Convergence of Allen-Cahn equations to multi-phase mean curvature flow

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1 The Allen–Cahn equation

This chapter follows [LS16], but since the authors decided to only sketch some proofs. we want to go into more detail.

Let $\Lambda > 0$ and define the flat torus $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$, where we work with periodic boundary conditions and write $\int \mathrm{d}x$ instead of $\int_{\mathbb{T}} \mathrm{d}x$. Then for $u \colon [0, \infty) \times \mathbb{T} \to \mathbb{R}^N$ and some potential $W \colon \mathbb{R}^N \to [0, \infty)$, the *Allen–Cahn equation* with parameter $\varepsilon > 0$ is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \tag{1.1}$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to u for a fixed time the real number

$$\mathrm{E}_\varepsilon(u) \coloneqq \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 \, \mathrm{d}x \,. \tag{1.2}$$

If everything is nice and smooth, we can compute that under the assumption that u satisfies equation (1.1), we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u) \,,\, \partial_t u \rangle + \varepsilon \langle \nabla u \,,\, \nabla \partial_t u \rangle \, \mathrm{d}x \\
&= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u \,,\, \partial_t u \right\rangle \mathrm{d}x \\
&= \int -\varepsilon |\partial_t u|^2 \, \mathrm{d}x \,. \qquad (1.1)
\end{aligned}$$

This calculation suggests that equation (1.1) is the L^2 gradient-flow (rescaled by $\sqrt{\varepsilon}$) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (1.1) via De Giorgis minimizing movements scheme, which we will do in theorem **??**.

But first we need to clarify what our potential W should look like. Classic examples in the scalar case are given by $W(u) = \left(u^2 - 1\right)^2$ or $W(u) = u^2 (u-1)^2$, and we call functions like these *doublewell potentials*, see also 1.

In higher dimensions, we want to accept the following potentials: $W \colon \mathbb{R}^N \to [0, \infty)$ has to be a a smooth multiwell potential with finitely many zeros at $u = \alpha_1, \ldots, \alpha_P \in \mathbb{R}^N$. Furthermore we aks for polynomial growth in the sense that there exists some $p \geq 2$ such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \tag{1.3}$$

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \tag{1.4}$$

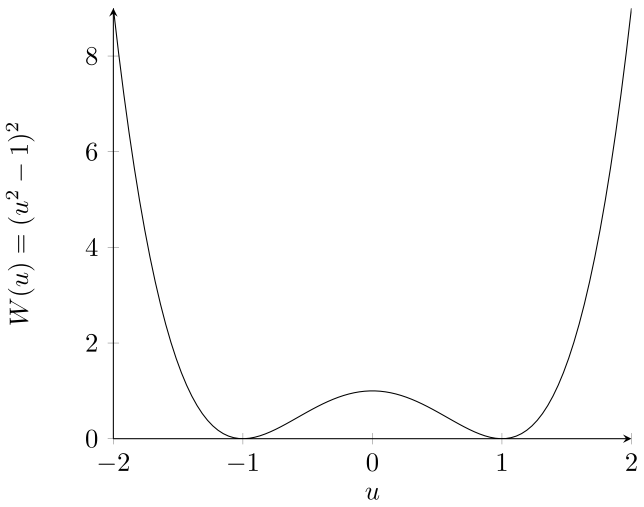

Figure 1.1: The graph of a doublewell potential

for all u sufficiently large. Lastly we want W to be convex up to a small perturbation in the sense that there exist smooth functions $W_{\text{conv}}, W_{\text{pert}} \colon \mathbb{R}^N \to [0, \infty)$ such that

$$W = W_{\text{conv}} + W_{\text{pert}}, \tag{1.5}$$

W_{conv} is convex and

$$\sup_{x \in \mathbb{R}^N} \left| \nabla^2 W_{\text{pert}} \right| < \infty. \tag{1.6}$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

Bibliography

[LS16] Tim Laux and Theresa Simon. "Convergence of the Allen-Cahn Equation to Multiphase Mean Curvature Flow". In: *Communications on Pure and Applied Mathematics* 71 (June 2016). DOI: 10.1002/cpa.21747.