads

$$\liminf_{\varepsilon\to 0} \frac{1}{2}\int_0^{T'}\int \varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \ge \frac{1}{2}\sum_{1\le i<j\le P}\sigma_{ij}\int_0^{T'}\int_{\Sigma_{ij}} V_i^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\,, \tag{5.3}$$

and the lower semicontinuity of the curvature term, which is given by

$$\liminf_{\varepsilon\to 0} \frac{1}{2}\int_0^{T'}\int \frac{1}{\varepsilon}\left|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla W(u_\varepsilon)\right|^2 \mathrm{d}x\,\mathrm{d}t \ge \frac{1}{2}\sum_{1\le i<j\le P}\sigma_{ij}\int_0^{T'}\int_{\Sigma_{ij}} |H|^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\,. \tag{5.4}$$

Moreover we have to show existence of the mean curvature function H. We could cheat in this step and simply use that by Theorem 4.2.2, we already know that the tangential convergence applied to ξ is given by the velocity, or in other words, that we already have $V_i = H$ on Σ_{ij}. But we want to present a more direct approach.

Consider the linear functional

$$L(\xi) \coloneqq -\sum_{1\le i<j\le P} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - \nu_i \otimes \nu_i\rangle \,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t$$

defined on test vector fields ξ. Then L is bounded with respect to $\mathrm{L}^2\left((0,T)\times\mathbb{T}, \mathrm{E}(\xi;\cdot);\mathbb{R}^d\right)$ since by the convergence of the curvature term observed in *Proposition* 4.2.7, we have

$$\begin{aligned}
|L(\xi)| &= \liminf_{\varepsilon\to 0} \left| -\int_0^T \int \left\langle \varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla W(u_\varepsilon)\,,\, \mathrm{D}\,u_\varepsilon \xi \right\rangle \mathrm{d}x\,\mathrm{d}t \right| \\
&\le \left(\int_0^T \int \frac{1}{\varepsilon} \left| \varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla W(u_\varepsilon)\right|^2 \mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\int_0^T \int \varepsilon |\mathrm{D}\,u_\varepsilon \xi|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \\
&\le \left(\int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\int_0^T \int \varepsilon |\nabla u_\varepsilon|^2 |\xi|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2}.
\end{aligned}$$

The first factor stays uniformly bounded due to the energy dissipation inequality (2.12), and by the equipartition of energies (Lemma 4.2.4), the second factor converges to the L^2-norm of ξ with respect to the energy measure, proving our claim. Therefore we can extend the functional to the square integrable functions with respect to the energy measure, and by Riesz representation theorem obtain the existence of the desired mean curvature vector H.

Let us now consider the lower semicontinuity of the curvature term. Let again ξ be some test vector field. Then for all $\varepsilon > 0$ and some fixed time, we have by Young's inequality and the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \frac{1}{2} \int \frac{1}{\varepsilon} \left| \varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla W(u_\varepsilon)\right|^2 \mathrm{d}x \\
\ge\ &\liminf_{\varepsilon\to 0} \int \left\langle \varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla W(u_\varepsilon)\,,\, \mathrm{D}\,u_\varepsilon \xi \right\rangle \mathrm{d}x - \frac{1}{2}\int \varepsilon |\nabla u_\varepsilon|^2 |\xi|^2 \,\mathrm{d}x \\
=\ & -\mathrm{E}\left(\chi; \langle H\,,\, \xi\rangle\right) - \frac{1}{2}\mathrm{E}\left(\chi; |\xi|^2\right).
\end{aligned}$$

Since this inequality holds for any test vector field, we may take a sequence of test vector fields satisfying

$$\lim_{n\to\infty} \|\xi_n + H\|_{\mathrm{L}^2(\mathbb{T},\mathrm{E}(\chi;\cdot);\mathbb{R}^d)} = 0.$$

This then yields the desired inequality (5.4) by applying Fatou's Lemma to pull the limes inferior into the time integral.

In principle the proof is now already done, since Theorem 4.2.2 already gives us that for almost every time t, we have $V_i\nu_i = -H$ on $\Sigma_{i,j}$ $\mathcal{H}^{d-1}$-almost everywhere. But since this makes heavy use of the previous arguments, we instead want to present another proof which directly proves the lower semicontinuity of the velocity term and may leave more room for future generalizations.

Let us first consider the scalar case $N = 1, P = 2$. By a similar duality argument as for the lower semicontinuity of the curvature term, we compute that for every test function φ, we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \frac{1}{2}\int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\geq \liminf_{\varepsilon\to 0} \int_0^T \int \partial_t u_\varepsilon \phi'(u_\varepsilon)\varphi \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^T \int \frac{1}{\varepsilon}\left(\phi'(u_\varepsilon)\varphi\right)^2 \mathrm{d}x\,\mathrm{d}t \\
&= \liminf_{\varepsilon\to 0} \int_0^T \int \partial_t \psi \varphi \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon)\varphi^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= \sigma \int_0^T \int_\Sigma \varphi V \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t - \frac{1}{2}\sigma \int_0^T \int_\Sigma \varphi^2 \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\,.
\end{aligned}$$

Since the inequality holds for any testfunction φ, we may plug in a sequence of testfunctions φ_n satisfying

$$\lim_{n\to\infty} \|\varphi_n - V\|_{\mathrm{L}^2\left((0,T)\times\mathbb{T},\mathrm{d}\,\mathcal{H}^{d-1}|_\Sigma \mathrm{d}t\right)} = 0$$

and thereby obtain the desired inequality (5.3).

For the multiphase case, we do not find an immediate generalization of this proof, but rather have to work with the usual localization argument to obtain a reduction to the scalar case.

As in the proof of Theorem 4.2.2, let $\delta > 0$ and $0 = T_0 < T_1 < \ldots < T_K = T'$ be a partition of $[0, T']$. Let $(g_k)_{k=1,\ldots,K}$ be a partition of unity with respect to the intervals $(T_0 - \delta, T_1 + \delta), \ldots, (T_{K-1} - \delta, T_K + \delta)$, let $r > 0$ and η_B as in Lemma 4.2.5. Then we estimate

$$\begin{aligned}
A &:= \liminf_{\varepsilon\to 0} \frac{1}{2}\int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= \liminf_{\varepsilon\to 0} \sum_{k=1}^K \sum_{B\in\mathcal{B}_r} \frac{1}{2}\int_0^T \int g_k \eta_B \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\geq \liminf_{\varepsilon\to 0} \sum_{k=1}^K \sum_{B\in\mathcal{B}_r} \max_{1\leq i\leq P} \sup_{\substack{\varphi\in \mathrm{C}_c^\infty((0,T)\times\mathbb{T})\\ |\varphi|\leq R}} \int_0^T \int g_k \eta_B \langle \nabla\phi_i(u_\varepsilon)\,,\, \partial_t u_\varepsilon\rangle \varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad - \frac{1}{2}\int_0^T \int g_k \eta_B \frac{1}{\varepsilon}|\nabla\phi_i(u_\varepsilon)|^2 \varphi^2 \,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

We identify that via the chain rule, we have $\langle \nabla\phi_i(u_\varepsilon)\,,\, \partial_t u_\varepsilon\rangle = \partial_t \psi_{\varepsilon,i}$ and moreover remember $|\nabla\phi_i| \leq \sqrt{2W}$. Pulling the limes inferior inside the double sum and the suprema, we thus obtain that this term can be estimated from below via

$$\sum_{k=1}^{K}\sum_{B\in\mathcal{B}_r}\max_{1\le i\le P}\sup_{\substack{\varphi\in \mathrm{C}_c^\infty((0,T)\times\mathbb{T})\\|\varphi|\le R}}\int_0^T\int g_k\eta_B\partial_t\psi_i\varphi\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{2}\int_0^T \mathrm{E}\left(\chi;g_k\eta_B\varphi^2\right)\mathrm{d}t$$

By an addition with zero, we have

$$\begin{aligned}&\int_0^T\int g_k\eta_B\partial_t\psi_i\varphi\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{2}\int_0^T \mathrm{E}\left(\chi;g_k\eta_B\varphi^2\right)\mathrm{d}t\\
&=\sum_{j=1}^{P}\left(\sigma_{ij}\int_0^T\int g_k\eta_B\varphi V_j|\nabla\chi_j|\,\mathrm{d}t-\frac{1}{2}\sigma_{ij}\int_0^T\int g_k\eta_B\varphi^2|\nabla\chi_j|\,\mathrm{d}t\right)\\
&\quad-\frac{1}{2}\left(\int_0^T \mathrm{E}\left(\chi;g_k\eta_B\varphi^2\right)-\int g_k\eta_B\varphi^2|\nabla\psi_i|\,\mathrm{d}t\right).\end{aligned}$$

Since no derivative has fallen on g_k, we may send δ and obtain by the dominated convergence theorem that we can replace g_k by $\mathbb{1}_{(T_{k-1},T_k)}$. We thus end up with a good summand consisting of

$$\sum_{k=1}^{K}\sum_{B\in\mathcal{B}_r}\max_{1\le i\le P}\sup_{\substack{\varphi\in \mathrm{C}_c^\infty((0,T)\times\mathbb{T})\\|\varphi|\le R}}\sum_{j=1}^{P}\sigma_{ij}\int_{T_{k-1}}^{T_k}\int\eta_B\varphi\left(V_j-\frac{1}{2}\varphi\right)|\nabla\chi_j|\,\mathrm{d}t \tag{5.5}$$

and an error summand given by

$$\begin{aligned}&\sum_{k=1}^{K}\sum_{B\in\mathcal{B}_r}\max_{1\le i\le P}\sup_{\substack{\varphi\in \mathrm{C}_c^\infty((0,T)\times\mathbb{T})\\|\varphi|\le R}}-\frac{1}{2}\int_{T_{k-1}}^{T_k}\left(\mathrm{E}\left(\chi;\eta_B\varphi^2\right)-\int\eta_B\varphi^2|\nabla\psi_i|\right)\mathrm{d}t\\
&\ge\sum_{k=1}^{K}\sum_{B\in\mathcal{B}_r}\max_{1\le i\le P}-\frac{R^2}{2}\int_{T_{k-1}}^{T_k}\left(\mathrm{E}\left(\chi;\eta_B\right)-\int\eta_B|\nabla\psi_i|\right)\mathrm{d}t\end{aligned} \tag{5.6}$$

By choosing a majority phase (i,j) we furthermore estimate the good summand (5.5) from below by

$$\begin{aligned}&\max_{1\le i\le P}\sup_{\substack{\varphi\in \mathrm{C}_c^\infty((0,T)\times\mathbb{T})\\|\varphi|\le R}}\sum_{j=1}^{P}\sigma_{ij}\int_{T_{k-1}}^{T_k}\int\eta_B\varphi\left(V_j-\frac{1}{2}\varphi\right)|\nabla\chi_j|\,\mathrm{d}t\\
&\ge\max_{i\ne j}\sup_{\substack{\varphi\in \mathrm{C}_c^\infty((0,T)\times\mathbb{T})\\|\varphi|\le R}}\sigma_{ij}\int_{T_{k-1}}^{T_k}\int\eta_B\varphi\left(V_j-\frac{1}{2}\varphi\right)|\nabla\chi_j|\,\mathrm{d}t\\
&\qquad\qquad-C\sum_{l\notin\{i,j\}}\int_{T_{k-1}}^{T_k}\int\eta_B\left(R|V_l|+R^2\right)|\nabla\chi_l|\,\mathrm{d}t.\end{aligned} \tag{5.7}$$

By Remark 4.2.6, we have

$$\frac{R^2}{2}\int_{T_{k-1}}^{T_k} \mathrm{E}\left(\chi;\eta_B\right) - \int \eta_B|\nabla\psi_i|\,\mathrm{d}t \lesssim R^2 \sum_{l\notin\{i,j\}} \int_{T_{k-1}}^{T_k}\int \eta_B|\nabla\chi_l|\,\mathrm{d}t\,,$$

which enables us to absorb the error (5.6) into the error term (5.7). By applying Young's inequality, we get that for every parameter $\alpha\in(0,1)$, we have

$$\int_{T_{k-1}}^{T_k}\int \eta_B R V_l|\nabla\chi_l|\,\mathrm{d}t \lesssim \alpha\int_{T_{k-1}}^{T_k}\int \eta_B V_l^2|\nabla\chi_l|\,\mathrm{d}t + \frac{R^2}{\alpha}\int_{T_{k-1}}^{T_k}\int \eta_B|\nabla\chi_l|\,\mathrm{d}t\,.$$

Collecting our estimates, we end up with an error term which can be estimated from below up to a constant by

$$-\alpha\sum_{l=1}^{P}\int_0^T\int V_l^2|\nabla\chi_l|\,\mathrm{d}t - \frac{R^2}{\alpha}\sum_{k=1}^{K}\sum_{B\in\mathcal{B}_r}\max_{i\neq j}\sum_{l\notin\{ij\}}\int_{T_{k-1}}^{T_k}\int\eta_B|\nabla\chi_l|\,\mathrm{d}t\,.$$

Moreover, we can choose for fixed k, B and tuple (i,j) a sequence of testfunctions φ_n with $|\varphi_n|\le R$ which approach $V_j\mathbb{1}_{|V_j\le R|}$ in the sense that

$$\lim_{n\to\infty}\left\|\varphi_n - V_j\mathbb{1}_{|V_j|\le R}\right\|_{\mathrm{L}^2((T_{k-1},T_k)\times\mathbb{T},|\nabla\chi_j|\mathrm{d}t)} = 0.$$

Combining these three arguments, we arrive at the estimate that for every parameter $\alpha\in(0,1)$, it holds that

$$\begin{aligned}
A \geq& -C\alpha\sum_{1\le l\le P}\int_0^T\int V_l^2|\nabla\chi_l|\,\mathrm{d}t\\
&+\sum_{k=1}^{K}\sum_{B\in\mathcal{B}_r}\max_{i\neq j}\int_{T_{k-1}}^{T_k}\frac{\sigma_{ij}}{2}\int\eta_B\left|V_j\mathbb{1}_{|V_j|\le R}\right|^2|\nabla\chi_j| - C\frac{R^2}{\alpha}\sum_{l\notin\{i,j\}}\int\eta_B|\nabla\chi_l|\,\mathrm{d}t\,.
\end{aligned}$$

With similar arguments as in the proof of Theorem 4.2.2, we can argue that by choosing partitions whose width tends to zero, we can pull the maximum inside the time integral to obtain

$$\begin{aligned}
A \geq& -\alpha C\sum_{1\le l\le P}\int_0^T\int V_l^2|\nabla\chi_l|\,\mathrm{d}t\\
&+\int_0^T\sum_{B\in\mathcal{B}_r}\max_{i\neq j}\frac{\sigma_{ij}}{2}\int\eta_B\left|V_j\mathbb{1}_{|V_j|\le R}\right|^2|\nabla\chi_j| - C\frac{R^2}{\alpha}\sum_{l\notin\{i,j\}}\int\eta_B|\nabla\chi_l|\,\mathrm{d}t\\
\geq& \frac{1}{2}\sum_{1\le i<j\le P}\sigma_{ij}\int_0^T\int_{\Sigma_{ij}} V_i^2\mathbb{1}_{|V_i|\le R}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\
&-C\left(\alpha\sum_{1\le l\le P}\int_0^T\int V_l^2|\nabla\chi_l|\,\mathrm{d}t + \frac{R^2}{\alpha}\int_0^T\sum_{B\in\mathcal{B}_r}\min_{i\neq j}\sum_{k\notin\{i,j\}}\int\eta_B|\nabla\chi_k|\,\mathrm{d}t\right).
\end{aligned}$$

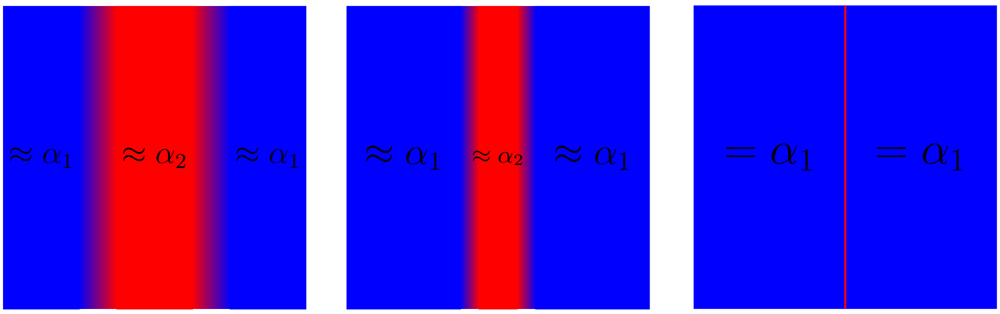
With the same argumentation as in the proof of Theorem 4.2.2, by first sending $r \to 0$ and then $\alpha \to 0$, we thus obtain that for all $R > 0$, we have

$$A = \liminf_{\varepsilon \to 0} \frac{1}{2} \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \geq \frac{1}{2} \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} V_i^2 \mathbb{1}_{|V_i| \leq R} \, \mathrm{d}\mathcal{H}^{d-1} \, \mathrm{d}t \, .$$

Thus the lower semicontinuity of the velocity term now follows form the monotone convergence theorem. □

5.2. De Giorgi type varifold solutions for mean curvature flow

Up until this point, we have always made the crucial assumption of energy convergence (4.1) for our proofs . However this is usually a very strong assumption. One thing which could go wrong is for example illustrated in Figure 5.1. There we see that two rough interfaces of the first and second phase collapse as ε tends to zero. This results in a loss of energy since the measure theoretic boundary of a set with finite perimeter does not see such lines.



(a) $1 > \varepsilon > 0$ (b) $1 \gg \varepsilon > 0$ (c) $\varepsilon = 0$

Figure 5.1.: Profile of the solution u_ε as ε tends to zero.

This leads us to the solution concept introduced by Hensel and Laux in [HL21]. For the twophase case, the definition is as follows.

Definition 5.2.1 (De Giorgi type varifold solution for two-phase mean curvature flow)**.** *Let $T < \infty$ be an arbitrary finite time horizon and let $\mu = \mathcal{L}^1 \otimes (\mu_t)_{t \in (0,T)}$ be a family of oriented varifolds $\mu_t \in \mathcal{M}\left(\mathbb{T} \times \mathbb{S}^{d-1}\right)$, $t \in (0,T)$, such that the map $t \mapsto \int_{\mathbb{T} \times \mathbb{S}^{d-1}} \eta(x,p,t) \mathrm{d}\mu_t(x,p)$ is measurable for all $\eta \in \mathrm{L}^1\left((0,T); \mathrm{C}\left(\mathbb{T}, \mathbb{S}^{d-1}\right)\right)$. Consider also a family $A = (A_t)_{t \in (0,T)}$ of subsets of $\mathbb{T}$ with finite perimeter such that the associated indicator function $\chi(x,t) = \chi_{A_t}(x)$ satisfies $\chi \in \mathrm{L}^\infty\left((0,T); \mathrm{BV}(\mathbb{T}; \{0,1\})\right)$. Let $\sigma > 0$ be a surface tension constant.*

Given an initial energy $\omega^0 \in \mathcal{M}(\mathbb{T})$ *and an initial phase indicator function* $\chi^0 \in \mathrm{BV}(\mathbb{T};\{0,1\})$*, we call the pair* (μ,χ) *a* De Giorgi type varifold solution for two-phase mean curvature flow with initial data (ω^0,χ^0) *if the following holds.*

1. *(Existence of a normal speed) Writing* $\mu_t = \omega_t \otimes (\lambda_{x,t})_{x\in\mathbb{T}}$ *for the disintegration of* μ_t*, we require the existence of some* $V \in \mathrm{L}^2((0,T)\times\mathbb{T},\omega_t)$ *encoding a normal speed in the sense of*
$$\sigma \int \chi(T',x)\varphi(T',x) - \chi^0(x)\varphi(0,x)\,\mathrm{d}x = \sigma\int_0^{T'}\int \chi\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int V\varphi\,\mathrm{d}\omega_t\,\mathrm{d}t \tag{5.8}$$
for almost every $T' \in (0,T)$ *and all* $\varphi \in \mathrm{C}_\mathrm{c}^\infty([0,T)\times\mathbb{T})$*.*

2. *(Existence of a generalized mean curvature vector) We require the existence of some* $H \in \mathrm{L}^2\left((0,T)\times\mathbb{T},\omega_t;\mathbb{R}^d\right)$ *encoding a generalized mean curvature vector by*
$$\int_0^T\int \langle H\,,\,\xi\rangle\,\mathrm{d}\omega_t\,\mathrm{d}t = -\int_0^T\int_{\mathbb{T}\times\mathbb{S}^{d-1}} \langle \xi\,,\,\mathrm{Id}-p\otimes p\rangle\,\mathrm{d}\mu_t\,\mathrm{d}t \tag{5.9}$$
for all $\xi \in \mathrm{C}_\mathrm{c}^\infty\left([0,T)\times\mathbb{T};\mathbb{R}^d\right)$*.*

3. *(De Giorgi type optimal energy dissipation inequality) A sharp energy dissipation inequality holds in form of*
$$\omega_{T'}(\mathbb{T}) + \frac{1}{2}\int_0^{T'}\int V^2 + |H|^2\,\mathrm{d}\omega_t\,\mathrm{d}t \le \omega^0(\mathbb{T}) \tag{5.10}$$
for almost every $T' \in (0,T)$*.*

4. *(Compatibility) For almost every* $t \in (0,T)$ *and all* $\xi \in \mathrm{C}^\infty\left(\mathbb{T};\mathbb{R}^d\right)$*, it holds that*
$$\sigma\int \langle \xi\,,\,\nabla\chi(t,\cdot)\rangle = \int_{\mathbb{T}\times\mathbb{S}^{d-1}} \langle \xi\,,\,p\rangle\,\mathrm{d}\mu_t\,. \tag{5.11}$$

We firstly want to discuss this definition. In the setting of Definition 5.1.1 and in the two-phase case, we can think of the oriented varifold μ as the measure
$$\mu = \sigma\,\mathcal{L}^1\,|_{(0,T)} \otimes |\nabla\chi(t)| \otimes \delta_{\nu(t,x)}, \tag{5.12}$$
and therefore the measure ω_t becomes the energy measure $\mathrm{E}(\chi(t),\cdot)$. The advantage of this formulation is that our new energy measure ω_t is not restricted to only seeing the measure theoretic boundary of χ, but can actually capture phenomenons as described in Figure 5.1. For example in such a scenario we would expect the measure μ_t to be defined by
$$\mu_t := 2\,\mathcal{H}^1\,|_l \otimes \left(\frac{1}{2}\delta_{(1,0)} + \frac{1}{2}\delta_{(-1,0)}\right)_{x\in\mathbb{T}}, \tag{5.13}$$
where l is the red line to which the phase of α_2 shrank down as ε approached zero. The factor 2 comes from the fact that we obtain energy from both the collapsing interfaces.

The equation (5.8) for the normal speed is simply motivated through the fundamental theorem of calculus. Assuming that everything is nice and smooth, we can compute that

$$\begin{aligned}
&\sigma \int \chi(T,x)\varphi(T,x) - \chi^0(x)\varphi(0,x)\,\mathrm{d}x \\
&= \sigma \int_0^T \int \partial_t\left(\chi(t,x)\varphi(t,x)\right)\mathrm{d}x\,\mathrm{d}t \\
&= \sigma \int_0^T \int \partial_t\chi(t,x)\varphi(t,x) + \chi(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int V\varphi\,\mathrm{d}\omega_t\,\mathrm{d}t + \sigma\int_0^T \int \chi(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}$$

where for the last equality, we used $\partial_t\chi = V|\nabla\chi|\,\mathrm{d}t$ and $\omega_t = \sigma|\nabla\chi(t)|$.

The equation (5.9) for the generalized mean curvature is also quite intuitive. In fact if we assume that the varifold is given through equation (5.12), then we have that the right hand side of equation (5.9) reads

$$\int_{\mathbb{T}\times\mathbb{S}^{d-1}} \langle \xi\,,\,\mathrm{Id} - p\otimes p\rangle\,\mathrm{d}\omega_t = \sigma\int \langle \xi\,,\,\mathrm{Id} - \nu\otimes\nu\rangle|\nabla\chi|,$$

which is the distributional formulation for the mean curvature vector.

De Giorgis inequality (5.10) is self-explanatory since ω_t is the energy measure. The compatibility condition (5.11) is necessary to couple the evolving set A_t to the varifold. Notice that in our above example (5.13), this condition is still satisfied even though we have that the energy measure ω_t sees the red strip and the measure theoretic boundary of the corresponding indicator function does not. In fact the term on the right hand side of (5.11) is zero since

$$\int_{\mathbb{T}\times\mathbb{S}^{d-1}} \langle \xi\,,\,p\rangle\,\mathrm{d}\mu_t = \int_l \frac{1}{2}\langle \xi\,,\,(1,0)+(-1,0)\rangle\,\mathrm{d}\mathcal{H}^1 = 0.$$

For the multiphase case, we propose the following solution concept, which generalizes [HL21, Def. 2] to the case of arbitrary surface tensions.

Definition 5.2.2 (De Giorgi type varifold solution for multiphase mean curvature flow)**.** *Let $T<\infty$ be an arbitrary finite time horizon and let $P\in\mathbb{N}_{\geq 2}$ be the number of phases. For each pair of phases $(i,j)\in\{1,\ldots,P\}$, let $\mu_{ij} = \mathcal{L}^1\lfloor_{(0,T)}\otimes(\mu_{t,ij})_{t\in(0,\infty)}$ be a family of oriented varifolds $\mu_{t,ij}\in\mathcal{M}\left(\mathbb{T}\times\mathbb{S}^{d-1}\right)$, $t\in(0,T)$, such that the map $t\mapsto\int_{\mathbb{T}\times\mathbb{S}^{d-1}}\eta(t,x,p)\,\mathrm{d}\mu_{t,ij}(x,p)$ is measurable for all functions $\eta\in\mathrm{L}^1\left((0,T);\mathrm{C}\left(\mathbb{T}\times\mathbb{S}^{d-1}\right)\right)$. Define evolving oriented varifolds $\mu_i = \mathcal{L}^1\lfloor_{(0,T)}\otimes(\mu_{t,i})_{t\in(0,T)}$, $i\in\{1,\ldots,P\}$ and $\mu = \mathcal{L}^1\lfloor_{(0,T)}\otimes(\mu_t)_{t\in(0,T)}$ by means of*

$$\mu_{t,i} \coloneqq 2\mu_{t,ii} + \sum_{j=1,j\neq i}^{P}\mu_{t,ij} \quad \textit{and} \quad \mu_t \coloneqq \frac{1}{2}\sum_{i=1}^{P}\mu_{t,i}.$$

The disintegration of $\mu_{t,ij}$ *is expressed in form of* $\mu_{t,ij} = \omega_{t,ij} \otimes (\lambda_{t,x,ij})_{x\in\mathbb{T}}$ *with expected value* $\langle \lambda_{t,x,ij}\rangle := \int_{\mathbb{S}^{d-1}} p \,\mathrm{d}\lambda_{t,x,ij}(p)$. *Analagous expressions are introduced for the disintegrations of* $\mu_{t,i}$ *and* μ_t.

Furthermore consider a family $A = (A_1, \dots, A_P)$ *such that for each phase* $1 \leq i \leq P$, *we have a family* $A_i = (A_i(t))_{t\in(0,T)}$ *of subsets of* $\mathbb{T}$ *with finite perimeter. We also require* $(A_1(t), \dots, A_P(t))$ *to be a partition of* $\mathbb{T}$ *for all* $t \in (0,T)$ *and that for each* $1 \leq i \leq P$, *the associated indicator function satisfies* $\chi_i \in \mathrm{L}^\infty\left((0,T); \mathrm{BV}\left(\mathbb{T}; \{0,1\}\right)\right)$. *We shortly write* $\chi = (\chi_1, \dots, \chi_P)$.

Given initial data $((\omega^0, (\chi_i^0)_{1\leq i\leq P})$ *of the above form and a* $P \times P$ *matrix of surface tensions* σ *such that* $\sigma_{ij} > 0$ *for* $i \neq j$, *we call the pair* (μ, χ) *a* De Giorgi type varifold solution for multiphase mean curvature flow with initial data (ω^0, χ^0) and surface tensions σ *if the following requirements hold true.*

1. *(Existence of normal speeds) For each phase* $1 \leq i \leq P$, *there exists a normal speed* $V_i \in \mathrm{L}^2\left((0,T) \times \mathbb{T}, \omega_i\right)$ *in the sense that*
$$\begin{aligned} &\int \chi_i(T', x)\varphi(T', x) - \chi_i^0(x)\varphi(0, x) \,\mathrm{d}x \\ &= \int_0^{T'} \int \chi_i \partial_t \varphi \,\mathrm{d}x \,\mathrm{d}t + \sum_{j=1, j\neq i}^{P} \frac{1}{\sigma_{ij}} \int_0^{T'} \int V_i \varphi \,\mathrm{d}\omega_{t,ij} \,\mathrm{d}t \end{aligned} \tag{5.14}$$
for almost every $T' \in (0,T)$ *and all* $\varphi \in \mathrm{C}_\mathrm{c}^\infty\left([0,T) \times \mathbb{T}\right)$.

2. *(Existence of a generalized mean curvature vector) There exists a generalized mean curvature vector* $H \in \mathrm{L}^2\left((0,T) \times \mathbb{T}, \omega; \mathbb{R}^d\right)$ *in the sense that*
$$\int_0^T \int \langle H\,,\, \xi\rangle \,\mathrm{d}\omega_t \,\mathrm{d}t = -\int_0^T \int_{\mathbb{T}\times\mathbb{S}^{d-1}} \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - p\otimes p\rangle \,\mathrm{d}\mu_t \,\mathrm{d}t \tag{5.15}$$
holds for all $\xi \in \mathrm{C}_\mathrm{c}^\infty\left([0,T) \times \mathbb{T}; \mathbb{R}^d\right)$.

3. *(De Giorgi type optimal energy dissipation inequality) A sharp energy dissipation inequality holds in form of*
$$\omega_{T'}(\mathbb{T}) + \frac{1}{2}\sum_{i=1}^{P} \int_0^{T'} \int V_i^2 \frac{1}{2} \,\mathrm{d}\omega_{t,i} \,\mathrm{d}t + \frac{1}{2}\int_0^{T'} \int |H|^2 \,\mathrm{d}\omega_t \,\mathrm{d}t \leq \omega^0(\mathbb{T}) \tag{5.16}$$
for almost every $T' \in (0,T)$.

4. *(Compatibility conditions) For all* $1 \leq i, j \leq P$, *we require*
$$\omega_{t,ij} = \omega_{t,ji} \tag{5.17}$$
for almost every $t \in (0,T)$ *and*
$$\langle \lambda_{t,x,ij}\rangle = -\langle \lambda_{t,x,ji}\rangle, \tag{5.18}$$
$$V_i(t,x) = -V_j(t,x), \tag{5.19}$$
$$|\langle \lambda_{t,x,ij}\rangle|^2 H(t,x) = \langle H(t,x)\,,\, \langle \lambda_{t,x,ij}\rangle\rangle \langle \lambda_{t,x}^{ij}\rangle \tag{5.20}$$

for almost every $t \in (0,T)$ and $\omega_{t,ij}$ almost every $x \in \mathbb{T}$. Finally for all $1 \leq i \leq P$, we have

$$\int \langle \xi \,,\, \nabla \chi_i(t,\cdot) \rangle = \sum_{j=1, j\neq i}^{P} \frac{1}{\sigma_{ij}} \int_{\mathbb{T} \times \mathbb{S}^{d-1}} \langle \xi \,,\, p \rangle \, \mathrm{d}\mu_{t,ij} \tag{5.21}$$

for almost every $t \in (0,T)$ and every $\xi \in \mathrm{C}^\infty\left(\mathbb{T}; \mathbb{R}^d\right)$.

As before, we first want to motivate this definition. In the simplest case, we can think of the varifold $\mu_{t,ij}$ for $i \neq j$ as

$$\mu_{t,ij} = \sigma_{ij} \, \mathrm{d}\, \mathcal{H}^{d-1} \lfloor_{\Sigma_{ij}} \otimes (\delta_{\nu_i})_{x \in \mathbb{T}}, \tag{5.22}$$

and $\mu_{t,ii} = 0$. But if approximate interfaces collapse as in Figure 5.1, then this is exactly captured by the varifolds $\mu_{t,ii}$, which, as in the twophase case, will be described by

$$\mu_{t,11} = 2\, \mathcal{H}^1 \lfloor_l \otimes \left(\frac{1}{2} \delta_{(1,0)} + \frac{1}{2} \delta_{(-1,0)} \right)_{x \in \mathbb{T}}. \tag{5.23}$$

In the definition of μ_t, the factor $1/2$ is present since every interface $\mu_{t,ij}$, $i \neq j$, is counted twice. Thus we need the factor 2 in front of $\mu_{t,ii}$ in the definition of $\mu_{t,i}$ since this interface is not counted twice.

The equation (5.14) is similar to the two-phase case. Note however that we only consider the energy measures $\omega_{t,ij}$ for $i \neq j$ since we do not expect $\omega_{t,ii}$ to be relevant for the motion of the i-th phase. Equation (5.15) is as in the two-phase case. For the energy dissipation inequality, we note that we need the factor $1/2$ in front of $\omega_{t,i}$ since each interface will be counted twice.

The first compatibility condition (5.17) follows simply from $\Sigma_{ij} = \Sigma_{ji}$ if we are in the setting of energy convergence. But it should even hold true in situations like Figure 5.1, since the equation becomes trivial for $i = j$. The second compatibility condition (5.18) states that in the setting of energy convergence, we have $\nu_i = -\nu_j$ on Σ_{ij} $\mathcal{H}^{d-1}$-almost everywhere. For $i = j$, the equation states that $\langle \lambda_{t,x,ii} \rangle = 0$, which is satisfied by equation (5.23). Similarly we explain the anti-symmetry of the velocities (5.19). The compatibility condition (5.20) encodes that the mean curvature vector should always point in direction of the inner unit normal, which is not necessarily the case for varifolds, as demonstrated in Example 5.2.4, but can always be assured in the setting of energy convergence, see the proof of Theorem 5.2.3. For the last compatibility condition (5.21), we again notice that it behaves similarly to its two-phase equivalent.

Theorem 5.2.3. *Every De Giorgi type* BV*-solution to multiphase mean curvature flow in the sense of Definition 5.1.1 is also a De Giorgi type varifold solutions for multiphase mean curvature flow in the sense of Definition 5.2.2 with varifold μ given by*

$$\mu_{t,ij} \coloneqq \sigma_{ij}\, \mathcal{H}^{d-1} \lfloor_{\Sigma_{ij}(t)} \otimes \left(\delta_{\nu_i(t,x)}\right)_{x \in \mathbb{T}}$$

for $i \neq j$, $\mu_{t,ii} = 0$ and initial energy $\omega^0 = \mathrm{E}_0$.

Proof. We first note that

$$\mu_{t,i} = \sum_{j\neq i} \sigma_{ij}\, \mathcal{H}^{d-1}_{\Sigma_{ij}} \otimes (\delta_{\nu_i})_{x\in\mathbb{T}}$$

and

$$\mu_t = \sum_{1\leq i<j\leq P} \sigma_{ij}\, \mathcal{H}^{d-1}\, |_{\Sigma_{ij}} \otimes \left(\frac{1}{2}\delta_{\nu_i} + \frac{1}{2}\delta_{\nu_j}\right)_{x\in\mathbb{T}}.$$

Thus the energy measures are given by

$$\omega_{t,i} = \sum_{j\neq i} \sigma_{ij}\, \mathcal{H}^{d-1}\, |_{\Sigma_{ij}(t)}$$

and

$$\omega_t = \sum_{1\leq i<j\leq P} \sigma_{ij}\, \mathcal{H}^{d-1}\, |_{\Sigma_{ij}(t)} = \mathrm{E}(\chi(t);\cdot).$$

The requirement $\chi_i \in \mathrm{L}^\infty\left((0,T);\mathrm{BV}(\mathbb{T};\{0,1\})\right)$ is an immediate consequence of the energy dissipation inequality. The existence of normal velocities is given, but we need to check that equation (5.14) holds. If we assume that φ is compactly supported, then the equation follows by approximating χ_i with $\rho_n * \chi_i$, where ρ_n is a sequence of radial symmetric standard mollifiers. In general, take a sequence η_n of non-decreasing smooth functions with compact support in $(0,T]$ which converge pointwise to the constant 1 function and satisfy $\eta(T) = 1$. Then we have

$$\begin{aligned}
&\int \chi_i(T',x)\varphi(T',x)\,\mathrm{d}x \\
=\ & \lim_{n\to\infty} \int \chi_i(T',x)\eta_n(T')\varphi(T',x)\,\mathrm{d}x \\
=\ & \lim_{n\to\infty} \int_0^{T'} \int \eta_n\varphi V_i \left|\nabla\chi^i\right| \mathrm{d}t + \int_0^{T'} \int \chi_i\eta_n\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'} \int \chi_i\partial_t\eta_n\varphi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for almost every $0 < T' < T$. Using the dominated convergence theorem, we see that the first two summands converge to the left hand side of the velocity equation (5.14). For the third summand, we compute that since $\int_0^T \eta'\,\mathrm{d}t = 1$, we have

$$\begin{aligned}
&\left| \int_0^{T'} \partial_t\eta_n \int \chi_i\varphi\,\mathrm{d}x\,\mathrm{d}t - \int \chi_i^0(x)\varphi(0,x)\,\mathrm{d}x \right| \\
=\ & \left| \int_0^{T'} \partial_t\eta_n \int \chi_i\varphi - \chi_i^0\varphi(0,x)\,\mathrm{d}x\,\mathrm{d}t \right| \\
\lesssim\ & \int_0^{T'} \partial_t\eta_n \left(\int \left|\chi^i\varphi - \chi_i^0\varphi(0,x)\right|^2 \mathrm{d}x \right)^{1/2} \mathrm{d}t\,,
\end{aligned}$$

which converges to zero since χ_i assumes the initial data continuously with respect to the L^2-norm. Therefore the velocity equation (5.14) follows.

The curvature equation (5.15) follows immediately from equation (5.1) since

$$\int_{\mathbb{T}\times\mathbb{S}^{d-1}} \langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-p\otimes p\rangle\,\mathrm{d}\mu_t = \sum_{1\le i<j\le P}\sigma_{ij}\int_{\Sigma_{ij}}\langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu_i\otimes\nu_i\rangle\,\mathrm{d}\,\mathcal{H}^{d-1}\,.$$

We used here that on Σ_{ij}, since $\nu_i=-\nu_j$, we have $\nu_i\otimes\nu_i=\nu_j\otimes\nu_j$. De Giorgi's optimal energy dissipation inequality is also identical.

The compatibility conditions (5.17)-(5.19) all hold restricted to Σ_{ij} $\mathcal{H}^{d-1}$ almost everywhere and are therefore satisfied. For the compatibility condition (5.20), we have to prove, since $\langle\lambda_{t,x,ij}\rangle=\nu_i(t,x)$ on Σ_{ij} $\mathcal{H}^{d-1}$-almost everywhere, that the mean curvature vector H points in the direction of ν_i $\mathcal{H}^{d-1}$-almost everywhere. This has been proven by Brakke in [Bra78, Thm. 5.8]. Note that we apply it (for a fixed time t) to the varifold $V=\mu_t$, which is strictly speaking no integer varifold, but all results still hold since

$$V=\sum_{1\le i<j\le P}\sigma_{ij}v(\Sigma_{ij})$$

is a finite sum. Here $v(\Sigma_{ij})$ is the naturally associated varifold to Σ_{ij} as defined by Brakke. The last compatibility condition (5.21) is a consequence of Lemma A.0.1, Item 2. This completes the proof. □

Example 5.2.4. *We want to present an example of a varifold where the mean curvature does not point in normal direction. Let $\mathbb{T}=[0,\Lambda)^2$ be the flat torus in two dimensions and let $\rho\colon\mathbb{R}\to(-\infty,\infty)$ be some positive smooth Λ-periodic function which is not constant. Then we consider the varifold given by $\mu=\rho(x_1)\,\mathcal{H}^1\,|_{[0,\Lambda)\times\{0\}}\otimes\delta_{(0,1)}$. We compute that for a given test vector field ξ, we have*

$$\begin{aligned}\int\langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-p\otimes p\rangle\,\mathrm{d}\mu &= \int_0^\Lambda \langle \mathrm{D}\,\xi(x_1,0)\,,\,(1,0)\otimes(1,0)\rangle\rho(x_1)\,\mathrm{d}x_1\\ &= \int_0^\Lambda \partial_{x_1}\xi^1\rho(x_1)\,\mathrm{d}x_1\\ &= -\int_0^\Lambda \xi^1\rho'(x_1)\,\mathrm{d}x_1\\ &= \int_0^\Lambda \langle H(x_1,0)\,,\,\xi(x_1,0)\rangle\rho(x_1)\,\mathrm{d}x_1\,,\end{aligned}$$

which implies that H is given on $[0,\Lambda)\times\{0\}$ $\mathcal{H}^1$ almost everywhere by

$$H(x)=\frac{\rho'(x_1)}{\rho(x_1)}(1,0),$$

which does not lie in the normal space of the varifold.

A. Appendix

Lemma A.0.1. *The following statments hold for Borel-measurable sets.*

1. *If $A, B, C \subset \mathbb{T}$ are mutually disjoint, then*
$$\mathcal{H}^{d-1}\left(\partial_*(A \cup B) \triangle \left(\partial_* A \triangle \partial_* B\right)\right) = 0 \quad \text{and} \quad \mathcal{H}^{d-1}\left(\partial_* A \cap \partial_* B \cap \partial_* C\right) = 0.$$

2. *If $(\Omega_i)_{i=1,\dots,P}$ is a partition of $\mathbb{T}$, then*
$$\mathcal{H}^{d-1}\left(\partial_*\Omega_i \triangle \bigcup_{j \neq i} \partial_*\Omega_i \cap \partial_*\Omega_j\right) = 0.$$

Remark A.0.2. In a maybe less confusing way, the first and third result yield that up to sets of $\mathcal{H}^{d-1}$-measure zero, we have $\partial_*(A \cup B) = \partial_* A \triangle \partial_* B$ and $\partial_*\Omega_i = \bigcup_{j\neq i} \partial_*\Omega_i \cap \partial_*\Omega_j$ under the given assumptions.

Proof. Let us first prove Item 1. Initially assume that $x \in \partial_*(A \cup B)$. By definition this implies that
$$\limsup_{r\to 0} \frac{\mathcal{L}^d(B_r(x) \cap (A \cup B))}{r^d} > 0.$$
Then we must either have
$$\limsup_{r\to 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{r^d} > 0 \quad \text{or} \quad \limsup_{r\to 0} \frac{\mathcal{L}^d(B_r(x) \cap B)}{r^d} > 0,$$
so let us without loss of generality assume the former. Since
$$\limsup_{r\to 0} \frac{\mathcal{L}^d(B_r(x) \setminus A)}{r^d} \geq \limsup_{r\to 0} \frac{\mathcal{L}^d(B_r(x) \setminus (A \cup B))}{r^d} > 0,$$
we thus have $x \in \partial_* A$. We now want to show that $x \notin \partial_* B$. Combining Thm. 5.14 and Lemma 5.5 in [EG15], we can rewrite for some measurable set N with $\mathcal{H}^{d-1}(N) = 0$ the measure theoretic boundary of A as
$$\partial_* A = \partial^{1/2} A \cup N := \left\{x : \lim_{r\to 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{\alpha(d) r^d} = \frac{1}{2} = \lim_{r\to 0} \frac{\mathcal{L}^d(B_r(x) \setminus A)}{\alpha(d) r^d}\right\} \cup N, \quad \text{(A.1)}$$
where $\alpha(d) := \mathcal{L}^d(B_1(0))$ is the volume of the unit ball in d dimensions. Thus assume $x \in \partial^{1/2} A \cap \partial^{1/2} B$. Then since A and B are disjoint, we would have
$$\limsup_{r\to 0} \frac{\mathcal{L}^d(B_r(x) \setminus (A \cup B))}{\alpha(d) r^d} = \limsup_{r\to 0} \frac{\mathcal{L}^d(B_r(x) \setminus A) - \mathcal{L}^d(B_r(x) \cap B)}{\alpha(d) r^d} = 0,$$

which contradicts $x \in \partial_*(A \cup B)$. Thus up to a set of $(d-1)$-dimensional Hausdorff-measure zero, we have $\partial_*(A \cup B) \subseteq \partial_* A \triangle \partial_* B$.

For the other inclusion, suppose that $x \in \partial_* A \triangle \partial_* B$ and without loss of generality that $x \in \partial_* A \setminus \partial_* B$. Then it must either hold that

$$\limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \cap B)}{r^d} = 0 \quad \text{or} \quad \limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \setminus B)}{r^d} = 0. \tag{A.2}$$

But since $x \in \partial_* A$, the latter can not be true since A and B are disjoint and therefore

$$0 < \limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{r^d} \leq \limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \setminus B)}{r^d},$$

thus the former from (A.2) has to be true. We now estimate

$$\limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \cap (A \cup B))}{r^d} \geq \limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{r^d} > 0$$

and

$$\begin{aligned}\limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \setminus (A \cup B))}{r^d} &= \limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \setminus A) - \mathcal{L}^d(B_r(x) \cap B)}{r^d} \\ &= \limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \setminus A)}{r^d} > 0,\end{aligned}$$

which proves $x \in \partial_*(A \cup B)$.

The second equality is an immediate consequence of observation (A.1).

Now let us prove Item 2. It follows immediately that

$$\partial_* \Omega_i \supseteq \bigcup_{j \neq i} \partial_* \Omega_i \cap \partial_* \Omega_j.$$

For the other inclusion, suppose $x \in \partial_* \Omega_i$. Since

$$0 < \limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \setminus \Omega_i)}{r^d} = \limsup_{r \to 0} \frac{\mathcal{L}^d\left(B_r(x) \cap \bigcup_{j \neq i} \Omega_j\right)}{r^d},$$

we find some $j \neq i$ such that

$$\limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \cap \Omega_j)}{r^d} > 0.$$

Since

$$\limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \setminus \Omega_j)}{r^d} \geq \limsup_{r \to 0} \frac{\mathcal{L}^d(B_r(x) \cap \Omega_i)}{r^d} > 0,$$

we therefore have $x \in \partial_* \Omega_j$, which finishes the proof. $\square$

Proof of Lemma 3.2.8. Without loss of generality we can assume that $i = 1$ and $\sigma_{1,j} \le \sigma_{1,j+1}$ for all $1 \le j \le P-1$. All equalities for sets in this proof hold up to a set of $(d-1)$-dimensional Hausdorff measure zero. First let us apply the Fleming–Rishel co-area formula to find that for a given open set $U \subseteq \mathbb{T}$, we have

$$\begin{aligned}
|\nabla\psi_1|(U) &= \int_0^\infty \mathcal{H}^{d-1}\left(\partial_*\left(\{x \in U : \psi_1 \le s\}\right)\right) \mathrm{d}s \\
&= \sum_{j=1}^{P-1} \int_{\sigma_{1,j}}^{\sigma_{1,j+1}} \mathcal{H}^{d-1}\left(\partial_*\left(\bigcup_{k=1}^{j} U \cap \Omega_k\right)\right) \mathrm{d}s \\
&= \sum_{j=1}^{P-1} (\sigma_{1,j+1} - \sigma_{1,j})\, \mathcal{H}^{d-1}\left(\partial_*\left(\bigcup_{k=1}^{j} U \cap \Omega_k\right)\right) =: I.
\end{aligned}$$

We claim that we have

$$\partial_*\left(\bigcup_{k=1}^{j} U \cap \Omega_k\right) = \bigcup_{k=1}^{j} \bigcup_{l=j+1}^{P} U \cap \partial_*\Omega_k \cap \partial_*\Omega_l. \tag{A.3}$$

Since $(\Omega_k \cap U)_k$ is again a partition of U, we omit taking the intersection with U for a briefer notation. The proof will be done via an induction over j. For $j = 1$, this is exactly Item 2 from Lemma A.0.1. For the induction step, assume that the claim (A.3) holds for all $j' \le j$. Again by Lemma A.0.1, Item 1 and Item 2, we can compute that

$$\begin{aligned}
\partial_*\left(\bigcup_{k=1}^{j+1} \Omega_k\right) &= \left(\partial_*\left(\bigcup_{k=1}^{j} \Omega_k\right) \cup \partial_*\Omega_{j+1}\right) \setminus \left(\partial_*\left(\bigcup_{k=1}^{j} \Omega_k\right) \cap \partial_*\Omega_{j+1}\right) \\
&= \left(\left(\bigcup_{k=1}^{j} \bigcup_{l=j+1}^{P} \partial_*\Omega_k \cap \partial_*\Omega_l\right) \cup \bigcup_{k \ne j+1} \partial_*\Omega_{j+1} \cap \partial_*\Omega_k\right) \setminus \left(\bigcup_{k=1}^{j} \bigcup_{l=j+1}^{P} \partial_*\Omega_k \cap \partial_*\Omega_l \cap \partial_*\Omega_{j+1}\right).
\end{aligned}$$

By Lemma A.0.1 Item 1, we can write the second term as

$$\bigcup_{k=1}^{j} \bigcup_{l=j+1}^{P} \partial_*\Omega_k \cap \partial_*\Omega_l \cap \Omega_{j+1} = \bigcup_{k=1}^{j} \partial_*\Omega_k \cap \partial_*\Omega_{j+1}.$$

By carefully considering both equations, we get

$$\begin{aligned}
&\partial_*\left(\bigcup_{k=1}^{j+1} \Omega_k\right) \\
&= \left(\left(\bigcup_{k=1}^{j} \bigcup_{l=j+2}^{P} \partial_*\Omega_k \cap \partial_*\Omega_l\right) \cup \bigcup_{k=j+2}^{P} \partial_*\Omega_{j+1} \cap \partial_*\Omega_k\right) \setminus \bigcup_{k=1}^{j} \partial_*\Omega_{j+1} \cap \partial_*\Omega_k \\
&= \left(\bigcup_{k=1}^{j+1} \bigcup_{l=j+2}^{P} \partial_*\Omega_k \cap \partial_*\Omega_l\right) \setminus \bigcup_{k=1}^{j} \partial_*\Omega_{j+1} \cap \partial_*\Omega_k.
\end{aligned} \tag{A.4}$$

Since by Lemma A.0.1 Item 1, we have for $1 \leq k \leq j+1$, $j+2 \leq l \leq P$ and $1 \leq m \leq j$ that

$$\mathcal{H}^{d-1}\left(\partial_* \Omega_k \cap \partial_* \Omega_l \cap \partial_* \Omega_{j+1} \cap \partial_* \Omega_m\right) = 0,$$

taking the complement in the term (A.4) becomes obsolete and thus we obtain the desired result (A.3). Therefore we have again by Lemma A.0.1 Item 1 that

$$\begin{aligned} I &= \sum_{j=1}^{P-1} (\sigma_{1,j+1} - \sigma_{1,j}) \left(\sum_{k=1}^{j} \sum_{l=j+1}^{P} \mathcal{H}^{d-1}\left(U \cap \partial_* \Omega_k \cap \partial_* \Omega_l\right) \right) \\ &= \sum_{1 \leq k < l \leq P} \mathcal{H}^{d-1}\left(U \cap \partial_* \Omega_k \cap \partial_* \Omega_l\right) \left(\sum_{j=k}^{l-1} \sigma_{1,j+1} - \sigma_{1,j} \right) \\ &= \sum_{1 \leq k < l \leq P} \mathcal{H}^{d-1}\left(U \cap \partial_* \Omega_k \cap \partial_* \Omega_l\right) \left(\sigma_{1,l} - \sigma_{1,k}\right). \end{aligned}$$

Since we assumed without loss of generality that $\sigma_{1,j} \leq \sigma_{1,j+1}$, we know that for $k < l$,

$$\sigma_{1,l} - \sigma_{1,k} = |\sigma_{1,l} - \sigma_{1,k}|$$

which finishes the proof. □

Lemma A.0.3. *Let μ be a regular positive Borel measure on some open set Ω and let $B_1, \ldots, B_m$ be μ-finite disjoint Borel subsets of Ω. Moreover let c_i^h for $1 \leq i \leq m$ and $1 \leq h \leq k$ be non-negative coefficients. Define the measures*

$$\mu_h := \sum_{i=1}^{m} c_i^h \mu|_{B_i} \quad \text{and} \quad \nu := \sum_{i=1}^{m} \max_h c_i^h \mu|_{B_i}.$$

Then we have

$$\nu = \bigvee_{h=1}^{k} \mu_h.$$

Proof. We first note that for all Borel sets A and all $\tilde{h}$, we have

$$\nu(A) = \sum_{i=1}^{m} \max_h c_i^h \mu(A \cap B_i) \geq \sum_{i=1}^{m} c_i^{\tilde{h}} \mu(A \cap B_i) = \mu_{\tilde{h}}(A).$$

On the other hand we have that for all $1 \leq i \leq m$, we find an index $h(i)$ such that $\max_h c_i^h = c_i^{h(i)}$. Since the sets B_i are disjoint, we therefore have

$$\nu(B_i) = \max_h c_i^h \mu(B_i) = \mu_{h(i)}(B_i),$$

thus we have for all $1 \leq i \leq m$ that

$$\nu(B_i) \leq \left(\bigvee_{h=1}^{k} \mu_h \right)(B_i).$$

Since the support of ν is contained in $\bigcup_{1 \leq i \leq m} B_i$, this finishes the proof. □

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