

# Convergence of Allen-Cahn equations to multiphase mean curvature flow

Pascal Maurice Steinke

Born 22.05.1999 in Gießen, Germany

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Advisor: Prof. Dr. Tim Laux

Second Advisor: Prof. Dr. Stefan Müller

INSTITUTE FOR APPLIED MATHEMATICS

MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT DER  
RHEINISCHEN FRIEDRICH-WILHELMS-UNIVERSITÄT BONN



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# 1. Gradient flows and mean curvature flow

## 1.1. Gradient flows

In the simplest case, a gradient flow of a given energy  $E: \mathbb{R}^N \rightarrow \mathbb{R}$  (with respect to the euclidean inner product) is a solution to the ordinary differential equation

$$\frac{d}{dt}x(t) = -\nabla E(x(t)), \quad (1.1)$$

where we usually prescribe some initial value  $x(0) = x_0 \in \mathbb{R}^N$ . The central structure here is that on the right hand side of the equation, we do not have any function, but the gradient of some function. What a solution  $x$  does is that it moves in the direction of the steepest descent of the energy  $E$ . Moreover this allows for the following computation given a solution  $x$  of (1.1):

$$\begin{aligned} \frac{d}{dt} E(x(t)) &= \langle \nabla E(x(t)), x'(t) \rangle = -|\nabla E(x(t))|^2 = -|x'(t)|^2 \\ &= -\frac{1}{2} \left( |x'(t)|^2 + |\nabla E(x(t))|^2 \right). \end{aligned}$$

We especially obtain that the the function  $E(x(t))$  is non-increasing, which coincides with our intuition that  $x$  moves along the steepest descent of the energy. But more precisely, we obtain from the fundamental theorem of calculus the *energy dissipation identity*

$$E(x(T)) + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla E(x(t))|^2 dt = E(x(0)). \quad (1.2)$$

One could now raise the question if this identity already characterizes the equation, which means that if some  $x$  satisfies equation (1.2), it should already be a solution to the ordinary differential equation (1.1). But as it turns out, we can go even one step further, namely we only ask for the *optimal energy dissipation inequality* given by

$$E(x(T)) + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla E(x(t))|^2 dt \leq E(x(0)). \quad (1.3)$$

We call this inequality optimal since as demonstrated before, we usually expect an equality to hold. Now let us assume that  $x$  satisfies (1.3) and has sufficient regularity.

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Then we can estimate again by the fundamental theorem that

$$\begin{aligned}
& \frac{1}{2} \int_0^T |\nabla E(x(t)) + x'(t)|^2 dt \\
&= \int_0^T \langle \nabla E(x(t)), x'(t) \rangle dt + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla E(x(t))|^2 dt \\
&= E(x(T)) - E(x(0)) + \frac{1}{2} \int_0^T \langle \nabla E(x(t)), x'(t) \rangle dt \leq 0.
\end{aligned}$$

Since we started with an integral over a non-negative function, this implies that for almost every time  $t$ , we have that  $x'(t) = -\nabla E(x(t))$ , which proves that  $x$  (again under sufficient regularity assumptions) is a gradient flow of the energy  $E$ .

The real strength of formulating the differential equation (1.1) via inequality (1.3) becomes clear if we want to consider gradient flows in a more complicated setting by replacing the space  $\mathbb{R}^N$ . In order to formulate equation (1.1), we need to have a notion of differentiation and a gradient in the target of  $x$ . Therefore we may use pre-Hilbert spaces or smooth Riemannian manifolds as a suitable substitutes for  $\mathbb{R}^N$ .

Examples for such more complicated gradient include the heat equation, which can be written as the  $L^2$ -gradient flow of the Dirichlet energy (REFERENZ EINFÜGEN), the Fokker-Planck equation (REFERENZ EINFÜGEN) or the Allen–Cahn equation and mean curvature flow, which of course both play an important role for us.

## 1.2. Mean curvature flow

Mean curvature flow describes the geometric evolution of a set  $(\Omega(t))_{t \geq 0}$  respectively the evolution of its boundary  $\Sigma(t) := \partial\Omega(t)$ . It is described by the equation

$$\frac{1}{\mu} V = -\sigma H \quad \text{on } \Sigma. \quad (1.4)$$

There are a lot of different notions of solutions to this equation (HIER EINFÜGEN). For us the central structure of this equation is its gradient flow structure. Formally we want to consider the space

$$\mathcal{M} := \{(d-1)\text{-dimensional surfaces on } \mathbb{T}\},$$

where the tangent space at a given  $\Sigma \in \mathcal{M}$  consists of the normal velocities on  $\Sigma$ . The metric tensor of two normal velocities is then given by

$$\langle V, W \rangle_\Sigma := \frac{1}{\mu} \int_\Sigma VW \, d\mathcal{H}^{d-1}$$

and our energy will simply be the rescaled perimeter functional

$$E(\Sigma) := \sigma \mathcal{H}^{d-1}(\Sigma).$$

Since the first inner variation of the perimeter functional is given by the mean curvature vector (see [Mag12, Thm. 17.5]), the gradient of the energy should simply be the mean curvature vector multiplied by  $\sigma\mu$ . Thus the mean curvature flow equation (1.4) corresponds exactly to the gradient flow equation (1.1).

In Section 1.1 we highlighted the importance of De Giorgi's optimal energy dissipation inequality (1.3). In our setting this translates to the inequality

$$\begin{aligned} & E(\Sigma(T)) + \frac{1}{2} \int_0^T \langle V, V \rangle + \langle \nabla E(\Sigma), \nabla E(\Sigma) \rangle \\ &= E(\Sigma(T)) + \frac{1}{2} \int_0^T \int \frac{1}{\mu} V^2 + \sigma^2 \mu H^2 \, d\mathcal{H}^{d-1} \, dt \leq E(\Sigma(0)) \end{aligned}$$

and it will be one of our first main results to deduce this inequality when passing to the limit in the Allen–Cahn equations.

Far more important for us shall however be multiphase mean curvature, which has been motivated in material science by grain growths in polycrystals (HIER NOCH MEHR SCHREIBEN). Essentially instead of just considering the evolution of one single set, we look at the evolution of a partition of the flat torus and require that at each interface of the sets, the mean curvature flow equation (1.4) is satisfied.

We say that a partition  $(\Omega_i(t))_{i=1,\dots,P}$  with  $\Sigma_{i,j} := \partial\Omega_i \cap \partial\Omega_j$  satisfies multiphase mean curvature with mobilities  $\mu_{i,j}$  and surface tension  $\sigma_{i,j}$  if

$$\begin{cases} \frac{1}{\mu_{i,j}} V_{i,j} = -\sigma_{i,j} H_{i,j} & \text{for all } i, j \\ \sum \sigma_{i,j} \nu_{i,j} = 0 & \text{at triple junctions.} \end{cases}$$

The second equation is a stability condition when more than two sets meet and called Herring's angle condition. Here  $\nu_{i,j}$  is the outer unit normal on  $\Sigma_{i,j}$  pointing from  $\Omega_i$  to  $\Omega_j$ . One can argue that we would also want stability conditions at for example quadruple junctions. In two dimensions though, such quadruple junctions are expected to immediately dissipate. In higher dimensions, quadruple junctions might become stable, but only on lower dimensional sets, and shall thus not be relevant for us.

As our space, we shall thus now consider tuples of  $(d-1)$ -dimensional surfaces on  $\mathbb{T}$ . The energy and metric tensor are given by

$$E(\Sigma) := \sum_{i < j} \sigma_{i,j} \mathcal{H}^{d-1}(\Sigma_{i,j})$$

and

$$\langle V, W \rangle_\Sigma := \sum_{i < j} \frac{1}{\mu_{i,j}} \int_{\Sigma_{i,j}} V_{i,j} W_{i,j} \, d\mathcal{H}^{d-1}.$$

Using a variant of the divergence theorem on surfaces ([Mag12, Thm. 11.8]) and again the computation for the first variation of the perimeter ([Mag12, Thm. 17.5]), we see that multiphase mean curvature has the desired gradient flow structure and that De Giorgi's inequality (1.3) translates to

$$E(\Sigma(T)) + \frac{1}{2} \sum_{i < j} \int_0^T \int_{\Sigma_{i,j}} \frac{1}{\mu_{i,j}} V_{i,j}^2 + \sigma_{i,j}^2 \mu_{i,j} H_{i,j}^2 \, d\mathcal{H}^{d-1} \, dt \leq E(\Sigma(0)).$$





## 2. The Allen–Cahn equation

### 2.1. Structure of the equation

This chapter follows [LS16], but since the authors decided to only sketch some of the proofs, we want to go into more detail.

Let  $\Lambda > 0$  and define the flat torus  $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$ , where we work with periodic boundary conditions and write  $\int dx$  instead of  $\int_{\mathbb{T}} dx$ . Then for  $u: [0, \infty) \times \mathbb{T} \rightarrow \mathbb{R}^N$  and some potential  $W: \mathbb{R}^N \rightarrow [0, \infty)$ , the *Allen–Cahn equation* with parameter  $\varepsilon > 0$  is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \quad (2.1)$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to  $u$  for a fixed time the real number

$$E_\varepsilon(u) := \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 dx. \quad (2.2)$$

If everything is nice and smooth, we can compute that under the assumption that  $u$  satisfies equation (2.1), we have that

$$\begin{aligned} \frac{d}{dt} E_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u), \partial_t u \rangle + \varepsilon \langle \nabla u, \nabla \partial_t u \rangle dx \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u, \partial_t u \right\rangle dx \\ &= \int -\varepsilon |\partial_t u|^2 dx. \end{aligned} \quad (2.1)$$

This calculation suggests that equation (2.1) is the  $L^2$  gradient flow (rescaled by  $\sqrt{\varepsilon}$ ) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (2.1) via De Giorgis minimizing movements scheme, which we will do in Theorem 2.2.1.

But first we need to clarify what our potential  $W$  should look like. Classic examples in the scalar case are given by  $W(u) = (u^2 - 1)^2$  or  $W(u) = u^2(u - 1)^2$ , and we call functions like these *doublewell potentials*, see also Figure 2.1.

In higher dimensions, we want to accept the following potentials:  $W: \mathbb{R}^N \rightarrow [0, \infty)$  has to be a smooth multiwell potential with finitely many zeros at  $u = \alpha_1, \dots, \alpha_P \in \mathbb{R}^N$ . Furthermore we ask for polynomial growth in the sense that there exists some  $p \geq 2$  such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \quad (2.3)$$

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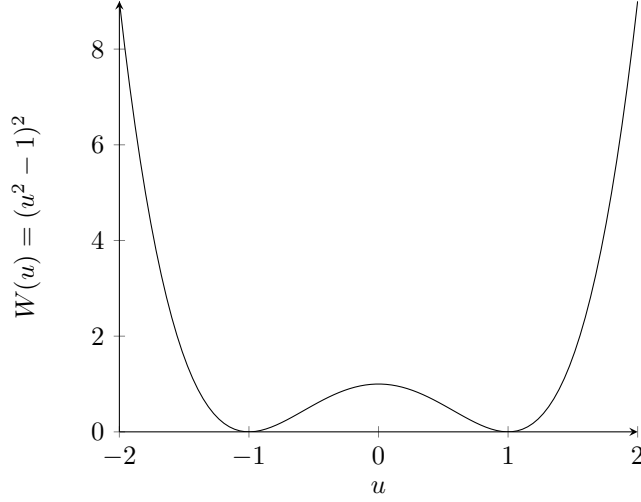


Figure 2.1.: The graph of a doublewell potential

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \quad (2.4)$$

for all  $u$  sufficiently large. Lastly we want  $W$  to be convex up to a small perturbation in the sense that there exist smooth functions  $W_{\text{conv}}, W_{\text{pert}}: \mathbb{R}^N \rightarrow [0, \infty)$  such that

$$W = W_{\text{conv}} + W_{\text{pert}}, \quad (2.5)$$

$W_{\text{conv}}$  is convex and

$$\sup_{x \in \mathbb{R}^N} |\nabla^2 W_{\text{pert}}| < \infty. \quad (2.6)$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen–Cahn equation (2.1) as maps from  $[0, T]$  into some suitable function space and thus use the following definition.

**Definition 2.1.1.** *We say that a function  $u_\varepsilon \in C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  which is also in  $L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$  is a weak solution of the Allen–Cahn equation (2.1) with parameter  $\varepsilon > 0$  and initial condition  $u_\varepsilon^0 \in L^2(\mathbb{T}; \mathbb{R}^N)$  if*

1. *the energy stays bounded, which means that*

$$\text{ess sup}_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t)) < \infty, \quad (2.7)$$

2. *its weak time derivative satisfies*

$$\partial_t u_\varepsilon \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N), \quad (2.8)$$

## 2.2. Existence of a solution

3. for almost every  $t \in [0, T]$  and every  $\xi \in L^p([0, T] \times \mathbb{T}; \mathbb{R}^N) \cap W^{1,2}([0, T] \times \mathbb{T}; \mathbb{R}^N)$ , we have

$$\int \left\langle \frac{1}{\varepsilon^2} \nabla W(u_\varepsilon(t)), \xi \right\rangle + \langle \nabla u_\varepsilon(t), \nabla \xi \rangle + \langle \partial_t u_\varepsilon(t), \xi \rangle dx = 0, \quad (2.9)$$

4. the initial conditions are achieved in the sense that  $u_\varepsilon(0) = u_\varepsilon^0$ .

*Remark.* Given our assumptions, we automatically have  $\nabla W(u_\varepsilon)(t) \in L^{p'}(\mathbb{T})$  since

$$|\nabla W(u_\varepsilon)|^{p/(p-1)} \lesssim 1 + |u_\varepsilon|^p \lesssim 1 + W(u), \quad (2.10)$$

which is integrable for almost every time  $t$  since we assume that the energy stays bounded, thus the integral in equation (2.9) is well defined.

Moreover we already obtain 1/2 Hölder-continuity in time from the embedding

$$W^{1,2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)) \hookrightarrow C^{1/2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)), \quad (2.11)$$

which follows from a generalized version of the fundamental theorem of calculus and Hölder's inequality.

## 2.2. Existence of a solution

With Definition 2.1.1, we are able to state our existence results for solutions of the Allen–Cahn equation. The proof uses De Giorgis minimizing movements scheme and arguments from the theory of gradient flows and has been briefly sketched in [LS16].

**Theorem 2.2.1.** *Let  $u_\varepsilon^0: \mathbb{T} \rightarrow \mathbb{R}^N$  be such that  $E_\varepsilon(u_\varepsilon^0) < \infty$ . Then there exists a weak solution  $u_\varepsilon$  to the Allen–Cahn equation (2.1) in the sense of Definition 2.1.1 with initial data  $u_\varepsilon^0$ . Furthermore the solution satisfies the energy dissipation inequality*

$$E_\varepsilon(u_\varepsilon(t)) + \int_0^t \int \varepsilon |\partial_t u_\varepsilon|^2 dx ds \leq E_\varepsilon(u_\varepsilon^0) \quad (2.12)$$

for every  $t \in [0, T]$  and we additionally have  $\partial_{i,j}^2 u_\varepsilon, \nabla W(u_\varepsilon) \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$  for all  $1 \leq i, j \leq d$ . In particular we can test the weak form (2.9) with  $\partial_{i,j}^2 u_\varepsilon$ .

*Proof. Step 1:* A minimization problem

Fix some  $h > 0$ ,  $u_{n-1} \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$  and consider the functional

$$\begin{aligned} \mathcal{F}: W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N) &\rightarrow \mathbb{R} \\ u &\mapsto E_\varepsilon(u) + \frac{1}{2h} \int \varepsilon |u - u_{n-1}|^2 dx. \end{aligned} \quad (2.13)$$

Then  $\mathcal{F}$  is coercive with respect to  $\|\cdot\|_{W^{1,2}}$  and bounded from below by zero, thus we may take a  $W^{1,2}$ -bounded sequence  $(v_k)_{k \in \mathbb{N}}$  in  $W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$  such that  $\mathcal{F}(v_k) \rightarrow \inf \mathcal{F}$

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as  $k \rightarrow \infty$ . These  $v_k$  have a non-relabelled subsequence which converges weakly in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$  and strongly in  $L^2(\mathbb{T}; \mathbb{R}^N)$  to some  $u \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . Moreover  $v_k$  is bounded in  $L^p(\mathbb{T}; \mathbb{R}^N)$  by the lower growth assumption (2.3) on  $W$  and thus obtain by a duality argument that  $u \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ .

Lastly we have

$$\frac{1}{2h} \int \varepsilon |u - u_{n-1}|^2 dx \leq \liminf_{k \rightarrow \infty} \frac{1}{2h} \int \varepsilon |v_k - u_{n-1}|^2 dx, \quad (2.14)$$

$$\int \frac{\varepsilon}{2} |\nabla u|^2 dx \leq \liminf_{k \rightarrow \infty} \int \frac{\varepsilon}{2} |\nabla v_k|^2 dx \quad (2.15)$$

by the weak convergence in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . By passing to another non-relabelled subsequence which converges pointwise almost everywhere, we moreover achieve by continuity of  $W$  that

$$\int \frac{1}{\varepsilon} W(u) dx = \int \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon} W(v_k) dx \leq \liminf_{k \rightarrow \infty} \int \frac{1}{\varepsilon} W(v_k) dx,$$

which proves that  $u$  is a minimizer of  $\mathcal{F}$ .

### Step 2: Minimizing movements scheme

By iteratively choosing minimizers  $u_n^h$  from Step 1, we obtain a sequence of functions  $u_\varepsilon^0, u_1^h, \dots$ . Thus we may define a function  $u \in C$  as the piecewise linear interpolation at the time-steps  $0, h, 2h, \dots$  of these functions.

### Step 3: Sharp energy dissipation inequality for $u_n^h$

We claim that there exists some constant  $C > 0$  such that for all  $h > 0$  and  $n \in \mathbb{N}$ , we have

$$E_\varepsilon(u_n^h) + \left( \frac{1}{h} - \frac{C}{2\varepsilon^2} \right) \int \varepsilon |u_n^h - u_{n-1}^h|^2 dx \leq E_\varepsilon(u_{n-1}^h). \quad (2.16)$$

In order to prove this inequality, we notice that since  $|\nabla^2 W_{\text{pert}}| \leq C$ , the function  $W_{\text{pert}} + C|u|^2/2$  is convex for  $C > 0$  sufficiently large, thus the functional

$$\tilde{E}_\varepsilon(u) := \int \frac{1}{\varepsilon} \left( W(u) + \frac{C}{2} |u|^2 \right) + \frac{\varepsilon}{2} |\nabla u|^2 dx$$

is convex on  $W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ . For a given  $\xi \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ , we thus have that the function  $t \mapsto \tilde{E}_\varepsilon(u_n^h + t\xi)$  is convex and differentiable, which yields that

$$\tilde{E}_\varepsilon(u_n^h + \xi) \geq \tilde{E}_\varepsilon(u_n^h) + \left. \frac{d}{dt} \right|_{t=0} \tilde{E}_\varepsilon(u_n^h + t\xi). \quad (2.17)$$

## 2.2. Existence of a solution

But since  $u_n^h$  is a minimizer of the functional  $\mathcal{F}$  defined by (2.13), we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \tilde{E}_\varepsilon(u_n^h + t\xi) &= \frac{d}{dt} \Big|_{t=0} \mathcal{F}(u_n^h + t\xi) \\ &+ \frac{C}{2\varepsilon} \int |u_n^h + t\xi|^2 dx - \frac{1}{2h} \int \varepsilon |u_n^h + t\xi - u_{n-1}^h|^2 dx \\ &= \int \frac{C}{\varepsilon} \langle u_n^h, \xi \rangle - \frac{1}{h} \varepsilon \langle u_n^h - u_{n-1}^h, \xi \rangle dx. \end{aligned}$$

Plugging  $\xi = u_{n-1}^h - u_n^h$  into this equation and using inequality (2.17) thus yields

$$\tilde{E}_\varepsilon(u_{n-1}^h) \geq \tilde{E}_\varepsilon(u_n^h) + \int \frac{C}{\varepsilon} \left( \langle u_{n-1}^h, u_n^h \rangle - |u_n^h|^2 \right) + \frac{1}{h} \varepsilon |u_n^h - u_{n-1}^h|^2 dx,$$

which is equivalent to

$$\begin{aligned} E_\varepsilon(u_{n-1}^h) &\geq E_\varepsilon(u_n^h) \\ &+ \int \left( \frac{C}{2\varepsilon} - \frac{C}{\varepsilon} \right) |u_n^h|^2 - \frac{C}{2\varepsilon} |u_{n-1}^h|^2 + \frac{C}{\varepsilon} \langle u_{n-1}^h, u_n^h \rangle + \frac{1}{h} \varepsilon |u_n^h - u_{n-1}^h|^2 dx \\ &= E_\varepsilon(u_n^h) + \left( \frac{1}{h} - \frac{C}{2\varepsilon^2} \right) \int \varepsilon |u_n^h - u_{n-1}^h|^2 dx, \end{aligned}$$

which is the claimed estimate (2.16).

**Step 4:** Hölder bounds for  $u_h$

From the energy estimate (2.16) we deduce via an induction that

$$\begin{aligned} &E_\varepsilon(u_n^h) + \left( 1 - \frac{Ch}{2\varepsilon^2} \right) \int_0^{nh} \int \varepsilon |\partial_t u^h|^2 dx dt \\ &= E_\varepsilon(u_n^h) + \left( h - \frac{Ch^2}{2\varepsilon^2} \right) \sum_{k=1}^n \int \varepsilon \left| \frac{u_n^h - u_{n-1}^h}{h} \right|^2 dx \\ &\leq E_\varepsilon(u_\varepsilon^0). \end{aligned} \tag{2.18}$$

This gives use with the use of Jensen's inequality for  $0 \leq s \leq t \leq T$  and  $h > 0$  sufficiently small that

$$\begin{aligned} \|u^h(t) - u^h(s)\|_{L^2} &= \left\| \int_s^t \partial_t u_h(\tau) d\tau \right\|_{L^2} \\ &\leq \sqrt{t-s} \left( \int_0^T \int |\partial_t u^h(\tau, x)|^2 dx d\tau \right)^{1/2} \\ &\leq \sqrt{t-s} \left( \varepsilon - \frac{Ch}{2\varepsilon} \right)^{-1/2} (E_\varepsilon(u_\varepsilon^0))^{1/2}, \end{aligned} \tag{2.19}$$

which gives us a uniform bound on the  $L^2$ -Hölder continuity of  $u^h$  in time as  $h$  tends to zero.

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### Step 5: Compactness

In order to apply Arzelà–Ascoli for the sequence  $(u_h)$  as  $h$  tends to zero, we need to check pointwise precompactness of the image and equicontinuity of the sequence. The equicontinuity follows from the previous estimate (2.19). In order to check the pointwise precompactness, we need to verify that for all  $t \in [0, T]$ , the set  $\{u^h(t)\}_{\delta > h > 0}$  is a precompact subset of  $L^2(\mathbb{T}; \mathbb{R}^N)$  (for  $\delta > 0$  sufficiently small). But this follows from the energy bound  $E_\varepsilon(u_n^h) \leq E_\varepsilon(u_0)$  (given by inequality (2.16)), which gives us a time-uniform bound on  $\|\nabla u^h(t)\|_{L^2(\mathbb{T}; \mathbb{R}^N)}$  and on  $\|u^h(t)\|_{L^p(\mathbb{T}; \mathbb{R}^N)}$  and thus on  $\|u^h(t)\|_{W^{1,2}(\mathbb{T}; \mathbb{R}^N)}$ . The compact embedding  $W^{1,2}(\mathbb{T}; \mathbb{R}^N) \hookrightarrow L^2(\mathbb{T}; \mathbb{R}^N)$  thus yields the desired pointwise precompactness.

Therefore we may apply Arzelà–Ascoli to obtain some  $u \in C^{1,2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  and some sequence  $h_n \rightarrow 0$  such that  $u^{h_n}$  converges uniformly to  $u$  on  $[0, T]$  with respect to  $\|\cdot\|_{L^2(\mathbb{T}; \mathbb{R}^N)}$ .

### Step 6: Additional regularity 1

We first want to argue that from our construction, one already obtains that  $u \in L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$ . For this we first notice that for a fixed  $t \in [0, T]$ , the sequence  $u^{h_n}(t)$  is by the energy bound (2.16) a bounded sequence in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ , thus we find some non-relabelled subsequence and  $v \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$  such that  $u^{h_n}(t)$  converges weakly to  $v$  in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . By uniqueness of the limit, we already have  $u(t) = v$  almost everywhere, which yields  $u(t) \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ , and by lower semicontinuity, we may also deduce that

$$\|u(t)\|_{W^{1,2}} \leq \liminf_{n \rightarrow \infty} \|u^{h_n}(t)\|_{W^{1,2}} \leq C E_\varepsilon,$$

from which we deduce that  $u \in L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$ .

Secondly the boundedness of the energies

$$\sup_{0 \leq t \leq T} E_\varepsilon(u(t)) < \infty$$

follows from the lower semicontinuity of the energy and the pointwise  $L^2$  convergence and pointwise weak convergence in  $W^{1,2}$  as described in step 1.

Lastly we want to argue that  $\partial_t u \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . From inequality (2.18) in step 4, we deduce that  $\partial_t u^h$  is a bounded sequence in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . Thus we find a non-relabelled subsequence of  $h_n$  and some  $w \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$  such that  $u^{h_n}$  converges weakly to  $w$  in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . But then  $w$  is already the weak time derivative of  $u$  since for any testfunction  $\xi$ , we have

$$\begin{aligned} \int_{[0, T] \times \mathbb{T}} \langle u, \partial_t \xi \rangle dx dt &= \lim_{n \rightarrow \infty} \int_{[0, T] \times \mathbb{T}} \langle u^{h_n}, \partial_t \xi \rangle dx dt \\ &= \lim_{n \rightarrow \infty} - \int_{[0, T] \times \mathbb{T}} \langle \partial_t u^{h_n}, \xi \rangle dx dt \\ &= - \int_{[0, T] \times \mathbb{T}} \langle w, \xi \rangle dx dt. \end{aligned}$$

## 2.2. Existence of a solution

**Step 7:**  $u$  is a weak solution

Going back to step 1, we see that  $u_n^h$  solves the Euler–Lagrange equation

$$\int \frac{1}{\varepsilon^2} \langle \nabla W(u_n^h), \xi \rangle + \langle \nabla u_n^h, \nabla \xi \rangle + \left\langle \frac{u_n^h - u_{n-1}^h}{h}, \xi \right\rangle dx = 0 \quad (2.20)$$

for any testfunction  $\xi \in C^\infty(\mathbb{T}; \mathbb{R}^N)$ . Let  $t \in [0, T]$ . Since  $u_h$  is defined as the pointwise linear interpolation of the functions  $u_n^h$ , we find for the sequence  $h_n$  corresponding sequences  $\lambda_n \in [0, 1]$  and  $k_n \in \mathbb{N}$  such that

$$t = \lambda_n(k_n - 1)h_n + (1 - \lambda_n)k_nh_n$$

and therefore we can write

$$u_h(t) = \frac{k_nh_n - t}{h_n} u_{k_n-1}^h + \frac{t - (k_n - 1)h_n}{h_n} u_{k_n}^{h_n}.$$

In order to pass to the limit in equation (2.20), we first note that  $u_{k_n}^{h_n}$  converges to  $u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  since

$$\begin{aligned} \|u(t) - u_{k_n}^{h_n}\|_{L^2(\mathbb{T}; \mathbb{R}^N)} &\leq \|u(t) - u^{h_n}(t)\|_{L^2(\mathbb{T}; \mathbb{R}^N)} + \|u^{h_n}(t) - u^{h_n}(k_nh_n)\|_{L^2(\mathbb{T}; \mathbb{R}^N)} \\ &\lesssim \|u(t) - u^{h_n}(t)\|_{L^2(\mathbb{T}; \mathbb{R}^N)} + \sqrt{h_n}, \end{aligned} \quad (2.19)$$

which goes to zero as  $n$  tends to infinity. This implies that

$$\begin{aligned} \int \langle \nabla u(t), \nabla \xi \rangle dx &= \int \langle u(t), \operatorname{div} \nabla \xi \rangle dx \\ &= \lim_{n \rightarrow \infty} \int \langle u_{k_n}^{h_n}, \operatorname{div} \nabla \xi \rangle dx \\ &= \lim_{n \rightarrow \infty} \int \langle \nabla u_{k_n}^{h_n}, \nabla \xi \rangle dx. \end{aligned}$$

In step 6, we moreover have shown that  $\partial_t u^h$  converges weakly to  $\partial_t u$  in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . This yields, by choosing cylindrical testfunctions, that  $\partial_t u^h(t)$  converges weakly to  $\partial_t u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  for almost every  $t \in [0, T]$ . Thus we obtain for almost every  $t \in [0, T]$  the convergence

$$\int \langle \partial_t u(t), \xi \rangle dx = \lim_{n \rightarrow \infty} \int \langle \partial_t u(t), \xi \rangle dx = \lim_{n \rightarrow \infty} \int \left\langle \frac{u_{k_n}^{h_n} - u_{k_n-1}^{h_n}}{h_n}, \xi \right\rangle dx.$$

To obtain the weak equation, we still need to prove convergence of remaining term. For this we note that

$$\frac{1}{\varepsilon^2} \left| \left\langle \nabla W(u_{k_n}^{h_n}), \xi \right\rangle \right| \lesssim \left( 1 + |u_{k_n}^{h_n}|^{p-1} \right) |\xi|,$$

## 2. The Allen–Cahn equation

which is a bounded sequence in  $L^{p'}(\mathbb{T}; \mathbb{R}^N)$ . Since we also may pass to a non-relabelled subsequence which converges almost everywhere, we thus obtain that

$$\int \langle \nabla W(u(t)), \xi \rangle dx = \lim_{n \rightarrow \infty} \int \langle \nabla W(u_{k_n}^{h_n}), \xi \rangle dx.$$

Therefore it follows from the Euler–Lagrange equation (2.20) that

$$\int \frac{1}{\varepsilon^2} \langle \nabla W(u(t)), \xi \rangle + \langle \nabla u, \nabla \xi \rangle + \langle \partial_t u, \xi \rangle dx = 0$$

for all  $\xi \in C^\infty(\mathbb{T}; \mathbb{R}^N)$ . By continuity this extends to all  $\xi \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ .

Thus the last thing to check is that

$$\sup_{0 \leq t \leq T} E_\varepsilon(u(t)) < \infty,$$

which follows from the next step.

### Step 8: Sharp energy dissipation inequality for $u$

Let  $0 \leq t \leq T$  and define  $k_n$  as in step 7, where we also established that  $u_{k_n}^{h_n}$  converges to  $u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$ , and thus we may pass to another non-relabelled subsequence to obtain pointwise convergence almost everywhere. By Fatou's Lemma, we thus obtain

$$\int \frac{1}{\varepsilon} W(u(t)) dx \leq \liminf_{n \rightarrow \infty} \int \frac{1}{\varepsilon} W(u_{k_n}^{h_n}) dx.$$

Moreover we can deduce from the  $L^2$ -convergence that  $\nabla u_{k_n}^{h_n}$  converges to  $\nabla u(t)$  in the distributional sense. But  $\nabla u_{k_n}^{h_n}$  is uniformly bounded in  $L^2(\mathbb{T}; \mathbb{R}^N)$  by the energy dissipation inequality (2.16), thus we already obtain that  $\nabla u_{k_n}^{h_n}$  converges weakly to  $\nabla u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  which yields

$$\int \frac{\varepsilon}{2} |\nabla u(t)|^2 dx \leq \liminf_{n \rightarrow \infty} \int \frac{\varepsilon}{2} |\nabla u_{k_n}^{h_n}|^2 dx.$$

Lastly we have by the weak convergence of  $\partial_t u^{h_n}$  to  $\partial_t u$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  proven in step 6 that

$$\begin{aligned} \int_0^t \int \varepsilon |\partial_t u|^2 dx dt &\leq \liminf_{n \rightarrow \infty} \left(1 - \frac{Ch_n}{2\varepsilon^2}\right) \int_0^t \int \varepsilon |\partial_t u^{h_n}|^2 dx dt \\ &\leq \liminf_{n \rightarrow \infty} \left(1 - \frac{Ch_n}{2\varepsilon^2}\right) \int_0^{k_n h_n} \int \varepsilon |\partial_t u^{h_n}|^2 dx dt. \end{aligned}$$

Summarizing these estimates, we obtain by the energy dissipation inequality for  $u^h$  (2.18) that for all  $0 \leq t \leq T$  we have

$$\begin{aligned} E_\varepsilon(u(t)) + \int_0^t \int \varepsilon |\partial_t u|^2 dx dt &\leq \liminf_{n \rightarrow \infty} E_\varepsilon(u_{k_n}) + \left(1 - \frac{Ch_n}{2\varepsilon^2}\right) \int_0^{k_n h_n} \int \varepsilon |\partial_t u^h|^2 dx dt \\ &\leq E_\varepsilon(u_\varepsilon^0). \end{aligned}$$



**Step 8:** Additional regularity 2

In order to complete the proof, we still have to show that  $\partial_{i,j}^2 u$  and  $\nabla W(u)$  are elements of  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . In order to show that the second partial derivatives of  $u$  are square-integrable, we test the weak formulation (2.9) with the finite differences. To this end, define the finite differences as

$$\Delta_h^+ v(t, x) := \frac{v(t, x + hv) - v(t, x)}{h}, \quad \Delta_h^- v(t, x) := \frac{v(t, x - hv) - v(t, x)}{h}$$

for some  $h > 0$  and  $v \in \mathbb{R}^d$ . Thus plugging  $\Delta_h^- \Delta_h^+ u$  into (2.9) yields by the transformation formula that

$$0 = \int_0^T \int \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u), \Delta_h^+ u \rangle + \langle \nabla \Delta_h^+ u, \nabla \Delta_h^+ u \rangle + \langle \partial_t \Delta_h^+ u, \Delta_h^+ u \rangle dx dt,$$

which is equivalent to

$$\begin{aligned} & \int_0^T \int |\Delta_h^+ \nabla u|^2 dx dt \\ &= - \int_0^T \int \partial_t \left( \frac{|\Delta_h^+ u|^2}{2} \right) + \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u), \Delta_h^+ u \rangle dx dt \\ &= \int \frac{|\Delta_h^+ u(0)|^2 - |\Delta_h^+ u(T)|^2}{2} dx \\ &\quad - \int_0^1 \int_0^T \int \langle D^2 W((1-s)u(t, x) + su(t, x + hv)) \Delta_h^+ u, \Delta_h^+ u \rangle dx dt ds. \end{aligned}$$

The first summand can be estimated by  $\|u\|_{L^\infty([0, T], W^{1,2}(\mathbb{T}; \mathbb{R}^N))}$ . For the second summand, we partition  $W$  into the sum of  $W_{\text{conv}}$  and  $W_{\text{pert}}$ . The term involving the convex summand can then be estimated by

$$\int_0^1 \int_0^T \int \langle D^2 W_{\text{conv}}((1-s)u(t, x) + su(t, x + hv)) \Delta_h^+ u, \Delta_h^+ u \rangle dx dt ds \geq 0$$

and for the perturbation term, we get via the bound on its second derivative that

$$\begin{aligned} & \left| \int_0^1 \int_0^T \int \langle D^2 W_{\text{pert}}((1-s)u(t, x) + su(t, x + hv)) \Delta_h^+ u, \Delta_h^+ u \rangle dx dt ds \right| \\ & \lesssim \int_0^T \int |\nabla u|^2 dx dt, \end{aligned}$$

which is also finite. Combining these estimates, we obtain that  $\int_0^T \int |\Delta_h^+ \nabla u|^2 dx dt$  is uniformly bounded in  $h$ . Applying our calculation to all directions  $v \in \mathbb{R}^d$ , we get by the finite-differences theorem for all  $1 \leq i, j \leq d$  that  $\partial_{i,j}^2 u \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ .

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In order to obtain  $\nabla W(u) \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ , we again consider the weak formulation (2.9) and notice that since we have already shown that both the time derivative and second space derivatives of  $u$  are square-integrable, our claim follows from a duality argument.  $\square$

*Remark.* The inequality (2.16) with the factor  $1/2h$  instead of  $1/h - C/2\varepsilon^2$  follows immediately from the definition of our optimization problem, but is not optimal for fixed  $\varepsilon$  if we want to study the behaviour as  $h$  tends to zero. Moreover this so called *sharp energy dissipation inequality* is important for later.

*Remark.* The energy dissipation inequality (2.12) can be deduced via the formal calculation

$$\begin{aligned} \frac{d}{dt} E_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u), \partial_t u \rangle + \varepsilon \langle \nabla u, \nabla \partial_t u \rangle \, dx \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u, \partial_t u \right\rangle \, dx \\ &= -\varepsilon \int |\partial_t u_\varepsilon|^2 \, dx. \end{aligned}$$

In order to make this calculation rigorous, we however need to show that  $t \mapsto E_\varepsilon(u(t))$  is absolutely continuous, which is non-trivial, but it would give us equality in energy dissipation inequality (2.12). Since we will only need the inequality, our proof will however suffice.

## 3. Conditional convergence of the Allen–Cahn equations

The main content of this chapter is to look at the behaviour of solutions of the Allen–Cahn equation (2.1) as the parameter  $\varepsilon$  tends to zero. As it turns out the scalar case is significantly easier to handle than the vectorial case, thus we shall first focus on the case  $N = 1$ , also called the *twophase case*.

### 3.1. Conditional convergence in the twophase case

#### 3.1.1. Convergence to an evolving set

When want to consider the behaviour of solutions  $u_\varepsilon : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$  to (2.1) for  $\varepsilon \rightarrow 0$ . Let us for now assume that the energies of the initial functions  $E_\varepsilon(u_\varepsilon^0)$  stay uniformly bounded as  $\varepsilon$  tends to zero. Then due to the energy dissipation inequality (2.12), we already obtain that for all  $0 \leq t \leq T$ , we have that  $E_\varepsilon(u_\varepsilon(t))$  stays uniformly bounded.

Another important observation for the convergence is the classic Modica Mortula trick (hier Referenz einfügen): let  $\alpha < \beta$  be the two distinct zeros of the doublewell potential  $W$ . Then we define a primitive of  $\sqrt{2W(u)}$  via

$$\phi(u) := \int_\alpha^u \sqrt{2W(s)} \, ds.$$

For  $\psi_\varepsilon := \phi \circ u_\varepsilon$ , we can show that  $\psi_\varepsilon \in W^{1,1}((0, T) \times \mathbb{T})$  with weak derivatives  $\nabla \psi_\varepsilon = \sqrt{2W(u_\varepsilon)} \nabla u_\varepsilon$  and  $\partial_t \psi_\varepsilon = \sqrt{2W(u_\varepsilon)} \partial_t u_\varepsilon$ . (we will later show this in more generality (Referenz einfügen)).

Thus via Young’s inequality, we can compute

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &= \int \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \, dx \\ &\geq \int \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx \\ &= \int |\nabla \psi_\varepsilon| \, dx, \end{aligned} \tag{3.1}$$

which suggests that we might hope for good compactness properties of  $\phi \circ u_\varepsilon$ . We combine these two observations into the following Proposition.

### 3. Conditional convergence of the Allen–Cahn equations

**Proposition 3.1.1.** *Given initial data  $u_\varepsilon^0$  whose energies stay uniformly bounded in the sense that*

$$\sup_{\varepsilon > 0} E_\varepsilon(u_\varepsilon^0) < \infty, \quad (3.2)$$

*there exists for any sequence  $\varepsilon \rightarrow 0$  some non-relabelled subsequence such that the solutions of the Allen–Cahn equation (2.1) with initial condition  $u_\varepsilon^0$  converge in  $L^1((0, T) \times \mathbb{T})$  to some  $u = \alpha(1 - \chi) + \beta\chi$  with  $\chi \in \text{BV}((0, T) \times \mathbb{T}; \{0, 1\})$ . Moreover the compositions  $\psi_\varepsilon$  are uniformly bounded in  $\text{BV}((0, T) \times \mathbb{T})$  and converge to  $\phi \circ u$  in  $L^1((0, T) \times \mathbb{T})$ .*

*Proof.* From the energy dissipation inequality (2.12) in Theorem 2.2.1 we infer that for all  $\varepsilon > 0$ , it holds that

$$\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t)) \leq E_\varepsilon(u_\varepsilon^0), \quad (3.3)$$

whose right hand side is by assumption uniformly bounded in  $\varepsilon$ . We want to use a similar calculation as (3.1) we thus infer that  $\nabla \psi_\varepsilon$  is uniformly bounded in  $L^1((0, T) \times \mathbb{T})$ . Moreover we may estimate

$$\begin{aligned} \int_0^T \int |\psi_\varepsilon| \, dx \, dt &= \int_0^T \int \left| \int_\alpha^{u_\varepsilon} \sqrt{2W(s)} \, ds \right| \, dx \, dt \\ &\leq \int_0^T \int |u_\varepsilon - \alpha| \sup_{s \in [\alpha, u_\varepsilon]} \sqrt{2W(s)} \, dx \, dt \\ &\lesssim 1 + \int_0^T \int |u_\varepsilon|^{1+p/2} \, dx \, dt \\ &\lesssim 1 + \int_0^T \int W(u_\varepsilon) \, dx \, dt, \end{aligned} \quad (3.4)$$

which is uniformly bounded via the energy bound. Here the last two inequalities follow from the upper respectively lower growth assumptions (2.3) on  $W$ . Thus we have indeed that  $\psi_\varepsilon$  is a bounded sequence in  $\text{BV}((0, T) \times \mathbb{T})$  and therefore there exists some non-relabelled subsequence and some  $\psi \in \text{BV}((0, T) \times \mathbb{T})$  such that  $\psi_\varepsilon$  converges to  $\psi$  in  $L^1((0, T) \times \mathbb{T})$ .

We notice that since  $W$  is non-negative and only has a discrete set of zeros, the function  $\phi$  is strictly increasing and continuous on  $\mathbb{R}$ , and is thus invertible. Moreover we may pass to a further non-relabelled subsequence of  $\psi_\varepsilon$  which converges almost everywhere to  $\psi$ . Thus defining  $u := \phi^{-1}(\psi)$ , we obtain that

$$u_\varepsilon = \phi^{-1}(\psi_\varepsilon) \rightarrow \phi^{-1}(\psi) = u$$

converges pointwise almost everywhere.

Moreover we notice that by Fatou's Lemma and the boundedness of the energies that

$$\int W(u) \, dx \leq \liminf_{\varepsilon \rightarrow 0} \int W(u_\varepsilon) \, dx \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon E_\varepsilon(u_\varepsilon) = 0.$$

Again using the non-negativity of  $W$ , this yields that  $W(u) = 0$  almost everywhere. Thus  $u \in \{\alpha, \beta\}$  almost everywhere and we may write  $u = \alpha(1 - \chi) + \beta\chi$  for some

### 3.1. Conditional convergence in the twophase case

$\chi: (0, T) \times \mathbb{T} \rightarrow \{0, 1\}$ . Looking again at the definition of  $u$ , we moreover obtain that

$$\psi = \phi(u) = \phi(\alpha)(1 - \chi) + \phi(\beta)\chi = \int_{\alpha}^{\beta} \sqrt{2W(s)} \, ds \, \chi =: \sigma \chi \quad (3.5)$$

and since  $\psi$  is a function of bounded variation, this implies that  $\chi$  is of bounded variation as well.

Finally from the energy bound and the estimate  $|u_{\varepsilon}| \lesssim 1 + W(u)$ , we infer that  $u_{\varepsilon}$  is  $L^p$ -bounded, and since  $u_{\varepsilon}$  converges pointwise almost everywhere to  $u$ , we obtain the desired  $L^1$ -convergence.  $\square$

Nextup, we want to make sure that  $u$  respectively  $\chi$  assume their initial data and on the way obtain a useful bound on the time derivative.

**Lemma 3.1.2.** *With the assumptions of Proposition 3.1.1, we have  $\psi_{\varepsilon} \in W^{1,2}([0, T]; L^1(\mathbb{T}))$  with the estimate*

$$\left( \int_0^T \left( \int |\partial_t \psi_{\varepsilon}| \, dx \right)^2 \, dt \right)^{1/2} \lesssim E_{\varepsilon}(u_{\varepsilon}^0). \quad (3.6)$$

Furthermore the sequence  $u_{\varepsilon}$  is precompact in  $C([0, T]; L^2(\mathbb{T}))$ .

*Proof. Step 1:*  $\psi_{\varepsilon} \in W^{1,2}([0, T]; L^1(\mathbb{T}))$  and satisfies the inequality (3.6)

From estimate (3.4) we can infer that

$$\int_0^T \left( \int |\psi_{\varepsilon}| \, dx \right)^2 \lesssim \int_0^T \left( 1 + \int W(u_{\varepsilon}) \, dx \right)^2 \, dt < \infty$$

via the uniform boundedness of the energies (3.3). For the desired bound (3.6), we estimate via Hölder's inequality and the uniform boundedness of the energies that

$$\begin{aligned} \int_0^T \left( \int |\partial_t \psi_{\varepsilon}| \, dx \right)^2 \, dt &\leq \int_0^T \left( \int \sqrt{2W(u_{\varepsilon})} |\partial_t u_{\varepsilon}| \, dx \right)^2 \, dt \\ &= \int_0^T \left( \int \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_{\varepsilon})} \sqrt{\varepsilon} |\partial_t u_{\varepsilon}| \, dx \right)^2 \, dt \\ &\leq \int_0^T \int \frac{1}{\varepsilon} 2W(u_{\varepsilon}) \, dx \int \varepsilon |\partial_t u_{\varepsilon}|^2 \, dx \, dt \\ &\leq 2 E_{\varepsilon}(u_{\varepsilon}^0) \int_0^T \int \varepsilon |\partial_t u_{\varepsilon}|^2 \, dx \, dt \quad (3.3) \\ &\leq 2 (E_{\varepsilon}(u_{\varepsilon}^0))^2, \quad (2.12) \end{aligned}$$

which completes step 1.

**Step 2:** The sequence  $\psi_{\varepsilon}$  is precompact in  $C([0, T]; L^1(\mathbb{T}))$ .

As noted in embedding (2.11), we have

$$W^{1,2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)) \hookrightarrow C^{1/2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$$

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from which the equicontinuity of the sequence follows with step 1. Moreover for fixed time  $t$ , estimate (3.1) yields that  $\nabla\psi_\varepsilon(t)$  is bounded in  $L^1(\mathbb{T})$ , and combined estimate (3.4) without integrating in time yields that  $\psi_\varepsilon(t)$  is a bounded sequence in  $W^{1,1}(\mathbb{T})$  and thus precompact in  $L^1(\mathbb{T})$ . Thus the Arzelà–Ascoli Theorem yields the desired precompactness.

**Step 3:** The sequence  $u_\varepsilon$  converges to  $u$  in measure uniformly in time.

From step 2, we infer that we already have

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int |\psi_\varepsilon(t, x) - \phi \circ u(t, x)| \, dx = 0.$$

It especially follows in combination with equation (3.5) that

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int (1 - \chi) |\psi_\varepsilon(t, x)| \, dx = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int (1 - \chi) |\psi_\varepsilon(t, x) - \phi \circ u(t, x)| \, dx = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int \chi |\psi_\varepsilon(t, x) - \sigma| \, dx = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int \chi |\psi_\varepsilon(t, x) - \phi \circ u(t, x)| \, dx = 0$$

Moreover we have by continuity and the  $p$ -growth of  $W$  (2.3) that for a given  $\delta < \beta - \alpha$ ,

$$\min\{|\phi(t)| : |t - \alpha| > \delta/2 \text{ or } |t - \beta| > \delta/2\} = \rho > 0.$$

Therefore we may estimate

$$\begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T} \mathcal{L}^d(\{x \in \mathbb{T} : |u_\varepsilon(t, x) - u(t, x)| > \delta\}) \\ & \leq \operatorname{ess\,sup}_{0 \leq t \leq T} \mathcal{L}^d(\{(1 - \chi(t, x))|u_\varepsilon(t, x) - \alpha| > \delta/2\}) + \mathcal{L}^d(\{\chi(t, x)|u_\varepsilon(t, x) - \beta| > \delta/2\}) \\ & \leq \operatorname{ess\,sup}_{0 \leq t \leq T} \mathcal{L}^d(\{(1 - \chi(t, x))|\psi_\varepsilon(t, x)| > \rho\}) + \mathcal{L}^d(\{\chi(t, x)|\psi_\varepsilon(t, x) - \sigma| > \rho\}) \\ & \leq \operatorname{ess\,sup}_{0 \leq t \leq T} \frac{1}{\rho} \int (1 - \chi(t, x)) |\psi_\varepsilon(t, x)| \, dx + \frac{1}{\rho} \int \chi(t, x) |\psi_\varepsilon(t, x) - \sigma| \, dx, \end{aligned}$$

which goes to zero as  $\varepsilon$  goes to zero, proving our claim.

**Step 4:**  $u_\varepsilon^2$  is equiintegrable uniformly in time

We have

$$0 \leq u_\varepsilon^2 \lesssim 1 + W(u_\varepsilon)$$

by the growth bounds (2.3) on  $W$ . Since  $W(u_\varepsilon)$  converges to 0 in  $L^1(\mathbb{T})$  uniformly in time by the energy bound (3.3), it is equiintegrable uniformly in time. Thus  $u_\varepsilon^2$  is equiintegrable uniformly in time as well.

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**Step 5:**  $u_\varepsilon$  converges in  $C([0, T]; L^2(\mathbb{T}))$ .

We somewhat repeat the proof the convergence in measure and integrability imply  $L^1$ -convergence, while keeping the uniformity in time.

Take some  $\delta > 0$ . Then we decompose the integral

$$\int |u_\varepsilon - u|^2 dx = \int_{\{|u_\varepsilon - u| \geq \delta\}} |u_\varepsilon - u|^2 dx + \int_{\{|u_\varepsilon - u| < \delta\}} |u_\varepsilon - u|^2 dx.$$

For the first summand, we notice that

$$\sup_{0 \leq t \leq T} \int_{\{|u_\varepsilon - u| \geq \delta\}} |u_\varepsilon - u|^2 dx \lesssim \sup_{0 \leq t \leq T} \int_{\{|u_\varepsilon - u| \geq \delta\}} 1 + u_\varepsilon^2 dx \rightarrow 0$$

as  $\varepsilon \rightarrow 0$  since  $\mathcal{L}^d(|u_\varepsilon - u| \geq \delta) \rightarrow 0$  uniformly in time by step 3 and  $u_\varepsilon^2 + 1$  is equiintegrable uniformly in time by step 4.

For the second summand, we simply estimate

$$\sup_{0 \leq t \leq T} \int_{\{|u_\varepsilon - u| < \delta\}} |u_\varepsilon - u|^2 dx = \delta^2 \Lambda^d$$

Taking the limes superior as  $\varepsilon$  tends to zero of this inequality yields that the right hand side can be made arbitrarily small, which yields

$$\sup_{0 \leq t \leq T} \int |u_\varepsilon - u|^2 dx \rightarrow 0$$

as  $\varepsilon$  tends to zero. □

*Remark.* From the previous Lemma, it follows that if the initial conditions  $u_\varepsilon^0$  converge in  $L^1$  or pointwise almost everywhere to the function  $\alpha(1 - \chi^0) + \beta\chi^0$  (and we know that the limit is of this form since the energies of the initial values stay bounded), then  $u$  also assumes this initial condition in  $L^2(\mathbb{T})$ .

#### 3.1.2. The energy convergence assumption

As often in the Calculus of Variations, convergence of energies boost our modulus of convergence and gives our limit therefore additional regularity. Thus let us assume

$$\int_0^T E_\varepsilon(u_\varepsilon) dt \rightarrow \int_0^T E(u) dt \text{ as } \varepsilon \rightarrow 0, \quad (3.7)$$

where the surface tension energy is defined for  $u = \alpha(1 - \chi) + \beta\chi$  by

$$E(u) := \sigma \int |\nabla \chi| \quad (3.8)$$

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already motivated in Section 1.2. Moreover we notice that the energy of  $u$  is exactly the total variation of  $\psi = \sigma\chi$ . Since for almost every time  $t$ , we have that  $\psi_\varepsilon(t)$  converges to  $\psi(t)$  in  $L^1(\mathbb{T})$  by Proposition 3.1.1, it follows from the lower semicontinuity of the variation measure and Young’s inequality that

$$E(u) \leq \liminf_{\varepsilon \rightarrow 0} \int |\nabla \psi_\varepsilon| \, dx \leq \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon).$$

Moreover by the energy dissipation inequality (2.12), the energies stay uniformly bounded in time. Thus using the dominated convergence theorem, we see that our time integrated energy convergence assumption is equivalent to saying that for almost every time  $t \in (0, T)$ , we have convergence of the energies  $E_\varepsilon(u_\varepsilon(t)) \rightarrow E(u(t))$ .

Since the energies themselves can be interpreted as measures on the flat torus, we define for a continuous function  $\varphi \in C(\mathbb{T})$  the corresponding energy measures by

$$\begin{aligned} E_\varepsilon(u_\varepsilon; \varphi) &:= \int \varphi \left( \frac{1}{\varepsilon} + \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \right) \, dx \quad \text{and} \\ E(u; \varphi) &:= \sigma \int \varphi |\nabla \chi|. \end{aligned}$$

Then from the energy convergence and lower semicontinuity just discussed, it follows that

$$\lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \varphi) = E(u; \varphi). \tag{3.9}$$

A first additional regularity result under the energy convergence assumption is the following Proposition, which ensures that we have square-integrable normal velocities.

**Proposition 3.1.3.** *In the setting of Proposition 3.1.1 and given the energy convergence assumption (3.7), the measure  $\partial_t \chi$  is absolutely continuous with respect to the measure  $|\nabla \chi| \, dt$  and the corresponding density  $V$  is square integrable with the estimate*

$$\int_0^T \int V^2 |\nabla \chi| \, dt \lesssim E^0,$$

where  $E^0 := \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon^0)$ .

*Proof.* Take a smooth test function  $\varphi \in C_c^\infty((0, T) \times \mathbb{T})$ . Then via the  $L^1$ -convergence



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of  $\psi_\varepsilon \rightarrow \psi$ , we have

$$\begin{aligned}
\partial_t \psi(\varphi) &= \liminf_{\varepsilon \rightarrow 0} \partial_t \psi_\varepsilon(\varphi) \\
&= \liminf_{\varepsilon \rightarrow 0} \int_0^T \int \sqrt{2W(u_\varepsilon)} \partial_t u_\varepsilon \varphi \, dx \, dt \\
&\leq \liminf_{\varepsilon \rightarrow 0} \left( \int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon) \varphi^2 \, dx \, dt \right)^{1/2} \left( \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt \right)^{1/2} \\
&\leq \liminf_{\varepsilon \rightarrow 0} \left( 2 \int_0^T E_\varepsilon(u_\varepsilon; \varphi^2) \, dt \right)^{1/2} (E_\varepsilon(u_\varepsilon))^{1/2} \tag{2.12} \\
&= \sqrt{2\sigma} \|\varphi\|_{L^2(\mathbb{T}, |\nabla \chi| dt)} \sqrt{E^0}. \tag{3.9}
\end{aligned}$$

This proves both the absolute continuity and via a duality argument the desired bound since  $\partial_t \psi = \sigma \partial_t \chi$  and  $\sigma > 0$ .  $\square$

We finish this section with a proof for the equipartition of the energy, which tells us that both the summand involving the potential  $W(u_\varepsilon)$  and the norm of the gradient contribute to the energy in similar parts.

**Lemma 3.1.4.** *Under the energy convergence assumption (3.7), we have for any continuous function  $\varphi \in C^\infty(\mathbb{T})$  that*

$$\begin{aligned}
E(u; \varphi) &= \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \varphi) = \lim_{\varepsilon \rightarrow 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx \\
&= \lim_{\varepsilon \rightarrow 0} \int \varphi \varepsilon |\nabla u_\varepsilon|^2 \, dx \\
&= \lim_{\varepsilon \rightarrow 0} \int \varphi \frac{1}{\varepsilon} 2W(u_\varepsilon) \, dx
\end{aligned}$$

for almost every time  $0 \leq t \leq T$ .

*Proof.* We have already established the first equality before. For the second equality, we first assume that  $\varphi \in C(\mathbb{T}; [0, \infty))$ . By the lower semicontinuity of the variation measure, we immediately obtain

$$\liminf_{\varepsilon \rightarrow 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx \geq E(u; \varphi).$$

But by Young's inequality, we also have

$$\limsup_{\varepsilon \rightarrow 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx \leq \limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \varphi) = E(u; \varphi).$$

For general  $\varphi \in C(\mathbb{T})$ , we decompose  $\varphi$  into its positive and negative part and apply the previous argument to both in order to get the claim.

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The third and fourth inequality follow for a given non-negative  $\varphi \in C(\mathbb{T}; [0, \infty))$  by the  $L^2$  estimate

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int \left| \sqrt{\varphi} \sqrt{\varepsilon} |\nabla u_\varepsilon| - \sqrt{\varphi} \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right|^2 dx \\ &= \lim_{\varepsilon \rightarrow 0} 2 E_\varepsilon(u_\varepsilon; \varphi) - 2 \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx = 0 \end{aligned}$$

which implies

$$\lim_{\varepsilon \rightarrow 0} \int \varphi \varepsilon |\nabla u_\varepsilon|^2 dx = \lim_{\varepsilon \rightarrow 0} \int \varphi \frac{1}{\varepsilon} 2W(u_\varepsilon) dx = \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \varphi),$$

finishing our proof.  $\square$

#### 3.1.3. Convergence to twophase mean curvature flow

We start by defining a BV-formulation for motion by mean curvature.

**Definition 3.1.5.** Fix some finite time horizon  $T < \infty$  and initial data  $\chi^0 \in \text{BV}(\mathbb{T}; \{0, 1\})$ , we say that

$$\chi \in C([0, T]; L^2(\mathbb{T}; \{0, 1\}))$$

with  $\text{ess sup}_{0 \leq t \leq T} E(\chi)$  moves by mean curvature if there is a normal velocity  $V \in L^2(|\nabla \chi| dt)$  such that

1. For all  $\xi \in C_c^\infty((0, T) \times \mathbb{T}; \mathbb{R}^d)$ , we have

$$\int_0^T \int V \langle \xi, \nu \rangle - \langle D\xi, \text{Id} - \nu \otimes \nu \rangle |\nabla \chi| dt = 0, \quad (3.10)$$

where  $\nu := \nabla \chi / |\nabla \chi|$  is the outer unit normal.

2. The function  $V$  is the normal velocity of  $\chi$  in the sense that

$$\partial_t \chi = V |\nabla \chi| dt$$

holds distributionally in  $(0, T) \times \mathbb{T}$ .

3. The initial data  $\chi^0$  is achieved in  $C([0, T]; L^2(\mathbb{T}))$ , which simply means that  $\chi(0) = \chi^0$  as functions in  $L^2(\mathbb{T})$ .

Our main goal in this section is now to show that the function  $\chi$  we have found in Proposition 3.1.1 moves by mean curvature. Thus our goal is to prove the following Theorem.

**Theorem 3.1.6.** Let a smooth doublewell potential  $W: \mathbb{R} \rightarrow [0, \infty)$  satisfy the assumptions (2.3)-(2.6). Let  $T < \infty$  be an arbitrary finite time horizon. Given a sequence of initial data  $u_\varepsilon^0: \mathbb{T} \rightarrow \mathbb{R}$  such that  $u_\varepsilon^0 \rightarrow u^0 = (1 - \chi^0)\alpha + \chi^0\beta$  and  $\limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon^0) < \infty$ ,

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we have that for some subsequence of solutions to (2.1)  $u_\varepsilon$ , there exists a pointwise almost everywhere limit  $u = (1 - \chi)\alpha + \chi\beta$  with  $\chi \in \text{BV}((0, T) \times \mathbb{T}; \{0, 1\})$  which assumes the initial data in  $C([0, T]; L^2(\mathbb{T}))$ . If we additionally assume that the time-integrated energies converge (3.7), then  $\chi$  moves by mean curvature in the sense of Definition 3.1.5.

Looking at equation (2.9) which reads

$$\int \frac{1}{\varepsilon^2} W'(u_\varepsilon(t)) \varphi + \nabla u_\varepsilon(t) \nabla \varphi + \partial_t u_\varepsilon(t) \varphi \, dx = 0, \quad (2.9)$$

we expect that for a suitable choice of testfunctions  $\varphi_\varepsilon$ , the following two terms converge:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \int \partial_t u_\varepsilon \varphi_\varepsilon &= \sigma \int_0^T \int V\langle \xi, \nu \rangle |\nabla \chi| \, dt, \\ \lim_{\varepsilon \rightarrow 0} \int_0^T \int \frac{1}{\varepsilon^2} W'(u_\varepsilon) \varphi_\varepsilon + \langle \nabla u_\varepsilon, \nabla \varphi_\varepsilon \rangle \, dx \, dt &= \sigma - \int_0^T \int \langle D\xi, \text{Id} - \nu \otimes \nu \rangle |\nabla \chi| \, dt. \end{aligned}$$

But how do we find these testfunctions? For this, we first note that the curvature term  $\int \langle D\xi, \text{Id} - \nu \otimes \nu \rangle |\nabla \chi| \, dt$  is by [Mag12, Thm. 17.5] the first inner variation with respect to  $\xi$  of the perimeter functional, which is just our energy  $E$  up to the surface tension constant  $\sigma > 0$ . Thus it is plausible to compute the first inner variation  $\frac{d}{ds} \Big|_{s=0} E(\rho_s)$  and then we can hopefully choose the testfunction  $\varphi_\varepsilon$  in such a way that it equals  $\int \frac{1}{\varepsilon^2} W'(u_\varepsilon(t)) \varphi + \nabla u_\varepsilon(t) \nabla \varphi \, dx$ .

Thus let  $(\rho_s)_s$  be functions which solve the ODE

$$\begin{cases} \partial_s \rho_s + \langle \xi, \nabla \rho_s \rangle &= 0 \\ \rho_0 &= u_\varepsilon. \end{cases}$$

Then we formally compute

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \int \frac{\varepsilon}{2} |\rho_s|^2 + \frac{1}{\varepsilon} W(\rho_s) \, dx &= \int \varepsilon \langle \nabla u_\varepsilon, \nabla (-\langle \xi, \nabla u_\varepsilon \rangle) \rangle + \frac{1}{\varepsilon} W'(u_\varepsilon) (-\langle \xi, \nabla u_\varepsilon \rangle) \, dx \\ &= \int \left( \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle \, dx. \end{aligned}$$

We therefore test equation (2.9) against  $\varphi_\varepsilon := \langle \xi, \nabla u_\varepsilon \rangle$ .

#### 3.1.4. Convergence of the curvature term

The goal of this section is to prove the convergence

$$\lim_{\varepsilon \rightarrow 0} \int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle \, dx = \sigma \int \langle D\xi, \text{Id} - \nu \otimes \nu \rangle |\nabla \chi|$$

for almost every time  $t$ . We directly follow the proof from Luckhaus and Modica in [LM89].

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Via an integration by parts, we obtain

$$\begin{aligned} \int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle &= \int -\varepsilon \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \left( \partial_{x_i} \xi^j \partial_{x_j} u_\varepsilon + \xi^j \partial_{x_i x_j}^2 u_\varepsilon \right) \\ &\quad + \frac{1}{\varepsilon} W(u_\varepsilon) \operatorname{div} \xi \, dx. \end{aligned} \quad (3.11)$$

Moreover by another integration by parts, we have

$$\begin{aligned} \int \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \xi^j \partial_{x_i x_j}^2 u_\varepsilon \, dx &= \int - \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \left( \partial_{x_i x_j}^2 u_\varepsilon \xi^j + \partial_{x_j} \xi^j \partial_{x_i} u_\varepsilon \right) \, dx \\ &= \int -|\nabla u_\varepsilon|^2 \operatorname{div} \xi - \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \partial_{x_i x_j}^2 u_\varepsilon \xi^j \, dx \end{aligned}$$

which is equivalent to

$$\int \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \xi^j \partial_{x_i x_j}^2 u_\varepsilon \, dx = -\frac{1}{2} \int |\nabla u_\varepsilon|^2 \operatorname{div} \xi \, dx.$$

Plugging this equation into the first equation (3.11), we obtain

$$\begin{aligned} &\int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle \, dx \\ &= \int -\varepsilon \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \partial_{x_j} u_\varepsilon \partial_{x_i} \xi^j + \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \operatorname{div} \xi + \frac{1}{\varepsilon} W(u_\varepsilon) \operatorname{div} \xi \, dx \\ &= \varepsilon \int |\nabla u_\varepsilon|^2 \operatorname{div} \xi - \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \partial_{x_j} u_\varepsilon \partial_{x_i} \xi^j \, dx \\ &\quad + \int \frac{1}{\varepsilon} W(u_\varepsilon) \operatorname{div} \xi - \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \operatorname{div} \xi \, dx. \end{aligned} \quad (3.12)$$

The last integral goes to zero by the equipartition of the energies (Lemma 3.1.4). Since  $\partial_{x_i} u_\varepsilon / |\nabla u_\varepsilon| = \partial_{x_i} \psi_\varepsilon / |\nabla \psi_\varepsilon|$  by the chain rule, the former integral can be written as

$$\int_{\mathbb{T}_\varepsilon} g(x, \nabla \psi_\varepsilon) \varepsilon |\nabla u_\varepsilon|^2 \, dx,$$

where

$$g(x, p) = \begin{cases} \sum_{i,j=1}^d -\frac{p_i}{|p|} \partial_{x_i} \xi^j \frac{p_j}{|p|} + \operatorname{div} \xi & \text{if } p \neq 0, \\ 0 & \text{else,} \end{cases}$$

and the set  $\mathbb{T}_\varepsilon$  is defined as

$$\mathbb{T}_\varepsilon := \{x \in \mathbb{T} : \nabla \psi_\varepsilon(x) \neq 0\} = \{x \in \mathbb{T} : \nabla u_\varepsilon(x) \neq 0\} \cap \{x \in \mathbb{T} : u_\varepsilon(x) \notin \{\alpha, \beta\}\}.$$

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For the representation (3.1.4), we also have to use that

$$\mathcal{L}^d(\{x \in \mathbb{T} : \nabla u_\varepsilon(x) \neq 0 \text{ and } u_\varepsilon(x) \in \{\alpha, \beta\}\}) = 0.$$

Again by the equipartition of energies (Lemma 3.1.4) and the boundedness of  $g$ , we can replace  $\varepsilon|\nabla u_\varepsilon|^2$  by  $\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|$  in the integral (3.1.4) via the estimate

$$\begin{aligned} & \int_{\mathbb{T}_\varepsilon} \left| \varepsilon|\nabla u_\varepsilon|^2 - \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| \right| dx \\ & \leq \left( \int_{\mathbb{T}_\varepsilon} \left| \sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)} \right|^2 dx \right)^{1/2} \left( \int_{\mathbb{T}_\varepsilon} \varepsilon|\nabla u_\varepsilon|^2 dx \right)^{1/2} \\ & \leq \left( \int_{\mathbb{T}_\varepsilon} \varepsilon|\nabla u_\varepsilon|^2 - 2\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| + \frac{1}{\varepsilon}2W(u_\varepsilon) dx \right) \sqrt{2E_\varepsilon(u_\varepsilon)} \end{aligned}$$

which vanishes as  $\varepsilon$  tends to zero by Lemma 3.1.4. Thus

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}_\varepsilon} g(x, \nabla \psi_\varepsilon) \varepsilon|\nabla u_\varepsilon|^2 dx &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}_\varepsilon} g(x, \nabla \psi_\varepsilon) \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}_\varepsilon} g(x, \nabla \psi_\varepsilon) |\nabla \psi_\varepsilon| dx \\ &= \lim_{\varepsilon \rightarrow 0} \int F(x, \nabla \psi_\varepsilon) dx, \end{aligned}$$

where  $F(x, p)$  is defined as  $g(x, p)|p|$  at points  $p$  not equal to 0, and defined as 0 elsewhere. Since  $F(x, \lambda p) = \lambda F(x, p)$  for positive  $\lambda$  and since  $F$  satisfies the periodic boundary conditions in  $x$ , we are in the position to apply a Theorem proven by Reshetnyak in [Res68] and again by Luckhaus and Modica in [LM89]. We will later see a quantitative version of this in ???. Here it yields that since  $|\nabla \psi_\varepsilon|(\mathbb{T}) \rightarrow |\nabla \psi|(\mathbb{T})$  by the equipartition of energies Lemma 3.1.4, we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int F(x, \nabla \psi_\varepsilon) dx &= \sigma \int F(x, \nu) |\nabla \chi| \\ &= \sigma \int \left( \sum_{i,j=1}^d -\frac{\nu_i}{|\nu|} \partial_{x_i} \xi^j \frac{\nu_j}{|\nu|} + \operatorname{div} \xi \right) |\nu| |\nabla \chi| \\ &= \sigma \int \langle D\xi, \operatorname{Id} - \nu \otimes \nu \rangle |\nabla \chi|, \end{aligned}$$

which finishes the proof. The time integrated version given by

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle dx dt = \sigma \int_0^T \int \langle D\xi, \operatorname{Id} - \nu \otimes \nu \rangle |\nabla \chi| dt$$

follows from the generalized dominated convergence theorem via the equality (3.12) which yields

$$\left| \int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle dx \right| \lesssim E_\varepsilon(u_\varepsilon).$$

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#### 3.1.5. Convergence of the velocity term

We now want to prove the convergence of the velocity term given by

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int \partial_t u_\varepsilon \langle \xi, \varepsilon \nabla u_\varepsilon \rangle dx dt = \sigma \int_0^T \int V \langle \xi, \nu \rangle |\nabla \chi| dt.$$

The difficulty here is that products of weakly converging sequences will in general not weakly converge. To be more precise, we only have  $\partial_t u_\varepsilon \rightharpoonup V |\nabla \chi| dt$  and  $\nu_\varepsilon := \varepsilon \nabla u_\varepsilon \approx \nu$  in a weak sense.

Therefore we try to freeze the normal in a fixed direction, apply the weak convergence of  $\partial_t u_\varepsilon$  and then unfreeze the normal. Freezing the approximate normal  $\varepsilon \nabla u_\varepsilon$  amounts to replacing  $\varepsilon \nabla u_\varepsilon$  by  $\varepsilon |\nabla u_\varepsilon| \nu^*$  for a suitably chosen  $\nu^* \in \mathbb{S}^{d-1}$ . Let  $\eta$  be a cutoff on the support of  $\xi$ . Then the error we make can be estimated for all  $\alpha > 0$  via Young's inequality by

$$\begin{aligned} & \left| \int_0^T \int \partial_t u_\varepsilon \langle \xi, \nabla u_\varepsilon \rangle dx dt - \int_0^T \int \partial_t u_\varepsilon \langle \xi, \varepsilon |\nabla u_\varepsilon| \nu^* \rangle dx dt \right| \\ & \leq \|\xi\|_{L^\infty} \int_0^T \int \eta \sqrt{\varepsilon} |\partial_t u_\varepsilon| \sqrt{\varepsilon} |\nabla u_\varepsilon| - |\nabla u_\varepsilon| \nu^* dx dt \\ & \leq \|\xi\|_{L^\infty} \left( \alpha \int_0^T \int \eta \varepsilon |\partial_t u_\varepsilon|^2 dx dt + \frac{1}{\alpha} \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 |\nu_\varepsilon - \nu^*|^2 dx dt \right) \\ & \leq \|\xi\|_{L^\infty} \left( \alpha \int_0^T \int \eta \varepsilon |\partial_t u_\varepsilon|^2 dx dt + \frac{1}{\alpha} \mathcal{E}_\varepsilon(\nu^*; \eta) \right). \end{aligned}$$

Here the approximate tilt excess in direction  $\nu^*$  is given by

$$\mathcal{E}_\varepsilon(\nu^*; \eta) := \int_0^T \int \varepsilon |\nabla u_\varepsilon|^2 |\nu_\varepsilon - \nu^*|^2 dx dt.$$

With our frozen normal, we now notice that via the equipartition of energies, we may replace  $\varepsilon |\nabla u_\varepsilon|$  by  $\sqrt{2W(u_\varepsilon)}$  via the estimate

$$\begin{aligned} & \int_0^T \int |\partial_t u_\varepsilon| \eta \left| \varepsilon |\nabla u_\varepsilon| - \sqrt{2W(u_\varepsilon)} \right| dx dt \\ & \leq \left( \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \right)^{1/2} \left( \int_0^T \int \eta^2 \left( \varepsilon |\nabla u_\varepsilon|^2 - 2 |\nabla u_\varepsilon| \sqrt{2W(u_\varepsilon)} + \frac{1}{\varepsilon} 2W(u_\varepsilon) \right) dx dt \right)^{1/2} \end{aligned}$$

The first factor is uniformly bounded by the energy dissipation inequality (2.12) and the second term vanishes as  $\varepsilon$  tends to zero by the equipartition of energies Lemma 3.1.4. But now we recognise the identity

$$\int_0^T \int \partial_t u_\varepsilon \sqrt{2W(u_\varepsilon)} \langle \xi, \nu^* \rangle dx dt = \int_0^T \int \partial_t \psi_\varepsilon \langle \xi, \nu^* \rangle dx dt,$$

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which converges as  $\varepsilon$  approaches zero to

$$\int_0^T \int \langle \xi, \nu^* \rangle \partial_t \psi = \sigma \int_0^T \int V \langle \xi, \nu^* \rangle |\nabla \chi| \, dt. \quad (3.13)$$

Finally we want to unfreeze the normal, which means that we want to replace  $\nu^*$  by  $\nu$  on the right hand side of equation (3.13). This can be estimated again by Young's inequality via the error

$$\|\xi\|_{L^\infty} \left( \alpha \int_0^T \int \eta V^2 |\nabla \chi| \, dt + \frac{1}{\alpha} \mathcal{E}(\nu^*; \eta) \right),$$

where the tilt excess is given by

$$\mathcal{E}(\nu^*; \eta) := \sigma \int_0^T \int \eta |\nu - \nu^*|^2 |\nabla \chi| \, dt.$$

This finishes the proof for the convergence of the velocity term up to arguing that the errors can be made arbitrarily small.

First we study the behaviour of the approximate tilt excess  $\mathcal{E}_\varepsilon$  by connecting it to  $\mathcal{E}$ . We notice that by expansion, we have

$$\begin{aligned} \mathcal{E}_\varepsilon(\nu^*; \eta) &= 2 \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \, dx \, dt - 2 \left\langle \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \nabla u_\varepsilon \, dx \, dt, \nu^* \right\rangle \\ \mathcal{E}(\nu^*; \eta) &= 2 \mathbf{E}(u; \eta) - 2\sigma \left\langle \int_0^T \int \eta \nu |\nabla \chi| \, dt, \nu^* \right\rangle. \end{aligned}$$

But by the equipartition of energies Lemma 3.1.4, we have that

$$2 \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \, dx \, dt \rightarrow 2 \mathbf{E}(u; \eta)$$

and recognizing that  $\sigma \int_0^T \int \eta \nu |\nabla \chi| \, dt = \int_0^T \int \eta \nabla \psi$ , we also obtain

$$\begin{aligned} & \left| \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \nabla u_\varepsilon \, dx \, dt - \sigma \int_0^T \int \eta \nu |\nabla \chi| \, dt \right| \\ & \leq \left| \int_0^T \int \eta \sqrt{2W(u_\varepsilon)} \nabla u_\varepsilon \, dx \, dt - \int_0^T \int \eta \nabla \psi \right| + \int_0^T \int \varepsilon |\nabla u_\varepsilon| \left| \nabla u_\varepsilon - \frac{1}{\varepsilon} \sqrt{2W(u_\varepsilon)} \right| \, dx \, dt \\ & \leq \left| \int_0^T \int \eta \nabla \psi_\varepsilon \, dx \, dt - \int_0^T \int \eta \nabla \psi \right| \\ & \quad + \left( \int_0^T \int \varepsilon |\nabla u_\varepsilon|^2 \, dx \, dt \right)^{1/2} \left( \int_0^T \int \left( \sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right)^2 \, dx \, dt \right)^{1/2}, \end{aligned}$$

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which vanishes as  $\varepsilon$  tends to zero by the weak convergence of  $\nabla\psi_\varepsilon \rightharpoonup^* \nabla\psi$  and the equipartition of the energies. Since we are taking  $\varepsilon \rightarrow 0$ , we thus only have to make sure that the tilt excess is sufficiently small since the approximate tilt excess converges to the tilt excess.

We are now in the position to argue why the error can be made arbitrarily small. Let  $\delta > 0$ . Then we first choose our  $\alpha > 0$  so small that

$$\limsup_{\varepsilon \rightarrow 0} \alpha \|\xi\|_{L^\infty} \left( \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt + \int_0^T \int V^2 |\nabla \chi| dt \right) < \frac{\delta}{2},$$

which is possible by the energy dissipation inequality (2.12) and Proposition 3.1.3. Then we choose a partition of unity  $(\eta_i)_{i=1,\dots,n}$  and approximate unit normals  $(\nu_i^*)_{i=1,\dots,n}$  such that

$$\frac{2}{\alpha} \|\xi\|_{L^\infty} \sigma \sum_{i=1}^n \int_0^T \int \eta_i |\nu - \nu_i^*|^2 |\nabla \chi| dt < \frac{\delta}{2}.$$

The existence of these can be seen by taking a smooth approximation of  $\nu$  with respect to the measure  $|\nabla \chi| dt$ . Collecting all of our errors, we obtain

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \left| \int_0^T \int \partial_t u_\varepsilon \langle \xi, \varepsilon \nabla u_\varepsilon \rangle dx dt - \sigma \int_0^T \int V \langle \xi, \nu \rangle |\nabla \chi| dt \right| \\ &= \limsup_{\varepsilon \rightarrow 0} \left| \sum_{i=1}^n \int_0^T \int \eta_i \partial_t u_\varepsilon \langle \xi, \varepsilon \nabla u_\varepsilon \rangle dx dt - \sigma \int_0^T \int \eta_i V \langle \xi, \nu \rangle |\nabla \chi| dt \right| \\ &\leq \limsup_{\varepsilon \rightarrow 0} \|\xi\|_{L^\infty} \left( \sum_{i=1}^n \alpha \int_0^T \int \eta_i \varepsilon |\partial_t u_\varepsilon|^2 dx - \int \eta_i V^2 |\nabla \chi| dt + \frac{2}{\alpha} \mathcal{E}(\nu_i^*; \eta_i) \right) < \delta, \end{aligned}$$

which finishes the proof.

## 3.2. Conditional convergence in the multiphase case

Let us now turn to the much more interesting and more challenging case where we consider systems of the Allen–Cahn equation (2.1), or in other words, we want to consider the case where  $u_\varepsilon$  maps to  $\mathbb{R}^N$  and therefore our potential  $W$  is a map from  $\mathbb{R}^N$  to  $[0, \infty)$ . The for us most relevant case is when  $W$  has exactly  $P = N + 1$  zeros given by  $\alpha_1, \dots, \alpha_P$ , but it is no limitation for us to allow more general amount of zeros.

### 3.2.1. Convergence to a moving partition

One of the many difficulties in the vectorial case is that that there is no easy choice of a primitive for  $\sqrt{2W(u)}$  compared to the scalar case. We there saw for example through the Modica–Mortola trick (3.1) that this provided a very powerful tool for us,



### 3.2. Conditional convergence in the multiphase case

and comparing the composition  $\phi \circ u_\varepsilon$  to  $u_\varepsilon$  was quite simple since the map  $\phi$  was invertible as a consequence of the non-negativity of  $W$ .

As a suitable replacement, we shall consider the *geodesic distance* defined as

$$d_W(u, v) := \inf \left\{ \int_0^1 \sqrt{2W(\gamma)} |\dot{\gamma}| dt : \gamma \in C^1([0, 1]; \mathbb{R}^N) \text{ with } \gamma(0) = u, \gamma(1) = v \right\}.$$

This indeed defines a metric on  $\mathbb{R}^N$ : If  $d_W(v, w) = 0$ , then by the continuity of  $W$  and since it only has a discrete set of zeros, we may deduce that  $v = w$ . Symmetry can be seen by reversing a given path between two points and the triangle inequality follows from concatenation of two paths and rescaling. Note moreover that by an approximation argument (for example through splines), we may also take paths  $\gamma$  which are piecewise continuously differentiable which makes constructions of paths easier.

The *geodesic distances* generated by  $W$  are defined as

$$\sigma_{i,j} := d_W(\alpha_i, \alpha_j)$$

and as a consequence of  $d_W$  being a metric satisfy

$$\sigma_{i,k} \leq \sigma_{i,j} + \sigma_{j,k},$$

$\sigma_{i,j} = 0$  if and only if  $i$  is equal to  $j$  and  $\sigma_{i,j} = \sigma_{j,i}$ .

Our replacement for the primitive  $\phi$  is now given for  $1 \leq i \leq P$  by the function

$$\phi_i(u) := d_W(\alpha_i, u).$$

Our first obstacle is the regularity of the function  $\psi_\varepsilon^i := \phi_i \circ u_\varepsilon$ . A priori we only know that  $\phi_i$  is locally Lipschitz continuous on  $\mathbb{R}^N$  and thus differentiable almost everywhere. If  $N = 1$ , then this would already suffice to deduce that  $\psi_\varepsilon^i$  is weakly differentiable, but in higher dimensions,  $u$  could for example move along a hyperplane where  $\phi_i$  could in theory be nowhere differentiable since it is a Lebesgue nullset. This can however be salvaged through the following chain rule for distributional derivatives by Ambrosio and Maso [AM90, Cor. 3.2].

**Theorem 3.2.1.** *Let  $\Omega \subseteq \mathbb{R}^d$  be an open set,  $p \in [0, \infty]$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^N)$  and let  $f: \mathbb{R}^N \rightarrow \mathbb{R}^k$  be a Lipschitz continuous function such that  $f(0) = 0$ . Then  $v := f \circ u \in W^{1,p}(\Omega, \mathbb{R}^k)$ . Furthermore for almost every  $x \in \Omega$  the restriction of  $f$  to the affine space*

$$T_x^u := \{y \in \mathbb{R}^N : y = u(x) + D u(x)z \text{ for } z \in \mathbb{R}^d\} = u(x) + \dot{T}_x^u$$

*is differentiable at  $u(x)$  and*

$$D v = D(f|_{T_x^u})(u) D u$$

*holds almost everywhere in  $\Omega$ .*

### 3. Conditional convergence of the Allen–Cahn equations

*Remark.* The matrix  $D(f|_{T_x^u})(u)$  can be interpreted as some matrix in  $\mathbb{R}^{k \times n}$  which acts on  $v \in \dot{T}_x^u$  by

$$D(f|_{T_x^u})(u(x))[v] = \lim_{h \rightarrow 0} \frac{f(u(x) + hv) - f(u(x))}{h}.$$

Thus we may choose a suitable representative since the product  $D(f|_{T_x^u})(u) Du$  will not change by definition of  $\dot{T}_x^u$ .

Moreover the assumption  $f(0) = 0$  can be left out on bounded domains by simply subtracting the constant  $f(0)$ .

We are not in the position to prove the following regularity result.

**Lemma 3.2.2.** *Let  $u \in C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  with*

$$\operatorname{ess\,sup}_{0 \leq t \leq T} E_\varepsilon(u) + \int_0^T \int \varepsilon |\partial_t u|^2 dx dt < \infty$$

*for some  $\varepsilon > 0$ . Then for all  $1 \leq i \leq P$  there exists a map*

$$\partial_u \phi_i(u): [0, T] \times \mathbb{T} \rightarrow \operatorname{Lin}(\mathbb{R}^N, \mathbb{R})$$

*such that the chain rule is valid with the pair  $\partial_u \phi_i(u)$  and  $(\partial_t, \nabla)u$ :*

*For almost every  $(t, x) \in [0, T] \times \mathbb{T}$  we have*

$$\nabla(\phi_i \circ u) = \partial_u \phi_i(u) \nabla u \quad \text{and} \quad \partial_t(\phi_i \circ u) = \partial_u \phi_i(u) \partial_t u.$$

*Furthermore we can control the modulus of  $\partial_u \phi_i(u)$  almost everywhere in time and space via the estimate*

$$|\partial_u \phi_i(u)| \leq \sqrt{2W(u)}. \quad (3.14)$$

*Additionally we have  $\phi_i \circ u \in L^\infty([0, T]; W^{1,1}(\mathbb{T})) \cap W^{1,1}([0, T] \times \mathbb{T})$  with the estimates*

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \int |\phi_i \circ u| dx \lesssim 1 + \operatorname{ess\,sup}_{0 \leq t \leq T} \varepsilon E_\varepsilon(u), \quad (3.15)$$

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \int |\nabla(\phi_i \circ u)| dx \leq \operatorname{ess\,sup}_{0 \leq t \leq T} E_\varepsilon(u) \quad (3.16)$$

*and*

$$\int_0^T \int |\partial_t(\phi_i \circ u)| dx dt \leq T \operatorname{ess\,sup}_{0 \leq t \leq T} E_\varepsilon(u) + \int_0^T \int \varepsilon |\partial_t u|^2 dx dt. \quad (3.17)$$

*Proof.* Since Theorem 3.2.1 requires Lipschitz continuity of  $\phi_i$ , but we only have local Lipschitz continuity, let us first assume that  $u$  is bounded in space and time. Then we may modify  $\phi_i$  outside of a compact set such that it is (globally) Lipschitz continuous and does not change on the image of  $u$ .

Since we have via the energy estimate that  $u \in W^{1,2}([0, T] \times \mathbb{T}; \mathbb{R}^N)$ , we obtain by the distributional chain rule Theorem 3.2.1 that  $\psi = \phi_i \circ u \in W^{1,2}([0, T] \times \mathbb{T})$ .

### 3.2. Conditional convergence in the multiphase case

Let  $\Pi(t, x)$  denote the orthogonal projection of  $\mathbb{R}^N$  onto  $\dot{T}_{t,x}^u$  and define

$$\partial_u \phi_i(u)(t, x)[v] := D(\phi_i|_{T_{t,x}^u})(u(t, x))[\Pi(t, x)v].$$

This defines a unique row vector and thus we can now proceed to prove inequality (3.14). Let  $v \in \dot{T}_{t,x}^u$ , a real number  $h \in \mathbb{R} \setminus \{0\}$  and  $\gamma: [0, 1] \rightarrow \mathbb{R}^N$  be a path connecting  $\alpha_i$  and  $u$ . Then we define the new path  $\tilde{\gamma}: [0, 1] \rightarrow \mathbb{R}^N$  by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(\frac{t}{2}) & , t \leq \frac{1}{2} \\ u + (t - \frac{1}{2}) 2hv & , t \geq \frac{1}{2}. \end{cases}$$

We observe that  $\tilde{\gamma}$  is a piecewise continuously differentiable path connecting  $\alpha_i$  and  $u + hv$ , thus we can estimate by a substitution that

$$\begin{aligned} & d_W(\alpha_i, u + hv) - \int_0^1 \sqrt{2W(\gamma(t))} |\gamma'(t)| dt \\ & \leq \int_0^1 \sqrt{2W(\tilde{\gamma}(t))} |\tilde{\gamma}'(t)| dt - \int_0^1 \sqrt{2W(\gamma(t))} |\gamma'(t)| dt \\ & = \int_0^1 \sqrt{2W(u + thv)} |hv| dt. \end{aligned}$$

Taking the infimum over all  $C^1$ -paths connecting  $\alpha_i$  and  $u$  yields

$$d_W(\alpha_i, u + hv) - d_W(\alpha_i, u) \leq \int_0^1 \sqrt{2W(u + thv)} |hv| dt.$$

Using a similar strategy but with a reversed path, we also obtain the inequality

$$d_W(\alpha_i, u) - d_W(\alpha_i, u + hv) \leq \int_0^1 \sqrt{2W(u + thv)} |hv| dt,$$

thus we obtain by the dominated convergence theorem

$$\limsup_{h \rightarrow 0} \left| \frac{\phi(u + hv) - \phi(u)}{h} \right| \leq \limsup_{h \rightarrow 0} \int_0^1 \sqrt{2W(u + thv)} |v| dt = \sqrt{2W(u)} |v|,$$

which yields

$$\left| D\phi_i|_{T_{t,x}^u}(u)[v] \right| \leq \sqrt{2W(u)} |v|$$

and thus gives us the desired inequality (3.14) since  $|\Pi(v)| \leq |v|$ .

Now let us consider the general case and denote by  $u_M$  the truncation of  $u$  defined by

$$u_M^j := \begin{cases} u & , \text{ if } |u^j| \leq M \\ M \frac{u^j}{|u^j|} & , \text{ else.} \end{cases}$$

Then we still have  $u_M \in W^{1,2}([0, T] \times \mathbb{T})$  and obtain by the previous step that  $\phi_i \circ u_M \in W^{1,2}([0, T] \times \mathbb{T})$  and that for almost every  $(t, x) \in [0, T] \times \mathbb{T}$ , the function

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$\phi_i$  is differentiable on  $T_{t,x}^{u_M}$ . Moreover if  $(t, x) \in u^{-1}([-M, M]^N)$ , then we obtain  $T_{t,x}^{u_M} = T_{t,x}^u$ . Next we want to show that  $\phi_i \circ u_M$  converges to  $\phi_i \circ u$  in a suitable sense. First we recognize that  $\phi_i \circ u_M$  converges to  $\phi_i \circ u$  pointwise almost everywhere. Moreover we find a majorant since

$$\begin{aligned} \phi_i(v) &\leq \int_0^1 \sqrt{2W(\alpha_i + s(v - \alpha_i))} |v - \alpha_i| \, ds \leq \left\| \sqrt{2W} \right\|_{L^\infty[v, \alpha_i]} |v - \alpha_i| \\ &\lesssim 1 + |v|^{1+p/2}, \end{aligned} \quad (3.18)$$

thus we have  $\phi_i \circ u_M \lesssim 1 + |u|^p$ , which is in integrable majorant. Therefore the dominated convergence theorem yields that  $\phi_i \circ u_M$  converges to  $\phi_i \circ u$  in  $L^1([0, T] \times \mathbb{T})$ . Moreover estimate (3.18) together with the  $p$ -growth of  $W$  (2.3) already yield the desired  $L^1$ -estimate (3.15).

Furthermore we recognize that on the set  $\{u_M = u\}$ , we already have

$$(\partial_t, \nabla)u_M = (\partial_t, \nabla)u$$

and the sets  $\{u_M = u\}$  are non-decreasing, thus

$$\lim_{M \rightarrow \infty} |\{u_M \neq u, (\partial_t, \nabla)u_M \neq (\partial_t, \nabla)u\}| = 0.$$

Moreover we have that for almost every  $(t, x)$ , the derivative  $\partial_u \phi_i(u_M(t, x))$  eventually becomes stationary. We denote its almost everywhere pointwise limit by  $\partial_u \phi_i(u)$  and it satisfies almost everywhere

$$|\partial_u \phi_i(u)| \leq \sqrt{2W(u)}.$$

In order to show that  $\phi_i \circ u$  is weakly differentiable with derivative

$$(\partial_t, \nabla)\phi_i \circ u = \partial_u \phi_i(u)(\partial_t u, \nabla u),$$

we compute for a testfunction  $\varphi$  that by the  $L^1$ -convergence, we have

$$\begin{aligned} \int_0^T \int \phi_i \circ u (\partial_t, \nabla) \varphi \, dx \, dt &= \lim_{M \rightarrow \infty} \int_0^T \int \phi_i \circ u_M (\partial_t, \nabla) u_M \varphi \, dx \, dt \\ &= - \lim_{M \rightarrow \infty} \int \partial_u \phi_i(u_M) (\partial_t, \nabla) u_M \varphi \, dx \, dt \\ &= - \lim_{M \rightarrow \infty} \int_{|u| \leq M} \partial_u \phi_i(u) (\partial_t, \nabla) u \varphi \, d(t, x) \\ &= \int_0^T \int \partial_u \phi_i(u) (\partial_t, \nabla) u \varphi \, dx \, dt. \end{aligned}$$

The last inequality is due to the dominated convergence theorem with majorant  $|\varphi| \sqrt{2W(u)} |(\partial_t, \nabla)u|$ . It remains to prove the estimates (3.16) and (3.17). These follow

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from applications of Young's inequality

$$\begin{aligned} \operatorname{ess\,sup}_{0 \leq t \leq T} \int |\nabla(\phi_i \circ u)| &= \operatorname{ess\,sup}_{0 \leq t \leq T} \int |\partial_u \phi_i(u) \nabla u| \, dx \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \int \sqrt{2W(u)} |\nabla u| \, dx \\ &\leq \sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon) \end{aligned}$$

and

$$\begin{aligned} \int_0^T \int |\partial_t(\phi_i \circ u)| \, dx \, dt &\leq \int_0^T \int \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u)} \sqrt{\varepsilon} |\partial_t u| \, dx \, dt \\ &\leq \int_0^T \int \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2} |\partial_t u|^2 \, dx \, dt, \end{aligned}$$

which finishes our proof.  $\square$

With this powerful tool on our hands, we are in the position to prove a similar result to Proposition 3.1.1 for the multiphase case.

**Proposition 3.2.3.** *Given initial data  $u_\varepsilon^0 = \sum_{i=1}^P \chi_i^0 \alpha_i$  whose energies satisfy*

$$E_\varepsilon(u_\varepsilon^0) \rightarrow E(u^0) =: E_0 < \infty,$$

*there exists for any sequence  $\varepsilon$  some non-relabelled subsequence such that the solutions of the Allen–Cahn equation (2.1) with initial condition  $u_\varepsilon^0$  converge in  $L^1((0, T) \times \mathbb{T}; \mathbb{R}^N)$  to some  $u = \sum_{i=1}^P \chi_i \alpha_i$  with a partition  $\chi \in \operatorname{BV}((0, T) \times \mathbb{T}; \{0, 1\}^P)$ . Furthermore we have*

$$\operatorname{ess\,sup}_{0 \leq t \leq T} E(u) \leq E_0 \quad (3.19)$$

*and for all  $1 \leq i \leq P$  the compositions  $\phi_i \circ u_\varepsilon$  are uniformly bounded in  $\operatorname{BV}((0, T) \times \mathbb{T})$  and converge to  $\phi_i \circ u$  in  $L^1((0, T) \times \mathbb{T})$ .*

*Remark.* The proof is quite similar to the proof of Proposition 3.1.1, but we have to work more in order to obtain the desired convergence to  $u$ .

*Proof.* By Theorem 2.2.1 the solution  $u_\varepsilon$  of (2.1) satisfies the assumptions of Lemma 3.2.2. Thus  $\psi_i^\varepsilon := \varphi_i \circ u_\varepsilon$  is uniformly bounded in  $\operatorname{BV}((0, T) \times \mathbb{T})$  as  $\varepsilon$  tends to zero for all  $1 \leq i \leq P$  and thus we find a non-relabelled subsequence and  $v_i \in \operatorname{BV}((0, T) \times \mathbb{T})$  such that  $\psi_i^\varepsilon$  converges to  $v_i$  in  $L^1((0, T) \times \mathbb{T})$  and pointwise almost everywhere.

We now want to show that we can write  $v_i = \phi_i \circ u$  for  $u = \sum \chi_j \alpha_j$ . An elegant proof of this presented in [FT89, Thm. 4.1] can be done through the fundamental theorem of Young measures ([Mül99, Thm. 3.1]). By passing to another non-relabelled subsequence of  $u_\varepsilon$ , we obtain that  $u_\varepsilon$  generates a Young measure  $\nu$ . Since  $u$  is  $L^p$ -bounded, we have that for almost every  $(t, x)$ , the measure  $\nu_{(t, x)}$  is a probability measure. Moreover the sequence  $W(u_\varepsilon)$  is uniformly integrable since

$$0 \leq W(u_\varepsilon) \leq \frac{1}{\varepsilon} W(u_\varepsilon) \rightarrow 0$$

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in  $L^1$ . Thus we obtain that for all  $\varphi \in L^\infty([0, T] \times \mathbb{T})$

$$0 = \lim_{\varepsilon \rightarrow 0} \int_0^T \int \varphi(t, x) W(u_\varepsilon(t, x)) \, dx \, dt = \int_0^T \int \varphi(t, x) \int_{\mathbb{R}^N} W(y) \, d\nu_{(t, x)}(y) \, dx \, dt,$$

which implies that for almost every  $(t, x)$ , the probability measure  $\nu_{(t, x)}$  is supported on the set  $\{\alpha_1, \dots, \alpha_P\}$ . Therefore we can write  $\nu_{(t, x)} = \sum_{j=1}^P \lambda_j \delta_{\alpha_j}$  for non-negative numbers  $\lambda_j \in [0, 1]$  with  $\sum \lambda_j = 1$ .

Now let  $1 \leq i \leq P$ . For every  $f \in C_0(\mathbb{R})$ , we have that  $f \circ \phi_i \in C_0(\mathbb{R}^N)$ , and thus we can compute that  $\phi_i \circ u_\varepsilon$  generates the Young measure given at almost every point  $(t, x)$  by  $\nu_{(t, x)} = \sum \lambda_j \delta_{\phi_i(\alpha_j)}$ . But  $\phi_i \circ u_\varepsilon$  converges to  $v_i$  in  $L^1$ , thus especially in measure, which implies by [Mül99, Cor. 3.2] that our convex combination already has to be trivial, or in other words, we can write

$$\nu_{(t, x)} = \sum_{j=1}^P \mathbb{1}_{\Omega_j(t)}(x) \delta_{\alpha_j}$$

for a time-dependent partition  $(\Omega_j(t))_{j=1, \dots, P}$  of  $\mathbb{T}$ . Thus we can write

$$\delta_{v_i(t, x)} = \sum_{j=1}^P \mathbb{1}_{\Omega_j(t)}(x) \delta_{\phi_i(\alpha_j)},$$

and by defining  $u := \sum_{j=1}^P \mathbb{1}_{\Omega_j(t)}(x) \alpha_j$ , we obtain that

$$v_i = \phi_i \circ u = \sum_{j=1}^P \mathbb{1}_{\Omega_j(t)}(x) \sigma_{i, j}.$$

What is left to show is the energy estimate (3.19) and that partition function  $\chi(t, x) := (\mathbb{1}_{\Omega_1(t)}(x), \dots, \mathbb{1}_{\Omega_P(t)}(x))$  is of bounded variation. The latter follows from the Fleming–Rishel co-area formula [FR60] which yields since  $\phi_i \circ u \in \text{BV}((0, T) \times \mathbb{T})$  that

$$\begin{aligned} \infty &> |(\partial_t, \nabla) \phi_i \circ u|((0, T) \times \mathbb{T}) \\ &= \int_0^\infty \mathcal{H}^d(\partial_* \{(t, x) : \phi_i(u(t, x)) \leq s\}) \, ds \\ &\geq \int_0^{\min_{i \neq j} \sigma_{i, j}} \mathcal{H}^d(\partial_* \{(t, x) \in (0, T) \times \mathbb{T} : \chi_i(t, x) = 1\}) \, ds \\ &= \min_{i \neq j} \sigma_{i, j} |(\partial_t, \nabla) \chi_i|((0, T) \times \mathbb{T}). \end{aligned}$$

Here we denote by  $\partial_* A$  the measure theoretic boundary of a given measurable set  $A$ , as defined for example in [EG15, Def. 5.7]. Since we have  $\sigma_{i, j} > 0$  for  $i \neq j$ , this proves that  $\chi_i \in \text{BV}((0, T) \times \mathbb{T})$ .

The energy estimate is a non-trivial consequence of the lower semicontinuity of the variation measure and has been proven by Baldo in [Bal90]. We will recap the proof since it provides some insight into the geometry of the phases.  $\square$

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**Definition 3.2.4.** Given a partition  $(\Omega_i)_{i=1,\dots,P}$  of the flat torus  $\mathbb{T}$ , where all sets  $\Omega_i$  are of finite perimeter, we define the  $(i, j)$ -th interface as  $\Sigma_{i,j} := \partial_* \Omega_i \cap \partial_* \Omega_j$ .

We already mentioned that we want to apply the lower semicontinuity of the variation measure. Since we can think of  $\phi_i$  as being locally the correct choice for suitable  $i$ , we introduce the following notion.

**Definition 3.2.5.** Let  $\mu, \nu$  be regular positive Borel measures on  $\mathbb{T}$ . Define the supremum  $\mu \vee \nu$  of  $\mu$  and  $\nu$  as the smallest regular positive measure which is greater or equal to  $\mu$  and  $\nu$  on a Borel subset of  $\mathbb{T}$ . By the regularity, we have

$$\mu \vee \nu(U) = \sup \{ \mu(V) + \mu(W) : V \cap W = \emptyset, V \cup W \subseteq U, V \text{ and } W \text{ are open subsets of } \mathbb{T} \}$$

for any open subset  $U \subseteq \mathbb{T}$ .

It is natural to ask how we can characterize the total variation of  $\psi_i = \phi_i \circ u = \sum \mathbb{1}_{\Omega_j} \sigma_{i,j}$ . For this we note that when going from set  $\Omega_j$  to  $\Omega_k$ , our function jumps from  $\sigma_{i,j}$  to  $\sigma_{i,k}$ . Thus we obtain the following intuitive result.

**Lemma 3.2.6.** In the setting of Proposition 3.2.3 we can write

$$|\nabla \psi_i| = \sum_{1 \leq j < k \leq P} |\sigma_{i,j} - \sigma_{i,k}| \, d\mathcal{H}^{d-1}|_{\Sigma_{j,k}}.$$

The proof can be found in the Appendix. If we now consider the supremum of the measures  $(|\nabla \psi_i|)_{i=1,\dots,P}$ , we notice that all of them are supported on the interfaces  $\Sigma_{j,k}$ , and that by the triangle inequality for the surface tensions, we have

$$\max_{1 \leq i \leq P} |\sigma_{i,j} - \sigma_{i,k}| = \sigma_{j,k}. \quad (3.20)$$

Thus, we obtain the following.

**Proposition 3.2.7.** In the situation of Proposition 3.2.3, we have

$$\left( \bigvee_{i=1}^P |\nabla \psi_i| \right) = \sum_{1 \leq i < j \leq P} \sigma_{i,j} \, d\mathcal{H}^{d-1}|_{\Sigma_{i,j}}.$$

*Proof.* This follows by combining Lemma 3.2.6 and Lemma A.0.2 with the above equation (3.20).  $\square$

Collecting our observations, we can finish our arguments.

*Continuation of the proof for Proposition 3.2.3.* We are left with proving the estimate (3.19). We first note that as for the proof of (3.16), it holds that for any open subset  $U$  of the flat torus, we have for almost every time  $t$  the estimate

$$\liminf_{\varepsilon \rightarrow 0} \int_U \frac{1}{\varepsilon} W(u_\varepsilon(t)) + \frac{\varepsilon}{2} |\nabla u_\varepsilon(t)|^2 \, dx \geq |\nabla \psi_i(t)|(U).$$

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Since

$$\left( \bigvee_{i=1}^P |\nabla \psi_i| \right) (\mathbb{T}) = \sup \left\{ \sum_{i=1}^P |\nabla \psi_i|(U_i) : (U_i)_i \text{ are disjoint open subsets of } \mathbb{T} \right\},$$

we finally deduce

$$\begin{aligned} \operatorname{ess\,sup}_{0 \leq t \leq T} E(u(t)) &= \operatorname{ess\,sup}_{0 \leq t \leq T} \sum_{1 \leq i < j \leq P} \sigma_{i,j} \mathcal{H}^{d-1}(\Sigma_{i,j}(t)) \\ &= \operatorname{ess\,sup}_{0 \leq t \leq T} \left( \bigvee_{i=1}^P |\nabla \psi_i(t)| \right) (\mathbb{T}) \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon(t)) \\ &\leq E_0. \end{aligned}$$

The last inequality is due the energy dissipation inequality (2.12).  $\square$

Nextup, we want to prove a stronger convergence of  $u_\varepsilon$  to  $u$ . This will let us prove that  $u$  achieves to initial data (and the existence of normal velocities?).

**Lemma 3.2.8.** *In the situation of Proposition 3.2.3 we have  $\psi_i^\varepsilon \in W^{1,2}([0, T]; L^1(\mathbb{T}))$  with the corresponding estimate*

$$\left( \int_0^T \left( \int |\partial_t \psi_i^\varepsilon| \, dx \right)^2 dt \right)^{1/2} \lesssim E_\varepsilon(u_\varepsilon(0)). \quad (3.21)$$

Furthermore the sequence  $u_\varepsilon$  is precompact in  $C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$ . In particular we get that  $u$  achieves the initial data in  $C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$ .

*Remark.* This statement is almost similar to its twophase equivalent Lemma 3.1.2 and the proof uses a similar strategy.

*Proof.* **Step 1:** Estimate (3.21) holds

We compute that by Hölder's inequality and the energy dissipation inequality (2.12)

$$\begin{aligned} \int_0^T \left( \int |\partial_t \psi_i^\varepsilon| \, dx \right)^2 dt &\leq \int_0^T \left( \int \sqrt{2W(u_\varepsilon)} |\partial_t u_\varepsilon| \, dx \right)^2 dt \\ &= \int_0^T \int \frac{2}{\varepsilon} W(u_\varepsilon) \, dx \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt \\ &\leq 2 E_\varepsilon(u_\varepsilon(0)) \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt \\ &\leq 2 E_\varepsilon(u_\varepsilon(0))^2. \end{aligned}$$

By estimate (3.16), we especially obtain  $\psi_i^\varepsilon \in L^2([0, T]; L^1(\mathbb{T}))$ .



### 3.2. Conditional convergence in the multiphase case

**Step 2:** The sequence  $\psi_i^\varepsilon$  is precompact in  $C([0, T]; L^1(\mathbb{T}))$ .

This follows exactly as in the twophase case from the embedding of  $W^{1,2}([0, T]; L^1(\mathbb{T}))$  into  $C^{1/2}([0, T]; L^1(\mathbb{T}))$ .

**Step 3:** The sequence  $u_\varepsilon$  converges to  $\sum_i \chi_i \mathbb{1}_{\Omega_i} \alpha_i$  in measure uniformly in time.

Let  $\rho > 0$ . Then by definition of  $\phi_i$ , there exists some  $\delta > 0$  such that if  $|v - \alpha_i| > \rho$  holds for all  $1 \leq i \leq P$ , then we already have  $\phi_v > \delta$  for all  $1 \leq i \leq P$ . Therefore we can estimate

$$\begin{aligned} \operatorname{ess\,sup}_{0 \leq t \leq T} \mathcal{L}^d \left( \left\{ x : \left| u_\varepsilon - \sum_i \mathbb{1}_{\Omega_i} \alpha_i \right| > \rho \right\} \right) &= \operatorname{ess\,sup}_{0 \leq t \leq T} \sum_{i=1}^P \mathcal{L}^d (\{x \in \Omega_i : |u_\varepsilon - \alpha_i| > \rho\}) \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \sum_{i=1}^P \mathcal{L}^d (\{x \in \Omega_i : \phi_i(u_\varepsilon) > \delta\}) \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \frac{1}{\delta} \sum_{i=1}^P \int_{\Omega_i} |\psi_i^\varepsilon| \, dx \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \frac{1}{\delta} \sum_{i=1}^P \int |\psi_i^\varepsilon - \psi_i| \, dx, \end{aligned}$$

which converges to 0 by Step 2.

**Step 4:** The sequence  $u_\varepsilon^2$  is equiintegrable uniformly in time

This follows as in the twophase case as well.

**Step 5:** The sequence  $u_\varepsilon$  converges in  $C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$ .

Here the proof does not change as well. □



## 4. i dont know yet

Baldo proved in his paper [Bal90] that the Cahn-Hilliard energies  $\Gamma$ -converge with respect to  $\|\cdot\|_{L^1}$  to an *optimal partition energy* given by

$$E(\chi) := \frac{1}{2} \sum_{1 \leq i, j \leq P} \sigma_{i,j} \int \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|) \quad (4.1)$$

for a partition  $\chi_1, \dots, \chi_P: \mathbb{T} \rightarrow \{0, 1\}$  satisfying  $\chi_{1 \leq i \leq P} \chi_i = 1$  almost everywhere. We may also define measurable sets  $\Omega_i$  through the relation  $\chi_i = \mathbb{1}_{\Omega_i}$ . The link between a sequence  $u_\varepsilon$  and  $\chi$  is given by  $u_\varepsilon \rightarrow u := \sum_{1 \leq i \leq P} \alpha_i \chi_i$  in  $L^1$ .

Moreover if we denote by  $\partial_* \Omega_i$  the reduced boundary of  $\Omega_i$  and by  $\Sigma_{i,j} := \partial_* \Omega_i \cap \partial_* \Omega_j$  the interface between  $\Omega_i$  and  $\Omega_j$ , then we may rewrite equation (4.1) as

$$E(\chi) = \frac{1}{2} \sum_{1 \leq i, j \leq P} \sigma_{i,j} \mathcal{H}^{d-1}(\Sigma_{i,j}).$$

Here, the surface tensions  $\sigma_{i,j}$  are the geodesic distances between the wells  $\alpha_i$  of  $W$  with respect to the metric  $2W(u)\langle \cdot, \cdot \rangle$ , which can be written out as

$$\sigma_{i,j} = d_W(\alpha_i, \alpha_j)$$

for the geodesic distance defined as

$$d_W(u, v) := \inf \left\{ \int_0^1 \sqrt{2W(\gamma)} |\dot{\gamma}| dt : \gamma \in C^1([0, 1]; \mathbb{R}^N) \text{ with } \gamma(0) = u, \gamma(1) = v \right\}. \quad (4.2)$$

Geometrically speaking, the partition energy  $E$  measures the surface tensions between the sets and penalizes large interfaces. Also observe that the factor  $1/2$  can be left out if we only count each interface once.



## A. appendix

**Lemma A.0.1.** *The following equalities hold up to set of  $d-1$ -dimensional Hausdorff-measure zero.*

1. *If  $A, B, C \subset \mathbb{T}$  are mutually disjoint, then*

$$\partial_*(A \cup B) = \partial_* A \triangle \partial_* B \quad \text{and} \quad \partial_* A \cap \partial_* B \cap \partial_* C = \emptyset.$$

2. *If  $(\Omega_i)_{i=1, \dots, P}$  is a partition of  $\mathbb{T}$ , then*

$$\partial_* \Omega_i = \bigcup_{j \neq i} \partial_* \Omega_i \cap \partial_* \Omega_j.$$

*Proof.* Let us first prove Item 1. Initially assume that  $x \in \partial_*(A \cup B)$ . By definition this implies that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap (A \cup B))}{r^d} > 0.$$

Then we must either have

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{r^d} > 0 \quad \text{or} \quad \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap B)}{r^d} > 0,$$

so let us without loss of generality assume the first inequality. Since

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A)}{r^d} \geq \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap (A \cup B))}{r^d} > 0,$$

we thus have  $x \in \partial_* A$ . Combining Thm. 5.14 and Lemma 5.5 in [EG15], we can rewrite for some measurable set  $N$  with  $\mathcal{H}^{d-1}(N) = 0$  the measure theoretic boundary of  $A$  as

$$\partial_* A = \partial^{1/2} A \cup N := \left\{ x : \lim_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{\alpha(d)r^d} = \frac{1}{2} = \lim_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A)}{\alpha(d)r^d} \right\} \cup N, \quad (\text{A.1})$$

where  $\alpha(d) := \mathcal{L}^d(B_1(0))$  is the volume of the unit ball in  $d$  dimensions. Thus assume  $x \in \partial^{1/2} A \cap \partial^{1/2} B$ . Then since  $A$  and  $B$  are disjoint, we would have

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus (A \cup B))}{\alpha(d)r^d} \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A) - \mathcal{L}^d(B_r(x) \cap B)}{\alpha(d)r^d} = 0,$$

which contradicts  $x \in \partial_*(A \cup B)$ . Thus up to a set of  $(d-1)$ -dimensional Hausdorff-measure zero, we have  $\partial_*(A \cup B) \subseteq \partial_* A \triangle \partial_* B$ .

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For the other inclusion, suppose that  $x \in \partial_* A \triangle \partial_* B$  and without loss of generality that  $x \in \partial_* A \setminus \partial_* B$ . Then it must either hold that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap B)}{r^d} = 0 \quad \text{or} \quad \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus B)}{r^d} = 0.$$

But since  $x \in \partial_* A$ , the latter can not be true since  $A$  and  $B$  are disjoint and therefore

$$0 < \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{r^d} = \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus B)}{r^d}.$$

We now estimate

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap (A \cup B))}{r^d} \geq \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{r^d} > 0$$

and

$$\begin{aligned} \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus (A \cup B))}{r^d} &= \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A) - \mathcal{L}^d(B_r(x) \cap B)}{r^d} \\ &= \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A)}{r^d} > 0, \end{aligned}$$

which proves  $x \in \partial_*(A \cup B)$ .

The second equality is an immediate consequence of observation (A.1).

Now let us prove Item 2. It follows immediately that

$$\partial_* \Omega_i \supseteq \bigcup_{j \neq i} \partial_* \Omega_i \cap \partial_* \Omega_j.$$

For the other inclusion, suppose  $x \in \partial_* \Omega_i$ . Since

$$0 < \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus \Omega_i)}{r^d} = \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap \bigcup_{j \neq i} \Omega_j)}{r^d},$$

we find some  $j \neq i$  such that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap \Omega_j)}{r^d} > 0.$$

Since

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus \Omega_j)}{r^d} \geq \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap \Omega_i)}{r^d} > 0,$$

we therefore have  $x \in \partial_* \Omega_j$ , which finishes the proof.  $\square$

*Proof of Lemma 3.2.6.* Without loss of generality we can assume that  $i = 1$  and  $\sigma_{1,j} \leq \sigma_{1,j+1}$  for all  $1 \leq j \leq P - 1$ . All equalities in this proof hold up to a set of

$(d - 1)$ -dimensional Hausdorff measure zero. First let us apply the Fleming–Rishel co-area formula to find that for a given open set  $U \subseteq \mathbb{T}$ , we have

$$\begin{aligned} |\nabla \psi_i|(U) &= \int_0^\infty \mathcal{H}^{d-1}(\partial_* (\{x \in U : \psi_i \leq s\})) \, ds \\ &= \sum_{j=1}^P \int_{\sigma_{1,j}}^{\sigma_{1,j+1}} \mathcal{H}^{d-1} \left( \partial_* \left( \bigcup_{j=1}^i U \cap \Omega_j \right) \right) \, ds \\ &= \sum_{j=1}^{P-1} (\sigma_{1,j+1} - \sigma_{1,j}) \mathcal{H}^{d-1} \left( \partial_* \left( \bigcup_{j=1}^i U \cap \Omega_j \right) \right) =: I. \end{aligned}$$

We claim that we have

$$\partial_* \left( \bigcup_{j=1}^i U \cap \Omega_j \right) = \bigcup_{j=1}^i \bigcup_{l=i+1}^P U \cap \partial_* \Omega_j \cap \partial_* \Omega_l. \quad (\text{A.2})$$

Since  $(\Omega_j \cap U)_j$  is again a partition of  $U$ , we omit taking the intersection with  $U$  for a briefer notation. The proof will be done via an induction over  $i$ . For  $i = 1$ , this is exactly Item 2 from Lemma A.0.1. For the induction step, assume that the claim (A.2) holds for all  $i' \leq i$ . Again by Lemma A.0.1 Item 1, we can compute that

$$\begin{aligned} \partial_* \left( \bigcup_{j=1}^{i+1} \Omega_j \right) &= \left( \partial_* \left( \bigcup_{j=1}^i \Omega_j \right) \cup \partial_* \Omega_{i+1} \right) \setminus \left( \partial_* \left( \bigcup_{j=1}^i \Omega_j \right) \cap \partial_* \Omega_{i+1} \right) \\ &= \left( \left( \bigcup_{j=1}^i \bigcup_{l=i+1}^P \partial_* \Omega_j \cap \partial_* \Omega_l \right) \cup \bigcup_{k \neq i+1} \partial_* \Omega_{i+1} \cap \partial_* \Omega_k \right) \setminus \left( \bigcup_{j=1}^i \bigcup_{l=i+1}^P \partial_* \Omega_j \cap \partial_* \Omega_l \cap \partial_* \Omega_{i+1} \right). \end{aligned}$$

By Lemma A.0.1 Item 1, we can write the second term as

$$\bigcup_{j=1}^i \bigcup_{l=i+1}^P \partial_* \Omega_j \cap \partial_* \Omega_l \cap \Omega_{i+1} = \bigcup_{j=1}^i \partial_* \Omega_j \cap \partial_* \Omega_{i+1}.$$

By carefully considering both equations, we get

$$\begin{aligned} &\partial_* \left( \bigcup_{j=1}^{i+1} \Omega_j \right) \\ &= \left( \left( \bigcup_{j=1}^i \bigcup_{l=i+2}^P \partial_* \Omega_j \cap \partial_* \Omega_l \right) \cup \bigcup_{k=i+2}^P \partial_* \Omega_{i+1} \cap \partial_* \Omega_k \right) \setminus \bigcup_{j=1}^i \partial_* \Omega_{i+1} \cap \partial_* \Omega_j \\ &= \left( \bigcup_{j=1}^{i+1} \bigcup_{l=i+2}^P \partial_* \Omega_j \cap \partial_* \Omega_l \right) \setminus \bigcup_{j=i}^i \partial_* \Omega_{i+1} \cap \partial_* \Omega_j. \end{aligned} \quad (\text{A.3})$$

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Since by Lemma A.0.1 Item 1, we have for  $1 \leq j \leq i+2$ ,  $i+2 \leq l \leq P$  and  $1 \leq k \leq i$  that

$$\partial_* \Omega_l \cap \partial_* \Omega_{i+1} \cap \partial_* \Omega_k$$

is a set of  $(d-1)$ -dimensional Hausdorff measure zero, taking the complement in the term (A.3) becomes obsolete and thus we obtain the desired result (A.2). Therefore we have again by Lemma A.0.1 Item 1 that

$$\begin{aligned} I &= \sum_{i=1}^{P-1} (\sigma_{1,i+1} - \sigma_{1,i}) \left( \sum_{j=1}^i \sum_{l=i+1}^P \mathcal{H}^{d-1}(U \cap \partial_* \Omega_j \cap \partial_* \Omega_l) \right) \\ &= \sum_{1 \leq j < l \leq P} \mathcal{H}^{d-1}(U \cap \partial_* \Omega_j \cap \partial_* \Omega_l) \left( \sum_{i=j}^{l-1} \sigma_{1,i+1} - \sigma_{1,i} \right) \\ &= \sum_{1 \leq j < l \leq P} \mathcal{H}^{d-1}(U \cap \partial_* \Omega_j \cap \partial_* \Omega_l) (\sigma_{1,l} - \sigma_{1,j}). \end{aligned}$$

Since we assumed without loss of generality that  $\sigma_{1,i} \leq \sigma_{1,j+1}$ , we know that

$$|\sigma_{1,l} - \sigma_{1,j}| = \sigma_{1,l} - \sigma_{1,j},$$

which finishes the proof.  $\square$

**Lemma A.0.2.** *Let  $\mu$  be a regular positive Borel measure on some open set  $\Omega$  and let  $B_1, \dots, B_m$  be  $\mu$ -finite disjoint Borel subsets of  $\Omega$ . Moreover let  $c_i^h$  for  $1 \leq i \leq m$  and  $1 \leq h \leq k$  be non-negative coefficients. Define the measures*

$$\mu_h := \sum_{i=1}^m c_i^h \mu|_{B_i} \quad \text{and} \quad \nu := \sum_{i=1}^m \max_h c_i^h \mu|_{B_i}.$$

Then we have

$$\nu = \bigvee_{h=1}^k \mu_h.$$

*Proof.* We first note that for all Borel sets  $A$  and all  $\tilde{h}$ , we have

$$\nu(A) = \sum_{i=1}^m \max_h c_i^h \mu(A \cap B_i) \geq \sum_{i=1}^m c_i^{\tilde{h}} \mu(A \cap B_i) = \mu_{\tilde{h}}(A).$$

On the other hand we have that for all  $1 \leq i \leq m$  that we find  $h(i)$  such that  $\max_h c_i^h = c_i^{h(i)}$ . But then we have

$$\nu(B_i) = \max_h c_i^h \mu(B_i) = \mu_{h(i)}(B_i),$$

thus we have for all  $1 \leq i \leq m$  that

$$\nu(B_i) \leq \left( \bigvee_{h=1}^k \mu_h \right) (B_i).$$

Since the support of  $\nu$  is contained in  $\bigcup_{1 \leq i \leq m} B_i$ , this finishes the proof.  $\square$



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