Convergence of Allen-Cahn equations to multi-phase mean curvature flow

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1 The Allen–Cahn equation

1.1 Structure of the equation

This chapter follows [LS16], but since the authors decided to only sketch some proofs. we want to go into more detail.

Let $\Lambda > 0$ and define the flat torus $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$, where we work with periodic boundary conditions and write $\int \mathrm{d}x$ instead of $\int_{\mathbb{T}} \mathrm{d}x$. Then for $u \colon [0, \infty) \times \mathbb{T} \to \mathbb{R}^N$ and some potential $W \colon \mathbb{R}^N \to [0, \infty)$, the *Allen–Cahn equation* with parameter $\varepsilon > 0$ is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \tag{1.1}$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to u for a fixed time the real number

$$\mathrm{E}_\varepsilon(u) := \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 \, \mathrm{d}x. \tag{1.2}$$

If everything is nice and smooth, we can compute that under the assumption that u satisfies equation (1.1), we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u) \,, \partial_t u \rangle + \varepsilon \langle \nabla u \,, \nabla \partial_t u \rangle \, \mathrm{d}x \\
&= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u \,, \partial_t u \right\rangle \mathrm{d}x \\
&= \int -\varepsilon |\partial_t u|^2 \, \mathrm{d}x. \tag{1.1}
\end{aligned}$$

This calculation suggests that equation (1.1) is the L^2 gradient-flow (rescaled by $\sqrt{\varepsilon}$) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (1.1) via De Giorgis minimizing movements scheme, which we will do in theorem Theorem 1.2.1.

But first we need to clarify what our potential W should look like. Classic examples in the scalar case are given by $W(u) = \left(u^2 - 1\right)^2$ or $W(u) = u^2(u - 1)^2$, and we call functions like these *doublewell potentials*, see also Figure 1.1.

In higher dimensions, we want to accept the following potentials: $W \colon \mathbb{R}^N \to [0, \infty)$ has to be a a smooth multiwell potential with finitely many zeros at $u = \alpha_1, \ldots, \alpha_P \in \mathbb{R}^N$. Furthermore we aks for polynomial growth in the sense that there exists some $p \geq 2$ such that

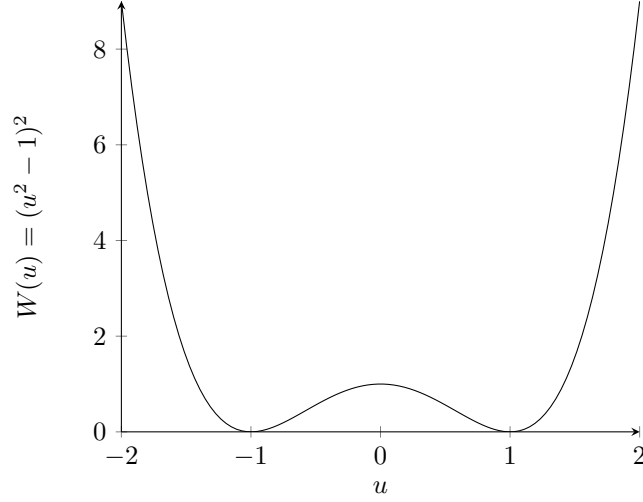$$|u|^p \lesssim W(u) \lesssim |u|^p \tag{1.3}$$

Figure 1.1: The graph of a doublewell potential

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \tag{1.4}$$

for all u sufficiently large. Lastly we want W to be convex up to a small perturbation in the sense that there exist smooth functions $W_{\mathrm{conv}}, W_{\mathrm{pert}} \colon \mathbb{R}^N \to [0, \infty)$ such that

$$W = W_{\mathrm{conv}} + W_{\mathrm{pert}}, \tag{1.5}$$

W_{conv} is convex and

$$\sup_{x \in \mathbb{R}^N} \left| \nabla^2 W_{\mathrm{pert}} \right| < \infty. \tag{1.6}$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen-Cahn equation (1.1) as maps from $[0, T]$ into some suitable function space and thus use the following definition.

Definition 1.1.1. *We say that a function $u_\varepsilon \in \mathrm{C}\left([0,T]; \mathrm{L}^2(\mathbb{T}; \mathbb{R}^N)\right)$ which is also in $\mathrm{L}^\infty\left([0,T]; \mathrm{W}^{1,2}(\mathbb{T}; \mathbb{R}^N)\right)$ is a weak solution of the Allen–Cahn equation (1.1) with parameter $\varepsilon > 0$ and initial condition $u_\varepsilon^0 \in \mathrm{L}^2(\mathbb{T}; \mathbb{R}^N)$ if*

   1. the energy stays bounded, which means that

$$\operatorname*{ess\,sup}_{0 \le t \le T} \mathrm{E}_\varepsilon(u_\varepsilon(t)) < \infty, \tag{1.7}$$

   2. its weak time derivative satisfies

$$\partial_t u_\varepsilon \in \mathrm{L}^2([0,T] \times \mathbb{T}; \mathbb{R}^N), \tag{1.8}$$

3. *for almost every $t \in [0, T]$ and every $\xi \in L^p([0,T] \times \mathbb{T}; \mathbb{R}^N) \cap W^{1,2}([0,T] \times \mathbb{T}; \mathbb{R}^N)$,
   we have*

$$\int \langle \partial_t u_\varepsilon(t), \, \xi \rangle + \langle \nabla u_\varepsilon(t), \, \nabla \xi \rangle + \left\langle \frac{1}{\varepsilon^2} \nabla W(u_\varepsilon(t)), \, \xi \right\rangle \mathrm{d}x = 0, \qquad (1.9)$$

4. *the initial conditions are achieved in the sense that $u_\varepsilon(0) = u_\varepsilon^0$.*

Remark. *Given our assumptions, we automatically have $\nabla W(u_\varepsilon)(t) \in L^{p'}(\mathbb{T})$ since*

$$|\nabla W(u_\varepsilon)|^{p/(p-1)} \lesssim 1 + |u_\varepsilon|^p \lesssim 1 + W(u), \qquad (1.10)$$

*which is integrable for almost every time t since we assume that the energy stays
bounded, thus the integral in equation (1.9) is well defined.*

Moreover we already obtain $1/2$ Hölder-continuity in time from the embedding

$$W^{1,2}([0,T]; L^2(\mathbb{T}; \mathbb{R}^N)) \hookrightarrow C^{1/2}([0,T]; L^2(\mathbb{T}; \mathbb{R}^N)), \qquad (1.11)$$

*which follows from a generalized version of the fundamental theorem of calculus and
Hölder's inequality.*

1.2 Existence of a solution

With Definition 1.1.1, we are able to state our existence results for solutions of the
Allen–Cahn equation. The proof uses De Giorgis minimizing movements scheme and
arguments from the theory of gradient flows.

Theorem 1.2.1. *Let $u_\varepsilon^0 \colon \mathbb{T} \to \mathbb{R}^N$ be such that $E_\varepsilon(u_\varepsilon^0) < \infty$. Then there exists a weak
solution u_ε to the Allen–Cahn equation (1.1) in the sense of Definition 1.1.1 with initial
data u_ε^0. Furthermore the solution satisfies the energy dissipation identity*

$$E_\varepsilon(u_\varepsilon(t)) + \int_0^t \int \varepsilon |\partial_t u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}s = E_\varepsilon(u_\varepsilon^0) \qquad (1.12)$$

*for almost every $t \in [0, T]$ and we additionally have $\partial_{i,j}^2 u_\varepsilon, \nabla W(u_\varepsilon) \in L^2([0,T] \times \mathbb{T}; \mathbb{R}^N)$
for all $1 \le i, j \le d$. In particular we can test the weak form (1.9) with $\partial_{i,j}^2 u_\varepsilon$.*

Proof. **Step 1:** A minimization problem

Fix some $h > 0$, $u_{n-1} \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ and consider the functional

$$\mathcal{F} \colon W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N) \to \mathbb{R}$$
$$u \mapsto E_\varepsilon(u) + \frac{1}{2h} \int |u - u_{n-1}|^2 \, \mathrm{d}x.$$

Then $\mathcal{F}$ is coercive with respect to $\|\cdot\|_{W^{1,2}}$ and bounded from below by zero, thus we may
take a $W^{1,2}$-bounded sequence $(v_k)_{k \in \mathbb{N}}$ in $W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$ such that $\mathcal{F}(v_k) \to \inf \mathcal{F}$

as $k \to \infty$. These v_k have a non-relabelled subsequence which converges weakly in $W^{1,2}(\mathbb{T}; \mathbb{R}^N)$ and strongly in $L^2(\mathbb{T}, \mathbb{R}^N)$ to some $u \in W^{1,2}(\mathbb{T}; \mathbb{R}^N)$. Moreover v_k is bounded in $L^p(\mathbb{T}; \mathbb{R}^N)$ by the lower growth assumption (1.3) on W and thus obtain by a duality argument that $u \in W^{1,2} \cap L^p(\mathbb{T}; \mathbb{R}^N)$.

Lastly we have

$$\frac{1}{2h} \int |u - u_{n-1}|^2 \, \mathrm{d}x \leq \liminf_{k \to \infty} \frac{1}{2h} \int |v_k - u_{n-1}|^2 \, \mathrm{d}x \,, \tag{1.13}$$

$$\int \frac{\varepsilon}{2} |\nabla u|^2 \, \mathrm{d}x \leq \liminf_{k \to \infty} \int \frac{\varepsilon}{2} |\nabla v_k|^2 \, \mathrm{d}x \tag{1.14}$$

by the weak convergence in $W^{1,2}(\mathbb{T}, \mathbb{R}^N)$. By passing to another non-relabelled subsequence which converges pointwise almost everywhere, we moreover achieve by continuity of W that

$$\int \frac{1}{\varepsilon} W(u) \, \mathrm{d}x = \int \liminf_{k \to \infty} \frac{1}{\varepsilon} W(v_k) \, \mathrm{d}x \leq \liminf_{k \to \infty} \int \frac{1}{\varepsilon} W(v_k) \, \mathrm{d}x \,,$$

which proves that u is a minimizer of $\mathcal{F}$.

Step 2: Minimizing movements scheme

By recursively choosing minimizers u_n^h from Step 1, we obtain a sequence of functions $u_0, u_1^h, \ldots$. Thus we may define a function $u \in C$ as the piecewise linear interpolation at the time-steps $0, h, 2h, \ldots$ of these functions.

Step 3: Estimates for u^h

We claim that there exists some constant $C > 0$ such that for all $h > 0$ and $n \in \mathbb{N}$, we have

$$\mathrm{E}_\varepsilon(u_n^h) + \left(\frac{1}{h} - \frac{C}{2\varepsilon}\right) \int |u_n - u_{n-1}|^2 \, \mathrm{d}x \leq \mathrm{E}_\varepsilon(u_{n-1}^h). \tag{1.15}$$

In order to prove this inequality, we notice that since $\left|\nabla^2 W_{\mathrm{pert}}\right| \leq C$, the function $W_{\mathrm{pert}} + C|u|^2/2$ is convex for $C > 0$ sufficiently large, thus the functional

$$\tilde{\mathrm{E}}_\varepsilon(u) := \int \frac{1}{\varepsilon} \left(W(u) + C \frac{|u|^2}{2} \right) + \frac{\varepsilon}{2} |\nabla u|^2 \, \mathrm{d}x$$

is convex on $W^{1,2} \cap L^p(\mathbb{T}, \mathbb{R}^N)$. For a given $\xi \in W^{1,2} \cap L^p(\mathbb{T}, \mathbb{R}^N)$, we thus have that the function $t \mapsto \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi)$ is convex and differentiable, which yields that

$$\tilde{\mathrm{E}}_\varepsilon(u_n^h + \xi) \geq \tilde{\mathrm{E}}_\varepsilon(u_n^h) + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi). \tag{1.16}$$

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But since u_n^h is a minimizer of $\mathcal{F}$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{F}(u_n^h + t\xi) \\
&\quad + \frac{C}{2\varepsilon}\int \left|u_n^h + t\xi\right|^2 \mathrm{d}x - \frac{1}{2h}\int \left|u_n^h + t\xi - u_{n-1}^h\right|^2 \mathrm{d}x \\
&= \int \frac{C}{\varepsilon}\langle u_n^h\,,\,\xi\rangle - \frac{1}{h}\langle u_n^h - u_{n-1}^h\,,\,\xi\rangle \,\mathrm{d}x\,.
\end{aligned}
$$

Plugging $\xi = u_{n-1}^h$

$\square$

Remark. *The inequality (1.15) with the factor $1/2h$ instead of $1/h - C/2\varepsilon$ follows immediately from the definition of our optimization problem, but is not optimal and this so called sharp energy dissipation inequality is important for later.*

2 Convergence of the Allen–Cahn equations

Baldo proved in his paper [Bal90] that the Cahn-Hilliard energies Γ-converge with respect to $\|\cdot\|_{L^1}$ to an *optimal partition energy* given by

$$\mathrm{E}(\chi) := \frac{1}{2} \sum_{1 \leq i,j \leq P} \sigma_{i,j} \int \frac{1}{2} \left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i + \chi_j)| \right) \tag{2.1}$$

for a partition $\chi_1, \ldots, \chi_P \colon \mathbb{T} \to \{0,1\}$ satisfying $\chi_{1 \leq i \leq P} \chi_i = 1$ almost everywhere. We may also define measurable sets Ω_i through the relation $\chi_i = \mathbb{1}_{\Omega_i}$. The link between a sequence u_ε and χ is given by $u_\varepsilon \to u := \sum_{1 \leq j \leq P} \alpha_i \chi_i$ in L^1.

Moreover if we denote by $\partial_* \Omega_i$ the reduced boundary of Ω_i and by $\Sigma_{i,j} := \partial_* \Omega_i \cap \Omega_j$ the interface between Ω_i and Ω_j, then we may rewrite equation (2.1) as

$$\mathrm{E}(\chi) = \frac{1}{2} \sum_{1 \leq i,j \leq P} \sigma_{i,j} \, \mathcal{H}^{d-1}(\Sigma_{i,j}).$$

Here, the surface tensions $\sigma_{i,j}$ are the geodesic distances between the wells α_i of W with respect to the metric $2W(u)\langle \cdot, \cdot \rangle$, which can be written out as

$$\sigma_{i,j} = \mathrm{d}_W(\alpha_i, \alpha_j)$$

for the geodesic distance defined as

$$\mathrm{d}_W(u,v) := \inf \left\{ \int_0^1 \sqrt{2W(\gamma)} |\dot\gamma| \, \mathrm{d}t \, : \, \gamma \in \mathrm{C}^1([0,1], \mathbb{R}^N) \text{ with } \gamma(0) = u, \, \gamma(1) = v \right\}. \tag{2.2}$$

Geometrically speaking, the partition energy E measures the surface tensions between the sets and penalizes larges interfaces. Also observe that the factor $1/2$ can be left out if we only count each interface once.

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