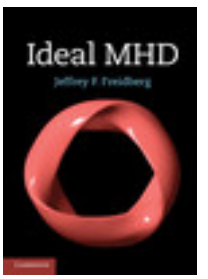

Ideal MHD

Jeffrey P. Freidberg

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9

Alternate MHD models

9.1 Introduction

Ideal MHD is the simplest model that describes the macroscopic equilibrium and stability of high-temperature fusion plasmas. A self-consistent derivation of the model has been presented in Chapter 2. The derivation requires that one restrict attention to the MHD length and time scales. The main assumptions for validity of ideal MHD are (1) small ion gyro radius, (2) high collisionality, and (3) negligible resistive diffusion. As pointed out, the high collisionality assumption is never satisfied in fusion-grade plasmas, which makes it perhaps surprising how accurate and reliable the model is in predicting experimental behavior.

The goal of Chapter 9 is to begin to address this unsettling situation. While there are a number of alternate MHD models in various regimes of collisionality, the strategy here is to focus on two specific models that describe MHD behavior in the collisionless regime. These are the models that are most relevant to fusion plasmas. Ultimately, in Chapter 10 a set of general stability "comparison theorems" is derived that enables one to make quantitative comparisons between the predictions of the collision dominated and collisionless models. The present chapter, however, focuses solely upon the introduction of the two alternate models. Also presented is a review of ideal MHD which serves as a reference. The specific models discussed are as follows:

- Ideal MHD: collision dominated fluid
- Kinetic MHD: collisionless kinetic plasma
- Double adiabatic theory: collisionless fluid.

As a general comment it is worth pointing out that as a species of plasma particles evolves from collision dominated to collisionless, its corresponding model switches from a fluid to a kinetic description. *All* models share the common feature of being based on the first two moments of the Boltzmann equation – mass and

momentum. The distinction between collisionless and collision dominated behavior arises in the approximations made to achieve closure. Fluid treatments require a macroscopic conservation of energy relation to close the system of equations. In contrast, the kinetic models require an approximate solution for the distribution function which is then used to directly calculate the pressure tensor, thereby providing closure.

The material in Chapter 9 begins with a more detailed analysis of the collision dominated assumption used in ideal MHD. This analysis suggests the different types of MHD models that need to be developed, in particular the models listed above. Next, the conservation of mass and momentum equations are derived. With this as a foundation, the alternate MHD models are then systematically developed.

The end goal is a closed formulation for each of the two alternate MHD models listed above.

9.2 Transition from collision dominated to collisionless regimes

The discussion begins with a re-examination of the collision dominated assumption used in the derivation of ideal MHD, the one that is always violated for fusion plasmas. Recall that the most stringent collisionality requirement used in the derivation requires temperature equilibration on a time scale fast compared to the MHD time scale.

For present purposes it is useful to separately consider three different collisionality regimes each characterized by an appropriate dimensionless parameter $\omega\tau$, where $\omega = V_{Ti}/a$ is the MHD frequency and τ is the collision time of interest. The three regimes are defined as follows:

$$\begin{aligned} &\omega\tau_{eq} \ll 1 && \text{temperatures equilibrate} \\ &\omega\tau_{ii} \ll 1 && \text{ions are collision dominated} \\ &\omega\tau_{ee} \ll 1 && \text{electrons are collision dominated} \end{aligned} \tag{9.1}$$

Here the characteristic collision times are chosen as $\tau_{ee} \approx \tau_e$, $\tau_{ii} \approx \tau_i = (2m_i/m_e)^{1/2}\tau_e$, and $\tau_{eq} = (m_i/2m_e)\tau_e$, where τ_e is given by Braginskii (1965)

$$\tau_e = 3(2\pi)^{3/2}\frac{\varepsilon_0^2 m_e^{1/2} T_e^{3/2}}{ne^4 \ln\Lambda} = 1.09\times 10^{-4}\frac{T_k^{3/2}}{n_{20}\ln\Lambda} \quad \text{sec} \tag{9.2}$$

One now sets the Coulomb logarithm to $\ln\Lambda = 19$ and the ion mass to that of deuterium. In order of decreasing stringency the collisionality boundaries reduce to

$$\begin{aligned} \omega\tau_{eq} &= 3.3\times 10^3\left(T_k^2/an_{20}\right) \ll 1 && \text{temperatures equilibrate} \\ \omega\tau_{ii} &= 1.5\times 10^2\left(T_k^2/an_{20}\right) \ll 1 && \text{ions are collision dominated} \\ \omega\tau_{ee} &= 1.8\left(T_k^2/an_{20}\right) \ll 1 && \text{electrons are collision dominated} \end{aligned} \tag{9.3}$$

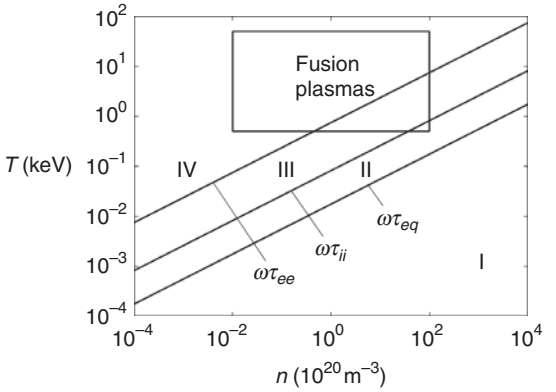

Figure 9.1 Regions of validity for the alternate MHD models in (n, T) space for the case $a = 1\,\mathrm{m}$.

As in Chapter 2, these quantities are plotted (in Fig. 9.1) as curves of T vs. n for the case of $a = 1\,\mathrm{m}$. Also shown is the rectangle corresponding to the region of fusion interest: $10^{18}\,\mathrm{m}^{-3} < n < 10^{22}\,\mathrm{m}^{-3}$ and $0.5\,\mathrm{keV} < T < 50\,\mathrm{keV}$.

Observe that there are four distinct collisionality regimes. Region I corresponds to the highest collisionality regime where the electron and ion temperatures have sufficient time to equilibrate: $\omega\tau_{eq} \ll 1$. This is the reference regime defining the single-fluid ideal MHD model.

In region II both the electrons and ions are collision dominated but there is not sufficient time for temperature equilibration: $\omega\tau_{ii} \ll 1 \ll \omega\tau_{eq}$. In this regime the electron and ion temperatures are decoupled and a separate fluid energy conservation relation is required for each species for closure. This is an intermediate collisionality regime.

As the collisionality continues to decrease one moves into region III where the electrons remain collision dominated but the ions become collisionless: $\omega\tau_{ee} \ll 1 \ll \omega\tau_{ii}$. Higher-density, lower-temperature fusion plasmas, often corresponding to the plasma edge, lie in this regime. For region III a fluid energy conservation relation is needed for closure of the electron model, while a kinetic treatment is required for the ions. This is also an intermediate collisionality regime.

Lastly, fully collisionless plasmas lie in region IV. Here, both the electrons and ions are collisionless: $1 \ll \omega\tau_{ee}$. The core of most fusion plasmas lies in this regime. Since both species are collisionless, each must be described by a kinetic model. A simple kinetic model can be derived by exploiting the small gyro radius assumption. It is known as kinetic MHD and is one of the models discussed below.

As shown, the kinetic MHD model is, in general, much easier to solve than the Vlasov equation but still much more difficult to solve than the ideal MHD model. This fact has led to attempts at fluid closures of the kinetic MHD model. One such

result is the Chew, Goldberger, Low (CGL) double adiabatic fluid model also discussed below. The double adiabatic model is far simpler to solve but cannot be justified by a consistent set of physical assumptions. Still, it is a very useful model when considering stability comparison theorems.

With this as background, the task now is to present a mathematical derivation for each of the two alternate models just described.

9.3 General formulation

The MHD models under consideration all have the same starting point consisting of an identical, simplified form of the mass and momentum moments of the Boltzmann–Maxwell kinetic equations. These simplified but accurate moments are coupled to the low-frequency, quasi-neutral form of Maxwell's equations.

The simplified moments are obtained from the exact moments by neglecting electron inertia in the electron momentum equation and neglecting resistivity in both momentum equations. Use is also made of the small gyro radius approximation: $r_{Li}/a \sim k_\perp r_{Li} \ll 1$. Under these conditions the starting equations are easily obtained from the analysis presented in Section 2.3 and can be written as

$$\begin{aligned} \frac{\partial n}{\partial t} + \nabla \cdot (n\,\mathbf{v}) &= 0 \\ \rho \frac{d\,\mathbf{v}}{dt} &= \mathbf{J} \times \mathbf{B} - \nabla \cdot (\mathbf{P}_e + \mathbf{P}_i) \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \\ n_e &= n_i = n \end{aligned} \tag{9.4}$$

Before proceeding, some discussion is required concerning Faraday's law. The simplified form given in Eq. (9.4) can be obtained by recalling that the electron fluid velocity is given by $\mathbf{u}_e = \mathbf{v} - \mathbf{J}/en$. After substituting this into the electron momentum equation one finds

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = (\mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}_e)/en \tag{9.5}$$

For the perpendicular component of $\mathbf{E}$, the right-hand side of Eq. (9.5) can be neglected because of the small gyro radius assumption. This implies that

$$\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B} \approx 0 \tag{9.6}$$

The parallel component of electric field is not affected by the $\mathbf{v} \times \mathbf{B}$ term. Its value is given by

$$E_{\|} = -\mathbf{b} \cdot (\nabla \cdot \mathbf{P}_e)/en \tag{9.7}$$

A simple scaling argument shows that $E_{\|}/|\mathbf{E}_{\perp}| \sim r_{Li}/a \ll 1$.

Therefore, with respect to Faraday's law $\mathbf{E} = \mathbf{E}_{\perp} + E_{\|}\mathbf{b} \approx \mathbf{E}_{\perp}$ and the electron momentum equation reduces to the ideal MHD Ohm's law $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$. Substituting $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ into Faraday's law leads to the simple form given in Eq. (9.4). This form of Faraday's law guarantees that the magnetic field lines are frozen into the plasma, the basic condition for ideal MHD behavior. However, one should keep in mind that there is a small non-zero $E_{\|}$ given by Eq. (9.7) which does not enter ideal MHD but will play a role in kinetic MHD. This is the main point of the above discussion.

Return now to Eq. (9.4). One sees that the model, as it now stands, is general, but has more unknowns than equations. Obtaining closure requires a prescription for determining the two pressure tensors $\mathbf{P}_e$ and $\mathbf{P}_i$. There are a variety of models that accomplish this task each corresponding to a different set of assumptions with respect to the collisionality of the electrons and ions. These different closure choices represent various alternate MHD models. In the discussion that follows an explicit mathematical formulation is presented for the MHD models of interest.

9.4 Ideal MHD closure

The closure assumptions for ideal MHD have already been discussed in Section 2.3 and are simply repeated here for completeness. In ideal MHD the ion and electron temperatures are equilibrated, $T_e = T_i \equiv T$, and the pressure is a scalar quantity, $p = p_e + p_i = 2nT$. Furthermore, the pressure satisfies the adiabatic conservation of energy relation leading to the following procedure for closure:

$$\begin{aligned} &\mathbf{P} = \mathbf{P}_e + \mathbf{P}_i = p\mathbf{I} \\ &\frac{d}{dt}\left(\frac{p}{n^{\gamma}}\right) = 0 \end{aligned} \tag{9.8}$$

where $\gamma = 5/3$.

9.5 Kinetic MHD

9.5.1 Basic assumptions

The kinetic MHD model lies in region IV of the T vs. n diagram (Fig. 9.1) and corresponds to the situation where both the electrons and ions are collisionless.

Many fusion experiments lie in this region. Since both species are collisionless it is necessary to treat each with a kinetic MHD model. To derive the model one needs to exploit both the MHD ordering, $\omega \sim V_{Ti}/a$, $k \sim 1/a$ and the small gyro radius approximation $r_{Li}/a \ll 1$, while ignoring collisions in the starting Boltzmann kinetic equation. That is, the analysis starts with the Vlasov equation which is a function of six phase space variables plus time: $\mathbf{r}$, $\mathbf{v}$, and t. The end result of the analysis is a simplified kinetic equation which is three-dimensional in physical space but only two-dimensional in velocity space: $\mathbf{r}$, ε, μ, and t. Also, the adiabatic invariant μ appears only as a parameter. This represents a substantial saving in complexity.

9.5.2 Derivation of the kinetic MHD model

The derivation of the kinetic MHD model from the Vlasov equation involves a somewhat lengthy analysis. The details are presented below for the ions and closely follow the excellent derivation by Hazeltine and Waelbroeck (1998). The electron equation is obtained by letting $m_i \to m_e$ and $e \to -e$.

The starting equations

The derivation of the kinetic MHD equation starts from the Vlasov equation, given by

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \nabla f + \frac{d\mathbf{u}}{dt} \cdot \nabla_u f = 0 \\ \frac{d\mathbf{r}}{dt} &= \mathbf{u} \\ \frac{d\mathbf{u}}{dt} &= \frac{e}{m_i}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \end{aligned} \tag{9.9}$$

Coordinate transformations

To obtain the kinetic MHD model new velocity variables are introduced into the Vlasov equation in order to exploit the small gyro radius approximation. Specifically, the single particle velocity vector $\mathbf{u} = u_x\mathbf{e}_x + u_y\mathbf{e}_y + u_z\mathbf{e}_z$ is replaced by

$$\mathbf{u} = \mathbf{w} + \mathbf{v}(\mathbf{r},t) \tag{9.10}$$

where $\mathbf{w}$ is the random component of the particle velocity (i.e., $\langle \mathbf{w} \rangle = 0$). Hopefully the notational confusion is tolerable even though a switch has been made from Chapter 2. Now, $\mathbf{u}$ represents the velocity variable in the Vlasov equation and $\mathbf{v}(\mathbf{r}, t)$ is the traditional MHD macroscopic fluid velocity. Also, recall that the MHD ordering implies that $\mathbf{v}_i = \mathbf{v}$ and $\mathbf{v}_e = \mathbf{v}_i - \mathbf{J}/en \approx \mathbf{v} + O(r_{Li}/a)$. To the order needed in kinetic MHD the ions and electrons have the same macroscopic velocity $\mathbf{v}(\mathbf{r}, t)$.

Next, $\mathbf{w}$ is decomposed as $\mathbf{w} = \mathbf{w}_\perp + w_\parallel \mathbf{b}$, where $\mathbf{b}(\mathbf{r}, t) = \mathbf{B}/B$ and

$$\begin{aligned} \mathbf{w}_\perp &= w_1 \mathbf{e}_1 + w_2 \mathbf{e}_2 \\ &= w_\perp \cos\alpha\, \mathbf{e}_1 + w_\perp \sin\alpha\, \mathbf{e}_2 \end{aligned} \tag{9.11}$$

Here, $\mathbf{e}_1(\mathbf{r}, t)$, $\mathbf{e}_2(\mathbf{r}, t)$ are a set of local orthogonal unit vectors perpendicular to each other as well as to $\mathbf{b}$: $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{b} = \mathbf{e}_2 \cdot \mathbf{b} = 0$. The independent velocity variables at this point are considered to be $w_\perp$, $w_\parallel$, and α, where $\alpha = \tan^{-1}(w_2/w_1)$ is the gyro phase angle.

Lastly, the perpendicular and parallel random velocities $w_\perp$ and $w_\parallel$ are replaced with two new variables ε and μ defined as follows:

$$\begin{aligned} \varepsilon &= \frac{1}{2} m_i \left(w_\perp^2 + w_\parallel^2 \right) \\ \mu &= \frac{m_i w_\perp^2}{2B(\mathbf{r},t)} \end{aligned} \tag{9.12}$$

Here, ε is the random particle kinetic energy and μ is the adiabatic invariant. These definitions define the overall velocity variable transformation:

$$u_x,\ u_y,\ u_z \rightarrow w_\perp,\ w_\parallel,\ \alpha \rightarrow \varepsilon,\ \mu,\ \alpha \tag{9.13}$$

The distribution function is now a function of $\mathbf{r}$, ε, μ, α, t and in terms of these variables the Vlasov equation becomes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} + \mathbf{w}) \cdot \nabla f + \dot{\varepsilon} \frac{\partial f}{\partial \varepsilon} + \dot{\mu} \frac{\partial f}{\partial \mu} + \dot{\alpha} \frac{\partial f}{\partial \alpha} = 0 \tag{9.14}$$

Expressions for $\dot{\varepsilon}, \dot{\mu}, \dot{\alpha}$

The next step is to derive expressions for $\dot{\varepsilon}, \dot{\mu}, \dot{\alpha}$ in terms of the new variables. Starting with the equations of motion given in Eq. (9.9) and forming various obvious combinations of the separate vector components one obtains after a straightforward calculation a set of complicated looking relations given by

$$\begin{aligned} \dot{\varepsilon} &= e w_\parallel E_\parallel + q\, \mathbf{w}_\perp \cdot (\mathbf{E}_\perp + \mathbf{v}_\perp \times \mathbf{B}) - m_i \mathbf{w} \cdot \frac{D\mathbf{v}}{Dt} \\ \dot{\mu} &= -\frac{\mu}{B} \frac{DB}{Dt} + \frac{e}{B} \mathbf{w}_\perp \cdot (\mathbf{E}_\perp + \mathbf{v}_\perp \times \mathbf{B}) - \frac{m_i}{B} \mathbf{w}_\perp \cdot \frac{D\mathbf{v}}{Dt} \\ &\quad + m_i \mathbf{w} \cdot \left(\cos\alpha \frac{D\mathbf{e}_1}{Dt} + \sin\alpha \frac{D\mathbf{e}_2}{Dt} \right) \\ \dot{\alpha} &= \omega_{ci} - \frac{e}{m_i w_\perp^2} \mathbf{b} \cdot \mathbf{w}_\perp \times (\mathbf{E}_\perp + \mathbf{v}_\perp \times \mathbf{B}) - \frac{1}{w_\perp^2} \mathbf{b} \cdot \mathbf{w}_\perp \times \frac{D\mathbf{v}}{Dt} \\ &\quad - \frac{1}{w_\perp} \mathbf{w} \cdot \left(\cos\alpha \frac{D\mathbf{e}_2}{Dt} - \sin\alpha \frac{D\mathbf{e}_1}{Dt} \right) \end{aligned} \tag{9.15}$$

where $\omega_{ci} = eB/m_i$. In the right-hand side of these equations it is understood that the random velocity $\mathbf{w}$ should be expressed in terms of ε, μ, α. Also the D/Dt operator acting on any macroscopic function of $\mathbf{r}$, t is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + (\mathbf{w}_\perp + w_\parallel \mathbf{b})\cdot\nabla \tag{9.16}$$

At this point in the analysis Eqs. (9.14) and (9.15) are still exact. In order to carry out the small gyro radius expansion it is necessary to separate each of the quantities $\dot{\varepsilon}, \dot{\mu}, \dot{\alpha}$ into a gyro averaged contribution plus a contribution that has zero gyro average. That is, each of the quantities $\dot{\varepsilon}, \dot{\mu}, \dot{\alpha}$ is written as

$$\begin{aligned} \dot{Q}(\mathbf{r},\varepsilon,\mu,\alpha,t) &= \overline{\dot{Q}}\,(\mathbf{r},\varepsilon,\mu,t) + \tilde{\dot{Q}}(\mathbf{r},\varepsilon,\mu,\alpha,t) \\ \overline{\dot{Q}} &= \frac{1}{2\pi}\int_0^{2\pi} \dot{Q}\,d\alpha \\ \tilde{\dot{Q}} &= \frac{1}{2\pi}\int_0^{2\pi} \left(\dot{Q} - \overline{\dot{Q}}\right) d\alpha = 0 \end{aligned} \tag{9.17}$$

When the gyro radius expansion is carried out it is shown that only the gyro averaged quantities $\overline{\dot{Q}}$ are needed. Calculating these averages is the next task.

Expressions for $\overline{\dot{\varepsilon}}, \overline{\dot{\mu}}, \overline{\dot{\alpha}}$

The required gyro averages are also straightforward to obtain but require some tedious algebra. The results are

$$\begin{aligned} \overline{\dot{\varepsilon}} &= e w_\parallel E_\parallel - m_i w_\parallel \mathbf{b}\cdot\frac{d\mathbf{v}}{dt} - \frac{m_i w_\perp^2}{2}\nabla\cdot\mathbf{v} - m_i\left(w_\parallel^2 - \frac{w_\perp^2}{2}\right)\mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\mathbf{v} \\ \overline{\dot{\mu}} &= 0 + O(r_{Li}/a) \\ \overline{\dot{\alpha}} &= \omega_{ci} + O(r_{Li}/a) \end{aligned} \tag{9.18}$$

where d/dt operating on a macroscopic function of $\mathbf{r}$, t is defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \tag{9.19}$$

The r_{Li}/a corrections to $\overline{\dot{\mu}}$ and $\overline{\dot{\alpha}}$ are complicated expressions which are not needed for the gyro radius expansion.

The gyro radius expansion

The groundwork has now been set for the gyro radius expansion. For the usual MHD ordering the largest term by far in Eq. (9.14) is the one proportional to $\dot{\alpha}$.

This follows because for the MHD ordering the dominant perpendicular fluid motion of both electrons and ions is the $\mathbf{E} \times \mathbf{B}/B^2$ drift: $\mathbf{E}_\perp + \mathbf{v}_\perp \times \mathbf{B} = O(r_L/a)$. Thus, if f is expanded in terms of $f = f_0 + f_1 + \cdots$ then the leading-order contribution to Eq. (9.14) is given by

$$\omega_{ci} \frac{\partial f_0}{\partial \alpha} = 0 \tag{9.20}$$

The solution is

$$f_0 = \overline{f}_0(\mathbf{r}, \varepsilon, \mu, t) \tag{9.21}$$

The first-order equation has the form

$$\omega_{ci} \frac{\partial f_1}{\partial \alpha} + \frac{\partial \overline{f}_0}{\partial t} + (\mathbf{v} + \mathbf{w}) \cdot \nabla \overline{f}_0 + \left(\overline{\dot{\varepsilon}} + \tilde{\dot{\varepsilon}}\right) \frac{\partial \overline{f}_0}{\partial \varepsilon} = 0 \tag{9.22}$$

For f_1 to be a physical solution it must be periodic in α. Therefore, averaging Eq. (9.22) over one period in α is equivalent to an integrability condition on $\overline{f}_0$. This condition defines the basic kinetic MHD equation. The resulting equations for ions and electrons (after dropping the "0" subscript) are given by

$$\text{Ions:} \qquad \frac{\partial \overline{f}_i}{\partial t} + \left(\mathbf{v} + w_\parallel \mathbf{b}\right) \cdot \nabla \overline{f}_i + \overline{\dot{\varepsilon}} \frac{\partial \overline{f}_i}{\partial \varepsilon} = 0$$

$$\overline{\dot{\varepsilon}} = e w_\parallel E_\parallel - m_i w_\parallel \mathbf{b} \cdot \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \mathbf{v} - \frac{m_i w_\perp^2}{2} \nabla \cdot \mathbf{v} - m_i \left(w_\parallel^2 - \frac{w_\perp^2}{2}\right) \mathbf{b} \cdot (\mathbf{b} \cdot \nabla) \mathbf{v} \tag{9.23}$$

$$\text{Electrons:} \qquad \frac{\partial \overline{f}_e}{\partial t} + \left(\mathbf{v} + w_\parallel \mathbf{b}\right) \cdot \nabla \overline{f}_e + \overline{\dot{\varepsilon}} \frac{\partial \overline{f}_e}{\partial \varepsilon} = 0$$

$$\overline{\dot{\varepsilon}} = -e w_\parallel E_\parallel - m_e w_\parallel \mathbf{b} \cdot \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \mathbf{v} - \frac{m_e w_\perp^2}{2} \nabla \cdot \mathbf{v} - m_e \left(w_\parallel^2 - \frac{w_\perp^2}{2}\right) \mathbf{b} \cdot (\mathbf{b} \cdot \nabla) \mathbf{v} \tag{9.24}$$

Observe that the same macroscopic fluid velocity $\mathbf{v}$ appears in both the ion and electron equations. These are the desired equations which at this point are still valid non-linearly.

An examination of Eqs. (9.23) and (9.24) shows that as expected the reduced distribution function is independent of the angle α, a consequence of gyro phase averaging. Furthermore, there are no terms containing a derivative with respect to μ since, on the MHD length and time scales, μ is an adiabatic constant of the

motion; that is, $\dot{\overline{\mu}} = 0$ and hence the term $\overline{\mu}(\partial \overline{f}/\partial \mu) = 0$. The implication is that the independent variable μ appears only as a parameter in the equation.

A further point to note is the explicit appearance of $E_\parallel$ in Eqs. (9.23) and (9.24). While $E_\parallel$ is small in Faraday's law because of the small gyro radius assumption, it is of comparable magnitude to the other terms in $\overline{\dot{\varepsilon}}$. It cannot be neglected. The appropriate way to determine $E_\parallel$ is to enforce the charge neutrality condition

$$n_e = n_i \rightarrow \int \overline{f}_e d\mathbf{w} = \int \overline{f}_i d\mathbf{w} \tag{9.25}$$

Final closure

The final step required to complete the discussion of the kinetic MHD model is the closure procedure. Specifically, assuming that a solution for $\overline{f}(\mathbf{r}, \varepsilon, \mu, t)$ has been found, how is closure obtained? The answer is as follows. Knowing $\overline{f}(\mathbf{r}, \varepsilon, \mu, t)$ one can directly calculate the unknown pressure tensors in Eq. (9.4) thereby closing the system of equations. For each species one finds

$$\begin{aligned} \mathbf{P} &= \begin{vmatrix} p_\perp & & \\ & p_\perp & \\ & & p_\parallel \end{vmatrix} = p_\perp \mathbf{I} + \Big(p_\parallel - p_\perp\Big)\mathbf{bb} \\ p_\perp &= \int \frac{m w_\perp^2}{2} \overline{f}\, d\mathbf{w} = \int \mu B \overline{f}\, d\mathbf{w} \\ p_\parallel &= \int m w_\parallel^2 \overline{f}\, d\mathbf{w} = \int 2(\varepsilon - \mu B)\overline{f}\, d\mathbf{w} \end{aligned} \tag{9.26}$$

where (with the $w_\parallel$ dependence explicitly shown)

$$Q(w_\parallel) d\mathbf{w} = \begin{cases} 2\pi Q(w_\parallel) w_\perp dw_\perp dw_\parallel & -\infty < w_\parallel < \infty \\ \dfrac{2^{1/2}\pi}{m^{3/2}} \dfrac{B}{(\varepsilon - \mu B)^{1/2}} [Q(w_\parallel) + Q(-w_\parallel)] d\varepsilon\, d\mu & 0 < w_\parallel < \infty \end{cases} \tag{9.27}$$

Observe that the pressure tensor is anisotropic but diagonal in structure. Once $p_\perp = p_{\perp i} + p_{\perp e}$ and $p_\parallel = p_{\parallel i} + p_{\parallel e}$ are known then the pressure tensor appearing in the momentum equation can be evaluated:

$$\nabla \cdot (\mathbf{P}_i + \mathbf{P}_e) = \nabla p_\perp + (p_\parallel - p_\perp)\boldsymbol{\kappa} + \mathbf{b}\, \mathbf{B} \cdot \nabla \left(\frac{p_\parallel - p_\perp}{B}\right) \tag{9.28}$$

Equation (9.28) is the required form of the pressure tensor, which when combined with the charge neutrality condition given by Eq. (9.25), formally closes the kinetic MHD model.

A last word of caution: one should *not* use the solution for $\overline{f}$ arising from the kinetic MHD equation to evaluate the moments corresponding to either $\mathbf{J}_\perp$ or $\mathbf{v}$. These quantities are determined directly from the fluid moment equations given by Eq. (9.4). They can in principle be derived from higher-order r_L/a corrections to the distribution function, but these are very complicated and unnecessary to calculate. In fact the need *not* to calculate these higher-order corrections is a major advantage of the kinetic MHD model.

From the practical mathematical point of view the unknown distribution function in kinetic MHD is a function of five phase space variables plus time, $\mathbf{r}$, ε, μ, t with μ appearing only as a parameter: $\overline{f} = \overline{f}(\mathbf{r}, \varepsilon, \mu, t)$. Clearly this is a substantial reduction in complexity as compared to the original Vlasov equation. Even so, the simplified kinetic MHD equation is still very complicated to solve in comparison to the ideal MHD fluid model. Nevertheless, it is possible, as shown in Chapter 10, to derive a general "stability comparison theorem" that allows one to compare the kinetic MHD and ideal MHD marginal stability boundaries.

9.6 The Chew, Goldberger, Low (CGL) double adiabatic model

Although the kinetic MHD model represents a substantial simplification of the Vlasov equation it is still quite difficult to solve theoretically or computationally because of the complicated single particle kinetic behavior parallel to the magnetic field. This is unfortunate since kinetic MHD is perhaps the most reliable model in terms of physics content for describing the MHD behavior of fusion plasmas.

These kinetic MHD difficulties have led to attempts at developing collisionless fluid closure models, almost a contradiction in terms. One such early model was developed by Chew, Goldberger, and Low (1956) and is often called the "CGL model" or, for reasons that will become apparent shortly, the "double adiabatic model." The goal in this section is to derive the original model plus a slightly modified form that is very useful when deriving stability comparison theorems.

The approach used consists of calculating the $mw_\perp^2/2$ and $mw_\parallel^2$ moments of the kinetic MHD equation. The result is a set of time evolution equations for $p_\perp$ and $p_\parallel$ in which, however, new unknown heat fluxes $h_\perp$ and $h_\parallel$ also appear. The system is closed by assuming that $h_\perp = h_\parallel = 0$. Neglect of the heat fluxes greatly simplifies the equations but cannot be justified by any rigorous mathematical or physical ordering procedure. The terms neglected are potentially of comparable or even larger size than those that are maintained. The lack of a sound physical basis supporting the model is a primary reason why it is not used very widely in fusion research.

Based on this observation one can ask whether it is actually worth the effort to derive the model. The answer is most definitely "yes" for the following reason. In spite of its lack of physical basis the simplicity of the double adiabatic model leads

to a form of δW which together with the ideal MHD δW, bracket the marginal stability boundaries of kinetic MHD. That is, the hard-to-calculate but reliable kinetic MHD predictions are bracketed by two much simpler models, ideal MHD which is a valid model but does not satisfy the collisionless requirement, and double adiabatic MHD which is collisionless but is not a valid model.

The bracketing of kinetic MHD stability predictions has been known for many years (Kruskal and Oberman, 1958; Rosenbluth and Rostoker, 1959). Surprisingly, relatively little work has been carried out to quantitatively determine whether the bracketing gaps are wide or narrow. In many cases, as shown in Chapters 11 and 12, the gaps are indeed narrow, helping to explain why ideal MHD works as well as it does.

For this reason it makes sense to derive the double adiabatic MHD model. In Chapter 10 various forms of δW are derived which explicitly show the bracketing of stability predictions.

9.6.1 Formulation of the problem

The basic strategy to derive double adiabatic MHD involves the following steps: (1) formulate an alternate version of the kinetic MHD equation by transforming the velocity coordinates from ε, μ to $w_\perp$, $w_\parallel$; (2) calculate the $mw_\parallel^2$ moment; (3) calculate the $mw_\perp^2/2$ moment; and (4) close the system of equations.

As is shown, the transformation of velocity coordinates substantially simplifies the derivation. The complete transformation (for ions) is defined by

$$\begin{aligned} t' &= t \\ \mathbf{r}' &= \mathbf{r} \\ w_\perp^2 &= \frac{2}{m_i}\mu B \\ w_\parallel^2 &= \frac{2}{m_i}(\varepsilon - \mu B) \end{aligned} \tag{9.29}$$

A straightforward calculation then shows that the derivatives appearing in the kinetic MHD equation are given by

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial t'} + \frac{1}{2B}\frac{\partial B}{\partial t'}\left(w_\perp \frac{\partial}{\partial w_\perp} - \frac{w_\perp^2}{w_\parallel}\frac{\partial}{\partial w_\parallel}\right) \\ \nabla &= \nabla' + \frac{1}{2B}\nabla' B\left(w_\perp \frac{\partial}{\partial w_\perp} - \frac{w_\perp^2}{w_\parallel}\frac{\partial}{\partial w_\parallel}\right) \\ \frac{\partial}{\partial \varepsilon} &= \frac{1}{m_i w_\parallel}\frac{\partial}{\partial w_\parallel} \end{aligned} \tag{9.30}$$

This transformation is substituted into Eq. (9.23). The result is the desired alternate but still complicated form of the kinetic MHD equation, which can be written as

$$\begin{aligned}
&\frac{\partial \bar{f}_i}{\partial t} + (\mathbf{v} + w_\parallel \mathbf{b}) \cdot \nabla \bar{f}_i + C_\perp \frac{\partial \bar{f}_i}{\partial w_\perp} + C_\parallel \frac{\partial \bar{f}_i}{\partial w_\parallel} = 0 \\
&C_\perp = \frac{w_\perp}{2B}\left(\frac{dB}{dt} + w_\parallel \mathbf{b} \cdot \nabla B\right) \\
&C_\parallel = \frac{eE_\parallel}{m_i} - \mathbf{b} \cdot \frac{d\mathbf{v}}{dt} - \frac{w_\perp^2}{2B}\mathbf{b} \cdot \nabla B - w_\parallel \mathbf{b} \cdot (\mathbf{b} \cdot \nabla)\mathbf{v}
\end{aligned} \tag{9.31}$$

Here, for simplicity the primes have been suppressed from t', $\mathbf{r}'$ and use has been made of the following relationship from Faraday's law:

$$\mathbf{b} \cdot \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B})\right] = 0 \quad \rightarrow \quad \frac{1}{B}\frac{dB}{dt} + \nabla \cdot \mathbf{v} - \mathbf{b} \cdot (\mathbf{b} \cdot \nabla)\mathbf{v} = 0 \tag{9.32}$$

An equation similar to Eq. (9.31) applies to the electrons.

The kinetic MHD equation is now in a convenient form to calculate the necessary moments.

9.6.2 Derivation of the double adiabatic model

The $m_i w_\parallel^2$ moment

The first required moment equation is obtained by multiplying Eq. (9.31) by $m_i w_\parallel^2$ and integrating over all velocities. To help carry out this calculation note that

$$\begin{aligned}
&d\mathbf{w} = 2\pi w_\perp dw_\perp dw_\parallel \\
&\int w_\parallel \bar{f}_i \, d\mathbf{w} = 0 \qquad (w_\parallel \text{ is a random velocity})
\end{aligned} \tag{9.33}$$

Also, the various macroscopic moments that appear in the derivation are defined as

$$\begin{aligned}
n &= \int \bar{f}_i \, d\mathbf{w} && \text{density} \\
p_{\perp i} &= \int \frac{m_i w_\perp^2}{2} \bar{f}_i \, d\mathbf{w} && \text{perpendicular pressure} \\
p_{\parallel i} &= \int m_i w_\parallel^2 \bar{f}_i \, d\mathbf{w} && \text{parallel pressure} \\
h_{\perp i} &= \int \frac{m_i w_\perp^2}{2} w_\parallel \bar{f}_i \, d\mathbf{w} && \text{perpendicular heat flux} \\
h_{\parallel i} &= \int m_i w_\parallel^3 \bar{f}_i \, d\mathbf{w} && \text{parallel heat flux}
\end{aligned} \tag{9.34}$$

After some standard integration by parts the $m_i w_\parallel^2$ moment can be evaluated leading to

$$\frac{dp_{\parallel i}}{dt} + \left(\frac{2}{B}\frac{dB}{dt} + 3\nabla\cdot\mathbf{v}\right)p_{\parallel i} - 2\left(\frac{\mathbf{b}\cdot\nabla B}{B}\right)h_{\perp i} + B\mathbf{b}\cdot\nabla\left(\frac{h_{\parallel i}}{B}\right) = 0 \qquad (9.35)$$

In the last step the ion and electron contributions are combined (e.g., $p_\perp = p_{\perp i} + p_{\parallel i}$, etc.) and use is made of the following identity from the conservation of mass equation:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) \quad \rightarrow \quad \nabla\cdot\mathbf{v} = -\frac{1}{n}\frac{dn}{dt} \qquad (9.36)$$

A short calculation then shows that Eq. (9.35) can be rewritten as

$$\frac{n^3}{B^2}\frac{d}{dt}\left(\frac{p_\parallel B^2}{n^3}\right) = 2\left(\frac{\mathbf{b}\cdot\nabla B}{B}\right)h_\perp - B\mathbf{b}\cdot\nabla\left(\frac{h_\parallel}{B}\right) \qquad (9.37)$$

Equation (9.37) is exact in the context of kinetic MHD and represents the first of the required moment relations.

The $m_i w_\perp^2/2$ moment

The equation describing the time evolution of $p_\perp$ is obtained by multiplying Eq. (9.31) by $m_i w_\perp^2/2$ and integrating over all velocities. An analogous calculation to the one just described for $p_\parallel$ yields (for the ions)

$$\frac{dp_{\perp i}}{dt} - \left(\frac{1}{B}\frac{dB}{dt} - \nabla\cdot\mathbf{v}\right)p_{\perp i} + B^2\mathbf{b}\cdot\nabla\left(\frac{h_{\perp i}}{B^2}\right) = 0 \qquad (9.38)$$

One again eliminates $\nabla\cdot\mathbf{v}$ by means of Eq. (9.36) and combines the ion plus electron contributions. A short calculation leads to

$$nB\frac{d}{dt}\left(\frac{p_\perp}{nB}\right) = -B^2\mathbf{b}\cdot\nabla\left(\frac{h_\perp}{B^2}\right) \qquad (9.39)$$

This is the second of the required moment relations.

Closure of the moment equations

As stated previously the closure prescription leading to double adiabatic MHD consists of setting $h_\perp = h_\parallel = 0$. This assumption cannot be justified on the basis of an asymptotic expansion that exploits the smallness of some physical parameter. In fact from a pure dimensional analysis it follows that after making the usual assumption $m_e w_e^2 \sim m_i w_i^2 \sim T_e \sim T_i$, the electron heat fluxes are large by $(m_i/m_e)^{1/2}$ compared to the terms that are kept. The situation may not be bad as this would suggest since by

symmetry the heat fluxes vanish for a pure Maxwellian distribution function. Still, the corrections to the Maxwellian can lead to comparably sized contributions to both the perpendicular and parallel pressure equations.

On the other hand, neglecting the heat fluxes does indeed greatly simplify the model. Plus some, if not all, of the collisionless anisotropic physics is correctly included in the model. In support of Chew, Goldberger, and Low one must appreciate that the double adiabatic model was derived during the very early days of the fusion program – it was important to learn how to walk before trying to run. Furthermore, although more reliable models such as kinetic MHD were also known since the early days of the program, these models were difficult to solve then and remain difficult to solve even now, including the large improvements in computing power.

In any event, following Chew, Goldberger, and Low, one makes the $h_\perp = h_\parallel = 0$ assumption in order to close the kinetic MHD moment equations. The final evolution equations for $p_\perp$ and $p_\parallel$ are thus given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{p_\parallel B^2}{n^3}\right) &= 0 \\ \frac{d}{dt}\left(\frac{p_\perp}{nB}\right) &= 0 \end{aligned} \tag{9.40}$$

For obvious reasons the model is referred to as "double adiabatic MHD."

9.6.3 The modified double adiabatic model

As is shown in Chapter 10 the double adiabatic model needs to be slightly modified in order to obtain a sharply defined upper bound for the stability predictions of kinetic MHD. The modification, first suggested by Rosenbluth and Rostoker (1959), requires the replacement of the parallel component of the momentum equation with a simpler relation, namely that the fluid acceleration parallel to the magnetic field be set to zero. Mathematically, the replacement corresponds to

$$\rho \mathbf{b} \cdot \frac{d\mathbf{v}}{dt} + \mathbf{b} \cdot [\nabla \cdot (\mathbf{P}_e + \mathbf{P}_i)] = 0 \quad \rightarrow \quad \mathbf{b} \cdot \frac{d\mathbf{v}}{dt} = 0 \tag{9.41}$$

As with the basic model itself there is no mathematical or physical assumption that can justify this replacement. The justification is purely one of convenience. The model remains simple and leads to a sharply defined upper bound for stability.

Hereafter, whenever the double adiabatic model appears in the discussion or analysis, it is the modified form that is being considered.

9.7 Summary

Two alternate models describing MHD behavior have been derived: kinetic MHD and double adiabatic MHD. Both models are collisionless, their goal being to overcome the unrealistic high collisionality assumption required for the derivation of ideal MHD.

As a general comment kinetic MHD is a mathematically self-consistent, physically reliable model that is difficult to solve because of kinetic effects. In contrast the double adiabatic model cannot be justified on physical grounds but is far easier to solve because of its fluid nature. It is worth re-emphasizing that in spite of its shaky physical underpinnings, the double adiabatic model still serves a very useful purpose because of (1) its simplicity and (2) the fact that in conjunction with ideal MHD the combined stability predictions bracket those of the more difficult-to-calculate kinetic MHD model.

For convenience, the final models are summarized below. All of the models satisfy the same basic conservation of mass and momentum equations and differ only in how closure is achieved. The basic equations are given by

$$\begin{aligned} \frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) &= 0 \\ \rho \frac{d\mathbf{v}}{dt} &= \mathbf{J} \times \mathbf{B} - \nabla \cdot (\mathbf{P}_i + \mathbf{P}_e) \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \\ n_e &= n_i \equiv n \end{aligned} \tag{9.42}$$

The various methods of closure are as follows.

9.7.1 Ideal MHD

In ideal MHD closure is achieved by writing the pressure tensor in terms of a single scalar pressure

$$\mathbf{P} = \mathbf{P}_i + \mathbf{P}_e = p\mathbf{I} \tag{9.43}$$

where p satisfies

$$\frac{d}{dt}\left(\frac{p}{n^\gamma}\right) = 0 \tag{9.44}$$

9.7.2 Kinetic MHD

As its name implies, in kinetic MHD the ions and electrons are treated kinetically. Their corresponding distribution functions satisfy

Ions:
$$\frac{\partial \overline{f}_i}{\partial t} + \left(\mathbf{v} + w_\parallel \mathbf{b}\right) \cdot \nabla \overline{f}_i + \dot{\overline{\varepsilon}}\, \frac{\partial \overline{f}_i}{\partial \varepsilon} = 0$$

$$\dot{\overline{\varepsilon}} = e w_\parallel E_\parallel - m_i w_\parallel \mathbf{b} \cdot \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} - \frac{m_i w_\perp^2}{2} \nabla \cdot \mathbf{v} - m_i \left(w_\parallel^2 - \frac{w_\perp^2}{2}\right) \mathbf{b} \cdot (\mathbf{b} \cdot \nabla)\mathbf{v} \tag{9.45}$$

Electrons:
$$\frac{\partial \overline{f}_e}{\partial t} + \left(\mathbf{v} + w_\parallel \mathbf{b}\right) \cdot \nabla \overline{f}_e + \dot{\overline{\varepsilon}}\, \frac{\partial \overline{f}_e}{\partial \varepsilon} = 0$$

$$\dot{\overline{\varepsilon}} = -e w_\parallel E_\parallel - m_e w_\parallel \mathbf{b} \cdot \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} - \frac{m_e w_\perp^2}{2} \nabla \cdot \mathbf{v} - m_e \left(w_\parallel^2 - \frac{w_\perp^2}{2}\right) \mathbf{b} \cdot (\mathbf{b} \cdot \nabla)\mathbf{v} \tag{9.46}$$

Here, if each species is denoted by $\alpha = e,\ i$, then $\overline{f}_\alpha = \overline{f}_\alpha(\mathbf{r}, \varepsilon, \mu, t)$ with $\varepsilon = (m_\alpha/2)(w_\perp^2 + w_\parallel^2)$, $\mu = m_\alpha w_\perp^2/2B$, and $\mathbf{v}_e \approx \mathbf{v}_i \equiv \mathbf{v}$ (the macroscopic fluid velocity).

Closure

Knowing $\overline{f}_i$ and $\overline{f}_e$ one then calculates (for $\alpha = i,\ e$)

$$\begin{aligned}
\mathbf{P}_\alpha &= p_{\perp\alpha}\, \mathbf{I} + \left(p_{\parallel\alpha} - p_{\perp\alpha}\right)\mathbf{bb} \\
p_{\perp\alpha} &= \int \frac{m_\alpha w_\perp^2}{2} \overline{f}_\alpha \, d\mathbf{w} = \int \mu B \overline{f}_\alpha \, d\mathbf{w} \\
p_{\parallel\alpha} &= \int m_\alpha w_\parallel^2 \overline{f}_\alpha \, d\mathbf{w} = \int 2(\varepsilon - \mu B)\overline{f}_\alpha \, d\mathbf{w} \\
n_\alpha &= \int \overline{f}_\alpha \, d\mathbf{w}
\end{aligned} \tag{9.47}$$

9.7.3 The double adiabatic model

Closure of the modified double adiabatic model requires two separate energy equations, one for $p_\perp$ and the other for $p_\parallel$. The closure procedure is given by

$$\mathbf{P} = \mathbf{P}_i + \mathbf{P}_e = p_\perp \mathbf{I} + \left(p_\parallel - p_\perp\right)\mathbf{bb} \tag{9.48}$$

where $p_\parallel$ and $p_\perp$ satisfy

$$\begin{aligned}
\frac{d}{dt}\left(\frac{p_\parallel B^2}{n^3}\right) &= 0 \\
\frac{d}{dt}\left(\frac{p_\perp}{nB}\right) &= 0
\end{aligned} \tag{9.49}$$

The momentum equation is slightly modified from the general form given in Eq. (9.42). The double adiabatic momentum equation is given by

$$\begin{aligned} \rho\mathbf{b}\times\left(\frac{d\mathbf{v}}{dt}\times\mathbf{b}\right) &= \mathbf{J}\times\mathbf{B}-\nabla_{\perp}p_{\perp}-(p_{\parallel}-p_{\perp})\boldsymbol{\kappa} \quad \text{perpendicular to } \mathbf{B} \\ \mathbf{b}\cdot\frac{d\mathbf{v}}{dt} &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{parallel to } \mathbf{B} \end{aligned} \tag{9.50}$$

This completes the summary of the models.

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Problems

The first three problems are aimed at understanding the modifications to the dispersion relation for waves in an infinite homogeneous plasma resulting from the new physics included in the alternate MHD models. Towards this goal assume that for each of the models under consideration the plasma equilibrium is given by

$$\begin{aligned} \mathbf{B} &= B_0\mathbf{e}_z \\ n &= n_0 \\ p &= p_0 \\ \mathbf{v} &= 0 \end{aligned}$$

where B_0, n_0, and p_0 are constants. Small amplitude waves are now allowed to propagate in the plasma with all perturbations varying as $Q(\mathbf{r},t)=\hat{Q}\exp(-i\omega t+\mathbf{k}\cdot\mathbf{r})$. Here, $\hat{Q}$ is a constant and

$$\mathbf{k}=k_{\perp}\mathbf{e}_y+k_{\parallel}\mathbf{e}_z$$

Derive the dispersion relation governing wave propagation for the alternate MHD models listed below and compare the results with those of ideal MHD by plotting ω vs. $k_{\|}$ for each of the three branches of the dispersion relation. Are any of the modes damped?

9.1 The original double adiabatic model.

9.2 The modified double adiabatic model.

9.3 The kinetic MHD model.

9.4 Repeat Problems 9.1 and 9.2 for the case of an anisotropic equilibrium using the double adiabatic model. Specifically assume that the equilibrium pressure is characterized by $p_{\perp} \neq p_{\|}$. Show that with sufficient anisotropy the plasma can become unstable. Derive the condition for stability.

9.5 Repeat Problem 9.4 for the kinetic MHD model.

9.6 Derive the general non-linear conservation of energy relation (analogous to Eq. (3.23)) for the double adiabatic model.

9.7 Derive Eq. (9.15).

9.8 Derive Eq. (9.18).

9.9 Derive Eq. (9.35).