Convergence of Allen-Cahn equations to multi-phase mean curvature flow

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1 The Allen–Cahn equation

1.1 Structure of the equation

This chapter follows [LS16], but since the authors decided to only sketch some proofs. we want to go into more detail.

Let $\Lambda > 0$ and define the flat torus $\mathbb{T} = [0,\Lambda)^d \subset \mathbb{R}^d$, where we work with periodic boundary conditions and write $\int \mathrm{d}x$ instead of $\int_{\mathbb{T}} \mathrm{d}x$. Then for $u\colon [0,\infty) \times \mathbb{T} \to \mathbb{R}^N$ and some potential $W\colon \mathbb{R}^N \to [0,\infty)$, the *Allen–Cahn equation* with parameter $\varepsilon > 0$ is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2}\nabla W(u). \tag{1.1}$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to u for a fixed time the real number

$$\mathrm{E}_\varepsilon(u) \coloneqq \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2}|\nabla u|^2 \,\mathrm{d}x\,. \tag{1.2}$$

If everything is nice and smooth, we can compute that under the assumption that u satisfies equation (1.1), we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon}\langle \nabla W(u)\,,\, \partial_t u\rangle + \varepsilon\langle \nabla u\,,\, \nabla\partial_t u\rangle \,\mathrm{d}x \\ &= \int \left\langle \frac{1}{\varepsilon}\nabla W(u) - \varepsilon\Delta u\,,\, \partial_t u \right\rangle \mathrm{d}x \\ &= \int -\varepsilon|\partial_t u|^2 \,\mathrm{d}x\,. \end{aligned} \tag{1.1}$$

This calculation suggests that equation (1.1) is the L^2 gradient-flow (rescaled by $\sqrt{\varepsilon}$) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (1.1) via De Giorgis minimizing movements scheme, which we will do in theorem Theorem 1.2.1.

But first we need to clarify what our potential W should look like. Classic examples in the scalar case are given by $W(u) = \left(u^2-1\right)^2$ or $W(u) = u^2(u-1)^2$, and we call functions like these *doublewell potentials*, see also Figure 1.1.

In higher dimensions, we want to accept the following potentials: $W\colon \mathbb{R}^N \to [0,\infty)$ has to be a a smooth multiwell potential with finitely many zeros at $u = \alpha_1, \dots, \alpha_P \in \mathbb{R}^N$. Furthermore we aks for polynomial growth in the sense that there exists some $p \geq 2$ such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \tag{1.3}$$

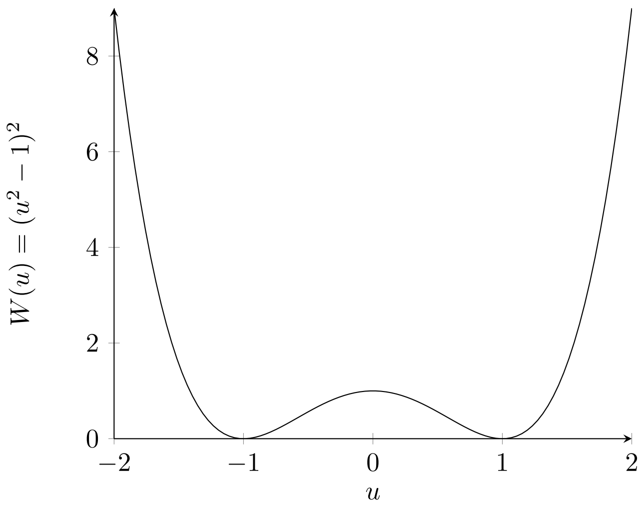

Figure 1.1: The graph of a doublewell potential

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \tag{1.4}$$

for all u sufficiently large. Lastly we want W to be convex up to a small perturbation in the sense that there exist smooth functions W_{conv}, $W_{\mathrm{pert}} : \mathbb{R}^N \to [0, \infty)$ such that

$$W = W_{\mathrm{conv}} + W_{\mathrm{pert}} \,, \tag{1.5}$$

W_{conv} is convex and

$$\sup_{x \in \mathbb{R}^N} \left| \nabla^2 W_{\mathrm{pert}} \right| < \infty. \tag{1.6}$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen-Cahn equation (1.1)as maps from $[0, T]$ into some suitable function space and thus use the following definition.

Definition 1.1.1. *We say that a function $u_\varepsilon \in \mathrm{C}\left([0,T]; \mathrm{L}^2(\mathbb{T}; \mathbb{R}^N)\right)$ which is also in $\mathrm{L}^\infty\left([0,T]; \mathrm{W}^{1,2}(\mathbb{T}; \mathbb{R}^N)\right)$ is a weak solution of the Allen–Cahn equation (1.1) with parameter $\varepsilon > 0$ and initial condition $u_\varepsilon^0 \in \mathrm{L}^2(\mathbb{T}; \mathbb{R}^N)$ if*

1. *the energy stays bounded, which means that*

$$\operatorname*{ess\,sup}_{0 \le t \le T} \mathrm{E}_\varepsilon(u_\varepsilon(t)) < \infty, \tag{1.7}$$

2. *its weak time derivative satisfies*

$$\partial_t u_\varepsilon \in \mathrm{L}^2([0,T] \times \mathbb{T}; \mathbb{R}^N), \tag{1.8}$$

3. *for almost every $t \in [0,T]$ and every $\xi \in \mathrm{L}^p([0,T]\times\mathbb{T};\mathbb{R}^N)\cap \mathrm{W}^{1,2}([0,T]\times\mathbb{T};\mathbb{R}^N)$, we have*

$$\int \left\langle \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon(t)) \, , \, \xi \right\rangle + \langle \nabla u_\varepsilon(t) \, , \, \nabla\xi\rangle + \langle \partial_t u_\varepsilon(t) \, , \, \xi\rangle \, \mathrm{d}x = 0, \tag{1.9}$$

4. *the initial conditions are achieved in the sense that $u_\varepsilon(0) = u_\varepsilon^0$.*

Remark. Given our assumptions, we automatically have $\nabla W(u_\varepsilon)(t) \in \mathrm{L}^{p'}(\mathbb{T})$ since

$$|\nabla W(u_\varepsilon)|^{p/(p-1)} \lesssim 1 + |u_\varepsilon|^p \lesssim 1 + W(u), \tag{1.10}$$

which is integrable for almost every time t since we assume that the energy stays bounded, thus the integral in equation (1.9) is well defined.

Moreover we already obtain $1/2$ Hölder-continuity in time from the embedding

$$\mathrm{W}^{1,2}([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)) \hookrightarrow \mathrm{C}^{1/2}([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)), \tag{1.11}$$

which follows from a generalized version of the fundamental theorem of calculus and Hölder's inequality.

1.2 Existence of a solution

With Definition 1.1.1, we are able to state our existence results for solutions of the Allen–Cahn equation. The proof uses De Giorgis minimizing movements scheme and arguments from the theory of gradient flows and has been briefly sketched in [LS16].

Theorem 1.2.1. *Let $u_\varepsilon^0 \colon \mathbb{T} \to \mathbb{R}^N$ be such that $\mathrm{E}_\varepsilon(u_\varepsilon^0) < \infty$. Then there exists a weak solution u_ε to the Allen–Cahn equation (1.1) in the sense of Definition 1.1.1 with initial data u_ε^0. Furthermore the solution satisfies the energy dissipation identity*

$$\mathrm{E}_\varepsilon(u_\varepsilon(t)) + \int_0^t \int \varepsilon|\partial_t u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}s \le \mathrm{E}_\varepsilon(u_\varepsilon^0) \tag{1.12}$$

for every $t \in [0,T]$ and we additionally have $\partial_{i,j}^2 u_\varepsilon, \nabla W(u_\varepsilon) \in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$ for all $1 \le i,j \le d$. In particular we can test the weak form (1.9) with $\partial_{i,j}^2 u_\varepsilon$.

Proof. **Step 1:** A minimization problem

Fix some $h > 0$, $u_{n-1} \in \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$ and consider the functional

$$\begin{aligned} \mathcal{F} \colon \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N) &\to \mathbb{R} \\ u &\mapsto \mathrm{E}_\varepsilon(u) + \frac{1}{2h}\int |u-u_{n-1}|^2 \, \mathrm{d}x \, . \end{aligned} \tag{1.13}$$

Then $\mathcal{F}$ is coercive with respect to $\|\cdot\|_{\mathrm{W}^{1,2}}$ and bounded from below by zero, thus we may take a $\mathrm{W}^{1,2}$-bounded sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$ such that $\mathcal{F}(v_k) \to \inf \mathcal{F}$

as $k \to \infty$. These v_k have a non-relabelled subsequence which converges weakly in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$ and strongly in $\mathrm{L}^2(\mathbb{T},\mathbb{R}^N)$ to some $u \in \mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$. Moreover v_k is bounded in $\mathrm{L}^p(\mathbb{T};\mathbb{R}^N$ by the lower growth assumption (1.3) on W and thus obtain by a duality argument that $u \in \mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$.

Lastly we have

$$\frac{1}{2h}\int |u-u_{n-1}|^2 \,\mathrm{d}x \le \liminf_{k\to\infty} \frac{1}{2h}\int |v_k-u_{n-1}|^2 \,\mathrm{d}x\,, \tag{1.14}$$

$$\int \frac{\varepsilon}{2}|\nabla u|^2 \,\mathrm{d}x \le \liminf_{k\to\infty} \int \frac{\varepsilon}{2}|\nabla v_k|^2 \,\mathrm{d}x \tag{1.15}$$

by the weak convergence in $\mathrm{W}^{1,2}(\mathbb{T},\mathbb{R}^N)$. By passing to another non-relabelled subsequence which converges pointwise almost everywhere, we moreover achieve by continuity of W that

$$\int \frac{1}{\varepsilon}W(u)\,\mathrm{d}x = \int \liminf_{k\to\infty} \frac{1}{\varepsilon}W(v_k)\,\mathrm{d}x \le \liminf_{k\to\infty}\int \frac{1}{\varepsilon}W(v_k)\,\mathrm{d}x\,,$$

which proves that u is a minimizer of $\mathcal{F}$.

Step 2: Minimizing movements scheme

By iteratively choosing minimizers u_n^h from Step 1, we obtain a sequence of functions $u_\varepsilon^0, u_1^h, \ldots$. Thus we may define a function $u \in \mathrm{C}$ as the piecewise linear interpolation at the time-steps $0, h, 2h, \ldots$ of these functions.

Step 3: Sharp energy dissipation inequality for u_n^h

We claim that there exists some constant $C > 0$ such that for all $h > 0$ and $n \in \mathbb{N}$, we have

$$\mathrm{E}_\varepsilon(u_n^h) + \left(\frac{1}{h} - \frac{C}{2\varepsilon}\right)\int \left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x \le \mathrm{E}_\varepsilon(u_{n-1}^h). \tag{1.16}$$

In order to prove this inequality, we notice that since $\left|\nabla^2 W_{\mathrm{pert}}\right| \le C$, the function $W_{\mathrm{pert}} + C|u|^2/2$ is convex for $C > 0$ sufficiently large, thus the functional

$$\tilde{\mathrm{E}}_\varepsilon(u) \coloneqq \int \frac{1}{\varepsilon}\left(W(u) + \frac{C}{2}|u|^2\right) + \frac{\varepsilon}{2}|\nabla u|^2 \,\mathrm{d}x$$

is convex on $\mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$. For a given $\xi \in \mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$, we thus have that the function $t \mapsto \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi)$ is convex and differentiable, which yields that

$$\tilde{\mathrm{E}}_\varepsilon(u_n^h + \xi) \ge \tilde{\mathrm{E}}_\varepsilon(u_n^h) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi). \tag{1.17}$$

But since u_n^h is a minimizer of the functional $\mathcal{F}$ defined by (1.13), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi) = & \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{F}(u_n^h + t\xi) \\ & + \frac{C}{2\varepsilon}\int \left|u_n^h + t\xi\right|^2 \mathrm{d}x - \frac{1}{2h}\int \left|u_n^h + t\xi - u_{n-1}^h\right|^2 \mathrm{d}x \\ = & \int \frac{C}{\varepsilon}\langle u_n^h \,,\, \xi\rangle - \frac{1}{h}\langle u_n^h - u_{n-1}^h \,,\, \xi\rangle \,\mathrm{d}x\,. \end{aligned}$$

Plugging $\xi = u_{n-1}^h - u_n^h$ into this equation and using inequality (1.17) thus yields

$$\tilde{\mathrm{E}}_\varepsilon(u_{n-1}^h) \geq \tilde{\mathrm{E}}_\varepsilon(u_n^h) + \int \frac{C}{\varepsilon}\left(\langle u_{n-1}^h \,,\, u_n^h\rangle - \left|u_n^h\right|^2\right) + \frac{1}{h}\left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x\,,$$

which is equivalent to

$$\begin{aligned} & \mathrm{E}_\varepsilon(u_{n-1}^h) \geq \mathrm{E}_\varepsilon(u_n^h) \\ & \qquad + \int \left(\frac{C}{2\varepsilon} - \frac{C}{\varepsilon}\right)\left|u_n^h\right|^2 - \frac{C}{2\varepsilon}\left|u_{n-1}^h\right|^2 + \frac{C}{\varepsilon}\langle u_{n-1}^h \,,\, u_n^h\rangle + \frac{1}{h}\left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x \\ & \quad = \mathrm{E}_\varepsilon(u_n^h) + \left(\frac{1}{h} - \frac{C}{2\varepsilon}\right)\int \left|u_n^h - u_{n-1}^h\right|^2 \mathrm{d}x\,, \end{aligned}$$

which is the claimed estimate (1.16).

Step 4: Hölder bounds for u_h

From the energy estimate (1.16) we deduce via an induction that

$$\begin{aligned} & \mathrm{E}_\varepsilon(u_n^h) + \left(1 - \frac{Ch}{2\varepsilon}\right)\int_0^{nh}\int \left|\partial_t u^h\right|^2 \mathrm{d}x\,\mathrm{d}t \\ = & \ \mathrm{E}_\varepsilon(u_n^h) + \left(h - \frac{Ch^2}{2\varepsilon}\right)\sum_{k=1}^{n}\int \left|\frac{u_n^h - u_{n-1}^h}{h}\right|^2 \mathrm{d}x \\ \leq & \ \mathrm{E}_\varepsilon(u_\varepsilon^0). \end{aligned} \tag{1.18}$$

This gives use with the use of Jensen's inequality for $0 \leq s \leq t \leq T$ and $h > 0$ sufficiently small that

$$\begin{aligned} \left\|u^h(t) - u^h(s)\right\|_{\mathrm{L}^2} &= \left\|\int_s^t \partial_t u_h(\tau)\,\mathrm{d}\tau\right\|_{\mathrm{L}^2} \\ &\leq \sqrt{t-s}\left(\int_0^T\int \left|\partial_t u^h(\tau,x)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\ &\leq \sqrt{t-s}\left(1 - \frac{Ch}{2\varepsilon}\right)^{-1/2}\left(E_\varepsilon(u_\varepsilon^0)\right)^{1/2}, \end{aligned} \tag{1.19}$$

which gives us a uniform bound on the L^2-Hölder continuity of u^h in time as h tends to zero.

Step 5: Compactness

In order to apply Arzelà-Ascoli for the sequence (u_h) as h tends to zero, we need to check pointwise precompactness of the image and equicontinuity of the sequence. The equicontinuity follows from the previous estimate (1.19). In order to check the pointwise precompactness, we need to verify that for all $t \in [0,T]$, the set $\{u^h(t)\}_{\delta>h>0}$ is a precompact subset of $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ (for $\delta > 0$ sufficiently small). But this follows from the energy bound $\mathrm{E}_\varepsilon(u_n^h) \le \mathrm{E}_\varepsilon(u_0)$ (given by inequality (1.16)), which gives us a time-uniform bound on $\left\|\nabla u^h(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)}$ and on $\left\|u^h(t)\right\|_{\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)}$ and thus on $\left\|u^h(t)\right\|_{\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)}$. The compact embedding $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N) \hookrightarrow \mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ thus yields the desired pointwise precompactness.

Therefore we may apply Arzelà–Ascoli to obtain some $u \in \mathrm{C}^{1,2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N\right)$ and some sequence $h_n \to 0$ such that u^{h_n} converges uniformly to u on $[0,T]$ with respect to $\|\cdot\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)}$.

Step 6: Additional regularity 1

We first want to argue that from our construction, one already obtains that $u \in \mathrm{L}^\infty\left([0,T];\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\right)$ For this we first notice that for a fixed $t \in [0,T]$, the sequence $u^{h_n}(t)$ is by the energy bound (1.16) a bounded sequence in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$, thus we find some non-relabelled subsequence and $v \in \mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$ such that $u^{n_h}(t)$ converges weakly to v in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$. By uniqueness of the limit, we already have $u(t) = v$ almost everywhere, which yields $u(t) \in \mathrm{W}^{1,2}(\mathbb{T},\mathbb{R}^N)$, and by lower semicontinuity, we may also deduce that

$$\|u(t)\|_{\mathrm{W}^{1,2}} \le \liminf_{n\to\infty} \left\|u^{h_n}(t)\right\|_{\mathrm{W}^{1,2}} \le C\,\mathrm{E}_\varepsilon,$$

from which we deduce that $u \in \mathrm{L}^\infty\left([0,T];\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\right)$.

Secondly the boundedness of the energies

$$\sup_{0\le t\le T} \mathrm{E}_\varepsilon(u(t)) < \infty$$

follows from the lower semicontinuity of the energy and the pointwise L^2 convergence and pointwise weak convergence in $\mathrm{W}^{1,2}$ as described in step 1.

Lastly we want to argue that $\partial_t u \in \mathrm{L}^2\left([0,T]\times\mathbb{T};\mathbb{R}^N\right)$. From inequality (1.18) in step 4, we deduce that $\partial_t u^h$ is a bounded sequence in $\mathrm{L}^2[0,T]\times\mathbb{T};\mathbb{R}^N)$. Thus we find a non-relabelled subsequence of h_n and some $w \in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$ such that u^{h_n} converges weakly to w in $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. But then w is already the weak time derivative of u since for any testfunction ξ, we have

$$\begin{aligned}\int_{[0,T]\times\mathbb{T}} \langle u\,,\,\partial_t\xi\rangle\,\mathrm{d}x\,\mathrm{d}t &= \lim_{n\to\infty}\int_{[0,T]\times\mathbb{T}} \left\langle u^{h_n}\,,\,\partial_t\xi\right\rangle\mathrm{d}x\,\mathrm{d}t\\ &= \lim_{n\to\infty} -\int_{[0,T]\times\mathbb{T}} \left\langle \partial_t u^{h_n}\,,\,\xi\right\rangle\mathrm{d}x\,\mathrm{d}t\\ &= -\int_{[0,T]\times\mathbb{T}} \langle w\,,\,\xi\rangle\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Step 7: u is a weak solution

Going back to step 1, we see that u_n^h solves the Euler–Lagrange equation

$$\int \frac{1}{\varepsilon}\langle \nabla W(u_n^h)\,,\,\xi\rangle + \varepsilon\langle \nabla u_n^h\,,\,\nabla\xi\rangle + \left\langle \frac{u_n^h - u_{n-1}^h}{h}\,,\,\xi\right\rangle \mathrm{d}x = 0 \tag{1.20}$$

for any testfunction $\xi \in \mathrm{C}^\infty(\mathbb{T};\mathbb{R}^N)$. Let $t \in [0,T]$. Since u_h is defined as the pointwise linear interpolation of the functions u_n^h, we find for the sequence h_n corresponding sequences $\lambda_n \in [0,1]$ and $k_n \in \mathbb{N}$ such that

$$t = \lambda_n(k_n - 1)h_n + (1-\lambda_n)k_n h_n$$

and therefore we can write

$$u_h(t) = \frac{k_n h_n - t}{h_n} u_{k_n-1}^h + \frac{t-(k_n-1)h_n}{h_n} u_{k_n}^{h_n}.$$

In order to pass to the limit in equation (1.20), we first note that $u_{k_n}^{h_n}$ converges to $u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ since

$$\begin{aligned}\left\| u(t) - u_{k_n}^{h_n}\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} &\le \left\| u(t) - u^{h_n}(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} + \left\| u^{h_n}(t) - u^{h_n}(k_n h_n)\right\|_{\mathrm{L}^2(\mathbb{T},\mathbb{R}^N)} \\ &\overset{(1.19)}{\lesssim} \left\| u(t) - u^{h_n}(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} + \sqrt{h_n},\end{aligned}$$

which goes to zero as n tends to infinity. This implies that

$$\begin{aligned}\int \varepsilon\langle \nabla u(t)\,,\,\nabla\xi\rangle\,\mathrm{d}x &= \int \varepsilon\langle u(t)\,,\,\operatorname{div}\nabla\xi\rangle\,\mathrm{d}x \\ &= \lim_{n\to\infty}\int \varepsilon\left\langle u_{k_n}^{h_n}\,,\,\operatorname{div}\nabla\xi\right\rangle \mathrm{d}x \\ &= \lim_{n\to\infty}\int \varepsilon\left\langle \nabla u_{k_n}^{h_n}\,,\,\nabla\xi\right\rangle \mathrm{d}x\,.\end{aligned}$$

In step 6, we moreover have shown that $\partial_t u^h$ converges weakly to $\partial_t u$ in $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. This yields, by choosing cylindrical testfunctions, that $\partial_t u^h(t)$ converges weakly to $\partial_t u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ for almost every $t \in [0,T]$. Thus we obtain for almost every $t \in [0,T]$ the convergence

$$\int \langle \partial_t u(t)\,,\,\xi\rangle\,\mathrm{d}x = \lim_{n\to\infty}\int \langle \partial_t u(t)\,,\,\xi\rangle\,\mathrm{d}x = \lim_{n\to\infty}\int \left\langle \frac{u_{k_n}^{h_n} - u_{k_n-1}^{h_n}}{h_n}\,,\,\xi\right\rangle \mathrm{d}x\,.$$

To obtain the weak equation, we still need to prove convergence of remaining term. For this we note that

$$\frac{1}{\varepsilon}\left|\left\langle \nabla W(u_{k_n}^{h_n})\,,\,\xi\right\rangle\right| \lesssim \left(1 + \left|u_{k_n}^{h_n}\right|^{p-1}\right)|\xi|,$$

which is a bounded sequence in $\mathrm{L}^{p'}(\mathbb{T};\mathbb{R}^N)$. Since we also may pass to a non-relabelled subsequence which converges almost everywhere, we thus obtain that

$$\int \langle \nabla W(u(t))\,,\,\xi\rangle\,\mathrm{d}x = \lim_{n\to\infty}\int \left\langle \nabla W(u_{k_n}^{h_n})\,,\,\xi\right\rangle \mathrm{d}x\,.$$

Thus it follows from the Euler–Lagrange equation (1.20) that

$$\int \frac{1}{\varepsilon}\langle \nabla W(u(t))\,,\,\xi\rangle + \varepsilon\langle \nabla u\,,\,\nabla\xi\rangle + \langle \partial_t u\,,\,\xi\rangle\,\mathrm{d}x = 0$$

for all $\xi \in \mathrm{C}^\infty(\mathbb{T};\mathbb{R}^N)$. By continuity this extends to all $\xi \in \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$.
Thus the last thing to check is that

$$\sup_{0\le t\le T} \mathrm{E}_\varepsilon(u(t)) < \infty,$$

which follows from the next step.

Step 8: Sharp energy dissipation inequality for u
Let $0 \le t \le T$ and define k_n as in step 7, where we also established that $u_{k_n}^{h_n}$ converges to $u(t)$ in $\mathrm{L}^2\left(\mathbb{T};\mathbb{R}^N\right)$, and thus we may pass to another non-relabelled subsequence to obtain pointwise convergence almost everywhere. By Fatou's Lemma, we thus obtain

$$\int \frac{1}{\varepsilon}W(u(t))\,\mathrm{d}x \le \liminf_{n\to\infty}\int \frac{1}{\varepsilon}W(u_{k_n}^{h_n})\,\mathrm{d}x\,.$$

Moreover we can deduce from the L^2-convergence that $\nabla u_{k_n}^{h_n}$ converges to $u(t)$ in the distributional sense. But $\nabla u_{k_n}^{h_n}$ is uniformly bounded in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ by the energy dissipation inequality (1.16), thus we already obtain that $\nabla u_{k_n}^{h_n}$ converges weakly to $\nabla u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ which yields

$$\int \frac{\varepsilon}{2}|\nabla u(t)|^2\,\mathrm{d}x \le \liminf_{n\to\infty}\int \frac{\varepsilon}{2}\left|\nabla u_{k_n}^{h_n}\right|^2\mathrm{d}x\,.$$

Lastly we have by the weak convergence of $\partial_t u^{h_n}$ to ∂tu in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ proven in step 6 that

$$\begin{aligned}\int_0^t\int |\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{n\to\infty}\left(1-\frac{Ch_n}{2\varepsilon}\right)\int_0^t\int \left|\partial_t u^{h_n}\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\le \liminf_{n\to\infty}\left(1-\frac{Ch_n}{2\varepsilon}\right)\int_0^{k_nh_n}\int \left|\partial_t u^{h_n}\right|^2\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Summarizing these estimates, we obtain by the energy dissipation inequality for u^h (1.18) that for all $0 \le t \le T$ we have

$$\begin{aligned}\mathrm{E}_\varepsilon(u(t)) + \int_0^t\int |\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{n\to\infty}\mathrm{E}_\varepsilon(u_{k_n}) + \left(1-\frac{Ch_n}{2\varepsilon}\right)\int_0^{k_nh_n}\int \left|\partial_t u^h\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\le \mathrm{E}_\varepsilon(u_\varepsilon^0).\end{aligned}$$

Step 8: Additional regularity 2

In order to complete the proof, we still have to show that $\partial^2_{i,j} u$ and$\nabla W(u)$ are elements of $\mathrm{L}^2([0,T] \times \mathbb{T}; \mathbb{R}^N)$. In order to show that the second partial derivatives of u are square-integrable, we test the weak formulation (1.9) with the finite differences. To this end, define the finite differences as

$$\Delta_h^+ v(t,x) \coloneqq \frac{v(t,x+hv) - v(t,x)}{h}, \quad \Delta_h^- v(t,x) \coloneqq \frac{v(t,x-hv) - v(t,x)}{h}$$

for some $h > 0$ and $v \in \mathbb{R}^d$. Thus plugging $\Delta_h^- \Delta_h^+ u$ into (1.9) yields by the transformation formula that

$$0 = \int_0^T \int \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u)\,,\, \Delta_h^+ u \rangle + \langle \nabla \Delta_h^+ u\,,\, \nabla \Delta_h^+ u \rangle + \langle \partial_t \Delta_h^+ u\,,\, \Delta_h^+ u \rangle \,\mathrm{d}x\,\mathrm{d}t\,,$$

which is equivalent to

$$\begin{aligned}
&\int_0^T \int \left|\Delta_h^+ \nabla u\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T \int \partial_t \left(\frac{\left|\Delta_h^+ u\right|^2}{2} \right) + \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u)\,,\, \Delta_h^+ u \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&= \int \frac{\left|\Delta_h^+ u(0)\right|^2 - \left|\Delta_h^+ u(T)\right|^2}{2} \,\mathrm{d}x \\
&\quad - \int_0^1 \int_0^T \int \langle \mathrm{D}^2\, W\left((1-s)u(t,x) + su(t,x+hv)\right)\right) \Delta_h^+ u\,,\, \Delta_h^+ u \rangle \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\,.
\end{aligned}$$

The first summand can be estimated by $\|u\|_{\mathrm{L}^\infty([0,T],\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N))}$. For the second summand, we partition W into the sum of W_{conv} and W_{pert}. The term involving the convex summand can then by estimated by

$$\int_0^1 \int_0^T \int \langle \mathrm{D}^2\, W_{\mathrm{conv}}\left((1-s)u(t,x) + su(t,x+hv)\right) \Delta_h^+ u\,,\, \Delta_h^+ u \rangle \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \geq 0$$

and the for the pertubation term, we get via the bound on its second derivative that

$$\begin{aligned}
&\left| \int_0^1 \int_0^T \int \langle \mathrm{D}^2\, W_{\mathrm{pert}}\left((1-s)u(t,x) + su(t,x+hv)\right) \Delta_h^+ u\,,\, \Delta_h^+ u \rangle \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \right| \\
&\lesssim \int_0^T \int |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}$$

which is also finite. Combining these estimates, we obtain that $\int_0^T \int \left|\Delta_h^+ \nabla u\right|^2 \mathrm{d}x\,\mathrm{d}t$ is uniformly bounded in h. Applying our calculation to all directions $v \in \mathbb{R}^d$, we get by the finite-differences theorem for all $1 \leq i,j \leq d$ that $\partial^2_{i,j} u \in \mathrm{L}^2([0,T] \times \mathbb{T}; \mathbb{R}^N)$.

In order to obtain $\nabla W(u) \in \mathrm{L}^2([0,T] \times \mathbb{T}; \mathbb{R}^N)$, we again consider the weak formulation (1.9) and notice that since we have already shown that both the time derivative and second space derivatives of u are square-integrable, our claim follows from a duality argument.

□

Remark. The inequality (1.16) with the factor $1/2h$ instead of $1/h - C/2\varepsilon$ follows immediately from the definition of our optimization problem, but is not optimal for fixed ε if we want to study the behaviour as h tends to zero. Moreover this so called *sharp energy dissipation inequality* is important for later.

Remark. The energy dissipation inequality (1.12) can be deduced via the formal calculation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon}\langle \nabla W(u)\,,\,\partial_t u\rangle + \varepsilon\langle \nabla u\,,\,\nabla\partial_t u\rangle\,\mathrm{d}x\\ &= \int \langle \frac{1}{\varepsilon}\nabla W(u) - \varepsilon\Delta u\,,\,\partial_t u\rangle\,\mathrm{d}x\\ &= -\varepsilon\int |\partial_t u_\varepsilon|^2\,\mathrm{d}x\,.\end{aligned}$$

In order to make this calculation rigorous, we however need to show that $t \mapsto \mathrm{E}_\varepsilon(u(t))$ is absolutely continuous, which is non-trivial, but it would give us equality in energy dissipation inequality (1.12) Since we will only need the inequality, our proof will however suffice.

2 Convergence of the Allen–Cahn equations

Baldo proved in his paper [Bal90] that the Cahn-Hilliard energies Γ-converge with respect to $\|\cdot\|_{\mathrm{L}^1}$ to an *optimal partition energy* given by

$$\mathrm{E}(\chi) \coloneqq \frac{1}{2} \sum_{1 \leq i,j \leq P} \sigma_{i,j} \int \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right) \tag{2.1}$$

for a partition $\chi_1, \ldots, \chi_P \colon \mathbb{T} \to \{0,1\}$ satisfying $\chi_{1 \leq i \leq P} \chi_i = 1$ almost everywhere. We may also define measurable sets Ω_i through the relation $\chi_i = \mathbb{1}_{\Omega_i}$. The link between a sequence u_ε and χ is given by $u_\varepsilon \to u \coloneqq \sum_{1 \leq i \leq P} \alpha_i \chi_i$ in L^1.

Moreover if we denote by $\partial_* \Omega_i$ the reduced boundary of Ω_i and by $\Sigma_{i,j} \coloneqq \partial_* \Omega_i \cap \Omega_j$ the interface between Ω_i and Ω_j, then we may rewrite equation (2.1) as

$$\mathrm{E}(\chi) = \frac{1}{2} \sum_{1 \leq i,j \leq P} \sigma_{i,j} \, \mathcal{H}^{d-1}(\Sigma_{i,j}).$$

Here, the surface tensions $\sigma_{i,j}$ are the geodesic distances between the wells α_i of W with respect to the metric $2W(u)\langle \cdot \, , \, \cdot \rangle$, which can be written out as

$$\sigma_{i,j} = \mathrm{d}_W(\alpha_i, \alpha_j)$$

for the geodesic distance defined as

$$\mathrm{d}_W(u,v) \coloneqq \inf \left\{ \int_0^1 \sqrt{2W(\gamma)} |\dot{\gamma}| \, \mathrm{d}t \; : \; \gamma \in \mathrm{C}^1([0,1], \mathbb{R}^N) \text{ with } \gamma(0) = u, \, \gamma(1) = v \right\}. \tag{2.2}$$

Geometrically speaking, the partition energy E measures the surface tensions between the sets and penalizes larges interfaces. Also observe that the factor $1/2$ can be left out if we only count each interface once.

Bibliography

[Bal90] Sisto Baldo. “Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids”. en. In: *Annales de l’I.H.P. Analyse non linéaire* 7.2 (1990), pp. 67–90. URL: http://www.numdam.org/item/AIHPC_1990__7_2_67_0/.

[LS16] Tim Laux and Theresa Simon. “Convergence of the Allen-Cahn Equation to Multiphase Mean Curvature Flow”. In: *Communications on Pure and Applied Mathematics* 71 (June 2016). DOI: 10.1002/cpa.21747.