Ideal MHD

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8

MHD stability – general considerations

8.1 Introduction

In the remainder of the book, it is assumed that an MHD equilibrium has been calculated, either analytically or numerically. The next basic question to ask is whether or not the equilibrium is MHD stable. Qualitatively, the question of stability can be stated as follows. The existence of an MHD equilibrium implies a plasma state in which the sum of all forces acting on the plasma is zero. Assume now that the plasma is perturbed from this state producing a set of corresponding perturbed forces. If the direction of these forces is such as to restore the plasma to its original equilibrium position then the plasma is stable. If, on the other hand, the direction of the forces tends to enhance the initial perturbation then the plasma is unstable.

The question of ideal MHD stability is a crucial one, since plasmas, in general, suffer serious degradation in performance, ranging from enhanced transport to catastrophic termination, as a consequence of such instabilities. Not surprisingly, there is consensus in the international fusion community that a plasma must be MHD stable to be viable in a fusion reactor. Indeed, it is fair to say that MHD stability considerations are a primary driver in the design of virtually all the magnetic geometries that have been proposed as fusion reactors.

The goal of Chapter 8 is to develop a basic understanding of the mechanisms that cause MHD instabilities and to discuss possible ways to avoid them. Several mathematical techniques are available to investigate ideal MHD stability. By far the most common technique is the study of linear stability which is often sufficient for practical situations. Linear theory is also amenable to analytic treatment although in many cases numerical calculations are required to ultimately obtain quantitatively accurate results. Non-linear techniques are more complicated and require numerical calculations from the outset. Often, although not always, non-linear ideal MHD stability analysis is not essential since the details of non-linear plasma degradation are less important than knowing how to avoid such instabilities in the first place. The conditions to excite or avoid MHD instabilities are well described by linear theory.

For this reason the basic discussion in Chapter 8 is focused on the linear theory of MHD instabilities. The treatment is entirely analytic. Important numerical results describing linear stability in realistic geometries are presented in Chapters 11 and 12 in conjunction with practical applications.

The material in Chapter 8 begins with a mathematical definition of stability that is particularly applicable to ideal MHD. Next, as a first application, the dispersion relation for ideal MHD waves in an infinite homogeneous magnetic field is derived. These waves, which are all MHD stable, provide valuable intuition into the dynamical behavior of a plasma. Following this is an extensive discussion of the formulation of the generalized linear stability problem. The discussion starts with the time-dependent equations of motion and culminates with the development of the Energy Principle, a powerful method for testing instability.

Two general results are then derived from the Energy Principle. First, the role of plasma compressibility on MHD stability is analyzed for arbitrary magnetic geometries. There are two types of behavior, the critical distinguishing feature being whether or not the configuration of interest has ergodic field lines or consists entirely of closed field lines. The second issue is subtle and is concerned with the question of whether the region outside the plasma core is best modeled by a vacuum or alternatively by a cold, but still perfectly conducting, force-free plasma. The last topic discussed is a general classification system that describes the different types of MHD instabilities that can be excited in a plasma.

8.2 Definition of MHD stability

To investigate "MHD stability" it is clearly necessary to have a sharp definition of stability. In practice there are several different definitions that might be applicable depending upon the particular physical properties of the system under consideration. Several possible mechanical analogs are illustrated in Fig. 8.1.

In Fig. 8.1a if the ball is moved a small distance away from its initial equilibrium position it simply oscillates indefinitely about this position, assuming an ideal frictionless system. Even though the ball never returns to rest at its equilibrium position the system can be considered to be stable; that is, the ball always remains close to its equilibrium position. This is the best that one can hope for in a system without dissipation. In contrast, Fig. 8.1b shows instability. A small perturbation off the top of the hill sets the ball rolling further and further away from its equilibrium position. Figure 8.1c is a transition point usually referred to as marginal stability or neutral stability. If the ball is placed a short distance away

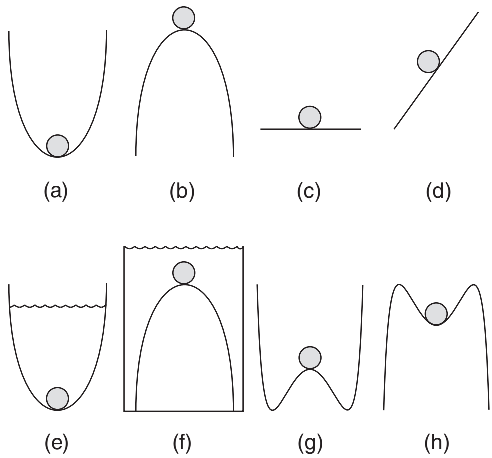

Figure 8.1 Mechanical analogs of various types of instabilities.

from its initial position it will just stay there, neither moving further away nor returning to its starting position. In most cases a change in some plasma parameter such as β or q transforms the system into one that is either stable or unstable. Therefore, marginal stability represents the stability boundary between stability and instability. Figure 8.1d shows a system not in equilbrium.

Some, but certainly not all, of the effects of dissipation can be ascertained by imagining that the ball is immersed in a viscous fluid such as oil. Figure 8.1e shows that a ball moved away from its original position will return and come to rest at its equilibrium position. The dissipation damps out the oscillations. On the other hand, Fig. 8.1f shows that while dissipation can slow down the motion of the ball it cannot convert an unstable system into a stable one. Figures 8.1g and 8.1h are examples of systems that are linearly unstable but non-linearly stable and vice versa, respectively.

To a good approximation ideal MHD is closely analogous to the set of examples illustrated in Figs. 8.1a–8.1c. There is no dissipation in ideal MHD. This recognition suggests the following definition of ideal MHD stability. Assume that all of the MHD quantities of interest are linearized about their equilibrium values

$$Q(\mathbf{r},t) = Q_0(\mathbf{r}) + \tilde{Q}_1(\mathbf{r},t) \tag{8.1}$$

Here, $Q_0(\mathbf{r})$ is the zeroth-order equilibrium value while $\tilde{Q}_1(\mathbf{r},t)$ is a small first-order perturbation satisfying $|\tilde{Q}_1/Q_0| \ll 1$. Now, since the equilibrium is by definition time independent, $\tilde{Q}_1(\mathbf{r},t)$ can in general be written as

$$\tilde{Q}_1(\mathbf{r},t) = Q_1(\mathbf{r})\,e^{-i\omega t} \tag{8.2}$$

where ω is an eigenvalue to be determined.

With very few exceptions the simplest and most reliable definition of MHD stability corresponds to "exponential stability." Specifically, if any of the eigenvalues of ω correspond to exponential growth the system is MHD unstable. If not the system is MHD stable:

$$\begin{aligned} &\mathrm{Im}(\omega) > 0 \qquad \text{exponential instability} \\ &\mathrm{Im}(\omega) \leq 0 \qquad \text{exponential stability} \end{aligned} \tag{8.3}$$

Implicit in this definition is the assumption that the modes are discrete with distinguishable eigenfrequencies. This is not always the case since MHD systems often contain continuous spectra. However, the continua lie in the stable part of the frequency spectrum and thus do not affect the existence of exponential instabilities. The frequency spectrum of ideal MHD is discussed further in Section 8.5.7. The conclusion is that if one is primarily interested in investigating MHD instabilities and the conditions for avoiding them (i.e., marginal stability) then attention should be focused on the unstable part of the spectrum. The definition Im $(\omega) > 0$ then provides a simple and reliable test for instability.

8.3 Waves in an infinite homogeneous plasma

As a first step towards understanding ideal MHD stability consider the simple case of an infinite homogeneous plasma with a uni-directional magnetic field. This configuration is obviously not toroidal and does not confine any plasma. Its "stability" actually corresponds to a determination of the basic waves that can propagate in an MHD plasma and, as such, forms a basic foundation upon which one can develop intuition that can be applied to more realistic magnetic geometries.

The "equilibrium" of the infinite homogeneous system is given by

$$\begin{aligned} \mathbf{B} &= B_0\mathbf{e}_z \\ p &= p_0 \\ \rho &= \rho_0 \\ \mathbf{J} &= 0 \\ \mathbf{v} &= 0 \end{aligned} \tag{8.4}$$

where $B_0, p_0,$ and ρ_0 are constants. Since $\nabla p = \nabla\rho = \nabla\times\mathbf{B} = \nabla\cdot\mathbf{B} = 0$, the solution described by Eq. (8.4) automatically satisfies the MHD equilibrium equations.

The stability of this system is determined by the linearization procedure described above. All quantities are expanded as $Q(\mathbf{r},t) = Q_0(\mathbf{r}) + \tilde{Q}_1(\mathbf{r},t)$ with $\tilde{Q}_1(\mathbf{r},t)$ being a small first-order perturbation. Since the equilibrium is independent of both time and space the most general form of the perturbation can be written as

$$\begin{aligned} \tilde{Q}_1(\mathbf{r},t) &= Q_1 \exp[-i(\omega t - \mathbf{k}\cdot\mathbf{r})] \\ \mathbf{k} &= k_\perp \mathbf{e}_y + k_\parallel \mathbf{e}_z \\ \mathbf{k}\cdot\mathbf{r} &= k_\perp y + k_\parallel z \end{aligned} \tag{8.5}$$

Here, without loss in generality it has been assumed that the coordinate system has been rotated around the z axis so that $\mathbf{k}$ lies in the y, z plane. Also, $\perp$ and $\|$ refer to perpendicular and parallel to the equilibrium field, respectively.

The next step is to substitute Eq. (8.5) into the linearized MHD equations, excluding for now the momentum equation. The results show that all perturbed quantities can be expressed in terms of the perturbed velocity $\mathbf{v}_1$,

$$\begin{aligned} \omega\rho_1 &= \rho_0(\mathbf{k}\cdot\mathbf{v}_1) && \text{conservation of mass} \\ \omega p_1 &= \gamma p_0(\mathbf{k}\cdot\mathbf{v}_1) && \text{conservation of energy} \\ \omega\mathbf{B}_1 &= -\mathbf{k}\times(\mathbf{v}_1\times\mathbf{B}_0) && \text{Faraday's law} \\ \omega\mu_0\mathbf{J}_1 &= -i\mathbf{k}\times[\mathbf{k}\times(\mathbf{v}_1\times\mathbf{B}_0)] && \text{Ampere's law} \end{aligned} \tag{8.6}$$

Note that the $\nabla\cdot\mathbf{B}=0$ equation reduces to $\mathbf{k}\cdot\mathbf{B}_1=0$ and is a redundant relation by virtue of Faraday's law. Substituting Eq. (8.6) into the linearized momentum equation yields the following three vector component relations:

$$\begin{aligned} \left(\omega^2 - k_\parallel^2 V_A^2\right) v_{1x} &= 0 \\ \left(\omega^2 - k_\perp^2 V_S^2 - k^2 V_A^2\right) v_{1y} - \left(k_\perp k_\parallel V_S^2\right) v_{1z} &= 0 \\ -\left(k_\perp k_\parallel V_S^2\right) v_{1y} + \left(\omega^2 - k_\parallel^2 V_S^2\right) v_{1z} &= 0 \end{aligned} \tag{8.7}$$

where $k^2 = k_\perp^2 + k_\parallel^2$, $V_A = (B_0^2/\mu_0\rho_0)^{1/2}$ is the Alfven speed and $V_S = (\gamma p_0/\rho_0)^{1/2}$ is the adiabatic sound speed. Setting the determinant of this system to zero yields the desired dispersion relation

$$\begin{aligned} \omega^2 &= k_\parallel^2 V_A^2 \\ \omega^2 &= \frac{1}{2}k^2\left(V_A^2+V_S^2\right)\left[1\pm(1-\alpha^2)^{1/2}\right] \end{aligned} \tag{8.8}$$

Here

$$\alpha^2 = 4\frac{k_\parallel^2}{k^2}\frac{V_S^2 V_A^2}{(V_S^2+V_A^2)^2} \tag{8.9}$$

Note that there are three branches to the dispersion relation. Since $0 \le \alpha^2 \le 1$ each corresponds to a purely oscillatory solution: $\mathrm{Im}(\omega) = 0$. As stated, the homogeneous magnetic field configuration is exponentially stable. This is not surprising since the system is in thermodynamic equilibrium and there are no sources of free energy available to drive instabilities.

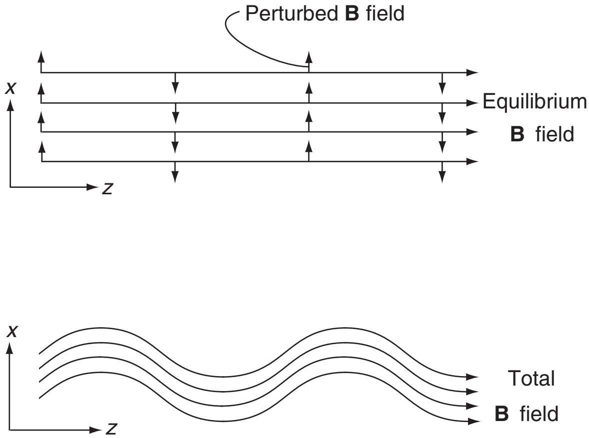

Figure 8.2 Magnetic perturbation for the shear Alfven wave shown as a separate vector component and combined with the total **B** field.

8.3.1 The shear Alfven wave

Consider now each branch of the dispersion relation. The first branch, $\omega_a^2 = k_\parallel^2 V_A^2$, is known as the shear Alfven wave and is independent of $k_\perp$ even when $k_\perp^2 \gg k_\parallel^2$. It is polarized so that the perturbed magnetic field B_{1x} and velocity v_{1x} are aligned and perpendicular to both $\mathbf{B}_0$ and $\mathbf{k}$; the wave is purely transverse. This causes the magnetic lines to bend. Plasma is carried with the magnetic perturbation by the $\mathbf{E} \times \mathbf{B}/B^2$ velocity. See Fig. 8.2. Furthermore, the quantities v_{1y}, v_{1z}, ρ_1, p_1, and $\nabla \cdot \mathbf{v}_1$ are all zero for the shear Alfven wave; the mode is incompressible and produces no density or pressure fluctuations. The shear Alfven wave describes a basic oscillation between perpendicular plasma kinetic energy and perpendicular magnetic energy; that is, a balance between inertial effects and the magnetic tension due to field line bending.

8.3.2 The fast magnetosonic wave

The second branch of the dispersion relation corresponding to the $+$ sign in Eq. (8.8) describes the fast magnetosonic wave, ω_f^2. A simple calculation shows that $\omega_f^2 \geq \omega_a^2$. This is a wave in which both the magnetic field and the plasma pressure are compressed so that $\nabla \cdot \mathbf{v}_1$ and p_1 are non-zero. Also, $\mathbf{B}_1$ has both a y and z component. See Fig. 8.3. In the interesting limit where $\beta \sim V_S^2/V_A^2 \ll 1$, the fast magnetosonic wave reduces to the compressional Alfven wave

$$\omega_f^2 \approx \left(k_\perp^2 + k_\parallel^2\right) V_A^2 \qquad (8.10)$$

In the low β limit, it can easily be shown that $\mu_0 p_1/B_0 B_{1z} \sim \beta \ll 1$ indicating that most of the compression involves the magnetic field and not the plasma. The

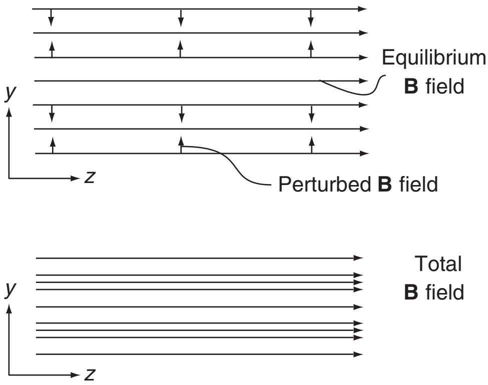

Figure 8.3 Magnetic perturbation for the compressional Alfven wave shown as a separate vector component and combined with the total **B** field.

compressional Alfven wave describes a basic oscillation between perpendicular plasma kinetic energy (plasma inertia) and parallel plus perpendicular magnetic energy. In other words, there is a balance between inertial effects and compression plus tension of the field lines. Also, since $v_{1z}/v_{1y} \sim \beta \ll 1$ the plasma motion is nearly transverse.

8.3.3 The slow magnetosonic wave

The third branch of the dispersion relation corresponds to the slow magnetosonic wave, ω_s^2. This wave always satisfies $\omega_s^2 \leq \omega_a^2$. As in the fast magnetosonic branch the wave is polarized so that both the plasma pressure and the magnetic field are compressed. However, for the slow magneto sonic wave it is the plasma rather than magnetic field that is primarily compressed. In the low β limit, $\beta \sim V_S^2/V_A^2 \ll 1$, the slow magnetosonic wave reduces to a sound wave,

$$\omega_s^2 \approx k_\parallel^2 V_S^2 \tag{8.11}$$

Observe that in this limit the mode is nearly longitudinal since $v_{1y}/v_{1z} \sim \beta \ll 1$ (see Fig. 8.4). Hence, the sound wave describes a basic oscillation between parallel plasma kinetic energy and plasma internal energy; that is, between inertial effects and plasma compression.

A subtle point concerning the sound wave is that the dispersion relation is identical to that of the two-fluid ion acoustic wave. There is an apparent paradox in that the ion acoustic wave is essentially a longitudinal electrostatic oscillation with $E_\parallel = -ik\phi$, while in ideal MHD $E_\parallel = 0$. Both modes are in fact the same mode and the paradox is resolved in Problem 8.2.

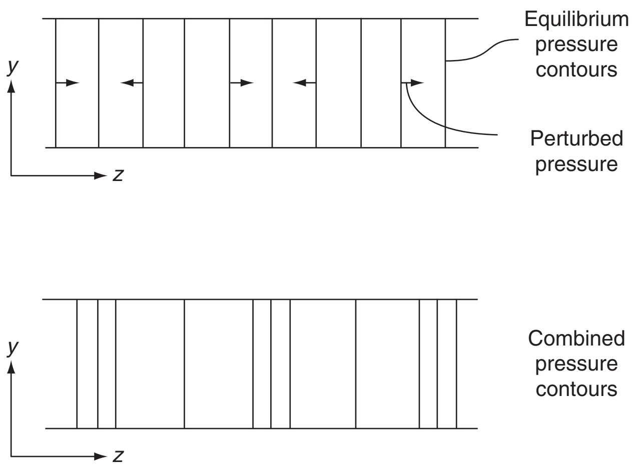

Figure 8.4 Pressure perturbation for the sound wave shown as a separate vector component and combined with the total **B** field.

8.3.4 Summary

To summarize, the three branches of the dispersion relation – the shear Alfven wave, and the fast and slow magnetosonic waves – describe the basic wave propagation characteristics of an ideal MHD plasma. In a homogeneous geometry, all are stable. In more physically interesting inhomogeneous geometries each of these waves is modified and can couple to one another. An important result related to stability is that, for reasons to be discussed later, the most unstable modes almost always couple to the shear Alfven wave.

8.4 General linearized stability equations

There exists an elegant and powerful theoretical formulation, known as the Energy Principle, that can test ideal MHD stability in an arbitrary three-dimensional geometry. To derive this Principle several steps are required. First, the stability problem is formulated as an initial value problem using the general linearized equations of motion. Second, these equations are cast into the form of a normal-mode eigenvalue problem. Next, the eigenmode formulation is transformed into a variational principle. Finally, the variational principle is reduced to the Energy Principle. Each of these steps is discussed in detail in the sections that follow.

8.4.1 Initial value formulation

To begin, assume that a static ideal MHD equilibrium, satisfying

$$\begin{aligned} \mathbf{J}_0 \times \mathbf{B}_0 &= \nabla p_0 \\ \mu_0 \mathbf{J}_0 &= \nabla \times \mathbf{B}_0 \\ \nabla \cdot \mathbf{B}_0 &= 0 \\ \mathbf{v}_0 &= 0 \end{aligned} \tag{8.12}$$

is given. All quantities are linearized about this background state: $Q(\mathbf{r},t) = Q_0(\mathbf{r}) + \tilde{Q}_1(\mathbf{r},t)$ with $|\tilde{Q}_1/Q_0| \ll 1$. When substituting into the MHD equations it is convenient to express all perturbed quantities in terms of a vector $\tilde{\xi}(\mathbf{r},t)$ defined by

$$\mathbf{v}_1 = \frac{\partial \tilde{\xi}}{\partial t} \tag{8.13}$$

The vector $\tilde{\xi}$ represents the displacement of the plasma away from its equilibrium position. The goal now is to express all perturbed quantities in terms of $\tilde{\xi}$ and then obtain a single equation describing the time evolution of $\tilde{\xi}$.

In an initial value formulation one needs to specify appropriate initial data. A very convenient choice of initial data for stability problems is as follows:

$$\begin{aligned} &\tilde{\xi}(\mathbf{r},0) = \tilde{\mathbf{B}}_1(\mathbf{r},0) = \tilde{\rho}_1(\mathbf{r},0) = \tilde{p}_1(\mathbf{r},0) = 0 \\ &\frac{\partial \tilde{\xi}(\mathbf{r},0)}{\partial t} = \tilde{\mathbf{v}}_1(\mathbf{r},0) \neq 0 \end{aligned} \tag{8.14}$$

This corresponds to the situation where at $t = 0$ the plasma is in its exact equilibrium position but is moving away with a small velocity $\tilde{\mathbf{v}}_1(\mathbf{r},0)$. Under these initial conditions the linearized form of the mass conservation equation, energy relation, and Faraday's law can be integrated with respect to time, yielding

$$\begin{aligned} \tilde{\rho}_1 &= -\nabla \cdot (\rho_0 \tilde{\xi}) \\ \tilde{p}_1 &= -\tilde{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \tilde{\xi} \\ \tilde{\mathbf{B}}_1 &= \nabla \times (\tilde{\xi} \times \mathbf{B}_0) \end{aligned} \tag{8.15}$$

Note that the relation $\nabla \cdot \tilde{\mathbf{B}}_1 = 0$ is trivially satisfied by virtue of Faraday's law.

These quantities are substituted into the momentum equation leading to a single vector equation for the displacement $\tilde{\xi}$:

$$\rho \frac{\partial^2 \tilde{\xi}}{\partial t^2} = \mathbf{F}(\tilde{\xi}) \tag{8.16}$$

where the force operator $\mathbf{F}(\tilde{\xi})$ is given by

$$\mathbf{F}(\tilde{\xi}) = \mathbf{J} \times \tilde{\mathbf{B}}_1 + \tilde{\mathbf{J}}_1 \times \mathbf{B} - \nabla \tilde{p}_1 \tag{8.17}$$

In Eq. (8.17) and hereafter, the zero subscript has been dropped from all equilibrium quantities. Equation (8.16), subject to $\tilde{\xi}(\mathbf{r},0) = 0$ and $\partial \tilde{\xi}(\mathbf{r},0)/\partial t = \tilde{\mathbf{v}}_1(\mathbf{r},0)$, plus appropriate boundary conditions (as discussed in Section 3.2) constitute the formulation of the general linearized stability equations as an initial value problem.

The initial value approach has the advantage of directly determining the actual time evolution of a given initial perturbation. The fastest growing mode becomes easily identifiable since it naturally dominates the numerically calculated evolution after a short period of time. The initial value approach is also useful in the numerical formulation of the full non-linear problem.

One drawback of the approach is that it often contains more information than is required to just determine stability boundaries, thereby requiring a substantial implementation effort. In addition, determining the marginal stability boundaries can be difficult since initial value codes have to be run for a long period of time to guarantee that a slow growing mode does not exist.

8.4.2 Normal mode formulation

A more efficient way to investigate linear stability is to reformulate Eq. (8.16) as a normal mode problem. This can easily be done by letting all perturbed quantifies vary as follows: $\tilde{Q}_1(\mathbf{r}, t) = Q_1(\mathbf{r})\exp(-i\omega t)$. Observe that the right-hand sides of Eqs. (8.15) and (8.16) contain no explicit time derivatives. Hence, the conservation of mass, conservation of energy, and Faraday's law reduce to

$$\begin{aligned} \rho_1 &= -\nabla \cdot (\rho_0 \boldsymbol{\xi}) \\ p_1 &= -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi} \\ \mathbf{B}_1 &= \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \end{aligned} \tag{8.18}$$

Upon substituting these relations into the momentum equation, one finds

$$-\omega^2 \rho \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}) \tag{8.19}$$

where

$$\mathbf{F}(\boldsymbol{\xi}) = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}_1 + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1) \times \mathbf{B} - \nabla p_1 \tag{8.20}$$

Equation (8.19) represents the normal mode formulation of the linearized MHD stability problem for general three-dimensional equilibria. In this approach only appropriate boundary conditions on $\boldsymbol{\xi}$ are required. Equation (8.19) can then be solved as an eigenvalue problem for the eigenvalue ω^2.

As compared to the initial value approach, the normal mode formulation is more amenable to analysis and more efficient with respect to numerical computations. As such it is often used in the study of ideal MHD stability. Even so, it too often contains more information than is needed to just determine a "yes or no" answer with respect to plasma stability. Thus, this is still not the most efficient way to determine marginal stability.

Another point worth noting is that the usefulness of normal mode approach is coupled to the assumption that for the problems of interest the eigenvalues are discrete and distinguishable so that the concept of exponential stability is valid. This is indeed true for the unstable part of the spectrum, although the full situation is more complicated. To obtain a more complete understanding, additional detailed knowledge of the properties of the force operator $\mathbf{F}(\xi)$ is required. This question is discussed in the next section.

Once this information is established one can proceed to the formulation of the Energy Principle, which is the most efficient way to investigate marginal stability.

8.5 Properties of the force operator $\mathbf{F}(\xi)$

The force operator $\mathbf{F}(\xi)$ possesses an important mathematical property that greatly aids in the analysis of linearized MHD stability. In particular, $\mathbf{F}(\xi)$ is a self-adjoint operator. A major consequence of this property is that each of the discrete eigenvalues ω^2 is purely real. Hence, stability transitions (i.e., $\mathrm{Im}(\omega) = 0$) always occur when ω^2 crosses zero (i.e., $\mathrm{Re}(\omega) = \mathrm{Im}(\omega) = 0$), rather than at some unknown and yet to be determined general point along the real axis (i.e., $\mathrm{Re}(\omega) \neq 0$). This fact ultimately leads to an elegant and efficient formulation for testing linear stability known as the Energy Principle.

A second important consequence of the self-adjoint property is that the discrete normal modes are orthogonal to each other. This property is useful for the following reason. After a short period of time, any arbitrary initial plasma perturbation will always be dominated by the single fastest growing mode as opposed to some combination of modes that would occur if the system were non-orthogonal. Stability is determined solely by learning how to avoid this fastest growing normal mode without regard to the shape of the initial perturbation.

Lastly, a further examination of $\mathbf{F}(\xi)$ shows that the frequency spectrum consists not only of discrete modes, but of continua as well. Even so, the continua lie on the stable side of the ω^2 axis (i.e. $\omega^2 > 0$) or at most reach the origin $\omega^2 = 0$. Thus when attention is focused on exponential instabilities, the difficulties associated with the continuous spectra are avoided.

Each of these issues is now discussed separately.

8.5.1 Self-adjointness of $F(\xi)$

As stated the self-adjointness of $\mathbf{F}(\xi)$ has a major impact on both the analytic and numerical formulation of linearized MHD stability. To demonstrate this property it is necessary to show that for any two arbitrary vectors $\boldsymbol{\xi}(\mathbf{r})$ and $\boldsymbol{\eta}(\mathbf{r})$ both satisfying the same well posed boundary conditions, such as those discussed in Section 3.2, the following relation holds:

$$\int \boldsymbol{\eta}\cdot\mathbf{F}(\xi)d\mathbf{r} = \int \xi\cdot\mathbf{F}(\boldsymbol{\eta})d\mathbf{r} \tag{8.21}$$

This is the definition of self-adjointness – switching ξ and $\boldsymbol{\eta}$ leaves the integrals unchanged. It is worth emphasizing that $\xi(\mathbf{r})$ and $\boldsymbol{\eta}(\mathbf{r})$ do not in general have to satisfy the actual eigenvalue equation and consequently are usually referred to as "trial functions."

The self-adjoint property, which is by no means obvious because of the complex structure of the force operator, is now directly demonstrated by a series of algebraic manipulations and integrations by parts involving $\mathbf{F}(\xi)$. In the process three separate forms of the integrals in Eq. (8.21) are derived. The first corresponds to the "standard form." This form is useful because by appropriate integrations by parts a boundary term is generated that leads to precisely the quantity needed to extend the analysis to include a vacuum region surrounding the plasma. The second form is known as the "intuitive form." Here, additional manipulations, which generate no further boundary terms, lead to a form where the individual contributions appearing in the integral can be given a simple physical interpretation. Note that the first and second forms do not obviously satisfy the self-adjoint property. The third form involves some further manipulations, again generating no new boundary terms. This final form is obviously self-adjoint by construction but still maintains the basic structure of the intuitive form. It is accurately described as the "intuitive self-adjoint form."

8.5.2 The "standard form" of δW

The starting point for the analysis is the left-hand integral in Eq. (8.21). This integral is usually multiplied by a mathematically unimportant factor "$-1/2$" and then defined as $\delta W(\boldsymbol{\eta}, \xi)$. The reason for the "$-1/2$" is given in Section 8.6, where it is shown that with this factor $\delta W(\xi^*, \xi)$ becomes equal to a physically relevant quantity, the perturbed potential energy of the plasma. The quantity $\delta W(\boldsymbol{\eta}, \xi)$ is thus defined as

$$\begin{aligned}\delta W(\boldsymbol{\eta},\xi) &= -\frac{1}{2}\int \boldsymbol{\eta}\cdot\mathbf{F}(\xi)\, d\mathbf{r}\\ &= -\frac{1}{2}\int \boldsymbol{\eta}\cdot\left[\frac{1}{\mu_0}(\nabla\times\mathbf{B}_1)\times\mathbf{B}+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}_1+\nabla(\boldsymbol{\xi}_\perp\cdot\nabla p+\gamma p\nabla\cdot\boldsymbol{\xi})\right]d\mathbf{r}\end{aligned} \tag{8.22}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel \mathbf{b}$, $\boldsymbol{\eta} = \boldsymbol{\eta}_\perp + \eta_\parallel \mathbf{b}$, $\mathbf{B}_1(\boldsymbol{\xi}_\perp) = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) = \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B})$ and use has been made of the relation $\boldsymbol{\xi} \cdot \nabla p = \boldsymbol{\xi}_\perp \cdot \nabla p$.

To obtain the standard form of δW two steps are needed. The first requires the integration by parts of the $\boldsymbol{\eta} \cdot (\nabla \times \mathbf{B}_1) \times \mathbf{B}$ and $\boldsymbol{\eta} \cdot \nabla(\gamma p \nabla \cdot \boldsymbol{\xi})$ terms. A short calculation yields

$$
\begin{aligned}
\delta W(\boldsymbol{\eta}, \boldsymbol{\xi}) &= \delta W_{F1} + BT1 \\
\delta W_{F1} &= \frac{1}{2} \int \left\{ \frac{\mathbf{B}_1(\boldsymbol{\eta}_\perp) \cdot \mathbf{B}_1(\boldsymbol{\xi}_\perp)}{\mu_0} + \gamma p (\nabla \cdot \boldsymbol{\eta})(\nabla \cdot \boldsymbol{\xi}) \right. \\
&\qquad \left. - \boldsymbol{\eta} \cdot [\mathbf{J} \times \mathbf{B}_1(\boldsymbol{\xi}_\perp) + \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p)] \right\} d\mathbf{r} \\
BT1 &= \frac{1}{2} \int (\mathbf{n} \cdot \boldsymbol{\eta}_\perp) \left[\frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B}_1(\boldsymbol{\xi}_\perp) - \gamma p \nabla \cdot \boldsymbol{\xi} \right] dS
\end{aligned} \tag{8.23}
$$

Here the δW_{F1} integral is over the plasma volume. The boundary term $BT1$ arises from the use of Gauss' theorem in the integration by parts. This integral is taken over the plasma surface $d\mathbf{S} = \mathbf{n} dS$ with $\mathbf{n}$ the outward normal vector.

A short calculation given below shows that $\eta_\parallel \mathbf{b} \cdot [\mathbf{J} \times \mathbf{B}_1(\boldsymbol{\xi}_\perp) + \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p)] = 0$:

$$
\begin{aligned}
\text{(a)}\ \mathbf{B} \cdot \mathbf{J} \times \mathbf{B}_1 &= -\mathbf{B}_1 \cdot \mathbf{J} \times \mathbf{B} = -\mathbf{B}_1 \cdot \nabla p \\
&= \nabla \cdot [\nabla p \times (\boldsymbol{\xi} \times \mathbf{B})] = -\nabla \cdot [(\boldsymbol{\xi} \cdot \nabla p)\mathbf{B}] \\
\text{(b)}\ \mathbf{B} \cdot \nabla(\boldsymbol{\xi} \cdot \nabla p) &= +\nabla \cdot [(\boldsymbol{\xi} \cdot \nabla p)\mathbf{B}]
\end{aligned} \tag{8.24}
$$

The term $\boldsymbol{\eta} \cdot [\mathbf{J} \times \mathbf{B}_1(\boldsymbol{\xi}_\perp) + \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p)]$ thus reduces to $\boldsymbol{\eta}_\perp \cdot [\mathbf{J} \times \mathbf{B}_1(\boldsymbol{\xi}_\perp) + \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p)]$. The $\boldsymbol{\eta}_\perp \cdot \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p)$ term is now integrated by parts. The expression for δW becomes

$$
\delta W(\boldsymbol{\eta}, \boldsymbol{\xi}) = \delta W_F + BT \tag{8.25}
$$

where

$$
\begin{aligned}
\delta W_F &= \frac{1}{2} \int \left[\frac{\mathbf{B}_1(\boldsymbol{\eta}_\perp) \cdot \mathbf{B}_1(\boldsymbol{\xi}_\perp)}{\mu_0} + \gamma p (\nabla \cdot \boldsymbol{\eta})(\nabla \cdot \boldsymbol{\xi}) \right. \\
&\qquad \left. - \boldsymbol{\eta}_\perp \cdot \mathbf{J} \times \mathbf{B}_1(\boldsymbol{\xi}_\perp) + (\boldsymbol{\xi}_\perp \cdot \nabla p) \nabla \cdot \boldsymbol{\eta}_\perp \right] d\mathbf{r} \\
BT &= \frac{1}{2} \int (\mathbf{n} \cdot \boldsymbol{\eta}_\perp) \left[\frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B}_1(\boldsymbol{\xi}_\perp) - \gamma p \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi}_\perp \cdot \nabla p \right] dS
\end{aligned} \tag{8.26}
$$

The quantity δW_F is known as the fluid energy and the expression given in Eq. (8.26) corresponds to the "standard form." Observe that the only appearance of $\xi_{\|}$ and $\eta_{\|}$ in δW_F occurs in the $\gamma p(\nabla \cdot \boldsymbol{\eta})(\nabla \cdot \xi)$ term. All the other terms are functions only of $\boldsymbol{\xi}_{\perp}$ and $\boldsymbol{\eta}_{\perp}$.

The boundary contribution BT contains precisely the right terms to account for a vacuum region surrounding the plasma. However, to temporarily simplify the algebra it is assumed at present that the plasma is surrounded by a perfectly conducting wall which requires setting $\mathbf{n} \cdot \xi_{\perp}(S) = \mathbf{n} \cdot \boldsymbol{\eta}_{\perp}(S) = 0$. Under this assumption it follows that $BT = 0$.

The boundary term is re-introduced in Section 8.8 when the vacuum region is considered. It is shown there that BT can also be written in a self-adjoint form thereby generalizing the proof given in the remainder of this section.

Under the $\mathbf{n} \cdot \boldsymbol{\xi}_{\perp}(S) = \mathbf{n} \cdot \boldsymbol{\eta}_{\perp}(S) = 0$ assumption, demonstration of the self-adjoint property involves only δW_F plus the requirement that further integration by parts should not produce any additional non-zero boundary terms when $\mathbf{n} \cdot \boldsymbol{\eta}_{\perp}(S) \neq 0$.

8.5.3 *The "intuitive form" of* δW

The next step in the proof of the self-adjointness of $\mathbf{F}(\xi)$ is to convert δW_F from the standard form to the intuitive form. This form is useful for providing physical insight into the nature of MHD instabilities. The intuitive form is an important result since it is used for most of the applications in the remainder of the book.

The starting point for the analysis is the standard form of δW_F given in Eq. (8.26). The conversion to the intuitive form is straightforward requiring only several algebraic manipulations. In the first step the perturbed magnetic field is separated into its perpendicular and parallel components,

$$\begin{aligned}\mathbf{B}_1(\boldsymbol{\xi}_{\perp}) &= [\mathbf{b} \times \nabla \times (\boldsymbol{\xi}_{\perp} \times \mathbf{B})] \times \mathbf{b} + [\mathbf{b} \cdot \nabla \times (\boldsymbol{\xi}_{\perp} \times \mathbf{B})]\,\mathbf{b} \\ &= \mathbf{Q}_{\perp}(\boldsymbol{\xi}_{\perp}) + Q_{\|}(\boldsymbol{\xi}_{\perp})\mathbf{b}\end{aligned} \tag{8.27}$$

The notation using $\mathbf{Q}$ for the perturbed magnetic field is largely historic in origin and fairly common in the literature. Equation (8.27) implies that the first term in the integrand of δW_F can be expressed as

$$\mathbf{B}_1(\boldsymbol{\eta}_{\perp}) \cdot \mathbf{B}_1(\boldsymbol{\xi}_{\perp}) = \mathbf{Q}_{\perp}(\boldsymbol{\eta}_{\perp}) \cdot \mathbf{Q}_{\perp}(\boldsymbol{\xi}_{\perp}) + Q_{\|}(\boldsymbol{\eta}_{\perp})Q_{\|}(\boldsymbol{\xi}_{\perp}) \tag{8.28}$$

Next, the third term in the integrand is rewritten as follows,

$$\boldsymbol{\eta}_{\perp} \cdot \mathbf{J} \times \mathbf{B}_1(\boldsymbol{\xi}_{\perp}) = J_{\|}\boldsymbol{\eta}_{\perp} \times \mathbf{b} \cdot \mathbf{Q}_{\perp}(\boldsymbol{\xi}_{\perp}) + Q_{\|}(\boldsymbol{\xi}_{\perp})\boldsymbol{\eta}_{\perp} \cdot \mathbf{J}_{\perp} \times \mathbf{b} \tag{8.29}$$

This term is simplified by noting that

$$
\begin{aligned}
&\text{(a)} \qquad \mathbf{J}_\perp = \frac{\mathbf{b}\times\nabla p}{B} \\
&\text{(b)} \quad Q_\parallel(\boldsymbol{\xi}_\perp) = \mathbf{b}\cdot\nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}) \\
&\qquad\qquad\quad = \mathbf{b}\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B} - \mathbf{B}\nabla\cdot\boldsymbol{\xi}_\perp) \\
&\qquad\qquad\quad = -B(\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) + \frac{\mu_0}{B}\boldsymbol{\xi}_\perp\cdot\nabla p
\end{aligned}
\tag{8.30}
$$

where as before $\boldsymbol{\kappa} = \mathbf{b}\cdot\nabla\mathbf{b}$ is the curvature vector.

These relations are substituted into Eqs. (8.28) and (8.29) which are then substituted back into Eq. (8.27). A short calculation yields

$$
\begin{aligned}
\delta W_F = \frac{1}{2\mu_0}\int \big[&\mathbf{Q}_\perp(\boldsymbol{\eta}_\perp)\cdot\mathbf{Q}_\perp(\boldsymbol{\xi}_\perp) + B^2(\nabla\cdot\boldsymbol{\eta}_\perp + 2\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})(\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) \\
&+ \mu_0\gamma p(\nabla\cdot\boldsymbol{\eta})(\nabla\cdot\boldsymbol{\xi}) - 2\mu_0(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa}) - \mu_0 J_\parallel\boldsymbol{\eta}_\perp\times\mathbf{b}\cdot\mathbf{Q}_\perp\big]d\mathbf{r}
\end{aligned}
\tag{8.31}
$$

This is the "intuitive form" of δW_F first suggested by Furth *et al.* (1965) and Greene and Johnson (1968). It may not be apparent exactly what is intuitive about this form as presently written. The intuitiveness can be made clearer by noting that since $\boldsymbol{\eta}$ is an arbitrary vector one can always choose $\boldsymbol{\eta} = \boldsymbol{\xi}^*$ as a special case. When this is done Eq. (8.31) can be rewritten as

$$
\begin{aligned}
\delta W_F = \frac{1}{2\mu_0}\int \Big[&|\mathbf{Q}_\perp|^2 && \text{shear Alfven wave} \\
&+ B^2|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2 && \text{compressional Alfven wave} \\
&+ \mu_0\gamma p|\nabla\cdot\boldsymbol{\xi}|^2 && \text{sound wave} \\
&- 2\mu_0(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa}) && \text{pressure-driven modes} \\
&- \mu_0 J_\parallel\boldsymbol{\xi}_\perp^*\times\mathbf{b}\cdot\mathbf{Q}_\perp(\boldsymbol{\xi}_\perp)\Big]\,d\mathbf{r} && \text{current-driven modes}
\end{aligned}
\tag{8.32}
$$

The terms have the following simple interpretation. The $|\mathbf{Q}_\perp|^2$ term represents the magnetic energy required to bend the magnetic field lines. For a homogeneous magnetic field this produces the shear Alfven wave. The second term corresponds to the energy required to compress the magnetic field. For a homogeneous magnetic field this term leads to the fast magnetosonic or compressional Alfven wave. The $\gamma p|\nabla\cdot\boldsymbol{\xi}|^2$ term represents the energy required to compress the plasma. In a homogeneous magnetic field this energy produces the slow magnetosonic or sound wave. Each of these terms is positive, which will be shown to correspond to stability.

The remaining two terms can be positive or negative and can therefore drive instabilities. The first of these is proportional to $\nabla p = \mathbf{J}_\perp\times\mathbf{B}$, while the second is proportional to $J_\parallel$. Thus, while a homogeneous vacuum magnetic field is MHD stable both perpendicular and parallel currents represent potential sources of instability. For obvious reasons instabilities driven by the ∇p term are often referred to as pressure-driven modes. Similarly instabilities driven by the $J_{\|}$ term are known as current-driven modes. A more detailed discussion of the classification of MHD instabilities is given in Section 8.11.

8.5.4 The "intuitive self-adjoint form" of δW

The last step in the analysis transforms the intuitive form of δW_F into the intuitive self-adjoint form. In this final form the self-adjoint property is obvious since each of the individual terms is symmetric in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Consequently, interchanging $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ leaves each term unchanged which is the definition of self-adjointness. The analysis requires a moderate amount of algebraic manipulations as follows.

First, by adding and subtracting appropriate terms the ∇p contribution is rewritten as a self-adjoint part plus a non-self-adjoint remainder R_1,

$$\begin{aligned} -2(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa}) &= -(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa}) - (\boldsymbol{\eta}_\perp\cdot\nabla p)(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) + R_1 \\ R_1 &= -(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa}) + (\boldsymbol{\eta}_\perp\cdot\nabla p)(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) \end{aligned} \tag{8.33}$$

Next, the remainder R_1 is simplified by some simple vector manipulations

$$\begin{aligned} R_1 &= -(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa}) + (\boldsymbol{\eta}_\perp\cdot\nabla p)(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) \\ &= \nabla p\cdot[\boldsymbol{\kappa}\times(\boldsymbol{\eta}_\perp\times\boldsymbol{\xi}_\perp)] \\ &= (\nabla p\times\boldsymbol{\kappa})\cdot(\boldsymbol{\eta}_\perp\times\boldsymbol{\xi}_\perp) \end{aligned} \tag{8.34}$$

Second, a similar but somewhat more complicated analysis is carried out for the $J_{\|}$ contribution,

$$\begin{aligned} -J_{\|}\boldsymbol{\eta}_\perp\times\mathbf{b}\cdot\mathbf{Q}_\perp(\boldsymbol{\xi}_\perp) &= -\frac{1}{2}\left[J_{\|}\boldsymbol{\eta}_\perp\times\mathbf{b}\cdot\mathbf{Q}_\perp(\boldsymbol{\xi}_\perp) + J_{\|}\boldsymbol{\xi}_\perp\times\mathbf{b}\cdot\mathbf{Q}_\perp(\boldsymbol{\eta}_\perp)\right] + R_2 \\ R_2 &= -\frac{1}{2}\left[J_{\|}\boldsymbol{\eta}_\perp\times\mathbf{b}\cdot\mathbf{Q}_\perp(\boldsymbol{\xi}_\perp) - J_{\|}\boldsymbol{\xi}_\perp\times\mathbf{b}\cdot\mathbf{Q}_\perp(\boldsymbol{\eta}_\perp)\right] \end{aligned} \tag{8.35}$$

The non-self-adjoint contribution R_2 can be rewritten as

$$\begin{aligned} R_2 &= -\frac{1}{2}\left[J_{\|}(\boldsymbol{\eta}_\perp\times\mathbf{b})\cdot\nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}) - J_{\|}(\boldsymbol{\xi}_\perp\times\mathbf{b})\cdot\nabla\times(\boldsymbol{\eta}_\perp\times\mathbf{B})\right] \\ &= -\frac{J_{\|}}{2B}\nabla\cdot[(\boldsymbol{\xi}_\perp\times\mathbf{B})\times(\boldsymbol{\eta}_\perp\times\mathbf{B})] \\ &= -\nabla\cdot\left\{\frac{J_{\|}}{2B}[(\boldsymbol{\xi}_\perp\times\mathbf{B})\times(\boldsymbol{\eta}_\perp\times\mathbf{B})]\right\} + \frac{1}{2}[(\boldsymbol{\xi}_\perp\times\mathbf{B})\times(\boldsymbol{\eta}_\perp\times\mathbf{B})]\cdot\nabla\left(\frac{J_{\|}}{B}\right) \end{aligned} \tag{8.36}$$

Now, note that $(\boldsymbol{\xi}_\perp \times \mathbf{B}) \times (\boldsymbol{\eta}_\perp \times \mathbf{B}) = -(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp \cdot \mathbf{B})\mathbf{B}$. Using this relation in Eq. (8.36) yields

$$R_2 = \nabla \cdot \left[\frac{J_\parallel}{2B}(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp \cdot \mathbf{B})\mathbf{B}\right] - \frac{1}{2}(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp \cdot \mathbf{B})\mathbf{B} \cdot \nabla\left(\frac{J_\parallel}{B}\right) \tag{8.37}$$

Since $\mathbf{n} \cdot \mathbf{B} = 0$ on the plasma surface, the divergence term in Eq. (8.37) integrates to zero by Gauss' theorem (even if $\mathbf{n} \cdot \boldsymbol{\eta}_\perp(S) \neq 0$). Hereafter, this term is suppressed. The remaining term is simplified by using the $\nabla \cdot \mathbf{J} = 0$ relation,

$$\begin{aligned} R_2 &= -\frac{1}{2}(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp \cdot \mathbf{B})\mathbf{B} \cdot \nabla\left(\frac{J_\parallel}{B}\right) \\ &= \frac{1}{2}(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp \cdot \mathbf{B})\nabla \cdot \mathbf{J}_\perp \\ &= \frac{1}{2}(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp \cdot \mathbf{B})\nabla \cdot \left(\frac{\mathbf{B} \times \nabla p}{B^2}\right) \\ &= -(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp \cdot \mathbf{b})(\mathbf{b} \times \nabla p \cdot \boldsymbol{\kappa}) \end{aligned} \tag{8.38}$$

Next, vector expand $\boldsymbol{\xi}_\perp = f_1\mathbf{n} + f_2\mathbf{n} \times \mathbf{b} + f_3\mathbf{b}$ and $\boldsymbol{\eta}_\perp = h_1\mathbf{n} + h_2\mathbf{n} \times \mathbf{b} + h_3\mathbf{b}$. The coefficients $f_3 = h_3 = 0$ since by definition $\boldsymbol{\xi}_\perp \cdot \mathbf{b} = \boldsymbol{\eta}_\perp \cdot \mathbf{b} = 0$. It then follows that $\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp = (f_1h_2 - f_2h_1)\mathbf{b}$. As might be expected $\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp$ only has a $\mathbf{b}$ component, which implies that $(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp \cdot \mathbf{b})\mathbf{b} = \boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp$. This result is substituted into Eq. (8.38) finally leading to

$$\begin{aligned} R_2 &= -(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp \cdot \mathbf{b})(\mathbf{b} \times \nabla p \cdot \boldsymbol{\kappa}) \\ &= -[(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp) \times \nabla p] \cdot \boldsymbol{\kappa} \\ &= -(\boldsymbol{\eta}_\perp \times \boldsymbol{\xi}_\perp) \cdot (\nabla p \times \boldsymbol{\kappa}) \end{aligned} \tag{8.39}$$

After this slightly lengthy calculation observe that $R_1 + R_2 = 0$. The terms exactly cancel. Collecting all the remaining terms yields the desired expression for "intuitive self-adjoint form" of δW_F

$$\begin{aligned} \delta W_F = \frac{1}{2\mu_0}\int \Big\{ & \mathbf{Q}_\perp(\boldsymbol{\eta}_\perp) \cdot \mathbf{Q}_\perp(\boldsymbol{\xi}_\perp) \\ & + B^2(\nabla \cdot \boldsymbol{\eta}_\perp + 2\boldsymbol{\eta}_\perp \cdot \boldsymbol{\kappa})(\nabla \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}) \\ & + \mu_0\gamma p(\nabla \cdot \boldsymbol{\eta})(\nabla \cdot \boldsymbol{\xi}) \\ & - \mu_0[(\boldsymbol{\xi}_\perp \cdot \nabla p)(\boldsymbol{\eta}_\perp \cdot \boldsymbol{\kappa}) + (\boldsymbol{\eta}_\perp \cdot \nabla p)(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})] \\ & - (\mu_0 J_\parallel/2)[\boldsymbol{\eta}_\perp \times \mathbf{b} \cdot \mathbf{Q}_\perp(\boldsymbol{\xi}_\perp) + \boldsymbol{\xi}_\perp \times \mathbf{b} \cdot \mathbf{Q}_\perp(\boldsymbol{\eta}_\perp)]\Big\}\, d\mathbf{r} \end{aligned} \tag{8.40}$$

As previously stated, all the terms are symmetric in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Therefore, δW_F is self-adjoint by construction and the proof is completed.

8.5.5 Real ω^2

By making use of the self-adjointness of $\mathbf{F}(\xi)$ it is straightforward to show that the eigenvalue ω^2 for any discrete normal mode is purely real. The proof is obtained by forming the dot product of Eq. (8.19) with $\boldsymbol{\xi}^*(\mathbf{r})$ and integrating over the plasma volume. The result is

$$\omega^2 \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r} = -\int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, d\mathbf{r} \tag{8.41}$$

The procedure is then repeated by forming the dot product of $\boldsymbol{\xi}(\mathbf{r})$ with the complex conjugate of Eq. (8.19),

$$(\omega^2)^* \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r} = -\int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}^*)\, d\mathbf{r} \tag{8.42}$$

The self-adjoint property of $\mathbf{F}$ implies that the right-hand sides of Eqs. (8.41) and (8.42) are equal. Therefore, subtracting the equations leads to

$$\left[\omega^2 - (\omega^2)^*\right] \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r} = 0 \tag{8.43}$$

or

$$\omega^2 = (\omega^2)^* \tag{8.44}$$

that is, ω^2 is purely real.

In terms of the definition of exponential stability a normal mode with $\omega^2 > 0$ corresponds to a pure oscillation and hence is considered stable. Conversely, for a mode with $\omega^2 < 0$, one branch must grow exponentially and thus is unstable. Clearly, the transition from stability to instability occurs when $\omega^2 = 0$. See Fig. 8.5. It is worthwhile emphasizing the importance of this result. In more general plasma models a stability transition occurs when $\mathrm{Im}(\omega) = 0$ but with $\mathrm{Re}(\omega) \neq 0$. The determination of marginal stability boundaries is therefore considerably more complicated to calculate since the value of $\mathrm{Re}(\omega)$ at the stability transition is unknown. It must be calculated separately but in parallel with the rest of the analysis. However, in ideal MHD the self-adjointness of $\mathbf{F}(\xi)$ guarantees that at any marginal stability boundary both $\mathrm{Im}(\omega)$ and the $\mathrm{Re}(\omega)$ must be zero simultaneously.

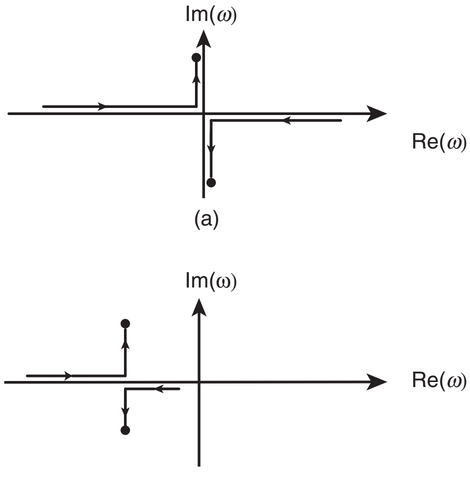

Figure 8.5 Stability transition (a) through $\omega^2 = 0$ in a self-adjoint system and (b) through a complex ω^2 in a non-self-adjoint system.

8.5.6 Orthogonality of the normal modes

A further consequence of the self-adjointness of $\mathbf{F}(\xi)$ is that the discrete normal modes are orthogonal. To show this consider two modes characterized by eigenfunctions and eigenvalues (ξ_m, ω_m^2) and (ξ_n, ω_n^2). These modes satisfy

$$\begin{aligned} -\omega_m^2 \rho \boldsymbol{\xi}_m &= \mathbf{F}(\boldsymbol{\xi}_m) \\ -\omega_n^2 \rho \boldsymbol{\xi}_n &= \mathbf{F}(\boldsymbol{\xi}_n) \end{aligned} \tag{8.45}$$

One now forms the dot product of the first equation with ξ_n and the second equation with ξ_m. Subtracting the equations and making use of the self-adjointness of $\mathbf{F}(\xi)$ yields

$$(\omega_m^2 - \omega_n^2) \int \rho \boldsymbol{\xi}_m \cdot \boldsymbol{\xi}_n \, d\mathbf{r} = 0 \tag{8.46}$$

Equation (8.46) shows that the non-degenerate discrete normal modes are orthogonal; that is, for two distinct modes with $\omega_m^2 \neq \omega_n^2$ it follows that

$$\int \rho \boldsymbol{\xi}_m \cdot \boldsymbol{\xi}_n \, d\mathbf{r} = 0 \tag{8.47}$$

The modes are orthogonal with weight function ρ.

8.5.7 Spectrum of $F(\xi)$

Because of the self-adjointness of $\mathbf{F}(\xi)$ one is strongly motivated to choose the normal-mode approach rather than the initial-value approach when considering

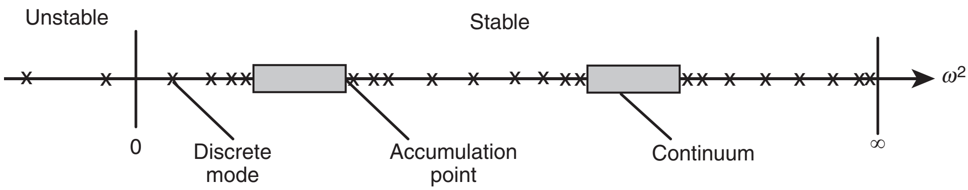

Figure 8.6 Typical ideal MHD spectrum where the continuum does not reach the origin.

the linear stability of ideal MHD plasmas. In fact, if the operator $\mathbf{F}(\boldsymbol{\xi})$ allowed only discrete eigenvalues, the concept of exponential stability could easily be extended to include both oscillatory and damped motions of the plasma. However, this is not the case. A detailed proof would require a complete spectral analysis of the force operator $\mathbf{F}(\boldsymbol{\xi})$, a task beyond the scope of the present work but which, nonetheless, has been investigated in the literature (Grad, 1973; Goedbloed, 1975; Goedbloed and Poedts, 2004). Based on these results, it is worthwhile to describe typical spectral properties that can occur and their influence on stability.

The spectral properties of $\mathbf{F}(\boldsymbol{\xi})$ follow from an examination of the operator $(\mathbf{F}/\rho - \lambda)^{-1}$ for all complex λ. If this operator exists and is bounded for a given λ, then the linearized inhomogeneous MHD equation $(\mathbf{F}/\rho - \lambda)\boldsymbol{\xi} = \mathbf{a}$ (which arises when solving an initial value problem by Laplace transforms) can be inverted yielding $\boldsymbol{\xi} = (\mathbf{F}/\rho - \lambda)^{-1}\mathbf{a}$.

The spectrum of $\mathbf{F}(\boldsymbol{\xi})$ consists of those values of λ for which the operator $(\mathbf{F}/\rho - \lambda)^{-1}$ cannot be inverted. There are two important cases. First is the familiar situation where λ is such that $(\mathbf{F}/\rho - \lambda)\boldsymbol{\xi} = 0$ possesses a non-trivial solution. These values of λ correspond to the point or discrete spectra of $\mathbf{F}(\boldsymbol{\xi})$ and represent the normal mode eigenvalues to be examined for exponential stability. In this case it is clear that the quantity $(\mathbf{F}/\rho - \lambda)^{-1}\mathbf{a}$ does not exist.

In the second case of interest λ is such that $(\mathbf{F}/\rho - \lambda)^{-1}\mathbf{a}$ exists, but $(\mathbf{F}/\rho - \lambda)^{-1}$ is itself unbounded over some continuous set of flux surfaces in the plasma. For example, in a cylindrical system one might find $\mathbf{F}/\rho - \lambda = k_{\|}^2 V_A^2(r) - \lambda$, where $V_A^2(r)$ is the local Alfven velocity. In these situations there is a continuous range of λ, defined by $\lambda = k_{\|}^2 V_A^2(r)$ for $0 < r < a$, over which the operator is ill-behaved. These values of λ constitute the continuous spectra.

A typical ideal MHD spectrum is illustrated schematically in Fig. 8.6. Notice that the continuous spectra would significantly increase the complexity of determining plasma stability if the continuum frequencies were located in that part of the complex plane where $\mathrm{Im}(\omega) > 0$. Fortunately, this appears not to be the case. In all the static ideal MHD equilibria thus far investigated the continuum frequencies, when they exist, are located on the real axis (i.e., $\omega^2 \geq 0$). This important result has

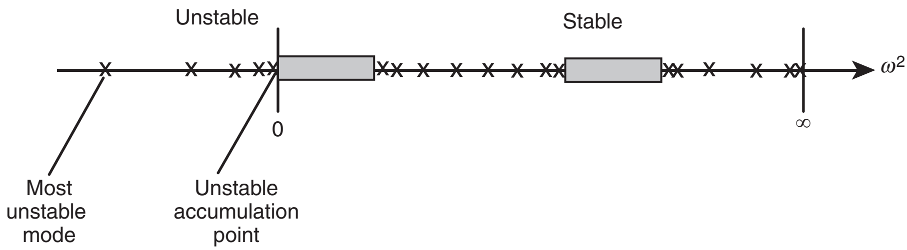

Figure 8.7 Typical ideal MHD spectrum showing a continuum reaching the origin with an accumulation of eigenvalues on the unstable side of the spectrum.

been explicitly proven for 1-D slab and cylindrical geometries and 2-D axisymmetric toroidal geometries (Goedbloed and Poedts, 2004).

One consequence of the existence of a continuum is that discrete modes can accumulate at either edge of its boundaries. If one boundary extends to the origin $\omega^2 = 0$, as it often does, it is possible to have an accumulation of discrete unstable modes with growth rates approaching zero (see Fig. 8.7). Although such behavior is perfectly acceptable mathematically, if not accounted for properly it can obscure the concept of marginal stability. On the other hand, if it is known that a continuum extends to the origin, it is often possible to derive an analytic criterion that depends only upon the equilibrium properties of the plasma that determines whether or not there is an accumulation of unstable modes. This procedure is far simpler than computing normal modes and gives rise to a necessary condition for stability.

In summary, the spectrum of the force operator $\mathbf{F}(\xi)$ contains both discrete eigenvalues and continua. However, the continua exist only for $\omega^2 \geq 0$. This result provides further motivation for examining plasma stability by the normal mode approach, restricting attention only to the question of whether or not exponentially growing modes exist. Even so, the situation can become somewhat complicated as $\omega^2 \to 0$ since unstable modes can either accumulate or make smooth transitions from instability to stability.

8.6 Variational formulation

Equation (8.19) represents the normal mode formulation of the general three-dimensional linearized MHD stability problem. These equations have the form of a set of three coupled homogeneous partial differential equations for the components of ξ with eigenvalue ω^2.

Because of the self-adjointness of $\mathbf{F}(\xi)$ the stability problem can be easily recast into the form of a variational principle (Bernstein *et al.*, 1958), which is an alternate but entirely equivalent integral representation of the linearized MHD partial differential equations. The variational formulation is an important step on the path to the Energy Principle. Below the ideal MHD variational integral is presented along with a demonstration of its equivalence to the differential form.

To begin, form the dot product of Eq. (8.19) with ξ^* and then integrate over the plasma volume. This yields

$$\omega^2 = \frac{\delta W(\xi^*,\xi)}{K(\xi^*,\xi)} \tag{8.48}$$

where

$$\begin{aligned} \delta W(\xi^*,\xi) &= -\frac{1}{2}\int \xi^* \cdot \mathbf{F}(\xi)\, d\mathbf{r} \\ K(\xi^*,\xi) &= \frac{1}{2}\int \rho|\xi|^2\, d\mathbf{r} \end{aligned} \tag{8.49}$$

Here, the unimportant numerical factor of "1/2" has been added to both terms so that $\omega^2 K$ is proportional to the kinetic energy of the plasma $\rho \mathbf{v}_1^2/2$. Although $\xi(\mathbf{r})$ and ω^2 have been shown to be real, $\xi(\mathbf{r})$ is treated as complex in Eq. (8.49) in anticipation of cases where, because of symmetry, several spatial coordinates can be Fourier analyzed (e.g., complex quantities appear when $\xi(\mathbf{r}) = \xi(r)\exp(im\theta + ikz)$).

The variational principle states that any allowable trial function $\xi(\mathbf{r})$ that produces an extremum (i.e., maximum, minimum, or saddle point) in the value of ω^2 is an actual eigenfunction of the ideal MHD normal mode equations with eigenvalue $\omega^2 = \delta W(\xi^*, \xi)/K(\xi^*, \xi)$. The proof follows by letting $\xi \to \xi + \delta\xi$, $\omega^2 \to \omega^2 + \delta\omega^2$ in Eq. (8.48). Here, $\delta\xi$ is an arbitrary perturbation away from the trial function ξ and $\delta\omega^2$ is the resulting change in ω^2. After substitution one finds that

$$\omega^2 + \delta\omega^2 = \frac{\delta W(\xi^*,\xi) + \delta W(\delta\xi^*,\xi) + \delta W(\xi^*,\delta\xi) + \delta W(\delta\xi^*,\delta\xi)}{K(\xi^*,\xi) + K(\delta\xi^*,\xi) + K(\xi^*,\delta\xi) + K(\delta\xi^*,\delta\xi)} \tag{8.50}$$

For small $\delta\xi$ and $\delta\omega^2$: (1) the leading-order terms yield an identity, (2) the terms quadratic in $\delta\xi$ can be neglected, and (3) the linear terms, including those obtained by Taylor expanding the denominator, lead to an expression for $\delta\omega^2$ that can be written as

$$\delta\omega^2 = \frac{\delta W(\delta\xi^*,\xi) + \delta W(\xi^*,\delta\xi) - \omega^2[K(\delta\xi^*,\xi) + K(\xi^*,\delta\xi)]}{K(\xi^*,\xi)} \tag{8.51}$$

Next, using the self-adjoint property of $\mathbf{F}(\xi)$ in Eq. (8.51), and setting $\delta\omega^2 = 0$, which is the requirement to form an extremum, leads to

$$\int d\mathbf{r}\{\delta\xi^* \cdot [\mathbf{F}(\xi) + \omega^2\rho\xi] + \delta\xi \cdot [\mathbf{F}(\xi^*) + \omega^2\rho\xi^*]\} = 0 \qquad (8.52)$$

Since Eq. (8.52) must be satisfied for any arbitrary $\delta\xi$, this can only occur if[1]

$$-\omega^2\rho\xi = \mathbf{F}(\xi) \qquad (8.53)$$

In order for the trial function ξ to produce an extremum in ω^2, it must satisfy the actual eigenvalue equation.

This completes the proof and demonstrates that the normal mode eigenvalue equation, Eq. (8.19), and the variational principle, Eq. (8.48), are equivalent formulations of the linearized ideal MHD stability problem.

Physically, the quantity $\delta W(\xi^*, \xi)$ represents the change in potential energy associated with the perturbation and is equal to the work done against the force $\mathbf{F}(\xi)$ in displacing the plasma by an amount ξ.

8.7 The Energy Principle

It is often of primary interest to determine whether a given MHD configuration is stable or unstable without being particularly concerned about the precise value of the growth rate or oscillation frequency for each mode. The reason, which is primarily concerned with instabilities, is that MHD growth times are typically on the order of $\tau < 50\,\mu\text{sec}$. This is much shorter than experimental pulse lengths, which are on the order of $\tau > 1\,\text{sec}$. Thus, it is usually far more important to determine the conditions to avoid instabilities than to calculate precise growth rates (which can easily be estimated). In such cases, the variational formulation just derived can be further simplified, leading to a powerful minimizing principle which determines the exact stability boundaries while providing reasonable estimates of the growth rates. The formulation is known as the Energy Principle and represents the most efficient and often the most intuitive method of determining plasma stability (Bernstein *et al.*, 1958). It receives wide application in the literature and is used extensively throughout the remainder of the book.

The Energy Principle applies to systems in which the plasma is surrounded by either a perfectly conducting wall or by a vacuum region which isolates it from a perfectly conducting wall. The Principle only requires that the force operator $\mathbf{F}(\xi)$ be self-adjoint. This property has been explicitly demonstrated for a plasma surrounded by a conducting wall. In Section 8.8 the self-adjoint property is generalized to include a vacuum region leading to the "Extended Energy Principle." For present purposes one does not have to distinguish

[1] The reality of $\mathbf{F}$ and ω^2 implies that both contributions in the square brackets yield the same result, namely Eq. (8.53).

between the wall vs. vacuum situations. All that is required is to assume that $\mathbf{F}(\xi)$ is self adjoint.

8.7.1 Statement of the Energy Principle

The physical basis for the Energy Principle is the fact that energy is exactly conserved in the ideal MHD model. As a consequence the particular extremum corresponding to the most negative eigenvalue of ω^2 actually represents the absolute minimum in potential energy δW. This in turn implies that the question of stability or instability can be determined by examining only the sign of $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})$ and not analyzing the full variational integral or normal-mode equations. Specifically, the Energy Principle states that an equilibrium is stable if and only if

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \geq 0 \tag{8.54}$$

for all allowable trial displacements (i.e., $\boldsymbol{\xi}$ bounded in energy and satisfying appropriate boundary conditions); that is, if the value of potential energy is positive for any and all trial displacements, the system is stable. If it is negative for any trial displacement, the system is unstable.

The first part of the statement is intuitively obvious. Since "any and all" trial displacements must include the actual eigenfunctions as special cases, then by assumption $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \geq 0$ for every eigenfunction implying that $\omega^2 \geq 0$ for all modes – the system is stable. The converse is not so obvious but is nonetheless true. If a trial displacement can be found that is not an actual eigenfunction but which makes $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) < 0$ the system is unstable. In other words, the (negative) value of ω^2 obtained by evaluating $\delta W/K$ using the trial $\boldsymbol{\xi}$ is less negative than the actual eigenvalue – the minimum, most unstable eigenvalue is always approached from above (i.e., the stable direction) when using trial functions.

For readers unfamiliar with the concept of an Energy Principle it may not be clear at this point how such a Principle actually simplifies the stability analysis of MHD systems. The simplifications are demonstrated by example in future chapters which examine the stability of various MHD configurations. In general, the most important simplification is that the determination of stability always results in an analysis in which the eigenvalue ω^2 does not explicitly appear. The corresponding equations are far simpler to analyze than the full eigenvalue equations as determined by either the normal-mode or variational formulations. A further discussion of the advantages of the Energy Principle is given in Section 8.7.3.

8.7.2 Proof of the Energy Principle

Proof assuming discrete normal modes

Consider now the proof of the Energy Principle. The proof would be straightforward if $\mathbf{F}(\xi)$ allowed only discrete normal modes which constituted a complete set of basis functions ξ_n, each satisfying

$$-\omega_n^2 \rho \xi_n = \mathbf{F}(\xi_n) \tag{8.55}$$

In this case, any arbitrary trial function $\xi(\mathbf{r})$ could be expanded as

$$\xi(\mathbf{r}) = \sum_0^\infty a_n \xi_n(\mathbf{r}) \tag{8.56}$$

Recall now that the ξ_n have been shown to be orthogonal (Section 8.5.6). Without loss in generality, they can be made orthonormal with respect to the density ρ by choosing their arbitrary amplitudes to satisfy

$$\int \rho \xi_m^* \cdot \xi_n \, d\mathbf{r} = \delta_{mn} \tag{8.57}$$

Then $\delta W(\xi^*, \xi)$ reduces to

$$\delta W(\xi^*, \xi) = \frac{1}{2} \sum_0^\infty |a_n|^2 \omega_n^2 \tag{8.58}$$

Thus, if a $\xi(\mathbf{r})$ could be found that makes $\delta W < 0$, then at least one $\omega_n^2 < 0$, indicating instability. Conversely, if $\delta W \geq 0$ for all $\xi(\mathbf{r})$ (i.e., any arbitrary choice of a_n) then each $\omega_n^2 \geq 0$ indicating exponential stability. Under the assumption that only discrete modes exist, the above analysis thus proves the validity of the Energy Principle.

A related proof demonstrating convergence of the most unstable eigenvalue from the stable direction follows by noting that any arbitrary trial function ξ can, without loss in generality, always be normalized so that the expansion coefficient $a_0 = 1$:

$$\xi = \xi_0 + \sum_1^\infty a_n \xi_n \tag{8.59}$$

Here, ξ_0 is the actual eigenfunction corresponding to the lowest (i.e., most unstable) eigenvalue ω_0^2. A simple calculation shows that the estimate for ω^2 using the trial function given by Eq. (8.59) has the value

$$\omega^2 = \omega_0^2 + \frac{\sum_1^\infty |a_n|^2 (\omega_n^2 - \omega_0^2)}{1 + \sum_1^\infty |a_n|^2} \geq \omega_0^2 \tag{8.60}$$

Equation (8.60) demonstrates that any estimate for ω^2 is always greater (i.e., more stable) that the actual lowest eigenvalue ω_0^2. Convergence using more terms in the trial function must occur from the stable direction.

General proof

Unfortunately, the proofs just given are not valid for the most general situation because of the existence of continua. An elegant proof of the Energy Principle, outlined below, has been given by Laval *et al*. (1965), which does not invoke the assumption of a complete set of discrete normal modes. The proof is based on the conservation of energy and is carried out in the real time domain. Consider the energy $H(t)$ given by

$$\begin{aligned} H(t) &= K\left(\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\right) + \delta W(\boldsymbol{\xi},\boldsymbol{\xi}) \\ &= \frac{1}{2}\int d\mathbf{r}\left[\rho\dot{\boldsymbol{\xi}}^2 - \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\right] \end{aligned} \tag{8.61}$$

where now $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{r}, t)$ is a real quantity.

A simple calculation that makes use of the self-adjoint property of $\mathbf{F}(\boldsymbol{\xi})$ yields

$$\frac{dH}{dt} = \int d\mathbf{r}\dot{\boldsymbol{\xi}}\cdot\left[\rho\ddot{\boldsymbol{\xi}} - \mathbf{F}(\boldsymbol{\xi})\right] = 0 \tag{8.62}$$

Equation (8.62) shows that $H(t) = H_0 =$ constant and corresponds to conservation of energy.

To show sufficiency of the Energy Principle, assume that $\delta W > 0$ for all allowable $\boldsymbol{\xi}(\mathbf{r}, t)$. Energy conservation (i.e., Eq. (8.61)) implies that

$$\delta W(\boldsymbol{\xi},\boldsymbol{\xi}) = H_0 - K(\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}) \tag{8.63}$$

Hence, unbounded growth of the kinetic energy $K(\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}) \to +\infty$ (e.g., exponential instability) would violate energy conservation since it has been assumed that $\delta W(\boldsymbol{\xi}, \boldsymbol{\xi}) > 0$: that is, $\delta W(\boldsymbol{\xi}, \boldsymbol{\xi}) > 0$ for all allowable $\boldsymbol{\xi}$ is sufficient for stability.

A simplified proof of necessity of the Energy Principle assumes that a perturbation $\boldsymbol{\eta}(\mathbf{r})$ exists that makes $\delta W(\boldsymbol{\eta}, \boldsymbol{\eta}) < 0$. Now, consider a displacement $\boldsymbol{\xi}(\mathbf{r}, t)$ satisfying initial conditions

$$\begin{aligned} \boldsymbol{\xi}(\mathbf{r},0) &= \boldsymbol{\eta}(\mathbf{r}) \\ \dot{\boldsymbol{\xi}}(\mathbf{r},0) &= 0 \end{aligned} \tag{8.64}$$

Energy conservation implies that

$$H_0 = \delta W(\boldsymbol{\xi},\boldsymbol{\xi}) + K(\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}) = (\delta W + K)_{t=0} = \delta W(\boldsymbol{\eta},\boldsymbol{\eta}) < 0 \tag{8.65}$$

One now calculates dI^2/dt^2 where

$$I(t) = K(\xi, \xi) = \frac{1}{2}\int d\mathbf{r}\rho\,\xi^2 \tag{8.66}$$

A simple calculation gives

$$\begin{aligned}\frac{d^2 I}{dt^2} &= \int d\mathbf{r}\left[\rho\dot{\xi}^2 + \xi\cdot\mathbf{F}(\xi)\right] \\ &= 2\left[K\left(\dot{\xi},\dot{\xi}\right) - \delta W(\xi,\xi)\right]\end{aligned} \tag{8.67}$$

Next, from Eq. (8.61) set $\delta W(\xi, \xi) = H_0 - K\left(\dot{\xi}, \dot{\xi}\right)$ and substitute into Eq. (8.67). The result is

$$\frac{d^2 I}{dt^2} = 2\left[2K\left(\dot{\xi},\dot{\xi}\right) - H_0\right] > -2H_0 > 0 \tag{8.68}$$

where the last inequality is a consequence of $H_0 < 0$ as shown by Eq. (8.65). Equation (8.68) implies that I grows without bound for large $t : I(t) \geq -H_0 t^2$ as $t \to \infty$ indicating that ξ increases as least as fast as t. This proves the necessity of the Energy Principle – any $\delta W(\boldsymbol{\eta}, \boldsymbol{\eta}) < 0$ leads to unbounded growth of $\boldsymbol{\xi}(\mathbf{r}, t)$.

Laval *et al.* (1965) derived a stronger result for necessity than Eq. (8.68) by a more sophisticated analysis. Specifically, they showed that if $\delta W(\boldsymbol{\eta}, \boldsymbol{\eta}) < 0$ then there exists a $\boldsymbol{\xi}$ that grows exponentially (i.e., $\xi \sim \exp(\omega_i t)$) with a growth rate at least as fast as

$$\omega_i = \left[-\frac{\delta W(\boldsymbol{\eta},\boldsymbol{\eta})}{K(\boldsymbol{\eta},\boldsymbol{\eta})}\right]^{1/2} \tag{8.69}$$

Their proof, which now follows, starts by considering a perturbation $\boldsymbol{\xi}$ satisfying a modified set of initial conditions given by

$$\begin{aligned}\xi(\mathbf{r},0) &= \boldsymbol{\eta}(\mathbf{r}) \\ \dot{\xi}(\mathbf{r},0) &= \omega_i\boldsymbol{\eta}(\mathbf{r})\end{aligned} \tag{8.70}$$

which is what one would expect for an exponentially growing mode. Conservation of energy as described by Eq. (8.62) still applies so that for the modified initial conditions the energy constant H_0 has the value

$$H_0 = \delta W(\xi,\xi) + K(\dot{\xi},\dot{\xi}) = (\delta W + K)_{t=0} = \delta W(\boldsymbol{\eta},\boldsymbol{\eta}) + \omega_i^2 K(\boldsymbol{\eta},\boldsymbol{\eta}) = 0 \tag{8.71}$$

The fact that $H_0 = 0$ implies that $\delta W(\xi, \xi) = -K(\dot{\xi}, \dot{\xi})$. Using this result and again introducing the integral $I(t)$ defined in Eq. (8.66) one finds that

$$\frac{d^2 I}{dt^2} = \int d\mathbf{r}\left[\rho\dot{\xi}^2 + \xi\cdot\mathbf{F}(\xi)\right] = 2\left[K\left(\dot{\xi},\dot{\xi}\right) - \delta W(\xi,\xi)\right] = 4K\left(\dot{\xi},\dot{\xi}\right) \tag{8.72}$$

The next step is to evaluate the quantity $I\ddot{I} - \dot{I}^2$. A short calculation yields

$$I\frac{d^2 I}{dt^2} - \left(\frac{dI}{dt}\right)^2 = \int \rho\xi^2 \, d\mathbf{r} \int \rho\dot{\xi}^2 \, d\mathbf{r} - \left[\int \rho\xi \cdot \dot{\xi} \, d\mathbf{r}\right]^2 \geq 0 \tag{8.73}$$

The right-hand side is positive because of Schwartz's inequality. Equation (8.73) can be rewritten as

$$\frac{d}{dt}\left(\frac{1}{I}\frac{dI}{dt}\right) \geq 0 \tag{8.74}$$

Both sides of the equation are integrated from 0 to t. Since integration is a summing operation, it does not change the sign of the inequality. Therefore, one obtains

$$\frac{1}{I(t)}\frac{dI(t)}{dt} - \frac{1}{I(0)}\frac{dI(0)}{dt} \geq 0 \tag{8.75}$$

Since $\dot{I}(0) = 2\omega_i I(0)$, Eq. (8.75) reduces to

$$\frac{1}{I(t)}\frac{dI(t)}{dt} \geq 2\omega_i \tag{8.76}$$

One further integration leads to the desired result

$$I(t) \geq I(0)\, e^{2\omega_i t} \tag{8.77}$$

which implies that $\boldsymbol{\xi}$ grows at least as fast as $\exp(\omega_i t)$.

The conclusion of this analysis is that if any perturbation $\boldsymbol{\eta}$ exists that makes $\delta W(\boldsymbol{\eta}, \boldsymbol{\eta}) < 0$, then the system is unstable – the displacement grows exponentially without bound. Consequently, $\delta W(\boldsymbol{\eta}, \boldsymbol{\eta}) > 0$ for all allowable trial functions is a necessary condition for stability.

This completes the proof of the Energy Principle.

8.7.3 Advantages of the Energy Principle

Consider now the advantages of using the Energy Principle. First, if one has some intuition about the form of an unstable perturbation, this form can be used as a trial function to evaluate δW. If the value of $\delta W < 0$, the Energy Principle guarantees that the actual eigenvalue must be more negative (i.e., more unstable) than the value $\omega^2 = \delta W/K$ calculated by using the trial function; that is, the existence of an allowable trial function that makes $\delta W < 0$ is sufficient for instability.

Second, when plasma parameters are such that a trial function produces an absolute minimum in δW exactly equal to zero then this trial function satisfies the marginal stability differential equation,

$$\mathbf{F}(\boldsymbol{\xi}) = 0 \tag{8.78}$$

This approach yields the exact marginal stability boundary.

Third, in the integral form of δW the various stabilizing and destabilizing mechanisms appear quite transparently, thus helping in the development of physical intuition.

Finally, there exists a very practical method for numerically testing MHD stability based on the Energy Principle. In this procedure one chooses a suitable set of simple but complete basis functions $\boldsymbol{\xi}_n$ and writes $\boldsymbol{\xi}$ as an arbitrary sum of these functions: $\boldsymbol{\xi} = \sum a_n \boldsymbol{\xi}_n$. Substituting into the expression for δW then yields $\delta W = \sum A_{mn} a_m a_n$, where the A_{mn} are well-defined, computable matrix elements. Once the A_{mn} are known, the expression for δW can be numerically minimized with respect to the coefficients a_n using standard linear algebra techniques. Clearly a clever choice and/or a sufficient number of basis functions gives an increasingly accurate indication of whether or not δW can be made negative. Note that the minimization with respect to the a_n can be carried out for any convenient choice of normalization, not necessarily the orthonormal one given by Eq. (8.57). When a trial function $\boldsymbol{\xi}$ is found that makes $\delta W < 0$ a reasonable estimate of the growth rate is obtained by setting $\omega^2 = \delta W/K$ using the trial $\boldsymbol{\xi}$. The numerical procedure just described for determining stability is very effective in multidimensional geometries where numerical shooting methods are not easy to implement.

8.8 The Extended Energy Principle

At this point in the discussion it has been shown that (1) the Energy Principle is valid when the force operator $\mathbf{F}(\boldsymbol{\xi})$ is self-adjoint, and (2) $\mathbf{F}(\boldsymbol{\xi})$ is self-adjoint for a plasma surrounded by a perfectly conducting wall (see Eq. (8.40)). This section extends the validity of the Energy Principle to plasmas that are separated from a conducting wall by a vacuum region. The resulting stability formulation is known as the "Extended Energy Principle." This is a very important result since the most dangerous MHD instabilities involve perturbations in which the plasma surface moves away from its equilibrium position. Note that such motions are not possible with a perfectly conducting wall surrounding the plasma since in this case, by definition, the surface is not allowed to move: $\mathbf{n} \cdot \boldsymbol{\xi}_\perp (S) = 0$.

The derivation of the Extended Energy Principle requires that $\mathbf{F}(\boldsymbol{\xi})$ remain self-adjoint when a vacuum region is included in the analysis. Once self-adjointness is demonstrated then, as stated, the analysis in Section 8.7 still applies, thereby extending the validity of the Energy Principle. The analysis proceeds as follows.

8.8.1 Statement of the problem

The potential energy of the plasma plus vacuum has been given by Eq. (8.25) and is repeated here for convenience,

$$\delta W(\boldsymbol{\eta}, \xi) = \delta W_F + BT \tag{8.79}$$

The fluid contribution δW_F has been shown to be self-adjoint in Eq. (8.40). Furthermore, this result did not require any knowledge of $\mathbf{n} \cdot \boldsymbol{\xi}_\perp\ (S)$. Thus, δW_F remains self-adjoint when a vacuum region is present. However, the boundary term BT was set to zero because of the simplifying assumption that the plasma was surrounded by a perfectly conducting wall: $\mathbf{n} \cdot \boldsymbol{\xi}_\perp\ (S) = 0$. This constraint is now relaxed and the goal is to show that the boundary term given in Eq. (8.26) and repeated here

$$\begin{aligned} BT &= \frac{1}{2}\int (\mathbf{n}\cdot\boldsymbol{\eta}_\perp)\left[\frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}_1(\xi_\perp) - \gamma p \nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}_\perp\cdot\nabla p\right] dS \\ &= \frac{1}{2}\int (\mathbf{n}\cdot\boldsymbol{\eta}_\perp)\left[\frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}_1(\xi_\perp) + p_1\right] dS \end{aligned} \tag{8.80}$$

is also self-adjoint when the plasma is isolated from the perfectly conducting wall by a vacuum region.

8.8.2 The boundary conditions

The first step in the transformation of Eq. (8.80) to a self-adjoint form is a careful statement of the boundary and jump conditions that must be satisfied by the perturbations. These conditions are required in order to obtain a well-posed problem for the vacuum magnetic field. The resulting solution for the field is expressed in terms of the plasma displacement evaluated on the plasma surface. This relationship is important because it connects the vacuum magnetic energy to the plasma displacement.

The simplest boundary condition is the one that must be satisfied on the rigid conducting wall surrounding the outer edge of the vacuum region. If the surface of the conducting wall is denoted by S_w then the appropriate boundary condition involves only the perturbed magnetic field and is given by

$$\left[\mathbf{n}\cdot\hat{\mathbf{B}}_1\right]_{S_w} = 0 \tag{8.81}$$

Here and below a caret signifies a vacuum quantity.

The next conditions apply to the plasma–vacuum interface. These have the form of jump conditions and are obtained by expanding the exact non-linear relations about the perturbed plasma surface $\mathbf{r} = \mathbf{r}_p + \boldsymbol{\xi}$. Two jump conditions are required, one involving $\nabla \cdot \mathbf{B} = 0$ and the other pressure balance across the surface. Consider first the exact $\nabla \cdot \mathbf{B} = 0$ jump condition given by Eq. (3.9). The linearized form can be written as

$$[\![\mathbf{n} \cdot \mathbf{B}]\!]_S = 0 \quad \rightarrow \quad \left[\mathbf{n}_0 \cdot \hat{\mathbf{B}}_1 + \mathbf{n}_1 \cdot \hat{\mathbf{B}}_0\right]_{S_p} - [\mathbf{n}_0 \cdot \mathbf{B}_1 + \mathbf{n}_1 \cdot \mathbf{B}_0]_{S_p} = 0 \qquad (8.82)$$

where S_p denotes the unperturbed plasma surface and $[\![\;]\!]$ denotes the jump from vacuum to plasma.

The goal now is to reduce this expression to one which relates $\mathbf{n}_0 \cdot \hat{\mathbf{B}}_1$ to $\boldsymbol{\xi}$. This is not as simple task as one might expect. The reason is that the perturbed normal vector $\mathbf{n}_1$ is quite complicated to calculate. There is a simpler approach that makes use of the fact that the plasma surface is a flux surface. This implies that just inside the plasma surface

$$[\mathbf{n} \cdot \mathbf{B}]_{S_-} = 0 \rightarrow [\mathbf{n}_0 \cdot \mathbf{B}_1 + \mathbf{n}_1 \cdot \mathbf{B}_0]_{S_{p-}} = 0 \qquad (8.83)$$

Thus, each of the two bracketed terms in the jump condition given by Eq. (8.82) is separately zero.

Within the plasma the relation between $\mathbf{B}$ and $\boldsymbol{\xi}$ is known: $\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B})$. Assume for the moment that there are no surface currents. Then all the components of $\mathbf{B}_1$ and $\hat{\mathbf{B}}_1$ are continuous across the interface leading to the condition $[\mathbf{n}_0 \cdot \hat{\mathbf{B}}_1]_{S_p} = [\mathbf{n}_0 \cdot \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B}_0)]_{S_p}$. For the general case where surface currents are allowed to flow one can imagine that they do so in a narrow layer just inside the plasma edge as illustrated in Fig. 8.8. In this case there is a jump in the tangential magnetic field but the normal component remains continuous. Therefore, the argument just given concerning continuity still applies. However, one must now use the magnetic field just outside the surface current layer in order to invoke continuity of the normal field across the actual plasma – vacuum interface: $\mathbf{B}_0(S_-) \rightarrow \hat{\mathbf{B}}_0(S_-)$ and $\hat{\mathbf{B}}_0(S_-) = \hat{\mathbf{B}}_0(S_+)$. Also as the layer width shrinks to zero the displacement $\boldsymbol{\xi}$ has the same value on both sides of the current sheet. The result is that the desired boundary condition relating $\mathbf{n}_0 \cdot \hat{\mathbf{B}}_1$ to $\boldsymbol{\xi}$ has the form

$$\left[\mathbf{n} \cdot \hat{\mathbf{B}}_1\right]_{S_p} = \left[\mathbf{n} \cdot \nabla \times (\boldsymbol{\xi}_\perp \times \hat{\mathbf{B}})\right]_{S_p} \qquad (8.84)$$

where for simplicity the zero subscript has been suppressed from the equilibrium quantities. Note that the condition has been derived without having had to calculate $\mathbf{n}_1$, although doing so and substituting directly into Eq. (8.82) would again yield Eq. (8.84).

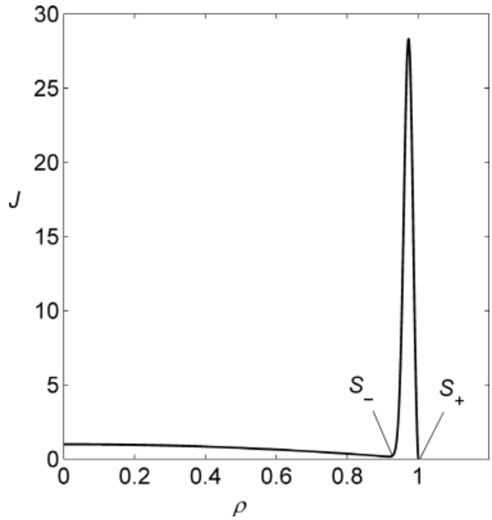

Figure 8.8 Surface currents flowing in a narrow layer at the plasma edge.

The second boundary condition is straightforward to derive but complicated to implement. It corresponds to the pressure balance jump condition also given by Eq. (3.9). The non-linear jump condition is given by

$$\left[\!\left[p + \frac{B^2}{2\mu_0} \right]\!\right]_S = 0 \tag{8.85}$$

The linearized form can be written as

$$\left[\frac{\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}_1}{\mu_0} + \boldsymbol{\xi} \cdot \nabla \left(\frac{\hat{B}^2}{2\mu_0} \right) \right]_{S_p} - \left[p_1 + \frac{\mathbf{B} \cdot \mathbf{B}_1}{\mu_0} + \boldsymbol{\xi} \cdot \nabla \left(p + \frac{B^2}{2\mu_0} \right) \right]_{S_p} = 0 \tag{8.86}$$

Equations (8.81), (8.84), and (8.86) are the required boundary and jump conditions coupling the field in the vacuum to the plasma displacement.

8.8.3 The natural boundary condition

Consider now the boundary term BT given by Eq. (8.80). There are two main problems to overcome. First, it is not in an obvious self-adjoint form. Second, even if self-adjoint, it is cumbersome to substitute trial functions to estimate stability. The reason is that it is very difficult to generate trial functions that automatically satisfy the complicated jump condition given by Eq. (8.86).

These difficulties are resolved by introducing a natural boundary condition into BT. The boundary term is then separated into two contributions. The first is the surface energy which arises when equilibrium surface currents are present. The second is the magnetic energy in the vacuum volume. Both contributions are self-adjoint by construction.

Most importantly, the principles of variational calculus prove that the introduction of a natural boundary condition eliminates the need to choose trial functions that explicitly satisfy the complicated pressure balance jump condition. In other words, it is perfectly acceptable to substitute a simple trial function consisting of a sum of terms none of which individually or collectively satisfy the boundary condition. As more terms are included in the sum the minimization of δW "naturally" drives the total trial function (i.e., the sum of all the terms) in a direction to satisfy the pressure balance jump condition. As the trial function approaches an actual eigenfunction, the pressure balance jump condition becomes exactly satisfied. For readers unfamiliar with natural boundary conditions, a short discussion is presented in Appendix G.

The introduction of the natural boundary condition into BT is a simple matter. Observe that the term in the square bracket of BT in Eq. (8.80) coincides with some of the terms in the pressure balance jump condition given by Eq. (8.86). These terms are eliminated leading to the following form of BT:

$$BT = \frac{1}{2\mu_0}\int (\mathbf{n}\cdot\boldsymbol{\eta}_\perp)\left[\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1(\boldsymbol{\xi}_\perp) + \boldsymbol{\xi}\cdot\nabla\frac{\hat{B}^2}{2} - \boldsymbol{\xi}\cdot\nabla\left(\frac{B^2}{2}+\mu_0 p\right)\right]dS \tag{8.87}$$

This is the natural boundary condition form of BT.

8.8.4 The surface energy

The terms in Eq. (8.87) are now rewritten in a form that is self-adjoint by construction. As stated, there are two contributions – the surface energy and the vacuum energy. The last two terms represent the surface energy δW_S, which can be written as

$$\begin{aligned}\delta W_S(\boldsymbol{\eta},\boldsymbol{\xi}) &= \frac{1}{2\mu_0}\int (\mathbf{n}\cdot\boldsymbol{\eta}_\perp)\left[\boldsymbol{\xi}\cdot\nabla\frac{\hat{B}^2}{2} - \boldsymbol{\xi}\cdot\nabla\left(\frac{B^2}{2}+\mu_0 p\right)\right]dS \\ &= \frac{1}{2\mu_0}\int (\mathbf{n}\cdot\boldsymbol{\eta}_\perp)\boldsymbol{\xi}\cdot\left[\!\left[\nabla\left(\frac{B^2}{2}+\mu_0 p\right)\right]\!\right]dS\end{aligned} \tag{8.88}$$

Keep in mind that even though $[\![B^2 + 2\mu_0 p]\!] = 0$ across the surface, the jump in its gradient is not necessarily zero. Specifically, $[\![\nabla(B^2 + 2\mu_0 p)]\!]$ is zero if, and only if there are no equilibrium surface currents. However, even when surface currents are present the fact that $[\![B^2 + 2\mu_0 p]\!] = 0$ remains valid implies that the tangential components of the gradient are continuous; that is, $\mathbf{b} \cdot [\![\nabla(B^2 + 2\mu_0 p)]\!] = \mathbf{b} \times \mathbf{n} \cdot [\![\nabla(B^2 + 2\mu_0 p)]\!] = 0$. The only possible non-zero contribution arises from the normal component of the gradient. Therefore, one can replace ξ in Eq. (8.88) with $(\mathbf{n} \cdot \xi)\mathbf{n} = (\mathbf{n} \cdot \xi_\perp)\,\mathbf{n}$. This leads to the usual form of the surface energy,

$$\delta W_S(\boldsymbol{\eta}_\perp, \xi_\perp) = \frac{1}{2\mu_0} \int (\mathbf{n} \cdot \boldsymbol{\eta}_\perp)(\mathbf{n} \cdot \xi_\perp)\mathbf{n} \cdot \left[\!\!\left[\nabla \left(\frac{B^2}{2} + \mu_0 p \right) \right]\!\!\right] dS \qquad (8.89)$$

The surface energy is clearly self-adjoint by construction. It also depends only on the perpendicular components of ξ and $\boldsymbol{\eta}$. If the plasma current is finite or smoothly approaches zero at the plasma edge then $\delta W_S = 0$. When surface currents flow on the plasma boundary then $\delta W_S \neq 0$ and Eq. (8.89) must be used to evaluate its contribution to the total potential energy. In general δW_S can be either positive or negative.

8.8.5 The vacuum energy

As the analysis now stands the natural boundary condition form of BT is given by

$$BT = \delta W_S + \frac{1}{2\mu_0} \int (\mathbf{n} \cdot \boldsymbol{\eta}_\perp) \left[\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}_1(\xi_\perp) \right] dS \qquad (8.90)$$

To complete the analysis it is necessary to show that the last term in Eq. (8.90) is equal to the vacuum magnetic energy. Several steps are required.

The starting point is the generalized definition of the vacuum magnetic energy, which is self-adjoint by construction,

$$\delta W_V(\boldsymbol{\eta}, \xi) = \frac{1}{2\mu_0} \int \hat{\mathbf{B}}_1(\boldsymbol{\eta}) \cdot \hat{\mathbf{B}}_1(\xi) d\mathbf{r} \qquad (8.91)$$

where the integration is over the vacuum volume. The derivation proceeds by introducing the vector potential:

$$\begin{aligned} \hat{\mathbf{B}}_1(\boldsymbol{\eta}) &= \nabla \times \hat{\mathbf{A}}_1(\boldsymbol{\eta}) \\ \hat{\mathbf{B}}_1(\xi) &= \nabla \times \hat{\mathbf{A}}_1(\xi) \end{aligned} \qquad (8.92)$$

In the vacuum region the vector potentials satisfy $\nabla \times \nabla \times \hat{\mathbf{A}}_1(\boldsymbol{\eta}) = \nabla \times \nabla \times \hat{\mathbf{A}}_1(\xi) = 0$. A convenient choice of gauge condition assumes that the scalar electric potentials satisfy $\phi_1(\boldsymbol{\eta}) = \phi_1(\xi) = 0$ implying that the perturbed electric fields are given by

$$\begin{aligned} \hat{\mathbf{E}}_1(\boldsymbol{\eta}) &= i\omega\hat{\mathbf{A}}_1(\boldsymbol{\eta}) \\ \hat{\mathbf{E}}_1(\xi) &= i\omega\hat{\mathbf{A}}_1(\xi) \end{aligned} \qquad (8.93)$$

Now, δW_V can be converted into a surface integral by using Gauss' theorem and the vacuum field identity $\nabla \cdot [\hat{\mathbf{A}}_1(\boldsymbol{\eta}) \times \nabla \times \hat{\mathbf{A}}_1(\xi)] = \nabla \times \hat{\mathbf{A}}_1(\boldsymbol{\eta}) \cdot \nabla \times \hat{\mathbf{A}}_1(\xi)$. The vacuum energy reduces to

$$\begin{aligned} \delta W_V(\boldsymbol{\eta}, \xi) &= \frac{1}{2\mu_0} \int \nabla \cdot [\hat{\mathbf{A}}_1(\boldsymbol{\eta}) \times \nabla \times \hat{\mathbf{A}}_1(\xi)] d\mathbf{r} \\ &= -\frac{1}{2\mu_0} \int \mathbf{n} \cdot \hat{\mathbf{A}}_1(\boldsymbol{\eta}) \times \hat{\mathbf{B}}_1(\xi) dS \\ &= -\frac{1}{2\mu_0} \int \mathbf{n} \times \hat{\mathbf{A}}_1(\boldsymbol{\eta}) \cdot \hat{\mathbf{B}}_1(\xi) dS \end{aligned} \tag{8.94}$$

The surface integral is over the plasma boundary with the negative sign appearing because $\mathbf{n}$ is defined as the outward pointing normal vector. Also the contribution from the outer perfectly conducting wall is zero since the vanishing of the normal component of magnetic field is equivalent to the vanishing of the tangential component of electric field. In other words, the boundary condition given by Eq. (8.81) is equivalent to $[\mathbf{n} \times \hat{\mathbf{A}}_1(\boldsymbol{\eta})]_{S_w} = 0$.

The next step is to relate the surface value of $\hat{\mathbf{A}}_1$ to the plasma displacement. This is accomplished by considering the region just inside the plasma surface. In this region the ideal Ohm's law still applies: $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$. The linearized form is $\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0 = 0$. If one introduces the vector potential in the plasma $\mathbf{A}_1(\boldsymbol{\eta})$ and the plasma displacement $\mathbf{v}_1 = -i\omega\boldsymbol{\eta}$ then the Ohm's law reduces to $\mathbf{A}_1(\boldsymbol{\eta}) - \boldsymbol{\eta} \times \mathbf{B}_0 = 0$. Here too the gauge condition $\phi_1(\boldsymbol{\eta}) = 0$ has been assumed. In the general case where surface currents are present the plasma magnetic field must be evaluated at the outer edge of the layer as has been shown in Fig. 8.8. This requires replacing $\mathbf{B}_0 \to \hat{\mathbf{B}}_0$ in order to exploit the interface continuity properties. Thus the general relation between $\mathbf{A}_1(\boldsymbol{\eta})$ and $\boldsymbol{\eta}$ at the plasma interface is given by $\mathbf{A}_1(\boldsymbol{\eta}) - \boldsymbol{\eta} \times \hat{\mathbf{B}}_0 = 0$. The key step now is to recognize that $\mathbf{A}_1$ is continuous across the interface: $\mathbf{A}_1(\boldsymbol{\eta}) = \hat{\mathbf{A}}_1(\boldsymbol{\eta})$. From this discussion one sees that the relation between $\hat{\mathbf{A}}_1$ and $\boldsymbol{\eta}$ on the unperturbed plasma surface can be written as

$$\mathbf{n} \times \hat{\mathbf{A}}_1(\boldsymbol{\eta}) = \mathbf{n} \times (\boldsymbol{\eta} \times \hat{\mathbf{B}}_0) = -(\mathbf{n} \cdot \boldsymbol{\eta})\hat{\mathbf{B}} \tag{8.95}$$

Equation (8.95) is substituted into Eq. (8.94) leading to

$$\delta W_V(\boldsymbol{\eta}, \xi) = \frac{1}{2\mu_0} \int \hat{\mathbf{B}}_1(\boldsymbol{\eta}) \cdot \hat{\mathbf{B}}_1(\xi) d\mathbf{r} = \frac{1}{2\mu_0} \int (\mathbf{n} \cdot \boldsymbol{\eta})[\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}_1(\xi)] dS \tag{8.96}$$

Observe that the last term in Eq. (8.96) is exactly equal to the second term in Eq. (8.90). The conclusion is that the boundary term can be written as

$$BT = \delta W_S + \delta W_V \tag{8.97}$$

where both δW_S and δW_V are self-adjoint by construction. Based on this result it follows that the operator $\mathbf{F}(\xi)$ is self-adjoint with either a conducting wall surrounding the plasma or when a vacuum region isolates the plasma from the conducting wall. The Energy Principle has thereby been generalized to the Extended Energy Principle.

8.8.6 Summary of the Extended Energy Principle

Having completed the derivation of the Extended Energy Principle it is convenient to collect and summarize the main results. These are as follows. A plasma is ideal MHD stable if and only if

$$\delta W(\xi^*, \xi) \geq 0 \tag{8.98}$$

for all allowable displacements (those whose energy is bounded and which satisfy appropriate boundary conditions). If any trial displacement results in $\delta W(\xi^*, \xi) < 0$ the plasma is unstable.

The potential energy comprises three self-adjoint contributions given by

$$\delta W(\xi^*, \xi) = \delta W_F + \delta W_S + \delta W_V \tag{8.99}$$

After setting $\boldsymbol{\eta} = \xi^*$ one finds that the separate contributions can be written as

The fluid energy

$$\begin{aligned}\delta W_F(\xi^*, \xi) = \frac{1}{2\mu_0}\int\limits_P \Big\{&|\mathbf{Q}_\perp|^2 + B^2|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\boldsymbol{\kappa}|^2 + \mu_0\gamma p|\nabla\cdot\xi|^2 \\ &- \mu_0[(\xi_\perp\cdot\nabla p)(\xi_\perp^*\cdot\boldsymbol{\kappa}) + (\xi_\perp^*\cdot\nabla p)(\xi_\perp\cdot\boldsymbol{\kappa})] \\ &- (\mu_0 J_\parallel/2)[\xi_\perp^*\times\mathbf{b}\cdot\mathbf{Q}_\perp + \xi_\perp\times\mathbf{b}\cdot\mathbf{Q}_\perp^*]\Big\}d\mathbf{r}\end{aligned} \tag{8.100}$$

The surface energy

$$\delta W_S(\xi_\perp^*, \xi_\perp) = \frac{1}{2\mu_0}\int\limits_{S_p} |\mathbf{n}\cdot\xi_\perp|^2\mathbf{n}\cdot\left[\!\!\left[\nabla\left(\frac{B^2}{2} + \mu_0 p\right)\right]\!\!\right] dS \tag{8.101}$$

The vacuum energy

$$\delta W_V(\xi_\perp^*, \xi_\perp) = \frac{1}{2\mu_0}\int\limits_V |\hat{\mathbf{B}}_1|^2 d\mathbf{r} \tag{8.102}$$

In these expressions $\mathbf{Q}(\xi_\perp) = \nabla\times(\xi_\perp\times\mathbf{B})$. The subscripts on the integrals correspond to the following: P = plasma volume, V = vacuum volume, S_p = plasma surface.

The fluid and surface contributions are expressed directly in terms of $\boldsymbol{\xi}$. The perturbed magnetic field in the vacuum $\hat{\mathbf{B}}_1$ is, as shown below, a function only of $\mathbf{n} \cdot \boldsymbol{\xi}_\perp$ on the plasma surface. To evaluate $\hat{\mathbf{B}}_1$ one must solve

$$\nabla \cdot \hat{\mathbf{B}}_1 = \nabla \times \hat{\mathbf{B}}_1 = 0 \tag{8.103}$$

The solutions can be found either by introducing a vector potential or a scalar magnetic potential. In practice the scalar magnetic potential is often the easiest to implement. The boundary conditions coupling $\hat{\mathbf{B}}_1$ to $\boldsymbol{\xi}_\perp$ are given by

$$\begin{aligned} \left[\mathbf{n} \cdot \hat{\mathbf{B}}_1\right]_{S_p} &= \left[\mathbf{n} \cdot \nabla \times (\boldsymbol{\xi}_\perp \times \hat{\mathbf{B}})\right]_{S_p} \\ \left[\mathbf{n} \cdot \hat{\mathbf{B}}_1\right]_{S_w} &= 0 \end{aligned} \tag{8.104}$$

Lastly, it is worth noting that the first boundary condition can be simplified by writing the equilibrium flux surfaces in the vicinity of the plasma surface as $\psi(\mathbf{r}) = \psi_p$ with $\psi_p =$ constant serving as the flux surface label and $\psi_p = 0$ corresponding to the actual plasma–vacuum interface. Next, recall that the normal vector on any surface is, by definition, defined as

$$\mathbf{n}(\mathbf{r}) = \nabla\psi / |\nabla\psi| \tag{8.105}$$

A short calculation then shows that the first boundary condition reduces to

$$\begin{aligned} \left[\mathbf{n} \cdot \hat{\mathbf{B}}_1\right]_{S_p} &= \frac{1}{|\nabla\psi|}\left[\hat{\mathbf{B}} \cdot \nabla(\nabla\psi \cdot \boldsymbol{\xi}_\perp)\right]_{S_p} \\ &= \left[\hat{\mathbf{B}} \cdot \nabla(\mathbf{n} \cdot \boldsymbol{\xi}_\perp) - (\mathbf{n} \cdot \boldsymbol{\xi}_\perp)\left(\mathbf{n} \cdot \nabla\hat{\mathbf{B}} \cdot \mathbf{n}\right)\right]_{S_p} \end{aligned} \tag{8.106}$$

As stated $\hat{\mathbf{B}}_1$ depends only on the normal component of $\boldsymbol{\xi}_\perp$ evaluated on the surface: $\nabla\psi \cdot \boldsymbol{\xi}_\perp = |\nabla\psi|\,(\mathbf{n} \cdot \boldsymbol{\xi}_\perp)$.

The summary is now complete and the Extended Energy Principle will be used extensively throughout the remainder of the book.

8.9 Incompressibility

8.9.1 The general minimizing condition

To test for MHD stability one must do as complete a job as possible finding a trial function that minimizes δW. Stability is determined by examining the sign of the resulting minimized δW. In general, the form of the minimizing trial function depends upon the profiles of the equilibrium fields. Also, keep in mind that the minimization must be carried out independently for each of the three vector components of $\boldsymbol{\xi}$.

However, because of the simple way in which $\xi_{||}$ appears in δW it is possible to minimize the potential energy once and for all with respect to $\xi_{||}$ for arbitrary geometry. The resulting form of δW is then only a function of $\boldsymbol{\xi}_\perp$ thereby reducing the number of ξ components to vary from three to two.

The general minimizing condition is obtained by recalling that the only appearance of $\xi_\|$ in δW occurs in the plasma compressibility term in δW_F. One can then separate δW_F into two components,

$$\delta W_F(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \delta W_\perp + \delta W_C \tag{8.107}$$

where

$$\begin{aligned}\delta W_\perp(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp) = \frac{1}{2\mu_0}\int_P \Big\{&|\mathbf{Q}_\perp|^2 + B^2|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2 \\ &-\mu_0[(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa}) + (\boldsymbol{\xi}_\perp^*\cdot\nabla p)(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})] \\ &-(\mu_0 J_\|/2)[\boldsymbol{\xi}_\perp^*\times\mathbf{b}\cdot\mathbf{Q}_\perp + \boldsymbol{\xi}_\perp\times\mathbf{b}\cdot\mathbf{Q}_\perp^*]\Big\}d\mathbf{r} \\ \delta W_C(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \frac{1}{2\mu_0}\int_P \mu_0\gamma p|\nabla\cdot\boldsymbol{\xi}|^2 d\mathbf{r}&\end{aligned} \tag{8.108}$$

The minimizing $\xi_\|$ is found by writing $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\|\mathbf{b}$ and letting $\xi_\| \to \xi_\| + \delta\xi_\|$ in δW_C, the only place where it appears. One then sets the corresponding variation in δW_C to zero:

$$\delta(\delta W_C) = \frac{1}{2}\int_P \gamma p\Big[(\nabla\cdot\boldsymbol{\xi}^*)(\nabla\cdot\delta\xi_\|\mathbf{b}) + (\nabla\cdot\boldsymbol{\xi})\Big(\nabla\cdot\delta\xi_\|^*\mathbf{b}\Big)\Big]d\mathbf{r} = 0 \tag{8.109}$$

Integrating the $\delta\xi_\|$ terms by parts yields

$$\delta(\delta W_C) = -\frac{1}{2}\int_P \Big[\delta\xi_\|\mathbf{b}\cdot\nabla(\gamma p\nabla\cdot\boldsymbol{\xi}^*) + \delta\xi_\|^*\mathbf{b}\cdot\nabla(\gamma p\nabla\cdot\boldsymbol{\xi})\Big]d\mathbf{r} = 0 \tag{8.110}$$

The boundary term vanishes since $\mathbf{n}\cdot\mathbf{b} = 0$ on the plasma surface. If the variation is set to zero for arbitrary $\delta\xi_\|$ and use is made of the fact that $\mathbf{B}\cdot\nabla p = 0$, then one obtains the general minimizing condition

$$\mathbf{B}\cdot\nabla(\nabla\cdot\boldsymbol{\xi}) = 0 \tag{8.111}$$

8.9.2 Ergodic systems

For most configurations, which are characterized by ergodic field lines, the operator $\mathbf{B}\cdot\nabla$ is non-singular and consequently Eq. (8.111) implies that the general minimizing condition reduces to

$$\nabla\cdot\boldsymbol{\xi} = 0 \tag{8.112}$$

That is, the most unstable perturbations (in the context of minimizing δW) are incompressible.

In a sense this is obvious from Eq. (8.108) since the only term containing $\xi_{\|}$ is positive. Thus, its smallest value is zero and is obtained by setting $\nabla \cdot \boldsymbol{\xi} = 0$. However, it is the components of $\boldsymbol{\xi}$ that are the physical quantities and not $\nabla \cdot \boldsymbol{\xi}$ itself. Therefore, setting $\nabla \cdot \boldsymbol{\xi} = 0$, requires that a physically allowable $\xi_{\|}$ be found that satisfies

$$\mathbf{B} \cdot \nabla \left(\frac{\xi_{\|}}{B} \right) = -\nabla \cdot \boldsymbol{\xi}_{\perp} \tag{8.113}$$

As for Eq. (8.111), physically allowable $\xi_{\|}$ are possible only if the operator $\mathbf{B} \cdot \nabla$ is non-singular. Even if $\mathbf{B} \cdot \nabla$ vanishes on isolated magnetic surfaces, a $\xi_{\|}$ can be constructed that is bounded in the vicinity of these surfaces but that makes a vanishingly small contribution to the plasma compressional energy.

8.9.3 Closed line systems

Although $\mathbf{B} \cdot \nabla$ is non-singular for most configurations there is one class of magnetic geometries for which the operator cannot be inverted. These are closed line configurations, for example a toroidal Z-pinch. For a closed line system the operator $\mathbf{B} \cdot \nabla$ is not automatically non-singular thus leading to $\nabla \cdot \boldsymbol{\xi} \neq 0$.

To demonstrate the problem consider the example of a pure cylindrical Z-pinch with equilibrium magnetic field $\mathbf{B} = B_\theta(r)\mathbf{e}_\theta$. If the perturbations are Fourier analyzed with respect to θ and z so that $\boldsymbol{\xi} = \boldsymbol{\xi}(r)\exp[i(m\theta + kz)]$ then for the $m = 0$ mode the operator $\mathbf{B} \cdot \nabla$ acting on any scalar identically vanishes. In this situation $\nabla \cdot \boldsymbol{\xi} = \nabla \cdot \boldsymbol{\xi}_{\perp} + \mathbf{B} \cdot \nabla(\xi_{\|}/B) = \nabla \cdot \boldsymbol{\xi}_{\perp}$; that is, there is no appearance of $\xi_{\|}$ in δW and the compressibility term $\gamma p|\nabla \cdot \boldsymbol{\xi}_{\perp}|^2$ must be maintained and included in the minimization with respect to $\boldsymbol{\xi}_{\perp}$.

For a general closed line configuration the operator $\mathbf{B} \cdot \nabla$ does not identically vanish but there is a periodicity constraint requiring $\xi_{\|}(l) = \xi_{\|}(l + L)$ on every magnetic line since the lines are closed. Here l is arc length, $\mathbf{B} \cdot \nabla = B(\partial/\partial l)$, and L is the total length of the field line under consideration. Note that the periodicity constraint only applies to perturbations that maintain the closed line symmetry of the equilibrium. For perturbations that break the closed line symmetry, the most unstable modes are incompressible. For modes which do not break the closed line symmetry it is necessary to add a homogeneous solution to Eq. (8.112) so that

$$\nabla \cdot \boldsymbol{\xi} = f(p) \tag{8.114}$$

where f is an arbitrary function of the pressure p which satisfies the homogeneous equation $\mathbf{B} \cdot \nabla p = 0$. Solving Eq. (8.114) for $\xi_{\|}$ yields

$$\frac{\xi_{\|}}{B} = -\int_0^l \frac{dl}{B} \nabla \cdot \boldsymbol{\xi}_{\perp} + f(p) \int_0^l \frac{dl}{B} \tag{8.115}$$

Note that $f(p)$ can be taken outside the integral since p is constant along a field line. The periodicity constraint requires that $f(p)$ be chosen as follows.

$$f(p) = \langle \nabla \cdot \boldsymbol{\xi}_{\perp} \rangle = \frac{\oint \frac{dl}{B} \nabla \cdot \boldsymbol{\xi}_{\perp}}{\oint \frac{dl}{B}} \tag{8.116}$$

The end result is that in closed line systems the minimum value of the plasma compressibility term cannot be made zero, but has the value

$$\frac{1}{2} \int_P \gamma p |\nabla \cdot \boldsymbol{\xi}|^2 d\mathbf{r} = \frac{1}{2} \int_P \gamma p |\langle \nabla \cdot \boldsymbol{\xi}_{\perp} \rangle|^2 d\mathbf{r} \tag{8.117}$$

8.9.4 Summary and discussion

The analysis just presented has shown that δW can be minimized with respect to $\xi_{\|}$ for arbitrary geometry. The displacement $\xi_{\|}$ appears only in δW_F, which, after minimization, has the form

$$\delta W_F(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \delta W_{\perp}(\boldsymbol{\xi}_{\perp}^*, \boldsymbol{\xi}_{\perp}) + \delta W_C(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \tag{8.118}$$

Here, $\delta W_{\perp}(\boldsymbol{\xi}_{\perp}^*, \boldsymbol{\xi}_{\perp})$ is given by Eq. (8.108) and

$$\delta W_C(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \begin{cases} 0 & \text{ergodic systems} \\ \dfrac{1}{2\mu_0} \displaystyle\int_P \mu_0 \gamma p |\langle \nabla \cdot \boldsymbol{\xi}_{\perp} \rangle|^2 d\mathbf{r} & \text{closed line systems} \end{cases} \tag{8.119}$$

Minimization of δW now only involves the two vector components of $\boldsymbol{\xi}_{\perp}$.

Based on these results there is an interesting conjecture that one can make concerning the accuracy of ideal MHD marginal stability predictions. Recall that the validity of ideal MHD requires that the plasma be collision dominated, a condition not satisfied in fusion plasmas. A primary consequence of this invalid assumption is that the ideal MHD energy equation has the form of an adiabatic fluid: $d(p/n^{\gamma})/dt = 0$. Note the appearance of the ratio of specific heats γ. Therefore, the appearance of γ in any marginal stability condition implies that the adiabatic energy equation has played an important role and the results should therefore be viewed with some suspicion.

In this connection observe that γ does not appear in δW for ergodic systems since the most unstable modes are incompressible – the adiabatic compressibility of the plasma does not play a role. Therefore, one can conjecture that ideal MHD marginal stability predictions may indeed be accurate for ergodic systems since the model is invalid only where it is unimportant. In contrast, for closed line systems γ appears in a stabilizing contribution to the marginal stability predictions and one should be cautious relying too heavily on plasma compressibility to provide stabilization. This last point is discussed further in Chapter 10.

8.10 Vacuum versus force-free plasma

8.10.1 The nature of the problem

An important but somewhat subtle issue is the dependence of stability boundaries on the assumption that the plasma is surrounded by a vacuum rather than a force-free plasma. The issue is as follows. Many confinement configurations consist of a hot core of plasma supporting most of the pressure gradient and carrying nearly all of the current. The edge of the core makes contact with either a limiter or divertor. The space between either of these first points of contact and the vacuum chamber (which for present purposes is assumed to be a perfectly conducting wall), is filled with a cold, low-density, low-pressure, current-free region of plasma. Since the electrical resistivity of this outer region is orders of magnitude larger than that in the core it might appear reasonable to treat this region as a perfect insulator – a vacuum.

However, it is more often the case that the resistivity of the outer region is still sufficiently low that its resistive diffusion time is long compared to the characteristic MHD time. To quantify this statement assume that the width of the region is of order Δr. Now, for the outer plasma the characteristic ideal MHD time is the ion thermal transit time, $\tau_{MHD} \sim \Delta r/V_{Ti}$, while the characteristic resistive time is the magnetic field diffusion time, $\tau_{RES} \sim \mu_0(\Delta r)^2/\eta_{\|}$. Using Spitzer resistivity and typical outer plasma parameters $\Delta r \approx 0.02\,\mathrm{m}$, $T_i \approx 10\,\mathrm{eV} = 0.01\,\mathrm{keV}$, one finds

$$\frac{\tau_{RES}}{\tau_{MHD}} \sim \frac{\mu_0(\Delta r)V_{Ti}}{\eta_{\|}} \sim 1.2 \times 10^7(\Delta r)T_k^2 \approx 24 \tag{8.120}$$

Since $\tau_{RES} \gg \tau_{MHD}$ the cold plasma still effectively behaves like a perfect conductor. It would thus seem more realistic to treat this region not as a vacuum, but as an ideal MHD plasma with low pressure (i.e., force-free) carrying no equilibrium current.

8.10.2 Vacuum vs. force-free plasma: the same results

The replacement of a perfectly insulating region with a perfectly conducting region might be expected to have a large effect on the overall stability. After all, the electrical resistivity has changed from $\eta = \infty$ to $\eta = 0$. While large effects may occur there are many cases where there is no effect at all. To understand the situation, compare the outer region contribution to δW for each case: (1) the vacuum given by Eq. (8.102) and (2) the force-free plasma with zero current given by the standard form of δW_F in Eq. (8.26) with $\mathbf{J} \to 0$, $p \to 0$,

$$\begin{aligned} \delta W_V &= \frac{1}{2\mu_0}\int_V |\mathbf{B}_1^2| d\mathbf{r} \qquad \text{vacuum} \\ \delta W_V &= \frac{1}{2\mu_0}\int_V |\mathbf{Q}^2| d\mathbf{r} \qquad \text{force-free plasma} \end{aligned} \tag{8.121}$$

Since $\mathbf{B}_1 = \mathbf{Q} = \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B})$ both energy contributions have the identical form. One might then be led to conclude that the magnetic perturbation $\mathbf{B}_1$, which minimizes the vacuum energy, also minimizes the force-free plasma energy. Therefore, both contributions to the overall δW should always be identical leading to the same marginal stability boundaries.

8.10.3 Vacuum vs. force-free plasma: different results

The above conclusion would be true if it were not for an additional constraint on the force-free plasma. Specifically, the plasma displacement resulting from the minimizing magnetic perturbation must correspond to a physically allowable motion, one in which $K(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp)$ is bounded. No such constraint exists in the vacuum since $\boldsymbol{\xi}_\perp$ has no meaning in this region.

For a well-defined, bounded magnetic perturbation $\mathbf{B}_1$, the force-free plasma topological constraint arises when attempting to invert the relationship $\mathbf{Q} = \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B})$ to determine $\boldsymbol{\xi}_\perp$. Whether or not the inversion is singular is closely related to the properties of the $\mathbf{B} \cdot \nabla$ operator. For example, in a general one-dimensional screw pinch the relationship between Q_r and ξ_r is given by $Q_r = \mathbf{B} \cdot \nabla \xi_r$. After Fourier analyzing with respect to θ and z one can invert this relationship yielding $\xi_r = -iQ_r/F$, $F = kB_z(r) + mB_\theta(r)/r$. Hence, if $F(r)$ vanishes anywhere in the outer region ξ_r is singular and $K(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp)$ is unbounded.

When this occurs, the stability of the force-free region must be recomputed with $\mathbf{n} \cdot \mathbf{Q}$ set to zero, not on the outer conductor, but on the singular surface where $F(r) = 0$ in order that ξ_r remain bounded. The force-free plasma is more stable than the vacuum since the new boundary condition is equivalent to moving the conducting boundary inward, a more constrained situation.

8.10.4 The real situation: a resistive region

Even with the above distinction between vacuum and force-free plasma there still remains one important physics issue. Each of the two cases discussed represents opposing limits of plasma resistivity. If, however, finite resistivity is included in the Ohm's law, the plasma can diffuse through the magnetic field implying that the ideal MHD frozen-in topological constraint, associated with perfect conductivity, no longer applies. Under this situation, the marginal stability boundaries become identical to the vacuum case (since no additional constraints are necessary) but the growth rates are much lower, depending upon the value of the resistivity at the singular surface.

In summary, the stability behavior of the outer region is as follows. The most pessimistic description corresponds to the vacuum case. Here, no additional constraints need to be imposed and the characteristic growth times are those of ideal MHD. If the vacuum region is replaced by an ideal force-free plasma the stability may or may not change depending upon whether the particular mode under consideration has a singular surface in the outer region. With no singular surface the situation is identical to the vacuum case. If there is a singular surface the force-free plasma is considerably more stable because of the ideal MHD topological constraint which effectively moves the conducting wall inward to the singular surface. Finally, when the outer region is filled with a resistive force-free plasma the topological constraint is eliminated. The marginal stability boundaries are again identical to the vacuum case. The growth rates are the same as for a vacuum when no singular surface is present. They are much smaller, but still finite, when a singular surface exists in the resistive plasma.

The resistive force-free plasma provides the most realistic description of the outer region. Even so, since the remainder of the book is primarily concerned with the determination of stability boundaries, whenever the outer region plays an important role in the stability of a given mode it suffices to treat this region as a vacuum.

8.11 Classification of MHD instabilities

This section describes a method of classifying MHD instabilities in terms of the general structure of the modes and the sources driving the instabilities. Also discussed are those properties of the equilibrium magnetic field that are effective in reducing or eliminating specific types of modes. The classification system divides naturally into two basic categories. First, one must distinguish whether a given instability is predominately an internal or external mode. Second, each instability can be characterized by its dominant driving source. Thus, a given mode can be either pressure driven or current driven. A detailed description of the classification is given below.

8.11.1 Internal/fixed boundary modes

Consider a magnetic configuration in which the plasma is surrounded by a vacuum. Instabilities whose mode structure does not require any motion of the plasma–vacuum interface away from its equilibrium position are called internal or fixed boundary modes. Oftentimes, an internal mode will have a singular surface (i.e., a surface where $\mathbf{B} \cdot \nabla$ vanishes) inside the plasma. The boundary condition applicable to such modes is $[\mathbf{n} \cdot \boldsymbol{\xi}_\perp]_{S_p} = 0$ and is equivalent to moving the conducting wall onto the plasma surface. For internal modes it is only necessary to minimize δW_F since $\delta W_S = \delta W_V = 0$.

8.11.2 External/free boundary modes

If the plasma–vacuum interface moves from its equilibrium position during an unstable MHD perturbation, the corresponding modes are known as external or free-boundary modes. These perturbations often possess a singular surface in the vacuum region. For such instabilities δW_S and δW_V must be evaluated as well as δW_F since $[\mathbf{n} \cdot \boldsymbol{\xi}_\perp]_{S_p} \neq 0$. Although an internal mode can be viewed as the special case of an external mode with $[\mathbf{n} \cdot \boldsymbol{\xi}_\perp]_{S_p} = 0$, the definition assumed here makes each mode mutually exclusive; that is, if $[\mathbf{n} \cdot \boldsymbol{\xi}_\perp]_{S_p} = 0$, the mode is an internal mode. If not, it is an external mode.

8.11.3 Pressure-driven modes

It was shown in Section 8.3 that a plasma immersed in an infinite, homogeneous, unidirectional magnetic field is always MHD stable. It was later shown by Eq. (8.100) that there are two possible sources of MHD instability, one proportional to ∇p and the other to $J_\|$. MHD instabilities in which the dominant destabilizing term is the one proportional to ∇p are known as pressure-driven modes. Since $\nabla p \propto \mathbf{J}_\perp$ these modes can exist even if no parallel currents are present in the plasma. Unstable pressure-driven instabilities in which $J_\|$ plays no role are usually internal modes with very short wavelengths perpendicular to the magnetic field and long wavelengths parallel to the field. These types of pressure-driven modes are traditionally subdivided into two categories: interchange and ballooning instabilities.

Interchange instabilities

Interchange instabilities are very similar in nature to the Rayleigh–Taylor instability (Chandraseker, 1961). Actually, except in special asymptotic limits the interchange perturbation is not a true mode of the system but represents a special trial function chosen to minimize the line bending contribution in δW_F (i.e., for an

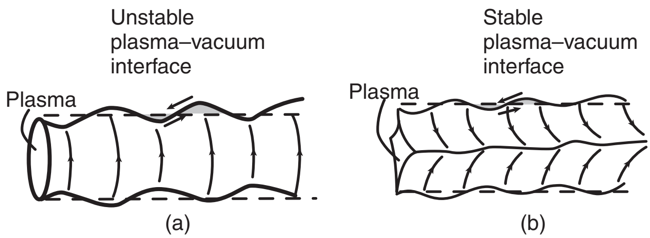

Figure 8.9 Illustration of configurations which are (a) unstable and (b) stable against interchange perturbations.

interchange, $\mathbf{Q}_\perp \approx 0$). The interchange instability is important in the RFP, the stellarator, and sometimes in tokamaks.

The interchange perturbation can lead to instability depending upon the relative sign of the magnetic field line curvature with respect to the pressure gradient. If the field lines bend towards the plasma their tension tends to make them shorten and collapse inward. The plasma pressure, on the other hand, has a natural tendency to expand outward. In such cases a perturbation that "interchanges" two flux tubes at different radii leads to a system with lower potential energy and hence instability (see Fig. 8.9). Because of the way the surfaces are fluted, the perturbations are also sometimes known as "flute modes." When the field lines bend away from the plasma, the system is stable to interchange perturbations.

From this description it follows that interchange instabilities represent plasma perturbations that are nearly constant along a field line (i.e., no line bending). Perpendicular to the field the most unstable perturbations have very rapid variation while parallel to the field the variation is much slower (i.e., $k_\perp a \gg 1$, $k_\parallel/k_\perp \ll 1$). Because of the short perpendicular wavelength the modes tend to be highly localized in radius. As such they are often called localized interchanges and are very amenable to analysis. In general, localized interchanges lead to necessary conditions for stability which can be expressed solely in terms of local values of the equilibrium quantities. Examples of such conditions are the Suydam criterion (one dimensional) and the Mercier criterion (two and three dimensional).

There are several properties of the magnetic geometry that can be effective in stabilizing interchanges. First, if there is shear in the magnetic field (i.e., the direction of $\mathbf{B}$ changes from one flux surface to another), it is not possible to interchange two neighboring flux tubes without at least a little bending of the magnetic lines, which is a stabilizing effect. When the pressure gradient is sufficiently small compared to the shear, the interchange can be stabilized. A second stabilizing method makes use of the fact that on the inside of a torus the local toroidal field line curvature is favorable – the lines bend away from the plasma. Thus, as a given magnetic line encircles the torus it in general passes through

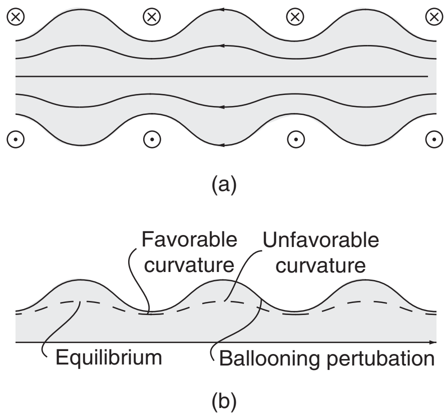

Figure 8.10 (a) Plasma with local regions of favorable and unfavorable curvature. (b) Plasma surface showing a ballooning perturbation.

alternating regions of favorable and unfavorable curvature. By carefully designing the configuration, the "average" curvature can be made favorable thereby stabilizing interchanges. Such configurations are said to possess a "magnetic well" or "average-favorable curvature."

Ballooning modes

Ballooning modes are internal pressure-driven instabilities that occur in toroidal or other multidimensional configurations. These instabilities are important because they determine one set of criteria which limits the maximum achievable value of β.

The designation "ballooning" refers to the fact that in multidimensional geometries the curvature of a magnetic field line often alternates between regions of favorable and unfavorable curvature. Thus, a perturbation that is not constant, but varies slowly along a field line in such a way that the mode is concentrated in the unfavorable curvature region, can lead to more unstable situations than the simple interchange perturbation. However, concentrating the mode also produces some stabilizing line bending (see Fig. 8.10). In effect, the "ballooning" nature of the perturbation in the unfavorable curvature region increases the pressure-driven destabilizing contribution to δW_F. If the localization is not too severe, the accompanying increase in stability from the line bending cannot compensate for this destabilizing effect.

Magnetic shear can be helpful in stabilizing ballooning modes. However, once in the regime where ballooning modes are important the most effective way to stability given magnetic field profiles is to keep β below some critical value.

Like interchanges, ballooning modes are quite amenable to analysis. The most unstable modes also occur for $k_\perp a \gg 1$, $k_\parallel / k_\perp \ll 1$. By exploiting the short perpendicular wavelength nature of the instabilities, the general multidimensional stability problem reduces to the solution of a one-dimensional differential equation along a field line on each flux surface. Substantial progress has been made in the study of ballooning modes in tokamaks and stellarators using this procedure. One property of the ballooning mode equation is that, in a special limit, it leads to the localized interchange criterion (i.e., the Mercier criterion).

8.11.4 Current-driven modes

A current-driven mode is one in which the dominant driving source of instability is proportional to the $J_{\|}$ term in Eq. (8.100). These modes are driven by parallel currents and can exist even in a zero-pressure force-free plasma. Current-driven instabilities are often known as "kink" modes. In general the most unstable perturbations have long parallel wavelengths $k_{\|}a \ll 1$, and perpendicular wavelengths that are macroscopic in scale $k_{\perp}a \sim 1$. Current-driven modes can have the form of either internal or external perturbations depending upon the location of the singular surface. As such, these modes are usually subdivided into two categories: external kinks and internal kinks.

External kink modes

The external kink mode is a very dangerous instability in tokamak and RFP experiments, often leading to major disruptions. At low β the main destabilizing effect for high m modes (m is the poloidal wave number) is the radial gradient of the parallel current near the plasma edge. For low m numbers, $m = 1$ in particular, the current profile is not as important as the total current itself. The basic form of the perturbation is such that the plasma surface "kinks" into a helix as illustrated in Fig. 8.11. The unstable modes occur for long parallel wavelength in order to minimize line bending. Usually the most dangerous kinks correspond to low perpendicular wave numbers.

At higher values of β the external kink mode develops a distinct ballooning structure. The harmonic content is concentrated at low m values, unlike the short-wavelength internal ballooning mode. At sufficiently high β the external kink mode can become unstable, even if the parallel current is less than the low β stability limit. This is the regime of the "external ballooning-kink" mode. The corresponding β limit is often the most severe stability criterion in the system.

There are several ways in which to improve the stability of a given configuration against external kink. First, for a prescribed geometry there is usually a critical parallel current, which depends on β, indicating the onset of instability. Stability can be achieved by keeping the parallel current below this value. Second, for a fixed parallel current a toroidal device with sufficiently small circumference will prohibit the formation of long-wavelength kinks. In other words, a tight aspect

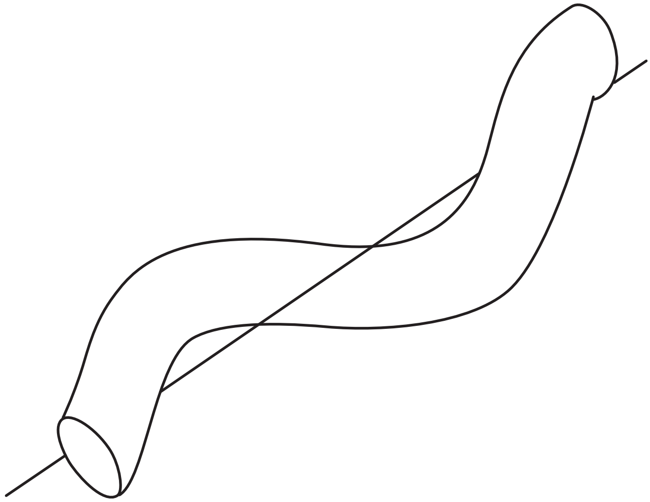

Figure 8.11 Plasma perturbation corresponding to a long wavelength $m = 1$ kink instability.

ratio is beneficial. Third, to the extent that one can experimentally control the current profiles, higher m kink modes can be stabilized by peaking the parallel current profile, keeping the total current fixed. This reduces the edge gradient. Fourth, since low m number kink perturbations have a broad radial extent, they can be stabilized if a perfectly conducting wall is placed sufficiently close to the plasma surface. Finally, one must guarantee that β does not become too large, or else a ballooning-kink instability would be excited.

Overall, the simplest practical advice is "keep the current low, keep the pressure low" if disruptions are to be avoided.

Internal kink modes

The internal kink has many properties similar to those of the external kink, although it is in general a weaker instability. In fact for a tokamak it is usually only $m = 1$ that can become unstable producing what is known as "sawtooth oscillations." As above, a combination of low current and/or tight aspect ratio (i.e., high q) can help to stabilize internal kink modes particularly in a tokamak. Neither of these methods is effective in the RFP since q is very small. However, a broad current profile helps to provide stability; that is, for internal modes, broadening the current profile at fixed total current effectively places a conducting wall closer to the current channel (due to the $\mathbf{n} \cdot \xi_a = 0$ boundary condition), thereby providing stability. In contrast broad profiles are detrimental for external kink stability for systems without a conducting wall. Stabilizing such modes in an RFP requires an actual close-fitting conducting wall plus feedback.

8.12 Summary

In Chapter 8 the general features of MHD stability theory have been described and are summarized below.

- **Exponential stability:** To determine whether or not a given configuration is MHD unstable, the following definition of instability is adopted. A plasma is said to be exponentially unstable if $\mathrm{Im}(\omega) > 0$ for any discrete normal mode of the system.
- **Waves in a homogeneous plasma:** As a simple test of stability, the infinite, homogeneous plasma in a uniform magnetic field is investigated. This system is shown to be stable. The dispersion relation predicts the existence of three distinct propagating waves: the shear Alfven wave, the fast magnetosonic wave (compressional Alfven wave), and the slow magnetosonic wave (sound wave). The shear Alfven wave plays a dominant role in plasma instabilities.
- **General linearized stability equations:** The general three-dimensional MHD stability analysis is formulated first as an initial value problem and then transformed into a normal mode eigenvalue problem. All of the perturbed quantities are expressed in terms of the plasma displacement ξ. The resulting system has the form of three coupled, homogeneous partial differential equations, with eigenvalue ω^2: $-\omega^2 \rho \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi})$.
- **Properties of the force operator $\mathbf{F}(\boldsymbol{\xi})$:** The force operator $\mathbf{F}(\xi)$ appearing in the normal mode equation is shown to possess an important mathematical property, self-adjointness. This leads to the conclusion that every eigenvalue ω^2 is purely real. Consequently, normal modes either oscillate, or grow purely exponentially. The self-adjointness also leads to the conclusion that the normal modes are orthogonal. As a general rule, the unstable part of the spectrum associated with $\mathbf{F}(\xi)$ contains only discrete modes. The stable part contains both discrete modes and continua.
- **Variational formulation:** By making use of the self-adjointness of $\mathbf{F}(\xi)$, the normal mode eigenvalue problem is recast into the form of a variational principle. The principle is an equivalent integral representation of the eigenvalue differential equation.
- **Energy Principle:** A further analysis of the variational principle shows that there exists a powerful minimizing principle, known as the Energy Principle, for testing MHD stability. The physical basis for this principle is the exact, non-linear conservation of energy in ideal MHD. The Energy Principle provides an exact test for the stability boundaries although it provides only an estimate of the actual growth rates. Stability is testing by examining the sign of δW for all allowable displacements. The system is stable if and only if $\delta W \geq 0$ for all displacements.
- **Incompressibility:** In attempting to find the most unstable trial functions that minimize δW, it is shown that the minimizing $\xi_{\|}$ can be calculated once and for

all for arbitrary geometrics. This reduces the number of unknowns to two, corresponding to the components of $\boldsymbol{\xi}_\perp$. For most systems, in which $\mathbf{B}\cdot\nabla$ is non-singular, or at most singular on isolated surfaces, $\xi_{||}$ must be chosen to make $\nabla\cdot\boldsymbol{\xi}=0$. For the special case involving a closed line geometry for equilibrium and perturbation there is an additional stabilizing term in δW_F due to the plasma compressibility. This contribution can be expressed explicitly in terms of $\boldsymbol{\xi}_\perp$.

- **Vacuum vs. force-free plasma:** It is shown that it is important to distinguish whether the region surrounding the main plasma core is a vacuum, an ideal MHD plasma, or a resistive MHD plasma. The vacuum case is the most pessimistic with respect to stability boundaries and growth rates. The ideal MHD plasma often predicts considerably improved stability boundaries because the class of allowable perturbations is restricted as a result of the perfect conductivity topological constraint. The most realistic, but more difficult to analyze situation, corresponds to the resistive plasma. This model gives rise to the identical stability boundaries as in the vacuum case, but with much lower growth rates. Since the primary concern is the stability boundaries it is sufficient to treat the outer region as a vacuum in the applications that follows.
- **Classification of MHD instabilities:** A discussion is presented describing the various types of MHD instabilities that can occur in a plasma. In general, instabilities can be driven by currents flowing parallel to the field (current-driven modes) or perpendicular to the field (pressure-driven modes). Modes are also distinguished by whether or not the unstable displacement perturbs the plasma surface: external or internal modes. These classifications are further sub-divided into (1) pressure-driven interchange and ballooning modes and (2) current-driven kinks and ballooning-kink modes. Depending on the mode in question equilibrium properties such as a conducting wall, maximum current, maximum β, magnetic shear, average favorable curvature, or tight aspect ratio may be used to improve stability. As a general feature essentially all MHD instabilities correspond to plasma displacements that are nearly constant along a field line ($k_{||}a \ll 1$ in order to minimize line bending). However, perpendicular to the field $\boldsymbol{\xi}$ can vary macroscopically for kinks ($k_\perp a \sim 1$) or very rapidly for interchange and ballooning modes ($k_\perp a \gg 1$).

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Further reading

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Problems

8.1

(a) Calculate the dispersion relation for waves propagating in an infinite homogeneous MHD plasma with uniform magnetic field $\mathbf{B} = B_0\mathbf{e}_z$. Include the effect of a small resistivity in Ohm's law: $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta\mathbf{J}$, $\eta = \text{const}$. Show that the shear Alfvén branch of the dispersion relation can be written as

$$\omega^2\left(1 + i\frac{\gamma_D}{\omega}\right) - k_\parallel^2 V_a^2 = 0$$

Similarly, show that the magnetosonic branch has the form

$$\omega^2\left(\omega^2 - k^2V_s^2\right)\left(1 + i\frac{\gamma_D}{\omega}\right) - \left(\omega^2 - k_\parallel^2 V_s^2\right)k^2V_a^2 = 0$$

where $\gamma_D = \eta k^2/\mu_0$.

(b) Prove that each of the three basic MHD waves becomes slightly damped in the presence of resistivity.

8.2 This problem investigates the apparent paradox associated with the ion acoustic wave in a homogeneous plasma. Specifically, the fact that both ideal MHD and two-fluid theory yield the same dispersion relation although $E_\parallel = 0$ in ideal MHD while $E_\parallel \neq 0$ is crucial in two-fluid theory. Consider the following simple two-fluid model: massless electrons, isothermal electrons and ions, quasineutrality, and negligible resistivity.

(a) Assume a background state $\mathbf{B} = B_0\mathbf{e}_z$, $\mathbf{v}_e = \mathbf{v}_i = 0$. Show that the linearized equations describing parallel propagation are given by

$$m_i \frac{\partial v_{zi}}{\partial t} = eE_z - \frac{T_i}{n_0}\frac{\partial n_i}{\partial z}$$

$$0 = -eE_z - \frac{T_e}{n_0}\frac{\partial n_e}{\partial z}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial z} n_0 v_{zi} = 0$$

$$n_e = n_i$$

$$\nabla \times (E_z \mathbf{e}_z) = 0$$

(b) Calculate the dispersion relation and show that

$$\omega^2 = k_\parallel^2 (T_e + T_i)/m_i.$$

(c) To obtain a single-fluid derivation, eliminate E_z from the momentum equations and set $n_i = n_e = n$. Note that the resulting mass and momentum equations are identical to those of ideal MHD for parallel propagation. Show that the mass and momentum equations, by themselves, yield the dispersion relation $\omega^2 = k_\parallel^2 (T_e + T_i)/m_i$. One thus concludes that the effects of $E_\parallel$ *are included* in the ideal MHD momentum equation. It is in Ohm's law that $E_\parallel$ is neglected, but this relation is not required for the ion acoustic mode. Thus, the paradox is resolved.

8.3 Consider the integral

$$I = \int_0^1 \eta F(\xi) dx$$

where $\eta(x)$, $\xi(x)$ are two complex functions and F is a differential operator. Determine which of the following forms of F corresponds to a self-adjoint operator:

(a) $F = 1$

(b) $F = \dfrac{\partial}{\partial x}$, $\quad \mathrm{BC}_1$

(c) $F = \dfrac{\partial^2}{\partial x^2}$, $\quad \mathrm{BC}_1$

(d) $F = \dfrac{\partial}{\partial x}\left(h \dfrac{\partial}{\partial x}\right)$, $\quad \mathrm{BC}_1$

(e) $F = \dfrac{\partial^2}{\partial x^2}$, $\quad BC_2$

Here, $h(x)$ is a known function and BC_1, BC_2 represent the following boundary conditions:

$$BC_1 : \xi(0) = \eta(0) = \xi(1) = \eta(1) = 0$$

$$BC_2 : \xi(0) = \eta(0) = 0, \xi'(1) = A\xi(1), \eta'(1) = A^*\eta(1)$$

8.4 Consider the Grad–Shafranov equation

$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi}$$

Derive a Lagrangian density, $\hat{L} = \hat{L}(\psi, \partial\psi/\partial R, \partial\psi/\partial Z, R, Z)$ such that the total Lagrangian

$$L = \int \hat{L} d\mathbf{r}$$

$$d\mathbf{r} = 2\pi R dR dZ$$

represents a variational principle for the Grad–Shafranov equation (i.e., letting $\psi \to \psi + \delta\psi$ in $\hat{L}$ and setting $\delta L = 0$ gives the Grad–Shafranov equation). State what boundary conditions you have assumed.

8.5 Consider the eigenvalue problem

$$y'' + \frac{y'}{x} + \lambda y = 0$$

$$y'(0) = 0, \quad y(1) = 0$$

(a) Multiply the equation by y and then integrate over the region $0 < x < 1$. Note that the integral relation obtained is correct but not variational. Substitute the trial function $y = 1 - x^2$ and evaluate λ.
(b) Derive a variational form for the eigenvalue problem.
(c) Substitute the trial function $y = 1 - x^2$ into the variational form and evaluate λ.
(d) Substitute the trial function $y = (1 - x^2)^\nu$ into the variational form and determine ν so that the Lagrangian is an extremum:

$$\partial L/\partial \nu = 0$$

(e) Compare these results with the exact answer.

8.6 To demonstrate the advantage of using natural boundary conditions consider the following problem:

$$\frac{d^2y}{dx^2} + \left(\lambda - x^2\right)y = 0$$

$$y'(0) = 0$$

$$y'(1) = 3y(1)$$

(a) Derive a Lagrangian with the property that the boundary condition at $x = 1$ appears as a natural boundary condition.
(b) Consider now the one-parameter trial function $y = 1 - \alpha x^2$. Choose α so that the boundary condition at $x = 1$ is satisfied and evaluate the eigenvalue λ.
(c) Recall that with natural boundary conditions one is not forced to choose a trial function such that the boundary condition at $x = 1$ is automatically satisfied. The variational principle will do "as good a job as possible" to ensure that this condition is satisfied. Instead, treat α as a variational parameter determined by setting $\delta L = 0$. Use this value of α and calculate the eigenvalue λ.
(d) Compare (b) and (c) with the exact result.

8.7 The boundary condition given by

$$\mathbf{n} \cdot \hat{\mathbf{B}}_1|_{r_p} = \mathbf{n} \cdot \nabla \times \left(\boldsymbol{\xi} \times \hat{\mathbf{B}}\right)|_{r_p}$$

is intuitively obvious although there are subtleties involved. In particular, the plasma displacement is not defined in the vacuum region. A more rigorous derivation starts from the basic condition $\mathbf{n} \cdot \hat{\mathbf{B}} = 0$. Linearize this relation about the equilibrium solution $\left(\mathbf{n} = \mathbf{n}_0 + \mathbf{n}_1, \hat{\mathbf{B}} = \hat{\mathbf{B}}_0 + \hat{\mathbf{B}}_1, \mathbf{r} = \mathbf{r}_\mathrm{p} + \boldsymbol{\xi}\right)$ and show that the equation (above) is indeed the correct boundary condition. Hint: Show that $\mathbf{n}_1 = -(\nabla\xi) \cdot \mathbf{n}_0 + \mathbf{n}_0\,[\mathbf{n}_0 \cdot (\nabla\xi) \cdot \mathbf{n}_0]$.

8.8 The boundary condition

$$\mathbf{n} \cdot \hat{\mathbf{B}}_1|_{r_p} = \mathbf{n} \cdot \nabla \times \left(\boldsymbol{\xi} \times \hat{\mathbf{B}}\right)|_{r_p}$$

can be expressed solely in terms of the normal component of the plasma displacement evaluated on the plasma surface: $\xi_n = \mathbf{n} \cdot \boldsymbol{\xi}$. Specifically, show that

$$\mathbf{n} \cdot \nabla \times (\boldsymbol{\xi} \times \mathbf{B})|_{r_p} = \mathbf{B} \cdot \nabla\xi_n - \xi_n[\mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\mathbf{B}]|_{r_p}$$

8.9 Show by the use of trial functions that the presence of a perfectly conducting wall is always stabilizing.