1 The Allen–Cahn equation

Let $\Lambda > 0$ and define the flat torus $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$, where we as usual assume periodic boundary conditions. Then for $u\colon [0, \infty) \times \mathbb{T} \to \mathbb{R}^N$ and some potential $W\colon \mathbb{R}^N \to [0, \infty)$, the *Allen–Cahn equation* with parameter $\varepsilon > 0$ is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \partial_u W(u). \tag{1.1}$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to u for a fixed time the real number

$$\mathrm{E}_\varepsilon(u) \coloneqq \int_{\mathbb{T}} \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} \|\nabla u\|^2 \,\mathrm{d}x\,. \tag{1.2}$$

If everything is nice and smooth, we can compute that under the assumption that u satisfies equation 1.1, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}_\varepsilon(u) = \int \frac{1}{\varepsilon} \partial_u W(u) \partial_t u + \varepsilon jj$$