

# Convergence of Allen-Cahn equations to multiphase mean curvature flow

Pascal Maurice Steinke

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Advisor: Prof. Dr. Tim Laux

Second Advisor: Prof. Dr. Stefan Müller

INSTITUTE FOR APPLIED MATHEMATICS

MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT DER  
RHEINISCHEN FRIEDRICH-WILHELMS-UNIVERSITÄT BONN



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# Introduction

Allgemeine Notizen: Etwas zu Eindeutigkeit sagen; wir gehen sehr oft zu Teilfolgen über. Hilft uns hierbei die Weak strong uniqueness? De Giorgis original paper finden?

## 1. Abstract

This thesis presents a conditional convergence result of the solutions to the Allen–Cahn equation to a De Giorgi type BV-solution to multiphase mean curvature flow for arbitrary potentials, but fixed mobilities. For this we will also recall the proof for the conditional convergence to BV-solutions in the sense of Laux and Simon. Lastly we show that De Giorgi type BV-solutions are De Giorgi type varifold solutions, and thus our solution concept exhibits a weak-strong uniqueness.

## 2. History and main goals

Multiphase mean curvature is an important geometric evolution equation which has been studied for a long time, bearing not only mathematical importance, but also for the applied sciences. Originally it was proposed to study the evolution of grain boundaries in annealed recrystallized metal, as described by Mullins in [Mul56], who cites Beck in [Bec52] as already having observed such a behaviour in 1952.

Over the years a number of different solution concepts for multiphase mean curvature flow have been proposed. Classically we have smooth solutions, where we require the evolution of sets to be smooth, for example described by Huisken in [Hui90]. Another description of smoothly evolving mean curvature flow can be found in the work of Gage and Hamilton [GH86], who proved the “shrinking conjecture” for convex planar curves. Brakke describes in his book [Bra78] the motion by mean curvature using varifolds, which yields a quite abstract and general notion for mean curvature flow and is based on the gradient flow structure of mean curvature flow. Luckhaus and Sturzenhecker introduced a distributional solution to mean curvature flow in their work [LS95]. Another solution concept is the viscosity solution concept, for example presented in ([CGG91], [ES91]), where it is shown that solutions of a certain parabolic equation have the property that if they are smooth, the corresponding level sets move by mean curvature.

The Allen–Cahn equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} \nabla W(u_\varepsilon) \quad (1)$$

## Introduction

is commonly used as a phase-field approximation for mean curvature flow and was first discovered by Allen and Cahn in their paper [AC79]. Here  $W: \mathbb{R}^N \rightarrow [0, \infty)$  is some smooth potential with finitely many zeros  $\alpha_1, \dots, \alpha_P$ . Since then it has been a topic of long study, both from a numerical and analytical standpoint.

If we consider the two-phase case, which corresponds to setting  $N = 1$  and  $P = 2$ , then the behaviour of the solutions to (1) are thoroughly researched. Chen has proven in [Che92] that as long as the interface of the limit evolves smoothly, we have convergence to mean curvature flow. The authors Bronsard and Kohn showed by utilization of the gradient flow structure of (1), more precisely by mainly utilizing the corresponding energy dissipation inequality, in [BK91] the compactness of solutions and regularity for the limit. Moreover they studied the behaviour for radially symmetric initial data and showed motion by mean curvature in this case. Ilmanen made fundamental contributions in [Ilm93] by proving the convergence to Brakke's mean curvature flow as long as the initial conditions are well-prepared by exploiting the equipartition of energies. However his methods seem to only work in the two-phase case since he uses a comparison principle whose generalization to the vectorial case is not clear.

The multiphase case for both the behaviour of the Allen–Cahn equation and mean curvature is much more involved and a topic of current research. For the convergence of the Allen–Cahn equation, asymptotic expansions with Neumann boundary conditions have been studied by Keller, Rubinstein and Sternberg in [RSK89] (NOCHMAL MIT TIM BESPREEHCEN). Moreover it has been shown by Bronsard and Reitich in [BR93] that for the three-phase case, we have short-time existence of a solution (in what sense????).

The analysis of multiphase mean curvature flow has been studied for example by Mantegazza, Novaga and Tortorelli in [MNT04], where they considered the planar case when only a single triple junction appears, and their work has been extended to several triple junctions in [Man+16]. Ilmanen, Neves and Schulze furthermore proved in [INS19] that even for non-regular initial data in the sense that not all angles at triple junctions are equal, we have short time existence by approximation through regular networks. For long time existence, it has been shown by Kim and Tonegawa in [KT17] through a modification of Brakke's approximation scheme that one can obtain non-trivial mean curvature flow even with singular initial data.

The first main goal of this thesis is to prove a conditional convergence result of solutions to the vectorial Allen–Cahn equation (1) to a De Giorgi type BV-solution of multiphase mean curvature flow in the sense of Definition 5.1.1. The proof is based mainly on a duality argument and the results of Laux and Simon in [LS16]. However the strong assumption (MIT TIM ÜBER SIEN PAPER DISKUTIEREN; WANN MAN DIES ZEIGEN KANN) we make here is that the Cahn–Hilliard energies (2.2) of the solutions to (1) converge to the perimeter functional (1.7) of the limit. This prevents that as  $\varepsilon$  tends to zero, approximate interfaces collapse, which would mean that energy dissipates, see also the discussion in Section 5.1.2. The energy convergence assumption provides us with the important equipartition of energies, whose proof under milder assumption was the main obstacle of Ilmanen in [Ilm93]. Moreover it is the key for our localization estimates and lets us localize on the different phases of mean curvature flow. Lastly it assures that the differential  $\nabla u_\varepsilon$  can locally up to an error be written as

### 3. Structure of the thesis

a rank-one matrix which is the tensor of the approximate frozen unit normal and the gradient of the geodesic distance function associated to the majority phase evaluated at  $u_\varepsilon$ , see the proof of Proposition 4.2.7.

The second main goal is to compare De Giorgi type BV-solutions to De Giorgi type varifold solutions, which were proposed by Hensel and Laux in [HL21]. We will show that our solution concept is stronger in the sense that every BV-solution is also a varifold solution. Since Hensel and Laux have shown weak-strong uniqueness for their varifold solution concept, it follows that we also obtain weak-strong uniqueness for our De Giorgi type BV-solution concept, which is the best we can expect. In fact it has been shown on a numerical basis by Angenent, Chopp and Ilmanen in [AIC95] (AUCH MIT TIM BESPREECHEN) that even in three dimensions, there exists a hypersurface whose evolution by mean curvature flow admits a singularity at a certain time after which we have nonuniqueness.

Let us also mention some of the closely related unanswered questions. For one Hensel and Laux have shown in [HL21] that in the two-phase case and under well prepared initial conditions, then solutions of the Allen–Cahn equation (1) converge to a De Giorgi type varifold solution. However their methods have no obvious generalization to the multiphase case without the energy convergence assumption, and one even struggles to find an approximate sequence which constructs the desired varifolds. And even then one would have to find suitable substitutions for the localization estimates explained in Section 4.2.2, which are based on De Giorgi’s structure theorem and thus only work for BV-functions. One other possible question would be how to generalize the results to the case of arbitrary mobilities: Throughout the thesis, and also the main background papers ([LS16], [HL21]), it is always assumed that the mobilities are fixed through the relation  $\mu_{ij} = 1/\sigma_{ij}$ , where  $\sigma_{ij}$  denotes the surface tension of the  $(i, j)$ -th interface, and  $\mu_{ij}$  its mobility. As proposed by Bretin, Danescu, Penuelas, and Masnou in [Bre+18], passing to arbitrary mobilities should amount to multiplying an appropriate “mobility matrix”  $M$  onto the right-hand side of (1) and changing the metric of the underlying space accordingly to  $\langle u, v \rangle = \int \langle Mu, v \rangle dx$ . The difference in their approach is that first uncouple their system so that they arrive at the scalar Allen–Cahn equation and then couple they components through a Lagrange-multiplier, which assures that the limit is a partition.

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# 1. Gradient flows and mean curvature flow

## 1.1. Gradient flows

In the simplest case, a gradient flow of a given energy  $E: \mathbb{R}^N \rightarrow \mathbb{R}$  (with respect to the euclidean inner product) is a solution to the ordinary differential equation

$$\frac{d}{dt}x(t) = -\nabla E(x(t)), \quad (1.1)$$

where we usually prescribe some initial value  $x(0) = x_0 \in \mathbb{R}^N$ . The central structure here is that on the right hand side of the equation, we do not have any function, but the gradient of some continuously differentiable function. What a solution  $x$  does is that it moves in the direction of the steepest descent of the energy  $E$ . Moreover this allows for the following computation given a continuously differentiable solution  $x$  of (1.1):

$$\begin{aligned} \frac{d}{dt} E(x(t)) &= \langle \nabla E(x(t)), x'(t) \rangle = -|\nabla E(x(t))|^2 = -|x'(t)|^2 \\ &= -\frac{1}{2} \left( |x'(t)|^2 + |\nabla E(x(t))|^2 \right). \end{aligned}$$

We especially obtain that the function  $E(x(t))$  is non-increasing, which coincides with our intuition that  $x$  moves along the steepest descent of the energy. But more precisely, we obtain from the fundamental theorem of calculus the *energy dissipation identity*

$$E(x(T)) + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla E(x(t))|^2 dt = E(x(0)). \quad (1.2)$$

One could now raise the question if this identity already characterizes equation (1.1). But as it turns out, we can go even one step further, namely we only ask for the *optimal energy dissipation inequality* given by

$$E(x(T)) + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla E(x(t))|^2 dt \leq E(x(0)). \quad (1.3)$$

We call this inequality optimal since as demonstrated before, we usually expect an equality to hold. Now let us assume that  $x$  satisfies (1.3) and has sufficient regularity.

### 1. Gradient flows and mean curvature flow

Then we can estimate again by the fundamental theorem that

$$\begin{aligned} & \frac{1}{2} \int_0^T |\nabla E(x(t)) + x'(t)|^2 dt \\ &= \int_0^T \langle \nabla E(x(t)), x'(t) \rangle dt + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla E(x(t))|^2 dt \\ &= E(x(T)) - E(x(0)) + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla E(x(t))|^2 dt \leq 0. \end{aligned}$$

Since we started with an integral over a non-negative function, this implies that for almost every time  $t$ , we have that  $x'(t) = -\nabla E(x(t))$ , which proves that  $x$  (again under sufficient regularity assumptions this then must already hold for every time) is a gradient flow of the energy  $E$ .

The real strength of formulating the differential equation (1.1) via inequality (1.3) becomes clear if we want to consider gradient flows in a more complicated setting. In order to formulate equation (1.1), we need to have a notion of differentiation and a gradient in the target of  $x$ . Therefore we may use pre-Hilbert spaces or smooth Riemannian manifolds as a suitable substitutes for  $\mathbb{R}^N$ .

Examples for such more complicated gradient flows include the heat equation, which can be written as the  $L^2$ -gradient flow of the Dirichlet energy, the Fokker-Planck equation, the Allen–Cahn equation and mean curvature flow. The two latter of course play a key role for us.

These observations have been around for a long time, and are credited to De Giorgi and his paper [De 93]. Sandier and Serfaty have also written an excellent paper on this topic, as well as Ambrosio, Gigli and Savaré [AGS05], to which we refer the interested reader.

## 1.2. Mean curvature flow

Mean curvature flow describes the geometric evolution of a set  $(\Omega(t))_{t \geq 0}$  respectively the evolution of its boundary  $\Sigma(t) := \partial\Omega(t)$ . It is formulated through the equation

$$\frac{1}{\mu} V = -\sigma H \quad \text{on } \Sigma. \quad (1.4)$$

Here  $\mu > 0$  is a positive constant which is called the *mobility* and  $\sigma > 0$  is a positive constant as well which we define as the *surface tension*. By  $V$  we denote the normal velocity of the set and by  $H$  its mean curvature, which is defined as the sum of the principle curvatures at a given point. There are a lot of different notions of solutions to this equation as already mentioned in the Introduction. For us the central structure of this equation is its gradient flow structure. Formally we want to consider the space

$$\mathcal{M} := \{(d-1)\text{-dimensional surfaces on } \mathbb{T}\},$$

where the tangent space at a given  $\Sigma \in \mathcal{M}$  consists of the normal velocities on  $\Sigma$ . The metric tensor at a surface  $\Sigma$  of two normal velocities is then given by the rescaled  $L^2$

inner product on  $\Sigma$

$$\langle V, W \rangle_\Sigma := \frac{1}{\mu} \int_\Sigma VW \, d\mathcal{H}^{d-1}$$

and our energy will simply be the rescaled perimeter functional

$$E(\Sigma) := \sigma \mathcal{H}^{d-1}(\Sigma).$$

Since the first inner variation of the perimeter functional is given by the mean curvature vector (see [Mag12, Thm. 17.5]), the gradient of the energy should simply be the mean curvature vector multiplied by  $\sigma\mu$ . Thus the mean curvature flow equation (1.4) corresponds exactly to the gradient flow equation (1.1). Note however that this metric tensor induces a degenerate metric in the sense that the distance between any two hypersurfaces is zero, which has been shown by Michor and Mumford in [MM06]. Roughly speaking, we can roughen up a hypersurface, then move it, and afterwards flatten it back, while only producing an arbitrarily small distance due to a scaling invariance.

In Section 1.1 we highlighted the importance of De Giorgi's optimal energy dissipation inequality (1.3). In our setting this translates to the inequality

$$\begin{aligned} E(\Sigma(T)) + \frac{1}{2} \int_0^T \langle V(t), V(t) \rangle_{\Sigma(t)} + \langle \nabla_{\Sigma(t)} E, \nabla_{\Sigma(t)} E \rangle_{\Sigma(t)} \, dt \\ = E(\Sigma(T)) + \frac{1}{2} \int_0^T \int_{\Sigma(t)} \frac{1}{\mu} V(t)^2 + \sigma^2 \mu H(t)^2 \, d\mathcal{H}^{d-1} \, dt \leq E(\Sigma(0)). \end{aligned}$$

This is the main motivation for the definition of De Giorgi type solutions to mean curvature flow in the two-phase case, see Definition 5.1.1 and Definition 5.1.4.

Far more important for us shall however be multiphase mean curvature flow, which is of high importance in the applied sciences and mathematically quite complex. Essentially instead of just considering the evolution of one single set, we look at the evolution of a partition of the flat torus and require that at each interface of the sets, the mean curvature flow equation (1.4) is satisfied.

We say that a partition  $(\Omega_i(t))_{i=1,\dots,P}$  with  $\Sigma_{ij} := \partial\Omega_i \cap \partial\Omega_j$  satisfies multiphase mean curvature with mobilities  $\mu_{ij}$  and surface tensions  $\sigma_{ij}$  if

$$\frac{1}{\mu_{ij}} V_{ij} = -\sigma_{ij} H_{ij} \quad \text{on } \Sigma_{ij} \text{ for all } i \neq j \text{ and} \quad (1.5)$$

$$\sigma_{ij} \nu_{ij} + \sigma_{jk} \nu_{jk} + \sigma_{ki} \nu_{ki} = 0 \quad \text{at triple junctions.} \quad (1.6)$$

The second equation is a stability condition when more than two sets meet and is called *Herring's angle condition*. Here  $\nu_{ij}$  is the outer unit normal on  $\Sigma_{i,j}$  pointing from  $\Omega_i$  to  $\Omega_j$ . One could argue that we would also want stability conditions at for example quadruple junctions. In two dimensions though, such quadruple junctions are expected to immediatly dissipate. In higher dimensions, quadruple junctions might become stable, but only on lower dimensional sets, and shall thus not be relevant for us.

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As our space, we shall thus now consider tuples of  $(d - 1)$ -dimensional surfaces on  $\mathbb{T}$ . The energy and metric tensor are given by

$$E(\Sigma) := \sum_{i < j} \sigma_{ij} \mathcal{H}^{d-1}(\Sigma_{ij}) \quad (1.7)$$

and

$$\langle V, W \rangle_{\Sigma} := \sum_{i < j} \frac{1}{\mu_{ij}} \int_{\Sigma_{ij}} V_{ij} W_{ij} \, d\mathcal{H}^{d-1}.$$

Of course this again produced a degenerate metric. Using a variant of the divergence theorem on surfaces ([Mag12, Thm. 11.8]) and again the computation for the first variation of the perimeter ([Mag12, Thm. 17.5]), we see that multiphase mean curvature has the desired gradient flow structure and that De Giorgi's inequality (1.3) translates to

$$E(\Sigma(T)) + \frac{1}{2} \sum_{i < j} \int_0^T \int_{\Sigma_{ij}} \frac{1}{\mu_{ij}} V_{ij}^2 + \sigma_{ij}^2 \mu_{ij} H_{ij}^2 \, d\mathcal{H}^{d-1} \, dt \leq E(\Sigma(0)).$$

If we make the simplifying assumption that the mobilities are already determined through the surface tensions by the relation  $\mu_{ij} = 1/\sigma_{ij}$ , then this inequality becomes

$$E(\Sigma(T)) + \frac{1}{2} \sum_{i < j} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} V_{ij}^2 + H_{ij}^2 \, d\mathcal{H}^{d-1} \, dt \leq E(\Sigma(0)),$$

which motivated the multiphase phase of Definition 5.1.1 and Definition 5.1.4.

## 2. The Allen–Cahn equation

### 2.1. Structure of the equation

This chapter follows [LS16], but since the authors decided to only sketch some of the proofs, we want to go into more detail.

Let  $\Lambda > 0$  and define the flat torus  $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$ , where we work with periodic boundary conditions and write  $\int dx$  instead of  $\int_{\mathbb{T}} dx$ . Then for  $u: [0, \infty) \times \mathbb{T} \rightarrow \mathbb{R}^N$  and some potential  $W: \mathbb{R}^N \rightarrow [0, \infty)$ , the *Allen–Cahn equation* with parameter  $\varepsilon > 0$  is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \quad (2.1)$$

To understand this equation better, we consider the *Cahn–Hilliard energy*, which assigns to  $u$  for a fixed time the real number

$$E_\varepsilon(u) := \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 dx. \quad (2.2)$$

If everything is nice and smooth, we can compute that under the assumption that  $u$  satisfies equation (2.1), we have that

$$\begin{aligned} \frac{d}{dt} E_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u), \partial_t u \rangle + \varepsilon \langle \nabla u, \nabla \partial_t u \rangle dx \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u, \partial_t u \right\rangle dx \\ &= \int -\varepsilon |\partial_t u|^2 dx. \end{aligned} \quad (2.1)$$

This calculation suggests that equation (2.1) is the  $L^2$  gradient flow (rescaled by  $\sqrt{\varepsilon}$ ) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (2.1) via De Giorgis minimizing movements scheme ([De 93]), which we will do in Theorem 2.2.1.

But first we need to clarify what our potential  $W$  should look like. Classic examples in the scalar case are given by  $W(u) = (u^2 - 1)^2$  or  $W(u) = u^2(u - 1)^2$ , and we call functions like these *doublewell potentials*, see also Figure 2.1. TIM MEINTE AUCH FÜR HÖHERE DIMENSION LEVEL SET GRAFIK ERSTELLEN:

In higher dimensions, we want to accept the following potentials:  $W: \mathbb{R}^N \rightarrow [0, \infty)$  has to be a smooth multiwell potential, meaning that it has finitely many zeros at  $u = \alpha_1, \dots, \alpha_P \in \mathbb{R}^N$ . Furthermore we ask for polynomial growth in the sense that there exists some  $p \geq 2$  such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \quad (2.3)$$

## 2. The Allen–Cahn equation

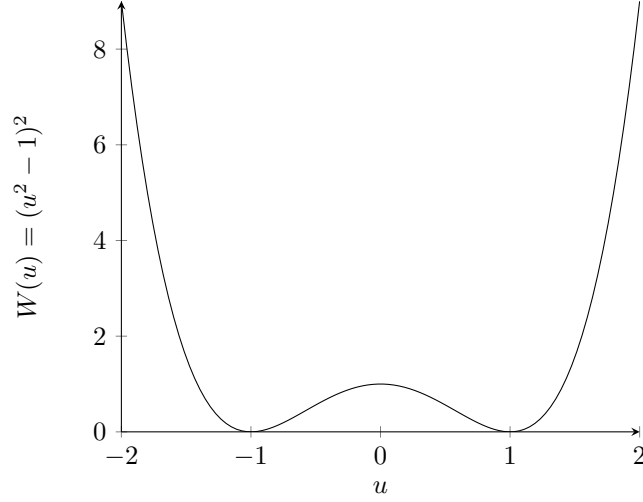


Figure 2.1.: The graph of a doublewell potential

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \quad (2.4)$$

for all  $u$  sufficiently large. Lastly we want  $W$  to be convex up to a small perturbation in the sense that there exist smooth functions  $W_{\text{conv}}, W_{\text{pert}}: \mathbb{R}^N \rightarrow [0, \infty)$  such that

$$W = W_{\text{conv}} + W_{\text{pert}}, \quad (2.5)$$

$W_{\text{conv}}$  is convex and

$$\sup_{x \in \mathbb{R}^N} |\nabla^2 W_{\text{pert}}| < \infty. \quad (2.6)$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen–Cahn equation (2.1) as maps from  $[0, T]$  into some suitable function space and thus use the following definition.

**Definition 2.1.1.** *We say that a function  $u_\varepsilon \in C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  which is also in  $L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$  is a weak solution of the Allen–Cahn equation (2.1) with parameter  $\varepsilon > 0$  and initial condition  $u_\varepsilon^0 \in L^2(\mathbb{T}; \mathbb{R}^N)$  if*

1. *the energy stays bounded, which means that*

$$\operatorname{ess\,sup}_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t)) < \infty, \quad (2.7)$$

2. *its weak time derivative satisfies*

$$\partial_t u_\varepsilon \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N), \quad (2.8)$$

## 2.2. Existence of a solution

3. for almost every  $t \in [0, T]$  and every  $\xi \in L^p([0, T] \times \mathbb{T}; \mathbb{R}^N) \cap W^{1,2}([0, T] \times \mathbb{T}; \mathbb{R}^N)$ , we have

$$\int \left\langle \frac{1}{\varepsilon^2} \nabla W(u_\varepsilon(t)), \xi \right\rangle + \langle \nabla u_\varepsilon(t), \nabla \xi \rangle + \langle \partial_t u_\varepsilon(t), \xi \rangle dx = 0, \quad (2.9)$$

4. the initial conditions are achieved in the sense that  $u_\varepsilon(0) = u_\varepsilon^0$ .

*Remark 2.1.2.* Given our assumptions, we automatically have  $\nabla W(u_\varepsilon)(t) \in L^{p'}(\mathbb{T})$  since

$$|\nabla W(u_\varepsilon)|^{p/(p-1)} \lesssim 1 + |u_\varepsilon|^p \lesssim 1 + W(u), \quad (2.10)$$

which is integrable for almost every time  $t$  since we assume that the energy stays bounded, thus the integral in equation (2.9) is well defined.

Moreover we already obtain  $1/2$  Hölder-continuity in time from the embedding

$$W^{1,2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)) \hookrightarrow C^{1/2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)), \quad (2.11)$$

which follows from a generalized version of the fundamental theorem of calculus and Hölder's inequality.

## 2.2. Existence of a solution

With Definition 2.1.1, we are able to state our existence results for solutions of the Allen–Cahn equation. The proof uses De Giorgis minimizing movements scheme and arguments from the theory of gradient flows and has been very briefly sketched in [LS16].

**Theorem 2.2.1.** *Let  $u^0: \mathbb{T} \rightarrow \mathbb{R}^N$  be such that  $E_\varepsilon(u^0) < \infty$ . Then there exists a weak solution  $u_\varepsilon$  to the Allen–Cahn equation (2.1) in the sense of Definition 2.1.1 with initial data  $u^0$ . Furthermore the solution satisfies the energy dissipation inequality*

$$E_\varepsilon(u_\varepsilon(t)) + \int_0^t \int \varepsilon |\partial_t u_\varepsilon|^2 dx ds \leq E_\varepsilon(u^0) \quad (2.12)$$

for every  $t \in [0, T]$  and we additionally have  $\partial_{i,j}^2 u_\varepsilon, \nabla W(u_\varepsilon) \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$  for all  $1 \leq i, j \leq d$ . In particular we can test the weak form (2.9) with  $\partial_{i,j}^2 u_\varepsilon$ .

*Proof. Step 1:* A minimization problem

Fix some  $h > 0$ ,  $u_{n-1} \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$  and consider the functional

$$\begin{aligned} \mathcal{F}: W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N) &\rightarrow \mathbb{R} \\ u &\mapsto E_\varepsilon(u) + \frac{1}{2h} \int \varepsilon |u - u_{n-1}|^2 dx. \end{aligned} \quad (2.13)$$

Then  $\mathcal{F}$  is coercive with respect to  $\|\cdot\|_{W^{1,2}}$  and bounded from below by zero, thus we may take a  $W^{1,2}$ -bounded sequence  $(v_k)_{k \in \mathbb{N}}$  in  $W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$  such that  $\mathcal{F}(v_k) \rightarrow \inf \mathcal{F}$

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as  $k \rightarrow \infty$ . These  $v_k$  have a non-relabelled subsequence which converges weakly in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$  and strongly in  $L^2(\mathbb{T}; \mathbb{R}^N)$  to some  $u \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . Moreover  $v_k$  is bounded in  $L^p(\mathbb{T}; \mathbb{R}^N)$  by the lower growth assumption (2.3) on  $W$  and thus we obtain by a duality argument that  $u \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ . Lastly we have

$$\begin{aligned} \frac{1}{2h} \int \varepsilon |u - u_{n-1}|^2 dx &\leq \liminf_{k \rightarrow \infty} \frac{1}{2h} \int \varepsilon |v_k - u_{n-1}|^2 dx, \\ \int \frac{\varepsilon}{2} |\nabla u|^2 dx &\leq \liminf_{k \rightarrow \infty} \int \frac{\varepsilon}{2} |\nabla v_k|^2 dx \end{aligned}$$

by the weak convergence in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . By passing to another non-relabelled subsequence which converges pointwise almost everywhere, we moreover achieve by the continuity of  $W$  that

$$\int \frac{1}{\varepsilon} W(u) dx = \int \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon} W(v_k) dx \leq \liminf_{k \rightarrow \infty} \int \frac{1}{\varepsilon} W(v_k) dx,$$

which proves that  $u$  is a minimizer of  $\mathcal{F}$ .

### Step 2: Minimizing movements scheme

By iteratively choosing minimizers  $u_n^h$  from Step 1, we obtain a sequence of functions  $u^0, u_1^h, \dots$ . Thus we may define a function  $u$  as the piecewise linear interpolation at the time-steps  $0, h, 2h, \dots$  of these functions.

### Step 3: Sharp energy dissipation inequality for $u_n^h$

We claim that there exists some constant  $C > 0$  such that for all  $h > 0$  and  $n \in \mathbb{N}$ , we have

$$E_\varepsilon(u_n^h) + \left( \frac{1}{h} - \frac{C}{2\varepsilon^2} \right) \int \varepsilon |u_n^h - u_{n-1}^h|^2 dx \leq E_\varepsilon(u_{n-1}^h). \quad (2.14)$$

In order to prove this inequality, we notice that since  $|\nabla^2 W_{\text{pert}}| \leq C$ , the function  $W_{\text{pert}} + C|u|^2/2$  is convex for  $C > 0$  sufficiently large, thus the functional

$$\tilde{E}_\varepsilon(u) := \int \frac{1}{\varepsilon} \left( W(u) + \frac{C}{2} |u|^2 \right) + \frac{\varepsilon}{2} |\nabla u|^2 dx$$

is convex on  $W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ . For a given  $\xi \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ , we thus have that the function  $t \mapsto \tilde{E}_\varepsilon(u_n^h + t\xi)$  is convex and differentiable, which yields that

$$\tilde{E}_\varepsilon(u_n^h + \xi) \geq \tilde{E}_\varepsilon(u_n^h) + \left. \frac{d}{dt} \right|_{t=0} \tilde{E}_\varepsilon(u_n^h + t\xi). \quad (2.15)$$

But since  $u_n^h$  is a minimizer of the functional  $\mathcal{F}$  defined by (2.13), we have

$$\begin{aligned} &\left. \frac{d}{dt} \right|_{t=0} \tilde{E}_\varepsilon(u_n^h + t\xi) \\ &= \left. \frac{d}{dt} \right|_{t=0} \left( \mathcal{F}(u_n^h + t\xi) + \frac{C}{2\varepsilon} \int |u_n^h + t\xi|^2 dx - \frac{1}{2h} \int \varepsilon |u_n^h + t\xi - u_{n-1}^h|^2 dx \right) \\ &= \int \frac{C}{\varepsilon} \langle u_n^h, \xi \rangle - \frac{1}{h} \varepsilon \langle u_n^h - u_{n-1}^h, \xi \rangle dx. \end{aligned}$$



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Plugging  $\xi = u_{n-1}^h - u_n^h$  into this equation and using inequality (2.15) thus yields

$$\tilde{E}_\varepsilon(u_{n-1}^h) \geq \tilde{E}_\varepsilon(u_n^h) + \int \frac{C}{\varepsilon} \left( \langle u_n^h, u_{n-1}^h \rangle - |u_n^h|^2 \right) + \frac{1}{h} \varepsilon |u_n^h - u_{n-1}^h|^2 dx,$$

which is equivalent to

$$\begin{aligned} & E_\varepsilon(u_{n-1}^h) \\ & \geq E_\varepsilon(u_n^h) + \int \left( \frac{C}{2\varepsilon} - \frac{C}{\varepsilon} \right) |u_n^h|^2 - \frac{C}{2\varepsilon} |u_{n-1}^h|^2 + \frac{C}{\varepsilon} \langle u_n^h, u_{n-1}^h \rangle + \frac{1}{h} \varepsilon |u_n^h - u_{n-1}^h|^2 dx \\ & = E_\varepsilon(u_n^h) + \left( \frac{1}{h} - \frac{C}{2\varepsilon^2} \right) \int \varepsilon |u_n^h - u_{n-1}^h|^2 dx. \end{aligned}$$

This is the claimed estimate (2.14).

### Step 4: Hölder bounds for $u_h$

From the energy estimate (2.14) we deduce via an induction that for every  $n \in \mathbb{N}$ , we have

$$\begin{aligned} & E_\varepsilon(u_n^h) + \left( 1 - \frac{Ch}{2\varepsilon^2} \right) \int_0^{nh} \int \varepsilon |\partial_t u^h|^2 dx dt \\ & = E_\varepsilon(u_n^h) + \left( h - \frac{Ch^2}{2\varepsilon^2} \right) \sum_{k=1}^n \int \varepsilon \left| \frac{u_k^h - u_{k-1}^h}{h} \right|^2 dx \\ & \leq E_\varepsilon(u^0). \end{aligned} \tag{2.16}$$

This gives us with the use of Jensen's inequality for  $0 \leq s \leq t \leq T$  and  $h > 0$  sufficiently small that

$$\begin{aligned} \|u^h(t) - u^h(s)\|_{L^2} &= \left\| \int_s^t \partial_t u_h(\tau) d\tau \right\|_{L^2} \\ &\leq \sqrt{t-s} \left( \int_0^T \int |\partial_t u^h(\tau, x)|^2 dx d\tau \right)^{1/2} \\ &\leq \sqrt{t-s} \left( \varepsilon - \frac{Ch}{2\varepsilon} \right)^{-1/2} (E_\varepsilon(u_\varepsilon^0))^{1/2}, \end{aligned} \tag{2.17}$$

which gives us a uniform bound on the  $L^2$ -Hölder continuity of  $u^h$  in time as  $h$  tends to zero since  $\varepsilon$  is fixed.

### Step 5: Compactness

In order to apply the Arzelà–Ascoli theorem for the sequence  $(u_h)$  in  $C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  as  $h$  tends to zero, we need to check pointwise precompactness of the image and equicontinuity of the sequence. The equicontinuity follows from the previous estimate (2.17). In order to check the pointwise precompactness, we need to verify that for all  $t \in [0, T]$ ,

## 2. The Allen–Cahn equation

the set  $\{u^h(t)\}_{\delta > h > 0}$  is a precompact subset of  $L^2(\mathbb{T}; \mathbb{R}^N)$  (for  $\delta > 0$  sufficiently small). But this follows from the energy bound  $E_\varepsilon(u_n^h) \leq E_\varepsilon(u_0)$  (given by inequality (2.14)), which gives us a time-uniform bound on  $\|\nabla u^h(t)\|_{L^2(\mathbb{T}; \mathbb{R}^N)}$  and on  $\|u^h(t)\|_{L^p(\mathbb{T}; \mathbb{R}^N)}$  and thus on  $\|u^h(t)\|_{W^{1,2}(\mathbb{T}; \mathbb{R}^N)}$ . The compact embedding  $W^{1,2}(\mathbb{T}; \mathbb{R}^N) \hookrightarrow L^2(\mathbb{T}; \mathbb{R}^N)$  thus yields the desired pointwise precompactness.

Therefore we may apply Arzelà–Ascoli to obtain some  $u \in C^{1,2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  and some sequence  $h_n \rightarrow 0$  such that  $u^{h_n}$  converges uniformly to  $u$  on  $[0, T]$  with respect to  $\|\cdot\|_{L^2(\mathbb{T}; \mathbb{R}^N)}$ .

### Step 6: Additional regularity 1

We first want to argue that from our construction, one already obtains that  $u \in L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$ . For this we first notice that for a fixed  $t \in [0, T]$ , we find by the pointwise precompactness some non-relabelled subsequence and  $v \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$  such that  $u^{h_n}(t)$  converges weakly to  $v$  in  $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . By uniqueness of the limit, we already have  $u(t) = v$  almost everywhere, which yields  $u(t) \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ . Applying the lower semicontinuity of the norm, we may also deduce that

$$\|u(t)\|_{W^{1,2}} \leq \liminf_{n \rightarrow \infty} \|u^{h_n}(t)\|_{W^{1,2}} \leq C E_\varepsilon(u^0),$$

from which we deduce that  $u \in L^\infty([0, T]; W^{1,2}(\mathbb{T}; \mathbb{R}^N))$ . Secondly the boundedness of the energies

$$\sup_{0 \leq t \leq T} E_\varepsilon(u(t)) < \infty$$

follows from the lower semicontinuity of the energy and the pointwise  $L^2$  convergence and pointwise weak convergence in  $W^{1,2}$  as described in Step 1. Lastly we want to argue that  $\partial_t u \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . From inequality (2.16) in step 4, we deduce that  $\partial_t u^h$  is a bounded sequence in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . Thus we find a non-relabelled subsequence of  $h_n$  and some  $w \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$  such that  $u^{h_n}$  converges weakly to  $w$  in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . But then  $w$  is already the weak time derivative of  $u$  since for any testfunction  $\xi$ , we have

$$\begin{aligned} \int_{[0, T] \times \mathbb{T}} \langle u, \partial_t \xi \rangle dx dt &= \lim_{n \rightarrow \infty} \int_{[0, T] \times \mathbb{T}} \langle u^{h_n}, \partial_t \xi \rangle dx dt \\ &= \lim_{n \rightarrow \infty} - \int_{[0, T] \times \mathbb{T}} \langle \partial_t u^{h_n}, \xi \rangle dx dt \\ &= - \int_{[0, T] \times \mathbb{T}} \langle w, \xi \rangle dx dt. \end{aligned}$$

### Step 7: $u$ is a weak solution

Going back to step 1, we see that  $u_n^h$  solves the Euler–Lagrange equation

$$\int \frac{1}{\varepsilon^2} \langle \nabla W(u_n^h), \xi \rangle + \langle \nabla u_n^h, \nabla \xi \rangle + \left\langle \frac{u_n^h - u_{n-1}^h}{h}, \xi \right\rangle dx = 0 \quad (2.18)$$

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for any testfunction  $\xi \in C^\infty(\mathbb{T}; \mathbb{R}^N)$ . Let  $t \in [0, T]$ . Since  $u_h$  is defined as the pointwise linear interpolation of the functions  $u_n^h$ , we find for the sequence  $h_n$  corresponding sequences  $\lambda_n \in [0, 1]$  and  $k_n \in \mathbb{N}$  such that

$$t = \lambda_n(k_n - 1)h_n + (1 - \lambda_n)k_nh_n$$

and therefore we can write

$$u_h(t) = \frac{k_nh_n - t}{h_n}u_{k_n-1}^h + \frac{t - (k_n - 1)h_n}{h_n}u_{k_n}^{h_n}.$$

In order to pass to the limit in equation (2.18), we first note that  $u_{k_n}^{h_n}$  converges to  $u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  since

$$\begin{aligned} \|u(t) - u_{k_n}^{h_n}\|_{L^2(\mathbb{T}; \mathbb{R}^N)} &\leq \|u(t) - u^{h_n}(t)\|_{L^2(\mathbb{T}; \mathbb{R}^N)} + \|u^{h_n}(t) - u^{h_n}(k_nh_n)\|_{L^2(\mathbb{T}; \mathbb{R}^N)} \\ &\lesssim \|u(t) - u^{h_n}(t)\|_{L^2(\mathbb{T}; \mathbb{R}^N)} + \sqrt{h_n}, \end{aligned}$$

which goes to zero as  $n$  tends to infinity. Here we used (2.17) for the second inequality. This implies that

$$\begin{aligned} \int \langle \nabla u(t), \nabla \xi \rangle dx &= - \int \langle u(t), \operatorname{div} \nabla \xi \rangle dx \\ &= - \lim_{n \rightarrow \infty} \int \langle u_{k_n}^{h_n}, \operatorname{div} \nabla \xi \rangle dx \\ &= \lim_{n \rightarrow \infty} \int \langle \nabla u_{k_n}^{h_n}, \nabla \xi \rangle dx. \end{aligned}$$

In Step 6, we moreover have shown that  $\partial_t u^{h_n}$  converges weakly to  $\partial_t u$  in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . This yields, by choosing cylindrical testfunctions, that  $\partial_t u^{h_n}(t)$  converges weakly to  $\partial_t u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  for almost every  $t \in [0, T]$ . Thus we obtain for almost every  $t \in [0, T]$  the convergence

$$\int \langle \partial_t u(t), \xi \rangle dx = \lim_{n \rightarrow \infty} \int \langle \partial_t u^{h_n}(t), \xi \rangle dx = \lim_{n \rightarrow \infty} \int \left\langle \frac{u_{k_n}^{h_n} - u_{k_n-1}^{h_n}}{h_n}, \xi \right\rangle dx.$$

To obtain the weak equation, we still need to prove convergence of remaining term. For this we note that

$$\left| \langle \nabla W(u_{k_n}^{h_n}), \xi \rangle \right| \lesssim \left( 1 + |u_{k_n}^{h_n}|^{p-1} \right) |\xi|,$$

which is a bounded sequence in  $L^{p'}(\mathbb{T}; \mathbb{R}^N)$ . Since we also may pass to a non-relabelled subsequence which converges almost everywhere, we thus obtain by the continuity of  $\nabla W$  that

$$\int \langle \nabla W(u(t)), \xi \rangle dx = \lim_{n \rightarrow \infty} \int \langle \nabla W(u_{k_n}^{h_n}), \xi \rangle dx.$$

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Therefore it follows from the Euler–Lagrange equation (2.18) that

$$\int \frac{1}{\varepsilon^2} \langle \nabla W(u(t)), \xi \rangle + \langle \nabla u(t), \nabla \xi \rangle + \langle \partial_t u(t), \xi \rangle dx = 0$$

for almost every  $t$  and all  $\xi \in C^\infty(\mathbb{T}; \mathbb{R}^N)$ . By continuity this extends to all  $\xi \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ .

**Step 8:** Sharp energy dissipation inequality for  $u$

Let  $0 \leq t \leq T$  and define  $k_n$  as in Step 7, where we also established that  $u_{k_n}^{h_n}$  converges to  $u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$ , and thus we may pass to another non-relabelled subsequence to obtain pointwise convergence almost everywhere. By Fatou's Lemma, we thus obtain

$$\int \frac{1}{\varepsilon} W(u(t)) dx \leq \liminf_{n \rightarrow \infty} \int \frac{1}{\varepsilon} W(u_{k_n}^{h_n}) dx.$$

Moreover we can deduce from the  $L^2$ -convergence that  $\nabla u_{k_n}^{h_n}$  converges to  $\nabla u(t)$  in the distributional sense. But  $\nabla u_{k_n}^{h_n}$  is uniformly bounded in  $L^2(\mathbb{T}; \mathbb{R}^N)$  by the energy dissipation inequality (2.14), thus we already obtain that  $\nabla u_{k_n}^{h_n}$  converges weakly to  $\nabla u(t)$  in  $L^2(\mathbb{T}; \mathbb{R}^N)$  which yields

$$\int \frac{\varepsilon}{2} |\nabla u(t)|^2 dx \leq \liminf_{n \rightarrow \infty} \int \frac{\varepsilon}{2} |\nabla u_{k_n}^{h_n}|^2 dx.$$

Lastly we have by the weak convergence of  $\partial_t u_{k_n}^{h_n}$  to  $\partial_t u$  in  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$  proven in Step 6 that

$$\begin{aligned} \int_0^t \int \varepsilon |\partial_t u|^2 dx dt &\leq \liminf_{n \rightarrow \infty} \left(1 - \frac{Ch_n}{2\varepsilon^2}\right) \int_0^t \int \varepsilon |\partial_t u_{k_n}^{h_n}|^2 dx dt \\ &\leq \liminf_{n \rightarrow \infty} \left(1 - \frac{Ch_n}{2\varepsilon^2}\right) \int_0^{k_n h_n} \int \varepsilon |\partial_t u_{k_n}^{h_n}|^2 dx dt. \end{aligned}$$

Summarizing these estimates, we obtain by the energy dissipation inequality for  $u^h$  (2.16) that for all  $0 \leq t \leq T$  we have

$$\begin{aligned} E_\varepsilon(u(t)) + \int_0^t \int \varepsilon |\partial_t u|^2 dx dt &\leq \liminf_{n \rightarrow \infty} E_\varepsilon(u_{k_n}) + \left(1 - \frac{Ch_n}{2\varepsilon^2}\right) \int_0^{k_n h_n} \int \varepsilon |\partial_t u_{k_n}^{h_n}|^2 dx dt \\ &\leq E_\varepsilon(u_\varepsilon^0). \end{aligned}$$

**Step 8:** Additional regularity 2

In order to complete the proof, we still have to show that  $\partial_{i,j}^2 u$  and  $\nabla W(u)$  are elements of  $L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ . In order to show that the second partial derivatives of  $u$  are square-integrable, we test the weak formulation (2.9) with the finite differences. To this end, define the finite differences as

$$\Delta_h^+ v(t, x) := \frac{v(t, x + hv) - v(t, x)}{h} \quad \text{and} \quad \Delta_h^- v(t, x) := \frac{v(t, x - hv) - v(t, x)}{h}$$

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for some  $h > 0$  and  $v \in \mathbb{R}^d$ . Thus plugging  $\Delta_h^- \Delta_h^+ u$  into (2.9) yields by the transformation formula that

$$0 = \int_0^T \int \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u), \Delta_h^+ u \rangle + \langle \nabla \Delta_h^+ u, \nabla \Delta_h^+ u \rangle + \langle \partial_t \Delta_h^+ u, \Delta_h^+ u \rangle dx dt,$$

which is equivalent to

$$\begin{aligned} & \int_0^T \int |\Delta_h^+ \nabla u|^2 dx dt \\ &= - \int_0^T \int \partial_t \left( \frac{|\Delta_h^+ u|^2}{2} \right) + \frac{1}{\varepsilon^2} \langle \Delta_h^+ \nabla W(u), \Delta_h^+ u \rangle dx dt \\ &= \int \frac{|\Delta_h^+ u(0)|^2 - |\Delta_h^+ u(T)|^2}{2} dx \\ &\quad - \int_0^1 \int_0^T \int \langle D^2 W((1-s)u(t, x) + su(t, x + hv)) \Delta_h^+ u, \Delta_h^+ u \rangle dx dt ds. \end{aligned}$$

The first summand can be estimated by  $\|u\|_{L^\infty([0, T], W^{1,2}(\mathbb{T}; \mathbb{R}^N))}$  (we are working with the representative chosen before). For the second summand, we partition  $W$  into the sum of  $W_{\text{conv}}$  and  $W_{\text{pert}}$ . The term involving the convex summand can then be estimated by

$$\int_0^1 \int_0^T \int \langle D^2 W_{\text{conv}}((1-s)u(t, x) + su(t, x + hv)) \Delta_h^+ u, \Delta_h^+ u \rangle dx dt ds \geq 0$$

and for the perturbation term, we get via the bound on its second derivative that

$$\begin{aligned} & \left| \int_0^1 \int_0^T \int \langle D^2 W_{\text{pert}}((1-s)u(t, x) + su(t, x + hv)) \Delta_h^+ u, \Delta_h^+ u \rangle dx dt ds \right| \\ & \lesssim \int_0^T \int |\nabla u|^2 dx dt, \end{aligned}$$

which is also finite. Combining these estimates, we obtain that  $\int_0^T \int |\Delta_h^+ \nabla u|^2 dx dt$  is uniformly bounded in  $h$ . Applying our calculation to all directions  $v \in \mathbb{R}^d$ , we get by the finite-differences theorem for all  $1 \leq i, j \leq d$  that  $\partial_{i,j}^2 u \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ .

In order to obtain  $\nabla W(u) \in L^2([0, T] \times \mathbb{T}; \mathbb{R}^N)$ , we again consider the weak formulation (2.9) and notice that since we have already shown that both the time derivative and second space derivatives of  $u$  are square-integrable, our claim follows from a duality argument.

This completes the proof.  $\square$

*Remark 2.2.2.* The inequality (2.14) with the factor  $1/2h$  instead of  $1/h - C/2\varepsilon^2$  follows immediately from the definition of our optimization problem, but is not optimal for fixed  $\varepsilon$  if we want to study the behaviour as  $h$  tends to zero.

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*Remark 2.2.3.* The energy dissipation inequality (2.12) can be deduced via the formal calculation

$$\begin{aligned}\frac{d}{dt} E_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u), \partial_t u \rangle + \varepsilon \langle \nabla u, \nabla \partial_t u \rangle \, dx \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u, \partial_t u \right\rangle \, dx \\ &= -\varepsilon \int |\partial_t u_\varepsilon|^2 \, dx.\end{aligned}$$

In order to make this calculation rigorous, we however need to show that  $t \mapsto E_\varepsilon(u(t))$  is absolutely continuous, which is non-trivial, but it would give us equality in energy dissipation inequality (2.12). Since we will only need the inequality, our proof will however suffice.

### 3. Convergence to an evolving partition

The main content of this chapter is to look at the behaviour of solutions of the Allen–Cahn equation (2.1) as the parameter  $\varepsilon$  tends to zero. As it turns out the scalar case is significantly easier to handle than the vectorial case, thus we shall first focus on the case  $N = 1$ , also called the *two-phase case*.

#### 3.1. Convergence in the twophase case

When want to consider the behaviour of solutions  $u_\varepsilon: [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$  to (2.1) for  $\varepsilon \rightarrow 0$ . Let us for now assume that the energies of the initial functions  $E_\varepsilon(u_\varepsilon^0)$  stay uniformly bounded as  $\varepsilon$  tends to zero. Then due to the energy dissipation inequality (2.12), we already obtain that for all  $0 \leq t \leq T$ , we have that  $E_\varepsilon(u_\varepsilon(t))$  stays uniformly bounded.

Another important observation for the convergence is the classic Modica Mortola trick (hier Referenz einfügen): Let  $\alpha < \beta$  be the two distinct zeros of the doublewell potential  $W$ . Then we define a primitive of  $\sqrt{2W(u)}$  via

$$\phi(u) := \int_\alpha^u \sqrt{2W(s)} \, ds.$$

For  $\psi_\varepsilon := \phi \circ u_\varepsilon$ , we can show that  $\psi_\varepsilon \in W^{1,1}((0, T) \times \mathbb{T})$  with weak derivatives  $\nabla \psi_\varepsilon = \sqrt{2W(u_\varepsilon)} \nabla u_\varepsilon$  and  $\partial_t \psi_\varepsilon = \sqrt{2W(u_\varepsilon)} \partial_t u_\varepsilon$ . (we will later show this in more generality (Referenz einfügen)).

Thus via Young's inequality, we can compute

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &= \int \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \, dx \\ &\geq \int \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx \\ &= \int |\nabla \psi_\varepsilon| \, dx, \end{aligned} \tag{3.1}$$

which suggests that we might hope for good compactness properties of  $\phi \circ u_\varepsilon$ . We combine these two observations into the following Proposition.

**Proposition 3.1.1.** *Given initial data  $u_\varepsilon^0$  whose energies stay uniformly bounded in the sense that*

$$\sup_{\varepsilon > 0} E_\varepsilon(u_\varepsilon^0) < \infty, \tag{3.2}$$

### 3. Convergence to an evolving partition

there exists for any sequence  $\varepsilon \rightarrow 0$  some non-relabelled subsequence such that the solutions of the Allen–Cahn equation (2.1) with initial condition  $u_\varepsilon^0$  converge in  $L^1((0, T) \times \mathbb{T})$  to some  $u = \alpha(1 - \chi) + \beta\chi$  with  $\chi \in \text{BV}((0, T) \times \mathbb{T}; \{0, 1\})$ . Moreover the compositions  $\psi_\varepsilon$  are uniformly bounded in  $\text{BV}((0, T) \times \mathbb{T})$  and converge to  $\phi \circ u$  in  $L^1((0, T) \times \mathbb{T})$ .

*Proof.* From the energy dissipation inequality (2.12) in Theorem 2.2.1 we infer that for all  $\varepsilon > 0$ , it holds that

$$\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t)) \leq E_\varepsilon(u_\varepsilon^0), \quad (3.3)$$

whose right hand side is by assumption uniformly bounded in  $\varepsilon$ . We want to use a similar calculation as (3.1) we thus infer that  $\nabla \psi_\varepsilon$  is uniformly bounded in  $L^1((0, T) \times \mathbb{T})$ . Moreover we may estimate

$$\begin{aligned} \int_0^T \int |\psi_\varepsilon| \, dx \, dt &= \int_0^T \int \left| \int_\alpha^{u_\varepsilon} \sqrt{2W(s)} \, ds \right| \, dx \, dt \\ &\leq \int_0^T \int |u_\varepsilon - \alpha| \sup_{s \in [\alpha, u_\varepsilon]} \sqrt{2W(s)} \, dx \, dt \\ &\lesssim 1 + \int_0^T \int |u_\varepsilon|^{1+p/2} \, dx \, dt \\ &\lesssim 1 + \int_0^T \int W(u_\varepsilon) \, dx \, dt, \end{aligned} \quad (3.4)$$

which is uniformly bounded via the energy bound. Here the last two inequalities follow from the upper respectively lower growth assumptions (2.3) on  $W$ . Thus we have indeed that  $\psi_\varepsilon$  is a bounded sequence in  $\text{BV}((0, T) \times \mathbb{T})$  and therefore there exists some non-relabelled subsequence and some  $\psi \in \text{BV}((0, T) \times \mathbb{T})$  such that  $\psi_\varepsilon$  converges to  $\psi$  in  $L^1((0, T) \times \mathbb{T})$ .

We notice that since  $W$  is non-negative and only has a discrete set of zeros, the function  $\phi$  is strictly increasing and continuous on  $\mathbb{R}$ , and is thus invertible. Moreover we may pass to a further non-relabelled subsequence of  $\psi_\varepsilon$  which converges almost everywhere to  $\psi$ . Thus defining  $u := \phi^{-1}(\psi)$ , we obtain that

$$u_\varepsilon = \phi^{-1}(\psi_\varepsilon) \rightarrow \phi^{-1}(\psi) = u$$

converges pointwise almost everywhere.

Moreover we notice that by Fatou's Lemma and the boundedness of the energies that

$$\int W(u) \, dx \leq \liminf_{\varepsilon \rightarrow 0} \int W(u_\varepsilon) \, dx \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon E_\varepsilon(u_\varepsilon) = 0.$$

Again using the non-negativity of  $W$ , this yields that  $W(u) = 0$  almost everywhere. Thus  $u \in \{\alpha, \beta\}$  almost everywhere and we may write  $u = \alpha(1 - \chi) + \beta\chi$  for some  $\chi: (0, T) \times \mathbb{T} \rightarrow \{0, 1\}$ . Looking again at the definition of  $u$ , we moreover obtain that

$$\psi = \phi(u) = \phi(\alpha)(1 - \chi) + \phi(\beta)\chi = \int_\alpha^\beta \sqrt{2W(s)} \, ds \, \chi =: \sigma \chi \quad (3.5)$$



### 3.1. Convergence in the twophase case

and since  $\psi$  is a function of bounded variation, this implies that  $\chi$  is of bounded variation as well.

Finally from the energy bound and the estimate  $|u_\varepsilon| \lesssim 1 + W(u)$ , we infer that  $u_\varepsilon$  is  $L^p$ -bounded, and since  $u_\varepsilon$  converges pointwise almost everywhere to  $u$ , we obtain the desired  $L^1$ -convergence.  $\square$

Nextup, we want to make sure that  $u$  respectively  $\chi$  assume their initial data and on the way obtain a useful bound on the time derivative.

**Lemma 3.1.2.** *With the assumptions of Proposition 3.1.1, we have  $\psi_\varepsilon \in W^{1,2}([0, T]; L^1(\mathbb{T}))$  with the estimate*

$$\left( \int_0^T \left( \int |\partial_t \psi_\varepsilon| dx \right)^2 dt \right)^{1/2} \lesssim E_\varepsilon(u_\varepsilon^0). \quad (3.6)$$

Furthermore the sequence  $u_\varepsilon$  is precompact in  $C([0, T]; L^2(\mathbb{T}))$ .

*Proof. Step 1:*  $\psi_\varepsilon \in W^{1,2}([0, T]; L^1(\mathbb{T}))$  and satisfies the inequality (3.6)

From estimate (3.4) we can infer that

$$\int_0^T \left( \int |\psi_\varepsilon| dx \right)^2 dt \lesssim \int_0^T \left( 1 + \int W(u_\varepsilon) dx \right)^2 dt < \infty$$

via the uniform boundedness of the energies (3.3). For the desired bound (3.6), we estimate via Hölder's inequality and the uniform boundedness of the energies that

$$\begin{aligned} \int_0^T \left( \int |\partial_t \psi_\varepsilon| dx \right)^2 dt &\leq \int_0^T \left( \int \sqrt{2W(u_\varepsilon)} |\partial_t u_\varepsilon| dx \right)^2 dt \\ &= \int_0^T \left( \int \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \sqrt{\varepsilon} |\partial_t u_\varepsilon| dx \right)^2 dt \\ &\leq \int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon) dx \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \\ &\leq 2 E_\varepsilon(u_\varepsilon^0) \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \quad (3.3) \\ &\leq 2 (E_\varepsilon(u_\varepsilon^0))^2, \quad (2.12) \end{aligned}$$

which completes step 1.

**Step 2:** The sequence  $\psi_\varepsilon$  is precompact in  $C([0, T]; L^1(\mathbb{T}))$ .

As noted in embedding (2.11), we have

$$W^{1,2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N)) \hookrightarrow C^{1/2}([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$$

from which the equicontinuity of the sequence follows with step 1. Moreover for fixed time  $t$ , estimate (3.1) yields that  $\nabla \psi_\varepsilon(t)$  is bounded in  $L^1(\mathbb{T})$ , and combined estimate (3.4) without integrating in time yields that  $\psi_\varepsilon(t)$  is a bounded sequence in  $W^{1,1}(\mathbb{T})$  and thus precompact in  $L^1(\mathbb{T})$ . Thus the Arzelà–Ascoli Theorem yields the desired precompactness.

### 3. Convergence to an evolving partition

**Step 3:** The sequence  $u_\varepsilon$  converges to  $u$  in measure uniformly in time.

From step 2, we infer that we already have

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int |\psi_\varepsilon(t, x) - \phi \circ u(t, x)| \, dx = 0.$$

It especially follows in combination with equation (3.5) that

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int (1 - \chi) |\psi_\varepsilon(t, x)| \, dx = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int (1 - \chi) |\psi_\varepsilon(t, x) - \phi \circ u(t, x)| \, dx = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int \chi |\psi_\varepsilon(t, x) - \sigma| \, dx = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{0 \leq t \leq T} \int \chi |\psi_\varepsilon(t, x) - \phi \circ u(t, x)| \, dx = 0$$

Moreover we have by continuity and the  $p$ -growth of  $W$  (2.3) that for a given  $\delta < \beta - \alpha$ ,

$$\min\{|\phi(t)| : |t - \alpha| > \delta/2 \text{ or } |t - \beta| > \delta/2\} = \rho > 0.$$

Therefore we may estimate

$$\begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T} \mathcal{L}^d(\{x \in \mathbb{T} : |u_\varepsilon(t, x) - u(t, x)| > \delta\}) \\ & \leq \operatorname{ess\,sup}_{0 \leq t \leq T} \mathcal{L}^d(\{(1 - \chi(t, x))|u_\varepsilon(t, x) - \alpha| > \delta/2\}) + \mathcal{L}^d(\{\chi(t, x)|u_\varepsilon(t, x) - \beta| > \delta/2\}) \\ & \leq \operatorname{ess\,sup}_{0 \leq t \leq T} \mathcal{L}^d(\{(1 - \chi(t, x))|\psi_\varepsilon(t, x)| > \rho\}) + \mathcal{L}^d(\{\chi(t, x)|\psi_\varepsilon(t, x) - \sigma| > \rho\}) \\ & \leq \operatorname{ess\,sup}_{0 \leq t \leq T} \frac{1}{\rho} \int (1 - \chi(t, x)) |\psi_\varepsilon(t, x)| \, dx + \frac{1}{\rho} \int \chi(t, x) |\psi_\varepsilon(t, x) - \sigma| \, dx, \end{aligned}$$

which goes to zero as  $\varepsilon$  goes to zero, proving our claim.

**Step 4:**  $u_\varepsilon^2$  is equiintegrable uniformly in time

We have

$$0 \leq u_\varepsilon^2 \lesssim 1 + W(u_\varepsilon)$$

by the growth bounds (2.3) on  $W$ . Since  $W(u_\varepsilon)$  converges to 0 in  $L^1(\mathbb{T})$  uniformly in time by the energy bound (3.3), it is equiintegrable uniformly in time. Thus  $u_\varepsilon^2$  is equiintegrable uniformly in time as well.

**Step 5:**  $u_\varepsilon$  converges in  $C([0, T]; L^2(\mathbb{T}))$ .

We somewhat repeat the proof the convergence in measure and integrability imply  $L^1$ -convergence, while keeping the uniformity in time.

Take some  $\delta > 0$ . Then we decompose the integral

$$\int |u_\varepsilon - u|^2 \, dx = \int_{\{|u_\varepsilon - u| \geq \delta\}} |u_\varepsilon - u|^2 \, dx + \int_{\{|u_\varepsilon - u| < \delta\}} |u_\varepsilon - u|^2 \, dx.$$

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For the first summand, we notice that

$$\sup_{0 \leq t \leq T} \int_{\{|u_\varepsilon - u| \geq \delta\}} |u_\varepsilon - u|^2 dx \lesssim \sup_{0 \leq t \leq T} \int_{\{|u_\varepsilon - u| \geq \delta\}} 1 + u_\varepsilon^2 dx \rightarrow 0$$

as  $\varepsilon \rightarrow 0$  since  $\mathcal{L}^d(|u_\varepsilon - u| \geq \delta) \rightarrow 0$  uniformly in time by step 3 and  $u_\varepsilon^2 + 1$  is equiintegrable uniformly in time by step 4.

For the second summand, we simply estimate

$$\sup_{0 \leq t \leq T} \int_{\{|u_\varepsilon - u| < \delta\}} |u_\varepsilon - u|^2 dx = \delta^2 \Lambda^d$$

Taking the limes superior as  $\varepsilon$  tends to zero of this inequality yields that the right hand side can be made arbitrarily small, which yields

$$\sup_{0 \leq t \leq T} \int |u_\varepsilon - u|^2 dx \rightarrow 0$$

as  $\varepsilon$  tends to zero. □

*Remark 3.1.3.* From the previous Lemma, it follows that if the initial conditions  $u_\varepsilon^0$  converge in  $L^1$  or pointwise almost everywhere to the function  $\alpha(1 - \chi^0) + \beta\chi^0$  (and we know that the limit is of this form since the energies of the initial values stay bounded), then  $u$  also assumes this initial condition in  $L^2(\mathbb{T})$ .

## 3.2. Convergence in the multiphase case

Let us now turn to the much more interesting and more challenging case where we consider systems of the Allen–Cahn equation (2.1), or in other words, we want to consider the case where  $u_\varepsilon$  maps to  $\mathbb{R}^N$  and therefore our potential  $W$  is a map from  $\mathbb{R}^N$  to  $[0, \infty)$ . The for us most relevant case is when  $W$  has exactly  $P = N + 1$  zeros given by  $\alpha_1, \dots, \alpha_P$ , but it is no limitation for us to allow more general amount of zeros.

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Let us also fix some notation for this section. For a function  $u = \sum_{i=1}^P \mathbf{1}_{\Omega_i} \alpha_i$  and the corresponding interfaces  $\Sigma_{i,j} := \partial_* \Omega_i \cap \partial_* \Omega_j$ , we define for a given testfunction  $\varphi \in C^\infty(\mathbb{T})$  the localized energies by

$$E_\varepsilon(u_\varepsilon; \varphi) := \int \varphi \left( \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \right) dx \quad (3.7)$$

and

$$E(u; \varphi) := \sum_{i < j} \sigma_{i,j} \int_{\Sigma_{i,j}} \varphi d\mathcal{H}^{d-1}. \quad (3.8)$$

### 3. Convergence to an evolving partition

One of the many difficulties in the vectorial case is that there is no easy choice of a primitive for  $\sqrt{2W(u)}$  compared to the scalar case. We there saw for example through the Modica-Mortola trick (3.1) that this provided a very powerful tool for us, and comparing the composition  $\phi \circ u_\varepsilon$  to  $u_\varepsilon$  was quite simple since the map  $\phi$  was invertible as a consequence of the non-negativity of  $W$ .

As a suitable replacement, we shall consider the *geodesic distance* defined as

$$d_W(u, v) := \inf \left\{ \int_0^1 \sqrt{2W(\gamma)} |\dot{\gamma}| dt : \gamma \in C^1([0, 1]; \mathbb{R}^N) \text{ with } \gamma(0) = u, \gamma(1) = v \right\}.$$

This indeed defines a metric on  $\mathbb{R}^N$ : If  $d_W(v, w) = 0$ , then by the continuity of  $W$  and since it only has a discrete set of zeros, we may deduce that  $v = w$ . Symmetry can be seen by reversing a given path between two points and the triangle inequality follows from concatenation of two paths and rescaling. Note moreover that by an approximation argument (for example through splines), we may also take paths  $\gamma$  which are piecewise continuously differentiable which makes constructions of paths easier.

The *geodesic distances* generated by  $W$  are defined as

$$\sigma_{i,j} := d_W(\alpha_i, \alpha_j)$$

and as a consequence of  $d_W$  being a metric satisfy

$$\sigma_{i,k} \leq \sigma_{i,j} + \sigma_{j,k},$$

$\sigma_{i,j} = 0$  if and only if  $i$  is equal to  $j$  and  $\sigma_{i,j} = \sigma_{j,i}$ .

Our replacement for the primitive  $\phi$  is now given for  $1 \leq i \leq P$  by the function

$$\phi_i(u) := d_W(\alpha_i, u).$$

Our first obstacle is the regularity of the function  $\psi_{\varepsilon,i} := \phi_i \circ u_\varepsilon$ . A priori we only know that  $\phi_i$  is locally Lipschitz continuous on  $\mathbb{R}^N$  and thus differentiable almost everywhere. If  $N = 1$ , then this would already suffice to deduce that  $\psi_{\varepsilon,i}$  is weakly differentiable, but in higher dimensions,  $u$  could for example move along a hyperplane where  $\phi_i$  could in theory be nowhere differentiable since it is a Lebesgue nullset. This can however be salvaged through the following chain rule for distributional derivatives by Ambrosio and Maso [AM90, Cor. 3.2].

**Theorem 3.2.1.** *Let  $\Omega \subseteq \mathbb{R}^d$  be an open set,  $p \in [0, \infty]$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^N)$  and let  $f: \mathbb{R}^N \rightarrow \mathbb{R}^k$  be a Lipschitz continuous function such that  $f(0) = 0$ . Then  $v := f \circ u \in W^{1,p}(\Omega, \mathbb{R}^k)$ . Furthermore for almost every  $x \in \Omega$  the restriction of  $f$  to the affine space*

$$T_x^u := \{y \in \mathbb{R}^N : y = u(x) + Du(x)z \text{ for } z \in \mathbb{R}^d\} = u(x) + \dot{T}_x^u$$

*is differentiable at  $u(x)$  and*

$$Dv = D(f|_{T_x^u})(u) Du$$

*holds almost everywhere in  $\Omega$ .*

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*Remark 3.2.2.* The matrix  $D(f|_{T_x^u})(u)$  can be interpreted as some matrix in  $\mathbb{R}^{k \times n}$  which acts on  $v \in \dot{T}_x^u$  by

$$D(f|_{T_x^u})(u(x))[v] = \lim_{h \rightarrow 0} \frac{f(u(x) + hv) - f(u(x))}{h}.$$

Thus we may choose a suitable representative since the product  $D(f|_{T_x^u})(u) Du$  will not change by definition of  $\dot{T}_x^u$ .

Moreover the assumption  $f(0) = 0$  can be left out on bounded domains by simply subtracting the constant  $f(0)$ .

We are now in the position to prove the following regularity result.

**Lemma 3.2.3.** *Let  $u \in C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$  with*

$$\operatorname{ess\,sup}_{0 \leq t \leq T} E_\varepsilon(u) + \int_0^T \int \varepsilon |\partial_t u|^2 \, dx \, dt < \infty$$

*for some  $\varepsilon > 0$ . Then for all  $1 \leq i \leq P$  there exists a map*

$$D\phi_i(u): [0, T] \times \mathbb{T} \rightarrow \operatorname{Lin}(\mathbb{R}^N, \mathbb{R})$$

*such that the chain rule is valid with the pair  $D\phi_i(u)$  and  $(\partial_t, D)u$ :*

*For almost every  $(t, x) \in [0, T] \times \mathbb{T}$  we have*

$$D(\phi_i \circ u) = D\phi_i(u) Du \quad \text{and} \quad \partial_t(\phi_i \circ u) = D\phi_i(u) \partial_t u.$$

*Furthermore we can control the modulus of  $\partial_u \phi_i(u)$  almost everywhere in time and space via the estimate*

$$|D\phi_i(u)| \leq \sqrt{2W(u)}. \quad (3.9)$$

*Additionally we have  $\phi_i \circ u \in L^\infty([0, T]; W^{1,1}(\mathbb{T})) \cap W^{1,1}([0, T] \times \mathbb{T})$  with the estimates*

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \int |\phi_i \circ u| \, dx \lesssim 1 + \operatorname{ess\,sup}_{0 \leq t \leq T} \varepsilon E_\varepsilon(u), \quad (3.10)$$

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \int |\nabla(\phi_i \circ u)| \, dx \leq \operatorname{ess\,sup}_{0 \leq t \leq T} E_\varepsilon(u) \quad (3.11)$$

*and*

$$\int_0^T \int |\partial_t(\phi_i \circ u)| \, dx \, dt \leq T \operatorname{ess\,sup}_{0 \leq t \leq T} E_\varepsilon(u) + \int_0^T \int \varepsilon |\partial_t u|^2 \, dx \, dt. \quad (3.12)$$

*Proof.* Since Theorem 3.2.1 requires Lipschitz continuity of  $\phi_i$ , but we only have local Lipschitz continuity, let us first assume that  $u$  is bounded in space and time. Then we may modify  $\phi_i$  outside of a compact set such that it is (globally) Lipschitz continuous and does not change on the image of  $u$ .

Since we have via the energy estimate that  $u \in W^{1,2}([0, T] \times \mathbb{T}; \mathbb{R}^N)$ , we obtain by the distributional chain rule Theorem 3.2.1 that  $\psi_i = \phi_i \circ u \in W^{1,2}([0, T] \times \mathbb{T})$ .

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Let  $\Pi(t, x)$  denote the orthogonal projection of  $\mathbb{R}^N$  onto  $\dot{T}_{x,t}^u$  and define

$$D\phi_i(u)(t, x)[v] := D(\phi_i|_{T_{t,x}^u})(u(t, x))[\Pi(t, x)v].$$

This defines a unique row vector and thus we can now proceed to prove inequality (3.9). Let  $v \in \dot{T}_{t,x}^u$ , a real number  $h \in \mathbb{R} \setminus \{0\}$  and  $\gamma: [0, 1] \rightarrow \mathbb{R}^N$  be a path connecting  $\alpha_i$  and  $u$ . Then we define the new path  $\tilde{\gamma}: [0, 1] \rightarrow \mathbb{R}^N$  by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(\frac{t}{2}) & , t \leq \frac{1}{2} \\ u + (t - \frac{1}{2})2hv & , t \geq \frac{1}{2}. \end{cases}$$

We observe that  $\tilde{\gamma}$  is a piecewise continuously differentiable path connecting  $\alpha_i$  and  $u + hv$ , thus we can estimate by a substitution that

$$\begin{aligned} & d_W(\alpha_i, u + hv) - \int_0^1 \sqrt{2W(\gamma(t))} |\gamma'(t)| dt \\ & \leq \int_0^1 \sqrt{2W(\tilde{\gamma}(t))} |\tilde{\gamma}'(t)| dt - \int_0^1 \sqrt{2W(\gamma(t))} |\gamma'(t)| dt \\ & = \int_0^1 \sqrt{2W(u + thv)} |hv| dt. \end{aligned}$$

Taking the infimum over all  $C^1$ -paths connecting  $\alpha_i$  and  $u$  yields

$$d_W(\alpha_i, u + hv) - d_W(\alpha_i, u) \leq \int_0^1 \sqrt{2W(u + thv)} |hv| dt.$$

Using a similar strategy but with a reversed path, we also obtain the inequality

$$d_W(\alpha_i, u) - d_W(\alpha_i, u + hv) \leq \int_0^1 \sqrt{2W(u + thv)} |hv| dt,$$

thus we obtain by the dominated convergence theorem

$$\limsup_{h \rightarrow 0} \left| \frac{\phi(u + hv) - \phi(u)}{h} \right| \leq \limsup_{h \rightarrow 0} \int_0^1 \sqrt{2W(u + thv)} |v| dt = \sqrt{2W(u)} |v|,$$

which yields

$$|D\phi_i|_{T_{t,x}^u}(u)[v]| \leq \sqrt{2W(u)} |v|$$

and thus gives us the desired inequality (3.9) since  $|\Pi(v)| \leq |v|$ .

Now let us consider the general case and denote by  $u_M$  the truncation of  $u$  defined by

$$u_M^j := \begin{cases} u & , \text{ if } |u^j| \leq M \\ M \frac{u^j}{|u^j|} & , \text{ else.} \end{cases}$$

Then we still have  $u_M \in W^{1,2}([0, T] \times \mathbb{T})$  and obtain by the previous step that  $\phi_i \circ u_M \in W^{1,2}([0, T] \times \mathbb{T})$  and that for almost every  $(t, x) \in [0, T] \times \mathbb{T}$ , the function

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$\phi_i$  is differentiable on  $T_{t,x}^{u_M}$ . Moreover if  $(t, x) \in u^{-1}([-M, M]^N)$ , then we obtain  $T_{t,x}^{u_M} = T_{t,x}^u$ . Next we want to show that  $\phi_i \circ u_M$  converges to  $\phi_i \circ u$  in a suitable sense. First we recognize that  $\phi_i \circ u_M$  converges to  $\phi_i \circ u$  pointwise almost everywhere. Moreover we find a majorant since

$$\begin{aligned} \phi_i(v) &\leq \int_0^1 \sqrt{2W(\alpha_i + s(v - \alpha_i))} |v - \alpha_i| ds \leq \left\| \sqrt{2W} \right\|_{L^\infty[v, \alpha_i]} |v - \alpha_i| \\ &\lesssim 1 + |v|^{1+p/2}, \end{aligned} \quad (3.13)$$

thus we have  $\phi_i \circ u_M \lesssim 1 + |u|^p$ , which is in integrable majorant. Therefore the dominated convergence theorem yields that  $\phi_i \circ u_M$  converges to  $\phi_i \circ u$  in  $L^1([0, T] \times \mathbb{T})$ . Moreover estimate (3.13) together with the  $p$ -growth of  $W$  (2.3) already yield the desired  $L^1$ -estimate (3.10).

Furthermore we recognize that on the set  $\{u_M = u\}$ , we already have

$$(\partial_t, D)u_M = (\partial_t, D)u$$

and the sets  $\{u_M = u\}$  are non-decreasing, thus

$$\lim_{M \rightarrow \infty} |\{u_M \neq u, (\partial_t, D)u_M \neq (\partial_t, D)u\}| = 0.$$

Moreover we have that for almost every  $(t, x)$ , the derivative  $D\phi_i(u_M(t, x))$  eventually becomes stationary. We denote its almost everywhere pointwise limit by  $D\phi_i(u)$  and it satisfies almost everywhere

$$|D\phi_i(u)| \leq \sqrt{2W(u)}.$$

In order to show that  $\phi_i \circ u$  is weakly differentiable with derivative

$$(\partial_t, D)\phi_i \circ u = D\phi_i(u)(\partial_t u, Du),$$

we compute for a testfunction  $\varphi$  that by the  $L^1$ -convergence, we have

$$\begin{aligned} \int_0^T \int \phi_i \circ u (\partial_t, D)\varphi dx dt &= \lim_{M \rightarrow \infty} \int_0^T \int \phi_i \circ u_M (\partial_t, D)\varphi dx dt \\ &= - \lim_{M \rightarrow \infty} \int D\phi_i(u_M)(\partial_t, D)u_M \varphi dx dt \\ &= - \lim_{M \rightarrow \infty} \int_{|u| \leq M} D\phi_i(u)(\partial_t, D)u \varphi d(t, x) \\ &= \int_0^T \int D\phi_i(u)(\partial_t, D)u \varphi dx dt. \end{aligned}$$

The last inequality is due to the dominated convergence theorem with majorant  $|\varphi| \sqrt{2W(u)} |(\partial_t, D)u|$ . It remains to prove the estimates (3.11) and (3.12). These follow

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from applications of Young's inequality

$$\begin{aligned} \operatorname{ess\,sup}_{0 \leq t \leq T} \int |\mathbf{D}(\phi_i \circ u)| &= \operatorname{ess\,sup}_{0 \leq t \leq T} \int |\mathbf{D} \phi_i(u) \mathbf{D} u| \, dx \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \int \sqrt{2W(u)} |\nabla u| \, dx \\ &\leq \sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon) \end{aligned}$$

and

$$\begin{aligned} \int_0^T \int |\partial_t(\phi_i \circ u)| \, dx \, dt &\leq \int_0^T \int \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u)} \sqrt{\varepsilon} |\partial_t u| \, dx \, dt \\ &\leq \int_0^T \int \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2} |\partial_t u|^2 \, dx \, dt, \end{aligned}$$

which finishes our proof.  $\square$

With this powerful tool on our hands, we are in the position to prove a similar result to Proposition 3.1.1 for the multiphase case.

**Proposition 3.2.4.** *Given initial data  $u_\varepsilon^0 = \sum_{i=1}^P \chi_i^0 \alpha_i$  whose energies satisfy*

$$E_\varepsilon(u_\varepsilon^0) \rightarrow E(u^0) =: E^0 < \infty,$$

*there exists for any sequence  $\varepsilon$  some non-relabelled subsequence such that the solutions of the Allen–Cahn equation (2.1) with initial condition  $u_\varepsilon^0$  converge in  $L^1((0, T) \times \mathbb{T}; \mathbb{R}^N)$  to some  $u = \sum_{i=1}^P \chi_i \alpha_i$  with a partition  $\chi \in \operatorname{BV}((0, T) \times \mathbb{T}; \{0, 1\}^P)$ . Furthermore we have*

$$\operatorname{ess\,sup}_{0 \leq t \leq T} E(u) \leq E^0 \tag{3.14}$$

*and for all  $1 \leq i \leq P$  the compositions  $\phi_i \circ u_\varepsilon$  are uniformly bounded in  $\operatorname{BV}((0, T) \times \mathbb{T})$  and converge to  $\phi_i \circ u$  in  $L^1((0, T) \times \mathbb{T})$ .*

*Remark 3.2.5.* The proof is quite similar to the proof of Proposition 3.1.1, but we have to work more in order to obtain the desired convergence to  $u$ .

*Proof.* By Theorem 2.2.1 the solution  $u_\varepsilon$  of (2.1) satisfies the assumptions of Lemma 3.2.3. Thus  $\psi_{\varepsilon,i} := \phi_i \circ u_\varepsilon$  is uniformly bounded in  $\operatorname{BV}((0, T) \times \mathbb{T})$  as  $\varepsilon$  tends to zero for all  $1 \leq i \leq P$  and thus we find a non-relabelled subsequence and  $v_i \in \operatorname{BV}((0, T) \times \mathbb{T})$  such that  $\psi_{\varepsilon,i}$  converges to  $v_i$  in  $L^1((0, T) \times \mathbb{T})$  and pointwise almost everywhere.

We now want to show that we can write  $v_i = \phi_i \circ u$  for  $u = \sum \chi_j \alpha_j$ . An elegant proof of this presented in [FT89, Thm. 4.1] can be done through the fundamental theorem of Young measures ([Mül99, Thm. 3.1]). By passing to another non-relabelled subsequence of  $u_\varepsilon$ , we obtain that  $u_\varepsilon$  generates a Young measure  $\nu$ . Since  $u$  is  $L^p$ -bounded, we have that for almost every  $(t, x)$ , the measure  $\nu_{(t,x)}$  is a probability measure. Moreover the sequence  $W(u_\varepsilon)$  is uniformly integrable since

$$0 \leq W(u_\varepsilon) \leq \frac{1}{\varepsilon} W(u_\varepsilon) \rightarrow 0$$



### 3.2. Convergence in the multiphase case

in  $L^1$ . Thus we obtain that for all  $\varphi \in L^\infty([0, T] \times \mathbb{T})$

$$0 = \lim_{\varepsilon \rightarrow 0} \int_0^T \int \varphi(t, x) W(u_\varepsilon(t, x)) \, dx \, dt = \int_0^T \int \varphi(t, x) \int_{\mathbb{R}^N} W(y) \, d\nu_{(t, x)}(y) \, dx \, dt,$$

which implies that for almost every  $(t, x)$ , the probability measure  $\nu_{(t, x)}$  is supported on the set  $\{\alpha_1, \dots, \alpha_P\}$ . Therefore we can write  $\nu_{(t, x)} = \sum_{j=1}^P \lambda_j \delta_{\alpha_j}$  for non-negative numbers  $\lambda_j \in [0, 1]$  with  $\sum \lambda_j = 1$ .

Now let  $1 \leq i \leq P$ . For every  $f \in C_0(\mathbb{R})$ , we have that  $f \circ \phi_i \in C_0(\mathbb{R}^N)$ , and thus we can compute that  $\phi_i \circ u_\varepsilon$  generates the Young measure given at almost every point  $(t, x)$  by  $\nu_{(t, x)} = \sum \lambda_j \delta_{\phi_i(\alpha_j)}$ . But  $\phi_i \circ u_\varepsilon$  converges to  $v_i$  in  $L^1$ , thus especially in measure, which implies by [Mül99, Cor. 3.2] that our convex combination already has to be trivial, or in other words, we can write

$$\nu_{(t, x)} = \sum_{j=1}^P \mathbb{1}_{\Omega_j(t)}(x) \delta_{\alpha_j}$$

for a time-dependent partition  $(\Omega_j(t))_{j=1, \dots, P}$  of  $\mathbb{T}$ . Thus we can write

$$\delta_{v_i(t, x)} = \sum_{j=1}^P \mathbb{1}_{\Omega_j(t)}(x) \delta_{\phi_i(\alpha_j)},$$

and by defining  $u := \sum_{j=1}^P \mathbb{1}_{\Omega_j(t)}(x) \alpha_j$ , we obtain that

$$v_i = \phi_i \circ u = \sum_{j=1}^P \mathbb{1}_{\Omega_j(t)}(x) \sigma_{i, j}.$$

What is left to show is the energy estimate (3.14) and that partition function  $\chi(t, x) := (\mathbb{1}_{\Omega_1(t)}(x), \dots, \mathbb{1}_{\Omega_P(t)}(x))$  is of bounded variation. The latter follows from the Fleming–Rishel co-area formula [FR60] which yields since  $\phi_i \circ u \in \text{BV}((0, T) \times \mathbb{T})$  that

$$\begin{aligned} & \infty > |(\partial_t, D)\phi_i \circ u|((0, T) \times \mathbb{T}) \\ &= \int_0^\infty \mathcal{H}^d(\partial_* \{(t, x) : \phi_i(u(t, x)) \leq s\}) \, ds \\ &\geq \int_0^{\min_{i \neq j} \sigma_{ij}} \mathcal{H}^d(\partial_* \{(t, x) \in (0, T) \times \mathbb{T} : \chi_i(t, x) = 1\}) \, ds \\ &= \min_{i \neq j} \sigma_{ij} |(\partial_t, D)\chi_i|((0, T) \times \mathbb{T}). \end{aligned}$$

Here we denote by  $\partial_* A$  the measure theoretic boundary of a given measurable set  $A$ , as defined for example in [EG15, Def. 5.7]. Since we have  $\sigma_{ij} > 0$  for  $i \neq j$ , this proves that  $\chi_i \in \text{BV}((0, T) \times \mathbb{T})$ .

The energy estimate is a non-trivial consequence of the lower semicontinuity of the variation measure and has been proven by Baldo in [Bal90]. We will recap the proof since it provides some insight into the geometry of the phases.  $\square$

### 3. Convergence to an evolving partition

**Definition 3.2.6.** Given a partition  $(\Omega_i)_{i=1,\dots,P}$  of the flat torus  $\mathbb{T}$ , where all sets  $\Omega_i$  are of finite perimeter, we define the  $(i, j)$ -th interface as  $\Sigma_{ij} := \partial_* \Omega_i \cap \partial_* \Omega_j$ .

We already mentioned that we want to apply the lower semicontinuity of the variation measure. Since we can think of  $\phi_i$  as being locally the correct choice for suitable  $i$ , we introduce the following notion.

**Definition 3.2.7.** Let  $\mu, \nu$  be regular positive Borel measures on  $\mathbb{T}$ . Define the supremum  $\mu \vee \nu$  of  $\mu$  and  $\nu$  as the smallest regular positive measure which is greater or equal to  $\mu$  and  $\nu$  on a Borel subset of  $\mathbb{T}$ . By the regularity, we have

$$\mu \vee \nu(U) = \sup \{ \mu(V) + \nu(W) : V \cap W = \emptyset, V \cup W \subseteq U, V \text{ and } W \text{ are open subsets of } \mathbb{T} \}$$

for any open subset  $U \subseteq \mathbb{T}$ .

It is natural to ask how we can characterize the total variation of  $\psi_i = \phi_i \circ u = \sum \mathbf{1}_{\Omega_j} \sigma_{ij}$ . For this we note that when going from set  $\Omega_j$  to  $\Omega_k$ , our function jumps from  $\sigma_{ij}$  to  $\sigma_{ik}$ . Thus we obtain the following intuitive result.

**Lemma 3.2.8.** In the setting of Proposition 3.2.4 we can write

$$|\nabla \psi_i| = \sum_{1 \leq j < k \leq P} |\sigma_{ij} - \sigma_{ik}| \, d\mathcal{H}^{d-1}|_{\Sigma_{jk}}.$$

The proof can be found in the Appendix. If we now consider the supremum of the measures  $(|\nabla \psi_i|)_{i=1,\dots,P}$ , we notice that all of them are supported on the interfaces  $\Sigma_{jk}$ , and that by the triangle inequality for the surface tensions, we have

$$\max_{1 \leq i \leq P} |\sigma_{ij} - \sigma_{ik}| = \sigma_{jk}. \quad (3.15)$$

Thus, we obtain the following.

**Proposition 3.2.9.** In the situation of Proposition 3.2.4, we have

$$\left( \bigvee_{i=1}^P |\nabla \psi_i| \right) = \sum_{1 \leq i < j \leq P} \sigma_{ij} \, d\mathcal{H}^{d-1}|_{\Sigma_{ij}}.$$

*Proof.* This follows by combining Lemma 3.2.8 and Lemma A.0.3 with the above equation (3.15).  $\square$

Collecting our observations, we can finish our arguments.

*Continuation of the proof for Proposition 3.2.4.* We are left with proving the estimate (3.14). We first note that as for the proof of (3.11), it holds that for any open subset  $U$  of the flat torus, we have for almost every time  $t$  the estimate

$$\liminf_{\varepsilon \rightarrow 0} \int_U \frac{1}{\varepsilon} W(u_\varepsilon(t)) + \frac{\varepsilon}{2} |\nabla u_\varepsilon(t)|^2 \, dx \geq |\nabla \psi_i(t)|(U).$$

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Since

$$\left( \bigvee_{i=1}^P |\nabla \psi_i| \right) (\mathbb{T}) = \sup \left\{ \sum_{i=1}^P |\nabla \psi_i|(U_i) : (U_i)_i \text{ are disjoint open subsets of } \mathbb{T} \right\},$$

we finally deduce

$$\begin{aligned} \operatorname{ess\,sup}_{0 \leq t \leq T} E(u(t)) &= \operatorname{ess\,sup}_{0 \leq t \leq T} \sum_{1 \leq i < j \leq P} \sigma_{ij} \mathcal{H}^{d-1}(\Sigma_{ij}(t)) \\ &= \operatorname{ess\,sup}_{0 \leq t \leq T} \left( \bigvee_{i=1}^P |\nabla \psi_i(t)| \right) (\mathbb{T}) \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon(t)) \\ &\leq E^0. \end{aligned}$$

The last inequality is due to the energy dissipation inequality (2.12).  $\square$

Nextup, we want to prove a stronger convergence of  $u_\varepsilon$  to  $u$ . This will let us prove that  $u$  achieves its initial data continuously in  $L^2$ .

**Lemma 3.2.10.** *In the situation of Proposition 3.2.4 we have  $\psi_{\varepsilon,i} \in W^{1,2}([0, T]; L^1(\mathbb{T}))$  with the corresponding estimate*

$$\left( \int_0^T \left( \int |\partial_t \psi_{\varepsilon,i}| \, dx \right)^2 dt \right)^{1/2} \lesssim E_\varepsilon(u_\varepsilon(0)). \quad (3.16)$$

Furthermore the sequence  $u_\varepsilon$  is precompact in  $C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$ . In particular we get that  $u$  achieves the initial data in  $C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$ .

*Remark 3.2.11.* This statement is almost similar to its twophase equivalent Lemma 3.1.2 and the proof uses a similar strategy.

*Proof. Step 1:* Estimate (3.16) holds

We compute that by Hölder's inequality and the energy dissipation inequality (2.12)

$$\begin{aligned} \int_0^T \left( \int |\partial_t \psi_{\varepsilon,i}| \, dx \right)^2 dt &\leq \int_0^T \left( \int \sqrt{2W(u_\varepsilon)} |\partial_t u_\varepsilon| \, dx \right)^2 dt \\ &= \int_0^T \int \frac{2}{\varepsilon} W(u_\varepsilon) \, dx \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt \\ &\leq 2 E_\varepsilon(u_\varepsilon(0)) \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt \\ &\leq 2 E_\varepsilon(u_\varepsilon(0))^2. \end{aligned}$$

By estimate (3.11), we especially obtain  $\psi_{\varepsilon,i} \in L^2([0, T]; L^1(\mathbb{T}))$ .

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**Step 2:** The sequence  $\psi_{\varepsilon,i}$  is precompact in  $C([0, T]; L^1(\mathbb{T}))$ .

This follows exactly as in the twophase case from the embedding of  $W^{1,2}([0, T]; L^1(\mathbb{T}))$  into  $C^{1/2}([0, T]; L^1(\mathbb{T}))$ .

**Step 3:** The sequence  $u_\varepsilon$  converges to  $\sum_i \chi_i \mathbb{1}_{\Omega_i} \alpha_i$  in measure uniformly in time.

Let  $\rho > 0$ . Then by definition of  $\phi_i$ , there exists some  $\delta > 0$  such that if  $|v - \alpha_i| > \rho$  holds for all  $1 \leq i \leq P$ , then we already have  $\phi_v > \delta$  for all  $1 \leq i \leq P$ . Therefore we can estimate

$$\begin{aligned} \operatorname{ess\,sup}_{0 \leq t \leq T} \mathcal{L}^d \left( \left\{ x : \left| u_\varepsilon - \sum_i \mathbb{1}_{\Omega_i} \alpha_i \right| > \rho \right\} \right) &= \operatorname{ess\,sup}_{0 \leq t \leq T} \sum_{i=1}^P \mathcal{L}^d (\{x \in \Omega_i : |u_\varepsilon - \alpha_i| > \rho\}) \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \sum_{i=1}^P \mathcal{L}^d (\{x \in \Omega_i : \phi_i(u_\varepsilon) > \delta\}) \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \frac{1}{\delta} \sum_{i=1}^P \int_{\Omega_i} |\psi_{\varepsilon,i}| \, dx \\ &\leq \operatorname{ess\,sup}_{0 \leq t \leq T} \frac{1}{\delta} \sum_{i=1}^P \int |\psi_{\varepsilon,i} - \psi_i| \, dx, \end{aligned}$$

which converges to 0 by Step 2.

**Step 4:** The sequence  $u_\varepsilon^2$  is equiintegrable uniformly in time

This follows as in the twophase case as well.

**Step 5:** The sequence  $u_\varepsilon$  converges in  $C([0, T]; L^2(\mathbb{T}; \mathbb{R}^N))$ .

Here the proof does not change as well.

This completes our arguments. □

## 4. Conditional convergence of the Allen–Cahn equation

### 4.1. Conditional convergence in the two-phase case

#### 4.1.1. The energy convergence assumption

As often in the Calculus of Variations, convergence of energies boost our modulus of convergence and gives our limit therefore additional regularity. Thus let us assume

$$\int_0^T E_\varepsilon(u_\varepsilon) dt \rightarrow \int_0^T E(u) dt \text{ as } \varepsilon \rightarrow 0, \quad (4.1)$$

where the surface tension energy is defined for  $u = \alpha(1 - \chi) + \beta\chi$  by

$$E(u) := \sigma \int |\nabla \chi| \quad (4.2)$$

already motivated in Section 1.2. Moreover we notice that the energy of  $u$  is exactly the total variation of  $\psi = \sigma\chi$ . Since for almost every time  $t$ , we have that  $\psi_\varepsilon(t)$  converges to  $\psi(t)$  in  $L^1(\mathbb{T})$  by Proposition 3.1.1, it follows from the lower semicontinuity of the variation measure and Young's inequality that

$$E(u) \leq \liminf_{\varepsilon \rightarrow 0} \int |\nabla \psi_\varepsilon| dx \leq \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon).$$

Moreover by the energy dissipation inequality (2.12), the energies stay uniformly bounded in time. Thus using the dominated convergence theorem, we see that our time integrated energy convergence assumption is equivalent to saying that for almost every time  $t \in (0, T)$ , we have convergence of the energies  $E_\varepsilon(u_\varepsilon(t)) \rightarrow E(u(t))$ .

Since the energies themselves can be interpreted as measures on the flat torus, we define for a continuous function  $\varphi \in C(\mathbb{T})$  the corresponding energy measures by

$$E_\varepsilon(u_\varepsilon; \varphi) := \int \varphi \left( \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \right) dx \text{ and}$$

$$E(u; \varphi) := \sigma \int \varphi |\nabla \chi|.$$

Then from the energy convergence and lower semicontinuity just discussed, it follows that

$$\lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \varphi) = E(u; \varphi). \quad (4.3)$$

#### 4. Conditional convergence of the Allen–Cahn equation

A first additional regularity result under the energy convergence assumption is the following Proposition, which ensures that we have square-integrable normal velocities.

**Proposition 4.1.1.** *In the setting of Proposition 3.1.1 and given the energy convergence assumption (4.1), the measure  $\partial_t \chi$  is absolutely continuous with respect to the measure  $|\nabla \chi| dt$  and the corresponding density  $V$  is square integrable with the estimate*

$$\int_0^T \int V^2 |\nabla \chi| dt \lesssim E^0,$$

where  $E^0 := \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon^0)$ .

*Remark 4.1.2.* By [AFP00, Thm. 3.103], we can disintegrate the measure  $|\nabla \chi|_{d+1}$ , which is the variation in space in time and space, into the measure  $|\nabla \chi|_d dt$ , which we will use by a slight abuse of notation and drop the subindex depending on the situation at hand. Here  $|\nabla \chi|_d$  simply denotes the variation in space for a fixed time.

*Proof.* Take a smooth test function  $\varphi \in C_C^\infty((0, T) \times \mathbb{T})$ . Then via the  $L^1$ -convergence of  $\psi_\varepsilon \rightarrow \psi$ , we have

$$\begin{aligned} \partial_t \psi(\varphi) &= \liminf_{\varepsilon \rightarrow 0} \partial_t \psi_\varepsilon(\varphi) \\ &= \liminf_{\varepsilon \rightarrow 0} \int_0^T \int \sqrt{2W(u_\varepsilon)} \partial_t u_\varepsilon \varphi dx dt \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left( \int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon) \varphi^2 dx dt \right)^{1/2} \left( \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \right)^{1/2} \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left( 2 \int_0^T E_\varepsilon(u_\varepsilon; \varphi^2) dt \right)^{1/2} (E_\varepsilon(u_\varepsilon))^{1/2} \tag{2.12} \\ &= \sqrt{2\sigma} \|\varphi\|_{L^2(\mathbb{T}, |\nabla \chi| dt)} \sqrt{E^0}. \tag{4.3} \end{aligned}$$

This proves both the absolute continuity and via a duality argument the desired bound since  $\partial_t \psi = \sigma \partial_t \chi$  and  $\sigma > 0$ .  $\square$

We finish this section with a proof for the equipartition of the energy, which tells us that both the summand involving the potential  $W(u_\varepsilon)$  and the norm of the gradient contribute to the energy in similar parts.

**Lemma 4.1.3.** *Under the energy convergence assumption (4.1), we have for any continuous function  $\varphi \in C^\infty(\mathbb{T})$  that*

$$\begin{aligned} E(u; \varphi) &= \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \varphi) = \lim_{\varepsilon \rightarrow 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx \\ &= \lim_{\varepsilon \rightarrow 0} \int \varphi \varepsilon |\nabla u_\varepsilon|^2 dx \\ &= \lim_{\varepsilon \rightarrow 0} \int \varphi \frac{1}{\varepsilon} 2W(u_\varepsilon) dx \end{aligned}$$

for almost every time  $0 \leq t \leq T$ .

#### 4.1. Conditional convergence in the two-phase case

*Proof.* We have already established the first equality before. For the second equality, we first assume that  $\varphi \in C(\mathbb{T}; [0, \infty))$ . By the lower semicontinuity of the variation measure, we immediately obtain

$$\liminf_{\varepsilon \rightarrow 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx \geq E(u; \varphi).$$

But by Young's inequality, we also have

$$\limsup_{\varepsilon \rightarrow 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx \leq \limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \varphi) = E(u; \varphi).$$

For general  $\varphi \in C(\mathbb{T})$ , we decompose  $\varphi$  into its positive and negative part and apply the previous argument to both in order to get the claim.

The third and fourth inequality follow for a given non-negative  $\varphi \in C(\mathbb{T}; [0, \infty))$  by the  $L^2$  estimate

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int \left| \sqrt{\varphi} \sqrt{\varepsilon} |\nabla u_\varepsilon| - \sqrt{\varphi} \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right|^2 dx \\ &= \lim_{\varepsilon \rightarrow 0} 2 E_\varepsilon(u_\varepsilon; \varphi) - 2 \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx = 0 \end{aligned}$$

which implies

$$\lim_{\varepsilon \rightarrow 0} \int \varphi \varepsilon |\nabla u_\varepsilon|^2 dx = \lim_{\varepsilon \rightarrow 0} \int \varphi \frac{1}{\varepsilon} 2W(u_\varepsilon) dx = \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \varphi),$$

finishing our proof.  $\square$

#### 4.1.2. Convergence to a BV-solution to twophase mean curvature flow

We start by defining a BV-formulation for motion by mean curvature.

**Definition 4.1.4** (BV-solution to twophase mean curvature flow). *Fix some finite time horizon  $T < \infty$  and initial data  $\chi^0 \in BV(\mathbb{T}; \{0, 1\})$ . We say that*

$$\chi \in C([0, T]; L^2(\mathbb{T}; \{0, 1\}))$$

*with  $\text{ess sup}_{0 \leq t \leq T} E(\chi)$  moves by mean curvature if there is a normal velocity  $V \in L^2(|\nabla \chi| dt)$  such that*

1. *For all  $\xi \in C_C^\infty((0, T) \times \mathbb{T}; \mathbb{R}^d)$ , we have*

$$\int_0^T \int V \langle \xi, \nu \rangle - \langle D\xi, \text{Id} - \nu \otimes \nu \rangle |\nabla \chi| dt = 0, \quad (4.4)$$

*where  $\nu := \nabla \chi / |\nabla \chi|$  is the inner unit normal.*

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2. The function  $V$  is the normal velocity of  $\chi$  in the sense that

$$\partial_t \chi = V |\nabla \chi| \, dt$$

holds distributionally in  $(0, T) \times \mathbb{T}$ .

3. The initial data  $\chi^0$  is achieved in  $C([0, T]; L^2(\mathbb{T}))$ , which simply means that  $\chi(0) = \chi^0$  as functions in  $L^2(\mathbb{T})$ .

Our main goal in this section is now to show that the function  $\chi$  we have found in Proposition 3.1.1 moves my mean curvature. Thus our goal is to prove the following Theorem.

**Theorem 4.1.5.** *Let a smooth doublewell potential  $W: \mathbb{R} \rightarrow [0, \infty)$  satisfy the assumptions (2.3)–(2.6). Let  $T < \infty$  be an arbitrary finite time horizon. Given a sequence of initial data  $u_\varepsilon^0: \mathbb{T} \rightarrow \mathbb{R}$  with  $\chi^0 \in \text{BV}(\mathbb{T}; \{0, 1\})$  such that  $u_\varepsilon^0 \rightarrow u^0 = (1 - \chi^0)\alpha + \chi^0\beta$  pointwise almost everywhere and*

$$E_0 := E(\chi^0) = \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon^0) < \infty,$$

*we have that for some subsequence of solutions to (2.1)  $u_\varepsilon$ , there exists a pointwise almost everywhere limit  $u = (1 - \chi)\alpha + \chi\beta$  with  $\chi \in \text{BV}((0, T) \times \mathbb{T}; \{0, 1\})$  and  $\chi \in C([0, T]; L^2(\mathbb{T}; \{0, 1\}))$  which assumes the initial data in  $C([0, T]; L^2(\mathbb{T}))$ . If we additionally assume that the time-integrated energies converge (4.1), then  $\chi$  moves by mean curvature in the sense of Definition 4.1.4.*

Looking at equation (2.9) which reads

$$\int \frac{1}{\varepsilon^2} W'(u_\varepsilon(t)) \varphi + \langle \nabla u_\varepsilon(t), \nabla \varphi \rangle + \partial_t u_\varepsilon(t) \varphi \, dx = 0, \quad (2.9)$$

we expect that for a suitable choice of testfunctions  $\varphi_\varepsilon$ , the following two terms converge:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \int \partial_t u_\varepsilon \varphi_\varepsilon &= \sigma \int_0^T \int V(\xi, \nu) |\nabla \chi| \, dt, \\ \lim_{\varepsilon \rightarrow 0} \int_0^T \int \frac{1}{\varepsilon^2} W'(u_\varepsilon) \varphi_\varepsilon + \langle \nabla u_\varepsilon, \nabla \varphi_\varepsilon \rangle \, dx \, dt &= \sigma - \int_0^T \int \langle D\xi, \text{Id} - \nu \otimes \nu \rangle |\nabla \chi| \, dt. \end{aligned}$$

But how do we find these testfunctions? For this, we first note that the curvature term  $\int \langle D\xi, \text{Id} - \nu \otimes \nu \rangle |\nabla \chi| \, dt$  is by [Mag12, Thm. 17.5] the first inner variation with respect to  $\xi$  of the perimeter functional, which is just our energy  $E$  up to the surface tension constant  $\sigma > 0$ . Since the energy  $E_\varepsilon$  converges to  $E$ , we may also hope that their first variations converge to each other. Thus it is plausible to compute the first inner variation  $\frac{d}{ds} \big|_{s=0} E_\varepsilon(\rho_s)$  and then we can hopefully choose the testfunction  $\varphi_\varepsilon$  in such a way that it equals  $\int \frac{1}{\varepsilon^2} W'(u_\varepsilon(t)) \varphi + \langle \nabla u_\varepsilon(t), \nabla \varphi \rangle \, dx$ .

Thus let  $(\rho_s)_s$  be functions which solve the ODE

$$\begin{cases} \partial_s \rho_s + \langle \xi, \nabla \rho_s \rangle &= 0 \\ \rho_0 &= u_\varepsilon. \end{cases}$$



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Then we formally compute

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \int \frac{\varepsilon}{2} |\rho_s|^2 + \frac{1}{\varepsilon} W(\rho_s) dx &= \int \varepsilon \langle \nabla u_\varepsilon, \nabla (-\langle \xi, \nabla u_\varepsilon \rangle) \rangle + \frac{1}{\varepsilon} W'(u_\varepsilon) (-\langle \xi, \nabla u_\varepsilon \rangle) dx \\ &= \int \left( \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle dx. \end{aligned}$$

We therefore test equation (2.9) against  $\varphi_\varepsilon := \langle \xi, \nabla u_\varepsilon \rangle$ .

##### 4.1.3. Convergence of the curvature term

The goal of this section is to prove the convergence

$$\lim_{\varepsilon \rightarrow 0} \int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle dx = \sigma \int \langle D\xi, \text{Id} - \nu \otimes \nu \rangle |\nabla \chi|$$

for almost every time  $t$ . We directly follow the proof from Luckhaus and Modica in [LM89].

Via an integration by parts, we obtain

$$\begin{aligned} \int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle &= \int -\varepsilon \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \left( \partial_{x_i} \xi^j \partial_{x_j} u_\varepsilon + \xi^j \partial_{x_i x_j}^2 u_\varepsilon \right) \\ &\quad + \frac{1}{\varepsilon} W(u_\varepsilon) \text{div } \xi dx. \end{aligned} \quad (4.5)$$

Moreover by another integration by parts, we have

$$\begin{aligned} \int \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \xi^j \partial_{x_i x_j}^2 u_\varepsilon dx &= \int - \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \left( \partial_{x_i x_j}^2 u_\varepsilon \xi^j + \partial_{x_j} \xi^j \partial_{x_i} u_\varepsilon \right) dx \\ &= \int -|\nabla u_\varepsilon|^2 \text{div } \xi - \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \partial_{x_i x_j}^2 u_\varepsilon \xi^j dx \end{aligned}$$

which is equivalent to

$$\int \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \xi^j \partial_{x_i x_j}^2 u_\varepsilon dx = -\frac{1}{2} \int |\nabla u_\varepsilon|^2 \text{div } \xi dx.$$

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Plugging this equation into the first equation (4.5), we obtain

$$\begin{aligned}
& \int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle dx \\
&= \int -\varepsilon \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \partial_{x_j} u_\varepsilon \partial_{x_i} \xi^j + \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \operatorname{div} \xi + \frac{1}{\varepsilon} W(u_\varepsilon) \operatorname{div} \xi dx \quad (4.6) \\
&= \varepsilon \int |\nabla u_\varepsilon|^2 \operatorname{div} \xi - \sum_{i,j=1}^d \partial_{x_i} u_\varepsilon \partial_{x_j} u_\varepsilon \partial_{x_i} \xi^j dx \\
&\quad + \int \frac{1}{\varepsilon} W(u_\varepsilon) \operatorname{div} \xi - \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \operatorname{div} \xi dx.
\end{aligned}$$

The last integral goes to zero by the equipartition of the energies (Lemma 4.1.3). Since  $\partial_{x_i} u_\varepsilon / |\nabla u_\varepsilon| = \partial_{x_i} \psi_\varepsilon / |\nabla \psi_\varepsilon|$  by the chain rule, the former integral can be written as

$$\int_{\mathbb{T}_\varepsilon} g(x, \nabla \psi_\varepsilon) \varepsilon |\nabla u_\varepsilon|^2 dx,$$

where

$$g(x, p) = \begin{cases} \sum_{i,j=1}^d -\frac{p_i}{|p|} \partial_{x_i} \xi^j \frac{p_j}{|p|} + \operatorname{div} \xi & \text{if } p \neq 0, \\ 0 & \text{else,} \end{cases}$$

and the set  $\mathbb{T}_\varepsilon$  is defined as

$$\mathbb{T}_\varepsilon := \{x \in \mathbb{T} : \nabla \psi_\varepsilon(x) \neq 0\} = \{x \in \mathbb{T} : \nabla u_\varepsilon(x) \neq 0\} \cap \{x \in \mathbb{T} : u_\varepsilon(x) \notin \{\alpha, \beta\}\}.$$

For the representation (4.1.3), we also have to use that

$$\mathcal{L}^d(\{x \in \mathbb{T} : \nabla u_\varepsilon(x) \neq 0 \text{ and } u_\varepsilon(x) \in \{\alpha, \beta\}\}) = 0.$$

Again by the equipartition of energies (Lemma 4.1.3) and the boundedness of  $g$ , we can replace  $\varepsilon |\nabla u_\varepsilon|^2$  by  $\sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon|$  in the integral (4.1.3) via the estimate

$$\begin{aligned}
& \int_{\mathbb{T}_\varepsilon} \left| \varepsilon |\nabla u_\varepsilon|^2 - \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \right| dx \\
& \leq \left( \int_{\mathbb{T}_\varepsilon} \left| \sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right|^2 dx \right)^{1/2} \left( \int_{\mathbb{T}_\varepsilon} \varepsilon |\nabla u_\varepsilon|^2 dx \right)^{1/2} \\
& \leq \left( \int \varepsilon |\nabla u_\varepsilon|^2 - 2\sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| + \frac{1}{\varepsilon} 2W(u_\varepsilon) dx \right) \sqrt{2E_\varepsilon(u_\varepsilon)}
\end{aligned}$$

which vanishes as  $\varepsilon$  tends to zero by Lemma 4.1.3. Thus

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}_\varepsilon} g(x, \nabla \psi_\varepsilon) \varepsilon |\nabla u_\varepsilon|^2 dx &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}_\varepsilon} g(x, \nabla \psi_\varepsilon) \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}_\varepsilon} g(x, \nabla \psi_\varepsilon) |\nabla \psi_\varepsilon| dx \\
&= \lim_{\varepsilon \rightarrow 0} \int F(x, \nabla \psi_\varepsilon) dx,
\end{aligned}$$

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where  $F(x, p)$  is defined as  $g(x, p)|p|$  at points  $p$  not equal to 0, and defined as 0 elsewhere. Since  $F(x, \lambda p) = \lambda F(x, p)$  for positive  $\lambda$  and since  $F$  satisfies the periodic boundary conditions in  $x$ , we are in the position to apply a Theorem proven by Reshetnyak in [Res68] and again by Luckhaus and Modica in [LM89]. We will later see a quantitative version of this in ???. Here it yields that since  $|\nabla \psi_\varepsilon|(\mathbb{T}) \rightarrow |\nabla \psi|(\mathbb{T})$  by the equipartition of energies Lemma 4.1.3, we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int F(x, \nabla \psi_\varepsilon) dx &= \sigma \int F(x, \nu) |\nabla \chi| \\ &= \sigma \int \left( \sum_{i,j=1}^d -\frac{\nu_i}{|\nu|} \partial_{x_i} \xi^j \frac{\nu_j}{|\nu|} + \operatorname{div} \xi \right) |\nu| |\nabla \chi| \\ &= \sigma \int \langle D \xi, \operatorname{Id} - \nu \otimes \nu \rangle |\nabla \chi|, \end{aligned}$$

which finishes the proof. The time integrated version given by

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle dx dt = \sigma \int_0^T \int \langle D \xi, \operatorname{Id} - \nu \otimes \nu \rangle |\nabla \chi| dt$$

follows from the generalized dominated convergence theorem via the equality (4.6) which yields

$$\left| \int \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \langle \xi, \nabla u_\varepsilon \rangle dx \right| \lesssim E_\varepsilon(u_\varepsilon).$$

##### 4.1.4. Convergence of the velocity term

We now want to prove the convergence of the velocity term given by

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int \partial_t u_\varepsilon \langle \xi, \varepsilon \nabla u_\varepsilon \rangle dx dt = \sigma \int_0^T \int V \langle \xi, \nu \rangle |\nabla \chi| dt.$$

The difficulty here is that products of weakly converging sequences will in general not weakly converge. To be more precise, we only have  $\partial_t u_\varepsilon \rightharpoonup V |\nabla \chi| dt$  and  $\nu_\varepsilon := \varepsilon \nabla u_\varepsilon \approx \nu$  in a weak sense.

Therefore we try to freeze the normal in a fixed direction, apply the weak convergence of  $\partial_t u_\varepsilon$  and then unfreeze the normal. Freezing the approximate normal  $\varepsilon \nabla u_\varepsilon$  amounts to replacing  $\varepsilon \nabla u_\varepsilon$  by  $\varepsilon |\nabla u_\varepsilon| \nu^*$  for a suitably chosen  $\nu^* \in \mathbb{S}^{d-1}$ . Let  $\eta$  be a cutoff on the support of  $\xi$ . Then the error we make can be estimated for all  $\alpha > 0$  via Young's

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inequality by

$$\begin{aligned}
& \left| \int_0^T \int \partial_t u_\varepsilon \langle \xi, \nabla u_\varepsilon \rangle dx dt - \int_0^T \int \partial_t u_\varepsilon \langle \xi, \varepsilon |\nabla u_\varepsilon| \nu^* \rangle dx dt \right| \\
& \leq \|\xi\|_{L^\infty} \int_0^T \int \eta \sqrt{\varepsilon} |\partial_t u_\varepsilon| \sqrt{\varepsilon} |\nabla u_\varepsilon| - |\nabla u_\varepsilon| \nu^* dx dt \\
& \leq \|\xi\|_{L^\infty} \left( \alpha \int_0^T \int \eta \varepsilon |\partial_t u_\varepsilon|^2 dx dt + \frac{1}{\alpha} \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 |\nu_\varepsilon - \nu^*|^2 dx dt \right) \\
& \leq \|\xi\|_{L^\infty} \left( \alpha \int_0^T \int \eta \varepsilon |\partial_t u_\varepsilon|^2 dx dt + \frac{1}{\alpha} \mathcal{E}_\varepsilon(\nu^*; \eta) \right).
\end{aligned}$$

Here the approximate tilt excess in direction  $\nu^*$  is given by

$$\mathcal{E}_\varepsilon(\nu^*; \eta) := \int_0^T \int \varepsilon |\nabla u_\varepsilon|^2 |\nu_\varepsilon - \nu^*|^2 dx dt.$$

With our frozen normal, we now notice that via the equipartition of energies, we may replace  $\varepsilon |\nabla u_\varepsilon|$  by  $\sqrt{2W(u_\varepsilon)}$  via the estimate

$$\begin{aligned}
& \int_0^T \int |\partial_t u_\varepsilon| \eta \left| \varepsilon |\nabla u_\varepsilon| - \sqrt{2W(u_\varepsilon)} \right| dx dt \\
& \leq \left( \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \right)^{1/2} \left( \int_0^T \int \eta^2 \left( \varepsilon |\nabla u_\varepsilon|^2 - 2 |\nabla u_\varepsilon| \sqrt{2W(u_\varepsilon)} + \frac{1}{\varepsilon} 2W(u_\varepsilon) \right) dx dt \right)^{1/2}
\end{aligned}$$

The first factor is uniformly bounded by the energy dissipation inequality (2.12) and the second term vanishes as  $\varepsilon$  tends to zero by the equipartition of energies Lemma 4.1.3. But now we recognise the identity

$$\int_0^T \int \partial_t u_\varepsilon \sqrt{2W(u_\varepsilon)} \langle \xi, \nu^* \rangle dx dt = \int_0^T \int \partial_t \psi_\varepsilon \langle \xi, \nu^* \rangle dx dt,$$

which converges as  $\varepsilon$  approaches zero to

$$\int_0^T \int \langle \xi, \nu^* \rangle \partial_t \psi = \sigma \int_0^T \int V \langle \xi, \nu^* \rangle |\nabla \chi| dt. \quad (4.7)$$

Finally we want to unfreeze the normal, which means that we want to replace  $\nu^*$  by  $\nu$  on the right hand side of equation (4.7). This can be estimated again by Young's inequality via the error

$$\|\xi\|_{L^\infty} \left( \alpha \int_0^T \int \eta V^2 |\nabla \chi| dt + \frac{1}{\alpha} \mathcal{E}(\nu^*; \eta) \right),$$

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where the tilt excess is given by

$$\mathcal{E}(\nu^*; \eta) := \sigma \int_0^T \int \eta |\nu - \nu^*|^2 |\nabla \chi| \, dt.$$

This finishes the proof for the convergence of the velocity term up to arguing that the errors can be made arbitrarily small.

First we study the behaviour of the approximate tilt excess  $\mathcal{E}_\varepsilon$  by connecting it to  $\mathcal{E}$ . We notice that by expansion, we have

$$\begin{aligned} \mathcal{E}_\varepsilon(\nu^*; \eta) &= 2 \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \, dx \, dt - 2 \left\langle \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \nabla u_\varepsilon \, dx \, dt, \nu^* \right\rangle \\ \mathcal{E}(\nu^*; \eta) &= 2 \mathbb{E}(u; \eta) - 2\sigma \left\langle \int_0^T \int \eta \nu |\nabla \chi| \, dt, \nu^* \right\rangle. \end{aligned}$$

But by the equipartition of energies Lemma 4.1.3, we have that

$$2 \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \, dx \, dt \rightarrow 2 \mathbb{E}(u; \eta)$$

and recognizing that  $\sigma \int_0^T \int \eta \nu |\nabla \chi| \, dt = \int_0^T \int \eta \nabla \psi$ , we also obtain

$$\begin{aligned} & \left| \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \nabla u_\varepsilon \, dx \, dt - \sigma \int_0^T \int \eta \nu |\nabla \chi| \, dt \right| \\ & \leq \left| \int_0^T \int \eta \sqrt{2W(u_\varepsilon)} \nabla u_\varepsilon \, dx \, dt - \int_0^T \int \eta \nabla \psi \right| + \int_0^T \int \varepsilon |\nabla u_\varepsilon| \left| \nabla u_\varepsilon - \frac{1}{\varepsilon} \sqrt{2W(u_\varepsilon)} \right| \, dx \, dt \\ & \leq \left| \int_0^T \int \eta \nabla \psi_\varepsilon \, dx \, dt - \int_0^T \int \eta \nabla \psi \right| \\ & \quad + \left( \int_0^T \int \varepsilon |\nabla u_\varepsilon|^2 \, dx \, dt \right)^{1/2} \left( \int_0^T \int \left( \sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right)^2 \, dx \, dt \right)^{1/2}, \end{aligned}$$

which vanishes as  $\varepsilon$  tends to zero by the weak convergence of  $\nabla \psi_\varepsilon \rightharpoonup^* \nabla \psi$  and the equipartition of the energies. Since we are taking  $\varepsilon \rightarrow 0$ , we thus only have to make sure that the tilt excess is sufficiently small since the approximate tilt excess converges to the tilt excess.

We are now in the position to argue why the error can be made arbitrarily small. Let  $\delta > 0$ . Then we first choose our  $\alpha > 0$  so small that

$$\limsup_{\varepsilon \rightarrow 0} \alpha \|\xi\|_{L^\infty} \left( \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt + \int_0^T \int V^2 |\nabla \chi| \, dt \right) < \frac{\delta}{2},$$

which is possible by the energy dissipation inequality (2.12) and Proposition 4.1.1. Then we choose a partition of unity  $(\eta_i)_{i=1, \dots, n}$  and approximate unit normals  $(\nu_i^*)_{i=1, \dots, n}$

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such that

$$\frac{2}{\alpha} \|\xi\|_{L^\infty} \sigma \sum_{i=1}^n \int_0^T \int \eta_i |\nu - \nu_i^*|^2 |\nabla \chi| dt < \frac{\delta}{2}.$$

The existence of these can be seen by taking a smooth approximation of  $\nu$  with respect to the measure  $|\nabla \chi| dt$ . Collecting all of our errors, we obtain

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \left| \int_0^T \int \partial_t u_\varepsilon \langle \xi, \varepsilon \nabla u_\varepsilon \rangle dx dt - \sigma \int_0^T \int V \langle \xi, \nu \rangle |\nabla \chi| dt \right| \\ &= \limsup_{\varepsilon \rightarrow 0} \left| \sum_{i=1}^n \int_0^T \int \eta_i \partial_t u_\varepsilon \langle \xi, \varepsilon \nabla u_\varepsilon \rangle dx dt - \sigma \int_0^T \int \eta_i V \langle \xi, \nu \rangle |\nabla \chi| dt \right| \\ &\leq \limsup_{\varepsilon \rightarrow 0} \|\xi\|_{L^\infty} \left( \sum_{i=1}^n \alpha \int_0^T \int \eta_i \varepsilon |\partial_t u_\varepsilon|^2 dx - \int \eta_i V^2 |\nabla \chi| dt + \frac{2}{\alpha} \mathcal{E}(\nu_i^*; \eta_i) \right) < \delta, \end{aligned}$$

which finishes the proof.

## 4.2. Conditional convergence in the multiphase case

### 4.2.1. Convergence to a BV-solution to multiphase mean curvature flow

We start by defining a BV-formulation for motion by multiphase mean curvature flow under the simplifying assumption that the mobilities are already fixed by the surface tensions through the equation

$$\mu_{ij} = \frac{1}{\sigma_{ij}}. \quad (4.8)$$

**Definition 4.2.1** (BV-solution to multiphase mean curvature flow). *Fix some finite time horizon  $T < \infty$ , a  $P \times P$  matrix of surface tensions  $\sigma$  and initial data  $\chi^0: \mathbb{T} \rightarrow \{0, 1\}^P$  with  $E_0 := (\chi^0)$  and  $\sum_{i=1}^P \chi_i^0 = 1$ . We say that*

$$\chi \in C([0, T]; L^2(\mathbb{T}; \{0, 1\}^P))$$

*with  $\text{ess sup}_{0 \leq t \leq T} E(\chi)$  and  $\sum_{i=1}^P \chi_i = 1$  moves by mean curvature with initial data  $\chi^0$  and surface tensions  $\sigma$  if there exist normal velocities  $V_i \in L^2(|\nabla \chi_i| dt)$  with*

$$\int_0^T \int V_i^2 |\nabla \chi_i| dt < \infty$$

*such that*

1. *For all  $\xi \in C_c^\infty((0, T) \times \mathbb{T}; \mathbb{R}^d)$ , we have*

$$\begin{aligned} & \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^T \int (V_i \langle \xi, \nu_i \rangle - \langle D \xi, \text{Id} - \nu_i \otimes \nu_i \rangle) \times \\ & \quad \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|) dt = 0, \end{aligned} \quad (4.9)$$

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where  $\nu_i$  is the inner unit normal of  $\chi_i$ .

2. The functions  $V_i$  are the normal velocities of the interfaces in the sense that

$$\partial_t \chi_i = V_i |\nabla \chi_i| \, dt$$

holds in the distributional sense on  $(0, T) \times \mathbb{T}$ .

3. The initial data is achieved in the space  $C([0, T]; L^2(\mathbb{T}))$ .

Since (4.9) is quite the long equation, we want to motivate it. Assuming that everything is nice and smooth, we have by the Gauss-Green theorem on surfaces [Mag12, Thm. 11.8] that

$$\int_{\Sigma_{ij}} \int \langle D\xi, \text{Id} - \nu \otimes \nu \rangle d\mathcal{H}^{d-1} = \int_{\Sigma_{ij}} \int H_{ij} \langle \xi, \nu_i \rangle d\mathcal{H}^{d-1} + \int_{\Gamma_{ij}} \int \langle \xi, \nu_{\Gamma_{ij}} \rangle d\mathcal{H}^{d-2}.$$

Here  $\Gamma_{ij}$  denotes the boundary of the surface  $\Sigma_{ij}$ , which we expect to only have significant contribution at triple junctions. But then Herring's angle condition (1.6) tells us that the boundary terms cancel out (HIER NOCHMAL TIM WEGEN DEN NORMALEN FRAGEN). But now we integrate at each interface over the quantity

$$(V_i - H_{ij}) \langle \xi, \nu_i \rangle,$$

which is zero by equation (1.5) combined with the assumption (4.8).

Our main goal will be to arrive at a similar conditional convergence result to multiphase mean curvature flow in the sense of the above definition. This is captured by the following.

**Theorem 4.2.2.** *Let a smooth multiwell potential  $W: \mathbb{R}^N \rightarrow [0, \infty)$  satisfy the assumptions (2.3)-(2.6). Let  $T < \infty$  be an arbitrary finite time horizon. Given a sequence of initial data  $u_\varepsilon^0: \mathbb{T} \rightarrow \mathbb{R}^N$  approximating a partition  $\chi^0 \in \text{BV}(\mathbb{T}; \{0, 1\}^P)$  in the sense that  $u_\varepsilon^0 \rightarrow u^0 = \sum_{1 \leq i \leq P} \chi_i^0 \alpha_i$  holds pointwise almost everywhere and*

$$E_0 := E(\chi^0) = \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon^0) < \infty,$$

*we have that for some subsequence of solutions to the Allen-Cahn equation (2.1)  $u_\varepsilon$  with initial datum  $u_\varepsilon^0$ , there exists a time-dependent partition  $\chi$  with  $\chi \in \text{BV}((0, T) \times \mathbb{T}; \{0, 1\}^P)$  and  $\chi \in C([0, T]; L^2(\mathbb{T}; \{0, 1\}^P))$  such that  $u_\varepsilon$  converges to  $u := \sum_{1 \leq i \leq P} \chi_i \alpha_i$  almost everywhere. Moreover  $u$  assumes the initial data  $u^0$  in  $C([0, T]; L^2(\mathbb{T}))$ . If we additionally assume that the time-integrated energies converge (4.1), then  $\chi$  moves by mean curvature in the sense of Definition 4.2.1.*

With these estimate, we can now prove the existence of normal velocities  $V_i$  for the evolving sets  $\Omega_i$  if we assume that the time-integrated energies converges, written out as in (4.1), but now for the multiphase energy given by (1.7).

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**Proposition 4.2.3.** *In the situation of Proposition 3.2.4 and under the energy convergence assumption (4.1), we have that for every  $1 \leq i \leq P$ , the signed measure  $\partial_t \chi_i$  is absolutely continuous with respect to the measure  $|\nabla \chi_i| dt$  and the corresponding density  $V_i$  is square integrable with the corresponding estimate*

$$\int_0^T \int V_i^2 |\nabla \chi_i| dt \lesssim E_0. \quad (4.10)$$

*Proof.* The proof proceeds in four steps. First we show that  $\partial_t \psi_i$  is absolutely continuous with respect to the energy measure. Afterwards, we show that we can already estimate  $|\partial_t \chi_i|$  by  $|\partial_t \psi_i|$ . To finish the argument, we then we have to argue that  $\partial_t \chi_i$  is already singular with respect to every part of the energy which is not contained in  $|\nabla \chi_i| dt$ .

**Step 1:** The signed measure  $\partial_t \psi_i$  is absolutely continuous with respect to the energy measure  $E(\cdot, u) dt$  with square-integrable density.

For this let  $\varphi$  be a testfunction. Then we can estimate by Hölder's inequality

$$\begin{aligned} \int_0^T \int \varphi \partial_t \psi_i &= \liminf_{\varepsilon \rightarrow 0} \int_0^T \int \varphi \partial_t \psi_{\varepsilon, i} dx dt \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left( \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \right)^{1/2} \left( \int_0^T \int \varphi^2 \frac{1}{\varepsilon} 2W(u_\varepsilon) dx dt \right)^{1/2} \\ &\leq \sqrt{2 E_0} \left( \int_0^T E(u; \varphi) dt \right)^{1/2}. \end{aligned}$$

Here the last inequality is due to the energy dissipation (2.12) and the proof of our claim is therefore complete.

**Step 2:** For  $d_i := \min_{j \neq i}$ , we have that  $d_i |\partial_t \chi_i| \leq |\partial_t \psi_i|$ .

Using the disintegration Theorem [AFP00, Thm. 3.103] and the Fleming–Rishel co-area formula, we have that for any open set  $U \subseteq [0, T] \times \mathbb{T}$  that

$$\begin{aligned} |\partial_t \psi_i|(U) &= \int |\partial_t \psi_i(\cdot, x)|(\pi_x(U)) dx \\ &= \int \int_0^\infty \mathcal{H}^0 \left( \left\{ t : \phi_i \left( \sum_j \chi_j \alpha_j \right) > s, (t, x) \in U \right\} \right) ds dx \\ &\geq \int \int_0^{d_i} \mathcal{H}^0 (\{t : \chi_i(t, x) = 1, (t, x) \in U\}) ds dx \\ &= d_i \int \int_0^\infty \mathcal{H}^0 (\{t : \chi_i(t, x) > s, (t, x) \in U\}) ds dx \\ &= d_i |\partial_t \chi_i|(U). \end{aligned}$$

Here the set  $\pi_x(U)$  denotes the set of all  $t$  such that  $(t, x)$  is an element of  $U$ . By an approximation argument, this inequality holds for every testfunction as well, proving the claim.



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**Step 3:** The signed measure  $\partial_t \chi_i$  is singular to the wrong parts of  $E(u; \cdot)$ . More precisely we claim that if  $1 \leq j < k \leq P$  and  $i \notin \{j, k\}$ , then the measures  $\mathcal{H}^{d-1}|_{\Sigma_{jk}} dt$  and  $|\partial_t \chi_i|$  are mutually singular.

We again write  $|\nabla \chi|_{d+1}$  for the space derivative in time and space and  $|\nabla \chi|_d$  for the space derivative for some fixed time  $t$ , which is defined for almost every time. Moreover, we denote by  $\tilde{\Omega}_i$  the set in time and space such that  $\chi_i(t, x) = \mathbb{1}_{\tilde{\Omega}_i}(t, x)$  and by  $\tilde{\Sigma}_{ij}$  the corresponding interfaces.

By Lemma A.0.1 we can then write

$$|\partial_t \chi_i| \leq |(\partial_t, \nabla) \chi_i| = \sum_{l \neq i} \mathcal{H}^d|_{\partial_* \tilde{\Omega}_i \cap \partial_* \tilde{\Omega}_l}.$$

But for all  $l \neq i$ , we have again by Lemma A.0.1 that either

$$|(\partial_t, \nabla) \chi_j|(\tilde{\Sigma}_{il}) = 0 \quad \text{or} \quad |(\partial_t, \nabla) \chi_k|(\tilde{\Sigma}_{il}) = 0.$$

Without loss of generality we assume the former. Then it follows that  $|\nabla \chi_j|_{d+1}(\tilde{\Sigma}_{il}) = 0$  and therefore we have

$$\begin{aligned} \mathcal{H}^{d-1}|_{\Sigma_{jk}} dt(\tilde{\Sigma}_{il}) &= \frac{1}{2} (|\nabla \chi_j|_d + |\nabla \chi_k|_d - |\nabla(\chi_j + \chi_k)|_d) dt(\tilde{\Sigma}_{il}) \\ &= \frac{1}{2} (|\nabla \chi_j|_{d+1} + |\nabla \chi_k|_{d+1} - |\nabla(\chi_j + \chi_k)|_{d+1})(\tilde{\Sigma}_{il}) \\ &= \frac{1}{2} (|\nabla \chi_k|_{d+1}(\tilde{\Sigma}_{i,l}) - |\nabla \chi_k|_{d+1})(\tilde{\Sigma}_{il}) = 0. \end{aligned}$$

This proves the desired singularity.

**Step 4:** We have  $|\partial_t \chi_i| \leq |\nabla \chi_i| dt$  and estimate (4.10) holds.

Combining Step 1 and Step 2, we obtain that  $|\partial_t \chi_i|$  is absolutely continuous with respect to the energy measure  $E(u; \cdot)$ . Moreover we can write

$$E(u; \cdot) dt = \sum_{j \neq i} \sigma_{ij} \mathcal{H}^{d-1}|_{\Sigma_{ij}} dt + \sum_{1 \leq j < k \leq P, j, k \neq i} \sigma_{jk} \mathcal{H}^{d-1}|_{\Sigma_{jk}} dt.$$

In the last term, the second summand is by Step 3 singular to  $|\partial_t \chi_i|$  and the first summand is bounded by  $|\nabla \chi_i| dt$ , which proves that  $\partial_t \chi_i$  is absolutely continuous with respect to  $|\nabla \chi_i| dt$ .

The estimate (4.10) is now a consequence of our previous arguments combined with a duality estimate. Fix some  $K > 0$  and let  $\varphi_n$  be a sequence of testfunctions which converges in  $L^2(|\nabla \chi_i| dt)$  to the function  $V_i \mathbb{1}_{|V_i| \leq K} \mathbb{1}_{\text{supp}|\partial_t \chi_i|}$ . Then combining the

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previous steps, we can estimate

$$\begin{aligned}
\int_0^T \int V_i^2 \mathbf{1}_{|V_i| \leq K} |\nabla \chi_i| \, dt &= \liminf_{n \rightarrow \infty} \left| \int_0^T \int V_i \varphi_n |\nabla \chi_i| \, dt \right| \\
&= \liminf_{n \rightarrow \infty} \left| \int_0^T \int \varphi_n \partial_t \chi_i \right| \\
&\lesssim \liminf_{n \rightarrow \infty} \int_0^T \int \varphi_n^2 |\partial_t \psi_i| \\
&\leq \sqrt{2 E_0} \liminf_{n \rightarrow \infty} \left( \int_0^T E(u; \varphi_n^2) \, dt \right)^{1/2} \\
&\lesssim \sqrt{2 E_0} \left( \int_0^T \int V_i^2 \mathbf{1}_{|V_i| \leq K} |\nabla \chi_i| \, dt \right)^{1/2}.
\end{aligned}$$

This proves by the monotone convergence theorem the desired claim.

Therefore our proof is complete.  $\square$

Nextup, we want to capture an important Lemma called the “equipartition of energy” which states that under the energy convergence assumption (4.1), it must already hold that

$$\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon} W(u_\varepsilon) \rightharpoonup^* 0$$

holds in the distributional sense. The proof is similar to the twophase case Lemma 4.1.3.

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**Lemma 4.2.4.** *In the situation of Proposition 3.2.4 and under the energy convergence assumption (4.1), we have for any continuous  $\varphi \in C(\mathbb{T})$  that*

$$\begin{aligned}
E(u; \varphi) &= \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \varphi) = \lim_{\varepsilon \rightarrow 0} \int \varphi \varepsilon |\nabla u_\varepsilon|^2 \, dx \\
&= \lim_{\varepsilon \rightarrow 0} \int \varphi \frac{1}{\varepsilon} 2W(u_\varepsilon) \, dx \\
&= \lim_{\varepsilon \rightarrow 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx
\end{aligned}$$

for almost every time  $0 \leq t \leq T$ .

#### 4.2.2. Localization estimates

In order to prove convergence of the curvature and velocity term, we want to reduce the multiphase case to the twophase case. The central idea here is to cover the flat torus with a suitable covering of balls and argue that, up to an arbitrarily small error as long as the balls are small enough, we can choose for each ball majority phases  $i, j$

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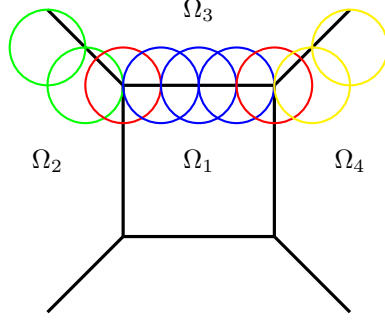


Figure 4.1.: Localization of multiphase mean curvature flow. For the green balls, we choose the majority phase (2,3), for the blue ones (1,3) and for the yellow balls (3,4). The red balls however give us an error.

such that the multiphase mean curvature looks like a twophase mean curvature on the ball, see Figure 4.1.

To formulate this rigorously let  $r > 0$  and define the covering  $\beta_r$  of  $\mathbb{T}$  by

$$\mathcal{B}_r := \{B_r(c) : c \in \mathcal{L}_r\},$$

where the set of centers  $\mathcal{L}_r$  is given by  $\mathcal{L}_r := \mathbb{T} \cap (r/\sqrt{d})\mathbb{Z}^d$ . Moreover let  $\rho_B$  be a smooth cutoff for the ball  $B$  with support in the ball with the same center, but double the radius.

Additionally to choosing a majority phase, we can also argue that along the chosen majority phase, we have a local flatness. Thus we may approximate the outer unit normal up to an arbitrarily small error by a constant unit vector. This is captured by the following Lemma.

**Lemma 4.2.5.** *For every  $\delta > 0$  and every partition  $\chi: \mathbb{T} \rightarrow \{0,1\}^P$  such that  $\chi_i \in \text{BV}(\mathbb{T})$  holds for all  $1 \leq i \leq P$ , there exist some  $r_0 > 0$  such that for all  $0 < r < r_0$ , we find for every ball  $B \in \mathcal{B}_r$  some unit vector  $\nu_B$  such that*

$$\sum_{B \in \mathcal{B}_r} \min_{i \neq j} \int \rho_B |\nu_i - \nu_B|^2 |\nabla \chi_i| + \int \rho_B |\nu_j + \nu_B|^2 |\nabla \chi_j| + \sum_{k \notin \{i,j\}} \int \rho_B |\nabla \chi_k| \lesssim \delta E(\chi)$$

Note that by Lemma A.0.1, integrating  $|\nabla \chi_k|$  over the sum of all  $k \notin \{i,j\}$  equates to summing over all interfaces which are not the  $(i,j)$ -th interface. Moreover the second summand is in some sense unnecessary, since the last summand provides us with a localization on the  $(i,j)$ -th interface, on which we have  $\nu_i = -\nu_j$ . However it is convenient to keep it so that we do not have to repeat this argument.

*Remark 4.2.6.* This estimates implies also the smallness of

$$\sum_{B \in \mathcal{B}_r} \min_i \left| E(\chi; \rho_B) - \int \rho_B |\nabla \psi_i| \right| = \sum_{B \in \mathcal{B}_r} E(\chi; \rho_B) - \max_i \int \rho_B |\nabla \psi_i|$$

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in the same sense as in the above Lemma 4.2.5. Notice that the equality follows for example from Proposition 3.2.9, which yields that  $|\nabla \psi_i| \leq E(\chi; \cdot)$ . The smallness follows since by Lemma 3.2.8, we have for every  $i \neq j$  that

$$\begin{aligned} & E(\chi; \rho_B) - \int \rho_B |\nabla \psi_i| \\ &= \sum_{1 \leq k < l \leq P} \sigma_{kl} \int_{\Sigma_{kl}} \rho_B \, d\mathcal{H}^{d-1} - \sum_{1 \leq k < l \leq P} |\sigma_{ik} - \sigma_{il}| \int_{\Sigma_{kl}} \rho_B \, d\mathcal{H}^{d-1} \\ &= \sum_{\substack{1 \leq k < l \leq P \\ \{k, l\} \neq \{i, j\}}} (\sigma_{kl} - |\sigma_{ik} - \sigma_{il}|) \int_{\Sigma_{kl}} \rho_B \, d\mathcal{H}^{d-1} \\ &\lesssim \sum_{k \notin \{i, j\}} \int \rho_B |\nabla \chi_k|. \end{aligned}$$

Thus we can estimate the error by the last summand of the error in Lemma 4.2.5.

#### 4.2.3. Convergence of the curvature term

**Proposition 4.2.7.** *In the situation of Proposition 3.2.4 and under the energy convergence assumption (4.1), we have that the first variations converge in the sense that for almost every time  $0 \leq t \leq T$ , we have*

$$\lim_{\varepsilon \rightarrow 0} \int \left\langle \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon), D u_\varepsilon \xi \right\rangle dx = \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_{\Sigma_{ij}} \langle D \xi, \text{Id} - \nu_i \otimes \nu_i \rangle d\mathcal{H}^{d-1}$$

and additionally we have the estimate

$$\int \left\langle \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon), D u_\varepsilon \xi \right\rangle dx \lesssim \|\nabla \xi\|_{\sup} E_\varepsilon(u_\varepsilon). \quad (4.11)$$

*Remark 4.2.8.* We can not proceed exactly as in the twophase case since we do not have one primitive  $\phi$  and more importantly, we can not expect the functions  $\psi_i^\varepsilon$  to satisfy

$$|\psi_{\varepsilon, i}|(\mathbb{T}) \rightarrow |\psi_i|(\mathbb{T}), \quad (4.12)$$

so the original theorem from Reshetnyak will not work here. However we have by the lower semicontinuity of the variation measure under the assumption of energy convergence (4.1) that

$$\begin{aligned} |\nabla \psi_i|(\mathbb{T}) &\leq \liminf_{\varepsilon \rightarrow 0} |\nabla \psi_{\varepsilon, i}|(\mathbb{T}) \\ &\leq \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) \\ &= E(u) \\ &= \sum_{1 \leq j < k \leq P} \sigma_{jk} \mathcal{H}^{d-1}(\Sigma_{jk}). \end{aligned}$$

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But by Lemma 3.2.8, we know that

$$|\nabla \psi_i| = \sum_{1 \leq j < k \leq P} |\sigma_{ij} - \sigma_{ik}| \mathcal{H}^{d-1}|_{\Sigma_{jk}},$$

which yields that localized on  $\partial_* \Omega_i$ , we have the necessary convergence  $|\nabla \psi_{\varepsilon,i}| \rightarrow |\nabla \psi_i|$ . Thus we will develop a quantitative version of Reshetnyaks theorem and apply it to our setting.

*Proof of Proposition 4.2.7.* Using the same calculations as in the twophase case, we obtain that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int \left\langle \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon), D u_\varepsilon \xi \right\rangle dx \\ &= \lim_{\varepsilon \rightarrow 0} \int \varepsilon \left( |\nabla u_\varepsilon|^2 \operatorname{div} \xi - \sum_{i=1}^N \sum_{j,k=1}^d \partial_{x_j} u_\varepsilon^i \partial_{x_j} \xi^k \partial_{x_k} u_\varepsilon^i \right) dx =: I \end{aligned}$$

Now however, we can not proceed as in the twophase case as explained above. Instead we rewrite

$$I = \int \varepsilon \langle D \xi, \operatorname{Id} - N_\varepsilon^\top N_\varepsilon \rangle |\nabla u_\varepsilon|^2 dx, \quad (4.13)$$

where we define  $N_\varepsilon := \nabla u_\varepsilon / |\nabla u_\varepsilon|$ . By the equipartition of energies Lemma 4.2.4 and using that  $|N_\varepsilon| \leq 1$ , we can replace  $\varepsilon |\nabla u_\varepsilon|^2$  by  $\sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon|$ .

Summarizing our results, it suffices by Lemma 4.2.4 and rescaling to show for all  $A \in C^\infty(\mathbb{T}; \mathbb{R}^{d \times d})$  with  $|A| \leq 1$  that

$$\lim_{\varepsilon \rightarrow 0} \int \langle A, N_\varepsilon^\top N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx = \sum_{1 \leq i < j \leq P} \sigma_{i,j} \int_{\Sigma_{i,j}} \langle A, \nu_i \otimes \nu_j \rangle d\mathcal{H}^{d-1}. \quad (4.14)$$

To this end, we will show the following three claims for a cutoff  $\eta$ .

Claim 1: We choose a majority phase by introducing the function  $\phi_i$  on the right hand side of equation (4.14). The corresponding error is given by

$$\limsup_{\varepsilon \rightarrow 0} \left| \int \eta \langle A, N_\varepsilon^\top N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx - \int \eta \langle A, \nu_{\varepsilon,i} \otimes \nu_{\varepsilon,i} \rangle |\nabla \psi_i| dx \right| \lesssim E(u; \eta) - \int \eta |\nabla \psi_i|,$$

where the approximate normal of the  $i$ -th phase is defined by  $\nu_i^\varepsilon := \nabla \psi_i^\varepsilon / |\nabla \psi_i^\varepsilon|$ .

Claim 2: By a quantitative Reshetnyak type argument, we have

$$\limsup_{\varepsilon \rightarrow 0} \left| \int \eta \langle A, \nu_{\varepsilon,i} \otimes \nu_{\varepsilon,i} \rangle |\nabla \psi_{\varepsilon,i}| dx - \int \eta \langle A, \theta_i \otimes \theta_i \rangle |\nabla \psi_i| \right| \lesssim E(u; \eta) - \int \eta |\nabla \psi_i|,$$

where  $\theta_i := \nabla \psi_i / |\nabla \psi_i|$ .

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Claim 3: We can undo the localization onto the majority phase. The corresponding error is given by

$$\left| \int \eta \langle A, \theta_i \otimes \theta_i \rangle |\nabla \psi_i| - \sum_{1 \leq j < k \leq P} \sigma_{jk} \int_{\Sigma_{jk}} \eta \langle A, \nu_j \otimes \nu_j \rangle d\mathcal{H}^{d-1} \right| \leq E(u; \eta) - \int \eta |\nabla \psi_i|,$$

where we remind that  $\nu_j$  is the inner unit normal of  $\Omega_i$ .

Let us first assume that we have proven those three claims and show how the desired convergence (4.14) follows from them. We take a partition of unity  $\eta_B$  of the flat torus with respect to the covering  $\mathcal{B}_r$  introduced in Section 4.2.2. Then

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \left| \int \langle A, N_\varepsilon^\top N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx - \sum_{1 \leq j < k \leq P} \sigma_{jk} \int_{\Sigma_{jk}} \langle A, \nu_j \otimes \nu_j \rangle d\mathcal{H}^{d-1} \right| \\ &= \left| \sum_{B \in \mathcal{B}_r} \int \langle A \eta_B, N_\varepsilon^\top N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx - \sum_{1 \leq j < k \leq P} \sigma_{jk} \int_{\Sigma_{jk}} \langle A \eta_B, \nu_i \otimes \nu_i \rangle d\mathcal{H}^{d-1} \right| \\ &\leq \sum_{B \in \mathcal{B}_r} \min_{1 \leq i \leq P} \left| \int \langle A \eta_B, N_\varepsilon^\top N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx - \int \langle A \eta_B, \nu_{\varepsilon, i} \otimes \nu_{\varepsilon, i} \rangle |\nabla \psi_{\varepsilon, i}| dx \right| \\ &\quad + \left| \int \langle A \eta_B, \nu_{\varepsilon, i} \otimes \nu_{\varepsilon, i} \rangle |\nabla \psi_{\varepsilon, i}| dx - \int \langle A \eta_B, \theta_i \otimes \theta_i \rangle |\nabla \psi_i| \right| \\ &\quad + \left| \int \langle A \eta_B, \theta_i \otimes \theta_i \rangle |\nabla \psi_i| - \sum_{1 \leq j < k \leq P} \sigma_{jk} \int_{\Sigma_{jk}} \langle A \eta_B, \nu_j \otimes \nu_j \rangle d\mathcal{H}^{d-1} \right| \\ &\lesssim \sum_{B \in \mathcal{B}_r} \min_{1 \leq i \leq P} E(u; \eta_B) - \int \eta_B |\nabla \psi_i|, \end{aligned}$$

which vanishes as  $r$  tends to zero by Remark 4.2.6. Thus let us now prove the three claims.

*Proof of Claim 1.* For simplicity, we drop the index  $i$  and for now also  $\varepsilon$ . First, we replace the matrix  $N$  by the matrix  $\pi N$ , where we define the rank-one matrix  $\pi$  by

$$\pi := \frac{\nabla \phi}{|\nabla \phi|} \otimes \frac{\nabla \phi}{|\nabla \phi|} \in \mathbb{R}^N.$$

The multiplication with  $\pi$  is an orthogonal projection  $\pi: \mathbb{R}^{N \times d} \rightarrow \mathbb{R}^{N \times d}$ . Moreover,

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we can compute that

$$\begin{aligned}
(\pi N^\top \pi N)_{ij} &= \sum_{k=1}^N (\pi N)_{ki} (\pi N)_{kj} = \sum_{k=1}^N \left( \sum_{l=1}^N \pi_{kl} N_{li} \right) \left( \sum_{r=1}^N \pi_{kr} N_{rj} \right) \\
&= \sum_{l,r=1}^N N_{li} N_{rj} \sum_{k=1}^N \pi_{kl} \pi_{kr} = \sum_{l,r=1}^N N_{li} N_{rj} \sum_{k=1}^N \frac{\partial_k \phi \partial_l \phi \partial_k \phi \partial_r \phi}{|\nabla \phi|^4} \\
&= \sum_{l,r=1}^N N_{li} \frac{\partial_l \phi}{|\nabla \phi|} N_{rj} \frac{\partial_r \phi}{|\nabla \phi|} = \frac{\partial_i \psi}{|\nabla u| |\nabla \phi|} \frac{\partial_j \psi}{|\nabla u| |\nabla \phi|}
\end{aligned}$$

Thus we infer that

$$\begin{aligned}
\langle A, \pi N_\varepsilon^\top \pi N_\varepsilon \rangle &= \sum_{i,j=1}^d A_{i,j} \frac{\partial_i \psi_\varepsilon}{|\nabla \phi| |\nabla u_\varepsilon|} \frac{\partial_j \psi_\varepsilon}{|\nabla \phi| |\nabla u_\varepsilon|} \\
&= \frac{|\nabla \psi_\varepsilon|^2}{|\nabla \phi|^2 |\nabla u_\varepsilon|^2} \left\langle A, \frac{\nabla \psi_\varepsilon}{|\nabla \psi_\varepsilon|} \otimes \frac{\nabla \psi_\varepsilon}{|\nabla \psi_\varepsilon|} \right\rangle = |\pi N_\varepsilon|^2 \langle A, \nu_\varepsilon \otimes \nu_\varepsilon \rangle,
\end{aligned}$$

where we used that

$$|\pi N_\varepsilon|^2 = \frac{|\nabla \psi_\varepsilon|^2}{|\nabla \phi|^2 |\nabla u_\varepsilon|^2}.$$

Moreover we recognize that since multiplication with  $\pi$  is an orthogonal projection, we have the Pythagorean Theorem

$$\begin{aligned}
N^\top N &= (\pi N + N - \pi N)^\top (\pi N + N - \pi N) \\
&= (\pi N)^\top \pi N + (N - \pi N)^\top (N - \pi N) + (\pi N)^\top (N - \pi N) + (N - \pi N)^\top \pi N \\
&= (\pi N)^\top \pi N + (N - \pi N)^\top (N - \pi N).
\end{aligned}$$

In order to prove the claim, we first get an error when replacing  $N_\varepsilon^\top N_\varepsilon$  with  $\nu_\varepsilon \otimes \nu_\varepsilon$  and the second error when replacing  $\sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon|$  by  $|\nabla \psi_\varepsilon|$ . For the first error we can estimate

$$\begin{aligned}
&\left| \int \eta \langle A, N_\varepsilon N_\varepsilon^\top \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx - \int \eta \langle A, \nu_\varepsilon \otimes \nu_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx \right| \\
&\leq \left| \int \eta \langle A, N_\varepsilon^\top N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx - \int \eta \langle A, \pi N_\varepsilon^\top \pi N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx \right| \\
&\quad + \left| \int \eta \langle A, \pi N_\varepsilon^\top \pi N_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx - \int \eta \langle A, \nu_\varepsilon \otimes \nu_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx \right| \\
&\leq \int \eta |N_\varepsilon - \pi N_\varepsilon|^2 \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx + \int \eta (1 - |\pi N_\varepsilon|)^2 \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| dx = I_\varepsilon.
\end{aligned}$$

Using again that multiplication with  $\pi$  is an orthogonal projection, we have

$$|(\text{Id} - \pi) N_\varepsilon|^2 = |N_\varepsilon|^2 - |\pi N_\varepsilon|^2 = 1 - |\pi N_\varepsilon|^2 \lesssim 1 - |\pi N_\varepsilon| = 1 - \left| \frac{\text{D} \phi}{|\nabla \phi|} N_\varepsilon \right|,$$

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where for the last identity, we used that  $|v^\top B| = |v \otimes v B|$  for all unit vectors  $v$  and matrices  $B$ . Therefore we can estimate

$$\begin{aligned} I_\varepsilon &\lesssim \int \eta \left( 1 - \left| \frac{D\phi}{|\nabla\phi|} N_\varepsilon \right| \right) \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx \\ &= \int \eta \left( \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| - |\nabla \psi_\varepsilon| \frac{\sqrt{2W(u_\varepsilon)}}{|\nabla \phi(u_\varepsilon)|} \right) \, dx \\ &\leq E_\varepsilon(u_\varepsilon; \eta) - \int \eta |\nabla \psi_\varepsilon| \, dx. \end{aligned}$$

The last inequality is due to Young's inequality and  $|\nabla \phi| \leq \sqrt{2W(u_\varepsilon)}$ . By the convergence of energies for almost every time and the lower semicontinuity of the variation measure, we thus have

$$\limsup_{\varepsilon \rightarrow 0} I_\varepsilon \lesssim E(u; \eta) - \int \eta |\nabla \psi|.$$

The second error is given by

$$\begin{aligned} &\left| \int \eta \langle A, \nu_\varepsilon \otimes \nu_\varepsilon \rangle \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx - \int \eta \langle A, \nu_\varepsilon \otimes \nu_\varepsilon \rangle |\nabla \psi_\varepsilon| \, dx \right| \\ &= \left| \int \eta \langle A, \nu_\varepsilon \otimes \nu_\varepsilon \rangle \left( \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| - |\nabla \psi_\varepsilon| \right) \, dx \right| \\ &\leq \int \eta \left( \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| - |\nabla \psi_\varepsilon| \right) \, dx \\ &\leq E_\varepsilon(u_\varepsilon; \eta) - \int \eta |\nabla \psi_\varepsilon| \, dx, \end{aligned}$$

which in the limes superior can be controlled by the desired term as above, finishing the proof for the first claim.  $\square$

*Proof of Claim 2.* Consider the sequence  $(\mu_\varepsilon)_\varepsilon$  of Radon measures on  $\mathbb{T} \times \mathbb{S}^{d-1}$  defined by

$$\mu_\varepsilon := \delta_{u_\varepsilon(x)} |\nabla \psi_\varepsilon| \, dx$$

which acts on continuous functions  $\varphi$  through

$$\int_{\mathbb{T} \times \mathbb{S}^{d-1}} \varphi(x, \nu) \, d\mu_\varepsilon(x, \nu) = \int_{\mathbb{T}} \varphi(x, \nu_\varepsilon(x)) |\nabla \psi_\varepsilon| \, dx.$$

By Young's inequality and the boundedness of energies,  $\mu_\varepsilon$  is a bounded sequence of Radon measures, thus we find a Radon measure  $\tilde{\mu}$  on  $\mathbb{T} \times \mathbb{S}^{d-1}$  and some non-relabelled subsequence such that  $\mu_\varepsilon \rightharpoonup^* \tilde{\mu}$  as Radon measures on  $\mathbb{T} \times \mathbb{S}^{d-1}$ . By [AFP00, Thm. 2.28] we can disintegrate the measure  $\tilde{\mu}$ , which means that we find probability measures  $(p_x)_{x \in \mathbb{T}}$  and a Radon measure  $\mu$  on  $\mathbb{T}$  such that  $x \mapsto p_x$  is  $\mu$ -measurable and we have the identity

$$\int_{\mathbb{T} \times \mathbb{S}^{d-1}} f(x, \tilde{\nu}) \, d\tilde{\mu}(x, \tilde{\nu}) = \int_{\mathbb{T}} \int_{\mathbb{S}^{d-1}} f(x, \tilde{\nu}) \, dp_x(\tilde{\nu}) \, d\mu(x)$$



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for all  $f \in L^1(\mathbb{T} \times \mathbb{S}^{d-1}, \tilde{\mu})$ . Thus we have

$$\lim_{\varepsilon \rightarrow 0} \int \varphi(x, \nu_\varepsilon) |\nabla \psi_\varepsilon| dx = \int_{\mathbb{T}} \int_{\mathbb{S}^{d-1}} \varphi(x, \tilde{\nu}) dp_x(\tilde{\nu}) d\mu(x)$$

for all continuous  $\varphi$  and plugging in  $\varphi(x, \tilde{\nu}) := \langle A(x), \tilde{\nu} \otimes \tilde{\nu} \rangle$  thus yields

$$\lim_{\varepsilon \rightarrow 0} \int \eta \langle A, \nu_\varepsilon \otimes \nu_\varepsilon \rangle |\nabla \psi_\varepsilon| dx = \int \eta \left\langle A(x), \int \tilde{\nu} \otimes \tilde{\nu} dp_x(\tilde{\nu}) \right\rangle d\mu(x).$$

We now would like to prove that up to an error bounded by  $E(u; \eta) - \int \eta |\nabla \psi|$ , the right-hand side is equal to  $\int \langle A, \theta \otimes \theta \rangle |\nabla \psi|$ .

On the one hand, we have by the lower semicontinuity of the variation measure that

$$\int \eta |\nabla \psi| \leq \liminf_{\varepsilon \rightarrow 0} \int \eta |\nabla \psi_\varepsilon| dx = \int \eta(x) \int 1 dp_x(\tilde{\nu}) d\mu(x) = \int \eta(x) d\mu(x), \quad (4.15)$$

or in other words, we have  $|\nabla \psi| \leq \mu$ .

On the other hand we have by the convergence of the energies and Young's inequality that we may estimate  $\mu$  via

$$\int \eta d\mu = \lim_{\varepsilon \rightarrow 0} \int \eta |\nabla \psi_\varepsilon| dx = \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \eta) = E(u; \eta). \quad (4.16)$$

Using that

$$|\tilde{\nu} \otimes \tilde{\nu} - \theta \otimes \theta| = |\tilde{\nu} \otimes (\tilde{\nu} - \theta) + \theta \otimes (\tilde{\nu} - \theta)| \leq 2|\tilde{\nu} - \theta|$$

and inequality (4.15), we can therefore estimate

$$\begin{aligned} & \left| \int \int_{\mathbb{S}^{d-1}} \eta \langle A, \tilde{\nu} \otimes \tilde{\nu} \rangle dp_x(\tilde{\nu}) d\mu - \int \eta \langle A, \theta \otimes \theta \rangle |\nabla \psi| \right| \\ & \leq \left| \int \eta \left\langle A, \int_{\mathbb{S}^{d-1}} \tilde{\nu} \otimes \tilde{\nu} dp_x(\tilde{\nu}) \right\rangle (d\mu - |\nabla \psi|) \right| + \left| \int \eta \left\langle A, \int_{\mathbb{S}^{d-1}} \tilde{\nu} \otimes \tilde{\nu} - \theta \otimes \theta dp_x(\tilde{\nu}) \right\rangle |\nabla \psi| \right| \\ & \leq \int \eta (d\mu - |\nabla \psi|) + 2 \int \eta \int_{\mathbb{S}^{d-1}} |\tilde{\nu} - \theta| dp_x(\tilde{\nu}) |\nabla \psi|. \end{aligned}$$

The first summand can by inequality (4.16) be estimated by  $E(u; \eta) - \int \eta |\nabla \psi|$ . For the second summand, we use a duality argument: Given a smooth vectorfield  $\xi$ , we have by the weak convergence  $\nabla \psi_\varepsilon \rightarrow \nabla \psi$  that

$$\int \langle \xi, \theta \rangle |\nabla \psi| = \lim_{\varepsilon \rightarrow 0} \int \langle \xi, \nu_\varepsilon \rangle |\nabla \psi_\varepsilon| dx = \int \left\langle \xi, \int \tilde{\nu} dp_x(\tilde{\nu}) \right\rangle d\mu,$$

from which we deduce that

$$\begin{aligned} \int \left\langle \xi, \int \theta - \tilde{\nu} dp_x(\tilde{\nu}) \right\rangle |\nabla \psi| &= \int \left\langle \xi, \int \tilde{\nu} dp_x(\tilde{\nu}) \right\rangle (d\mu - |\nabla \psi|) \\ &\leq \int |\xi| (d\mu - |\nabla \psi|) \\ &\leq E(u; |\xi|) - \int |\xi| |\nabla \psi|, \end{aligned}$$

which finishes the proof.  $\square$

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*Proof of Claim 3.* First we notice that since  $\psi_i = \sum_j \sigma_{i,j} \mathbf{1}_{\Omega_j}$ , we have  $\theta_i = \pm \nu_j$  on  $\Sigma_{j,k}$   $|\nabla \psi_i|$  almost everywhere. Additionally using the representation of  $|\nabla \psi_i|$  given in Lemma 3.2.8, we thus have

$$\begin{aligned} & \left| \int \eta \langle A, \theta_i \otimes \theta_i \rangle |\nabla \psi_i| - \sum_{1 \leq j < k \leq P} \sigma_{jk} \int_{\Sigma_{jk}} \eta \langle A, \nu_j \otimes \nu_j \rangle d\mathcal{H}^{d-1} \right| \\ &= \left| \sum_{1 \leq j < k \leq P} (|\sigma_{ij} - \sigma_{ik}| - \sigma_{jk}) \int_{\Sigma_{jk}} \eta \langle A, \nu_j \otimes \nu_j \rangle d\mathcal{H}^{d-1} \right| \\ &\leq \sum_{1 \leq j < k \leq P} \int_{\Sigma_{jk}} \eta (\sigma_{jk} - |\sigma_{ij} - \sigma_{ik}|) d\mathcal{H}^{d-1} \\ &= E(u; \eta) - \int \eta |\nabla \psi_i|, \end{aligned}$$

where for the inequality, we used that by the triangle inequality for the surface tensions, we have  $|\sigma_{ij} - \sigma_{ik}| \leq \sigma_{jk}$ .  $\square$

This finishes the proofs for the three claims. Lastly we notice that we obtain the pointwise in time bound (4.11) from the rewritten term (4.13).  $\square$

#### 4.2.4. Convergence of the velocity term

As in the previous proof for the convergence of the curvature term we want to localize and argue as in the twophase case. Moreover we remember from the twophase case that we had to freeze the unit normal. However we now have the important difference that  $\nabla u_\varepsilon$  describes both the change in physical space (domain) and the state space (codomain). Since the supposed limit  $\nu_i$  only describes the change in physical space, we should only freeze the normal in this direction. Using the equipartition of energies Lemma 4.2.4 which tells us that  $\varepsilon/2 |\nabla u_\varepsilon|^2$  roughly equals  $1/\varepsilon W(u_\varepsilon)$ , we come up with the following definition which is similar to the tilt-excess defined in the twophase case.

**Definition 4.2.9.** Let  $\nu^* \in \mathbb{S}^{d-1}$  and  $\eta \in C^\infty([0, T] \times \mathbb{T}; [0, 1])$ . For  $\varepsilon > 0$  and a function  $u_\varepsilon \in W^{1,2}((0, T) \times \mathbb{T}; \mathbb{R}^N)$ , the approximative localized tilt-excess of the  $i$ -th phase is given by

$$\mathcal{E}_i^\varepsilon(\nu^*; \eta, u_\varepsilon) := \int_0^T \int \eta \frac{1}{\varepsilon} |\varepsilon \nabla u_\varepsilon + \nabla \phi_i(u_\varepsilon) \otimes \nu^*|^2 dx dt.$$

In the limit  $\varepsilon = 0$  and for a partition  $\chi_i = \mathbf{1}_{\Omega_i} \in \text{BV}((0, T) \times \mathbb{T}; \{0, 1\})$  with  $\sum_i \chi_i = 1$ , we define the tilt-excess for  $1 \leq i, j \leq P$ ,  $i \neq j$  to be

$$\mathcal{E}_{ij}(\nu^*; \eta, u) := \int_0^T \int \eta |\nu_i - \nu^*|^2 |\nabla \chi_i| dt + \int_0^T \int \eta |\nu_j + \nu^*|^2 |\nabla \chi_j| dt + \sum_{k \in \{i, j\}} \int_0^T \int \eta |\nabla \chi_k| dt,$$

where  $u = \sum_{1 \leq i \leq P} \chi_i \alpha_i$ .

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Notice that we use  $+\nabla\phi_i(u_\varepsilon) \otimes \nu^*$  in the definition of the approximative tilt excess instead of its negative since the normal of  $\chi_i$  points inwards with respect to  $\Omega_i$ , but  $\nabla\phi_i/|\nabla\phi_i|$  points away from  $\Omega_i$ .

Moreover we want to point out that the first two summands of the tilt-excess  $\mathcal{E}_{ij}$  measure the local flatness of the boundary, and the last summand measure if mostly the  $(i, j)$ -th phase is present, see also the discussion in Section 4.2.2.

As in the twophase case, we first want to argue that when  $\varepsilon$  approaches zero, the approximate tilt-excess can be bounded by the tilt-excess of the limit.

**Lemma 4.2.10.** *Assume that we are in the situation of Proposition 3.2.4 and that the time-integrated energies converge (4.1). Then for every  $1 \leq i, j \leq P$  with  $i \neq j$ , every unit vector  $\nu^* \in \mathbb{S}^{d-1}$  and  $\eta \in C^\infty([0, T] \times \mathbb{T}; [0, 1])$ , we have*

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_i^\varepsilon(\nu^*; \eta, u_\varepsilon) \lesssim \mathcal{E}_{ij}(\nu^*; \eta, \chi).$$

*Proof.* By expanding the square, we see that we can rewrite the approximate tilt-excess as

$$\begin{aligned} & \mathcal{E}_i^\varepsilon(\nu^*; \eta, u_\varepsilon) \\ &= \int_0^T \int \eta \frac{1}{\varepsilon} \left( \varepsilon^2 |\nabla u_\varepsilon|^2 + 2\varepsilon \langle \nabla u_\varepsilon, \nabla \phi_i(u_\varepsilon) \otimes \nu^* \rangle + |\nabla \phi_i(u_\varepsilon) \otimes \nu^*|^2 \right) dx dt \\ &=: A_\varepsilon + B_\varepsilon + C_\varepsilon. \end{aligned}$$

Using the equipartition of energies Lemma 4.2.4, we immediately get that

$$\limsup_{\varepsilon \rightarrow 0} A_\varepsilon = \int_0^T E(u; \eta) dt$$

and

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} C_\varepsilon &\leq \limsup_{\varepsilon \rightarrow 0} \int_0^T \int \eta \frac{1}{\varepsilon} |\nabla \phi_i(u_\varepsilon)|^2 dx dt \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_0^T \int \eta \frac{1}{\varepsilon} 2W(u_\varepsilon) dx dt \\ &= \int_0^T E(u; \eta) dt. \end{aligned}$$

For the remaining summand, we note that

$$\limsup_{\varepsilon \rightarrow 0} B_\varepsilon = \limsup_{\varepsilon \rightarrow 0} \int_0^T \int 2\eta \langle \nabla \psi_{\varepsilon, i}, \nu^* \rangle dx dt = 2 \int_0^T \int \eta \langle \nu^*, \nabla \psi_i \rangle dt.$$

Summarizing these estimates, we have

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_i^\varepsilon(\nu^*; \eta, u_\varepsilon) \leq 2 \int_0^T E(u; \eta) dt + 2 \int_0^T \int \eta \langle \nu^*, \nabla \psi_i \rangle dt.$$

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By focusing on the  $(i, j)$ -th interface, we estimate the energy term by

$$\int_0^T E(u; \eta) \, dt \leq \int_0^T \sigma_{ij} \int \eta |\nabla \chi_j| \, dt + C \sum_{k \notin \{i, j\}} \int_0^T \int \eta |\nabla \chi_k| \, dt$$

and the other term via

$$\begin{aligned} \int_0^T \left\langle \nu^*, \int \eta \nabla \psi_i \right\rangle dt &= \sum_{1 \leq k \leq P} \sigma_{ik} \int_0^T \left\langle \nu^*, \int \eta \nabla \chi_k \right\rangle dt \\ &\leq \int_0^T \sigma_{ij} \int \eta \langle \nu^*, \nu \rangle |\nabla \chi_j| + C \sum_{k \notin \{i, j\}} \int_0^T \int \eta |\nabla \chi_k| \, dt, \end{aligned}$$

which yields that

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_i^\varepsilon(\nu^*; \eta, u_\varepsilon) \leq 2\sigma_{ij} \int_0^T \int \eta (1 + \langle \nu^*, \nu_j \rangle) |\nabla \chi_j| \, dt + C \sum_{k \notin \{i, j\}} \int_0^T \int \eta |\nabla \chi_k| \, dt.$$

Since  $1 + \langle \nu^*, \nu_j \rangle = \frac{1}{2} |\nu^* + \nu_j|^2$ , this finishes the proof.  $\square$

Note that we have actually proven a seemingly stronger estimate, since the summand  $|\nu_i - \nu^*|$  is not included on the right-hand side of the last estimate. However it serves to symmetrize the multiphase excess.

Using Lemma 4.2.10, we can now prove the convergence of the velocity term.

**Proposition 4.2.11.** *In the situation of Proposition 3.2.4 and under the assumption that the time-integrated energies converge (4.1), there exists a finite Radon measure  $\mu$  on  $[0, T] \times \mathbb{T}$  such that for any  $1 \leq i, j \leq P$  with  $i \neq j$  and any parameter  $\alpha \in (0, 1)$ , any unit vector  $\nu^* \in \mathbb{S}^{d-1}$  and any test vector field  $\xi \in C_c^\infty((0, T) \times \mathbb{T}; \mathbb{R}^d)$ , we have*

$$\begin{aligned} &\limsup_{\varepsilon \rightarrow 0} \left| \int_0^T \int \varepsilon \langle \partial_t u_\varepsilon, D u_\varepsilon \xi \rangle \, dx \, dt - \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle \xi, \nu_i \rangle V_i \, d\mathcal{H}^{d-1} \, dt \right| \\ &\lesssim \|\xi\|_{\sup} \left( \frac{1}{\alpha} \mathcal{E}_{ij}(\nu^*; \rho, u) + \alpha \mu(\rho) \right). \end{aligned}$$

Here  $\rho \in C^\infty([0, T] \times \mathbb{T}; \mathbb{R}^d)$  is any smooth cutoff for the support of  $\xi$ .

*Proof.* On a technical note, we first pass to a non-relabelled subsequence such that the limes superior becomes the regular limit. As in the twophase case, we first choose a majority phase and then freeze the approximate normal  $\varepsilon \nabla u_\varepsilon$  by replacing it with

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$-\nabla\phi_i(u_\varepsilon) \otimes \nu^*$ . The error we can be estimated by Young's inequality through

$$\begin{aligned} & \left| \int_0^T \int \varepsilon \langle \partial_t u_\varepsilon, D u_\varepsilon \xi \rangle dx dt + \int_0^T \int \langle \partial_t u_\varepsilon, \nabla \phi_i(u_\varepsilon) \otimes \nu^* \xi \rangle dx dt \right| \\ &= \left| \int_0^T \int \sqrt{\varepsilon} \left\langle \partial_t u_\varepsilon, \left( \sqrt{\varepsilon} \nabla u_\varepsilon + \frac{1}{\sqrt{\varepsilon}} \nabla \phi_i \otimes \nu^* \right) \xi \right\rangle dx dt \right| \\ &\leq \frac{1}{2} \|\xi\|_{\sup} \left( \alpha \int_0^T \int \rho \varepsilon |\partial_t u_\varepsilon|^2 dx dt + \frac{1}{\alpha} \mathcal{E}_i^\varepsilon(\nu^*; \rho, u_\varepsilon) \right). \end{aligned}$$

We thus have arrived at the term

$$- \int_0^T \int \langle \partial_t u_\varepsilon, \nabla \phi_i(u_\varepsilon) \otimes \nu^* \xi \rangle dx dt = - \int_0^T \int \partial_t \psi_{\varepsilon,i} \langle \xi, \nu^* \rangle dx dt,$$

which by the weak convergence  $\partial_t \psi_{\varepsilon,i} \rightharpoonup \partial_t \psi_i$  converges to

$$- \sum_{1 \leq j \leq P} \sigma_{ij} \int_0^T \int \langle \xi, \nu^* \rangle V_j |\nabla \chi_j| dt.$$

Thus

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \left| - \int_0^T \int \langle \partial_t u_\varepsilon, \nabla \phi_i(u_\varepsilon) \otimes \nu^* \xi \rangle dx dt - \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle \xi, \nu_i \rangle V_i d\mathcal{H}^{d-1} dt \right| \\ &= \left| \sum_{1 \leq k \leq P} \sigma_{ik} \int_0^T \int \langle \xi, \nu^* \rangle V_k |\nabla \chi_k| dt + \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle \xi, \nu_i \rangle V_i d\mathcal{H}^{d-1} dt \right| \\ &\lesssim \|\xi\|_{\sup} \sum_{k \notin \{i,j\}} \int_0^T \int \rho |V_k| |\nabla \chi_k| dt + \left| \int_0^T \int_{\Sigma_{ij}} (\langle \xi, \nu^* \rangle + \langle \xi, \nu_i \rangle) V_i d\mathcal{H}^{d-1} dt \right|. \end{aligned}$$

By applying Young's inequality twice and using that on  $\Sigma_{ij}$ , we have  $V_i = -V_j$ , we furthermore estimate

$$\begin{aligned} & \|\xi\|_{\sup} \left( \sum_{k \notin \{i,j\}} \alpha \int_0^T \int \rho |V_k|^2 |\nabla \chi_k| dt + \frac{1}{\alpha} \int_0^T \int \rho |\nabla \chi_k| dt + \int_0^T \int_{\Sigma_{ij}} |V_i| |\nu^* - \nu_i| d\mathcal{H}^{d-1} dt \right) \\ &\leq \|\xi\|_{\sup} \left( \alpha \sum_{1 \leq k \leq P} \int_0^T \int \rho |V_k|^2 |\nabla \chi_k| dt + \frac{1}{\alpha} \mathcal{E}_{ij}(\nu^*; \rho, u) \right). \end{aligned}$$

We therefore define the measure  $\mu$  through

$$\mu := \sum_{1 \leq k \leq P} |V_k|^2 |\nabla \chi_k| dt + \mu_2,$$

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where  $\mu_2$  is the weak  $*$  limit of some non-relabelled subsequence of the Radon measures  $\varepsilon |\partial_t u_\varepsilon|^2 dx dt$ , which stay bounded due to the energy dissipation inequality (2.12). This defines a finite Radon measure on  $[0, T] \times \mathbb{T}$  by the square-integrability of the velocities observed in Proposition 4.2.3 and thus, the proof is complete.  $\square$

We are now in the position to prove the main result Theorem 4.2.2.

*Proof of Theorem 4.2.2.* Let us first collect all previous results. Proposition 3.2.4 guarantees that a limit  $\chi$  as described in Theorem 4.2.2 exists. After that Lemma 3.2.10 ensures that  $\chi$  is continuous in time with respect to the  $L^2$ -norm and therefore assumes the initial data. The existence of square-integrable normal velocities is proven in Proposition 4.2.3, and the distributional equation for motion uses convergence of the convergence term Proposition 4.2.7. Moreover we still need the full convergence of the velocity term, and the bulk of the work has already been done in Proposition 4.2.11. We see that the only thing left to show is the full convergence of the velocity term, which states that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int \varepsilon \langle \partial_t u_\varepsilon, D u_\varepsilon \xi \rangle dx dt = \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle \xi, \nu_i \rangle V_i d\mathcal{H}^{d-1} dt.$$

The key problem we are facing is that the localization estimate Lemma 4.2.5 states that if we can choose the majority phase for a fixed time, then we expect smallness. However we see that the definition of the tilt-excess yields only a time-integrated error. Thus we want to take a partition of unity also in time and control the error.

To this end, let  $0 = T_0 < \dots < T_K = T$  be a partition of  $[0, T]$  and for a given  $\delta > 0$ , let  $(g_k)_{k=1, \dots, K}$  be a partition of unity with respect to the intervals  $((T_{k-1} - \delta, T_k + \delta))$ . As before let  $\eta_B$  be a partition of unity in space with respect to the covering  $\mathcal{B}_r$ . Then for any parameter  $\alpha \in (0, 1)$ , we compute that by Proposition 4.2.11, we have

$$\begin{aligned} A &:= \limsup_{\varepsilon \rightarrow 0} \left| \int_0^T \int \varepsilon \langle \partial_t u_\varepsilon, D u_\varepsilon \xi \rangle dx dt - \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle \xi, \nu_i \rangle V_i d\mathcal{H}^{d-1} dt \right| \\ &\leq \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \limsup_{\varepsilon \rightarrow 0} \left| \int_0^T \int \varepsilon g_k \eta_B \langle \partial_t u_\varepsilon, D u_\varepsilon \xi \rangle dx dt \right. \\ &\quad \left. - \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} g_k \eta_B \langle \xi, \nu_i \rangle V_i d\mathcal{H}^{d-1} dt \right| \\ &\lesssim \|\xi\|_{\sup} \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \min_{i \neq j} \min_{\nu^* \in \mathbb{S}^{d-1}} \frac{1}{\alpha} \mathcal{E}_{ij}(\nu^*; g_k \rho_B, u) + \alpha \mu(g_k \rho_B) \end{aligned}$$

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Since no derivative falls on  $g_k$ , this term converges as  $\delta$  tends to zero to

$$\begin{aligned} \|\xi\|_{\sup} \alpha \mu([0, T] \times \mathbb{T}) + \|\xi\|_{\sup} \frac{1}{\alpha} \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \min_{i \neq j} \min_{\nu^* \in \mathbb{S}^{d-1}} \int_{T_{k-1}}^{T_k} \int \rho_B |\nu_i - \nu^*|^2 |\nabla \chi_i| \\ + \int \rho_B |\nu_j + \nu^*|^2 |\nabla \chi_j| + \sum_{k \notin \{i, j\}} \int \rho_B |\nabla \chi_k| dt. \end{aligned} \quad (4.17)$$

Our problem is now that we would like to take the majority phase approximative inner normal dependent on time, which equates to pulling both minima inside the time integral. If our partition  $\chi$  is smooth, then by choosing partitions  $(T_k)_{k=1, \dots, K}$  whose width goes to zero, this would be possible. Since our partition will in general not be smooth, we instead choose a smooth approximation  $\chi^n$  which converges to  $\chi$  with respect to the strict metric as  $n$  tends to infinity. We estimate that for every  $1 \leq i \leq P$ , every unit vector  $\nu^* \in \mathbb{S}^{d-1}$ , fixed index  $k$  and ball  $B$ , we have that when replacing  $\chi$  with  $\chi^n$  in a summand of equation (4.17), we get an error of size at most

$$\begin{aligned} \left| \int_{T_{k-1}}^{T_k} \left| \int \rho_B |\nu_i - \nu^*|^2 |\nabla \chi_i| - \int \rho_B |\nu_i^n - \nu^*|^2 |\nabla \chi_i^n| \right| \right. \\ \left. + \left| \int \rho_B |\nu_j + \nu^*|^2 |\nabla \chi_j| - \int \rho_B |\nu_j^n + \nu^*|^2 |\nabla \chi_j^n| \right| + \sum_{l \notin \{i, j\}} \left| \int \rho_B |\nabla \chi_l| - \int \rho_B |\nabla \chi_l^n| \right| dt \right| \\ \lesssim \sum_{l=1}^P \int_{T_{k-1}}^{T_k} \left| \int \rho_B |\nabla \chi_l| - \int \rho_B |\nabla \chi_l^n| \right| + \left| \int \rho_B \nu_l |\nabla \chi_l| - \int \rho_B \nu_l^n |\nabla \chi_l^n| \right| dt. \end{aligned}$$

Here  $\nu_i^n$  is defined by  $\nabla \chi_i^n / |\nabla \chi_i^n|$ , and for the inequality we used that  $|\theta - \gamma|^2 = 2(1 - \langle \theta, \gamma \rangle)$  for unit vectors  $\theta$  and  $\gamma$ . Thus when replacing  $\chi$  by  $\chi^n$  in (4.17), we get an error of size at most

$$\frac{1}{\alpha} \sum_{B \in \mathcal{B}_r} \int_0^T \sum_{1 \leq l \leq P} \left| \int \rho_B |\nabla \chi_l| - \int \rho_B |\nabla \chi_l^n| \right| + \left| \int \rho_B \nu_l |\nabla \chi_l| - \int \rho_B \nu_l^n |\nabla \chi_l^n| \right| dt,$$

which can be made arbitrarily small independent of the partition of  $[0, T]$  we have chosen when we let  $n$  tend to infinity. Since  $\chi^n$  is smooth, we can now make the width of our partition arbitrarily small and therefore obtain that

$$\begin{aligned} A \lesssim \|\xi\|_{\sup} \alpha \mu([0, T] \times \mathbb{T}) \\ + \|\xi\|_{\sup} \frac{1}{\alpha} \int_0^T \sum_{B \in \mathcal{B}_r} \min_{i \neq j} \min_{\nu^* \in \mathbb{S}^{d-1}} \int \rho_B |\nu_i^n - \nu^*|^2 |\nabla \chi_i^n| + \int \rho_B |\nu_j^n + \nu^*|^2 |\nabla \chi_j^n| \\ + \sum_{k \notin \{i, j\}} \int \rho_B |\nabla \chi_k^n| dt. \end{aligned}$$

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Now letting  $n$  tend to infinity yields by the generalized dominated convergence theorem with majorant  $C \sum_i \int \eta_B |\nabla \chi_i^n|$  (whose time integral converges due to the strict convergence) that

$$\begin{aligned} A &\lesssim \|\xi\|_{\sup} \alpha \mu([0, T] \times \mathbb{T}) \\ &\quad + \|\xi\|_{\sup} \frac{1}{\alpha} \int_0^T \sum_{B \in \mathcal{B}_r} \min_{i \neq j} \min_{\nu^* \in \mathbb{S}^{d-1}} \int \rho_B |\nu_i - \nu^*|^2 |\nabla \chi_i| + \int \rho_B |\nu_j + \nu^*|^2 |\nabla \chi_j| \\ &\quad + \sum_{k \notin \{i, j\}} \int \rho_B |\nabla \chi_k| \, dt. \end{aligned}$$

Using the localization result Lemma 4.2.5 and the energy dissipation inequality (2.12) together with the dominated convergence theorem, the second summand vanishes as  $r$  tends to zero. Then sending  $\alpha$  to zero completes the proof.  $\square$



## 5. De Giorgi's mean curvature flow

In this chapter, we build on the results of the previous chapter and introduce a different solution concept for mean curvature flow, namely a De Giorgi type BV-solution to mean curvature. A similar solution concept has been introduced in [LL21, Def. 1], but in the context of convergence of the thresholding scheme to mean curvature flow. Moreover, we will also compare it to the solution concept [HL21, Def. 1], which instead of using functions of bounded variation also permits the more general varifolds to be solutions to mean curvature flow.

### 5.1. Conditional convergence to De Giorgi's mean curvature flow

In this section, we shall state our solution concept and prove convergence to the aforementioned.

#### 5.1.1. Definition of De Giorgi type BV-solutions to mean curvature flow

**Definition 5.1.1** (De Giorgi type BV-solution to multiphase mean curvature flow). *Fix some finite time horizon  $T < \infty$ , a  $P \times P$  matrix of surface tensions  $\sigma$  and initial data  $\chi^0: \mathbb{T} \rightarrow \{0, 1\}^P$  with  $E_0 := E(\chi^0)$  and  $\sum_{i=1}^P \chi_i^0 = 1$ . We say that*

$$\chi \in C([0, T]; L^2(\mathbb{T}; \{0, 1\}^P))$$

*with  $\text{ess sup}_{0 \leq t \leq T} E(\chi)$  and  $\sum_{i=1}^P \chi_i = \sum_{i=1}^P \mathbf{1}_{\Omega_i} = 1$  is a De Giorgi type BV-solution to multiphase mean curvature flow with initial data  $\chi^0$  and surface tensions  $\sigma$  if the following holds.*

1. *For all  $1 \leq i \leq P$ , there exist normal velocities  $V_i \in L^2(|\nabla \chi_i| dt)$  of the interfaces in the sense that*

$$\partial_t \chi_i = V_i |\nabla \chi_i| dt$$

*holds in the distributional sense on  $(0, T) \times \mathbb{T}$ .*

2. *There exist a mean curvature vector  $H \in L^2(E(u; \cdot) dt; \mathbb{R}^d)$  which satisfies*

$$\int_0^T E(\chi; \langle H, \xi \rangle) dt = - \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle D\xi, \text{Id} - \nu_i \otimes \nu_i \rangle d\mathcal{H}^{d-1} dt \quad (5.1)$$

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for all test vector fields  $\xi \in C_c^\infty((0, T) \times \mathbb{T}; \mathbb{R}^d)$ , where  $\nu_i := \nabla \chi_i / |\nabla \chi_i|$  are the inner unit normals and  $\Sigma_{i,j} := \partial_* \Omega_i \cap \partial_* \Omega_j$  is the  $(i, j)$ -th interface.

3. The partition  $\chi$  satisfies a De Giorgi type optimal energy dissipation inequality in the sense that for almost every time  $0 < T' < T$ , we have

$$E(\chi(T')) + \frac{1}{2} \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^{T'} \int_{\Sigma_{ij}} V_i^2 + H^2 d\mathcal{H}^{d-1} dt \leq E_0.$$

4. The initial data is achieved in the space  $C([0, T]; L^2(\mathbb{T}))$ .

**Theorem 5.1.2.** *Let a smooth multiwell potential  $W: \mathbb{R}^N \rightarrow [0, \infty)$  satisfy the assumptions (2.3)-(2.6). Let  $T < \infty$  be an arbitrary finite time horizon. Given a sequence of initial data  $u_\varepsilon^0: \mathbb{T} \rightarrow \mathbb{R}^N$  approximating a partition  $\chi^0 \in \text{BV}(\mathbb{T}; \{0, 1\}^P)$  in the sense that  $u_\varepsilon^0 \rightarrow u^0 = \sum_{1 \leq i \leq P} \chi_i^0 \alpha_i$  holds pointwise almost everywhere and*

$$E_0 := E(\chi^0) = \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon^0) < \infty,$$

*we have that for some subsequence of solutions to the Allen–Cahn equation (2.1)  $u_\varepsilon$  with initial datum  $u_\varepsilon^0$ , there exists a time-dependent partition  $\chi$  with  $\chi \in \text{BV}((0, T) \times \mathbb{T}; \{0, 1\}^P)$  and  $\chi \in C([0, T]; L^2(\mathbb{T}; \{0, 1\}^P))$  such that  $u_\varepsilon$  converges to  $u := \sum_{1 \leq i \leq P} \chi_i \alpha_i$  almost everywhere. Moreover  $u$  assumes the initial data  $u^0$  in  $C([0, T]; L^2(\mathbb{T}))$ . If we additionally assume that the time-integrated energies converge (4.1), then  $\chi$  is a BV-solution to De Giorgis mean curvature flow in the sense of Definition 5.1.1.*

This result is similar to Theorem 4.2.2 and we only need to prove that  $\chi$  is a BV-solution to De Giorgis mean curvature flow.

*Proof.* The idea of the proof is that we already have an optimal energy dissipation inequality for the Allen–Cahn equation given by (2.12). If we use that additionally use the Allen–Cahn equation once, we arrive at the reformulated version of the optimal energy dissipation inequality given by

$$E_\varepsilon(u_\varepsilon(T')) + \frac{1}{2} \int_0^{T'} \int \varepsilon |\partial_t u_\varepsilon|^2 + \frac{1}{\varepsilon} \left| \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon) \right|^2 dx dt \leq E_\varepsilon(u_\varepsilon^0).$$

Our hope is that as  $\varepsilon$  tends to zero, we can pass to the optimal energy dissipation inequality for  $\chi$  through lower semicontinuity. Since we assume the convergence of the initial energies and energy convergence for almost every time, the only terms we have to care about are the lower semicontinuity of the velocity term, which reads

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \int_0^{T'} \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \geq \frac{1}{2} \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^{T'} \int_{\Sigma_{ij}} V_i^2 d\mathcal{H}^{d-1} dt, \quad (5.2)$$

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and the lower semicontinuity of the curvature term, which reads

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \int_0^{T'} \int \frac{1}{\varepsilon} \left| \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon) \right|^2 dx dt \geq \frac{1}{2} \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^{T'} \int_{\Sigma_{ij}} H^2 d\mathcal{H}^{d-1} dt. \quad (5.3)$$

Moreover we have to show existence of the mean curvature function  $H$ . We could cheat in this step and simply use that by Theorem 4.2.2, we already know that the tangential convergence applied to  $\xi$  is given by the velocity, or in other words, that we already have  $V_i = H$  on  $\Sigma_{ij}$ . But we want to present a more direct approach.

Consider the linear functional

$$L(\xi) := - \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_0^T \int_{\Sigma_{ij}} \langle D\xi, \text{Id} - \nu_i \otimes \nu_i \rangle d\mathcal{H}^{d-1} dt$$

defined on test vector fields  $\xi$ . Then  $L$  is bounded with respect to  $L^2((0, T) \times \mathbb{T}; \mathbb{R}^d; E(\xi; \cdot))$  since by the convergence of the curvature term observed in Proposition 4.2.7, we have

$$\begin{aligned} |L(\xi)| &= \liminf_{\varepsilon \rightarrow 0} \left| - \int_0^T \int \left\langle \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon), D u_\varepsilon \xi \right\rangle dx dt \right| \\ &\leq \left( \int_0^T \int \frac{1}{\varepsilon} \left| \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon) \right|^2 dx dt \right)^{1/2} \left( \int_0^T \int \varepsilon |\nabla u_\varepsilon|^2 |\xi|^2 dx dt \right)^{1/2} \\ &\leq \left( \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \right)^{1/2} \left( \int_0^T \int \varepsilon |\nabla u_\varepsilon|^2 |\xi|^2 dx dt \right)^{1/2}. \end{aligned}$$

The first factor stays uniformly bounded due to the energy dissipation inequality (2.12), and by the equipartition of energies (Lemma 4.2.4), the second factor converges to the  $L^2$ -norm of  $\xi$  with respect to the energy measure, proving our claim. But therefore we can extend the functional to the square integrable functions with respect to the energy measure, and by Riesz representation theorem obtain the existence of the desired mean curvature vector  $H$ .

Let us moreover firstly consider the lower semicontinuity of the curvature term. Let again  $\xi$  be some test vector field. Then for all  $\varepsilon > 0$  and some fixed time, we have by Young's inequality and Cauchy–Schwarz that

$$\begin{aligned} &\liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \int \frac{1}{\varepsilon} \left| \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon) \right|^2 dx \\ &\geq \liminf_{\varepsilon \rightarrow 0} \int \left\langle \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \nabla W(u_\varepsilon), D u_\varepsilon \xi \right\rangle dx - \frac{1}{2} \int \varepsilon |\nabla u_\varepsilon|^2 |\xi|^2 dx \\ &= -E(\chi; \langle H, \xi \rangle) - \frac{1}{2} E(\chi; |\xi|^2). \end{aligned}$$

Since this inequality holds for any test vector field, we may take a sequence of test vector fields satisfying

$$\lim_{n \rightarrow \infty} \|\xi_n + H\|_{L^2(\mathbb{T}; E(\chi; \cdot))} = 0.$$

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This then yields the desired inequality (5.3) by applying Fatou's Lemma to pull the limes inferior into the time integral.

In principle the proof is now already done, since Theorem 4.2.2 already gives us that for almost every time  $t$ , we already have  $V_i \nu_i = -H$  on  $\Sigma_{i,j}$   $\mathcal{H}^{d-1}$ -almost everywhere. But since this makes heavy use of the previous arguments, we instead want to present another proof which directly proves the lower semicontinuity of the velocity term.

Let us first consider the scalar case  $N = 1, P = 2$ . By a similar duality argument as for the lower semicontinuity of the curvature term, we compute that for every test function  $\varphi$ , we have

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt \\ & \geq \liminf_{\varepsilon \rightarrow 0} \int_0^T \int \partial_t u_\varepsilon \phi'(u_\varepsilon) \varphi \, dx \, dt - \frac{1}{2} \int_0^T \int \frac{1}{\varepsilon} (\phi'(u_\varepsilon) \varphi)^2 \, dx \, dt \\ & \geq \liminf_{\varepsilon \rightarrow 0} \int_0^T \int \partial_t \psi \varphi \, dx \, dt - \frac{1}{2} \int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon) \varphi^2 \, dx \, dt \\ & = \sigma \int_0^T \int_\Sigma \varphi V \, d\mathcal{H}^{d-1} \, dt - \frac{1}{2} \sigma \int_0^T \int_\Sigma \varphi^2 \, d\mathcal{H}^{d-1} \, dt. \end{aligned}$$

Since again the inequality holds for any testfunction  $\varphi$ , we may plug in a sequence of testfunctions  $\varphi_n$  satisfying

$$\lim_{n \rightarrow \infty} \|\varphi_n - V\|_{L^2((0,T) \times \mathbb{T}; d\mathcal{H}^{d-1}|_\Sigma dt)} = 0$$

and thereby obtain the desired inequality (5.2).

For the multiphase case, we do not find an immediate generalization of this proof, but rather have to work with the usual localization argument to obtain a reduction to the scalar case.

As in the proof of Theorem 4.2.2, let  $\delta > 0$  and  $0 = T_0 < T_1 < \dots < T_K = T'$  be a partition of  $[0, T']$ . Let  $(g_k)_{k=1, \dots, K}$  be a partition of unity with respect to the intervals  $(T_0 - \delta, T_1 + \delta), \dots, (T_{K-1} - \delta, T_K + \delta)$  and let  $r > 0$ , where  $\eta_B$  and  $\rho_B$  are

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as in Lemma 4.2.5. Then we estimate

$$\begin{aligned}
A &:= \liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 dx dt \\
&= \liminf_{\varepsilon \rightarrow 0} \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \frac{1}{2} \int_0^T \int g_k \eta_B \varepsilon |\partial_t u_\varepsilon|^2 dx dt \\
&\geq \liminf_{\varepsilon \rightarrow 0} \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \max_{1 \leq i \leq P} \sup_{\substack{\varphi \in C_c^\infty((0,T) \times \mathbb{T}) \\ |\varphi| \leq R}} \int_0^T \int g_k \eta_B \langle \nabla \phi_i(u_\varepsilon), \partial_t u_\varepsilon \rangle \varphi dx dt \\
&\quad - \frac{1}{2} \int_0^T \int g_k \eta_B \frac{1}{\varepsilon} |\nabla \phi_i(u_\varepsilon)|^2 \varphi^2 dx dt \\
&\geq \liminf_{\varepsilon \rightarrow 0} \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \max_{1 \leq i \leq P} \sup_{\substack{\varphi \in C_c^\infty((0,T) \times \mathbb{T}) \\ |\varphi| \leq R}} \int_0^T \int g_k \eta_B \partial_t \psi_{\varepsilon,i} \varphi dx dt \\
&\quad - \frac{1}{2} \int_0^T \int g_k \eta_B \frac{1}{\varepsilon} 2W(u_\varepsilon) \varphi^2 dx dt
\end{aligned}$$

Pulling the limes inferior inside the double sum and the suprema, we obtain that this term can be estimated from below via

$$\begin{aligned}
&\sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \max_{1 \leq i \leq P} \sup_{\substack{\varphi \in C_c^\infty((0,T) \times \mathbb{T}) \\ |\varphi| \leq R}} \int_0^T \int g_k \eta_B \partial_t \psi_i \varphi dx dt - \frac{1}{2} \int_0^T E(\chi; g_k \eta_B \varphi^2) dt \\
&= \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \max_{1 \leq i \leq P} \sup_{\substack{\varphi \in C_c^\infty((0,T) \times \mathbb{T}) \\ |\varphi| \leq R}} \sum_{j=1}^P \left( \sigma_{ij} \int_0^T \int g_k \eta_B \varphi V_j |\nabla \chi_j| dt - \frac{1}{2} \sigma_{ij} \int_0^T \int g_k \eta_B \varphi^2 |\nabla \chi_j| dt \right) \\
&\quad - \frac{1}{2} \left( \int_0^T E(\chi; g_k \eta_B \varphi^2) - \int g_k \eta_B \varphi^2 |\nabla \psi_i| dt \right).
\end{aligned}$$

Since no derivative has fallen on  $g_k$ , we may send  $\delta$  to zero and obtain via the dominated

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convergence theorem that

$$\begin{aligned}
A &\geq \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \max_{1 \leq i \leq P} \sup_{\substack{\varphi \in C_c^\infty((0,T) \times \mathbb{T}) \\ |\varphi| \leq R}} \sum_{j=1}^P \left( \sigma_{ij} \int_{T_{k-1}}^{T_k} \int \eta_B \varphi V_j |\nabla \chi_j| \, dt - \frac{1}{2} \sigma_{ij} \int_{T_{k-1}}^{T_k} \int \eta_B \varphi^2 |\nabla \chi_j| \, dt \right) \\
&\quad - \frac{1}{2} \left( \int_{T_{k-1}}^{T_k} \mathbb{E}(\chi; \eta_B \varphi^2) - \int \eta_B \varphi^2 |\nabla \psi_i| \, dt \right) \\
&\geq \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \max_{i \neq j} \sup_{\substack{\varphi \in C_c^\infty((0,T) \times \mathbb{T}) \\ |\varphi| \leq R}} \sigma_{ij} \int_{T_{k-1}}^{T_k} \int \eta_B \varphi \left( V_j - \frac{1}{2} \varphi \right) |\nabla \chi_j| \, dt - C \sum_{l \notin \{i,j\}} \int_{T_{k-1}}^{T_k} \int \eta_B (R|V_l| + R^2) \\
&\quad - \frac{R^2}{2} \int_{T_{k-1}}^{T_k} \mathbb{E}(\chi; \eta_B) - \int \eta_B |\nabla \psi_i| \, dt
\end{aligned}$$

By Remark 4.2.6, we have

$$\frac{R^2}{2} \int_{T_{k-1}}^{T_k} \mathbb{E}(\chi; \eta_B) - \int \eta_B |\nabla \psi_i| \, dt \lesssim R^2 \sum_{l \notin \{i,j\}} \int_{T_{k-1}}^{T_k} \int \eta_B |\nabla \chi_l| \, dt,$$

and by applying Young's inequality, we get that for every parameter  $\alpha \in (0, 1)$ , we have

$$\begin{aligned}
&\sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \min_{i \neq j} \sum_{l \notin \{i,j\}} \int_{T_{k-1}}^{T_k} \int \eta_B R V_l |\nabla \chi_l| \, dt \\
&\lesssim \alpha \sum_{1 \leq l \leq P} \int_{T_{k-1}}^{T_k} \int V_l^2 |\nabla \chi_l| \, dt + \frac{R^2}{\alpha} \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \min_{l \notin \{i,j\}} \int_{T_{k-1}}^{T_k} \int \eta_B |\nabla \chi_l|.
\end{aligned}$$

Moreover, we can choose for fixed  $k, B$  and tuple  $(i, j)$  a sequence of testfunctions  $\varphi_n$  with  $|\varphi_n| \leq R$  which approach  $V_j \mathbb{1}_{|V_j| \leq R}$  in the sense that

$$\lim_{n \rightarrow \infty} \left\| \varphi_n - V_j \mathbb{1}_{|V_j| \leq R} \right\|_{L^2((T_{k-1}, T_k) \times \mathbb{T}; |\nabla \chi_j| \, dt)} = 0.$$

Combining these three arguments, we arrive at the estimate that for every parameter  $\alpha \in (0, 1)$ , it holds that

$$\begin{aligned}
A &\geq -\alpha C \sum_{1 \leq l \leq P} \int_0^T \int V_l^2 |\nabla \chi_l| \, dt \\
&\quad + \sum_{k=1}^K \sum_{B \in \mathcal{B}_r} \max_{i \neq j} \frac{\sigma_{ij}}{2} \int_{T_{k-1}}^{T_k} \int \eta_B |V_j \mathbb{1}_{|V_j| \leq R}|^2 |\nabla \chi_j| \, dt - C \frac{R^2}{\alpha} \sum_{l \notin \{i,j\}} \int_{T_{k-1}}^{T_k} \int \eta_B |\nabla \chi_l| \, dt.
\end{aligned}$$

With similar arguments as in the proof of Theorem 4.2.2, we can argue that by choosing partitions whose width tends to zero, we can pull the maximum inside the time integral

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to obtain

$$\begin{aligned}
A &\geq -\alpha C \sum_{1 \leq l \leq P} \int_0^T \int V_l^2 |\nabla \chi_l| \, dt \\
&\quad + \int_0^T \sum_{B \in \mathcal{B}_r} \max_{i \neq j} \frac{\sigma_{ij}}{2} \int \eta_B |V_j \mathbb{1}_{|V_j| \leq R}|^2 |\nabla \chi_j| - C \frac{R^2}{\alpha} \sum_{l \notin \{i,j\}} \int \eta_B |\nabla \chi_l| \, dt \\
&\geq \frac{1}{2} \sum_{1 \leq i < j \leq P} \int_0^T \int_{\Sigma_{ij}} V_i^2 \mathbb{1}_{|V_i| \leq R} \, d\mathcal{H}^{d-1} \, dt \\
&\quad - C \left( \alpha \sum_{1 \leq l \leq P} \int_0^T \int V_l^2 |\nabla \chi_l| \, dt + \frac{R^2}{\alpha} \int_0^T \sum_{B \in \mathcal{B}_r} \min_{i \neq j} \sum_{k \notin \{i,j\}} \int \eta_B |\nabla \chi_k| \, dt \right).
\end{aligned}$$

With the same argumentation as in the proof of Theorem 4.2.2, by first sending  $r \rightarrow 0$  and then  $\alpha \rightarrow 0$ , we thus obtain that for all  $R > 0$ , we have

$$A = \liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt \geq \frac{1}{2} \sum_{1 \leq i < j \leq P} \int_0^T \int_{\Sigma_{i,j}} V_i^2 \mathbb{1}_{|V_i| \leq R} \, d\mathcal{H}^{d-1} \, dt,$$

and thus the lower semicontinuity of the velocity term now follows from the monotone convergence theorem.  $\square$

#### 5.1.2. De Giorgi type varifold solutions for mean curvature flow

Up until this point, we have always made the crucial assumption of energy convergence for our proofs (HIER VLT. EINEN TEIL DER THESIS VERLINKEN, WO ICH MehrR AUF DIESE ANNAHME EINGEHE; AM BESTEN BEVOR ICH SIE DAS ERSTE MAL BENUTZE). However this is usually a very strong assumption and we should not be able to expect it. One thing which could go wrong is for example illustrated in Figure 5.1. There we see that two rough interfaces of the first and second phase collapse as  $\varepsilon$  tend to zero, which results in a loss of energy, since the measure theoretic boundary of a set of bounded variation does not see such lines.

This leads us to the solution concept introduced by Hensel and Laux in [HL21]. For the twophase case, the definition is as follows.

**Definition 5.1.3** (De Giorgi type varifold solution for twophase mean curvature flow). *Let  $T < \infty$  be an arbitrary finite time horizon and let  $\mu = \mathcal{L}^1 \otimes (\mu_t)_{t \in (0,T)}$  be a family of oriented varifolds  $\mu_t \in \mathcal{M}(\mathbb{T} \times \mathbb{S}^{d-1})$ ,  $t \in (0, T)$ , such that the map  $t \mapsto \int_{\mathbb{T} \times \mathbb{S}^{d-1}} \eta(x, p, t) d\mu_t(x, p)$  is measurable for all  $\eta \in L^1((0, T); C(\mathbb{T}, \mathbb{S}^{d-1}))$ . Consider also a family  $A = (A_t)_{t \in (0,T)}$  of subsets of  $\mathbb{T}$  with finite perimeter such that the associated indicator function  $\chi(x, t) = \chi_{A_t}(x)$  satisfies  $\chi \in L^\infty((0, T); \text{BV}(\mathbb{T}; \{0, 1\}))$ . Let  $\sigma > 0$  be a surface tension constant.*

*Given an initial energy  $\omega^0 \in \mathcal{M}(\mathbb{T})$  and an initial phase indicator function  $\chi^0 \in \text{BV}(\mathbb{T}; \{0, 1\})$ , we call the pair  $(\mu, \chi)$  a De Giorgi type varifold solution for twophase mean curvature flow with initial data  $(\omega^0, \chi^0)$  if the following holds.*

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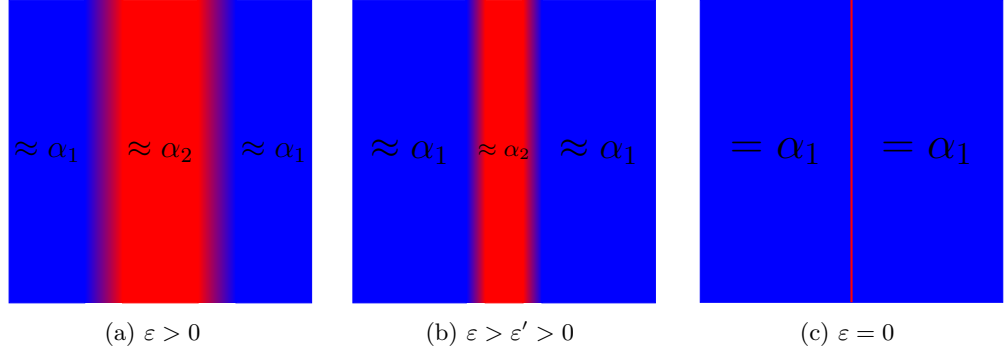


Figure 5.1.: Profile of the solution  $u_\varepsilon$  as  $\varepsilon$  tends to zero.

1. *(Existence of a normal speed)* Writing  $\mu_t = \omega_t \otimes (\lambda_{x,t})_{x \in \mathbb{T}}$  for the disintegration of  $\mu_t$ , we require the existence of some  $V \in L^2((0, T); L^2(\mathbb{T}; \omega_t))$  encoding a normal speed in the sense of

$$\sigma \int \chi(T', x) \varphi(T', x) - \chi^0(x) \varphi(0, x) dx = \sigma \int_0^{T'} \int \chi \partial_t \varphi dx dt + \int_0^{T'} \int V \varphi d\omega_t dt \quad (5.4)$$

for almost every  $T' \in (0, T)$  and all  $\varphi \in C_c^\infty([0, T] \times \mathbb{T})$ .

2. *(Existence of a generalized mean curvature vector)* We require the existence of some  $H \in L^2((0, T); L^2(\mathbb{T}, \omega_t; \mathbb{R}^d))$  encoding a generalized mean curvature vector by

$$\int_0^T \int \langle H, \xi \rangle d\omega_t dt = - \int_0^T \int_{\mathbb{T} \times \mathbb{S}^{d-1}} \langle \xi, \text{Id} - p \otimes p \rangle d\mu_t dt \quad (5.5)$$

for all  $\xi \in C_c^\infty([0, T] \times \mathbb{T}; \mathbb{R}^d)$ .

3. *(De Giorgi type optimal energy dissipation inequality)* A sharp energy dissipation inequality holds in form of

$$\omega_{T'}(\mathbb{T}) + \frac{1}{2} \int_0^{T'} \int V^2 + |H|^2 d\omega_t dt \leq \omega^0(\mathbb{T}) \quad (5.6)$$

for almost every  $T' \in (0, T)$ .

4. *(Compatibility)* For almost every  $t \in (0, T)$  and all  $\xi \in C^\infty(\mathbb{T}; \mathbb{R}^d)$ , it holds that

$$\sigma \int \langle \xi, \nabla \chi(t, \cdot) \rangle = \int_{\mathbb{T} \times \mathbb{S}^{d-1}} \langle \xi, p \rangle d\mu_t. \quad (5.7)$$

We firstly want to discuss this definition. In the setting of Definition 5.1.1 and in the twophase case, we can think of the oriented varifold  $\mu$  as the measure

$$\mu = \sigma \mathcal{L}^1|_{(0, T)} \otimes |\nabla \chi(t)| \otimes \delta_{\nu(t, x)}, \quad (5.8)$$



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and therefore the measure  $\omega_t$  becomes the energy measure  $E(\chi(t), \cdot)$ . The advantage of this formulation is that our new energy measure  $\omega_t$  is not restricted to only seeing the measure theoretic boundary of  $\chi$ , but can actually capture phenomena as described in Figure 5.1. For example in such a scenario we would expect the measure  $\mu_t$  to be defined by

$$\mu_t := 2 \mathcal{H}^1 \llcorner_l \otimes \frac{1}{2} (\delta_{(1,0)} + \delta_{(-1,0)}), \quad (5.9)$$

where  $l$  is the red line to which the phase of  $\alpha_2$  shrank down as  $\varepsilon$  approached zero. The factor 2 comes from the fact that we obtain energy from both the collapsing interfaces.

The equation (5.4) for the normal speed is simply motivated through the fundamental theorem of calculus. Assuming that everything is nice and smooth, we can compute that

$$\begin{aligned} \sigma \int \chi(T, x) \varphi(T, x) - \chi^0(x) \varphi(0, x) dx &= \sigma \int_0^T \int \partial_t (\chi(t, x) \varphi(t, x)) dx dt \\ &= \sigma \int_0^T \int \partial_t \chi(t, x) \varphi(t, x) \chi(t, x) \partial_t \varphi(t, x) dx dt \\ &= \sigma \int_0^T \int \chi(t, x) \partial_t (t, x) dx dt + \int_0^T \int V \varphi d\omega_t dt, \end{aligned}$$

where for the last equality, we used  $\partial_t \chi = V |\nabla \chi| dt$  and  $\omega_t = \sigma |\nabla \chi(t)|$ .

The equation (5.5) for the generalized mean curvature is also quite intuitive since assuming that the simple case of the varifold (5.8) holds, we have that the right hand side of equation (5.5) reads

$$\int_{\mathbb{T} \times \mathbb{S}^{d-1}} \langle \xi, \text{Id} - p \otimes p \rangle d\omega_t = \sigma \int \langle \xi, \text{Id} - \nu \otimes \nu \rangle |\nabla \chi|,$$

which is the standard distributional formulation for the mean curvature vector.

De Giorgi inequality (5.6) is self-explanatory since  $\omega_t$  is the energy measure. The compatibility condition is necessary to couple the evolving set  $A_t$  to the varifold. Notice that in our above example (5.9), we have that even though the energy measure  $\omega_t$  sees the red strip and the measure theoretic boundary of the corresponding indicator function does not, the term on the right hand side of (5.7) is zero since

$$\int_{\mathbb{T} \times \mathbb{S}^{d-1}} \langle \xi, p \rangle d\mu_t = \int_l \frac{1}{2} \langle \xi, (1, 0) + (-1, 0) \rangle d\mathcal{H}^1 = 0,$$

and thus we have no problem.

For the multiphase case, we propose the following solution concept, which generalized [HL21, Def. 2] to the case of arbitrary surface tensions and removes the varifold  $\mu_{ii}$ .

**Definition 5.1.4** (De Giorgi type varifold solution for multiphase mean curvature flow). *Let  $T < \infty$  be an arbitrary finite time horizon and let  $P \in \mathbb{N}_{\geq 2}$  be the number of phases. For each pair of phases  $(i, j) \in \{1, \dots, P\}$ ,  $i \neq j$ , let  $\mu_{ij} = \mathcal{L}^1 \llcorner_{(0, T)}$  be a family of oriented varifolds  $\mu_{t, ij} \in \mathcal{M}(\mathbb{T} \times \mathbb{S}^{d-1})$ ,  $t \in (0, T)$ , such that the map*

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$t \mapsto \int_{\mathbb{T} \times \mathbb{S}^{d-1}} \eta(t, x, p) d\mu_{t,ij}(x, p)$  is measurable for all test functions  $\eta \in L^1((0, T); C(\mathbb{T} \times \mathbb{S}^{d-1}))$ . Define evolving oriented varifolds  $\mu_i = \mathcal{L}^1|_{(0,T)} \otimes (\mu_{t,i})_{t \in (0,T)}$ ,  $i \in \{1, \dots, P\}$  and  $\mu = \mathcal{L}^1|_{(0,T)} \otimes (\mu_t)_{t \in (0,T)}$  by means of

$$\mu_{t,i} := \sum_{j=1, j \neq i}^P \mu_{t,ij} \quad \text{and} \quad \mu_t := \frac{1}{2} \sum_{i=1}^P \mu_{t,i}.$$

The disintegration of  $\mu_{t,ij}$  is expressed in form of  $\mu_{t,ij} = \omega_{t,ij} \otimes (\lambda_{t,x,ij})_{x \in \mathbb{T}}$  with expected value  $\langle \lambda_{t,x,ij} \rangle := \int_{\mathbb{S}^{d-1}} p d\lambda_{t,x,ij}(p)$ . Analogous expressions are introduced for the disintegrations of  $\mu_{t,i}$  and  $\mu_t$ .

Furthermore consider a family  $A = (A_1, \dots, A_P)$  such that for each phase  $1 \leq i \leq P$ , we have a family  $A_i = (A_i(t))_{t \in (0,T)}$  of subsets of  $\mathbb{T}$  with finite perimeter. We also require  $(A_1(t), \dots, A_P(t))$  to be a partition of  $\mathbb{T}$  for all  $t \in (0, T)$  and that for each  $1 \leq i \leq P$ , the associated indicator function satisfies  $\chi_i \in L^\infty((0, T); BV(\mathbb{T}; \{0, 1\}))$ . We shortly write  $\chi = (\chi_1, \dots, \chi_P)$ .

Given initial data  $((\omega_{ij}^0)_{1 \leq i, j \leq P, i \neq j}, (\chi_i^0)_{1 \leq i \leq P})$  of the above form and a  $P \times P$  matrix of surface tensions  $\sigma$  such that  $\sigma_{ij} > 0$  for  $i \neq j$ , we call the pair  $(\mu, \chi)$  a De Giorgi type varifold solution for multiphase mean curvature flow with initial data  $(\omega^0, \chi^0)$  and surface tensions  $\sigma$  if the following requirements hold true.

1. (Existence of normal speeds) For each phase  $1 \leq i \leq P$ , there exists a normal speed  $V_i \in L^2((0, T) \times \mathbb{T}, \omega_i)$  in the sense that

$$\int \chi_i(T', x) \varphi(T', x) - \chi_i^0(x) \varphi(0, x) dx = \int_0^{T'} \int \chi_i \partial_t \varphi dx dt + \sum_{j=1, j \neq i}^P \frac{1}{\sigma_{ij}} \int_0^{T'} \int V_i \varphi d\omega_{t,ij} dt \quad (5.10)$$

for almost every  $T' \in (0, T)$  and all  $\varphi \in C_c^\infty([0, T) \times \mathbb{T})$ .

2. (Existence of a generalized mean curvature vector) There exists a generalized mean curvature vector  $H \in L^2((0, T) \times \mathbb{T}, \omega; \mathbb{R}^d)$  in the sense that

$$\int_0^{T'} \int \langle H, \xi \rangle d\omega_t dt = - \int_0^{T'} \int_{\mathbb{T} \times \mathbb{S}^{d-1}} \langle D\xi, \text{Id} - p \otimes p \rangle d\mu_t dt \quad (5.11)$$

holds for almost every  $T' \in (0, T)$  and all  $\xi \in C_c^\infty([0, T) \times \mathbb{T}; \mathbb{R}^d)$ .

3. (De Giorgi type optimal energy dissipation inequality) A sharp energy dissipation inequality holds in form of

$$\omega_{T'}(\mathbb{T}) + \frac{1}{2} \sum_{i=1}^P \int_0^{T'} \int V_i^2 \frac{1}{2} d\omega_{t,i} dt + \frac{1}{2} \int_0^{T'} \int |H|^2 d\omega_t dt \leq \omega^0(\mathbb{T}) \quad (5.12)$$

for almost every  $T' \in (0, T)$ .

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4. (Compatibility conditions) For all  $1 \leq i, j \leq P$  with  $i \neq j$ , we require

$$\omega_{t,ij} = \omega_{t,ji} \quad (5.13)$$

for almost every  $t \in (0, T)$  and

$$\langle \lambda_{t,x,ij} \rangle = -\langle \lambda_{t,x,ji} \rangle, \quad (5.14)$$

$$V_i(t, x) = -V_j(t, x), \quad (5.15)$$

$$|\langle \lambda_{t,x,ij} \rangle|^2 H(t, x) = \langle H(t, x), \langle \lambda_{t,x,ij} \rangle \rangle \langle \lambda_{t,x}^{ij} \rangle \quad (5.16)$$

for almost every  $t \in (0, T)$  and  $\omega_{t,ij}$  almost every  $x \in \mathbb{T}$ . Finally for all  $1 \leq i \leq P$ , we have

$$\int \langle \xi, d\nabla \chi_i(t, \cdot) \rangle = \sum_{j=1, j \neq i}^P \frac{1}{\sigma_{ij}} \int_{\mathbb{T} \times \mathbb{S}^{d-1}} \langle \xi, p \rangle d\mu_{t,ij} \quad (5.17)$$

for almost every  $t \in (0, T)$  and every  $\xi \in C^\infty(\mathbb{T}; \mathbb{R}^d)$ .

As before, we first want to motivate this definition. In the simplest case, we can think of the varifold  $\mu_{t,ij}$  for  $i \neq j$  as

$$\mu_{t,ij} = \sigma_{ij} d\mathcal{H}^{d-1}|_{\Sigma_{ij}} \otimes (\delta_{\nu_i})_{x \in \mathbb{T}}. \quad (5.18)$$

But if approximate interfaces collapse as in Figure 5.1, then this is exactly captured by the varifolds  $\mu_{t,ij}$  and  $\mu_{t,ji}$ , which, as in the twophase case, will be described by

$$\mu_{t,ij} = \mathcal{H}^1|_l \otimes (\delta_{(-1,0)})_{x \in \mathbb{T}} \quad \text{and} \quad \mu_{t,ji} = \mathcal{H}^1|_l \otimes (\delta_{(1,0)})_{x \in \mathbb{T}}, \quad (5.19)$$

where we assume that  $(i, j) = (1, 2)$ .

Equations (5.10) and (5.11) are as in the twophase case. For the energy dissipation inequality, we note that we need the factor  $1/2$  in front of  $\omega_{t,i}$  since each interface will be counted twice.

The first compatibility condition (5.13) follows simply from  $\Sigma_{ij} = \Sigma_{ji}$  if we are in the setting of energy convergence, but should even hold true in situations like Figure 5.1, which follows from equation (5.19). The second compatibility condition (5.14) states that in the setting of energy convergence, we have  $\nu_i = -\nu_j$  on  $\Sigma_{ij}$   $\mathcal{H}^{d-1}$ -almost everywhere. Nonetheless, we see again that by equation (5.19), it should still hold true if interfaces collapse. Similarly we explain the anti-symmetry of the velocities (5.15). The compatibility condition (5.16) encodes that the mean curvature vector should always point in direction of the inner unit normal, which is not necessarily the case for varifolds, as demonstrated in Example 5.1.6. For the last compatibility condition (5.17), we again notice that it behaves similarly to its twophase equivalent.

**Theorem 5.1.5.** *Every De Giorgi type BV-solution to multiphase mean curvature flow in the sense of Definition 5.1.1 is also a De Giorgi type varifold solutions for multiphase mean curvature flow in the sense of Definition 5.1.4 with varifold  $\mu$  given by*

$$\mu_{t,ij} := \sigma_{ij} \mathcal{H}^{d-1}|_{\Sigma_{ij}(t)} \otimes (\delta_{\nu_i(t,x)})_{x \in \mathbb{T}}$$

for  $i \neq j$ .

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*Proof.* We first note that

$$\mu_{t,i} = \sum_{j \neq i} \sigma_{ij} \mathcal{H}_{\Sigma_{ij}}^{d-1} \otimes (\delta_{\nu_i})_{x \in \mathbb{T}}$$

and

$$\mu_t = \sum_{1 \leq i < j \leq P} \sigma_{ij} \mathcal{H}^{d-1}|_{\Sigma_{ij}} \otimes (\delta_{\nu_i})_{x \in \mathbb{T}}.$$

Therefore the measure  $\omega_t$  is exactly the energy measure  $E(\chi(t), \cdot)$ . The requirement  $\chi_i \in L^\infty((0, T); BV(\mathbb{T}; \{0, 1\}))$  is an immediate consequence of the energy dissipation inequality. The existence of normal velocities is given, but we need to check that equation (5.10) holds. If we assume that  $\varphi$  is compactly supported, then the equation follows by approximating  $\chi_i$  with  $\rho_n * \chi_i$ , where  $\rho_n$  is a sequence of radial symmetric standard mollifiers. In general, take a sequence  $\eta_n$  of smooth functions with compact support in  $(0, T]$  which converge pointwise to the constant 1 function satisfying  $\eta(T) = 1$  and  $\eta_n$  is non-decreasing. Then we have

$$\begin{aligned} \int \chi_i(T', x) \varphi(T', x) dx &= \lim_{n \rightarrow \infty} \int \chi_i(T', x) \eta_n(T') \varphi(T', x) dx \\ &= \int_0^{T'} \int \eta_n \varphi V_i |\nabla \chi^i| dt + \int_0^{T'} \int \chi_i \eta_n \partial_t \varphi dx dt + \int_0^{T'} \int \chi_i \partial_t \eta_n \varphi dx dt \end{aligned}$$

for almost every  $0 < T' < T$ . Using the dominated convergence theorem, we see that the first two summands converge to the left hand side of the velocity equation (5.10). For the third summand, we compute that since  $\int_0^T \eta' dt = 1$ , we have

$$\begin{aligned} \left| \int_0^{T'} \partial_t \eta_n \int \chi_i \varphi dx - \int \chi_i^0(x) \varphi(0, x) dx \right| &= \left| \int_0^{T'} \partial_t \eta_n \int \chi_i \varphi - \chi_i^0 \varphi(0, x) dx dt \right| \\ &\lesssim \int_0^{T'} \partial_t \eta_n \left( \int |\chi^i \varphi - \chi_i^0 \varphi(0, x)|^2 dx \right)^{1/2} dt, \end{aligned}$$

which converges to zero since  $\chi_i$  assumes the initial data continuously with respect to the  $L^2$ -norm. Therefore equation (5.10) follows.

Equation (5.11) follows immediately from equation (5.1) since

$$\int_{\mathbb{T} \times \mathbb{S}^{d-1}} \langle D\xi, \text{Id} - p \otimes p \rangle d\mu_t = \sum_{1 \leq i < j \leq P} \sigma_{ij} \int_{\Sigma_{ij}} \langle D\xi, \text{Id} - \nu_i \otimes \nu_i \rangle d\mathcal{H}^{d-1},$$

and De Giorgi's optimal energy dissipation inequality is also identical.

The compatibility conditions (5.13)-(5.15) all hold restricted to  $\Sigma_{ij} \mathcal{H}^{d-1}$  almost everywhere and are therefore satisfied. For the compatibility condition (5.16), we have to prove, since  $\langle \lambda_{t,x,ij} \rangle = \nu_i(t, x)$  on  $\Sigma_{ij} \mathcal{H}^{d-1}$ -almost everywhere, that the mean curvature vector  $H$  points in the direction of  $\nu_i \mathcal{H}^{d-1}$ -almost everywhere. This has been proven by Brakke in [Bra78, Thm. 5.8]. Note that we apply it (for a fixed time  $t$ )

### 5.1. Conditional convergence to De Giorgi's mean curvature flow

to the varifold  $V = \mu_t$ , which is strictly speaking no integer varifold, but all results still hold since

$$V = \sum_{1 \leq i < j \leq P} \sigma_{ij} v(\Sigma_{ij})$$

is a finite sum. Here  $v(\Sigma_{ij})$  is the naturally associated varifold to  $\Sigma_{ij}$  as defined by Brakke. The last compatibility condition (5.17) is a consequence of Item 2. This completes the proof.  $\square$

**Example 5.1.6.** *We want to present an example of a varifold where the mean curvature does not point in normal direction. Let  $\mathbb{T} = [0, \Lambda)^2$  be the flat torus in two dimensions and let  $\rho: \mathbb{R} \rightarrow (-\infty, \infty)$  be some positive smooth  $\Lambda$ -periodic function which is not constant. Then we consider the varifold given by  $\mu = \rho(x_1) \mathcal{H}^1|_{[0, \Lambda) \times \{0\}} \otimes \delta_{(0,1)}$ . We compute that for a given test vector field  $\xi$ , we have*

$$\begin{aligned} \int \langle D\xi, \text{Id} - p \otimes p \rangle d\mu &= \int_0^\Lambda \langle D\xi(x_1, 0), (1, 0) \otimes (1, 0) \rangle \rho(x_1) dx_1 \\ &= \int_0^\Lambda \partial_{x_1} \xi^1 \rho(x_1) dx_1 \\ &= - \int_0^\Lambda \xi^1 \rho'(x_1) dx_1, \end{aligned}$$

which implies that  $H$  is given (almost everywhere) by

$$H(x) = \frac{\rho'(x_1)}{\rho(x_1)} (1, 0),$$

which does not lie in the normal space of the varifold.



# A. Appendix

**Lemma A.0.1.** *The following statments hold for Borel-measurable sets.*

1. *If  $A, B, C \subset \mathbb{T}$  are mutually disjoint, then*

$$\mathcal{H}^{d-1}(\partial_*(A \cup B) \triangle (\partial_* A \triangle \partial_* B)) = 0 \quad \text{and} \quad \mathcal{H}^{d-1}(\partial_* A \cap \partial_* B \cap \partial_* C) = 0.$$

2. *If  $(\Omega_i)_{i=1, \dots, P}$  is a partition of  $\mathbb{T}$ , then*

$$\mathcal{H}^{d-1} \left( \partial_* \Omega_i \triangle \bigcup_{j \neq i} \partial_* \Omega_i \cap \partial_* \Omega_j \right) = 0.$$

*Remark A.0.2.* In a maybe less confusing way, the first and third result yield that up to sets of  $\mathcal{H}^{d-1}$ -measure zero, we have  $\partial_*(A \cup B) = \partial_* A \triangle \partial_* B$  and  $\partial_* \Omega_i = \bigcup_{j \neq i} \partial_* \Omega_i \cap \partial_* \Omega_j$  under the given assumptions.

*Proof.* Let us first prove Item 1. Initially assume that  $x \in \partial_*(A \cup B)$ . By definition this implies that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap (A \cup B))}{r^d} > 0.$$

Then we must either have

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{r^d} > 0 \quad \text{or} \quad \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap B)}{r^d} > 0,$$

so let us without loss of generality assume the first inequality. Since

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A)}{r^d} \geq \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap (A \cup B))}{r^d} > 0,$$

we thus have  $x \in \partial_* A$ . Combining Thm. 5.14 and Lemma 5.5 in [EG15], we can rewrite for some measurable set  $N$  with  $\mathcal{H}^{d-1}(N) = 0$  the measure theoretic boundary of  $A$  as

$$\partial_* A = \partial^{1/2} A \cup N := \left\{ x : \lim_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{\alpha(d)r^d} = \frac{1}{2} = \lim_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A)}{\alpha(d)r^d} \right\} \cup N, \quad (\text{A.1})$$

where  $\alpha(d) := \mathcal{L}^d(B_1(0))$  is the volume of the unit ball in  $d$  dimensions. Thus assume  $x \in \partial^{1/2} A \cap \partial^{1/2} B$ . Then since  $A$  and  $B$  are disjoint, we would have

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus (A \cup B))}{\alpha(d)r^d} \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A) - \mathcal{L}^d(B_r(x) \cap B)}{\alpha(d)r^d} = 0,$$

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which contradicts  $x \in \partial_*(A \cup B)$ . Thus up to a set of  $(d-1)$ -dimensional Hausdorff-measure zero, we have  $\partial_*(A \cup B) \subseteq \partial_*A \Delta \partial_*B$ .

For the other inclusion, suppose that  $x \in \partial_*A \Delta \partial_*B$  and without loss of generality that  $x \in \partial_*A \setminus \partial_*B$ . Then it must either hold that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap B)}{r^d} = 0 \quad \text{or} \quad \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus B)}{r^d} = 0.$$

But since  $x \in \partial_*A$ , the latter can not be true since  $A$  and  $B$  are disjoint and therefore

$$0 < \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{r^d} = \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus B)}{r^d},$$

thus the former has to be true. We now estimate

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap (A \cup B))}{r^d} \geq \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap A)}{r^d} > 0$$

and

$$\begin{aligned} \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus (A \cup B))}{r^d} &= \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A) - \mathcal{L}^d(B_r(x) \cap B)}{r^d} \\ &= \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus A)}{r^d} > 0, \end{aligned}$$

which proves  $x \in \partial_*(A \cup B)$ .

The second equality is an immediate consequence of observation (A.1).

Now let us prove Item 2. It follows immediately that

$$\partial_*\Omega_i \supseteq \bigcup_{j \neq i} \partial_*\Omega_i \cap \partial_*\Omega_j.$$

For the other inclusion, suppose  $x \in \partial_*\Omega_i$ . Since

$$0 < \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus \Omega_i)}{r^d} = \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap \bigcup_{j \neq i} \Omega_j)}{r^d},$$

we find some  $j \neq i$  such that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap \Omega_j)}{r^d} > 0.$$

Since

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \setminus \Omega_j)}{r^d} \geq \limsup_{r \rightarrow 0} \frac{\mathcal{L}^d(B_r(x) \cap \Omega_i)}{r^d} > 0,$$

we therefore have  $x \in \partial_*\Omega_j$ , which finishes the proof.  $\square$



*Proof of Lemma 3.2.8.* Without loss of generality we can assume that  $i = 1$  and  $\sigma_{1,j} \leq \sigma_{1,j+1}$  for all  $1 \leq j \leq P-1$ . All equalities in this proof hold up to a set of  $(d-1)$ -dimensional Hausdorff measure zero. First let us apply the Fleming–Rishel co-area formula to find that for a given open set  $U \subseteq \mathbb{T}$ , we have

$$\begin{aligned} |\nabla \psi_i|(U) &= \int_0^\infty \mathcal{H}^{d-1}(\partial_* (\{x \in U : \psi_i \leq s\})) \, ds \\ &= \sum_{j=1}^P \int_{\sigma_{1,j}}^{\sigma_{1,j+1}} \mathcal{H}^{d-1} \left( \partial_* \left( \bigcup_{j=1}^i U \cap \Omega_j \right) \right) \, ds \\ &= \sum_{j=1}^{P-1} (\sigma_{1,j+1} - \sigma_{1,j}) \mathcal{H}^{d-1} \left( \partial_* \left( \bigcup_{j=1}^i U \cap \Omega_j \right) \right) =: I. \end{aligned}$$

We claim that we have

$$\partial_* \left( \bigcup_{j=1}^i U \cap \Omega_j \right) = \bigcup_{j=1}^i \bigcup_{l=i+1}^P U \cap \partial_* \Omega_j \cap \partial_* \Omega_l. \quad (\text{A.2})$$

Since  $(\Omega_j \cap U)_j$  is again a partition of  $U$ , we omit taking the intersection with  $U$  for a briefer notation. The proof will be done via an induction over  $i$ . For  $i = 1$ , this is exactly Item 2 from Lemma A.0.1. For the induction step, assume that the claim (A.2) holds for all  $i' \leq i$ . Again by Lemma A.0.1 Item 1, we can compute that

$$\begin{aligned} \partial_* \left( \bigcup_{j=1}^{i+1} \Omega_j \right) &= \left( \partial_* \left( \bigcup_{j=1}^i \Omega_j \right) \cup \partial_* \Omega_{i+1} \right) \setminus \left( \partial_* \left( \bigcup_{j=1}^i \Omega_j \right) \cap \partial_* \Omega_{i+1} \right) \\ &= \left( \left( \bigcup_{j=1}^i \bigcup_{l=i+1}^P \partial_* \Omega_j \cap \partial_* \Omega_l \right) \cup \bigcup_{k \neq i+1} \partial_* \Omega_{i+1} \cap \partial_* \Omega_k \right) \setminus \left( \bigcup_{j=1}^i \bigcup_{l=i+1}^P \partial_* \Omega_j \cap \partial_* \Omega_l \cap \partial_* \Omega_{i+1} \right). \end{aligned}$$

By Lemma A.0.1 Item 1, we can write the second term as

$$\bigcup_{j=1}^i \bigcup_{l=i+1}^P \partial_* \Omega_j \cap \partial_* \Omega_l \cap \Omega_{i+1} = \bigcup_{j=1}^i \partial_* \Omega_j \cap \partial_* \Omega_{i+1}.$$

By carefully considering both equations, we get

$$\begin{aligned} &\partial_* \left( \bigcup_{j=1}^{i+1} \Omega_j \right) \\ &= \left( \left( \bigcup_{j=1}^i \bigcup_{l=i+2}^P \partial_* \Omega_j \cap \partial_* \Omega_l \right) \cup \bigcup_{k=i+2}^P \partial_* \Omega_{i+1} \cap \partial_* \Omega_k \right) \setminus \bigcup_{j=1}^i \partial_* \Omega_{i+1} \cap \partial_* \Omega_j \\ &= \left( \bigcup_{j=1}^{i+1} \bigcup_{l=i+2}^P \partial_* \Omega_j \cap \partial_* \Omega_l \right) \setminus \bigcup_{j=i}^i \partial_* \Omega_{i+1} \cap \partial_* \Omega_j. \end{aligned} \quad (\text{A.3})$$

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Since by Lemma A.0.1 Item 1, we have for  $1 \leq j \leq i+2$ ,  $i+2 \leq l \leq P$  and  $1 \leq k \leq i$  that

$$\partial_* \Omega_l \cap \partial_* \Omega_{i+1} \cap \partial_* \Omega_k$$

is a set of  $(d-1)$ -dimensional Hausdorff measure zero, taking the complement in the term (A.3) becomes obsolete and thus we obtain the desired result (A.2). Therefore we have again by Lemma A.0.1 Item 1 that

$$\begin{aligned} I &= \sum_{i=1}^{P-1} (\sigma_{1,i+1} - \sigma_{1,i}) \left( \sum_{j=1}^i \sum_{l=i+1}^P \mathcal{H}^{d-1}(U \cap \partial_* \Omega_j \cap \partial_* \Omega_l) \right) \\ &= \sum_{1 \leq j < l \leq P} \mathcal{H}^{d-1}(U \cap \partial_* \Omega_j \cap \partial_* \Omega_l) \left( \sum_{i=j}^{l-1} \sigma_{1,i+1} - \sigma_{1,i} \right) \\ &= \sum_{1 \leq j < l \leq P} \mathcal{H}^{d-1}(U \cap \partial_* \Omega_j \cap \partial_* \Omega_l) (\sigma_{1,l} - \sigma_{1,j}). \end{aligned}$$

Since we assumed without loss of generality that  $\sigma_{1,i} \leq \sigma_{1,j+1}$ , we know that

$$|\sigma_{1,l} - \sigma_{1,j}| = \sigma_{1,l} - \sigma_{1,j},$$

which finishes the proof.  $\square$

**Lemma A.0.3.** *Let  $\mu$  be a regular positive Borel measure on some open set  $\Omega$  and let  $B_1, \dots, B_m$  be  $\mu$ -finite disjoint Borel subsets of  $\Omega$ . Moreover let  $c_i^h$  for  $1 \leq i \leq m$  and  $1 \leq h \leq k$  be non-negative coefficients. Define the measures*

$$\mu_h := \sum_{i=1}^m c_i^h \mu|_{B_i} \quad \text{and} \quad \nu := \sum_{i=1}^m \max_h c_i^h \mu|_{B_i}.$$

Then we have

$$\nu = \bigvee_{h=1}^k \mu_h.$$

*Proof.* We first note that for all Borel sets  $A$  and all  $\tilde{h}$ , we have

$$\nu(A) = \sum_{i=1}^m \max_h c_i^h \mu(A \cap B_i) \geq \sum_{i=1}^m c_i^{\tilde{h}} \mu(A \cap B_i) = \mu_{\tilde{h}}(A).$$

On the other hand we have that for all  $1 \leq i \leq m$  that we find  $h(i)$  such that  $\max_h c_i^h = c_i^{h(i)}$ . But then we have

$$\nu(B_i) = \max_h c_i^h \mu(B_i) = \mu_{h(i)}(B_i),$$

thus we have for all  $1 \leq i \leq m$  that

$$\nu(B_i) \leq \left( \bigvee_{h=1}^k \mu_h \right) (B_i).$$

Since the support of  $\nu$  is contained in  $\bigcup_{1 \leq i \leq m} B_i$ , this finishes the proof.  $\square$

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