

# Convergence of Allen-Cahn equations to multi-phase mean curvature flow

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# 1 The Allen–Cahn equation

## 1.1 Structure of the equation

This chapter follows [LS16], but since the authors decided to only sketch some proofs, we want to go into more detail.

Let  $\Lambda > 0$  and define the flat torus  $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$ , where we work with periodic boundary conditions and write  $\int dx$  instead of  $\int_{\mathbb{T}} dx$ . Then for  $u: [0, \infty) \times \mathbb{T} \rightarrow \mathbb{R}^N$  and some potential  $W: \mathbb{R}^N \rightarrow [0, \infty)$ , the *Allen–Cahn equation* with parameter  $\varepsilon > 0$  is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \quad (1.1)$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to  $u$  for a fixed time the real number

$$E_\varepsilon(u) := \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 dx. \quad (1.2)$$

If everything is nice and smooth, we can compute that under the assumption that  $u$  satisfies equation (1.1), we have that

$$\begin{aligned} \frac{d}{dt} E_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u), \partial_t u \rangle + \varepsilon \langle \nabla u, \nabla \partial_t u \rangle dx \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u, \partial_t u \right\rangle dx \\ &= \int -\varepsilon |\partial_t u|^2 dx. \end{aligned} \quad (1.1)$$

This calculation suggests that equation (1.1) is the  $L^2$  gradient-flow (rescaled by  $\sqrt{\varepsilon}$ ) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (1.1) via De Giorgis minimizing movements scheme, which we will do in theorem Theorem 1.2.1.

But first we need to clarify what our potential  $W$  should look like. Classic examples in the scalar case are given by  $W(u) = (u^2 - 1)^2$  or  $W(u) = u^2(u - 1)^2$ , and we call functions like these *doublewell potentials*, see also Figure 1.1.

In higher dimensions, we want to accept the following potentials:  $W: \mathbb{R}^N \rightarrow [0, \infty)$  has to be a smooth multiwell potential with finitely many zeros at  $u = \alpha_1, \dots, \alpha_P \in \mathbb{R}^N$ . Furthermore we ask for polynomial growth in the sense that there exists some  $p \geq 2$  such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \quad (1.3)$$

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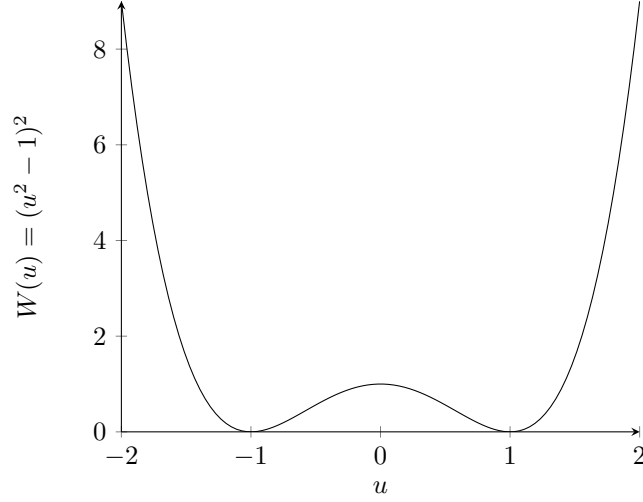


Figure 1.1: The graph of a doublewell potential

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \quad (1.4)$$

for all  $u$  sufficiently large. Lastly we want  $W$  to be convex up to a small perturbation in the sense that there exist smooth functions  $W_{\text{conv}}, W_{\text{pert}}: \mathbb{R}^N \rightarrow [0, \infty)$  such that

$$W = W_{\text{conv}} + W_{\text{pert}}, \quad (1.5)$$

$W_{\text{conv}}$  is convex and

$$\sup_{x \in \mathbb{R}^N} |\nabla^2 W_{\text{pert}}| < \infty. \quad (1.6)$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen–Cahn equation (1.1) as maps from  $[0, T]$  into some suitable function space and thus use the following definition.

**Definition 1.1.1.** *We say that a function  $u_\varepsilon \in C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  which is also in  $L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$  is a weak solution of the Allen–Cahn equation (1.1) with parameter  $\varepsilon > 0$  and initial condition  $u_\varepsilon^0 \in L^2(\mathbb{T}; \mathbb{R}^N)$  if*

1. *the energy stays bounded, which means that*

$$\operatorname{ess\,sup}_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t)) < \infty, \quad (1.7)$$

2. *its weak time derivative satisfies*

$$\partial_t u_\varepsilon \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N), \quad (1.8)$$

## 1.2 Existence of a solution

3. for almost every  $t \in [0, T]$  and every  $\xi \in L^p([0, T] \times \mathbb{T}; \mathbb{R}^N) \cap W^{1,2}([0, T] \times \mathbb{T}; \mathbb{R}^N)$ , we have

$$\int \langle \partial_t u_\varepsilon(t), \xi \rangle + \langle \nabla u_\varepsilon(t), \nabla \xi \rangle + \left\langle \frac{1}{\varepsilon^2} \nabla W(u_\varepsilon(t)), \xi \right\rangle dx = 0, \quad (1.9)$$

4. the initial conditions are achieved in the sense that  $u_\varepsilon(0) = u_\varepsilon^0$ .

**Remark.** Given our assumptions, we automatically have  $\nabla W(u_\varepsilon)(t) \in L^{p'}(\mathbb{T})$  since

$$|\nabla W(u_\varepsilon)|^{p/(p-1)} \lesssim 1 + |u_\varepsilon|^p \lesssim 1 + W(u), \quad (1.10)$$

which is integrable for almost every time  $t$  since we assume that the energy stays bounded, thus the integral in equation (1.9) is well defined.

Moreover we already obtain  $1/2$  Hölder-continuity in time from the embedding

$$W^{1,2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)) \hookrightarrow C^{1/2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)), \quad (1.11)$$

which follows from a generalized version of the fundamental theorem of calculus and Hölder's inequality.

## 1.2 Existence of a solution

With Definition 1.1.1, we are able to state our existence results for solutions of the Allen–Cahn equation. The proof uses De Giorgis minimizing movements scheme and arguments from the theory of gradient flows.

**Theorem 1.2.1.** *Let  $u_\varepsilon^0: \mathbb{T} \rightarrow \mathbb{R}^N$  be such that  $E_\varepsilon(u_\varepsilon^0) < \infty$ . Then there exists a weak solution  $u_\varepsilon$  to the Allen–Cahn equation (1.1) in the sense of Definition 1.1.1 with initial data  $u_\varepsilon^0$ . Furthermore the solution satisfies the energy dissipation identity*

$$E_\varepsilon(u_\varepsilon(t)) + \int_0^t \int \varepsilon |\partial_t u_\varepsilon|^2 dx ds = E_\varepsilon(u_\varepsilon^0) \quad (1.12)$$

for almost every  $t \in [0, T]$  and we additionally have  $\partial_{i,j}^2 u_\varepsilon, \nabla W(u_\varepsilon) \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$  for all  $1 \leq i, j \leq d$ . In particular we can test the weak form (1.9) with  $\partial_{i,j}^2 u_\varepsilon$ .

**Proof. Step 1: A minimization problem** Fix some  $h > 0$ ,  $u_{n-1} \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$  and consider the functional

$$\begin{aligned} \mathcal{F}: W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N) &\rightarrow \mathbb{R} \\ u &\mapsto E_\varepsilon + \frac{1}{2h} \int |u - u_{n-1}|^2 dx. \end{aligned}$$

Then  $\mathcal{F}$  is coercive with respect to  $\|\cdot\|_{W^{1,2}}$  and bounded from below by zero, thus we may take a  $W^{1,2}$ -bounded sequence  $(v_k)_{k \in \mathbb{N}}$  in  $W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$  such that  $\mathcal{F}(v_k) \rightarrow \inf \mathcal{F}$

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as  $k \rightarrow \infty$ . These  $v_k$  have a non-relabelled subsequence which converges weakly in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$  and strongly in  $L^2(\mathbb{T}, \mathbb{R}^N)$  to some  $u \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . Moreover  $v_k$  is bounded in  $L^p(\mathbb{T}; \mathbb{R}^N)$  by the lower growth assumption (1.3) on  $W$  and thus obtain by a duality argument that  $u \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ .

Lastly we have

$$\frac{1}{2h} \int |u - u_{n-1}|^2 dx \leq \liminf_{k \rightarrow \infty} \frac{1}{2h} \int |v_k - u_{n-1}|^2 dx, \quad (1.13)$$

$$\int \frac{\varepsilon}{2} |\nabla u|^2 dx \leq \liminf_{k \rightarrow \infty} \int \frac{\varepsilon}{2} |\nabla v_k|^2 dx \quad (1.14)$$

by the weak convergence in  $W^{1,2}(\mathbb{T}, \mathbb{R}^N)$ . By passing to another non-relabelled subsequence which converges pointwise almost everywhere, we moreover achieve by continuity of  $W$  that

$$\int \frac{1}{\varepsilon} W(u) dx = \int \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon} W(v_k) dx \leq \liminf_{k \rightarrow \infty} \int \frac{1}{\varepsilon} W(v_k) dx,$$

which proves that  $u$  is a minimizer of  $\mathcal{F}$ .

**Step 2: Minimizing movements scheme** By recursively choosing minimizers  $u_n^h$  from Step 1, we obtain a sequence of functions  $u_0, u_1^h, \dots$ . Thus we may define a function  $u \in C$   $\square$



## 2 Convergence of the Allen–Cahn equations

Baldo proved in his paper [Bal90] that the Cahn–Hilliard energies  $\Gamma$ -converge with respect to  $\|\cdot\|_{L^1}$  to an *optimal partition energy* given by

$$E(\chi) := \frac{1}{2} \sum_{1 \leq i, j \leq P} \sigma_{i,j} \int \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|) \quad (2.1)$$

for a partition  $\chi_1, \dots, \chi_P: \mathbb{T} \rightarrow \{0, 1\}$  satisfying  $\chi_{1 \leq i \leq P} \chi_i = 1$  almost everywhere. We may also define measurable sets  $\Omega_i$  through the relation  $\chi_i = \mathbb{1}_{\Omega_i}$ . The link between a sequence  $u_\varepsilon$  and  $\chi$  is given by  $u_\varepsilon \rightarrow u := \sum_{1 \leq j \leq P} \alpha_j \chi_j$  in  $L^1$ .

Moreover if we denote by  $\partial_* \Omega_i$  the reduced boundary of  $\Omega_i$  and by  $\Sigma_{i,j} := \partial_* \Omega_i \cap \Omega_j$  the interface between  $\Omega_i$  and  $\Omega_j$ , then we may rewrite equation (2.1) as

$$E(\chi) = \frac{1}{2} \sum_{1 \leq i, j \leq P} \sigma_{i,j} \mathcal{H}^{d-1}(\Sigma_{i,j}).$$

Here, the surface tensions  $\sigma_{i,j}$  are the geodesic distances between the wells  $\alpha_i$  of  $W$  with respect to the metric  $2W(u)\langle \cdot, \cdot \rangle$ , which can be written out as

$$\sigma_{i,j} = d_W(\alpha_i, \alpha_j)$$

for the geodesic distance defined as

$$d_W(u, v) := \inf \left\{ \int_0^1 \sqrt{2W(\gamma)} |\dot{\gamma}| dt : \gamma \in C^1([0, 1], \mathbb{R}^N) \text{ with } \gamma(0) = u, \gamma(1) = v \right\}. \quad (2.2)$$

Geometrically speaking, the partition energy  $E$  measures the surface tensions between the sets and penalizes large interfaces. Also observe that the factor  $1/2$  can be left out if we only count each interface once.



# Bibliography

- [Bal90] Sisto Baldo. “Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids”. en. In: *Annales de l’I.H.P. Analyse non linéaire* 7.2 (1990), pp. 67–90. URL: [http://www.numdam.org/item/AIHPC\\_1990\\_\\_7\\_2\\_67\\_0/](http://www.numdam.org/item/AIHPC_1990__7_2_67_0/).
- [LS16] Tim Laux and Theresa Simon. “Convergence of the Allen-Cahn Equation to Multiphase Mean Curvature Flow”. In: *Communications on Pure and Applied Mathematics* 71 (June 2016). DOI: 10.1002/cpa.21747.