Convergence of Allen-Cahn equations to multiphase mean curvature flow

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1.1 Gradient flows

In the simplest case, a gradient flow of a given energy $\mathrm{E}\colon \mathbb{R}^N \to \mathbb{R}$ (with respect to the euclidean inner product) is a solution to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\nabla \mathrm{E}(x(t)), \tag{1.1}$$

where we usually prescribe some initial value $x(0) = x_0 \in \mathbb{R}^N$. The central structure here is that on the right hand side of the equation, we do not have any function, but the gradient of some function. What a solution x does is that it moves in the direction of the steepest descent of the energy E. Moreover this allows for the following computation given a solution x of (1.1):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}(x(t)) &= \langle \nabla \mathrm{E}(x(t)) \, , \, x'(t) \rangle = -|\nabla \mathrm{E}(x(t))|^2 = -|x'(t)|^2 \\ &= -\frac{1}{2} \left(|x'(t)|^2 + |\nabla \mathrm{E}(x(t))|^2 \right). \end{aligned}$$

We especially obtain that the the function $\mathrm{E}(x(t))$ is non-increasing, which coincides with our intuition that x moves along the steepest descent of the energy. But more precisely, we obtain from the fundamental theorem of calculus the *energy dissipation identity*

$$\mathrm{E}(x(T)) + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla \mathrm{E}(x(t))|^2 \, \mathrm{d}t = \mathrm{E}(x(0)). \tag{1.2}$$

One could now raise the question if this is identity already characterizes the equation, which means that if some x satisfies equation (1.2), it should already be a solution to the ordinary differential equation (1.1). But as it turns out, we can go even one step further, namely we only ask for the *optimal energy dissipation inequality* given by

$$\mathrm{E}(x(T)) + \frac{1}{2} \int_0^T |x'(t)|^2 + |\nabla \mathrm{E}(x(t))|^2 \, \mathrm{d}t \leq \mathrm{E}(x(0)). \tag{1.3}$$

We call this inequality optimal since as demonstrated before, we usually expect an equality to hold. Now let us assume that x satisfies (1.3) and has sufficient regularity.

Then we can estimate again by the fundamental theorem that

$$
\begin{aligned}
&\frac{1}{2}\int_0^T |\nabla \mathrm{E}(x(t)) + x'(t)|^2 \,\mathrm{d}t \\
= &\int_0^T \langle \nabla \mathrm{E}(x(t)) \,,\, x'(t)\rangle \,\mathrm{d}t + \frac{1}{2}\int_0^T |x'(t)|^2 + |\nabla \mathrm{E}(x(t))|^2 \,\mathrm{d}t \\
= &\ \mathrm{E}(x(T)) - \mathrm{E}(x(0)) + \frac{1}{2}\int_0^T \langle \nabla \mathrm{E}(x(t)) \,,\, x'(t)\rangle \,\mathrm{d}t \leq 0.
\end{aligned}
$$

Since we started with an integral over a non-negative function, this implies that for almost every time t, we have that $x'(t) = -\nabla \mathrm{E}(x(t))$, which proves that x (again under sufficient regularity assumptions) is a gradient flow of the energy E.

The real strength of formulating the differential equation (1.1) via inequality (1.3) becomes clear if we want to consider gradient flows in a more complicated setting by replacing the space $\mathbb{R}^N$. In order to formulate equation (1.1), we need to have a notion of differentiation and a gradient in the target of x. Therefore we may use pre-Hilbert spaces or smooth Riemannian manifolds as a suitable substitutes for $\mathbb{R}^N$.

Examples for such more complicated gradient include the heat equation, which can be written as the L^2-gradient flow of the Dirichlet energy (REFERENZ EINFÜGEN), the Fokker-Planck equation (REFERENZ EINFÜGEN) or the Allen–Cahn equation and mean curvature flow, which of course both play an important role for us.

1.2 Mean curvature flow

Mean curvature flow describes the geometric evolution of a set $(\Omega(t))_{t\geq 0}$ respectively the evolution of its boundary $\Sigma(t) := \partial\Omega(t)$. It is described by the equation

$$
\frac{1}{\mu} V = -\sigma H \quad \text{on } \Sigma. \tag{1.4}
$$

There are a lot of different notions of solutions to this equation(HIER EINFÜGEN). For us the central structure of this equation is its gradient flow structure. Formally we want to consider the space

$$
\mathcal{M} := \{(d-1)\text{-dimensional surfaces on } \mathbb{T}\},
$$

where the tangent space at a given $\Sigma \in \mathcal{M}$ consists of the normal velocities on Σ. The metric tensor of two normal velocities is then given by

$$
\langle V \,,\, W\rangle_\Sigma := \frac{1}{\mu}\int_\Sigma V W \,\mathrm{d}\,\mathcal{H}^{d-1}
$$

and our energy will simply be the rescaled perimeter functional

$$
\mathrm{E}(\Sigma) := \sigma\,\mathcal{H}^{d-1}(\Sigma).
$$

Since the first inner variation of the perimeter functional is given by the mean curvature vector (see [Mag12, Thm. 17.5]), the gradient of the energy should simply be the mean curvature vector multiplied by $\sigma\mu$. Thus the mean curvature flow equation (1.4) corresponds exactly to the gradient flow equation (1.1).

In Section 1.1 we highlighted the importance of De Giorgi's optimal energy dissipation inequality (1.3). In our setting this translates to the inequality

$$\begin{aligned}&\mathrm{E}(\Sigma(T)) + \frac{1}{2}\int_0^T \langle V\,,\,V\rangle + \langle \nabla E(\Sigma)\,,\,\nabla E(\Sigma)\rangle\\ =\;& \mathrm{E}(\Sigma(T)) + \frac{1}{2}\int_0^T \int \frac{1}{\mu}V^2 + \sigma^2\mu H^2 \,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t \leq \mathrm{E}(\Sigma(0))\end{aligned}$$

and it will be one our first main results to deduce this inequality when passing to the limit in the Allen–Cahn equations.

Far more important for us shall however be multiphase mean curvature, which has been motivated in material science by grain growths in polycristals (HIER NOCH MEHR SCHREIBEN). Essentially instead of just considering the evolution of one single set, we look at the evolution of a partition of the flat torus and require that at each interface of the sets, the mean curvature flow equation (1.4) is satisfied.

We say that a partition $(\Omega_i(t))_{i=1,\dots,P}$ with $\Sigma_{i,j} \coloneqq \partial\Omega_i \cap \partial\Omega_j$ satisfies multiphase mean curvature with mobilities $\mu_{i,j}$ and surface tension $\sigma_{i,j}$ if

$$\begin{cases}\frac{1}{\mu_{i,j}}V_{i,j} = -\sigma_{i,j}H_{i,j} & \text{for all } i,j\\ \sum \sigma_{i,j}\nu_{i,j} = 0 & \text{at triple junctions.}\end{cases}$$

The second equation is a stability condition when more than two sets meet and called Herring's angle condition. Here $\nu_{i,j}$ is the outer unit normal on $\Sigma_{i,j}$ pointing from Ω_i to Ω_j. One can argue that we would also want stability conditions at for example quadruple junctions. In two dimensions though, such quadruple junctions are expected to immediatly dissipate. In higher dimensions, quadruple junctions might become stable, but only on lower dimensional sets, and shall thus not be relevant for us.

As our space, we shall thus now consider tuples of $(d-1)$-dimensional surfaces on $\mathbb{T}$. The energy and metric tensor are given by

$$\mathrm{E}(\Sigma) \coloneqq \sum_{i<j} \sigma_{i,j}\,\mathcal{H}^{d-1}(\Sigma_{i,j})$$

and

$$\langle V\,,\,W\rangle_\Sigma \coloneqq \sum_{i<j}\frac{1}{\mu_{i,j}}\int_{\Sigma_{i,j}} V_{i,j}W_{i,j}\,\mathrm{d}\,\mathcal{H}^{d-1}\,.$$

Using a variant of the divergence theorem on surfaces ([Mag12, Thm. 11.8]) and again the computation for the first variation of the perimeter ([Mag12, Thm. 17.5]), we see that multiphase mean curvature has the desired gradient flow structure and that De Giorgi's inequality (1.3) translates to

$$\mathrm{E}(\Sigma(T)) + \frac{1}{2}\sum_{i<j}\int_0^T\int_{\Sigma_{i,j}} \frac{1}{\mu_{i,j}}V_{i,j}^2 + \sigma_{i,j}^2\mu_{i,j}H_{i,j}^2\,\mathrm{d}\,\mathcal{H}^{d-1}\,\mathrm{d}t \leq \mathrm{E}(\Sigma(0)).$$

2 The Allen–Cahn equation

2.1 Structure of the equation

This chapter follows [LS16], but since the authors decided to only sketch some of the proofs, we want to go into more detail.

Let $\Lambda > 0$ and define the flat torus $\mathbb{T} = [0, \Lambda)^d \subset \mathbb{R}^d$, where we work with periodic boundary conditions and write $\int \mathrm{d}x$ instead of $\int_{\mathbb{T}} \mathrm{d}x$. Then for $u\colon [0,\infty) \times \mathbb{T} \to \mathbb{R}^N$ and some potential $W\colon \mathbb{R}^N \to [0,\infty)$, the *Allen–Cahn equation* with parameter $\varepsilon > 0$ is given by

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \nabla W(u). \tag{2.1}$$

To understand this equation better, we consider the *Cahn–Hilliard energy* which assigns to u for a fixed time the real number

$$\mathrm{E}_\varepsilon(u) := \int \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |\nabla u|^2 \,\mathrm{d}x\,. \tag{2.2}$$

If everything is nice and smooth, we can compute that under the assumption that u satisfies equation (2.1), we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon} \langle \nabla W(u)\,,\, \partial_t u \rangle + \varepsilon \langle \nabla u\,,\, \nabla \partial_t u \rangle \,\mathrm{d}x \\ &= \int \left\langle \frac{1}{\varepsilon} \nabla W(u) - \varepsilon \Delta u\,,\, \partial_t u \right\rangle \mathrm{d}x \\ &= \int -\varepsilon |\partial_t u|^2 \,\mathrm{d}x\,. \end{aligned} \tag{2.1}$$

This calculation suggests that equation (2.1) is the L^2 gradient flow (rescaled by $\sqrt{\varepsilon}$) of the Cahn–Hilliard energy. Thus we can try to construct a solution to the PDE (2.1) via De Giorgis minimizing movements scheme, which we will do in Theorem 2.2.1.

But first we need to clarify what our potential W should look like. Classic examples in the scalar case are given by $W(u) = \left(u^2 - 1\right)^2$ or $W(u) = u^2(u-1)^2$, and we call functions like these *doublewell potentials*, see also Figure 2.1.

In higher dimensions, we want to accept the following potentials: $W\colon \mathbb{R}^N \to [0,\infty)$ has to be a a smooth multiwell potential with finitely many zeros at $u = \alpha_1, \ldots, \alpha_P \in \mathbb{R}^N$. Furthermore we aks for polynomial growth in the sense that there exists some $p \geq 2$ such that

$$|u|^p \lesssim W(u) \lesssim |u|^p \tag{2.3}$$

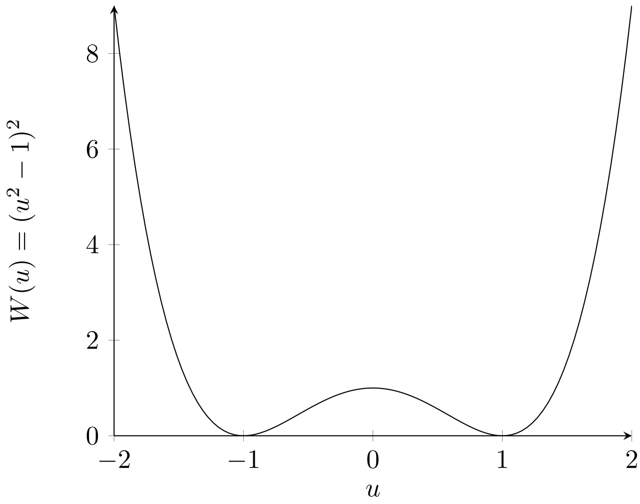

Figure 2.1: The graph of a doublewell potential

and

$$|\nabla W(u)| \lesssim |u|^{p-1} \tag{2.4}$$

for all u sufficiently large. Lastly we want W to be convex up to a small perturbation in the sense that there exist smooth functions $W_{\mathrm{conv}}, W_{\mathrm{pert}} \colon \mathbb{R}^N \to [0, \infty)$ such that

$$W = W_{\mathrm{conv}} + W_{\mathrm{pert}}\,, \tag{2.5}$$

W_{conv} is convex and

$$\sup_{x \in \mathbb{R}^N} \left| \nabla^2 W_{\mathrm{pert}} \right| < \infty. \tag{2.6}$$

These assumptions are in particular satisfied by our two examples for doublewell potentials and therefore seem to be plausible.

As it is custom for parabolic PDEs, we view solutions of the Allen-Cahn equation (2.1) as maps from $[0, T]$ into some suitable function space and thus use the following definition.

Definition 2.1.1. *We say that a function $u_\varepsilon \in \mathrm{C}\left([0,T]; \mathrm{L}^2\left(\mathbb{T}; \mathbb{R}^N\right)\right)$ which is also in $\mathrm{L}^\infty\left([0,T]; \mathrm{W}^{1,2}(\mathbb{T}; \mathbb{R}^N)\right)$ is a weak solution of the Allen–Cahn equation (2.1) with parameter $\varepsilon > 0$ and initial condition $u_\varepsilon^0 \in \mathrm{L}^2(\mathbb{T}; \mathbb{R}^N)$ if*

1. *the energy stays bounded, which means that*

$$\operatorname*{ess\,sup}_{0 \le t \le T} \mathrm{E}_\varepsilon(u_\varepsilon(t)) < \infty, \tag{2.7}$$

2. *its weak time derivative satisfies*

$$\partial_t u_\varepsilon \in \mathrm{L}^2\left([0,T] \times \mathbb{T}; \mathbb{R}^N\right), \tag{2.8}$$

3. *for almost every $t \in [0,T]$ and every $\xi \in \mathrm{L}^p([0,T]\times\mathbb{T};\mathbb{R}^N)\cap\mathrm{W}^{1,2}([0,T]\times\mathbb{T};\mathbb{R}^N)$, we have*

$$\int \left\langle \frac{1}{\varepsilon^2}\nabla W(u_\varepsilon(t))\,,\,\xi \right\rangle + \langle \nabla u_\varepsilon(t)\,,\,\nabla\xi\rangle + \langle \partial_t u_\varepsilon(t)\,,\,\xi\rangle \,\mathrm{d}x = 0, \tag{2.9}$$

4. *the initial conditions are achieved in the sense that $u_\varepsilon(0) = u_\varepsilon^0$.*

Remark. Given our assumptions, we automatically have $\nabla W(u_\varepsilon)(t) \in \mathrm{L}^{p'}(\mathbb{T})$ since

$$|\nabla W(u_\varepsilon)|^{p/(p-1)} \lesssim 1 + |u_\varepsilon|^p \lesssim 1 + W(u), \tag{2.10}$$

which is integrable for almost every time t since we assume that the energy stays bounded, thus the integral in equation (2.9) is well defined.

Moreover we already obtain $1/2$ Hölder-continuity in time from the embedding

$$\mathrm{W}^{1,2}([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)) \hookrightarrow \mathrm{C}^{1/2}([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)), \tag{2.11}$$

which follows from a generalized version of the fundamental theorem of calculus and Hölder's inequality.

2.2 Existence of a solution

With Definition 2.1.1, we are able to state our existence results for solutions of the Allen–Cahn equation. The proof uses De Giorgis minimizing movements scheme and arguments from the theory of gradient flows and has been briefly sketched in [LS16].

Theorem 2.2.1. *Let $u_\varepsilon^0\colon\,\mathbb{T}\to\mathbb{R}^N$ be such that $\mathrm{E}_\varepsilon(u_\varepsilon^0) < \infty$. Then there exists a weak solution u_ε to the Allen–Cahn equation (2.1) in the sense of Definition 2.1.1 with initial data u_ε^0. Furthermore the solution satisfies the energy dissipation inequality*

$$\mathrm{E}_\varepsilon(u_\varepsilon(t)) + \int_0^t \int \varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s \leq \mathrm{E}_\varepsilon(u_\varepsilon^0) \tag{2.12}$$

for every $t\in[0,T]$ and we additionally have $\partial^2_{i,j}u_\varepsilon, \nabla W(u_\varepsilon) \in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$ for all $1\leq i,j\leq d$. In particular we can test the weak form (2.9) with $\partial^2_{i,j}u_\varepsilon$.

Proof. **Step 1:** A minimization problem

Fix some $h>0$, $u_{n-1}\in \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$ and consider the functional

$$\begin{aligned}\mathcal{F}\colon\ \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N) &\to \mathbb{R} \\ u &\mapsto \mathrm{E}_\varepsilon(u) + \frac{1}{2h}\int \varepsilon|u-u_{n-1}|^2\,\mathrm{d}x\,.\end{aligned} \tag{2.13}$$

Then $\mathcal{F}$ is coercive with respect to $\|\cdot\|_{\mathrm{W}^{1,2}}$ and bounded from below by zero, thus we may take a $\mathrm{W}^{1,2}$-bounded sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$ such that $\mathcal{F}(v_k)\to\inf\mathcal{F}$

as $k \to \infty$. These v_k have a non-relabelled subsequence which converges weakly in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$ and strongly in $\mathrm{L}^2(\mathbb{T},\mathbb{R}^N)$ to some $u \in \mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$. Moreover v_k is bounded in $\mathrm{L}^p\left(\mathbb{T};\mathbb{R}^N\right)$ by the lower growth assumption (2.3) on W and thus obtain by a duality argument that $u \in \mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$.

Lastly we have

$$\frac{1}{2h}\int \varepsilon|u-u_{n-1}|^2\,\mathrm{d}x \le \liminf_{k\to\infty}\frac{1}{2h}\int \varepsilon|v_k-u_{n-1}|^2\,\mathrm{d}x\,, \tag{2.14}$$
$$\int \frac{\varepsilon}{2}|\nabla u|^2\,\mathrm{d}x \le \liminf_{k\to\infty}\int \frac{\varepsilon}{2}|\nabla v_k|^2\,\mathrm{d}x \tag{2.15}$$

by the weak convergence in $\mathrm{W}^{1,2}(\mathbb{T},\mathbb{R}^N)$. By passing to another non-relabelled subsequence which converges pointwise almost everywhere, we moreover achieve by continuity of W that

$$\int \frac{1}{\varepsilon}W(u)\,\mathrm{d}x = \int \liminf_{k\to\infty}\frac{1}{\varepsilon}W(v_k)\,\mathrm{d}x \le \liminf_{k\to\infty}\int \frac{1}{\varepsilon}W(v_k)\,\mathrm{d}x\,,$$

which proves that u is a minimizer of $\mathcal{F}$.

Step 2: Minimizing movements scheme

By iteratively choosing minimizers u_n^h from Step 1, we obtain a sequence of functions $u_\varepsilon^0, u_1^h, \dots$. Thus we may define a function $u \in \mathrm{C}$ as the piecewise linear interpolation at the time-steps $0, h, 2h, \dots$ of these functions.

Step 3: Sharp energy dissipation inequality for u_n^h

We claim that there exists some constant $C > 0$ such that for all $h > 0$ and $n \in \mathbb{N}$, we have

$$\mathrm{E}_\varepsilon(u_n^h) + \left(\frac{1}{h} - \frac{C}{2\varepsilon^2}\right)\int \varepsilon\left|u_n^h - u_{n-1}^h\right|^2\mathrm{d}x \le \mathrm{E}_\varepsilon(u_{n-1}^h). \tag{2.16}$$

In order to prove this inequality, we notice that since $\left|\nabla^2 W_{\mathrm{pert}}\right| \le C$, the function $W_{\mathrm{pert}} + C|u|^2/2$ is convex for $C > 0$ sufficiently large, thus the functional

$$\tilde{\mathrm{E}}_\varepsilon(u) := \int \frac{1}{\varepsilon}\left(W(u) + \frac{C}{2}|u|^2\right) + \frac{\varepsilon}{2}|\nabla u|^2\,\mathrm{d}x$$

is convex on $\mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$. For a given $\xi \in \mathrm{W}^{1,2} \cap \mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$, we thus have that the function $t \mapsto \tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi)$ is convex and differentiable, which yields that

$$\tilde{\mathrm{E}}_\varepsilon(u_n^h + \xi) \ge \tilde{\mathrm{E}}_\varepsilon(u_n^h) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\tilde{\mathrm{E}}_\varepsilon(u_n^h + t\xi). \tag{2.17}$$

But since u_n^h is a minimizer of the functional $\mathcal{F}$ defined by (2.13), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\tilde{\mathrm{E}}_\varepsilon(u_n^h+t\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}&\mathcal{F}(u_n^h+t\xi)\\ &+\frac{C}{2\varepsilon}\int\left|u_n^h+t\xi\right|^2\mathrm{d}x-\frac{1}{2h}\int\varepsilon\left|u_n^h+t\xi-u_{n-1}^h\right|^2\mathrm{d}x\\ =&\int\frac{C}{\varepsilon}\langle u_n^h\,,\,\xi\rangle-\frac{1}{h}\varepsilon\langle u_n^h-u_{n-1}^h\,,\,\xi\rangle\,\mathrm{d}x\,.\end{aligned}$$

Plugging $\xi = u_{n-1}^h - u_n^h$ into this equation and using inequality (2.17) thus yields

$$\tilde{\mathrm{E}}_\varepsilon(u_{n-1}^h)\geq\tilde{\mathrm{E}}_\varepsilon(u_n^h)+\int\frac{C}{\varepsilon}\left(\langle u_{n-1}^h\,,\,u_n^h\rangle-\left|u_n^h\right|^2\right)+\frac{1}{h}\varepsilon\left|u_n^h-u_{n-1}^h\right|^2\mathrm{d}x\,,$$

which is equivalent to

$$\begin{aligned}\mathrm{E}_\varepsilon(u_{n-1}^h)\geq\ &\mathrm{E}_\varepsilon(u_n^h)\\ &+\int\left(\frac{C}{2\varepsilon}-\frac{C}{\varepsilon}\right)\left|u_n^h\right|^2-\frac{C}{2\varepsilon}\left|u_{n-1}^h\right|^2+\frac{C}{\varepsilon}\langle u_{n-1}^h\,,\,u_n^h\rangle+\frac{1}{h}\varepsilon\left|u_n^h-u_{n-1}^h\right|^2\mathrm{d}x\\ =\ &\mathrm{E}_\varepsilon(u_n^h)+\left(\frac{1}{h}-\frac{C}{2\varepsilon^2}\right)\int\varepsilon\left|u_n^h-u_{n-1}^h\right|^2\mathrm{d}x\,,\end{aligned}$$

which is the claimed estimate (2.16).

Step 4: Hölder bounds for u_h

From the energy estimate (2.16) we deduce via an induction that

$$\begin{aligned}&\mathrm{E}_\varepsilon(u_n^h)+\left(1-\frac{Ch}{2\varepsilon^2}\right)\int_0^{nh}\int\varepsilon\left|\partial_t u^h\right|^2\mathrm{d}x\,\mathrm{d}t\\ =\ &\mathrm{E}_\varepsilon(u_n^h)+\left(h-\frac{Ch^2}{2\varepsilon^2}\right)\sum_{k=1}^{n}\int\varepsilon\left|\frac{u_n^h-u_{n-1}^h}{h}\right|^2\mathrm{d}x\\ \leq\ &\mathrm{E}_\varepsilon(u_\varepsilon^0).\end{aligned}\tag{2.18}$$

This gives use with the use of Jensen's inequality for $0\leq s\leq t\leq T$ and $h>0$ sufficiently small that

$$\begin{aligned}\left\|u^h(t)-u^h(s)\right\|_{\mathrm{L}^2}&=\left\|\int_s^t\partial_t u_h(\tau)\,\mathrm{d}\tau\right\|_{\mathrm{L}^2}\\ &\leq\sqrt{t-s}\left(\int_0^T\int\left|\partial_t u^h(\tau,x)\right|^2\mathrm{d}x\,\mathrm{d}\tau\right)^{1/2}\\ &\leq\sqrt{t-s}\left(\varepsilon-\frac{Ch}{2\varepsilon}\right)^{-1/2}\left(E_\varepsilon(u_\varepsilon^0)\right)^{1/2},\end{aligned}\tag{2.19}$$

which gives us a uniform bound on the L^2-Hölder continuity of u^h in time as h tends to zero.

Step 5: Compactness

In order to apply Arzelà-Ascoli for the sequence (u_h) as h tends to zero, we need to check pointwise precompactness of the image and equicontinuity of the sequence. The equicontinuity follows from the previous estimate (2.19). In order to check the pointwise precompactness, we need to verify that for all $t \in [0,T]$, the set $\{u^h(t)\}_{\delta>h>0}$ is a precompact subset of $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ (for $\delta > 0$ sufficiently small). But this follows from the energy bound $\mathrm{E}_\varepsilon(u_n^h) \leq \mathrm{E}_\varepsilon(u_0)$ (given by inequality (2.16)), which gives us a time-uniform bound on $\left\|\nabla u^h(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)}$ and on $\left\|u^h(t)\right\|_{\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)}$ and thus on $\left\|u^h(t)\right\|_{\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)}$. The compact embedding $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N) \hookrightarrow \mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ thus yields the desired pointwise precompactness.

Therefore we may apply Arzelà–Ascoli to obtain some $u \in \mathrm{C}^{1,2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N\right)$ and some sequence $h_n \to 0$ such that u^{h_n} converges uniformly to u on $[0,T]$ with respect to $\|\cdot\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)}$.

Step 6: Additional regularity 1

We first want to argue that from our construction, one already obtains that $u \in \mathrm{L}^\infty\left([0,T];\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\right)$ For this we first notice that for a fixed $t \in [0,T]$, the sequence $u^{h_n}(t)$ is by the energy bound (2.16) a bounded sequence in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$, thus we find some non-relabelled subsequence and $v \in \mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$ such that $u^{n_h}(t)$ converges weakly to v in $\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)$. By uniqueness of the limit, we already have $u(t) = v$ almost everywhere, which yields $u(t) \in \mathrm{W}^{1,2}(\mathbb{T},\mathbb{R}^N)$, and by lower semicontinuity, we may also deduce that

$$\|u(t)\|_{\mathrm{W}^{1,2}} \leq \liminf_{n\to\infty} \left\|u^{h_n}(t)\right\|_{\mathrm{W}^{1,2}} \leq C\,\mathrm{E}_\varepsilon,$$

from which we deduce that $u \in \mathrm{L}^\infty\left([0,T];\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N)\right)$.

Secondly the boundedness of the energies

$$\sup_{0\leq t\leq T} \mathrm{E}_\varepsilon(u(t)) < \infty$$

follows from the lower semicontinuity of the energy and the pointwise L^2 convergence and pointwise weak convergence in $\mathrm{W}^{1,2}$ as described in step 1.

Lastly we want to argue that $\partial_t u \in \mathrm{L}^2\left([0,T]\times\mathbb{T};\mathbb{R}^N\right)$. From inequality (2.18) in step 4, we deduce that $\partial_t u^h$ is a bounded sequence in $\mathrm{L}^2[0,T]\times\mathbb{T};\mathbb{R}^N)$. Thus we find a non-relabelled subsequence of h_n and some $w \in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$ such that u^{h_n} converges weakly to w in $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. But then w is already the weak time derivative of u since for any testfunction ξ, we have

$$\begin{aligned}
\int_{[0,T]\times\mathbb{T}} \langle u\,,\,\partial_t\xi\rangle\,\mathrm{d}x\,\mathrm{d}t &= \lim_{n\to\infty}\int_{[0,T]\times\mathbb{T}} \left\langle u^{h_n}\,,\,\partial_t\xi\right\rangle\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{n\to\infty} -\int_{[0,T]\times\mathbb{T}} \left\langle \partial_t u^{h_n}\,,\,\xi\right\rangle\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{[0,T]\times\mathbb{T}} \langle w\,,\,\xi\rangle\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

Step 7: u is a weak solution

Going back to step 1, we see that u_n^h solves the Euler–Lagrange equation

$$\int \frac{1}{\varepsilon^2}\langle \nabla W(u_n^h)\,,\,\xi\rangle + \langle \nabla u_n^h\,,\,\nabla\xi\rangle + \left\langle \frac{u_n^h - u_{n-1}^h}{h}\,,\,\xi\right\rangle \mathrm{d}x = 0 \tag{2.20}$$

for any testfunction $\xi \in \mathrm{C}^\infty(\mathbb{T};\mathbb{R}^N)$. Let $t \in [0,T]$. Since u_h is defined as the pointwise linear interpolation of the functions u_n^h, we find for the sequence h_n corresponding sequences $\lambda_n \in [0,1]$ and $k_n \in \mathbb{N}$ such that

$$t = \lambda_n(k_n-1)h_n + (1-\lambda_n)k_n h_n$$

and therefore we can write

$$u_h(t) = \frac{k_n h_n - t}{h_n} u_{k_n-1}^h + \frac{t-(k_n-1)h_n}{h_n} u_{k_n}^{h_n}.$$

In order to pass to the limit in equation (2.20), we first note that $u_{k_n}^{h_n}$ converges to $u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ since

$$\begin{aligned}\left\|u(t) - u_{k_n}^{h_n}\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} &\le \left\|u(t) - u^{h_n}(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} + \left\|u^{h_n}(t) - u^{h_n}(k_n h_n)\right\|_{\mathrm{L}^2(\mathbb{T},\mathbb{R}^N)} \\ &\overset{(2.19)}{\lesssim} \left\|u(t) - u^{h_n}(t)\right\|_{\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)} + \sqrt{h_n},\end{aligned}$$

which goes to zero as n tends to infinity. This implies that

$$\begin{aligned}\int \langle \nabla u(t)\,,\,\nabla\xi\rangle \,\mathrm{d}x &= \int \langle u(t)\,,\,\operatorname{div}\nabla\xi\rangle \,\mathrm{d}x \\ &= \lim_{n\to\infty}\int \left\langle u_{k_n}^{h_n}\,,\,\operatorname{div}\nabla\xi\right\rangle \mathrm{d}x \\ &= \lim_{n\to\infty}\int \left\langle \nabla u_{k_n}^{h_n}\,,\,\nabla\xi\right\rangle \mathrm{d}x\,.\end{aligned}$$

In step 6, we moreover have shown that $\partial_t u^h$ converges weakly to $\partial_t u$ in $\mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$. This yields, by choosing cylindrical testfunctions, that $\partial_t u^h(t)$ converges weakly to $\partial_t u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ for almost every $t \in [0,T]$. Thus we obtain for almost every $t \in [0,T]$ the convergence

$$\int \langle \partial_t u(t)\,,\,\xi\rangle \,\mathrm{d}x = \lim_{n\to\infty}\int \langle \partial_t u(t)\,,\,\xi\rangle \,\mathrm{d}x = \lim_{n\to\infty}\int \left\langle \frac{u_{k_n}^{h_n} - u_{k_n-1}^{h_n}}{h_n}\,,\,\xi\right\rangle \mathrm{d}x\,.$$

To obtain the weak equation, we still need to prove convergence of remaining term. For this we note that

$$\frac{1}{\varepsilon^2}\left|\left\langle \nabla W(u_{k_n}^{h_n})\,,\,\xi\right\rangle\right| \lesssim \left(1 + \left|u_{k_n}^{h_n}\right|^{p-1}\right)|\xi|,$$

which is a bounded sequence in $\mathrm{L}^{p'}(\mathbb{T};\mathbb{R}^N)$. Since we also may pass to a non-relabelled subsequence which converges almost everywhere, we thus obtain that

$$\int \langle \nabla W(u(t))\,,\,\xi\rangle\,\mathrm{d}x = \lim_{n\to\infty}\int \left\langle \nabla W(u_{k_n}^{h_n})\,,\,\xi\right\rangle \mathrm{d}x\,.$$

Therefore it follows from the Euler–Lagrange equation (2.20) that

$$\int \frac{1}{\varepsilon^2}\langle \nabla W(u(t))\,,\,\xi\rangle + \langle \nabla u\,,\,\nabla\xi\rangle + \langle \partial_t u\,,\,\xi\rangle\,\mathrm{d}x = 0$$

for all $\xi \in \mathrm{C}^\infty(\mathbb{T};\mathbb{R}^N)$. By continuity this extends to all $\xi \in \mathrm{W}^{1,2}\cap\mathrm{L}^p(\mathbb{T};\mathbb{R}^N)$.
Thus the last thing to check is that

$$\sup_{0\le t\le T}\mathrm{E}_\varepsilon(u(t)) < \infty,$$

which follows from the next step.

Step 8: Sharp energy dissipation inequality for u
Let $0 \le t \le T$ and define k_n as in step 7, where we also established that $u_{k_n}^{h_n}$ converges to $u(t)$ in $\mathrm{L}^2\left(\mathbb{T};\mathbb{R}^N\right)$, and thus we may pass to another non-relabelled subsequence to obtain pointwise convergence almost everywhere. By Fatou's Lemma, we thus obtain

$$\int \frac{1}{\varepsilon}W(u(t))\,\mathrm{d}x \le \liminf_{n\to\infty}\int \frac{1}{\varepsilon}W(u_{k_n}^{h_n})\,\mathrm{d}x\,.$$

Moreover we can deduce from the L^2-convergence that $\nabla u_{k_n}^{h_n}$ converges to $u(t)$ in the distributional sense. But $\nabla u_{k_n}^{h_n}$ is uniformly bounded in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ by the energy dissipation inequality (2.16), thus we already obtain that $\nabla u_{k_n}^{h_n}$ converges weakly to $\nabla u(t)$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ which yields

$$\int \frac{\varepsilon}{2}|\nabla u(t)|^2\,\mathrm{d}x \le \liminf_{n\to\infty}\int \frac{\varepsilon}{2}\left|\nabla u_{k_n}^{h_n}\right|^2\mathrm{d}x\,.$$

Lastly we have by the weak convergence of $\partial_t u^{h_n}$ to $\partial t u$ in $\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)$ proven in step 6 that

$$\begin{aligned}\int_0^t\int \varepsilon|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{n\to\infty}\left(1-\frac{Ch_n}{2\varepsilon^2}\right)\int_0^t\int \varepsilon\left|\partial_t u^{h_n}\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\le \liminf_{n\to\infty}\left(1-\frac{Ch_n}{2\varepsilon^2}\right)\int_0^{k_n h_n}\int \varepsilon\left|\partial_t u^{h_n}\right|^2\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Summarizing these estimates, we obtain by the energy dissipation inequality for u^h (2.18) that for all $0\le t\le T$ we have

$$\begin{aligned}\mathrm{E}_\varepsilon(u(t)) + \int_0^t\int \varepsilon|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{n\to\infty}\mathrm{E}_\varepsilon(u_{k_n}) + \left(1-\frac{Ch_n}{2\varepsilon^2}\right)\int_0^{k_n h_n}\int \varepsilon\left|\partial_t u^h\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\le \mathrm{E}_\varepsilon(u_\varepsilon^0).\end{aligned}$$

Step 8: Additional regularity 2

In order to complete the proof, we still have to show that $\partial^2_{i,j} u$ and$\nabla W(u)$ are elements of $\mathrm{L}^2([0,T] \times \mathbb{T}; \mathbb{R}^N)$. In order to show that the second partial derivatives of u are square-integrable, we test the weak formulation (2.9) with the finite differences. To this end, define the finite differences as

$$\Delta_h^+ v(t,x) := \frac{v(t,x+hv) - v(t,x)}{h}, \quad \Delta_h^- v(t,x) := \frac{v(t,x-hv) - v(t,x)}{h}$$

for some $h > 0$ and $v \in \mathbb{R}^d$. Thus plugging $\Delta_h^- \Delta_h^+ u$ into (2.9) yields by the transformation formula that

$$0 = \int_0^T \int \frac{1}{\varepsilon^2} \left\langle \Delta_h^+ \nabla W(u)\,,\, \Delta_h^+ u \right\rangle + \left\langle \nabla \Delta_h^+ u\,,\, \nabla \Delta_h^+ u \right\rangle + \left\langle \partial_t \Delta_h^+ u\,,\, \Delta_h^+ u \right\rangle \mathrm{d}x\, \mathrm{d}t\,,$$

which is equivalent to

$$\begin{aligned}
&\int_0^T \int \left|\Delta_h^+ \nabla u\right|^2 \mathrm{d}x\, \mathrm{d}t \\
&= -\int_0^T \int \partial_t \left(\frac{\left|\Delta_h^+ u\right|^2}{2} \right) + \frac{1}{\varepsilon^2} \left\langle \Delta_h^+ \nabla W(u)\,,\, \Delta_h^+ u \right\rangle \mathrm{d}x\, \mathrm{d}t \\
&= \int \frac{\left|\Delta_h^+ u(0)\right|^2 - \left|\Delta_h^+ u(T)\right|^2}{2} \mathrm{d}x \\
&\quad - \int_0^1 \int_0^T \int \left\langle \mathrm{D}^2\, W\left((1-s)u(t,x) + su(t,x+hv)\right)\right) \Delta_h^+ u\,,\, \Delta_h^+ u \right\rangle \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s\,.
\end{aligned}$$

The first summand can be estimated by $\|u\|_{\mathrm{L}^\infty([0,T],\mathrm{W}^{1,2}(\mathbb{T};\mathbb{R}^N))}$. For the second summand, we partition W into the sum of W_{conv} and W_{pert}. The term involving the convex summand can then by estimated by

$$\int_0^1 \int_0^T \int \left\langle \mathrm{D}^2\, W_{\mathrm{conv}}\left((1-s)u(t,x) + su(t,x+hv)\right) \Delta_h^+ u\,,\, \Delta_h^+ u \right\rangle \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \geq 0$$

and the for the pertubation term, we get via the bound on its second derivative that

$$\begin{aligned}
&\left| \int_0^1 \int_0^T \int \left\langle \mathrm{D}^2\, W_{\mathrm{pert}}\left((1-s)u(t,x) + su(t,x+hv)\right) \Delta_h^+ u\,,\, \Delta_h^+ u \right\rangle \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \right| \\
&\lesssim \int_0^T \int |\nabla u|^2 \,\mathrm{d}x\, \mathrm{d}t\,,
\end{aligned}$$

which is also finite. Combining these estimates, we obtain that $\int_0^T \int \left|\Delta_h^+ \nabla u\right|^2 \mathrm{d}x\, \mathrm{d}t$ is uniformly bounded in h. Applying our calculation to all directions $v \in \mathbb{R}^d$, we get by the finite-differences theorem for all $1 \leq i,j \leq d$ that $\partial^2_{i,j} u \in \mathrm{L}^2([0,T] \times \mathbb{T}; \mathbb{R}^N)$.

In order to obtain $\nabla W(u) \in \mathrm{L}^2([0,T]\times\mathbb{T};\mathbb{R}^N)$, we again consider the weak formulation (2.9) and notice that since we have already shown that both the time derivative and second space derivatives of u are square-integrable, our claim follows from a duality argument.

$\square$

Remark. The inequality (2.16) with the factor $1/2h$ instead of $1/h - C/2\varepsilon^2$ follows immediately from the definition of our optimization problem, but is not optimal for fixed ε if we want to study the behaviour as h tends to zero. Moreover this so called *sharp energy dissipation inequality* is important for later.

Remark. The energy dissipation inequality (2.12) can be deduced via the formal calculation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}_\varepsilon(u) &= \int \frac{1}{\varepsilon}\langle \nabla W(u)\,,\, \partial_t u\rangle + \varepsilon\langle \nabla u\,,\, \nabla\partial_t u\rangle\,\mathrm{d}x \\ &= \int \left\langle \frac{1}{\varepsilon}\nabla W(u) - \varepsilon\Delta u\,,\, \partial_t u\right\rangle \mathrm{d}x \\ &= -\varepsilon\int |\partial_t u_\varepsilon|^2\,\mathrm{d}x\,.\end{aligned}$$

In order to make this calculation rigorous, we however need to show that $t \mapsto \mathrm{E}_\varepsilon(u(t))$ is absolutely continuous, which is non-trivial, but it would give us equality in energy dissipation inequality (2.12). Since we will only need the inequality, our proof will however suffice.

3 Conditional convergence of the Allen–Cahn equations

The main content of this chapter is to look at the behaviour of solutions of the Allen–Cahn equation (2.1) as the parameter ε tends to zero. As it turns out the scalar case is significantly easier to handle than the vectorial case, thus we shall first focus on the case $N = 1$, also called the *twophase case*.

3.1 Conditional convergence in the twophase case

3.1.1 Convergence to a moving partition

When want to consider the behaviour of solutions $u_\varepsilon \colon [0,T] \times \mathbb{T} \to \mathbb{R}$ to (2.1) for $\varepsilon \to 0$. Let us for now assume that the energies of the initial functions $\mathrm{E}_\varepsilon(u_\varepsilon^0)$ stay uniformly bounded as ε tends to zero. Then due to the energy dissipation inequality (2.12), we already obtain that for all $0 \le t \le T$, we have that $\mathrm{E}_\varepsilon(u_\varepsilon(t))$ stays uniformly bounded.

Another important observation for the convergence is the classic Modica Mortula trick (hier Referenz einfügen): let $\alpha < \beta$ be the two distinct zeros of the doublewell potential W. Then we define a primitive of $\sqrt{2W(u)}$ via

$$\phi(u) := \int_\alpha^u \sqrt{2W(s)}\,\mathrm{d}s\,.$$

For $\psi_\varepsilon := \phi \circ u_\varepsilon$, we can show that $\psi_\varepsilon \in \mathrm{W}^{1,1}((0,T) \times \mathbb{T})$ with weak derivatives $\nabla\psi_\varepsilon = \sqrt{2W(u_\varepsilon)}\nabla u_\varepsilon$ and $\partial_t\psi_\varepsilon = \sqrt{2W(u_\varepsilon)}\partial_t u_\varepsilon$. (we will later show this in more generality (Referenz einfügen)).

Thus via Young's inequality, we can compute

$$\begin{aligned}\mathrm{E}_\varepsilon(u_\varepsilon) &= \int \frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\,\mathrm{d}x \\ &\ge \int \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x \\ &= \int |\nabla\psi_\varepsilon|\,\mathrm{d}x\,, \end{aligned} \tag{3.1}$$

which suggests that we might hope for good compactness properties of $\phi \circ u_\varepsilon$. We combine these two observations into the following Proposition.

Proposition 3.1.1. *Given initial data u_ε^0 whose energies stay uniformly bounded in the sense that*

$$\sup_{\varepsilon>0} \mathrm{E}_\varepsilon(u_\varepsilon^0) < \infty, \tag{3.2}$$

there exists for any sequence $\varepsilon \to 0$ some non-relabelled subsequence such that the solutions of the Allen–Cahn equation (2.1) with initial condition u_ε^0 converge in $\mathrm{L}^1((0,T)\times\mathbb{T})$ to some $u = \alpha(1-\chi)+\beta\chi$ with $\chi \in \mathrm{BV}\left((0,T)\times\mathbb{T};\{0,1\}\right)$. Moreover the compositions ψ_ε are uniformly bounded in $\mathrm{BV}\left((0,T)\times\mathbb{T}\right)$ and converge to $\phi\circ u$ in $\mathrm{L}^1((0,T)\times\mathbb{T})$.

Proof. From the energy dissipation inequality (2.12) in Theorem 2.2.1 we infer that for all $\varepsilon > 0$, it holds that

$$\sup_{0\le t\le T} \mathrm{E}_\varepsilon(u_\varepsilon(t)) \le \mathrm{E}_\varepsilon(u_\varepsilon^0), \tag{3.3}$$

whose right hand side is by assumption uniformly bounded in ε. We want to use a similar calculation as (3.1) we thus infer that $\nabla\psi_\varepsilon$ is uniformly bounded in $\mathrm{L}^1\left((0,T)\times\mathbb{T}\right)$. Moreover we may estimate

$$\begin{aligned}
\int_0^T \int |\psi_\varepsilon| \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int \left| \int_\alpha^{u_\varepsilon} \sqrt{2W(s)} \right| \mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_0^T \int |u_\varepsilon - \alpha| \sup_{s\in[\alpha,u_\varepsilon]} \sqrt{2W(s)}\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim 1 + \int_0^T \int |u_\varepsilon|^{1+p/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim 1 + \int_0^T \int W(u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned} \tag{3.4}$$

which is uniformly bounded via the energy bound. Here the last two inequalities follow from the upper respectively lower growth assumptions (2.3) on W. Thus we have indeed that ψ_ε is a bounded sequence in $\mathrm{BV}((0,T)\times\mathbb{T})$ and therefore there exists some non-relabelled subsequence and some $\psi \in \mathrm{BV}((0,T)\times\mathbb{T})$ such that ψ_ε converges to ψ in $\mathrm{L}^1((0,T)\times\mathbb{T})$.

We notice that since W is non-negative and only has a discrete set of zeros, the function ϕ is strictly increasing and continuous on $\mathbb{R}$, and is thus invertible. Moreover we may pass to a further non-relabelled subsequence of ψ_ε which converges almost everywhere to ψ. Thus defining $u := \phi^{-1}(\psi)$, we obtain that

$$u_\varepsilon = \phi^{-1}(\psi_\varepsilon) \to \phi^{-1}(\psi) = u$$

converges pointwise almost everywhere.

Moreover we notice that by Fatou's Lemma and the boundedness of the energies that

$$\int W(u)\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \int W(u_\varepsilon)\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \varepsilon\,\mathrm{E}_\varepsilon(u_\varepsilon) = 0.$$

Again using the non-negativity of W, this yields that $W(u) = 0$ almost everywhere. Thus $u \in \{\alpha,\beta\}$ almost everywhere and we may write $u = \alpha(1-\chi)+\beta\chi$ for some

$\chi\colon (0,T)\times\mathbb{T}\to\{0,1\}$. Looking again at the definition of u, we moreover obtain that

$$\psi=\phi(u)=\phi(\alpha)(1-\chi)+\phi(\beta)\chi=\int_\alpha^\beta\sqrt{2W(s)}\,\mathrm{d}s\,\chi=:\sigma\chi \tag{3.5}$$

and since ψ is a function of bounded variation, this implies that χ is of bounded variation as well.

Finally from the energy bound and the estimate $|u_\varepsilon|\lesssim 1+W(u)$, we infer that u_ε is L^p-bounded, and since u_ε converges pointwise almost everywhere to u, we obtain the desired L^1-convergence. □

Nextup, we want to make sure that u respectively χ assume their initial data and on the way obtain a useful bound on the time derivative.

Lemma 3.1.2. *With the assumptions of Proposition 3.1.1, we have* $\psi_\varepsilon\in\mathrm{W}^{1,2}([0,T];\mathrm{L}^1(\mathbb{T}))$ *with the estimate*

$$\left(\int_0^T\left(\int|\partial_t\psi_\varepsilon|\,\mathrm{d}x\right)^2\mathrm{d}t\right)^{1/2}\lesssim\mathrm{E}_\varepsilon(u_\varepsilon^0). \tag{3.6}$$

Furthermore the sequence u_ε *is precompact in* $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$.

Proof. **Step 1:** $\psi_\varepsilon\in\mathrm{W}^{1,2}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$ and satisfies the inequality (3.6)

From estimate (3.4) we can infer that

$$\int_0^T\left(\int|\psi_\varepsilon|\,\mathrm{d}x\right)^2\lesssim\int_0^T\left(1+\int W(u_\varepsilon)\,\mathrm{d}x\right)^2\mathrm{d}t<\infty$$

via the uniform boundedness of the energies (3.3). For the desired bound (3.6), we estimate via Hölder's inequality and the uniform boundedness of the energies that

$$\begin{aligned}
\int_0^T\left(\int|\partial_t\psi_\varepsilon|\,\mathrm{d}x\right)^2\mathrm{d}t&\le\int_0^T\left(\int\sqrt{2W(u_\varepsilon)}|\partial_t u_\varepsilon|\,\mathrm{d}x\right)^2\mathrm{d}t\\
&=\int_0^T\left(\int\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\sqrt{\varepsilon}|\partial_t u_\varepsilon|\,\mathrm{d}x\right)^2\mathrm{d}t\\
&\le\int_0^T\int\frac{1}{\varepsilon}2W(u_\varepsilon)\,\mathrm{d}x\int\varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le 2\,\mathrm{E}_\varepsilon(u_\varepsilon^0)\int_0^T\int\varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t && (3.3)\\
&\le 2\left(\mathrm{E}_\varepsilon(u_\varepsilon^0)\right)^2, && (2.12)
\end{aligned}$$

which completes step 1.

Step 2: The sequence ψ_ε is precompact in $\mathrm{C}\left([0,T];\mathrm{L}^1(\mathbb{T})\right)$.

As noted in embedding (2.11), we have

$$\mathrm{W}^{1,2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)\hookrightarrow\mathrm{C}^{1/2}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)$$

from which the equicontinuity of the sequence follows with step 1. Moreover for fixed time t, estimate (3.1) yields that $\nabla\psi_\varepsilon(t)$ is bounded in $\mathrm{L}^1(\mathbb{T})$, and combined estimate (3.4) without integrating in time yields that $\psi_\varepsilon(t)$ is a bounded sequence in $\mathrm{W}^{1,1}(\mathbb{T})$ and thus precompact in $\mathrm{L}^1(\mathbb{T})$. Thus the Arzelà–Ascoli Theorem yields the desired precompactness.

Step 3: The sequence u_ε converges to u in measure uniformly in time.

From step 2, we infer that we already have

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int |\psi_\varepsilon(t,x) - \phi\circ u(t,x)| \,\mathrm{d}x = 0.$$

It especially follows in combination with equation (3.5) that

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int (1-\chi)|\psi_\varepsilon(t,x)| \,\mathrm{d}x = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int (1-\chi)|\psi_\varepsilon(t,x) - \phi\circ u(t,x)| \,\mathrm{d}x = 0$$

and

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int \chi|\psi_\varepsilon(t,x) - \sigma| \,\mathrm{d}x = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{0\le t\le T} \int \chi|\psi_\varepsilon(t,x) - \phi\circ u(t,x)| \,\mathrm{d}x = 0$$

Moreover we have by continuity and the p-growth of W (2.3) that for a given $\delta < \beta - \alpha$,

$$\min\{|\phi(t)| : \ |t-\alpha| > \delta/2 \text{ or } |t-\beta| > \delta/2\} = \rho > 0.$$

Therefore we may estimate

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\{x\in\mathbb{T} : \ |u_\varepsilon(t,x) - u(t,x)| > \delta\}\right) \\
\le\ &\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\{(1-\chi(t,x))|u_\varepsilon(t,x) - \alpha| > \delta/2\}\right) + \mathcal{L}^d\left(\{\chi(t,x)|u_\varepsilon(t,x) - \beta| > \delta/2\}\right) \\
\le\ &\operatorname*{ess\,sup}_{0\le t\le T} \mathcal{L}^d\left(\{(1-\chi(t,x))|\psi_\varepsilon(t,x)| > \rho\}\right) + \mathcal{L}^d\left(\{\chi(t,x)|\psi_\varepsilon(t,x) - \sigma| > \rho\}\right) \\
\le\ &\operatorname*{ess\,sup}_{0\le t\le T} \frac{1}{\rho}\int (1-\chi(t,x))|\psi_\varepsilon(t,x)| \,\mathrm{d}x + \frac{1}{\rho}\int \chi(t,x)|\psi_\varepsilon(t,x) - \sigma| \,\mathrm{d}x\,,
\end{aligned}$$

which goes to zero as ε goes to zero, proving our claim.

Step 4: u_ε^2 is equiintegrable uniformly in time

We have

$$0 \le u_\varepsilon^2 \lesssim 1 + W(u_\varepsilon)$$

by the growth bounds (2.3) on W. Since $W(u_\varepsilon)$ converges to 0 in $\mathrm{L}^1(\mathbb{T})$ uniformly in time by the energy bound (3.3), it is equiintegrable uniformly in time. Thus u_ε^2 is equiintegrable uniformly in time as well.

Step 5: u_ε converges in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$.

We somewhat repeat the proof the convergence in measure and integrability imply L^1-convergence, while keeping the uniformity in time.

Take some $\delta > 0$. Then we decompose the integral

$$\int |u_\varepsilon - u|^2 \,\mathrm{d}x = \int_{\{|u_\varepsilon - u| \geq \delta\}} |u_\varepsilon - u|^2 \,\mathrm{d}x + \int_{\{|u_\varepsilon - u| < \delta\}} |u_\varepsilon - u|^2 \,\mathrm{d}x\,.$$

For the first summand, we notice that

$$\sup_{0\leq t\leq T} \int_{\{|u_\varepsilon - u| \geq \delta\}} |u_\varepsilon - u|^2 \,\mathrm{d}x \lesssim \sup_{0\leq t\leq T} \int_{\{|u_\varepsilon - u| \geq \delta\}} 1 + u_\varepsilon^2 \,\mathrm{d}x \to 0$$

as $\varepsilon \to 0$ since $\mathcal{L}^d\left(|u_\varepsilon - u| \geq \delta\right) \to 0$ uniformly in time by step 3 and $u_\varepsilon^2 + 1$ is equiintegrable uniformly in time by step 4.

For the second summand, we simply estimate

$$\sup_{0\leq t\leq T} \int_{\{|u_\varepsilon - u| < \delta\}} |u_\varepsilon - u|^2 \,\mathrm{d}x = \delta^2 \Lambda^d$$

Taking the limes superior as ε tends to zero of this inequality yields that the right hand side can be made arbitrarily small, which yields

$$\sup_{0\leq t\leq T} \int |u_\varepsilon - u|^2 \,\mathrm{d}x \to 0$$

as ε tends to zero.

□

Remark. From the previous Lemma, it follows that if the initial conditions u_ε^0 converge in L^1 or pointwise almost everywhere to the function $\alpha(1-\chi^0) + \beta\chi^0$ (and we know that the limit is of this form since the energies of the initial values stay bounded), then u also assumes this initial condition in $\mathrm{L}^2(\mathbb{T})$.

3.1.2 The energy convergence assumption

As often in the Calculus of Variations, convergence of energies boost our modulus of convergence and gives our limit therefore additional regularity. Thus let us assume

$$\int_0^T \mathrm{E}_\varepsilon(u_\varepsilon)\,\mathrm{d}t \to \int_0^T \mathrm{E}(u)\,\mathrm{d}t \ \text{ as } \varepsilon \to 0, \tag{3.7}$$

where the surface tension energy is defined for $u = \alpha(1-\chi) + \beta\chi$ by

$$\mathrm{E}(u) \coloneqq \sigma \int |\nabla \chi| \tag{3.8}$$

already motivated in Section 1.2. Moreover we notice that the energy of u is exactly the total variation of $\psi = \sigma\chi$. Since for almost every time t, we have that $\psi_\varepsilon(t)$ converges to $\psi(t)$ in $\mathrm{L}^1(\mathbb{T})$ by Proposition 3.1.1, it follows from the lower semicontinuity of the variation measure and Young's inequality that

$$\mathrm{E}(u) \leq \liminf_{\varepsilon\to 0} \int |\nabla\psi_\varepsilon| \,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon).$$

Moreover by the energy dissipation inequality (2.12), the energies stay uniformly bounded in time. Thus using the dominated convergence theorem, we see that our time integrated energy convergence assumption is equivalent to saying that for almost every time $t \in (0,T)$, we have convergence of the energies $\mathrm{E}_\varepsilon(u_\varepsilon(t)) \to \mathrm{E}(u(t))$.

Since the energies themselves can be interpreted as measures on the flat torus, we define for a continuous function $\varphi \in \mathrm{C}(\mathbb{T})$ the corresponding energy measures by

$$\begin{aligned}\mathrm{E}_\varepsilon(u_\varepsilon;\varphi) &:= \int \varphi\left(\frac{1}{\varepsilon} + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\right)\mathrm{d}x \text{ and}\\ \mathrm{E}(u;\varphi) &:= \sigma\int \varphi|\nabla\chi|.\end{aligned}$$

Then from the energy convergence and lower semicontinuity just discussed, it follows that

$$\lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) = \mathrm{E}(u;\varphi). \tag{3.9}$$

A first additional regularity result under the energy convergence assumption is the following Proposition, which ensures that we have square-integrable normal velocities.

Proposition 3.1.3. *In the setting of Proposition 3.1.1 and given the energy convergence assumption (3.7), the measure $\partial_t\chi$ is absolutely continuous with respect to the measure $|\nabla\chi|\,\mathrm{d}t$ and the corresponding density V is square integrable with the estimate*

$$\int_0^T \int V^2 |\nabla\chi| \,\mathrm{d}t \lesssim \mathrm{E}^0,$$

where $\mathrm{E}^0 := \liminf_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon^0)$.

Proof. Take a smooth test function $\varphi \in \mathrm{C}^\infty_\mathrm{C}((0,T)\times\mathbb{T})$. Then via the L^1-convergence of $\psi_\varepsilon \to \psi$, we have

$$\begin{aligned}
\partial_t \psi(\varphi) &= \liminf_{\varepsilon\to 0} \partial_t \psi_\varepsilon(\varphi) \\
&= \liminf_{\varepsilon\to 0} \int_0^T \int \sqrt{2W(u_\varepsilon)} \partial_t u_\varepsilon \varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\le \liminf_{\varepsilon\to 0} \left(\int_0^T \int \frac{1}{\varepsilon} 2W(u_\varepsilon)\varphi^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\int_0^T \int \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \\
&\le \liminf_{\varepsilon\to 0} \left(2 \int_0^T \mathrm{E}_\varepsilon \left(u_\varepsilon; \varphi^2\right) \mathrm{d}t \right)^{1/2} \left(\mathrm{E}_\varepsilon(u_\varepsilon)\right)^{1/2} && (2.12) \\
&= \sqrt{2\sigma} \|\varphi\|_{\mathrm{L}^2(\mathbb{T}, |\nabla\chi| \mathrm{d}t)} \sqrt{\mathrm{E}^0}. && (3.9)
\end{aligned}$$

This proves both the absolute continuity and via a duality argument the desired bound since $\partial_t \psi = \sigma \partial_t \chi$ and $\sigma > 0$. □

We finish this section with a proof for the equipartition of the energy, which tells us that both the summand involving the potential $W(u_\varepsilon$ and the norm of the gradient contribute to the energy in similar parts.

Lemma 3.1.4. *Under the energy convergence assumption (3.7), we have for any continuous function $\varphi \in \mathrm{C}^\infty(\mathbb{T})$ that*

$$\begin{aligned}
\mathrm{E}(u;\varphi) = \lim_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) &= \lim_{\varepsilon\to 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int \varphi \varepsilon |\nabla u_\varepsilon|^2 \,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int \varphi \frac{1}{\varepsilon} 2W(u_\varepsilon) \,\mathrm{d}x
\end{aligned}$$

for almost every time $0 \le t \le T$.

Proof. We have already established the first equality before. For the second equality, we first assume that $\varphi \in \mathrm{C}(\mathbb{T}; [0,\infty))$. By the lower semicontinuity of the variation measure, we immediately obtain

$$\liminf_{\varepsilon\to 0} \int \varphi \sqrt{2W(u_\varepsilon} |\nabla u_\varepsilon| \,\mathrm{d}x \ge \mathrm{E}(u;\varphi).$$

But by Young's inequality, we also have

$$\limsup_{\varepsilon\to 0} \int \varphi \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon| \,\mathrm{d}x \le \limsup_{\varepsilon\to 0} \mathrm{E}_\varepsilon(u_\varepsilon;\varphi) = \mathrm{E}(u;\varphi).$$

For general $\varphi \in \mathrm{C}(\mathbb{T})$, we decompose φ into its positive and negative part and apply the previous argument to both in order to get the claim.

The third and fourth inequality follow for a given non-negative $\varphi \in \mathrm{C}(\mathbb{T};[0,\infty))$ by the L^2 estimate

$$\begin{aligned}&\lim_{\varepsilon\to0}\int\left|\sqrt{\varphi}\sqrt{\varepsilon}|\nabla u_\varepsilon| - \sqrt{\varphi}\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right|^2\mathrm{d}x\\ &=\lim_{\varepsilon\to0} 2\,\mathrm{E}_\varepsilon(u_\varepsilon;\varphi) - 2\int\varphi\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,\mathrm{d}x = 0\end{aligned}$$

which implies

$$\lim_{\varepsilon\to0}\int\varphi\varepsilon|\nabla u_\varepsilon|^2\,\mathrm{d}x = \lim_{\varepsilon\to0}\int\varphi\frac{1}{\varepsilon}2W(u_\varepsilon)\,\mathrm{d}x = \lim_{\varepsilon\to0}\mathrm{E}_\varepsilon(u_\varepsilon;\varphi),$$

finishing our proof. □

3.1.3 Convergence to twophase mean curvature flow

We start by definining a BV-formulation for motion by mean curvature.

Definition 3.1.5. *Fix some finite time horizon $T<\infty$ and initial data $\chi^0 \in \mathrm{BV}(\mathbb{T};\{0,1\})$, we say that*

$$\chi \in \mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T};\{0,1\})\right)$$

with $\operatorname{ess\,sup}_{0\le t\le T}\mathrm{E}(\chi)$ moves by mean curvature *if there is a normal velocity $V\in\mathrm{L}^2(|\nabla\chi|\,\mathrm{d}t)$ such that*

1. *For all $\xi\in\mathrm{C}^\infty_\mathrm{C}\left((0,T)\times\mathbb{T};\mathbb{R}^d\right)$, we have*
$$\int_0^T\int V\langle\xi\,,\,\nu\rangle - \langle\mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu\otimes\nu\rangle|\nabla\chi|\,\mathrm{d}t = 0, \tag{3.10}$$
where $\nu := \nabla\chi/|\nabla\chi|$ is the outer unit normal.
2. *The function V is the normal velocity of χ in the sense that*
$$\partial_t\chi = V|\nabla\chi|\,\mathrm{d}t$$
holds distributionally in $(0,T)\times\mathbb{T}$.
3. *The initial data χ^0 is achieved in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$, which simply means that $\chi(0)=\chi^0$ as functions in $\mathrm{L}^2(\mathbb{T})$.*

Our main goal in this section is now to show that the function χ we have found in Proposition 3.1.1 moves my mean curvature. Thus our goal is to prove the following Theorem.

Theorem 3.1.6. *Let a smooth doublewell potential $W\colon\mathbb{R}\to[0,\infty)$ satisfy the assumptions (2.3)-(2.6). Let $T<\infty$ be an arbitrary finite time horizon. Given a sequence of initial data $u^0_\varepsilon\colon\ \mathbb{T}\to\mathbb{R}$ such that $u^0_\varepsilon\to u^0=(1-\chi^0)\alpha+\chi^0\beta$ and $\limsup_{\varepsilon\to0}\mathrm{E}_\varepsilon(u^0_\varepsilon)<\infty$,* *we have that that for some subsequence of solutions to (2.1) u_ε, there exists a pointwise almost everywhere limit $u = (1-\chi)\alpha + \chi\beta$ with $\chi \in \mathrm{BV}\left((0,T)\times\mathbb{T};\{0,1\}\right)$ which assumes the initial data in $\mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T})\right)$. If we additionally assume that the time-integrated energies converge (3.7), then χ moves by mean curvature in the sense of Definition 3.1.5.*

Looking at equation (2.9) which reads

$$\int \frac{1}{\varepsilon^2} W'(u_\varepsilon(t))\varphi + \nabla u_\varepsilon(t)\nabla\varphi + \partial_t u_\varepsilon(t)\varphi \,\mathrm{d}x = 0, \tag{2.9}$$

we expect that for a suitable choice of testfunctions φ_ε, the following two terms converge:

$$\lim_{\varepsilon\to 0}\int_0^T\int \partial_t u_\varepsilon \varphi_\varepsilon = \sigma\int_0^T\int V\langle \xi\,,\,\nu\rangle|\nabla\chi|\,\mathrm{d}t\,,$$
$$\lim_{\varepsilon\to 0}\int_0^T\int \frac{1}{\varepsilon^2}W'(u_\varepsilon)\varphi_\varepsilon + \langle\nabla u_\varepsilon\,,\,\nabla\varphi_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t = \sigma - \int_0^T\int\langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu\otimes\nu\rangle|\nabla\chi|\,\mathrm{d}t\,.$$

But how do we find these testfunctions? For this, we first note that the curvature term $\int\langle \mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu\otimes\nu\rangle|\nabla\chi|\,\mathrm{d}t$ is by [Mag12, Thm. 17.5] the first inner variation with respect to ξ of the perimeter functional, which is just our energy E up to the surface tension constant $\sigma > 0$. Thus it is plausible to compute the first inner variation $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} E(\rho_s)$ and then we can hopefully choose the testfunction φ_ε in such a way that it equals $\int \frac{1}{\varepsilon^2}W'(u_\varepsilon(t))\varphi + \nabla u_\varepsilon(t)\nabla\varphi\,\mathrm{d}x$.

Thus let $(\rho_s)_s$ be functions which solve the ODE

$$\begin{cases}\partial_s\rho_s + \langle\xi\,,\,\nabla\rho_s\rangle & = 0\\ \rho_0 & = u_\varepsilon.\end{cases}$$

Then we formally compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\int\frac{\varepsilon}{2}|\rho_s|^2 + \frac{1}{\varepsilon}W(\rho_s)\,\mathrm{d}x &= \int \varepsilon\langle\nabla u_\varepsilon\,,\,\nabla\left(-\langle\xi\,,\,\nabla u_\varepsilon\rangle\right)\rangle + \frac{1}{\varepsilon}W'(u_\varepsilon)(-\langle\xi\,,\,\nabla u_\varepsilon\rangle)\,\mathrm{d}x\\ &= \int\left(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle\xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x\,.\end{aligned}$$

We therefore test equation (2.9) against $\varphi_\varepsilon \coloneqq \langle\xi\,,\,\nabla u_\varepsilon\rangle$.

3.1.4 Convergence of the curvature term

The goal of this section is to prove the convergence

$$\lim_{\varepsilon\to 0}\int\left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle\xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x = \sigma\int\langle\mathrm{D}\,\xi\,,\,\mathrm{Id}-\nu\otimes\nu\rangle|\nabla\chi|$$

for almost every time t. We directly follow the proof from Luckhaus and Modica in [LM89].

Via an integration by parts, we obtain

$$\int \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle \xi\,,\,\nabla u_\varepsilon\rangle = \int -\,\varepsilon\sum_{i,j=1}^{d}\partial_{x_i}u_\varepsilon\left(\partial_{x_i}\xi^j\partial_{x_j}u_\varepsilon + \xi^j\partial^2_{x_ix_j}u_\varepsilon\right) + \frac{1}{\varepsilon}W(u_\varepsilon)\operatorname{div}\xi\,\mathrm{d}x\,. \tag{3.11}$$

Moreover by another integration by parts, we have

$$\begin{aligned}\int\sum_{i,j=1}^{d}\partial_{x_i}u_\varepsilon\xi^j\partial^2_{x_ix_j}u_\varepsilon\,\mathrm{d}x &= \int -\sum_{i,j=1}^{d}\partial_{x_i}u_\varepsilon\left(\partial^2_{x_ix_j}u_\varepsilon\xi^j + \partial_{x_j}\xi^j\partial_{x_i}u_\varepsilon\right)\mathrm{d}x\\ &= \int -|\nabla u_\varepsilon|^2\operatorname{div}\xi - \sum_{i,j=1}^{d}\partial_{x_i}u_\varepsilon\partial^2_{x_ix_j}u_\varepsilon\xi^j\,\mathrm{d}x\end{aligned}$$

which is equivalent to

$$\int\sum_{i,j=1}^{d}\partial_{x_i}u_\varepsilon\xi^j\partial^2_{x_ix_j}u_\varepsilon\,\mathrm{d}x = -\frac{1}{2}\int|\nabla u_\varepsilon|^2\operatorname{div}\xi\,\mathrm{d}x\,.$$

Plugging this equation into the first equation (3.11), we obtain

$$\begin{aligned}&\int\left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\langle\xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x\\ &= \int -\varepsilon\sum_{i,j=1}^{d}\partial_{x_i}u_\varepsilon\partial_{x_j}u_\varepsilon\partial_{x_i}\xi^j + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\operatorname{div}\xi + \frac{1}{\varepsilon}W(u_\varepsilon)\operatorname{div}\xi\,\mathrm{d}x \qquad (3.12)\\ &= \varepsilon\int|\nabla u_\varepsilon|^2\operatorname{div}\xi - \sum_{i,j=1}^{d}\partial_{x_i}u_\varepsilon\partial_{x_j}u_\varepsilon\partial_{x_i}\xi^j\,\mathrm{d}x\\ &\quad + \int\frac{1}{\varepsilon}W(u_\varepsilon)\operatorname{div}\xi - \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\operatorname{div}\xi\,\mathrm{d}x\,.\end{aligned}$$

The last integral goes to zero by the equipartition of the energies (Lemma 3.1.4). Since $\partial_{x_i}u_\varepsilon/|\nabla u_\varepsilon| = \partial_{x_i}\psi_\varepsilon/|\nabla\psi_\varepsilon|$ by the chain rule, the former integral can be written as

$$\int_{\mathbb{T}_\varepsilon} g(x,\nabla\psi_\varepsilon)\varepsilon|\nabla u_\varepsilon|^2\,\mathrm{d}x\,,$$

where

$$g(x,p) = \begin{cases}\sum_{i,j=1}^{d} -\frac{p_i}{|p|}\partial_{x_i}\xi^j\frac{p_j}{|p|} + \operatorname{div}\xi & \text{if } p\neq 0,\\ 0 & \text{else,}\end{cases}$$

and the set $\mathbb{T}_\varepsilon$ is defined as

$$\mathbb{T}_\varepsilon \coloneqq \{x\in\mathbb{T}:\ \nabla\psi_\varepsilon(x)\neq 0\} = \{x\in\mathbb{T}:\ \nabla u_\varepsilon(x)\neq 0\}\cap\{x\in\mathbb{T}:\ u_\varepsilon(x)\notin\{\alpha,\beta\}\}\,.$$

For the representation (3.1.4), we also have to use that

$$\mathcal{L}^d\left(\{x \in \mathbb{T} :\ \nabla u_\varepsilon(x) \neq 0 \text{ and } u_\varepsilon(x) \in \{\alpha, \beta\}\}\right) = 0.$$

Again by the equipartition of energies (Lemma 3.1.4) and the boundedness of g, we can replace $\varepsilon|\nabla u_\varepsilon|^2$ by $\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|$ in the integral (3.1.4) via the estimate

$$\begin{aligned}
&\int_{\mathbb{T}_\varepsilon} \left|\varepsilon|\nabla u_\varepsilon|^2 - \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\right| \mathrm{d}x \\
&\leq \left(\int_{\mathbb{T}_\varepsilon} \left|\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right|^2 \mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{T}_\varepsilon} \varepsilon|\nabla u_\varepsilon|^2 \,\mathrm{d}x\right)^{1/2} \\
&\leq \left(\int \varepsilon|\nabla u_\varepsilon|^2 - 2\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon| + \frac{1}{\varepsilon}2W(u_\varepsilon)\,\mathrm{d}x\right) \sqrt{2\,\mathrm{E}_\varepsilon(u_\varepsilon)}
\end{aligned}$$

which vanishes as ε tends to zero by Lemma 3.1.4. Thus

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\mathbb{T}_\varepsilon} g(x, \nabla\psi_\varepsilon)\varepsilon|\nabla u_\varepsilon|^2 \,\mathrm{d}x &= \lim_{\varepsilon\to 0} \int_{\mathbb{T}_\varepsilon} g(x, \nabla\psi_\varepsilon)\sqrt{2W(u_\varepsilon)}|u_\varepsilon| \,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{T}_\varepsilon} g(x, \nabla\psi_\varepsilon)|\nabla\psi_\varepsilon| \,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int F(x, \nabla\psi_\varepsilon) \,\mathrm{d}x\,,
\end{aligned}$$

where $F(x, p)$ is defined as $g(x, p)|p|$ at points p not equal to 0, and defined as 0 elsewhere. Since $F(x, \lambda p) = \lambda F(x, p)$ for positive λ and since F satisfies the periodic boundary conditions in x, we are in the position the apply a Theorem proven by Reshetnyak in [Res68] and again by Luckhaus and Modica in [LM89]. We will later see a quantitative version of this in **??**. Here it yields that since $|\nabla\psi_\varepsilon|(\mathbb{T}) \to |\nabla\psi|(\mathbb{T})$ by the equipartition of energies Lemma 3.1.4, we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int F(x, \nabla\psi_\varepsilon) \,\mathrm{d}x = \sigma \int F(x, \nu)|\nabla\chi| \\
&= \sigma \int \left(\sum_{i,j=1}^{d} -\frac{\nu_i}{|\nu|}\partial_{x_i}\xi^j \frac{\nu_j}{|\nu|} + \operatorname{div}\xi\right) |\nu||\nabla\chi| \\
&= \sigma \int \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - \nu\otimes\nu\rangle|\nabla\chi|,
\end{aligned}$$

which finishes the proof. The time integrated version given by

$$\lim_{\varepsilon\to 0} \int_0^T \int \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right) \langle \xi\,,\, \nabla u_\varepsilon\rangle \,\mathrm{d}x\,\mathrm{d}t = \sigma \int_0^T \int \langle \mathrm{D}\,\xi\,,\, \mathrm{Id} - \nu\otimes\nu\rangle|\nabla\chi| \,\mathrm{d}t$$

follows from the generalized dominated convergence theorem via the equality (3.12) which yields

$$\left|\int \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right) \langle \xi\,,\, \nabla u_\varepsilon\rangle \,\mathrm{d}x\right| \lesssim \mathrm{E}_\varepsilon(u_\varepsilon).$$

3.1.5 Convergence of the velocity term

We now want to prove the convergence of the velocity term given by

$$\lim_{\varepsilon\to 0}\int_0^T\int \partial_t u_\varepsilon\langle\xi\,,\,\varepsilon\nabla u_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t=\sigma\int_0^T\int V\langle\xi\,,\,\nu\rangle|\nabla\chi|\,\mathrm{d}t\,.$$

The difficulty here is that products of weakly converging sequences will in general not weakly converge. To be more precise, we only have $\partial_t u_\varepsilon \rightharpoonup V|\nabla\chi|\,\mathrm{d}t$ and $\nu_\varepsilon := \varepsilon\nabla u_\varepsilon \approx \nu$ in a weak sense.

Therefore we try to freeze the normal in a fixed direction, apply the weak convergence of $\partial_t u_\varepsilon$ and then unfreeze the normal. Freezing the approximate normal $\varepsilon\nabla u_\varepsilon$ amounts to replacing $\varepsilon\nabla u_\varepsilon$ by $\varepsilon|\nabla u_\varepsilon|\nu^*$ for a suitably chosen $\nu^*\in\mathbb{S}^{d-1}$. Let η be a cutoff on the support of ξ. Then the error we make can be estimated for all $\alpha>0$ via Young's inequality by

$$\begin{aligned}
&\left|\int_0^T\int \partial_t u_\varepsilon\langle\xi\,,\,\nabla u_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int \partial_t u_\varepsilon\langle\xi\,,\,\varepsilon|\nabla u_\varepsilon|\nu^*\rangle\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\le \|\xi\|_{\mathrm{L}^\infty}\int_0^T\int \eta\sqrt{\varepsilon}|\partial_t u_\varepsilon|\sqrt{\varepsilon}|\nabla u_\varepsilon-|\nabla u_\varepsilon|\nu^*|\,\mathrm{d}x\,\mathrm{d}t\\
&\le \|\xi\|_{\mathrm{L}^\infty}\left(\alpha\int_0^T\int\eta\varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\alpha}\int_0^T\int\eta\varepsilon|\nabla u_\varepsilon|^2|\nu_\varepsilon-\nu^*|\,\mathrm{d}x\,\mathrm{d}t\right)\\
&\le \|\xi\|_{\mathrm{L}^\infty}\left(\alpha\int_0^T\int\eta\varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\alpha}\,\mathcal{E}_\varepsilon(\nu^*;\eta)\right).
\end{aligned}$$

Here the approximate tilt excess in direction ν^* is given by

$$\mathcal{E}_\varepsilon(\nu^*;\eta):=\int_0^T\int\varepsilon|\nabla u_\varepsilon|^2|\nu_\varepsilon-\nu^*|^2\,\mathrm{d}x\,\mathrm{d}t\,.$$

With our frozen normal, we now notice that via the equipartition of energies, we may replace $\varepsilon|\nabla u_\varepsilon|$ by $\sqrt{2W(u_\varepsilon)}$ via the estimate

$$\begin{aligned}
&\int_0^T\int|\partial_t u_\varepsilon|\eta\left|\varepsilon|\nabla u_\varepsilon|-\sqrt{2W(u_\varepsilon)}\right|\mathrm{d}x\,\mathrm{d}t\\
&\le\left(\int_0^T\int\varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int\eta^2\left(\varepsilon|\nabla u_\varepsilon|^2-2|\nabla u_\varepsilon|\sqrt{2W(u_\varepsilon)}+\frac{1}{\varepsilon}2W(u_\varepsilon)\right)\mathrm{d}x\,\mathrm{d}t\right)^{1/2}
\end{aligned}$$

The first factor is uniformly bounded by the energy dissipation inequality (2.12) and the second term vanishes as ε tends to zero by the equipartition of energies Lemma 3.1.4. But now we recognise the identity

$$\int_0^T\int\partial_t u_\varepsilon\sqrt{2W(u_\varepsilon)}\langle\xi\,,\,\nu^*\rangle\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int\partial_t\psi_\varepsilon\langle\xi\,,\,\nu^*\rangle\,\mathrm{d}x\,\mathrm{d}t\,,$$

which converges as ε approaches zero to

$$\int_0^T \int \langle \xi \,,\, \nu^* \rangle \partial_t \psi = \sigma \int_0^T \int V \langle \xi \,,\, \nu^* \rangle |\nabla \chi| \, \mathrm{d}t \,. \tag{3.13}$$

Finally we want to unfreeze the normal, which means that we want to replace ν^* by ν on the right hand side of equation (3.13). This can be estimated again by Young's inequality via the error

$$\|\xi\|_{\mathrm{L}^\infty} \left(\alpha \int_0^T \int \eta V^2 |\nabla \chi| \, \mathrm{d}t + \frac{1}{\alpha} \, \mathcal{E}(\nu^*; \eta) \right),$$

where the tilt excess is given by

$$\mathcal{E}(\nu^*; \eta) := \sigma \int_0^T \int \eta |\nu - \nu^*|^2 |\nabla \chi| \, \mathrm{d}t \,.$$

This finishes the proof for the convergence of the velocity term up to arguing that the errors can be made arbitrarily small.

First we study the behaviour of the approximate tilt excess $\mathcal{E}_\varepsilon$ by connecting it to $\mathcal{E}$. We notice that by expansion, we have

$$\begin{aligned}
\mathcal{E}_\varepsilon(\nu^*; \eta) &= 2 \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t - 2 \left\langle \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \nabla u_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \;,\; \nu^* \right\rangle \\
\mathcal{E}(\nu^*; \eta) &= 2 \, \mathrm{E}(u; \eta) - 2\sigma \left\langle \int_0^T \int \eta \nu |\nabla \chi| \, \mathrm{d}t \;,\; \nu^* \right\rangle.
\end{aligned}$$

But by the equipartition of energies Lemma 3.1.4, we have that

$$2 \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \to 2 \, \mathrm{E}(u; \eta)$$

and recognizing that $\sigma \int_0^T \int \eta \nu |\nabla \chi| \, \mathrm{d}t = \int_0^T \int \eta \nabla \psi$, we also obtain

$$\begin{aligned}
&\left| \int_0^T \int \eta \varepsilon |\nabla u_\varepsilon| \nabla u_\varepsilon \, \mathrm{d}x \, \mathrm{d}t - \sigma \int_0^T \int \eta \nu |\nabla \chi| \, \mathrm{d}t \right| \\
&\leq \left| \int_0^T \int \eta \sqrt{2W(u_\varepsilon)} \nabla u_\varepsilon \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int \eta \nabla \psi \right| + \int_0^T \int \varepsilon |\nabla u_\varepsilon| \left| \nabla u_\varepsilon - \frac{1}{\varepsilon} \sqrt{2W(u_\varepsilon)} \right| \mathrm{d}x \, \mathrm{d}t \\
&\leq \left| \int_0^T \int \eta \nabla \psi_\varepsilon \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int \eta \nabla \psi \right| \\
&\quad + \left(\int_0^T \int \varepsilon |\nabla u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \left(\int_0^T \int \left(\sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right)^2 \mathrm{d}x \, \mathrm{d}t \right)^{1/2},
\end{aligned}$$

which vanishes as ε tends to zero by the weak convergence of $\nabla\psi_\varepsilon \rightharpoonup^* \nabla\psi$ and the equipartition of the energies. Since we are taking $\varepsilon \to 0$, we thus only have to make sure that the tilt excess is sufficiently small since the approximate tilt excess converges to the tilt excess.

We are now in the position to argue why the error can be made arbitrarily small. Let $\delta > 0$. Then we first choose our $\alpha > 0$ so small that

$$\limsup_{\varepsilon\to 0} \alpha\|\xi\|_{\mathrm{L}^\infty}\left(\int_0^T\int \varepsilon|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int V^2|\nabla\chi|\,\mathrm{d}t\right) < \frac{\delta}{2},$$

which is possible by the energy dissipation inequality (2.12) and Proposition 3.1.3. Then we choose a partition of unity $(\eta_i)_{i=1,\dots,n}$ and approximate unit normals $(\nu_i^*)_{i=1,\dots,n}$ such that

$$\frac{2}{\alpha}\|\xi\|_{\mathrm{L}^\infty}\sigma\sum_{i=1}^n\int_0^T\int \eta_i|\nu-\nu_i^*|^2|\nabla\chi|\,\mathrm{d}t < \frac{\delta}{2}.$$

The existence of these can be seen by taking a smooth approximation of ν with respect to the measure $|\nabla\chi|\,\mathrm{d}t$. Collecting all of our errors, we obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left|\int_0^T\int \partial_t u_\varepsilon\langle\xi\,,\,\varepsilon\nabla u_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^T\int V\langle\xi\,,\,\nu\rangle|\nabla\chi|\,\mathrm{d}t\right|\\
&= \limsup_{\varepsilon\to 0}\left|\sum_{i=1}^n\int_0^T\int \eta_i\partial_t u_\varepsilon\langle\xi\,,\,\varepsilon\nabla u_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^T\int \eta_i V\langle\xi\,,\,\nu\rangle|\nabla\chi|\,\mathrm{d}t\right|\\
&\le \limsup_{\varepsilon\to 0}\|\xi\|_{\mathrm{L}^\infty}\left(\sum_{i=1}^n\alpha\int_0^T\int \eta_i\varepsilon|\partial_t u_\varepsilon|^2\,\mathrm{d}x - \int \eta_i V^2|\nabla\chi|\,\mathrm{d}t + \frac{2}{\alpha}\,\mathcal{E}(\nu_i^*;\eta_i)\right) < \delta,
\end{aligned}$$

which finishes the proof.

3.2 Conditional convergence in the multiphase case

Let us now turn to the much more interesting and more challenging case where we consider systems of the Allen–Cahn equation (2.1), or in other words, we want to consider the case where u_ε maps to $\mathbb{R}^N$ and therefore our potential W is a map from $\mathbb{R}^N$ to $[0,\infty)$. The for us most relevant case is when W has exactly $P = N+1$ zeros given by $\alpha_1,\dots,\alpha_P$, but it is no limitation for us to allow more general amount of zeros.

One of the many difficulties in the vectorial case is that that there is no easy choice of a primitive for $\sqrt{2W(u)}$ compared to the scalar case. We there saw for example through the Modica-Mortula trick (3.1) that this provided a very powerful tool for us, and comparing the composition $\phi\circ u_\varepsilon$ to u_ε was quite simple since the map ϕ was invertible as a consequence of the non-negativity of W.

As a suitable replacement, we shall consider the *geodesic distance* defined as

$$\mathrm{d_W}(u,v) \coloneqq \inf\left\{\int_0^1\sqrt{2W(\gamma)}|\dot\gamma|\,\mathrm{d}t \,:\, \gamma\in\mathrm{C}^1\left([0,1];\mathbb{R}^N\right) \text{ with } \gamma(0)=u,\ \gamma(1)=v\right\}.$$

This indeed defines a metric on $\mathbb{R}^N$: If $\mathrm{d_W}(v, w) = 0$, then by the continuity of W and since it only has a discrete set of zeros, we may deduce that $v = w$. Symmetry can be seen by reversing a given path between two points and the triangle inequality follows from concatenation (and smoothing) of two paths and rescaling. Note moreover that by an approximation argument (for example through splines), we may also take paths γ which are piecewise continuously differentiable which makes constructions of paths easier.

The *geodesic distances* generated by W are defined as

$$\sigma_{i,j} \coloneqq \mathrm{d_W}(\alpha_i, \alpha_j)$$

and as a consequence of $\mathrm{d_W}$ being a metric satisfy

$$\sigma_{i,k} \leq \sigma_{i,j} + \sigma_{j,k},$$

$\sigma_{i,j} = 0$ if and only if i is equal to j and $\sigma_{i,j} = \sigma_{j,i}$.

Our replacement for the primitive ϕ is now given for $1 \leq i \leq P$ by the function

$$\phi_i(u) \coloneqq \mathrm{d_W}(\alpha_i, u).$$

Our first obstacle is the regularity of the function $\psi_\varepsilon^i \coloneqq \phi_i \circ u_\varepsilon$. A priori we only know that ϕ_i is locally Lipschitz continuous on $\mathbb{R}^N$ and thus differentiable almost everywhere. If $N = 1$, then this would already suffice to deduce that ψ_ε^i is weakly differentiable, but in higher dimensions, u could for example move along a hyperplane where ϕ_i could in theory be nowhere differentiable since it is a Lebesgue nullset. This can however be salvaged through the following chain rule for distributional derivatives by Ambrosio and Maso [AM90, Cor. 3.2].

Theorem 3.2.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set, $p \in [0, \infty]$, $u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^N)$ and let $f \colon \mathbb{R}^N \to \mathbb{R}^k$ be a Lipschitz continuous function such that $f(0) = 0$. Then $v \coloneqq f \circ u \in \mathrm{W}^{1,p}\left(\Omega, \mathbb{R}^k\right)$. Furthermore for almost every $x \in \Omega$ the restriction of f to the affine space*

$$T_x^u \coloneqq \left\{ y \in \mathbb{R}^N : \ y = u(x) + \mathrm{D}\, u(x) z \ \text{for } z \in \mathbb{R}^d \right\} = u(x) + \dot{T}_x^u$$

is differentiable at $u(x)$ and

$$\mathrm{D}\, v = \mathrm{D}\left(f|_{T_x^u}\right)(u)\, \mathrm{D}\, u$$

holds almost everywhere in Ω.

Remark. The matrix $\mathrm{D}\left(f|_{T_x^u}\right)(u)$ can be interpreted as some matrix in $\mathbb{R}^{k \times n}$ which acts on $v \in \dot{T}_x^u$ by

$$\mathrm{D}\left(f|_{T_x^u}\right)(u(x))[v] = \lim_{h \to 0} \frac{f(u(x) + hv) - f(u(x))}{h}.$$

Thus we may choose a suitable representative since the product $\mathrm{D}\left(f|_{T_x^u}\right)(u)\, \mathrm{D}\, u$ will not change by definition of $\dot{T}_x^u$.

Moreover the assumption $f(0) = 0$ can be left out on bounded domains by simply subtracting the constant $f(0)$.

We are not in the position to prove the following regularity result.

Lemma 3.2.2. *Let* $u \in \mathrm{C}\left([0,T];\mathrm{L}^2(\mathbb{T};\mathbb{R}^N)\right)$ *with*

$$\operatorname*{ess\,sup}_{0\le t\le T} \mathrm{E}_\varepsilon(u) + \int_0^T \int \varepsilon|\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty$$

for some $\varepsilon > 0$. *Then for all* $1 \le i \le P$ *there exists a map*

$$\partial_u \phi_i(u)\colon [0,T]\times\mathbb{T} \to \operatorname{Lin}\left(\mathbb{R}^N,\mathbb{R}\right)$$

such that the chain rule is valid with the pair $\partial_u\phi_i(u)$ *and* $(\partial_t,\nabla u)u$*:*
For almost every $(t,x)\in[0,T]\times\mathbb{T}$ *we have*

$$\nabla(\phi_i\circ u) = \partial_u\phi_i(u)\nabla u \quad \text{and} \quad \partial_t(\phi_i\circ u) = \partial_u\phi_i(u)\partial_t u.$$

Furthermore we can control the modulus of $\partial_u\phi_i(u)$ *almost everywhere in time and space via the estimate*

$$|\partial_u\phi_i(u)| \le \sqrt{2W(u)}. \tag{3.14}$$

Additionally we have $\phi_i\circ u \in \mathrm{L}^\infty\left([0,T];\mathrm{W}^{1,1}(\mathbb{T})\right)\cap\mathrm{W}^{1,1}([0,T]\times\mathbb{T})$ *with the estimates*

$$\operatorname*{ess\,sup}_{0\le t\le T}\int |\phi_i\circ u|\,\mathrm{d}x \lesssim 1 + \operatorname*{ess\,sup}_{0\le t\le T}\varepsilon\,\mathrm{E}_\varepsilon(u) \tag{3.15}$$

$$\operatorname*{ess\,sup}_{0\le t\le T}\int |\nabla(\phi_i\circ u)|\,\mathrm{d}x \le \operatorname*{ess\,sup}_{0\le t\le T}\mathrm{E}_\varepsilon(u) \tag{3.16}$$

and

$$\int_0^T\int |\partial_t(\phi_i\circ u)|\,\mathrm{d}x\,\mathrm{d}t \lesssim T\operatorname*{ess\,sup}_{0\le t\le T}\mathrm{E}_\varepsilon(u) + \int_0^T\int \varepsilon|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t\,. \tag{3.17}$$

Proof. Since Theorem 3.2.1 requires Lipschitz continuity of ϕ_i, but we only have local Lipschitz continuity, let us first assume that u is bounded in space and time. Then we may modify ϕ_i outside of a compact set such that it is (globally) Lipschitz continuous and does not change on the image of u.

Since we have via the energy estimate that $u\in\mathrm{W}^{1,2}([0,T]\times\mathbb{T};\mathbb{R}^N)$, we obtain by the distributional chain rule Theorem 3.2.1 that $\psi = \phi_i\circ u\in\mathrm{W}^{1,2}([0,T]\times\mathbb{T})$.

Let $\Pi(t,x)$ denote the orthogonal projection of $\mathbb{R}^N$ onto $\dot{T}^u_{x,t}$ and define

$$\partial_u\phi_i(u)(t,x)[v] \coloneqq \mathrm{D}(\phi_i|_{T^u_{t,x}})(u(t,x))[\Pi(t,x)v].$$

This defines a unique row vector and thus we can now proceed to prove inequality (3.14). Let $v\in\dot{T}^u_{t,x}$, $h\mathbb{R}\setminus\{0\}$ and $\gamma\colon[0,1]\to\mathbb{R}^N$ be a path connecting α_i and u. Then we define the new path $\tilde\gamma\colon[0,1]\to\mathbb{R}^N$ by

$$\tilde\gamma(t) = \begin{cases}\gamma(\frac{t}{2}) & ,t\le\frac12\\ u+\left(t-\frac12\right)2hv & ,t\ge\frac12.\end{cases}$$

We observe that $\tilde{\gamma}$ is a piecewise continuously differentiable path connecting α_i and $u+hv$, thus we can estimate by a substitution that

$$\begin{aligned}\mathrm{d_W}(\alpha_i, u+hv) - \int_0^1 \sqrt{2W(\gamma(t)}|\gamma'(t)|\,\mathrm{d}t &\le \int_0^1 \sqrt{2W(\tilde{\gamma}(t)}|\tilde{\gamma}'(t)|\,\mathrm{d}t - \int_0^1 \sqrt{2W(\gamma(t)}|\gamma'(t)|\,\mathrm{d}t \\ &= \int_0^1 \sqrt{2W(u+thv)}|hv|\,\mathrm{d}t\,.\end{aligned}$$

Taking the infimum over all C^1-paths connecting α_i and u yields

$$\mathrm{d_W}(\alpha_i, u+hv) - \mathrm{d_W}(\alpha_i, u) \le \int_0^1 \sqrt{2W(u+thv)}|hv|\,\mathrm{d}t\,.$$

Using a similar strategy but with a reversed path, we also obtain the inequality

$$\mathrm{d_W}(\alpha_i, u) - \mathrm{d_W}(\alpha_i, u+hv) \le \int_0^1 \sqrt{2W(u+thv)}|hv|\,\mathrm{d}t\,,$$

thus we obtain by the dominated convergence theorem

$$\limsup_{h\to 0}\left|\frac{\phi(u+hv)-\phi(u)}{h}\right| \le \limsup_{h\to 0}\int_0^1 \sqrt{2W(u+thv)}|v|\,\mathrm{d}t = \sqrt{2W(u)}|v|,$$

which yields

$$\left|\mathrm{D}\,\phi_i|_{T^u_{t,x}}(u)[v]\right| \le \sqrt{2W(u)}|v|$$

and thus gives us the desired inequality (3.14) since $|\Pi(v)| \le |v|$.

Now let us consider the general case and denote by u_M the truncation of u defined by

$$u^j_M := \begin{cases} u & \text{, if}\left|u^j\right| \le M \\ M\frac{u^j}{|u_j|} & \text{, else.}\end{cases}$$

Then we still have $u_M \in \mathrm{W}^{1,2}([0,T]\times\mathbb{T})$ and obtain by the previous step that $\phi_i \circ u_M \in \mathrm{W}^{1,2}([0,T]\times\mathbb{T})$ and that for almost every $(t,x)\in[0,T]\times\mathbb{T}$, the function ϕ_i is differentiable on $T^{u_M}_{t,x}$. Moreover if $(t,x) \in u_{-1}([-M,M]^N)$, then we obtain $T^{u_M}_{t,x} = T^u_{t,x}$. Next we we want to show that $\phi_i \circ u_M$ converges to $\phi_i \circ u$ in a suitable sense. First we recognize that $\phi_i \circ u_M$ converges to $\phi_i \circ u$ pointwise almost everywhere. Moreover we find a majorant since

$$\begin{aligned}\phi_i(v) &\le \int_0^1 \sqrt{2W(\alpha_i + s(v-\alpha_i))}|v-\alpha_i|\,\mathrm{d}s \le \left\|\sqrt{2W}\right\|_{\mathrm{L}^\infty[v,\alpha_i]}|\alpha_i - v| \\ &\lesssim 1+|v|^{1+p/2},\end{aligned}$$

thus we have $\phi_i \circ u_M \lesssim 1+|u|^p$, which is in integrable majorant. Thus the dominated convergence theorem yields that $\phi_i \circ u_M$ converges to $\phi_i \circ u$ in $\mathrm{L}^1([0,T]\times\mathbb{T})$. ☐

4 i dont know yet

Baldo proved in his paper [Bal90] that the Cahn-Hilliard energies Γ-converge with respect to $\|\cdot\|_{\mathrm{L}^1}$ to an *optimal partition energy* given by

$$\mathrm{E}(\chi) := \frac{1}{2} \sum_{1 \le i,j \le P} \sigma_{i,j} \int \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right) \tag{4.1}$$

for a partition $\chi_1, \ldots, \chi_P \colon \mathbb{T} \to \{0,1\}$ satisfying $\chi_{1 \le i \le P} \chi_i = 1$ almost everywhere. We may also define measurable sets Ω_i through the relation $\chi_i = \mathbb{1}_{\Omega_i}$. The link between a sequence u_ε and χ is given by $u_\varepsilon \to u := \sum_{1 \le i \le P} \alpha_i \chi_i$ in L^1.

Moreover if we denote by $\partial_* \Omega_i$ the reduced boundary of Ω_i and by $\Sigma_{i,j} := \partial_* \Omega_i \cap \partial_* \Omega_j$ the interface between Ω_i and Ω_j, then we may rewrite equation (4.1) as

$$\mathrm{E}(\chi) = \frac{1}{2} \sum_{1 \le i,j \le P} \sigma_{i,j} \, \mathcal{H}^{d-1}(\Sigma_{i,j}).$$

Here, the surface tensions $\sigma_{i,j}$ are the geodesic distances between the wells α_i of W with respect to the metric $2W(u)\langle \cdot \, , \, \cdot \rangle$, which can be written out as

$$\sigma_{i,j} = \mathrm{d}_W(\alpha_i, \alpha_j)$$

for the geodesic distance defined as

$$\mathrm{d}_W(u,v) := \inf \left\{ \int_0^1 \sqrt{2W(\gamma)} |\dot{\gamma}| \, \mathrm{d}t \; : \; \gamma \in \mathrm{C}^1\left([0,1]; \mathbb{R}^N\right) \text{ with } \gamma(0) = u, \, \gamma(1) = v \right\}. \tag{4.2}$$

Geometrically speaking, the partition energy E measures the surface tensions between the sets and penalizes larges interfaces. Also observe that the factor 1/2 can be left out if we only count each interface once.

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