

# Convergence of Allen-Cahn equations to multi-phase mean curvature flow

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# 1 The Allen–Cahn equation

## 1.1 Structure of the equation

This chapter follows [LS16], but since the authors decided to only sketch some of the proofs, we want to go into more detail.

Let  $\Lambda > 0$  and define the flat torus  $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$ , where we work with periodic boundary conditions and write  $\int dx$  instead of  $\int_{\mathbb{T}} dx$ . Then for  $u: [0, \infty) \times \mathbb{T} \rightarrow \mathbb{R}^N$  and some potential  $W: \mathbb{R}^N \rightarrow [0, \infty)$ , the *Allen–Cahn equation* with parameter  $\varepsilon > 0$  is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \quad (1.1)$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to  $u$  for a fixed time the real number

$$E_\varepsilon(u) := \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 dx. \quad (1.2)$$

If everything is nice and smooth, we can compute that under the assumption that  $u$  satisfies equation (1.1), we have that

$$\begin{aligned} \frac{d}{dt} E_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u), \partial_t u \rangle + \varepsilon \langle \nabla u, \nabla \partial_t u \rangle dx \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u, \partial_t u \right\rangle dx \\ &= \int -\varepsilon |\partial_t u|^2 dx. \end{aligned} \quad (1.1)$$

This calculation suggests that equation (1.1) is the  $L^2$  gradient flow (rescaled by  $\sqrt{\varepsilon}$ ) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (1.1) via De Giorgis minimizing movements scheme, which we will do in Theorem 1.2.1.

But first we need to clarify what our potential  $W$  should look like. Classic examples in the scalar case are given by  $W(u) = (u^2 - 1)^2$  or  $W(u) = u^2(u - 1)^2$ , and we call functions like these *doublewell potentials*, see also Figure 1.1.

In higher dimensions, we want to accept the following potentials:  $W: \mathbb{R}^N \rightarrow [0, \infty)$  has to be a smooth multiwell potential with finitely many zeros at  $u = \alpha_1, \dots, \alpha_P \in \mathbb{R}^N$ . Furthermore we ask for polynomial growth in the sense that there exists some  $p \geq 2$  such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \quad (1.3)$$

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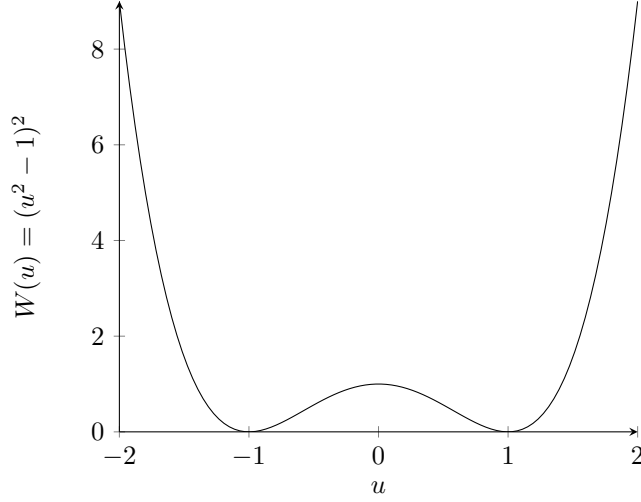


Figure 1.1: The graph of a doublewell potential

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \quad (1.4)$$

for all  $u$  sufficiently large. Lastly we want  $W$  to be convex up to a small perturbation in the sense that there exist smooth functions  $W_{\text{conv}}, W_{\text{pert}}: \mathbb{R}^N \rightarrow [0, \infty)$  such that

$$W = W_{\text{conv}} + W_{\text{pert}}, \quad (1.5)$$

$W_{\text{conv}}$  is convex and

$$\sup_{x \in \mathbb{R}^N} |\nabla^2 W_{\text{pert}}| < \infty. \quad (1.6)$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen–Cahn equation (1.1) as maps from  $[0, T]$  into some suitable function space and thus use the following definition.

**Definition 1.1.1.** *We say that a function  $u_\varepsilon \in C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  which is also in  $L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$  is a weak solution of the Allen–Cahn equation (1.1) with parameter  $\varepsilon > 0$  and initial condition  $u_\varepsilon^0 \in L^2(\mathbb{T}; \mathbb{R}^N)$  if*

1. *the energy stays bounded, which means that*

$$\text{ess sup}_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t)) < \infty, \quad (1.7)$$

2. *its weak time derivative satisfies*

$$\partial_t u_\varepsilon \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N), \quad (1.8)$$

## 1.2 Existence of a solution

3. for almost every  $t \in [0, T]$  and every  $\xi \in L^p([0, T] \times \mathbb{T}; \mathbb{R}^N) \cap W^{1,2}([0, T] \times \mathbb{T}; \mathbb{R}^N)$ , we have

$$\int \left\langle \frac{1}{\varepsilon^2} \nabla W(u_\varepsilon(t)), \xi \right\rangle + \langle \nabla u_\varepsilon(t), \nabla \xi \rangle + \langle \partial_t u_\varepsilon(t), \xi \rangle dx = 0, \quad (1.9)$$

4. the initial conditions are achieved in the sense that  $u_\varepsilon(0) = u_\varepsilon^0$ .

*Remark.* Given our assumptions, we automatically have  $\nabla W(u_\varepsilon)(t) \in L^{p'}(\mathbb{T})$  since

$$|\nabla W(u_\varepsilon)|^{p/(p-1)} \lesssim 1 + |u_\varepsilon|^p \lesssim 1 + W(u), \quad (1.10)$$

which is integrable for almost every time  $t$  since we assume that the energy stays bounded, thus the integral in equation (1.9) is well defined.

Moreover we already obtain 1/2 Hölder-continuity in time from the embedding

$$W^{1,2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)) \hookrightarrow C^{1/2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)), \quad (1.11)$$

which follows from a generalized version of the fundamental theorem of calculus and Hölder's inequality.

## 1.2 Existence of a solution

With Definition 1.1.1, we are able to state our existence results for solutions of the Allen–Cahn equation. The proof uses De Giorgis minimizing movements scheme and arguments from the theory of gradient flows and has been briefly sketched in [LS16].

**Theorem 1.2.1.** *Let  $u_\varepsilon^0: \mathbb{T} \rightarrow \mathbb{R}^N$  be such that  $E_\varepsilon(u_\varepsilon^0) < \infty$ . Then there exists a weak solution  $u_\varepsilon$  to the Allen–Cahn equation (1.1) in the sense of Definition 1.1.1 with initial data  $u_\varepsilon^0$ . Furthermore the solution satisfies the energy dissipation inequality*

$$E_\varepsilon(u_\varepsilon(t)) + \int_0^t \int \varepsilon |\partial_t u_\varepsilon|^2 dx ds \leq E_\varepsilon(u_\varepsilon^0) \quad (1.12)$$

for every  $t \in [0, T]$  and we additionally have  $\partial_{i,j}^2 u_\varepsilon, \nabla W(u_\varepsilon) \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$  for all  $1 \leq i, j \leq d$ . In particular we can test the weak form (1.9) with  $\partial_{i,j}^2 u_\varepsilon$ .

*Proof. Step 1:* A minimization problem

Fix some  $h > 0$ ,  $u_{n-1} \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$  and consider the functional

$$\begin{aligned} \mathcal{F}: W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N) &\rightarrow \mathbb{R} \\ u &\mapsto E_\varepsilon(u) + \frac{1}{2h} \int |u - u_{n-1}|^2 dx. \end{aligned} \quad (1.13)$$

Then  $\mathcal{F}$  is coercive with respect to  $\|\cdot\|_{W^{1,2}}$  and bounded from below by zero, thus we may take a  $W^{1,2}$ -bounded sequence  $(v_k)_{k \in \mathbb{N}}$  in  $W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$  such that  $\mathcal{F}(v_k) \rightarrow \inf \mathcal{F}$

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as  $k \rightarrow \infty$ . These  $v_k$  have a non-relabelled subsequence which converges weakly in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$  and strongly in  $L^2(\mathbb{T}; \mathbb{R}^N)$  to some  $u \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . Moreover  $v_k$  is bounded in  $L^p(\mathbb{T}; \mathbb{R}^N)$  by the lower growth assumption (1.3) on  $W$  and thus obtain by a duality argument that  $u \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ .

Lastly we have

$$\frac{1}{2h} \int |u - u_{n-1}|^2 dx \leq \liminf_{k \rightarrow \infty} \frac{1}{2h} \int |v_k - u_{n-1}|^2 dx, \quad (1.14)$$

$$\int \frac{\varepsilon}{2} |\nabla u|^2 dx \leq \liminf_{k \rightarrow \infty} \int \frac{\varepsilon}{2} |\nabla v_k|^2 dx \quad (1.15)$$

by the weak convergence in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . By passing to another non-relabelled subsequence which converges pointwise almost everywhere, we moreover achieve by continuity of  $W$  that

$$\int \frac{1}{\varepsilon} W(u) dx = \int \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon} W(v_k) dx \leq \liminf_{k \rightarrow \infty} \int \frac{1}{\varepsilon} W(v_k) dx,$$

which proves that  $u$  is a minimizer of  $\mathcal{F}$ .

#### Step 2: Minimizing movements scheme

By iteratively choosing minimizers  $u_n^h$  from Step 1, we obtain a sequence of functions  $u_\varepsilon^0, u_1^h, \dots$ . Thus we may define a function  $u \in C$  as the piecewise linear interpolation at the time-steps  $0, h, 2h, \dots$  of these functions.

#### Step 3: Sharp energy dissipation inequality for $u_n^h$

We claim that there exists some constant  $C > 0$  such that for all  $h > 0$  and  $n \in \mathbb{N}$ , we have

$$E_\varepsilon(u_n^h) + \left( \frac{1}{h} - \frac{C}{2\varepsilon} \right) \int |u_n^h - u_{n-1}^h|^2 dx \leq E_\varepsilon(u_{n-1}^h). \quad (1.16)$$

In order to prove this inequality, we notice that since  $|\nabla^2 W_{\text{pert}}| \leq C$ , the function  $W_{\text{pert}} + C|u|^2/2$  is convex for  $C > 0$  sufficiently large, thus the functional

$$\tilde{E}_\varepsilon(u) := \int \frac{1}{\varepsilon} \left( W(u) + \frac{C}{2} |u|^2 \right) + \frac{\varepsilon}{2} |\nabla u|^2 dx$$

is convex on  $W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ . For a given  $\xi \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ , we thus have that the function  $t \mapsto \tilde{E}_\varepsilon(u_n^h + t\xi)$  is convex and differentiable, which yields that

$$\tilde{E}_\varepsilon(u_n^h + \xi) \geq \tilde{E}_\varepsilon(u_n^h) + \left. \frac{d}{dt} \right|_{t=0} \tilde{E}_\varepsilon(u_n^h + t\xi). \quad (1.17)$$



But since  $u_n^h$  is a minimizer of the functional  $\mathcal{F}$  defined by (1.13), we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \tilde{E}_\varepsilon(u_n^h + t\xi) &= \frac{d}{dt} \Big|_{t=0} \mathcal{F}(u_n^h + t\xi) \\ &+ \frac{C}{2\varepsilon} \int |u_n^h + t\xi|^2 dx - \frac{1}{2h} \int |u_n^h + t\xi - u_{n-1}^h|^2 dx \\ &= \int \frac{C}{\varepsilon} \langle u_n^h, \xi \rangle - \frac{1}{h} \langle u_n^h - u_{n-1}^h, \xi \rangle dx. \end{aligned}$$

Plugging  $\xi = u_{n-1}^h - u_n^h$  into this equation and using inequality (1.17) thus yields

$$\tilde{E}_\varepsilon(u_{n-1}^h) \geq \tilde{E}_\varepsilon(u_n^h) + \int \frac{C}{\varepsilon} \left( \langle u_{n-1}^h, u_n^h \rangle - |u_n^h|^2 \right) + \frac{1}{h} |u_n^h - u_{n-1}^h|^2 dx,$$

which is equivalent to

$$\begin{aligned} E_\varepsilon(u_{n-1}^h) &\geq E_\varepsilon(u_n^h) \\ &+ \int \left( \frac{C}{2\varepsilon} - \frac{C}{\varepsilon} \right) |u_n^h|^2 - \frac{C}{2\varepsilon} |u_{n-1}^h|^2 + \frac{C}{\varepsilon} \langle u_{n-1}^h, u_n^h \rangle + \frac{1}{h} |u_n^h - u_{n-1}^h|^2 dx \\ &= E_\varepsilon(u_n^h) + \left( \frac{1}{h} - \frac{C}{2\varepsilon} \right) \int |u_n^h - u_{n-1}^h|^2 dx, \end{aligned}$$

which is the claimed estimate (1.16).

**Step 4:** Hölder bounds for  $u_h$

From the energy estimate (1.16) we deduce via an induction that

$$\begin{aligned} &E_\varepsilon(u_n^h) + \left( 1 - \frac{Ch}{2\varepsilon} \right) \int_0^{nh} \int |\partial_t u^h|^2 dx dt \\ &= E_\varepsilon(u_n^h) + \left( h - \frac{Ch^2}{2\varepsilon} \right) \sum_{k=1}^n \int \left| \frac{u_n^h - u_{n-1}^h}{h} \right|^2 dx \\ &\leq E_\varepsilon(u_\varepsilon^0). \end{aligned} \tag{1.18}$$

This gives use with the use of Jensen's inequality for  $0 \leq s \leq t \leq T$  and  $h > 0$  sufficiently small that

$$\begin{aligned} \|u^h(t) - u^h(s)\|_{L^2} &= \left\| \int_s^t \partial_t u_h(\tau) d\tau \right\|_{L^2} \\ &\leq \sqrt{t-s} \left( \int_0^T \int |\partial_t u^h(\tau, x)|^2 dx dt \right)^{1/2} \\ &\leq \sqrt{t-s} \left( 1 - \frac{Ch}{2\varepsilon} \right)^{-1/2} (E_\varepsilon(u_\varepsilon^0))^{1/2}, \end{aligned} \tag{1.19}$$

which gives us a uniform bound on the  $L^2$ -Hölder continuity of  $u^h$  in time as  $h$  tends to zero.

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### Step 5: Compactness

In order to apply Arzelà–Ascoli for the sequence  $(u_h)$  as  $h$  tends to zero, we need to check pointwise precompactness of the image and equicontinuity of the sequence. The equicontinuity follows from the previous estimate (1.19). In order to check the pointwise precompactness, we need to verify that for all  $t \in [0, T]$ , the set  $\{u^h(t)\}_{\delta > h > 0}$  is a precompact subset of  $L^2(\mathbb{T}; \mathbb{R}^N)$  (for  $\delta > 0$  sufficiently small). But this follows from the energy bound  $E_\varepsilon(u_n^h) \leq E_\varepsilon(u_0)$  (given by inequality (1.16)), which gives us a time-uniform bound on  $\|\nabla u^h(t)\|_{L^2(\mathbb{T}; \mathbb{R}^N)}$  and on  $\|u^h(t)\|_{L^p(\mathbb{T}; \mathbb{R}^N)}$  and thus on  $\|u^h(t)\|_{W^{1,2}(\mathbb{T}; \mathbb{R}^N)}$ . The compact embedding  $W^{1,2}(\mathbb{T}; \mathbb{R}^N) \hookrightarrow L^2(\mathbb{T}; \mathbb{R}^N)$  thus yields the desired pointwise precompactness.

Therefore we may apply Arzelà–Ascoli to obtain some  $u \in C^{1,2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  and some sequence  $h_n \rightarrow 0$  such that  $u^{h_n}$  converges uniformly to  $u$  on  $[0, T]$  with respect to  $\|\cdot\|_{L^2(\mathbb{T}; \mathbb{R}^N)}$ .

### Step 6: Additional regularity 1

We first want to argue that from our construction, one already obtains that  $u \in L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$ . For this we first notice that for a fixed  $t \in [0, T]$ , the sequence  $u^{h_n}(t)$  is by the energy bound (1.16) a bounded sequence in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ , thus we find some non-relabelled subsequence and  $v \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$  such that  $u^{h_n}(t)$  converges weakly to  $v$  in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . By uniqueness of the limit, we already have  $u(t) = v$  almost everywhere, which yields  $u(t) \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ , and by lower semicontinuity, we may also deduce that

$$\|u(t)\|_{W^{1,2}} \leq \liminf_{n \rightarrow \infty} \|u^{h_n}(t)\|_{W^{1,2}} \leq C E_\varepsilon,$$

from which we deduce that  $u \in L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$ .

Secondly the boundedness of the energies

$$\sup_{0 \leq t \leq T} E_\varepsilon(u(t)) < \infty$$

follows from the lower semicontinuity of the energy and the pointwise  $L^2$  convergence and pointwise weak convergence in  $W^{1,2}$  as described in step 1.

Lastly we want to argue that  $\partial_t u \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . From inequality (1.18) in step 4, we deduce that  $\partial_t u^h$  is a bounded sequence in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . Thus we find a non-relabelled subsequence of  $h_n$  and some  $w \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$  such that  $u^{h_n}$  converges weakly to  $w$  in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . But then  $w$  is already the weak time derivative of  $u$  since for any testfunction  $\xi$ , we have

$$\begin{aligned} \int_{[0, T] \times \mathbb{T}} \langle u, \partial_t \xi \rangle dx dt &= \lim_{n \rightarrow \infty} \int_{[0, T] \times \mathbb{T}} \langle u^{h_n}, \partial_t \xi \rangle dx dt \\ &= \lim_{n \rightarrow \infty} - \int_{[0, T] \times \mathbb{T}} \langle \partial_t u^{h_n}, \xi \rangle dx dt \\ &= - \int_{[0, T] \times \mathbb{T}} \langle w, \xi \rangle dx dt. \end{aligned}$$

**Step 7:**  $u$  is a weak solution

Going back to step 1, we see that  $u_n^h$  solves the Euler–Lagrange equation

$$\int \frac{1}{\varepsilon} \langle \nabla W(u_n^h), \xi \rangle + \varepsilon \langle \nabla u_n^h, \nabla \xi \rangle + \left\langle \frac{u_n^h - u_{n-1}^h}{h}, \xi \right\rangle dx = 0 \quad (1.20)$$

for any testfunction  $\xi \in C^\infty(\mathbb{T}; \mathbb{R}^N)$ . Let  $t \in [0, T]$ . Since  $u_h$  is defined as the pointwise linear interpolation of the functions  $u_n^h$ , we find for the sequence  $h_n$  corresponding sequences  $\lambda_n \in [0, 1]$  and  $k_n \in \mathbb{N}$  such that

$$t = \lambda_n(k_n - 1)h_n + (1 - \lambda_n)k_n h_n$$

and therefore we can write

$$u_h(t) = \frac{k_n h_n - t}{h_n} u_{k_n-1}^h + \frac{t - (k_n - 1)h_n}{h_n} u_{k_n}^{h_n}.$$

In order to pass to the limit in equation (1.20), we first note that  $u_{k_n}^{h_n}$  converges to  $u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  since

$$\begin{aligned} \|u(t) - u_{k_n}^{h_n}\|_{L^2(\mathbb{T}; \mathbb{R}^N)} &\leq \|u(t) - u^{h_n}(t)\|_{L^2(\mathbb{T}; \mathbb{R}^N)} + \|u^{h_n}(t) - u^{h_n}(k_n h_n)\|_{L^2(\mathbb{T}; \mathbb{R}^N)} \\ &\lesssim \|u(t) - u^{h_n}(t)\|_{L^2(\mathbb{T}; \mathbb{R}^N)} + \sqrt{h_n}, \end{aligned} \quad (1.19)$$

which goes to zero as  $n$  tends to infinity. This implies that

$$\begin{aligned} \int \varepsilon \langle \nabla u(t), \nabla \xi \rangle dx &= \int \varepsilon \langle u(t), \operatorname{div} \nabla \xi \rangle dx \\ &= \lim_{n \rightarrow \infty} \int \varepsilon \langle u_{k_n}^{h_n}, \operatorname{div} \nabla \xi \rangle dx \\ &= \lim_{n \rightarrow \infty} \int \varepsilon \langle \nabla u_{k_n}^{h_n}, \nabla \xi \rangle dx. \end{aligned}$$

In step 6, we moreover have shown that  $\partial_t u^h$  converges weakly to  $\partial_t u$  in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . This yields, by choosing cylindrical testfunctions, that  $\partial_t u^h(t)$  converges weakly to  $\partial_t u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  for almost every  $t \in [0, T]$ . Thus we obtain for almost every  $t \in [0, T]$  the convergence

$$\int \langle \partial_t u(t), \xi \rangle dx = \lim_{n \rightarrow \infty} \int \langle \partial_t u(t), \xi \rangle dx = \lim_{n \rightarrow \infty} \int \left\langle \frac{u_{k_n}^{h_n} - u_{k_n-1}^{h_n}}{h_n}, \xi \right\rangle dx.$$

To obtain the weak equation, we still need to prove convergence of remaining term. For this we note that

$$\frac{1}{\varepsilon} \left| \langle \nabla W(u_{k_n}^{h_n}), \xi \rangle \right| \lesssim \left( 1 + |u_{k_n}^{h_n}|^{p-1} \right) |\xi|,$$

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which is a bounded sequence in  $L^{p'}(\mathbb{T}; \mathbb{R}^N)$ . Since we also may pass to a non-relabelled subsequence which converges almost everywhere, we thus obtain that

$$\int \langle \nabla W(u(t)), \xi \rangle dx = \lim_{n \rightarrow \infty} \int \langle \nabla W(u_{k_n}^{h_n}), \xi \rangle dx.$$

Therefore it follows from the Euler–Lagrange equation (1.20) that

$$\int \frac{1}{\varepsilon} \langle \nabla W(u(t)), \xi \rangle + \varepsilon \langle \nabla u, \nabla \xi \rangle + \langle \partial_t u, \xi \rangle dx = 0$$

for all  $\xi \in C^\infty(\mathbb{T}; \mathbb{R}^N)$ . By continuity this extends to all  $\xi \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ .

Thus the last thing to check is that

$$\sup_{0 \leq t \leq T} E_\varepsilon(u(t)) < \infty,$$

which follows from the next step.

#### Step 8: Sharp energy dissipation inequality for $u$

Let  $0 \leq t \leq T$  and define  $k_n$  as in step 7, where we also established that  $u_{k_n}^{h_n}$  converges to  $u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$ , and thus we may pass to another non-relabelled subsequence to obtain pointwise convergence almost everywhere. By Fatou's Lemma, we thus obtain

$$\int \frac{1}{\varepsilon} W(u(t)) dx \leq \liminf_{n \rightarrow \infty} \int \frac{1}{\varepsilon} W(u_{k_n}^{h_n}) dx.$$

Moreover we can deduce from the  $L^2$ -convergence that  $\nabla u_{k_n}^{h_n}$  converges to  $\nabla u(t)$  in the distributional sense. But  $\nabla u_{k_n}^{h_n}$  is uniformly bounded in  $L^2(\mathbb{T}; \mathbb{R}^N)$  by the energy dissipation inequality (1.16), thus we already obtain that  $\nabla u_{k_n}^{h_n}$  converges weakly to  $\nabla u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  which yields

$$\int \frac{\varepsilon}{2} |\nabla u(t)|^2 dx \leq \liminf_{n \rightarrow \infty} \int \frac{\varepsilon}{2} |\nabla u_{k_n}^{h_n}|^2 dx.$$

Lastly we have by the weak convergence of  $\partial_t u^{h_n}$  to  $\partial_t u$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  proven in step 6 that

$$\begin{aligned} \int_0^t \int |\partial_t u|^2 dx dt &\leq \liminf_{n \rightarrow \infty} \left(1 - \frac{Ch_n}{2\varepsilon}\right) \int_0^t \int |\partial_t u_{k_n}^{h_n}|^2 dx dt \\ &\leq \liminf_{n \rightarrow \infty} \left(1 - \frac{Ch_n}{2\varepsilon}\right) \int_0^{k_n h_n} \int |\partial_t u_{k_n}^{h_n}|^2 dx dt. \end{aligned}$$

Summarizing these estimates, we obtain by the energy dissipation inequality for  $u^h$  (1.18) that for all  $0 \leq t \leq T$  we have

$$\begin{aligned} E_\varepsilon(u(t)) + \int_0^t \int |\partial_t u|^2 dx dt &\leq \liminf_{n \rightarrow \infty} E_\varepsilon(u_{k_n}) + \left(1 - \frac{Ch_n}{2\varepsilon}\right) \int_0^{k_n h_n} \int |\partial_t u_{k_n}^{h_n}|^2 dx dt \\ &\leq E_\varepsilon(u_\varepsilon^0). \end{aligned}$$

**Step 8:** Additional regularity 2

In order to complete the proof, we still have to show that  $\partial_{i,j}^2 u$  and  $\nabla W(u)$  are elements of  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . In order to show that the second partial derivatives of  $u$  are square-integrable, we test the weak formulation (1.9) with the finite differences. To this end, define the finite differences as

$$\Delta_h^+ v(t, x) := \frac{v(t, x + hv) - v(t, x)}{h}, \quad \Delta_h^- v(t, x) := \frac{v(t, x - hv) - v(t, x)}{h}$$

for some  $h > 0$  and  $v \in \mathbb{R}^d$ . Thus plugging  $\Delta_h^- \Delta_h^+ u$  into (1.9) yields by the transformation formula that

$$0 = \int_0^T \int \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u), \Delta_h^+ u \rangle + \langle \nabla \Delta_h^+ u, \nabla \Delta_h^+ u \rangle + \langle \partial_t \Delta_h^+ u, \Delta_h^+ u \rangle dx dt,$$

which is equivalent to

$$\begin{aligned} & \int_0^T \int |\Delta_h^+ \nabla u|^2 dx dt \\ &= - \int_0^T \int \partial_t \left( \frac{|\Delta_h^+ u|^2}{2} \right) + \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u), \Delta_h^+ u \rangle dx dt \\ &= \int \frac{|\Delta_h^+ u(0)|^2 - |\Delta_h^+ u(T)|^2}{2} dx \\ &\quad - \int_0^1 \int_0^T \int \langle D^2 W((1-s)u(t, x) + su(t, x + hv)) \Delta_h^+ u, \Delta_h^+ u \rangle dx dt ds. \end{aligned}$$

The first summand can be estimated by  $\|u\|_{L^\infty([0, T], W^{1,2}(\mathbb{T}; \mathbb{R}^N))}$ . For the second summand, we partition  $W$  into the sum of  $W_{\text{conv}}$  and  $W_{\text{pert}}$ . The term involving the convex summand can then be estimated by

$$\int_0^1 \int_0^T \int \langle D^2 W_{\text{conv}}((1-s)u(t, x) + su(t, x + hv)) \Delta_h^+ u, \Delta_h^+ u \rangle dx dt ds \geq 0$$

and for the perturbation term, we get via the bound on its second derivative that

$$\begin{aligned} & \left| \int_0^1 \int_0^T \int \langle D^2 W_{\text{pert}}((1-s)u(t, x) + su(t, x + hv)) \Delta_h^+ u, \Delta_h^+ u \rangle dx dt ds \right| \\ & \lesssim \int_0^T \int |\nabla u|^2 dx dt, \end{aligned}$$

which is also finite. Combining these estimates, we obtain that  $\int_0^T \int |\Delta_h^+ \nabla u|^2 dx dt$  is uniformly bounded in  $h$ . Applying our calculation to all directions  $v \in \mathbb{R}^d$ , we get by the finite-differences theorem for all  $1 \leq i, j \leq d$  that  $\partial_{i,j}^2 u \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ .

## 1 The Allen–Cahn equation

In order to obtain  $\nabla W(u) \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ , we again consider the weak formulation (1.9) and notice that since we have already shown that both the time derivative and second space derivatives of  $u$  are square-integrable, our claim follows from a duality argument.  $\square$

*Remark.* The inequality (1.16) with the factor  $1/2h$  instead of  $1/h - C/2\varepsilon$  follows immediately from the definition of our optimization problem, but is not optimal for fixed  $\varepsilon$  if we want to study the behaviour as  $h$  tends to zero. Moreover this so called *sharp energy dissipation inequality* is important for later.

*Remark.* The energy dissipation inequality (1.12) can be deduced via the formal calculation

$$\begin{aligned} \frac{d}{dt} E_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u), \partial_t u \rangle + \varepsilon \langle \nabla u, \nabla \partial_t u \rangle \, dx \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u, \partial_t u \right\rangle \, dx \\ &= -\varepsilon \int |\partial_t u_\varepsilon|^2 \, dx. \end{aligned}$$

In order to make this calculation rigorous, we however need to show that  $t \mapsto E_\varepsilon(u(t))$  is absolutely continuous, which is non-trivial, but it would give us equality in energy dissipation inequality (1.12). Since we will only need the inequality, our proof will however suffice.

## 2 Convergence of the Allen–Cahn equations

The main content of this chapter is to look at the behaviour of solutions of the Allen–Cahn equation (1.1) as the parameter  $\varepsilon$  tends to zero. As it turns out the scalar case is significantly easier to handle than the vectorial case, thus we shall first focus on the case  $N = 1$ , also called the *twophase case*.

### 2.1 The twophase case

When want to consider the behaviour of solutions  $u_\varepsilon: [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$  to (1.1) for  $\varepsilon \rightarrow 0$ . Let us for now assume that the energies of the initial functions  $E_\varepsilon(u_\varepsilon^0)$  stay uniformly bounded as  $\varepsilon$  tends to zero. Then due to the energy dissipation inequality (1.12), we already obtain that for all  $0 \leq t \leq T$ , we have that  $E_\varepsilon(u_\varepsilon(t))$  stays uniformly bounded.

Another important observation for the convergence is the classic Modica Mortola trick (hier Referenz einfügen): let  $\alpha < \beta$  be the two distinct zeros of the doublewell potential  $W$ . Then we define a primitive of  $\sqrt{2W(u)}$  via

$$\phi(u) := \int_\alpha^u \sqrt{2W(s)} \, ds.$$

For  $\psi_\varepsilon := \phi \circ u_\varepsilon$ , we can show that  $\psi_\varepsilon \in W^{1,1}((0, T) \times \mathbb{T})$  with weak derivatives  $\nabla \psi_\varepsilon = \sqrt{2W(u_\varepsilon)} \nabla u_\varepsilon$  and  $\partial_t \psi_\varepsilon = \sqrt{2W(u_\varepsilon)} \partial_t u_\varepsilon$ . (we will later show this in more generality (Referenz einfügen)).

Thus via Young's inequality, we can compute

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &= \int \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \, dx \\ &\geq \int \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx \\ &= \int |\nabla \psi_\varepsilon| \, dx, \end{aligned} \tag{2.1}$$

which suggests that we might hope for good compactness properties of  $\phi \circ u_\varepsilon$ . We combine these two observations into the following Proposition.

**Proposition 2.1.1.** *Given initial data  $u_\varepsilon^0$  whose energies stay uniformly bounded in the sense that*

$$\sup_{\varepsilon > 0} E_\varepsilon(u_\varepsilon^0) < \infty, \tag{2.2}$$

## 2 Convergence of the Allen–Cahn equations

there exists for any sequence  $\varepsilon \rightarrow 0$  some non-relabelled subsequence such that the solutions of the Allen–Cahn equation (1.1) with initial condition  $u_\varepsilon^0$  converge in  $L^1((0, T) \times \mathbb{T})$  to some  $u = \alpha(1 - \chi) + \beta\chi$  with  $\chi \in \text{BV}((0, T) \times \mathbb{T}; \{0, 1\})$ . Moreover the compositions  $\psi_\varepsilon$  are uniformly bounded in  $\text{BV}((0, T) \times \mathbb{T})$  and converge to  $\phi \circ u$  in  $L^1((0, T) \times \mathbb{T})$ .

*Proof.* From the energy dissipation inequality (1.12) in Theorem 1.2.1 we infer that for all  $\varepsilon > 0$ , it holds that

$$\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t)) \leq E_\varepsilon(u_\varepsilon^0), \quad (2.3)$$

whose right hand side is by assumption uniformly bounded in  $\varepsilon$ . We want to use a similar calculation as (2.1) we thus infer that  $\nabla \psi_\varepsilon$  is uniformly bounded in  $L^1((0, T) \times \mathbb{T})$ . Moreover we may estimate

$$\begin{aligned} \int_0^T \int |\psi_\varepsilon| \, dx \, dt &= \int_0^T \int \left| \int_\alpha^{u_\varepsilon} \sqrt{2W(s)} \, ds \right| \, dx \, dt \\ &\leq \int_0^T \int |u_\varepsilon - \alpha| \sup_{s \in [\alpha, u_\varepsilon]} \sqrt{2W(s)} \, dx \, dt \\ &\lesssim 1 + \int_0^T \int |u_\varepsilon|^{1+p/2} \, dx \, dt \\ &\lesssim 1 + \int_0^T \int W(u_\varepsilon) \, dx \, dt, \end{aligned} \quad (2.4)$$

which is uniformly bounded via the energy bound. Here the last two inequalities follow from the upper respectively lower growth assumptions (1.3) on  $W$ . Thus we have indeed that  $\psi_\varepsilon$  is a bounded sequence in  $\text{BV}((0, T) \times \mathbb{T})$  and therefore there exists some non-relabelled subsequence and some  $v \in \text{BV}((0, T) \times \mathbb{T})$  such that  $\psi_\varepsilon$  converges to  $v$  in  $L^1((0, T) \times \mathbb{T})$ .

We notice that since  $W$  is non-negative and only has two zeros, the function  $\phi$  is strictly increasing and continuous on  $\mathbb{R}$ , and is thus invertible. Moreover we may pass to a further non-relabelled subsequence of  $\psi_\varepsilon$  which converges almost everywhere to  $v$ . Thus defining  $u := \phi^{-1}(v)$ , we obtain that

$$u_\varepsilon = \phi^{-1}(\psi_\varepsilon) \rightarrow \phi^{-1}(v) = u$$

converges pointwise almost everywhere.

Moreover we notice that by Fatou's Lemma and the boundedness of the energies that

$$\int W(u) \, dx \leq \liminf_{\varepsilon \rightarrow 0} \int W(u_\varepsilon) \, dx \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon E_\varepsilon(u_\varepsilon) = 0.$$

Again using the non-negativity of  $W$ , this yields that  $W(u) = 0$  almost everywhere. Thus  $u \in \{\alpha, \beta\}$  almost everywhere and we may write  $u = \alpha(1 - \chi) + \beta\chi$  for some  $\chi: (0, T) \times \mathbb{T} \rightarrow \{0, 1\}$ . Looking again at the definition of  $u$ , we moreover obtain that

$$v = \phi(u) = (1 - \chi)\phi(\alpha) + \chi\phi(\beta) = \chi \int_\alpha^\beta \sqrt{2W(s)} \, ds$$



## 2.1 The twophase case

and since  $v$  is a function of bounded variation, this implies that  $\chi$  is of bounded variation as well.

Finally from the energy bound and the estimate  $|u_\varepsilon| \lesssim 1 + W(u)$ , we infer that  $u_\varepsilon$  is  $L^p$ -bounded, and since  $u_\varepsilon$  converges pointwise almost everywhere to  $u$ , we obtain the desired  $L^1$ -convergence.  $\square$

Nextup, we want to make sure that  $u$  respectively  $\chi$  assume their initial data and on the way obtain a useful bound on the time derivative.

**Lemma 2.1.2.** *With the assumptions of Proposition 2.1.1, we have  $\psi_\varepsilon \in W^{1,2}([0, T]; L^1(\mathbb{T}))$  with the estimate*

$$\left( \int_0^T \left( \int |\partial_t \psi_\varepsilon| dx \right)^2 dt \right)^{1/2} \lesssim E_\varepsilon(u_\varepsilon^0). \quad (2.5)$$

Furthermore the sequence  $u_\varepsilon$  is precompact in  $C([0, T]; L^2(\mathbb{T}))$ . In particular  $\chi$  achieves the initial condition (??check what we need).

*Proof.* **Step 1**  $\psi_\varepsilon \in W^{1,2}([0, T]; L^1(\mathbb{T}))$  and satisfies the inequality (2.5)

From estimate (2.4) we can infer that

$$\int_0^T \left( \int |\psi_\varepsilon| dx \right)^2 \lesssim \int_0^T \left( 1 + \int W(u_\varepsilon) dx \right)^2 dt < \infty$$

via the uniform boundedness of the energies (2.3). For the desired bound (2.5), we estimate via Hölder's inequality and the uniform boundedness of the energies that

$$\begin{aligned} \int_0^T \left( \int |\partial_t \psi_\varepsilon| dx \right)^2 dt &\leq \int_0^T \left( \int \sqrt{2W(u_\varepsilon)} |\partial_t u_\varepsilon| dx \right)^2 dt \\ &= \int_0^T \left( \int \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \sqrt{\varepsilon} |\partial_t u_\varepsilon| dx \right)^2 dt \\ &\leq \int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon) dx \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \\ &\leq 2 E_\varepsilon(u_\varepsilon^0) \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \quad (2.3) \\ &\leq 2 (E_\varepsilon(u_\varepsilon^0))^2, \quad (1.12) \end{aligned}$$

which completes step 1.

**Step 2 :** The sequence  $\psi_\varepsilon$  is precompact in  $L^\infty([0, T]; L^1(\mathbb{T}))$ .

As noted in embedding (1.11)

$\square$



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