Convergence of Allen-Cahn equations to multi-phase mean curvature flow

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1 The Allen–Cahn equation

This chapter follows [LS16], but since the authors decided to only sketch some proofs. we want to go into more detail.

Let $\Lambda > 0$ and define the flat torus $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$, where we work with periodic boundary conditions and write $\int \mathrm{d}x$ instead of $\int_{\mathbb{T}} \mathrm{d}x$. Then for $u\colon [0, \infty) \times \mathbb{T} \to \mathbb{R}^N$ and some potential $W\colon \mathbb{R}^N \to [0, \infty)$, the *Allen–Cahn equation* with parameter $\varepsilon > 0$ is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \tag{1.1}$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to u for a fixed time the real number

$$\mathrm{E}_\varepsilon(u) \coloneqq \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 \, \mathrm{d}x \,. \tag{1.2}$$

If everything is nice and smooth, we can compute that under the assumption that u satisfies equation (1.1), we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u) \,,\, \partial_t u \rangle + \varepsilon \langle \nabla u \,,\, \nabla \partial_t u \rangle \, \mathrm{d}x \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u \,,\, \partial_t u \right\rangle \mathrm{d}x \\ &= \int -\varepsilon |\partial_t u|^2 \, \mathrm{d}x \,. \end{aligned} \tag{1.1}$$

This calculation suggests that equation (1.1) is the L^2 gradient-flow (rescaled by $\sqrt{\varepsilon}$) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (1.1) via De Giorgis minimizing movements scheme, which we will do in theorem **??**.

But first we need to clarify what our potential W should look like. Classic examples in the scalar case are given by $W(u) = \left(u^2 - 1\right)^2$ or $W(u) = u^2(u-1)^2$, and we call functions like these *doublewell potentials*, see also Figure 1.1.

In higher dimensions, we want to accept the following potentials: $W\colon \mathbb{R}^N \to [0, \infty)$ has to be a a smooth multiwell potential with finitely many zeros at $u = \alpha_1, \dots, \alpha_P \in \mathbb{R}^N$. Furthermore we aks for polynomial growth in the sense that there exists some $p \geq 2$ such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \tag{1.3}$$

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \tag{1.4}$$

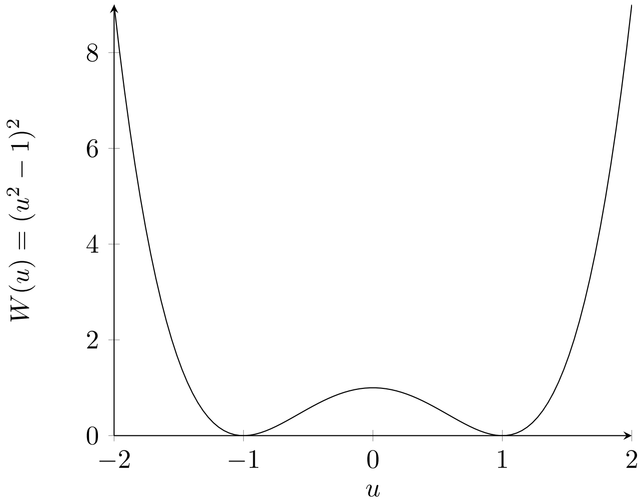

Figure 1.1: The graph of a doublewell potential

for all u sufficiently large. Lastly we want W to be convex up to a small perturbation in the sense that there exist smooth functions $W_{\text{conv}}, W_{\text{pert}} \colon \mathbb{R}^N \to [0, \infty)$ such that

$$W = W_{\text{conv}} + W_{\text{pert}}, \tag{1.5}$$

W_{conv} is convex and

$$\sup_{x \in \mathbb{R}^N} \left|\nabla^2 W_{\text{pert}}\right| < \infty. \tag{1.6}$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen-Cahn equation (1.1)as maps from $[0, T]$ into some suitable function space and thus use the following definition.

Definition 1.0.1. *We say that a function $u_\varepsilon \in \mathrm{C}\left([0, T]; \mathrm{L}^2(\mathbb{T}; \mathbb{R}^N)\right)$ which is also in $\mathrm{L}^\infty\left([0, T]; \mathrm{W}^{1,2}(\mathbb{T}; \mathbb{R}^N)\right)$ is a weak solution of the Allen–Cahn equation (1.1) with parameter $\varepsilon > 0$ if*

1. *the energy stays bounded, which means that*

$$\sup_{0 \le t \le T} \mathrm{E}_\varepsilon(u_\varepsilon(t)) < \infty, \tag{1.7}$$

2. *its weak time derivative satisfies*

$$\partial_t u_\varepsilon \in \mathrm{L}^2([0, T] \times \mathbb{T}), \tag{1.8}$$

2 Convergence of the Allen–Cahn equations

Baldo proved in his paper [Bal90] that the Cahn-Hilliard energies Γ-converge with respect to $\|\cdot\|_{\mathrm{L}^1}$ to an *optimal partition energy* given by

$$\mathrm{E}(\chi) := \frac{1}{2} \sum_{1\le i,j\le P} \sigma_{i,j} \int \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) \tag{2.1}$$

for a partition $\chi_1, \cdots, \chi_P \colon \mathbb{T} \to \{0,1\}$ satisfying $\chi_{1\le i\le P}\chi_i = 1$ almost everywhere. We may also define measurable sets Ω_i through the relation $\chi_i = \mathbb{1}_{\Omega_i}$. The link between a sequence u_ε and χ is given by $u_\varepsilon \to u := \sum_{1\le j\le P} \alpha_i \chi_i$ in L^1.

Moreover if we denote by $\partial_* \Omega_i$ the reduced boundary of Ω_i and by $\Sigma_{i,j} := \partial_* \Omega_i \cap \Omega_j$ the interface between Ω_i and Ω_j, then we may rewrite equation (2.1) as

$$\mathrm{E}(\chi) = \frac{1}{2} \sum_{1\le i,j\le P} \sigma_{i,j}\, \mathcal{H}^{d-1}(\Sigma_{i,j}).$$

Here, the surface tensions $\sigma_{i,j}$ are the geodesic distances between the wells α_i of W with respect to the metric $2W(u)\langle\cdot\,,\,\cdot\rangle$, which can be written out as

$$\sigma_{i,j} = \mathrm{d}_W(\alpha_i, \alpha_j)$$

for the geodesic distance defined as

$$\mathrm{d}_W(u,v) := \inf\left\{ \int_0^1 \sqrt{2W(\gamma)}|\dot\gamma|\,\mathrm{d}t \;:\; \gamma \in \mathrm{C}^1([0,1],\mathbb{R}^N) \text{ with } \gamma(0) = u,\, \gamma(1) = v \right\}. \tag{2.2}$$

Geometrically speaking, the partition energy E measures the surface tensions between the sets and penalizes larges interfaces. Also observe that the factor $1/2$ can be left out if we only count each interface once.

Bibliography

[Bal90] Sisto Baldo. "Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids". en. In: *Annales de l'I.H.P. Analyse non linéaire* 7.2 (1990), pp. 67–90. URL: http://www.numdam.org/item/AIHPC_1990__7_2_67_0/.

[LS16] Tim Laux and Theresa Simon. "Convergence of the Allen-Cahn Equation to Multiphase Mean Curvature Flow". In: *Communications on Pure and Applied Mathematics* 71 (June 2016). DOI: 10.1002/cpa.21747.