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CS 2100: Discrete Structures

Homework 4: Combinatorics

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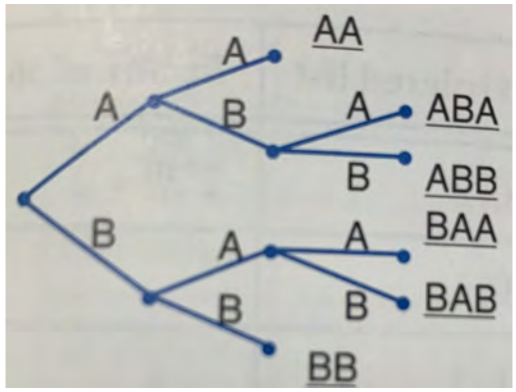
1. **Exercise 10 on page 383**
The tree for a "best of 3" tennis match between two players A and B is given below:



This can be extended to a "best of 5" match in which the first player to win three sets wins the match. Use this "best of 5" match to answer the following questions:

(a) Give a tree to represent this "best of 5" match between players A and B.

(b) How many ways are there for the match to end after three games?

(c) How many ways are there for Player A to win in four games?

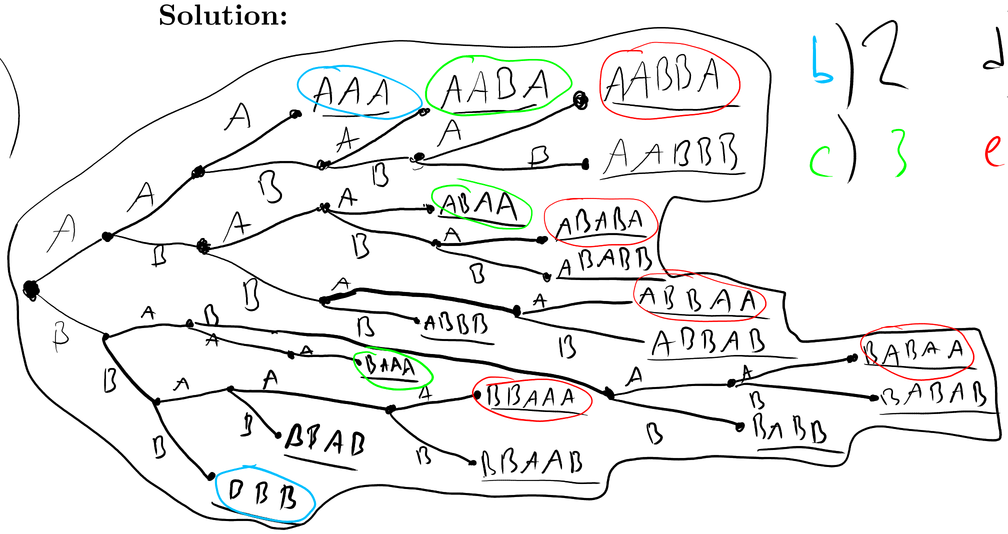
(d) Explain why the answer for part (c) is the same as the answer to the question "How many ordered lists of length 3 taken from $\{A, B\}$ have exactly two A's?"

(e) How many ways are there for Player A to win in exactly five games?

(f) Fill in the blanks:
The answer to part (e) is the same as the answer to the question "How many ordered lists of length __4__ taken from $\{A, B\}$ have exactly __2__ A's?"

Solution:

a)



b) 2

c) 3

d) The 4 games must end with A winning, so there are 3 games from which there must be 2 A's. There must be one B, and there are 3 places for a B to be, therefore 3 valid outcomes.

e) 5

2. **Exercise 19 on page 384**
The following two questions have the same answer:

- How many eight-digit binary sequences have three 1's, no two of which are adjacent?
- How many three element subsets of $\{1, 2, 3, 4, 5, 6\}$ are there?

Here is a function that demonstrates this fact: Given an eight-digit binary sequence containing three 1's, not two of which are adjacent, write down the three positions containing the 1's. Decrease the second number you wrote by 1, and decrease the third number by 2. Answer the following two questions:

- Describe in words the reverse function.
- Fill in following table to illustrate the correspondence.

10101000	$\{1, 2, 3\}$
01010100	$\{2, 3, 4\}$
00100101	$\{3, 5, 6\}$
00010101	$\{4, 5, 6\}$
10010010	$\{1, 3, 5\}$
10010001	$\{1, 3, 6\}$
00101010	$\{3, 4, 5\}$
01001001	$\{2, 4, 6\}$

Solution:

Reverse function: Given a 3 element subset of {1,2,3,4,5,6} of the pattern {x,y,z}, create an 8 digit binary sequence such that x indicates the position of the first one, y+1 is the position of the second 1, and z+2 is the position of the last 1. The remaining digits should be 0s.

3. **Exercise 20 on page 384**
   Explain why (i) and (ii) have the same answer in each of the following pairs of questions below. You do not need to actually answer either question.

   (a) (i) How many distinguishable arrangements of the letters in $ABABA$ are there?
       (ii) How many five-digit binary sequences are there with exactly three 1's?

   (b) (i) How many two-element subsets of the set $\{1, 2, 3, 4, 5\}$ are there?
       (ii) How many three-element subsets of the set $\{1, 2, 3, 4, 5\}$ are there?

   (c) (i) How many permutations of all five of the objects in $\{a, 1, x, 3, 9\}$ are there?
       (ii) How many permutations of length 4 of the objects in $\{a, 1, x, 3, 9\}$ are there?

   (d) (i) How many ways are there to flip three heads in five tosses of a coin?
       (ii) How many three-element subsets of $\{1, 2, 3, 4, 5\}$ are there?

   Solution:

a) replacing As with 1s and Bs with 0s creates the same problem.
b) choosing 2 elements to include from a 5 element set is the same as choosing 3 elements from the set to not include. Combination is symmetric.
c) the set being drawn from is 5 elements long. Choosing 4 elements and excluding a fifth is analogous to choosing those 4 elements and appending the remaining element to the end, creating a 5 element permutation.
d) the 3 elements of the set {1,2,3,4,5} indicate the position of the 3 heads flipped.

4. **Exercise 21 on page 396**
The professor in the your class asks each of the 30 students in the class his or her birthday, writing their responses as an ordered list of length 30. Assume no one is born on February 29th.

   (a) How many different results are possible?

   (b) Of these, for how many are there no duplicate birthdays listed?

   (c) What percent of the possible results have a duplicate birthday? Is this percent higher than you expected, or lower than you expected, or about what you expected?

Solution:

a) 365^{30}

b) $\frac{365!}{(365-30)!}$

c) $\frac{\frac{365!}{(365-30)!}}{365!} = \frac{1}{335!} = 3.98 \cdot 10^{-779} = 3.98 \cdot 10^{-777}\ \%$

about what I expected.

5. **Exercise 24.a on page 396**
   A certain club is forming a recruitment committee consisting of five of its members. This club has two members named Jack and Jill. They have calculated that 2,380 of the possible committees have Jack on them, 2,380 have Jill, 1,820 have Jack but not Jill, 1,820 have Jill but not Jack, and 560 have both Jack and Jill.

   (a) How many committees have either Jack or Jill?

   Solution:

   4,200

6. **Exercise 18 on page 407**
Suppose a shipment of 100 computers contains four defective computers, and we choose a sample of six computers.

   (a) How many different samples are there?

   (b) Of these, how many sample contain all four defective computers? What percent of the total does this represent?

   (c) How many samples contain one or more defective computers? What percent of the total does this represent?

Solution:

a) $C(100,6) = 1,192,052,400$

b) $\underset{1}{\cancel{C(4,4)}} \cdot C(96,2) = 4,560$

$$\frac{4560}{1,192,052,400} \cdot 100 = 0.000\,382\,533\,5195\,\%$$

c) $C(4,1) \cdot C(96,5) + C(4,2) \cdot C(96,4) + C(4,3) \cdot C(96,3) + C(4,4) \cdot C(96,2)$

$= 4 \cdot 61,124,064 + 6 \cdot 3,321,960 + 4 \cdot 142,880 + 1 \cdot 4,560$

$= 265,004,096$

$$\frac{265,004,096 \cdot 100}{1,192,052,400} = 22.2309\%$$

7. **Exercise 22 on page 407**
Three members (Mary, Sue, and Tom) of a 20-Person office are carpooling, so they insist on never working separately. That is, whenever one of them is on a committee, all three must be. How many committees of size 7 meet this requirement?

Solution:

$$C(17,7) + C(3,3)\cdot C(17,4) = 19{,}448 + 1 \cdot 2{,}380$$

$$= 21{,}828$$

8. **Exercise 4 on page 417**
You are tracking your favorite baseball player by writing down his performance in 10 successive plate appearances using an ordered list of length 10.

(a) If you use H for a hit, S for a strikeout, B for a base-on-balls, and O for anything else, how many results are possible?

(b) Of the total, how many have exactly three H's?

(c) Of the total, how many have exactly four H's, exactly one S, exactly two B's, and exactly three O's?

Solution:

a) $4^{10} = 1,048,576$

b) $C(10,3) \cdot 3^7 = 262,440$

c) $C(10,4) \cdot C(6,1) \cdot C(5,2) \cdot C(3,3)$

$= 210 \cdot 6 \cdot 10 \cdot 1 = 12,600$

9. **Exercise 6 on page 417**
Compute exact numerical answers for (a) and (b), and then answer (c) using complete sentences.

   (a) How many binary strings of a length at most 5 have exactly three 1's?
   (b) How many binary sequences of length 6 have exactly four 1's?
   (c) Why do these two questions have the exact same answer?

Solution:

a) $C(3,3)+C(4,3)+C(5,3)=15$

b) $C(6,4)=15$

c) ?

10. **Exercise 22 on page 418**
How many different bags of produce can I bring back from the store, assuming that there are apples, bananas, oranges, and peaches available; I buy at least one of each; and I purchase exactly 15 pieces of fruit altogether?

Solution:

$$a + b + o + p = 15$$

$a' = a - 1$
$b' = b - 1$
$o' = o - 1$
$p' = p - 1$

$$a' + 1 + b' + 1 + o' + 1 + p' + 1 = 15$$

$$a' + b' + o' + p' = 11$$

$n = 4, r = 11$

$$C(4 + 11 - 1, 11) = C(14, 11) = \boxed{364}$$

11. **Exercise 28 on page 418**
How many ways are there to distribute 200 apples among 43 women and 47 men if each woman must get at least two apples and each man must get at least on apple?

Solution:

$200 - 43 \cdot 2 - 47 = 67$ apples after obligatory distribution,

90 people

$$p_1 + p_2 + p_3 \cdots + p_{90} = 67$$

$n = 90$
$r = 67$

$$C(n+r-1, r) = C(90+67-1, 67) = C(156, 67)$$

$= 1,240,909,534,734,936,778,800,041,680,588,057,270,093,009,000$