Ideal MHD

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The ideal MHD model

2.1 Introduction

The goal of Chapter 2 is to provide a physical understanding of the ideal MHD model. Included in the discussion are (1) a basic description of the model, (2) a derivation starting from a more fundamental kinetic model, and, most importantly, (3) an examination of its range of validity.

In particular, it is shown that ideal MHD is the simplest fluid model that describes the macroscopic equilibrium and stability properties of a plasma. The claim of "simplest" is justified by a discussion of the large number of important plasma phenomena *not* covered by the model. However, in spite of its simplicity it is still a difficult model to solve analytically or even computationally because of the geometrical complexities associated with the two and three dimensionality of the configurations of fusion interest.

The derivation of the MHD model follows from the standard procedure of starting with a more fundamental and inclusive kinetic description of the plasma which describes the behavior of the electron and ion distribution functions. The mass, momentum, and energy moments of the kinetic equations are then evaluated. By introducing the characteristic length and time scales of ideal MHD, and making several corresponding ordering approximations, one is then able to close the system. The end result is the set of ideal MHD fluid equations.

The validity of the model is then assessed by examining the ordering assumptions used for closure to see whether or not they are consistent with the actual properties of fusion plasmas. This is a crucial step since ideal MHD is widely used in the design and interpretation of fusion experiments and one must be sure to understand the limits on the validity of the model. The assessment shows that while the basic derivation of MHD is straightforward there are several hidden surprises and subtleties.

Questions arise for two reasons. First, one of the basic assumptions used in the derivation, i.e., that the plasma is collision dominated, is *never* satisfied in plasmas of fusion interest. Even so, there is overwhelming empirical evidence that MHD provides an accurate description of macroscopic plasma behavior. This apparent good fortune is not a lucky coincidence but the consequence of some subtle physics; namely, those parts of the MHD model that are not valid because of violation of the collision dominated assumption are not directly involved in many if not most phenomena of interest. In other words, the model is only incorrect when it is unimportant. An attempt is made to clarify these issues in Chapter 9 by the introduction of several more sophisticated, low-collisionality plasma models whose regimes of validity are more closely aligned with actual experimental operating conditions. These models are more difficult to solve mathematically. However, several general equilibrium and stability comparison theorems are derived in Chapter 10 that help explain why ideal MHD works as well as it does.

The second subtle MHD issue concerns the following. Ideal MHD is an asymptotic model in the sense that specific length and time scales must be assumed for the derivation to be valid. In addition certain naturally appearing dimensionless parameters involving the MHD length and time scales must be ordered as small, medium, or large in order to close the system. For instance, high collisionality is represented by one such parameter. The issue here is that the multiple criteria defining the regime of validity arise from the need to simultaneously satisfy each assumption used in the derivation. However, a certain subset of phenomena described by the model requires only a corresponding subset of criteria to be satisfied, and consequently can have a much wider range of validity. One important example is MHD equilibrium. This important and useful information is discussed as the analysis progresses.

With these subtleties in mind attention is now focused on providing an in-depth description of the ideal MHD model.

2.2 Description of the model

The ideal MHD model provides a single-fluid description of long-wavelength, low-frequency, macroscopic plasma behavior. To put the model in perspective, it is perhaps useful to first discuss those plasma phenomena *not* described by ideal MHD.

Regarding physics in general, it has been pointed out that the three major discoveries of modern physics during the last two centuries, namely:

- Maxwell's equations with wave propagation
- relativity
- quantum mechanics

are each eliminated in the derivation of MHD.

Within the narrower confines of plasma physics itself, there are a variety of phenomena important in fusion plasmas. Among them are:

- radiation
- RF heating and current drive
- resonant particle effects
- micro instabilities
- classical and anomalous transport
- plasma–wall interactions
- resistive instabilities
- α-particle behavior.

Similarly, none of these phenomena is adequately described by ideal MHD.

Although the apparent lack of physical content is humbling, the one crucial phenomenon simply but accurately described by the model is the effect of magnetic geometry on the macroscopic equilibrium and stability of fusion plasmas. Specifically, ideal MHD answers such basic questions as: How does a given magnetic geometry provide forces to hold a plasma in equilibrium? Why are certain magnetic geometries more stable against macroscopic disturbances than others? Why do fusion configurations have such technologically undesirable shapes as a torus or a toroidal-helix?

One should be aware that in spite of the simplicity implied by its limited physical content, the ideal MHD model is still too difficult to solve in most geometries of interest. This will become evident as the text progresses by noting the many sophisticated expansions required to obtain analytic insight into the MHD behavior of various magnetic configurations. Attempts to solve similar problems using more comprehensive kinetic models are extremely difficult, even numerically, in realistic two- and three-dimensional geometries.

With this perspective the ideal MHD model is given by

$$
\begin{aligned}
&\text{Mass:} && \frac{\partial \rho}{\partial \mathrm{t}} + \nabla \cdot (\rho \mathbf{v}) = 0 \\
&\text{Momentum:} && \rho \frac{d\mathbf{v}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p \\
&\text{Energy:} && \frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0 \\
&\text{Ohm's law:} && \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \\
&\text{Maxwell:} && \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
& && \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\
& && \nabla \cdot \mathbf{B} = 0
\end{aligned}
\tag{2.1}
$$

In these equations, the electromagnetic variables are the electric field **E**, the magnetic field **B**, and the current density **J**. The fluid variables are the mass density ρ, the fluid velocity **v**, and the pressure p. Also, $\gamma = 5/3$ is the ratio of specific heats and $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is the convective derivative.

Observe that in ideal MHD the electromagnetic behavior is governed by the low-frequency, pre-Maxwell equations. The MHD fluid equations describe the time evolution of mass, momentum, and energy.

The mass equation implies that the total number of plasma particles is conserved; phenomena such as ionization, recombination, charge exchange, and unfortunately fuel depletion by fusion reactions, are negligible to a high order of accuracy on the MHD time scale.

The basic physics of the momentum equation corresponds to that of a fluid with three interacting forces: the pressure gradient force ∇p, the magnetic force $\mathbf{J} \times \mathbf{B}$, and the inertial force $\rho d\mathbf{v}/dt$. In static equilibrium it is the $\mathbf{J} \times \mathbf{B}$ force that balances the ∇p force, thereby confining the plasma. Dynamically, one must examine the stability of any such equilibrium to determine whether or not the plasma remains in place.

The energy equation expresses an adiabatic evolution characterized by a ratio of specific heats, $\gamma = 5/3$. The remaining relation is Ohm's law, which implies that in a reference frame moving with plasma the electric field is zero; that is, the plasma is a perfect conductor. It is the perfect conductivity assumption of Ohm's law that gives rise to the name "ideal" MHD.

As stated previously, the conditions for validity of the ideal MHD model imply that the phenomena of interest correspond to certain length and time scales. For macroscopic behavior the characteristic length scale is that of the overall plasma dimension. Denoting this dimension by a, then typically, for present day high-performance experiments, $a \sim 1$ m. The characteristic speed with which MHD phenomena occur is the thermal velocity of the plasma ions: $V_{Ti} = (2T_i/m_i)^{1/2}$, where T_i is the ion temperature and m_i is the ion mass. This gives rise to a characteristic MHD time $\tau_M \equiv a/V_{Ti}$. For m_i equivalent to deuterium and $T_i = 3$ keV then $\tau_M \sim 2$ μsec. The MHD length and time scales are compared with those of other basic plasma physics phenomena in Tables 2.1 and 2.2. In computing these values it has been assumed that $a = 1$ m, $T_e = T_i = 3$ keV, $B = 5$ T, $n = 10^{20}$ m^{-3} (particle number density) and m_i equivalent to deuterium. Also the Coulomb logarithm has been set to $\ln \Lambda = 19$.

The richness of plasma physics is clearly evidenced by the large number and wide range of length and time scales. Among these, ideal MHD lies midway between a variety of high-frequency microscopic phenomena and low-frequency collisional transport phenomena. This is the regime of macroscopic equilibrium and stability.

Table 2.1 *Comparison of the characteristic MHD time with those of other basic plasma physics phenomena. In practical formulas $T_e = T_i \equiv T$ is expressed in keV and n in* $10^{20}\,m^{-3}$.

Plasma physics time scales	Formulas	Values (sec)
Electron gyro period	$\tau_{ce} = 2\pi/\omega_{ce} = 2\pi m_e/eB$	7.1×10^{-12}
Electron plasma period	$\tau_{pe} = 2\pi/\omega_{pe} = 2\pi(m_e\varepsilon_0/ne^2)^{1/2}$	1.1×10^{-11}
Ion plasma period	$\tau_{pi} = (m_i/m_e)^{1/2}\tau_{pe}$	6.7×10^{-10}
Ion gyro period	$\tau_{ci} = (m_i/m_e)\tau_{ce}$	2.8×10^{-8}
MHD time	$\tau_M = a/V_{Ti}$	1.9×10^{-6}
Electron–electron collision time	$\tau_{ee} = 6.7 \times 10^{-6}\ T^{3/2}/n$	3.0×10^{-5}
Ion–ion collision time	$\tau_{ii} = (2m_i/m_e)^{1/2}\tau_{ee}$	2.6×10^{-3}
Energy equilibration time	$\tau_{eq} \approx (m_i/2m_e)\tau_{ee}$	5.5×10^{-2}
Energy confinement time for ignition	$\tau_E = 1.7/n$	1.7
Resistive diffusion time	$\tau_D = \mu_0 a^2/\eta = 40\ a^2T^{3/2}$	2.1×10^2

Table 2.2 *Comparison of the characteristic MHD length with those of other basic plasma physics phenomena.*

Plasma physics length scales	Formulas	Values (m)
Electron gyro radius	$r_{Le} = V_{Te}/\omega_{ce}$	3.7×10^{-5}
Debye length	$\lambda_D = V_{Te}/\omega_{pe}$	5.8×10^{-5}
Electron skin depth	$\delta_e = c/\omega_{pe}$	5.3×10^{-4}
Ion gyro radius	$r_{Li} = (m_i/m_e)^{1/2} r_{Le}$	2.2×10^{-3}
Ion skin depth	$\delta_i = (m_i/m_e)^{1/2}\delta_e$	3.2×10^{-2}
MHD length	a	1
Ion–ion mfp	$\lambda_{ii} = V_{Ti}\tau_{ii}$	1.4×10^3
Electron–electron mfp	$\lambda_{ee} = \lambda_{ii}$	1.4×10^3

Observe that while typical MHD times (on the order of microseconds) are much shorter than typical experimental times (on the order of seconds), this does not imply that MHD can be applied only during a small fraction of the time of interest. The widest use of MHD involves the repeated calculation of equilibrium and stability over many small time increments during the slow evolution of a discharge.

For example, illustrated in Fig. 2.1 is a curve of the time evolution of the current in a tokamak. Also shown is a narrow time increment Δt during which MHD is valid. If at the beginning of Δt the external fields and the plasma current and pressure are given, then the MHD model can be used to calculate the equilibrium and stability of the system. This procedure can then be repeated many times over a continuing sequence of small time increments as the pressure and current slowly evolve on the transport time scale, an evolution whose physics is not described by

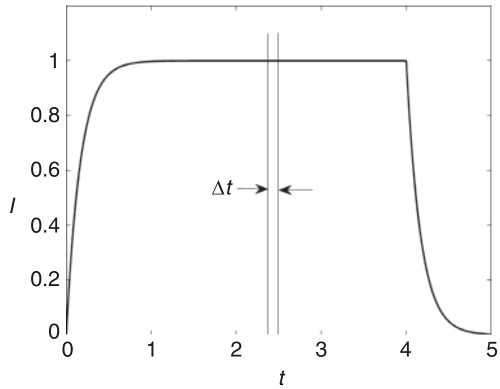

Figure 2.1 Diagram of a 5 sec current pulse in a typical tokamak. The interval $\Delta t \sim 50\,\mu$sec corresponds to one of a continual sequence of time slices during which ideal MHD is valid.

ideal MHD. In other words, the plasma conditions at the start of each MHD increment must be provided by experimental measurements or by theoretical transport modeling.

There is a critical implication, however, that during each successive time increment Δt, the plasma is MHD stable. If not, the plasma properties would likely change in a rapid and unfavorable way (e.g. a large increase in anomalous transport or complete destruction of the plasma) within several MHD times because of the strength of the instabilities. Stated differently, it is often more important to learn how to avoid MHD instabilities during the long time transport evolution of a plasma rather than to learn the precise details of what happens after an instability is excited – invariably such an instability leads to a major degradation in performance.

In summary, although many important plasma physics processes are neglected in the derivation of ideal MHD, the one critical phenomenon that remains is the self-consistent treatment of macroscopic equilibrium and stability in multidimensional magnetic geometries.

2.3 Derivation of the ideal MHD model

2.3.1 Starting equations

In order to more fully appreciate the physics content of ideal MHD as well as the subtleties involved, a derivation of the model is presented starting from basic principles. A number of such derivations exist in the literature and each, including the one presented here, follows the same general procedure: fluid moments are calculated from a general kinetic model, and then a number of assumptions are made to obtain closure of the system.

Many would agree that Braginskii's (1965) classic calculation represents one of the earliest and most rigorous derivations of plasma fluid equations in the collision dominated regime. The analysis presented here is noticeably simpler because the end goal is much narrower in scope; that is, the derivation is focused solely on the ideal MHD model with little attention given to the general question of determining transport coefficients in different collisionality regimes. Even so, some of the results of Braginskii are quoted to demonstrate the smallness of certain terms neglected in the ideal MHD model.

The starting point for the present derivation is the full set of Maxwell's equations coupled to the Boltzmann kinetic model for the plasma. Specifically, the kinetic model for each species and its coupling to Maxwell's equations is given by

$$\begin{aligned} \frac{df_\alpha}{dt} \equiv \frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha + \frac{Z_\alpha e}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_\alpha &= \left(\frac{\partial f_\alpha}{\partial t}\right)_c \\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\times\mathbf{B} &= \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \\ \nabla\cdot\mathbf{E} &= \frac{\sigma}{\varepsilon_0} \\ \nabla\cdot\mathbf{B} &= 0 \end{aligned} \tag{2.2}$$

with the current and charge densities defined as

$$\begin{aligned} \mathbf{J} &= \sum_\alpha Z_\alpha e\int \mathbf{v} f_\alpha d\mathbf{v} \\ \sigma &= \sum_\alpha Z_\alpha e\int f_\alpha d\mathbf{v} \end{aligned} \tag{2.3}$$

Here, $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ is the distribution function for each species α, and is in general a function of spatial coordinate, velocity, and time. It is assumed that the plasma is fully ionized and consists of two species, electrons and singly charged ions. Hence, $\alpha = e, i$ and $Z_i = 1$, $Z_e = -1$. (A derivation of the starting kinetic equation can be found in many textbooks. See the Further reading at the end of the chapter. A heuristic derivation is given in Appendix A.)

In the kinetic description there are two types of forces that act on the particles. First there are the long-range Lorentz forces, $Z_\alpha e\,\mathbf{E}$ and $Z_\alpha e\,\mathbf{v}\times\mathbf{B}$, in which $\mathbf{E}$ and $\mathbf{B}$ are smoothly behaved fields calculated from the averaged current and charge densities as indicated in Eq. (2.3). Second, the right-hand side of the kinetic equation represents the forces due to short-range interactions, or collisions. In the derivation presented here, the details of the collision operator are not of major importance. Only certain global conservation relations are needed. For plasmas of fusion interest, the dominant collisions are elastic Coulomb collisions between both like and unlike particles. The conservation laws for elastic collisions can be summarized as follows. If the collision operator is defined in the usual way:

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = \sum_\beta C_{\alpha\beta} \tag{2.4}$$

where $C_{\alpha\beta}$ represents collisions of particles of species α with particles of species β, then:

- Conservation of particles between like and unlike particle collisions implies

$$\int C_{ee} d\mathbf{v} = \int C_{ii} d\mathbf{v} = \int C_{ei} d\mathbf{v} = \int C_{ie} d\mathbf{v} = 0 \tag{2.5}$$

- Conservation of momentum and energy between like particle collisions implies

$$\begin{aligned} \int m_e \mathbf{v} C_{ee} d\mathbf{v} &= \int m_i \mathbf{v} C_{ii} d\mathbf{v} = 0 \\ \int \frac{1}{2} m_e v^2 C_{ee} d\mathbf{v} &= \int \frac{1}{2} m_i v^2 C_{ii} d\mathbf{v} = 0 \end{aligned} \tag{2.6}$$

- Conservation of total momentum and energy between unlike particle collisions implies

$$\begin{aligned} \int (m_e \mathbf{v} C_{ei} + m_i \mathbf{v} C_{ie}) d\mathbf{v} &= 0 \\ \int \frac{1}{2} \left(m_e v^2 C_{ei} + m_i v^2 C_{ie}\right) d\mathbf{v} &= 0 \end{aligned} \tag{2.7}$$

More information on collisions in a plasma can be obtained from many excellent textbooks listed at the end of the chapter.

The full set of kinetic-Maxwell equations provides a detailed and complete description of plasma behavior. At one end of the spectrum, it contains microscopic information about the orbits of individual charged particles on the very short gyro time scale and gyro radius length scale. At the other end, it accurately describes the macroscopic behavior of large plasma experiments including MHD equilibrium and stability as well as very slow transport phenomena. Not surprisingly, the complexity arising from this breadth of information makes it virtually impossible to solve, even numerically, the kinetic-Maxwell system of equations in any non-trivial geometry. This realization has led to the development of several simpler models with narrower physics content. Ideal MHD is one such model.

2.3.2 Two-fluid equations

The derivation of the ideal MHD equations begins by taking moments of the kinetic equation. This calculation transforms a single equation for f_α in seven variables ($\mathbf{r}$, $\mathbf{v}$, t) into an infinite set of fluid equations in four variables ($\mathbf{r}$, t). The procedure is straightforward although somewhat tedious. Once the fluid equations are obtained, physical variables such as density, velocity, and pressure are introduced. Then, the physical regime of interest is defined, allowing various approximations and expansions to be made, ultimately leading to truncation and closure of the infinite set of equations. Since the truncated system is by design focused on a specific regime of physics its total information content is much less than that of the original kinetic equation. Even so, the virtue of the fluid equations is that they are enormously simpler to solve.

To derive the ideal MHD model, the moments that are required correspond to mass, momentum, and energy; that is, starting with the kinetic equation one evaluates

$$\int g_i \left[\frac{df_\alpha}{dt} - \left(\frac{\partial f_\alpha}{\partial t} \right)_c \right] d\mathbf{v} = 0 \tag{2.8}$$

for $i = 1-3$ with $g_i(\mathbf{v})$ given by

$$\begin{aligned} g_1 &= 1 && \text{(mass)} \\ g_2 &= m_\alpha \mathbf{v} && \text{(momentum)} \\ g_3 &= m_\alpha v^2/2 && \text{(energy)} \end{aligned} \tag{2.9}$$

After a straightforward calculation the fluid equations for each species can be written as

$$\begin{gathered} \frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0 \\ \frac{\partial}{\partial t}(m_\alpha n_\alpha \mathbf{u}_\alpha) + \nabla \cdot (m_\alpha n_\alpha \langle \mathbf{v}\mathbf{v} \rangle) - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = \int m_\alpha \mathbf{v}\, C_{\alpha\beta} d\mathbf{v} \\ \frac{\partial}{\partial t}\left(\frac{1}{2} m_\alpha n_\alpha \langle v^2 \rangle \right) + \nabla \cdot \left(\frac{1}{2} m_\alpha n_\alpha \langle v^2 \mathbf{v} \rangle \right) - Z_\alpha e n_\alpha \mathbf{u}_\alpha \cdot \mathbf{E} = \int \frac{1}{2} m_\alpha v^2\, C_{\alpha\beta} d\mathbf{v} \end{gathered} \tag{2.10}$$

Here, n_α and $\mathbf{u}_\alpha$ are the macroscopic number density and fluid velocity defined by

$$\begin{aligned} n_\alpha &= \int f_\alpha d\mathbf{v} \\ \mathbf{u}_\alpha &= \frac{1}{n_\alpha} \int \mathbf{v} f_\alpha d\mathbf{v} \end{aligned} \tag{2.11}$$

and the only non-zero contributions to the collision terms are from unlike particle interactions, $\alpha \neq \beta$. The quantities $\langle \mathbf{vv} \rangle$, $\langle v^2 \rangle$, and $\langle v^2\mathbf{v} \rangle$ are higher moments of the distribution function defined by the general relation

$$\langle h \rangle = \frac{1}{n_\alpha} \int h f_\alpha d\mathbf{v} \tag{2.12}$$

The next step in the derivation is to introduce a new independent velocity variable, $\mathbf{w} = \mathbf{v} - \mathbf{u}_\alpha(\mathbf{r}, t)$, representing the random thermal motion of the particles, which by definition satisfies $\langle \mathbf{w} \rangle = 0$. By introducing this variable into Eq. (2.10) one can write the fluid equations in terms of more physical macroscopic quantities. In particular, the quantities of interest are the scalar pressure,

$$p_\alpha = \frac{1}{3} m_\alpha n_\alpha \langle w^2 \rangle \tag{2.13}$$

the total pressure tensor,

$$\mathsf{P} = m_\alpha n_\alpha \langle \mathbf{ww} \rangle \tag{2.14}$$

the anisotropic part of the pressure tensor,

$$\mathsf{\Pi}_\alpha = \mathsf{P}_\alpha - p_\alpha \mathsf{I} \tag{2.15}$$

the temperature,

$$T_\alpha = p_\alpha / n_\alpha \tag{2.16}$$

the heat flux due to random motion,

$$\mathbf{q}_\alpha = \frac{1}{2} m_\alpha n_\alpha \langle w^2 \mathbf{w} \rangle \tag{2.17}$$

the mean momentum transferred between unlike particles due to the friction of collisions,

$$\mathbf{R}_\alpha = \int m_\alpha \mathbf{w} C_{\alpha\beta} d\mathbf{w} \tag{2.18}$$

and the heat generated and transferred between unlike particles due to collisional dissipation,

$$Q_\alpha = \int \frac{1}{2} m_\alpha w^2 C_{\alpha\beta} d\mathbf{w} \tag{2.19}$$

Substituting these definitions into Eq. (2.10) and making use of the mass continuity relation leads to the following form of the moment equations:

$$
\begin{aligned}
\left(\frac{dn_\alpha}{dt}\right)_\alpha + n_\alpha \nabla \cdot \mathbf{u}_\alpha &= 0 \\
m_\alpha n_\alpha \left(\frac{d\mathbf{u}_\alpha}{dt}\right)_\alpha - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \nabla \cdot \mathbf{P}_\alpha &= \mathbf{R}_\alpha \\
n_\alpha \left[\frac{d}{dt}\left(\frac{1}{2} m_\alpha u_\alpha^2 + \frac{3}{2} T_\alpha\right)\right]_\alpha - Z_\alpha e n_\alpha \mathbf{u}_\alpha \cdot \mathbf{E} + \nabla \cdot (\mathbf{u}_\alpha \cdot \mathbf{P}_\alpha + \mathbf{q}_\alpha) &= Q_\alpha + \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha
\end{aligned}
\tag{2.20}
$$

where

$$
\left(\frac{d}{dt}\right)_\alpha \equiv \frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \tag{2.21}
$$

is the convective derivative for the species α.

A further reduction can be made leading to a simpler form of the energy equation. This form is obtained by evaluating the dot product of the momentum equation with $\mathbf{u}_\alpha$ and subtracting the result from the energy equation. Upon replacing the energy equation with the simplified form one obtains the final set of two-fluid equations given by

$$
\begin{aligned}
&\left(\frac{dn_\alpha}{dt}\right)_\alpha + n_\alpha \nabla \cdot \mathbf{u}_\alpha = 0 \\
&m_\alpha n_\alpha \left(\frac{d\mathbf{u}_\alpha}{dt}\right)_\alpha - Z_\alpha e n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \nabla \cdot \mathbf{P}_\alpha = \mathbf{R}_\alpha \\
&\frac{3}{2} n_\alpha \left(\frac{dT_\alpha}{dt}\right)_\alpha + \mathbf{P}_\alpha : \nabla \mathbf{u}_\alpha + \nabla \cdot \mathbf{q}_\alpha = Q_\alpha \\
&\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
&\nabla \times \mathbf{B} = \mu_0 e (n_i \mathbf{u}_i - n_e \mathbf{u}_e) + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \\
&\nabla \cdot \mathbf{E} = \frac{e}{\varepsilon_0} (n_i - n_e) \\
&\nabla \cdot \mathbf{B} = 0
\end{aligned}
\tag{2.22}
$$

where, as stated, the ions are singly charged so that $Z_i = -Z_e = 1$.

This set of equations is exact if not very useful since there is as yet no prescription for closing the system: there are more unknowns than equations. The particular prescription that leads to the ideal MHD equations consists of the following steps. First, certain asymptotic orderings are introduced that eliminate the very high-frequency, short-wavelength information in the model. These orderings are well satisfied for the phenomena of interest, i.e., those involving the macroscopic behavior of fusion plasmas. Next, the equations are rewritten as a set of single-fluid equations by the introduction of appropriate single-fluid variables. The system still has more unknowns than equations. A crucial step follows. The plasma is assumed to be collision dominated, thereby enabling the higher-order moments to be approximated in terms of the basic variables by means of Braginskii's well-known transport theory (1965). The result is a complicated but closed set of equations. It is at this point that the characteristic MHD length and time scales are introduced. The various terms appearing in the equations can now be ordered as small, medium, or large in terms of several dimensionless parameters. One can then define a set of validity conditions in terms of these parameters such that the remaining, greatly simplified set of equations corresponds to ideal MHD.

A remarkable feature of the analysis is that while the model is derived on the assumption that the plasma is collision dominated, the final ideal MHD equations do not explicitly depend on the details of the collisions. The main purpose of the collisional transport theory is to provide the appropriate scaling of the transport coefficients. The corresponding terms are then systematically ordered out of the model by comparing them with other terms dominated by the length and time scales associated with macroscopic MHD behavior.

2.3.3 Low-frequency, long-wavelength, asymptotic expansions

There are two important approximations that can be made, leading to substantial simplifications in the two-fluid equations. These approximations serve to eliminate the high-frequency, short-wavelength information from the model. Each approximation has the form of an asymptotic expansion which eliminates a leading-order time derivative and thus alters the basic mathematical structure of the model.

The first approximation represents the transformation of the full Maxwell's equations to the low-frequency pre-Maxwell's equations. This limit is formally accomplished by letting $\varepsilon_0 \to 0$. As a result, the displacement current $\varepsilon_0 \partial \mathbf{E}/\partial t$ and the net charge $\varepsilon_0 \nabla \cdot \mathbf{E}$ can both be neglected. The resulting set of equations is easily shown to be Galilean invariant.

The neglect of the displacement current requires that the electromagnetic waves of interest have phase velocities much slower than the speed of light (i.e., $\omega/k \ll c$), and that the characteristic thermal velocities be non-relativistic (i.e., $V_{Ti} \ll V_{Te} \ll c$). The neglect of the net charge restricts attention to plasma behavior whose characteristic frequency is much less than the electron plasma frequency (i.e., $\omega \ll \omega_{pe}$)

and whose characteristic length is much longer than the Debye length (i.e., $a \gg \lambda_D$). An examination of Tables 2.1 and 2.2 shows that these assumptions are satisfied by a large margin when applied to the MHD behavior of fusion plasmas.

The neglect of $\varepsilon_0 \nabla \cdot \mathbf{E}$ implies that

$$n_i = n_e \equiv n \tag{2.23}$$

This relationship is called the quasineutral approximation. It should be emphasized that Eq. (2.23) does not imply that $\mathbf{E} = 0$ or $\nabla \cdot \mathbf{E} = 0$, only that $\varepsilon_0 \nabla \cdot \mathbf{E}/en \ll 1$. For any low-frequency macroscopic charge separation that tends to develop, the electrons have more than an adequate time to respond, creating an electric field whose direction is such as to cancel the charge imbalance. This maintains the plasma in local quasineutrality.

As an example, consider an electrostatic problem with $\mathbf{E} = -\nabla\phi$ and $n_e = n_e(\phi, \mathbf{r})$, $n_i = n_i(\phi, \mathbf{r})$. Equating n_e to n_i and inverting the relationship then gives $\phi = \phi(\mathbf{r})$. Quasineutrality implies that the electric field calculated from $\mathbf{E} = -\nabla\phi$ satisfies $\varepsilon_0 \nabla \cdot \mathbf{E}/en \ll 1$.

The second asymptotic assumption neglects electron inertia in the electron momentum equation and is accomplished formally by letting $m_e \to 0$. This implies that on time scales of MHD interest the electrons have an infinitely fast response time because of their small mass. Specifically, time scales long compared to those of the electron plasma frequency, ω_{pe}, and the electron cyclotron frequency, ω_{ce}, are required. Similarly, the length scales must be long compared to the Debye length, λ_D, and the electron gyro radius, r_{Le}. As before, an examination of Tables 2.1 and 2.2 shows that these conditions are easily satisfied for macroscopic phenomena in fusion plasmas.

2.3.4 The single-fluid equations

Using the asymptotic approximations just described, one can derive a set of single-fluid equations by introducing a new set of fluid variables. To begin, it is customary to introduce a mass density rather than a number density. Since $m_e \to 0$ and $n_i = n_e \equiv n$, the mass density corresponds to that of the ions and is defined as

$$\rho = m_i n \tag{2.24}$$

Likewise, the momentum of the fluid is also carried by the ions, so that the appropriate definition of the fluid velocity is given by

$$\mathbf{v} = \mathbf{u}_i \tag{2.25}$$

The current density is proportional to the difference in flow velocity between electrons and ions,

$$\begin{aligned}\mathbf{J} &= en(\mathbf{u}_i - \mathbf{u}_e)\\ \mathbf{u}_e &= \mathbf{v} - \mathbf{J}/en\end{aligned} \tag{2.26}$$

The final definitions required are for the total pressure and temperature,

$$\begin{aligned}p &= p_i + p_e = 2nT\\ T &= (T_i + T_e)/2\end{aligned} \tag{2.27}$$

Equations (2.24)–(2.27) relate the single-fluid variables ρ, $\mathbf{v}$, $\mathbf{J}$, p, T to the two-fluid variables n, $\mathbf{u}_i$, $\mathbf{u}_e$, p_i, p_e, T_i, T_e.

Two points require discussion before proceeding with the single-fluid equations. First, care must be exercised to conserve the total information content of the starting equations. For instance, several of the single-fluid equations are obtained by combining various equations of the two-fluid model. If these manipulations involve two equations from the starting model, then two equivalent equations must appear in the final model.

This is not as trivial as it might seem, as evidenced by the second point of discussion: namely, that if one counts fluid variables then the two-fluid model has 11 unknowns while the single-fluid model has 9 unknowns. A careful accounting of the information content in the two-fluid equations is shown to explain and eliminate this imbalance.

Consider now the derivation of the single-fluid model. The first equation is obtained from the mass conservation equations. Multiplying the ion mass equation by m_i one finds

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.28}$$

which is identical to the ideal MHD mass conservation relation. The other information contained in these equations is obtained by multiplying the electron and ion equations by e and then subtracting. The result is

$$\nabla \cdot \mathbf{J} = 0 \tag{2.29}$$

Equation (2.29) is redundant with charge conservation in the low-frequency form of Maxwell's equations: $\nabla \cdot (\nabla \times \mathbf{B} - \mu_0 \mathbf{J}) = \mu_0 \nabla \cdot \mathbf{J} = 0$.

The next set of single-fluid equations follows from the momentum equations. The electron and ion equations are first added together. Making use of the fact that $\mathbf{R}_e = -\mathbf{R}_i$ leads to the relationship

$$\rho \frac{d\mathbf{v}}{dt} - \mathbf{J} \times \mathbf{B} + \nabla p = -\nabla \cdot (\mathbf{\Pi}_i + \mathbf{\Pi}_e) \tag{2.30}$$

Here, $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ now represents the convective derivative moving with the (ion) fluid. The left-hand side of Eq. (2.30) corresponds to the ideal MHD momentum equation. The condition under which the right-hand side is negligible defines one constraint on the parameter regime for which the MHD equations are valid and will be discussed shortly. Note that there is no electric force, $\sigma\mathbf{E}$, acting on the fluid, since $\sigma = 0$ as a result of the quasineutral approximation.

The second piece of information contained in the two-fluid momentum equations is obtained by simply rewriting the electron equation in terms of the single-fluid variables. This leads to the following relation:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{en}(\mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \mathbf{\Pi}_e + \mathbf{R}_e) \tag{2.31}$$

The left-hand side of Eq. (2.31) corresponds to the ideal MHD Ohm's law. The conditions under which the right-hand side is negligible determine additional constraints on the parameter regime for the validity of MHD and are also discussed shortly.

The third set of single-fluid equations follows from the two-fluid energy equations. After some straightforward algebra one obtains

$$\begin{aligned}
\frac{d}{dt}\left(\frac{p_i}{\rho^\gamma}\right) &= \frac{2}{3\rho^\gamma}(Q_i - \nabla \cdot \mathbf{q}_i - \mathbf{\Pi}_i : \nabla \mathbf{v}) \\
\frac{d}{dt}\left(\frac{p_e}{\rho^\gamma}\right) &= \frac{2}{3\rho^\gamma}\left[Q_e - \nabla \cdot \mathbf{q}_e - \mathbf{\Pi}_e : \nabla\left(\mathbf{v} - \frac{\mathbf{J}}{en}\right)\right] + \frac{1}{en}\mathbf{J} \cdot \nabla\left(\frac{p_e}{\rho^\gamma}\right)
\end{aligned} \tag{2.32}$$

where $\gamma = 5/3$. The last term in the electron component of Eq. (2.32) results from the convective derivative relation $(d/dt)_e = (d/dt) - (\mathbf{J}/en) \cdot \nabla$. The left-hand sides of Eq. (2.32) are closely related to the ideal MHD equation of state. There are a number of terms on the right-hand side, which in order to be neglected define additional conditions on the parameter regime of validity. Furthermore, since MHD is a single-fluid model, other assumptions must be made to couple the individual electron and ion temperatures and pressures into single-fluid variables.

The final equations of the single-fluid model are the remaining low-frequency Maxwell equations given by

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned} \tag{2.33}$$

As stated previously, Eq. (2.29) is consistent with the low-frequency form of Ampere's law.

The single-fluid model is described by Eqs. (2.28)–(2.33). No assumptions other than the asymptotic approximations have been made at this point. Although the left-hand sides of the fluid equations are identical to the MHD model, the full set of equations is still not closed because of the presence of the as yet undefined higher moments and the unresolved coupling of the electron and ion energies.

2.3.5 The ideal MHD limit

This section contains a description of the critical assumptions required for the single-fluid model to reduce to ideal MHD; in particular, quantitative conditions are derived that determine when the right-hand sides of Eqs. (2.30)–(2.32) are negligible.

The basic requirement for the validity of ideal MHD is that both the electrons and ions be collision dominated. This is the usual requirement for a fluid model to be useful. If there are sufficient collisions, a given particle remains reasonably close to its neighboring particles during the time scales of interest. In this case the division of the plasma into small identifiable fluid elements provides a good description of the physics.

The question as to whether a given model describes collision dominated or collisionless behavior rests largely in the evolution of the pressure tensor; that is, if one considers the exact mass and momentum equations it is only the pressure tensor that remains as an undefined higher moment. A comparison between Eq. (2.30) and the ideal MHD momentum equation indicates that for the models to be equivalent, $\mathbf{\Pi}_i$ and $\mathbf{\Pi}_e$ must be negligible. The implication is that the full pressure tensor $\mathbf{P}$ reduces to a simple scalar isotropic pressure p.

One intuitively expects an isotropic pressure to arise in systems where many collisions take place on a time scale which is short compared to those of interest. The collisions rapidly randomize the distribution function into a Maxwellian form giving rise to an isotropic pressure. This is the basic assumption of the scalar pressure ideal MHD model. In collisionless models closure of the pressure tensor is more complicated, requiring the solution of reduced kinetic equations. A perhaps surprising feature of MHD is that for most problems involving equilibrium and stability the results are not overly sensitive to the specific model used to describe the evolution of $\mathbf{P}$. It is this fact that is ultimately responsible for the unexpectedly reliable predictions of ideal MHD well outside its regime of validity.

The collision dominated assumption enters into the single-fluid equations as follows. In this limit the distribution functions for electrons and ions are nearly locally Maxwellians. As a consequence, one can refer to well-established theories, such as given by Braginskii (1965), in order to obtain expressions for the higher-order fluid moments in terms of appropriate transport coefficients. By defining the characteristic length and time scales of ideal MHD it is then possible to compare the MHD terms with the transport terms. This then determines a set of conditions for the right-hand sides of the single-fluid equations to be negligible. These are the validity conditions for ideal MHD.

A convenient place to begin the analysis is to recall the characteristic MHD length and time scales. Since the main goal of ideal MHD is the investigation of macroscopic phenomena, the length scales of interest correspond to the macroscopic dimensions of the plasma denoted by a. The typical time scale of MHD interest corresponds to a/V_{Ti}, the ion thermal transit time across a macroscopic plasma dimension. This time scale is characteristic of many MHD plasma instabilities and represents the fastest time scale in which macroscopic plasma motion can occur. In determining the scaling relations it is helpful to introduce the characteristic MHD frequency ω and wave number k as follows:

$$\begin{aligned} \omega &\sim \frac{\partial}{\partial t} \sim \frac{V_{Ti}}{a} \\ k &\sim |\nabla| \sim \frac{1}{a} \end{aligned} \tag{2.34}$$

and, similarly, the resulting velocity

$$\frac{\omega}{k} \sim |\mathbf{v}| \sim V_{Ti} \tag{2.35}$$

The next step in the analysis is to consider the conditions for the collision-dominated transport theory to be valid. There are two such conditions. The first requires that during the MHD time scale of interest, each species has sufficient collisions to make the distribution function nearly Maxwellian. For the ions the dominant collision mechanism is due to ion–ion interactions, characterized by a collision time τ_{ii}. The electrons become Maxwellian by colliding with either other electrons or ions. Since $\tau_{ee} \sim \tau_{ei}$ it is not important to make this distinction. Hence, the condition that each species be collision dominated is given by

$$\begin{array}{ll} \text{Ions} & \omega\tau_{ii} \sim V_{Ti}\tau_{ii}/a \ll 1 \\ \text{Electrons} & \omega\tau_{ee} \sim (m_e/m_i)^{1/2} V_{Ti}\tau_{ii}/a \ll 1 \end{array} \tag{2.36}$$

Here, use has been made of the fact that $\tau_{ee} \sim (m_e/m_i)^{1/2}\,\tau_{ii}$ when $T_i \sim T_e$. As might be expected, this condition is more restrictive for ions than electrons.

The second condition for the collision dominated theory to be valid requires that the macroscopic scale length be much longer than the mean free path. Noting that the mean free path for each species is given by $\lambda_\alpha \sim V_{T\alpha}\tau_{\alpha\alpha}$, the condition $\lambda_\alpha \ll a$ reduces to $V_{Ti}\tau_{ii}/a \ll 1$ for both electrons and ions; that is, for the ions both the time scale and length scale requirements yield the same condition for high collisionality. For the electrons, the length scale requirement is more restrictive and yields the same condition as for the ions. Thus, the conditions for a collision dominated plasma can be summarized as follows:

$$V_{Ti}\tau_{ii}/a \sim V_{Te}\tau_{ee}/a \ll 1 \tag{2.37}$$

The collision dominated assumption is now used to estimate the higher-order moments, which enter as transport terms, in the single-fluid equations. (Readers unfamiliar with the evaluation of transport coefficients should attempt Problems 2.2–2.4, which demonstrate a simple procedure for deriving the basic scaling of such coefficients.) For present purposes the transport coefficients used in the analysis are those derived by Braginskii (1965) and are summarized for convenience in Appendix B.

The analysis begins with Eq. (2.31), the momentum equation. The matrix elements for $\mathbf{\Pi}_i$ and $\mathbf{\Pi}_e$ are a rather complicated series of terms. The leading-order effect, however, is viscosity. Moreover, the ion viscosity coefficient is larger than that of the electrons by a factor $(m_i/m_e)^{1/2}$. From Braginskii (1965) it follows that the largest elements of the $\mathbf{\Pi}_i$ tensor have the form (in rectangular coordinates)

$$\Pi_{jj} \sim \mu\left(2\nabla_{\parallel}\cdot\mathbf{v}_{\parallel} - \frac{2}{3}\nabla\cdot\mathbf{v}\right) \sim \mu\frac{V_{Ti}}{a} \tag{2.38}$$

where the viscosity coefficient μ is given by

$$\mu \sim nT_i\tau_{ii} \tag{2.39}$$

If the right-hand side of Eq. (2.30) is now compared with the ∇p term one finds

$$|\nabla\cdot\mathbf{\Pi}_i/\nabla p| \sim V_{Ti}\tau_{ii}/a \ll 1 \tag{2.40}$$

Therefore, if the collision dominated assumption is satisfied, the viscosity is negligible and the momentum equation reduces to that of ideal MHD.

The next equation to consider is Ohm's law, given by Eq. (2.31). From the argument just given it is clear that the $\mathbf{\Pi}_e$ term can be neglected compared to the ∇p_e term. From the momentum equation it also follows that the $\mathbf{J}\times\mathbf{B}$ and ∇p_e terms are comparable. The $\mathbf{J}\times\mathbf{B}$ term in Ohm's law represents the Hall effect, while the ∇p_e term represents the effect of the electron diamagnetic drift. Comparing either of these terms with the $\mathbf{v}\times\mathbf{B}$ term yields

$$\frac{|\nabla p_e/en|}{|\mathbf{v}\times\mathbf{B}|} \sim \frac{r_{Li}}{a} \tag{2.41}$$

where $r_{Li} = V_{Ti}/\omega_{ci}$ is the ion gyro radius. Thus, if one makes the additional assumption that

$$r_{Li}/a \ll 1 \tag{2.42}$$

which is well satisfied in fusion experiments, then the $\mathbf{J} \times \mathbf{B}$ and ∇p_e terms can be neglected in the Ohm's laws. Note that Eq. (2.42) implies that

$$\omega/\omega_{ci} \sim r_{Li}/a \ll 1 \tag{2.43}$$

MHD frequencies are much lower than the ion gyro frequency.

The remaining term on the right-hand side of Eq. (2.31), $\mathbf{R}_e/en$, represents the momentum transfer due to the friction of collisions between electrons and ions. The dominant contribution to $\mathbf{R}_e$ is electrical resistivity. Using the results from Braginskii, one can express $\mathbf{R}_e$ as

$$\mathbf{R}_e/en \sim \eta \mathbf{J} \tag{2.44}$$

where the electrical resistivity η is given by

$$\eta \sim \frac{m_e}{ne^2 \tau_{ei}} \tag{2.45}$$

Substituting the scaling relation (from the momentum equation) $|\mathbf{J}| \sim |\nabla p_e|/|\mathbf{B}|$ leads to the following requirement for the $\eta\mathbf{J}$ term to be negligible compared to the $\mathbf{v} \times \mathbf{B}$ term:

$$\frac{|\eta \mathbf{J}|}{|\mathbf{v} \times \mathbf{B}|} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \left(\frac{r_{Li}}{a}\right)^2 \left(\frac{a}{V_{Ti}\tau_{ii}}\right) \ll 1 \tag{2.46}$$

Equation (2.46) implies that while τ_{ii} must be sufficiently small for the collision dominated approximation to hold it cannot be so small that the plasma will be dominated by resistive diffusion. Alternatively, one can view Eq. (2.46) as a requirement to make the macroscopic dimension a large enough so that the resistive diffusion time is long compared to the characteristic MHD time. Thus, in order for the ideal MHD Ohm's law to apply, the small gyro radius and small resistivity conditions must both be satisfied. Practically, the condition given by Eq. (2.46) is well satisfied in all fusion experiments.

Before proceeding to the energy equations there are several subtleties related to the electric field and Ohm's law that should be discussed. (1) The ideal MHD Ohm's law $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ implies that $|\mathbf{E}_\perp| \sim V_{Ti}B$. However, in many MHD stability analyses the equilibrium ion flow is assumed to be zero (i.e., $\mathbf{v} = 0$). In this case the equilibrium ion pressure is confined by an electrostatic electric field $\mathbf{E} = -\nabla\phi$ that scales as $|\mathbf{E}_\perp| \sim |\nabla p_i|/en \sim (r_{Li}/a)V_{Ti}B$. The equilibrium $\mathbf{E}_\perp$ is incorrectly omitted from ideal MHD equilibrium analyses (since it is being compared to zero) because of the $r_{Li}/a \ll 1$ assumption. (2) The electric field associated with fast time-varying MHD motions is treated correctly. Specifically, in terms of the vector potential it follows that $\mathbf{E}_\perp \sim i\omega\mathbf{A}_\perp$ and $\mathbf{B} \sim i\mathbf{k} \times \mathbf{A}$ which, using the MHD ordering, implies that $|\mathbf{E}_\perp| \sim V_{Ti}|\mathbf{B}|$. This is consistent with the ideal MHD

Ohm's law. The situation can be summarized as follows: ideal MHD treats the inductive part of the electric field correctly and the electrostatic part incorrectly. The error, nevertheless, is not important since the only way the electric field is used in ideal MHD is in Faraday's law, which requires the evaluation of $\nabla \times \mathbf{E}$. Therefore, even if the electrostatic part of $\mathbf{E}$ (i.e., the $\nabla\phi$ part) is calculated incorrectly the error is annihilated by the curl operation.

Lastly, consider the energy conservation relations given by Eq. (2.32). With the assumptions already made, most of the terms on the right-hand sides are already negligible. In particular,

$$\begin{aligned}
\frac{\mathbf{\Pi}_e : \nabla(\mathbf{J}/en)}{\partial p_e/\partial t} &\sim \left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{r_{Li}}{a}\right)\left(\frac{V_{Ti}\tau_{ii}}{a}\right) \ll 1 \\
\frac{(\mathbf{J}\cdot\nabla p_e)/en}{\partial p_e/\partial t} &\sim \left(\frac{r_{Li}}{a}\right) \ll 1 \\
\frac{\mathbf{\Pi}_e : \nabla\mathbf{v}}{\partial p_e/\partial t} &\sim \left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{V_{Ti}\tau_{ii}}{a}\right) \ll 1 \\
\frac{\mathbf{\Pi}_i : \nabla\mathbf{v}}{\partial p_i/\partial t} &\sim \left(\frac{V_{Ti}\tau_{ii}}{a}\right) \ll 1
\end{aligned} \tag{2.47}$$

The remaining terms contain the heat flux $\mathbf{q}_\alpha$ and the collisional heating Q_α. The largest contribution to $\mathbf{q}_\alpha$ is due to thermal conductivity. There are separate conductivity coefficients parallel and perpendicular to the field and for electrons and ions. By far, the largest coefficients are those parallel to the field so that $\mathbf{q}_\alpha \approx -\kappa_{\|\alpha}\nabla_\| T_\alpha$. The main contributions to the collisional heating Q_α are joule heating and electron and ion energy equilibration. If the condition to neglect resistive diffusion given by Eq. (2.46) is satisfied, then joule heating is also negligible. Therefore, what remains of the energy equations is as follows:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{p_i}{\rho^\gamma}\right) &= \frac{2}{3\rho^\gamma}\left[\nabla_\| \cdot \left(\kappa_{\|i}\nabla_\| T_i\right) + \frac{n(T_e - T_i)}{\tau_{eq}}\right] \\
\frac{d}{dt}\left(\frac{p_e}{\rho^\gamma}\right) &= \frac{2}{3\rho^\gamma}\left[\nabla_\| \cdot \left(\kappa_{\|e}\nabla_\| T_e\right) - \frac{n(T_e - T_i)}{\tau_{eq}}\right]
\end{aligned} \tag{2.48}$$

where τ_{eq} is the energy equilibration time. As they now stand, these equations describe the separate time evolution of the electrons and ions. In order to obtain the MHD energy equation a further assumption is required which couples the electron and ion energies together. This assumption corresponds to the condition that the energy equilibration time be short compared to the characteristic time of interest, so that $T_e \approx T_i$. This can be expressed as $\omega\tau_{eq} \ll 1$, or, using the relation $\tau_{eq} \sim (m_i/m_e)^{1/2}\,\tau_{ii}$,

$$\omega\tau_{eq} \sim \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{V_{Ti}\tau_{ii}}{a}\right) \ll 1 \tag{2.49}$$

Because the energy equilibration time is long compared to the momentum exchange time, Eq. (2.49) is even more restrictive in terms of collisionality than Eq. (2.37), the basic condition for the collision dominated expansion to be valid.

If Eq. (2.49) is satisfied, then

$$\begin{aligned} T_i &\approx T_e \equiv T \\ p_i &\approx p_e = nT \equiv p/2 \end{aligned} \tag{2.50}$$

Note that the strong collisionality assumption resolves the problem of the unequal number of fluid variables in the two-fluid and ideal MHD models as only a single temperature and pressure rather than two separate ones for each quantity are now required.

Both the electron and ion energy equations yield the identical piece of information – namely that $T_i \approx T_e$ in the limit of small τ_{eq}. The second piece of independent information follows from annihilating the $T_i \approx T_e$ redundancy by adding the equations and setting $T_i = T_e = T$. The result is

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\nabla_\parallel \cdot \left[(\kappa_{\parallel i} + \kappa_{\parallel e})\nabla_\parallel T\right] \tag{2.51}$$

The ideal MHD equation of state follows when the right-hand side of Eq. (2.51) can be neglected. Braginskii's transport theory shows that the parallel electron thermal conductivity $\kappa_{\parallel e} \sim nT_e\tau_{ee}/m_e$ is larger by $(m_i/m_e)^{1/2}$ than that of the ions. Consequently, the right-hand side is negligible when

$$\frac{\nabla_\parallel \cdot (\kappa_{\parallel e}\nabla_\parallel T)}{\partial p/\partial t} \sim \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{V_{Ti}\tau_{ii}}{a}\right) \ll 1 \tag{2.52}$$

which is identical to Eq. (2.49).

This completes the derivation of the ideal MHD equations. In summary, the validity of ideal MHD imposes several conditions on the plasma: (1) high collisionality; (2) characteristic dimensions much larger than an ion gyro radius; and (3) sufficient size that resistive diffusion is negligible, despite the high collisionality.

2.4 Region of validity

2.4.1 Overall criteria

With the somewhat tedious analysis completed, the next important issue is to focus on the most restrictive assumptions required for the derivation and to discuss their regions of validity. Furthermore, there is additional useful information to be extracted by examining the conditions for validity, one equation at a time. This provides insight into which specific phenomena are not accurately described by ideal MHD; perhaps more importantly, it also indicates those phenomena that will still be reliably treated even if a particular validity condition is violated.

These goals are accomplished in several steps. The starting point is the introduction of dimensionless variables

$$\begin{aligned} y &= \left(\frac{r_{Li}}{a}\right) \\ x &= \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{V_{Ti}\tau_{ii}}{a}\right) \end{aligned} \tag{2.53}$$

As just discussed, there are three independent conditions that must be satisfied for ideal MHD to be valid. They are

$$\begin{array}{lll} (1)\ \text{High collisionality} & x \ll 1 \\ (2)\ \text{Small gyro radius} & y \ll 1 \\ (3)\ \text{Small resistivity} & y^2/x \ll 1 \end{array} \tag{2.54}$$

These conditions are illustrated in Fig. 2.2. In the region labeled "ideal MHD" all three conditions are simultaneously satisfied. Although a significant number of approximations have been made in the derivation, there is a substantial region of parameter space where all the assumptions are satisfied and the ideal MHD equations are valid.

The next question to be asked is whether plasmas of fusion interest lie in the region of MHD validity. This can be answered by transforming the (x, y) diagram into a (T, n) diagram and observing whether values of n and T of fusion interest lie in the region of validity.

The first step is to define the parameter range of fusion interest. Past experiments and extrapolations to future fusion reactors indicate that the densities and temperatures of fusion plasmas lie in the range

$$\begin{gathered} 10^{18}\ \mathrm{m}^{-3} < n < 10^{22}\ \mathrm{m}^{-3} \\ 0.5\ \mathrm{keV} < T < 50\ \mathrm{keV} \end{gathered} \tag{2.55}$$

These conditions describe a rectangle in the (n, T) diagram.

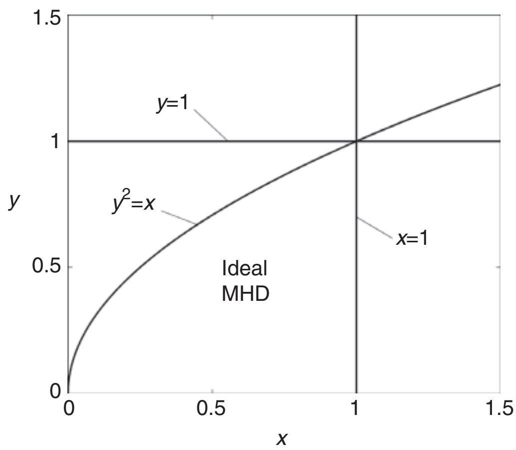

Figure 2.2 Region of validity for the ideal MHD model in terms of the normalized variables $y = r_{Li}/a$ and $x = (m_i/m_e)^{1/2}(V_{Ti}\tau_{ii}/a)$. In the region labeled "Ideal MHD" the validity conditions are satisfied.

The second step is to rewrite the conditions in Eq. (2.54) in terms of n and T. Since the B field explicitly appears, a prescription is needed to specify how B varies with n and T. A reasonable choice is to assume that $\beta = 4\mu_0 nT/B^2$ is held fixed. The parameter β measures the ratio of plasma energy to magnetic energy. It is a dimensionless quantity whose value is important in fusion reactor designs and is often limited by MHD instabilities. Consequently, treating β and the scale length a as parameters leads to the following expressions for the three MHD criteria of validity:

$$\begin{array}{lll} (1)\ \text{High collisionality} & x = 9.2 \times 10^3 (T^2/an) \ll 1 & \\ (2)\ \text{Small gyro radius} & y = 2.3 \times 10^{-2} (\beta/na^2)^{1/2} \ll 1 & (2.56) \\ (3)\ \text{Small resistivity} & y^2/x = 5.6 \times 10^{-8} (\beta/aT^2) \ll 1 & \end{array}$$

In these expressions the units are a (m), T (keV), and n (10^{20} m^{-3}). The characteristic collision times are chosen as $\tau_{ee} \approx \tau_e$ and $\tau_{ii} \approx \tau_i$ where τ_e and τ_i are given by Braginskii (see Appendix B). Also, the Coulomb logarithm has been set to $\ln \Lambda = 19$, and the ion mass corresponds to deuterium. Equation (2.56) is illustrated in Fig. 2.3 for the case $a = 1$ m and $\beta = 0.05$. Observe that the conditions of small gyro radius and small resistivity are well satisfied for plasmas of fusion interest. Note, however, that the high collisionality assumption is never satisfied. *Thus, the region in which ideal MHD is valid completely excludes plasmas of fusion interest!* This disconcerting conclusion contradicts the overwhelming empirical experimental evidence which demonstrates that ideal MHD provides a very accurate description of most macroscopic plasma behavior.

The question may now be asked whether or not this is coincidence or, alternatively, the result of some subtle and perhaps unexpected physics? The resolution

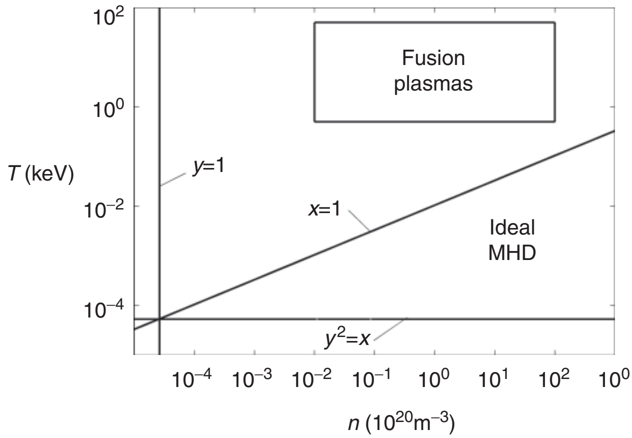

Figure 2.3 Region of validity for the ideal MHD model in (n, T) parameter space for the case $\beta = 0.05$ and $a = 1$ m. In the region labeled "Ideal MHD" the validity conditions are satisfied.

follows from an examination of the validity conditions, one equation at a time. The results show that there are indeed subtle physics issues at play which are only rigorously resolved by examining certain general stability predictions of more realistic but complicated models. Several such models are described in Chapter 9 and the stability predictions are covered in Chapter 10.

2.4.2 Conservation of mass

To begin, consider the conservation of mass relation given in Eq. (2.1). Since none of the MHD validity conditions is required to obtain this equation, its range of applicability extends far beyond that of ideal MHD. It is valid for collisionless or collision dominated systems regardless of the size of the ion gyro radius or the influence of plasma resistivity. Modifications are only required when phenomena such as recombination, ionization, charge exchange, fusion fuel depletion, or source terms due to gas puffing or pellet injection are present.

2.4.3 Momentum equation

The validity of the momentum equation is also much wider than the ideal MHD conditions would imply, although the reasons involve considerably more subtle physics. Recall that the collision dominated assumption is required to neglect $\mathbf{\Pi}_i$ and $\mathbf{\Pi}_e$. In a collisionless plasma the magnetic field in a certain sense plays the role of collisions for the perpendicular motion; that is, perpendicular to the field, particles are confined to the vicinity of a given field line executing nearly (two-dimensional) isotropic motion if their gyro radius is much smaller than the characteristic plasma dimension. In fact, a calculation of $\mathbf{\Pi}_{\perp i}$ in the collisionless regime (see, for instance, Bowers and Haines (1971)) shows that for the MHD ordering $\mathbf{\Pi}_{\perp i}/p_i \sim r_{Li}/a \ll 1$. Therefore, the perpendicular motion is fluid-like, implying that the perpendicular components of the momentum equation provide an excellent description of plasma behavior in either the collision dominated or collisionless regimes.

The general situation parallel to the field is much more complicated and it is here that ideal MHD treats the physics very inaccurately. In a collisionless plasma there is no reason a priori to assume that any simple relationship exists between the perpendicular and parallel pressures, $p_\perp$ and $p_\parallel$; perpendicular to the **B** field, particles execute well-confined gyro orbits, while along the field they execute free-streaming kinetic motion.

Even so, in practice the situation parallel to the field simplifies considerably for the following subtle reason. Consider the application of the MHD model over the narrow increment Δt as discussed in connection with Fig. 2.1. By assumption, the plasma has evolved through a series of macroscopically stable states over a long period of time up until the beginning of Δt. During this period there is more than adequate time for collisions to isotropize the pressure. Consequently, at the beginning of Δt it is a good approximation to set $p_\perp = p_\parallel$ as an *initial condition*. However, as time progresses during the short MHD increment Δt, the two pressures will in general evolve quite differently because of the different physics involved perpendicular and parallel to the field. During this short increment there is not enough time for collisions to re-isotropize the plasma.

Now, it turns out that a not immediately obvious property of most MHD instabilities is that the plasma pressure neither compresses nor expands as the modes grow; instead, the perpendicular and parallel pressures are simply convected with the fluid motion. The reason is that compression and expansion correspond to very stable motions and would suppress most instabilities if excited to any appreciable level. This crucial fact of plasma behavior, demonstrated explicitly in Chapter 8, implies that neither $p_\perp$ nor $p_\parallel$ changes significantly from its initial value during the increment Δt. For fast MHD instabilities in a collisionless plasma, $p_\perp$ and $p_\parallel$ thus remain isotropic if they are so initially.

Under the situation just described, it is accurate to use the ideal MHD momentum equation in both the collision dominated and collisionless regimes. In the collisionless regime that part of the physics treated inaccurately by ideal MHD does not matter: the nature of MHD instabilities is such that the dynamical motions where the inaccuracies would be important are not excited.

Finally, as a caution, it should be noted that this conclusion applies to most but not all fusion configurations of interest. Specifically, for magnetic configurations with closed field lines certain modes are strongly affected by plasma compression and for these cases the ideal MHD and collisionless models can give substantially different results.

2.4.4 Energy equation

The ideal MHD adiabatic energy equation is never valid for plasmas of fusion interest. Like the parallel motion in the momentum equation, it too, for similar reasons, has little effect on the applicability of the model.

Even assuming that evolution over the long period prior to Δt leads to temperature equilibration (i.e., $T_e = T_i = T$), one still requires negligible parallel thermal conductivity for the adiabatic energy relation to be valid. Substituting numerical values from fusion experiments one can show that it would be far more accurate to treat the parallel thermal conductivity as infinite rather than zero. Therefore, a more appropriate energy equation follows from setting $\nabla_{\|} \cdot (\kappa_{\|} \nabla_{\|} T) = 0$ for both electrons and ions, yielding

$$\mathbf{B} \cdot \nabla T = 0 \tag{2.57}$$

If the assumption is again made that MHD instabilities are incompressible for most fusion configurations of interest, it can then be proven that the "incorrect" adiabatic energy relation yields the same results as the "infinite parallel thermal conductivity" constraint given by Eq. (2.57). The proof of this statement is as follows.

By definition, if the plasma is incompressible the density is just convected with the fluid as it moves under the action of an MHD instability. Density convection implies that

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0 \tag{2.58}$$

The conservation of mass equation shows that this relation is equivalent to the well-known result that incompressible motions are characterized by the constraint

$$\nabla \cdot \mathbf{v} = 0 \tag{2.59}$$

The adiabatic energy equation, before incompressibility is assumed, can with the aid of the mass conservation relation be written as

$$\frac{dp}{dt} + \gamma p \nabla \cdot \mathbf{v} = 0 \tag{2.60}$$

Invoking incompressibility gives

$$\frac{dp}{dt} = 0 \tag{2.61}$$

implying that the energy as well as the mass is convected with the fluid.

The same result follows from the collisionless equation of state. Taking the total time derivative of Eq. (2.57) and using Faraday's law yields

$$\frac{d}{dt}\left[\mathbf{B}\cdot\nabla\left(\frac{p}{\rho}\right)\right] = \mathbf{B}\cdot\nabla\left[\frac{d}{dt}\left(\frac{p}{\rho}\right)\right] - (\nabla\cdot\mathbf{v})\left[\mathbf{B}\cdot\nabla\left(\frac{p}{\rho}\right)\right] \tag{2.62}$$

Now assume the motions are incompressible. Then $\nabla\cdot\mathbf{v} = 0$ and from mass conservation $d\rho/dt = 0$, implying that Eq. (2.62) reduces to

$$\frac{dp}{dt} = 0 \tag{2.63}$$

The collision dominated and collisionless equations of state are identical when the plasma motions of interest are incompressible. The only hidden assumption in the analysis is the requirement that the operator $\mathbf{B}\cdot\nabla \neq 0$ over the entire plasma, a requirement that is satisfied by almost all fusion configurations of interest. The one exception involves certain modes in closed line systems in which case, as with the parallel momentum equation, the results from a more sophisticated kinetic model can differ substantially from ideal MHD. This too is discussed in Chapters 9 and 10.

A corollary of the analysis presented above is the following. Any calculation using the ideal MHD model in which the results depend explicitly on the ratio of specific heats γ is unreliable. Equation (2.60) implies that such results must involve compressibility. Therefore, they make explicit use of the collision dominated assumption, which is not valid for fusion plasmas.

2.4.5 Ohm's law

The final equation to consider is Ohm's law. Here, the main terms neglected (i.e., the Hall effect, the electron diamagnetic drift, and electrical resistivity) do not depend on the collision dominated assumption. Thus, Ohm's law is approximately valid in the collisionless regime for phenomena on the MHD time scale. Even so, there is one important subtlety that should be noted.

The subtlety involves the resistivity. Although its magnitude is very small it nonetheless can play an important role. This follows because the ideal MHD Ohm's law implies that $E_{\parallel} = 0$ for any situation. With resistivity included, $E_{\parallel} = \eta J_{\parallel}$. While $E_{\parallel}$ can be shown to be small (i.e., $E_{\parallel}/E_{\perp} \ll 1$) it must really be compared to zero since it occurs in the vector component of Ohm's law that does not contain $\mathbf{E}_{\perp}$ or $\mathbf{v}\times\mathbf{B}$.

The critical feature is that unlike the Hall and diamagnetic effects discussed earlier, the resistivity contribution does not arise from an electrostatic potential. As such, it has the important property that, while small, it allows a new class of motions to occur that are entirely excluded from ideal MHD: the tearing and reconnecting of magnetic field lines. These phenomena occur on a hybrid time scale related to a geometric-like mean of the short ideal MHD and long resistive diffusion time scales. Since the hybrid scale is much longer than the characteristic MHD time, resistivity has little effect on ideal MHD behavior. However, the hybrid time scale sets a well-defined upper limit to the incremental width of time Δt over which ideal MHD is valid. Typical numerical values for the hybrid scale are on the order of 1 ms.

2.4.6 Summary of validity conditions

The ideal MHD model follows from a straightforward, self-consistent closure of the two-fluid moment equations. The conditions for its validity are (1) high collisionality, (2) small ion gyro radius, and (3) small resistivity. Conditions (2) and (3) are well satisfied in fusion plasmas. In contrast, condition (1) is never satisfied. A more subtle examination of ideal MHD, one equation at a time, shows that in most situations the model is still reliable even though the collision dominated assumption is not satisfied. Crucial to this argument is the claim that for problems involving MHD equilibrium and stability, the plasma motions of interest are incompressible. Specifically, the conservation of mass and the perpendicular momentum equation are shown to be valid in the collisionless limit even when $\nabla \cdot \mathbf{v} \neq 0$. However, both the parallel momentum equation and the energy equation are incorrect in this regime. Nevertheless, when $\nabla \cdot \mathbf{v} = 0$ neither of these equations, which though incorrect, plays an important role.

An analysis of Ohm's law demonstrates that ideal MHD accurately calculates the inductive part of the electric field, but incorrectly predicts the electrostatic part. Even so, for the low-frequency Maxwell's equations, this has no effect on the evaluation of the remaining fluid and field variables since only $\nabla \times \mathbf{E}$ enters the analysis. Finally, resistivity, while small, can play an important role in that it allows new motions to occur that are prohibited in ideal MHD. These phenomena occur on the slower MHD-resistive diffusion hybrid time scale and thus set a practical upper limit on the maximum time period Δt over which ideal MHD can be considered reliable.

The overall conclusion is that ideal MHD predictions are actually more accurate than one might expect. The reasons are subtle and associated with stability results that are stated but not (at this point) proved. The proofs will be given in Chapters 9 and 10, after development of the general equilibrium and stability properties of ideal MHD which then serve as a point of reference.

2.5 Overall summary

The ideal MHD model is a single-fluid model that describes the effects of magnetic geometry on the macroscopic equilibrium and stability properties of fusion plasmas. The MHD length and time scales of interest are a, the macroscopic plasma dimension, and a/V_{Ti}, the ion thermal transit time across the plasma.

The model is derived in a straightforward manner by forming the mass, momentum, and energy moments of the kinetic-Maxwell equations. The moment equations reduce to ideal MHD with the introduction of three critical assumptions: high collisionality, small ion gyro radius, and small resistivity. An analysis of the validity conditions shows that the collision dominated assumption is never satisfied in plasmas of fusion interest. The remaining two conditions are satisfied by a wide margin.

A careful examination of the collision dominated assumption shows that those particular parts of ideal MHD treated inaccurately (i.e., the parallel momentum and energy equations) play little if any practical role in most MHD equilibrium and stability phenomena; that is, these equations primarily describe the compression and expansion of a plasma whereas most MHD instabilities involve incompressible motions. The question on plasma compressibility is examined in Chapters 9 and 10 by means of several alternate, collisionless models.

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Further reading

Most of the material covered in this chapter has been known for many years. Listed below are some early and more recent references covering the material of interest.

The Boltzmann kinetic equation

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Problems

2.1 Starting from the kinetic-Maxwell equations, and following the steps outlined in the text, derive the general set of two-fluid equations given by Eq. (2.22).

2.2 This problem, and the ones below, demonstrate a simple procedure for estimating the transport coefficients in a collision-dominated plasma. The required calculations are based on a simplified form of the Krook collision operator discussed in Appendix A. Also, to minimize the algebra, only the dominant terms in the kinetic equation are maintained which focus on the transport coefficient of interest. The first problem involves the parallel plasma resistivity. The relevant kinetic equation is given by

$$-\frac{e}{m_e}E_z\frac{\partial f}{\partial v_z} = -\nu_{ei}(f - f_M)$$
$$f_M = n\left(\frac{m_e}{2\pi T}\right)^{3/2}\exp\left(-\frac{m_e v^2}{2T}\right)$$

Assume that n, T, E_z are constants. Here and below the collision frequencies are also treated as constants. Consider the collision dominated limit corresponding to large ν_{ei} and solve for the distribution function by expanding as follows:

$$f = f_M + f_1 + f_2 + \cdots$$

where $f_n/f_M \sim O(1/\nu_{ei}^n)$.

(a) Using this expansion calculate f_1.
(b) Substitute the solution for f_1 into the definition of $J_z = -e\int v_z f\, d\mathbf{v}$. The result should be of the form $J_z = (1/\eta_\|)E_z$. Find the parallel resistivity $\eta_\|$.

2.3 Consider next the parallel electron thermal conductivity which arises from both electron–electron and electron–ion collisions. In this case the relevant limit of the kinetic equation can be written as

$$v_z \frac{\partial f}{\partial z} = -(\nu_{ee} + \nu_{ei})(f - f_M)$$

$$f_M = n\left(\frac{m_e}{2\pi T}\right)^{3/2} \exp\left(-\frac{m_e v^2}{2T}\right)$$

Here it has been assumed that n is a constant and $T_e = T_i \equiv T(z)$.

(a) Calculate f_1 using the expansion described in Problem 2.2.
(b) Substitute the solution for f_1 into the definition for the heat flux. Show that the result is of the form $\mathbf{q}_e = -\kappa_{\|}(\partial T/\partial z)\mathbf{e}_z$. Find the parallel electron thermal conductivity $\kappa_{\|}$.

2.4 The last transport coefficient of interest is the parallel viscosity which is largest for the ions. The appropriate limit of the kinetic equation is given by

$$v_z \frac{\partial f}{\partial z} = -\nu_{ii}(f - f_M)$$

$$f_M = n\left(\frac{m_i}{2\pi T}\right)^{3/2} \exp\left[-\frac{m_i(\mathbf{v} - u_z\mathbf{e}_z)^2}{2T}\right]$$

Assume that n and T are constants and note that the distribution function corresponds to a shifted Maxwellian whose fluid velocity is given by $\mathbf{u}_i = u_z(z)\,\mathbf{e}_z$.

(a) Solve for f_1 using the expansion described in Problem 2.2.
(b) Substitute the solution into the definition of Π_{zz} and show that the result is of the form $\Pi_{zz} = -\mu_{\|}(\partial u_z/\partial z)$. Find the parallel viscosity $\mu_{\|}$.

2.5 Consider a system in which each species of particles is acted upon by an additional force arising from an external potential; that is, Newton's law for each particle is given by

$$m_\alpha \frac{d\mathbf{u}_\alpha}{dt} = Z_\alpha e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla\phi_\alpha$$

where $\phi_\alpha = \phi_\alpha(\mathbf{r})$ is a known fixed potential function.

(a) Using the same assumptions made in the derivation of ideal MHD, derive an equivalent set of closed, single-fluid equations which include the effects of ϕ_e and ϕ_i. In order to complete this derivation some assumptions must be made regarding the size ϕ_e and ϕ_i. Order ϕ_e and ϕ_i so that their contributions to the momentum equation are comparable to the $\mathbf{J} \times \mathbf{B}$ force. What ordering gives this result?
(b) Assume the external potential represents a fixed gravitational force in the $\mathbf{e}_x$ direction of magnitude g. What are the corresponding functional dependences

of ϕ_e and ϕ_i? Write down the modified MHD equations including gravity for $m_e \ll m_i$.

2.6 Does the local Maxwellian distribution function

$$\begin{aligned} f &= n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right) \\ n &= n(\mathbf{r}, t) \\ T &= T(\mathbf{r}, t) \end{aligned}$$

satisfy the Vlasov equation (i.e. the kinetic equation neglecting collisions)?

2.7 Consider a system in steady state $\partial/\partial t = 0$, having y symmetry $\partial/\partial y = 0$. The ions are described by a distribution function of the form

$$\begin{aligned} f &= n_0\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{\varepsilon + \alpha p^2/m}{T_0}\right) \\ n_0,\ T_0,\ \alpha &= \text{constants} \\ \varepsilon &= \frac{1}{2}mv^2 + e\phi(x, z) \\ p &= mv_y + eA_y(x, z) \end{aligned}$$

Does this distribution function satisfy the Vlasov equation? Here, ϕ and A_y are the scalar and vector potential, respectively. For this distribution function calculate n, $\mathbf{u}$, p, $\mathbf{\Pi}$, and $\mathbf{q}$ as functions of ϕ and A_y.

2.8 This problem shows that a similar set of equilibrium equations can exist for an ideal MHD plasma and a Vlasov plasma despite the opposing limits on collisionality. To illustrate this point derive the following exact result. Consider a quasineutral plasma described by a hybrid model. The ions satisfy the Vlasov equation. The electrons satisfy the massless fluid equation $0 = -e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nabla p_e/n$.

(a) For static equilibria ($\partial/\partial t = 0$) show that any distribution function $f_i = f_i(\varepsilon)$ with $\varepsilon = m_i v^2/2 + e\phi(\mathbf{r})$ satisfies the Vlasov equation for an arbitrary three-dimensional geometry.

(b) Calculate $\mathbf{J} = e\int \mathbf{v} f_i d\mathbf{v} - en\mathbf{u}_e$ and show that

$$\begin{aligned} \mathbf{J} \times \mathbf{B} &= \nabla p \\ p &= p_e + \int \frac{mv^2}{3} f_i \, d\mathbf{v} \end{aligned}$$

Note that this equation is identical to the ideal MHD static equilibrium relation.

(c) Compare the equilibrium electric field in ideal MHD and the hybrid model.