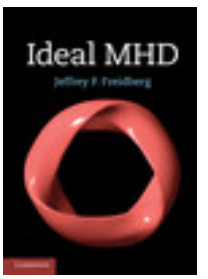

Ideal MHD

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10

MHD stability comparison theorems

10.1 Introduction

Three models have been introduced to investigate the MHD equilibrium and stability properties of a general multidimensional magnetic fusion configuration: ideal MHD, kinetic MHD, and double adiabatic MHD. Ideal MHD is by far the most widely used model although there is concern since the collision dominated assumption used in the derivation is not satisfied in fusion-grade plasmas. The collisionless kinetic MHD model provides the most reliable description of the physics but is difficult to solve in realistic geometries because of the complex kinetic behavior parallel to the magnetic field. Double adiabatic MHD is a collisionless fluid model that is much easier to solve than kinetic MHD but the closure assumptions cannot be justified by any rigorous mathematical or physical arguments.

Based on this assessment one sees that the situation is not very satisfactory from a theoretical point of view. In practice, ideal MHD, because of its mathematical simplicity, is the model that is most widely used to design, predict, and interpret fusion experiments. Many years of experience have shown, perhaps surprisingly, that the model is far more accurate and reliable than one might have anticipated.

Chapter 10 attempts to provide a partial theoretical explanation for this apparent good fortune. The goal is accomplished by examining two basic aspects of MHD behavior. The first issue is MHD equilibrium. The analysis shows that for the special choice of isotropic pressure (i.e., $p_\perp = p_\parallel = p$), both kinetic MHD and double adiabatic MHD lead to the same equilibrium equations as ideal MHD. Therefore, when comparing the stability predictions of each model one is starting with identical equilibria. This is an important property that helps makes the comparisons meaningful.

The second issue is MHD stability. In the analysis, equivalent forms of δW are derived for kinetic MHD and double adiabatic MHD which are then

compared to the ideal MHD δW. By focusing on marginal stability boundaries (as opposed to growth rates) it is shown that there is a well-defined hierarchy of stability predictions with ideal MHD being the most pessimistic, double adiabatic MHD being the most optimistic, and kinetic MHD always lying in between. Furthermore, detailed comparisons in Chapters 11 and 12 show that the gaps in stability predictions between ideal and double adiabatic MHD are often not very large. The closeness of the stability predictions, which bracket the intermediate predictions of kinetic MHD, provides a good explanation of why ideal MHD works as well as it does.

The analysis in Chapter 10 is presented as follows. (1) The ideal MHD stability predictions are briefly reviewed in order to establish a point of reference. (2) An Energy Principle is derived for double adiabatic MHD. This is straightforward and closely follows the corresponding derivation for ideal MHD. (3) An Energy Relation is derived for kinetic MHD. This requires a complicated calculation and distinctions must be made between cylindrical and toroidal configurations as well as between ergodic and closed line magnetic geometries. The kinetic MHD stability operator is not self-adjoint (because of kinetic effects) so that only an Energy Relation and not an Energy Principle can be derived. Still, the required stability information can be extracted from the Energy Relation. (4) The results from all three models are collected from which the desired hierarchy of stability predictions can then be easily obtained.

10.2 Ideal MHD equilibrium and stability

Ideal MHD serves as the reference case. The relevant results have been derived in Chapter 8 and are summarized as follows. In ideal MHD any equilibrium must satisfy

$$\begin{aligned} \mathbf{J} \times \mathbf{B} &= \nabla p \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{10.1}$$

Stability can be described in terms of the variational integral (Bernstein *et al.*, 1958),

$$\omega^2 = \frac{\delta W_{MHD}}{K_{MHD}} \tag{10.2}$$

The subscript "MHD" denotes expressions corresponding to ideal MHD. Also, for convenience in deriving the comparison theorems, it is useful to separate δW_{MHD} and K_{MHD} into contributions that depend solely on $\boldsymbol{\xi}_\perp$ and those that contain $\xi_\parallel$. Therefore, K_{MHD} is given by

$$
\begin{aligned}
K_{MHD}(\boldsymbol{\xi}^*, \boldsymbol{\xi}) &= K_\perp + K_\parallel \\
K_\perp(\boldsymbol{\xi}^*_\perp, \boldsymbol{\xi}_\perp) &= \frac{1}{2}\int \rho |\boldsymbol{\xi}_\perp|^2 \, d\mathbf{r} \\
K_\parallel(\xi^*_\parallel, \xi_\parallel) &= \frac{1}{2}\int \rho |\xi_\parallel|^2 \, d\mathbf{r}
\end{aligned}
\tag{10.3}
$$

Similarly δW_{MHD} can be written in terms of the "standard" form as follows:

$$
\begin{aligned}
\delta W_{MHD}(\boldsymbol{\xi}^*, \boldsymbol{\xi}) &= \delta W_\perp + \delta W_C \\
\delta W_\perp(\boldsymbol{\xi}^*_\perp, \boldsymbol{\xi}_\perp) &= \frac{1}{2\mu_0}\int \left[|\mathbf{Q}|^2 - \mu_0 \boldsymbol{\xi}^*_\perp \cdot \mathbf{J} \times \mathbf{Q} + \mu_0 (\boldsymbol{\xi}_\perp \cdot \nabla p) \nabla \cdot \boldsymbol{\xi}^*_\perp \right] d\mathbf{r} \\
&\quad + \delta W_S + \delta W_V \\
\delta W_S(\boldsymbol{\xi}^*_\perp, \boldsymbol{\xi}_\perp) + \delta W_V(\boldsymbol{\xi}^*_\perp, \boldsymbol{\xi}_\perp) &= \frac{1}{2\mu_0}\int (\mathbf{n} \cdot \boldsymbol{\xi}^*_\perp)(\mathbf{B} \cdot \mathbf{Q} + \mu_0 p_1) \, dS \\
\delta W_C(\boldsymbol{\xi}^*, \boldsymbol{\xi}) &= \frac{1}{2}\int \frac{5}{3} p |\nabla \cdot \boldsymbol{\xi}|^2 \, d\mathbf{r}
\end{aligned}
\tag{10.4}
$$

where δW_C is the contribution due to plasma compressibility and γ, the ratio of specific heats, has been set to $\gamma = 5/3$. As has been shown in Chapter 8 the standard form of δW_{MHD} can, after a series of algebraic manipulations, be rewritten in a self-adjoint form.

The Energy Principle states that a necessary and sufficient condition for MHD stability is that

$$
\delta W_{MHD}(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \geq 0 \tag{10.5}
$$

for all allowable displacements. A similar set of relations is now derived for double adiabatic MHD and kinetic MHD which then sets the stage to deduce quantitative comparison theorems.

10.3 Double adiabatic MHD equilibrium and stability

As previously stated, in order to make a fair stability comparison with ideal MHD one must, as a special case, choose the double adiabatic MHD equilibrium pressures to be isotropic: $p_\perp = p_\parallel = p$. Under this assumption the double adiabatic pressure tensor reduces to $\mathbf{P} = p_\perp \mathbf{I} + (p_\parallel - p_\perp)\mathbf{bb} = p\mathbf{I}$ and $\nabla \cdot \mathbf{P} = \nabla p$. Clearly, equilibria are then identical in both models. Even so, it is important to keep in mind that the perturbed pressures in double adiabatic MHD are anisotropic.

Consider next the formulation of the stability problem (Kruskal and Oberman, 1958; Rosenbluth and Rostoker, 1959). The analysis closely follows the procedure used for ideal MHD in Chapter 8 with only slightly more algebra. The details are given below.

10.3.1 Relation between the perturbed quantities and ξ

The analysis is again carried out in terms of the perturbed plasma displacement $\boldsymbol{\xi}$ defined as $\mathbf{v}_1 = -i\omega\boldsymbol{\xi}$. First note that for the modified double adiabatic model, the parallel component of the momentum equation, $\mathbf{b} \cdot (d\mathbf{v}/dt) = 0$, reduces to

$$\xi_{\parallel} = 0 \tag{10.6}$$

Second, from Eq. (9.42) it follows that the perturbed density, magnetic field, and current are identical to ideal MHD with only a slight modification in n_1 (i.e., $\nabla \cdot \boldsymbol{\xi} \to \nabla \cdot \boldsymbol{\xi}_{\perp}$):

$$\begin{aligned} n_1 &= -\boldsymbol{\xi}_{\perp} \cdot \nabla n - n\nabla \cdot \boldsymbol{\xi}_{\perp} \\ \mathbf{B}_1 &= \nabla \times (\boldsymbol{\xi}_{\perp} \times \mathbf{B}) \\ \mu_0 \mathbf{J}_1 &= \nabla \times [\nabla \times (\boldsymbol{\xi}_{\perp} \times \mathbf{B})] \end{aligned} \tag{10.7}$$

Also needed for the double adiabatic analysis is an expression for B_1 which is easily obtained from Eq. (10.7),

$$\begin{aligned} B_1 = \mathbf{b} \cdot \mathbf{B}_1 &= B\,\mathbf{b} \cdot (\mathbf{b} \cdot \nabla)\boldsymbol{\xi}_{\perp} - \mathbf{b} \cdot (\boldsymbol{\xi}_{\perp} \cdot \nabla)\mathbf{B} - B\nabla \cdot \boldsymbol{\xi}_{\perp} \\ &= -B\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa} - \boldsymbol{\xi}_{\perp} \cdot \nabla B - B\nabla \cdot \boldsymbol{\xi}_{\perp} \end{aligned} \tag{10.8}$$

The remaining quantities required are the perturbed pressures which are derived from Eq. (9.49). For the parallel pressure the non-linear adiabatic energy equation can be rewritten as

$$\frac{d}{dt}\left(\frac{p_{\parallel}B^2}{n^3}\right) = 0 \quad \to \quad \frac{1}{p_{\parallel}}\frac{dp_{\parallel}}{dt} + \frac{2}{B}\frac{dB}{dt} - \frac{3}{n}\frac{dn}{dt} = 0 \tag{10.9}$$

Each of these quantities is now linearized: $Q = Q_0 + Q_1$. Using the facts that $dQ_0/dt = 0$ and $(dQ/dt)_1 = -i\omega(Q_1 + \boldsymbol{\xi}_{\perp} \cdot \nabla Q_0)$ one finds that

$$\frac{1}{p}\left(\frac{dp_{\parallel}}{dt}\right)_1 + \frac{2}{B}\left(\frac{dB}{dt}\right)_1 - \frac{3}{n}\left(\frac{dn}{dt}\right)_1 = 0 \tag{10.10}$$

where

$$\begin{aligned} \left(\frac{dn}{dt}\right)_1 &= -i\omega(n_1 + \boldsymbol{\xi}_{\perp} \cdot \nabla n) = i\omega n \nabla \cdot \boldsymbol{\xi}_{\perp} \\ \left(\frac{dB}{dt}\right)_1 &= -i\omega(B_1 + \boldsymbol{\xi}_{\perp} \cdot \nabla B) = i\omega B(\nabla \cdot \boldsymbol{\xi}_{\perp} + \boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa}) \\ \left(\frac{dp_{\parallel}}{dt}\right)_1 &= -i\omega\left(p_{\parallel 1} + \boldsymbol{\xi}_{\perp} \cdot \nabla p\right) \end{aligned} \tag{10.11}$$

Straightforward substitution then yields

$$p_{\|1} = -\boldsymbol{\xi}_\perp \cdot \nabla p - p\nabla \cdot \boldsymbol{\xi}_\perp + 2p\, \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} \tag{10.12}$$

Similarly, the non-linear relation for the perpendicular pressure can be rewritten as

$$\frac{d}{dt}\left(\frac{p_\perp}{nB}\right) = 0 \;\rightarrow\; \frac{1}{p_\perp}\frac{dp_\perp}{dt} - \frac{1}{B}\frac{dB}{dt} - \frac{1}{n}\frac{dn}{dt} = 0 \tag{10.13}$$

Linearization leads to an expression for $p_{\perp 1}$ given by

$$p_{\perp 1} = -\boldsymbol{\xi}_\perp \cdot \nabla p - 2p\nabla \cdot \boldsymbol{\xi}_\perp - p\, \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} \tag{10.14}$$

An examination of Eqs. (10.12) and (10.14) indeed shows that the perturbed pressures are anisotropic.

10.3.2 The double adiabatic MHD Energy Principle

All of the perturbed quantities have now been expressed in terms of $\boldsymbol{\xi}_\perp$. These expressions are to be substituted into the linearized form of the double adiabatic perpendicular momentum equation obtained from Eq. (9.42),

$$-\omega^2 \rho \boldsymbol{\xi}_\perp = \mathbf{J}_1 \times \mathbf{B} + \mathbf{J} \times \mathbf{Q} - \nabla_\perp p_{\perp 1} - (p_{\|1} - p_{\perp 1})\boldsymbol{\kappa} \tag{10.15}$$

The next step is to multiply this expression by $\boldsymbol{\xi}_\perp^*$ and integrate over the plasma volume leading to

$$\omega^2 = \frac{\delta W_{CGL}}{K_{CGL}} \tag{10.16}$$

where

$$\begin{aligned} \delta W_{CGL} &= -\frac{1}{2}\int \boldsymbol{\xi}_\perp^* \cdot \left[\mathbf{J}_1 \times \mathbf{B} + \mathbf{J} \times \mathbf{Q} - \nabla_\perp p_{\perp 1} - (p_{\|1} - p_{\perp 1})\boldsymbol{\kappa}\right] d\mathbf{r} \\ K_{CGL} &= K_\perp = \frac{1}{2}\int \rho\, |\boldsymbol{\xi}_\perp|^2 d\mathbf{r} \end{aligned} \tag{10.17}$$

As with ideal MHD, several algebraic manipulations and integrations by parts are required to recast δW_{CGL} in a form suitable for comparisons. The steps are summarized below.

- **The $\boldsymbol{\xi}_\perp^* \cdot \mathbf{J}_1 \times \mathbf{B}$ term**

$$-\int \boldsymbol{\xi}_\perp^* \cdot \mathbf{J}_1 \times \mathbf{B}\, d\mathbf{r} = \frac{1}{\mu_0}\int |\mathbf{Q}|^2\, d\mathbf{r} + \frac{1}{\mu_0}\int (\mathbf{n} \cdot \boldsymbol{\xi}_\perp^*)(\mathbf{B} \cdot \mathbf{Q})\, dS \tag{10.18}$$

- **The $\boldsymbol{\xi}_\perp^* \cdot \nabla_\perp p_{\perp 1}$ term**

$$\int \boldsymbol{\xi}_\perp^* \cdot \nabla_\perp p_{\perp 1}\, d\mathbf{r} = \int \left[2p|\nabla \cdot \boldsymbol{\xi}_\perp|^2 - \boldsymbol{\xi}_\perp^* \cdot \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p + p\, \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\right] d\mathbf{r} - \int (\mathbf{n} \cdot \boldsymbol{\xi}_\perp^*)(2p \nabla \cdot \boldsymbol{\xi}_\perp)\, dS \tag{10.19}$$

Several terms are now integrated by parts.

- **The $\boldsymbol{\xi}_\perp^* \cdot \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p)$ term**

$$-\int \boldsymbol{\xi}_\perp^* \cdot \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p)\, d\mathbf{r} = \int (\nabla \cdot \boldsymbol{\xi}_\perp^*)(\boldsymbol{\xi}_\perp \cdot \nabla p)\, d\mathbf{r} - \int (\mathbf{n} \cdot \boldsymbol{\xi}_\perp^*)(\boldsymbol{\xi}_\perp \cdot \nabla p)\, dS \tag{10.20}$$

- **The $\boldsymbol{\xi}_\perp^* \cdot \nabla(p\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})$ term**

$$-\int \boldsymbol{\xi}_\perp^* \cdot \nabla(p\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\, d\mathbf{r} = \int (\nabla \cdot \boldsymbol{\xi}_\perp^*)(p\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\, d\mathbf{r} - \int (\mathbf{n} \cdot \boldsymbol{\xi}_\perp^*)(p\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\, dS \tag{10.21}$$

The contributions from these steps are combined leading to an expression for δW_{CGL} that can be written as

$$\delta W_{CGL} = \frac{1}{2\mu_0} \int \Big[|\mathbf{Q}|^2 + 2\mu_0 p|\nabla \cdot \boldsymbol{\xi}_\perp|^2 - \mu_0 \boldsymbol{\xi}_\perp^* \cdot \mathbf{J} \times \mathbf{Q} + \mu_0 (\nabla \cdot \boldsymbol{\xi}_\perp^*)(\boldsymbol{\xi}_\perp \cdot \nabla p) + \mu_0 p (\nabla \cdot \boldsymbol{\xi}_\perp^*)(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}) + \mu_0 (p_{\parallel 1} - p_{\perp 1})(\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa})\Big]\, d\mathbf{r} + BT \tag{10.22}$$

Here, the boundary term BT is given by

$$\begin{aligned} BT &= \frac{1}{2\mu_0} \int (\mathbf{n} \cdot \boldsymbol{\xi}_\perp^*)(\mathbf{B} \cdot \mathbf{Q} - \mu_0 \boldsymbol{\xi}_\perp \cdot \nabla p - 2\mu_0 p \nabla \cdot \boldsymbol{\xi}_\perp - \mu_0 p \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\, dS \\ &= \frac{1}{2\mu_0} \int (\mathbf{n} \cdot \boldsymbol{\xi}_\perp^*)(\mathbf{B} \cdot \mathbf{Q} + \mu_0 p_{\perp 1})\, dS \end{aligned} \tag{10.23}$$

Observe that BT is in the identical form as for ideal MHD as given by Eq. (8.80) or equivalently Eq. (10.4). Thus, one is again allowed to use the natural boundary condition analysis described in Section 8.8. The result is that

$$BT = \delta W_S + \delta W_V \tag{10.24}$$

where δW_S and δW_V have the same form as the ideal MHD surface and vacuum energies summarized in Eqs. (8.101) and (8.102).

The last two terms in the fluid contribution to δW_{CGL} are simplified by substituting for $p_{\parallel 1}$, $p_{\perp 1}$ from Eqs. (10.12) and (10.14). A short calculation yields

$$p(\nabla \cdot \boldsymbol{\xi}_\perp^*)(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}) + (p_{\parallel 1} - p_{\perp 1})(\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa}) = \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_\perp + 3\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}|^2 - \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_\perp|^2 \tag{10.25}$$

Equation (10.25) is substituted into Eq. (10.22) leading to an expression for δW_{CGL} that can be written as

$$\delta W_{CGL} = \frac{1}{2\mu_0}\int \left[|\mathbf{Q}|^2 + \frac{5}{3}\mu_0 p|\nabla \cdot \boldsymbol{\xi}_\perp|^2 - \mu_0 \boldsymbol{\xi}_\perp^* \cdot \mathbf{J} \times \mathbf{Q} + \mu_0(\nabla \cdot \boldsymbol{\xi}_\perp^*)(\boldsymbol{\xi}_\perp \cdot \nabla p) \right.$$
$$\left. + \frac{\mu_0 p}{3}|\nabla \cdot \boldsymbol{\xi}_\perp + 3\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}|^2 \right] d\mathbf{r} + \delta W_S + \delta W_V \tag{10.26}$$

The final step is to rewrite this equation as

$$\delta W_{CGL} = \delta W_\perp + \delta Q_{CGL} \tag{10.27}$$

where

$$\delta Q_{CGL} = \frac{1}{2}\int \left[\frac{5}{3}p|\nabla \cdot \boldsymbol{\xi}_\perp|^2 + \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_\perp + 3\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}|^2 \right] d\mathbf{r} \tag{10.28}$$

Equation (10.27) represents the desired potential energy relation for double adiabatic MHD.

10.3.3 Summary of CGL stability

To summarize, the variational formulation of the double adiabatic model, originally given by Eq. (10.16), can now be rewritten as

$$\omega^2 = \frac{\delta W_{CGL}}{K_{CGL}} = \frac{\delta W_\perp + \delta Q_{CGL}}{K_\perp} \tag{10.29}$$

The quantity δQ_{CGL} is the modification to the potential energy due to the plasma compressibility and pressure anisotropy. Since δW_{CGL} is self-adjoint by construction, then an Energy Principle also applies. Specifically $\delta W_{CGL}(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \geq 0$ for all allowable displacements is a necessary and sufficient condition for stability in the modified double adiabatic model.

10.4 Kinetic MHD equilibrium and stability

The goal in this subsection is to derive an Energy Relation describing the stability of the kinetic MHD model (Kruskal and Oberman, 1958; Rosenbluth

and Rostoker, 1959). The analysis for general geometry is complicated and lengthy requiring many steps and several clever mathematical tricks. The strategy to reach the final goal has three parts. The first part discusses general equilibrium and is simple and straightforward. The second part focuses on systems with cylindrical symmetry. The symmetry leads to a closed form analytic solution for the perturbed distribution function from which much insight can be obtained. In carrying out this analysis it is necessary to distinguish between ergodic and closed line systems. Overall, the cylindrical part of the analysis is not overly complicated.

The third part of the analysis describes the stability of general toroidal systems where again one must distinguish between ergodic and closed line configurations. The part of the formulation that leads to a general Energy Relation is straightforward although lengthy. It is described in the main text. A considerably more complicated analysis is required to obtain upper and lower bounds on certain terms appearing in the Energy Relation that are needed to make comparisons with the other models. The corresponding details are presented in Appendix H.

It is worth noting at the outset that in order to obtain analytic results for the general toroidal case one must restrict attention to the marginal stability limit $\omega \to 0$. The reason is as follows. Since kinetic MHD is not a self-adjoint model because of the presence of resonant particles it is by no means obvious that $\omega_r = 0$ when $\omega_i \to 0$ at marginal stability; that is, resonant particles lead to Landau damping which in turn leads to a complex dispersion relation. The analysis shows that this damping vanishes when $\omega_r = 0$. Therefore, only by focusing on $\omega \to 0$ does the analysis become sufficiently simplified that it is possible to obtain the necessary upper and lower bounds. The end result is a kinetic MHD Energy Relation from which one can deduce stability comparison theorems.

10.4.1 Equilibrium

To compare kinetic MHD stability with ideal MHD and double adiabatic MHD one again needs to focus on isotropic equilibria. In the kinetic MHD model this is easily accomplished for general geometry by choosing the equilibrium distribution functions for each species to be of the form

$$\overline{f}_0(\mathbf{r}, \varepsilon, \mu) = \overline{f}_0(\varepsilon, \psi) \tag{10.30}$$

where $\psi(\mathbf{r})$ is the equilibrium flux function satisfying $\mathbf{B} \cdot \nabla\psi = 0$. Here and below the species subscript α is suppressed for simplicity except where explicitly needed. This distribution function automatically satisfies the non-linear kinetic MHD equation for $\overline{f}$ and leads to

$$p_{\perp 0}(\mathbf{r}) = p_{\parallel 0}(\mathbf{r}) = p_0(\psi) \tag{10.31}$$

which is identical to ideal MHD. Also, the equilibrium parallel electric field vanishes since

$$E_{\parallel 0} = -\frac{\mathbf{b}_0 \cdot (\nabla \cdot \mathbf{P}_{0e})}{en_0} = -\frac{\mathbf{b}_0 \cdot \nabla p_{0e}(\psi)}{en_0} = -\frac{dp_{0e}}{d\psi}\frac{\mathbf{b}_0 \cdot \nabla \psi}{en_0} = 0 \tag{10.32}$$

The equilibria are isotropic with a one-to-one correspondence to ideal MHD.

10.4.2 Stability of a closed line cylindrical system

The first stability problem considered corresponds to a closed line cylindrically symmetric system subject to perturbations which maintain the closed line symmetry. In practical terms the configuration of interest is a pure Z-pinch subject to an $m = 0$ perturbation where $\boldsymbol{\xi}(\mathbf{r}) = \boldsymbol{\xi}(r)\exp(im\theta + ikz)$. This is the simplest problem to study. The reason is that, as a result of the special symmetries involved, the perturbed kinetic MHD distribution function has only fluid-like contributions – all kinetic effects cancel.

The procedure used to obtain the stability results is to (1) calculate the perturbed distribution function $\overline{f}_1$, (2) from $\overline{f}_1$ calculate the perturbed parallel and perpendicular pressures $p_{\parallel 1}$ and $p_{\perp 1}$, and (3) use $p_{\parallel 1}$ and $p_{\perp 1}$ to obtain the kinetic MHD Energy Relation. The analysis proceeds as follows.

The linearized stability equations

For the Z-pinch it is convenient to solve for the perturbed distribution functions in terms of the general $w_\perp$, $w_\parallel$ formulation of the kinetic MHD equations given by Eq. (9.31) and repeated here for convenience (for a species with mass m and charge q):

$$\begin{aligned}
&\frac{\partial \overline{f}}{\partial t} + (\mathbf{v} + w_\parallel \mathbf{b}) \cdot \nabla \overline{f} + C_\perp \frac{\partial \overline{f}}{\partial w_\perp} + C_\parallel \frac{\partial \overline{f}}{\partial w_\parallel} = 0 \\
&C_\perp = \frac{w_\perp}{2B}\left(\frac{dB}{dt} + w_\parallel \mathbf{b} \cdot \nabla B\right) \\
&C_\parallel = \frac{qE_\parallel}{m} - \mathbf{b} \cdot \frac{d\mathbf{v}}{dt} - \frac{w_\perp^2}{2B}\mathbf{b} \cdot \nabla B - w_\parallel \mathbf{b} \cdot (\mathbf{b} \cdot \nabla)\mathbf{v}
\end{aligned} \tag{10.33}$$

Here, $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$.

Equation (10.33) can be easily linearized for generalized geometry by again using the relations $\mathbf{v}_1 = -i\omega\boldsymbol{\xi}$ and $\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B})$. The result is

$$-i\omega\overline{f}_1 + w_\parallel \mathbf{b}_0\cdot\nabla\overline{f}_1 + (-i\omega\boldsymbol{\xi} + w_\parallel\mathbf{b}_1)\cdot\nabla\overline{f}_0 + C_{\perp 1}\frac{\partial\overline{f}_0}{\partial w_\perp} + C_{\parallel 1}\frac{\partial\overline{f}_0}{\partial w_\parallel} = 0$$

$$C_{\perp 1} = \frac{w_\perp}{2B}\left[\left(\frac{dB}{dt}\right)_1 + w_\parallel(\mathbf{b}\cdot\nabla B)_1\right] \tag{10.34}$$

$$C_{\parallel 1} = \frac{qE_{\parallel 1}}{m} - \omega^2\xi_\parallel - \frac{w_\perp^2}{2B}(\mathbf{b}\cdot\nabla B)_1 + i\omega w_\parallel \mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}$$

These complicated expressions can be simplified as follows. First, by making use of the fact that $\overline{f}_0 = \overline{f}_0(w_\perp^2 + w_\parallel^2, \psi)$ one finds that

$$\begin{aligned}(\mathbf{b}\cdot\nabla B)_1\left(w_\parallel w_\perp\frac{\partial\overline{f}_0}{\partial w_\perp} - w_\perp^2\frac{\partial\overline{f}_0}{\partial w_\parallel}\right) &= 0\\ \mathbf{b}_1\cdot\nabla\overline{f}_0 = \frac{1}{B_0}\nabla\cdot[(\boldsymbol{\xi}_\perp\times\mathbf{B}_0)\times\nabla\overline{f}_0] &= \mathbf{b}_0\cdot\nabla(\boldsymbol{\xi}_\perp\cdot\nabla\overline{f}_0)\end{aligned} \tag{10.35}$$

Second, $E_{\parallel 1}$ can be simplified by noting that the total electric field in the plasma can be written in two different but equivalent forms:

$$\begin{aligned}\mathbf{E}_1 &= i\omega\boldsymbol{\xi}_\perp\times\mathbf{B}_0 + E_{\parallel 1}\mathbf{b}_0\\ &= -\nabla\phi_1 + i\omega\mathbf{A}_{\perp 1} + i\omega A_{\parallel 1}\mathbf{b}_0\end{aligned} \tag{10.36}$$

Flux coordinates ψ, χ, l are now introduced (as discussed in Chapter 7), where $\mathbf{B}_0 = \nabla\psi\times\nabla\chi$ and $\nabla l = \mathbf{b}_0$. The perpendicular and parallel components of Eq. (10.36) are equated leading to

$$\begin{aligned}i\omega\boldsymbol{\xi}_\perp\times\mathbf{B}_0 &= -\frac{\partial\phi_1}{\partial\psi}\nabla\psi - \frac{\partial\phi_1}{\partial\chi}\nabla\chi + i\omega\mathbf{A}_{\perp 1}\\ E_{\parallel 1} &= -\mathbf{b}_0\cdot\nabla\phi_1 + i\omega A_{\parallel 1}\end{aligned} \tag{10.37}$$

Equation (10.37) is valid for all ω including $\omega\to 0$. Taking this limit implies that ϕ_1 must be of the form $\phi_1 = \overline{\phi}_1(l) + i\omega\tilde{\phi}_1(\psi,\chi,l)$. Furthermore, $\overline{\phi}_1(l)$ must vanish since it has an incompatible spatial dependence with respect to the mode under consideration. For example, for the $m = 0$ mode in a Z-pinch, $\boldsymbol{\xi} = \boldsymbol{\xi}(r)\exp(ikz)$ while $\overline{\phi}_1 = \overline{\phi}_1(\theta)$. Lastly, the $\tilde{\phi}_1$ contribution can easily be incorporated into $A_{\parallel 1}$ by means of a gauge transformation. The end result is that $E_{\parallel 1}$ can be written as

$$E_{\parallel 1} = i\omega A_{\parallel 1} \tag{10.38}$$

which is the desired simplification.

The last step is to write $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel\mathbf{b}_0$ and to then collect all the $\xi_\parallel$ terms. After substituting all of the simplifications into Eq. (10.34) one obtains a correspondingly reduced form of the kinetic MHD equations,

$$(-i\omega + w_\parallel \mathbf{b}\cdot\nabla)\left(\bar{f}_1 + \boldsymbol{\xi}_\perp\cdot\nabla\bar{f} + i\omega\boldsymbol{\xi}_\parallel\frac{\partial\bar{f}}{\partial w_\parallel}\right) + i\omega\left(\hat{C}_{\perp 1}\frac{\partial\bar{f}}{\partial w_\perp} + \hat{C}_{\parallel 1}\frac{\partial\bar{f}}{\partial w_\parallel}\right) = 0$$

$$\hat{C}_{\perp 1} = \frac{w_\perp}{2}(\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})$$

$$\hat{C}_{\parallel 1} = \frac{qA_{\parallel 1}}{m} - w_\parallel\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}$$

(10.39)

Here, the "0" subscript has been suppressed from all equilibrium quantities. At this point Eq. (10.39) is still valid for general geometry.

Solution for the m = *0 mode in a Z-pinch*

In a cylindrical Z-pinch the perturbations can be Fourier analyzed in θ and z : $\boldsymbol{\xi} = \boldsymbol{\xi}(r)\exp(im\theta + ikz)$. For the $m = 0$ mode the closed line symmetry of the equilibrium is preserved since $B_{1r} = B_{1z} = 0$. Both the equilibrium and perturbed magnetic fields point only in the θ direction. The $m = 0$ mode is thus the mode of interest in the present subsection.

Focusing on $m = 0$ in a cylindrical geometry greatly simplifies the analysis. Specifically, for this mode the operator $\mathbf{b}\cdot\nabla S = imS/r = 0$ where S is any scalar quantity. The kinetic MHD equation for each species reduces to a simple algebraic equation for $\bar{f}_{1\alpha}$ whose solution is given by

$$\bar{f}_{1\alpha} = -\boldsymbol{\xi}_\perp\cdot\nabla\bar{f}_\alpha - i\omega\boldsymbol{\xi}_\parallel\frac{\partial\bar{f}_\alpha}{\partial w_\parallel} + \frac{w_\perp}{2}(\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})\frac{\partial\bar{f}_\alpha}{\partial w_\perp} + \left(\frac{q_\alpha A_{\parallel 1}}{m_\alpha} - w_\parallel\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right)\frac{\partial\bar{f}_\alpha}{\partial w_\parallel} \tag{10.40}$$

Knowing $\bar{f}_{1\alpha}$ it is then straightforward to calculate the perturbed pressures. A short calculation yields

$$p_{\parallel 1} = \sum_\alpha \tilde{p}_{\parallel\alpha} = \sum_\alpha \int m_\alpha w_\parallel^2\bar{f}_{1\alpha}\,d\mathbf{w} = -\boldsymbol{\xi}_\perp\cdot\nabla p + 2p\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} - p\nabla\cdot\boldsymbol{\xi}_\perp$$

$$p_{\perp 1} = \sum_\alpha \tilde{p}_{\perp\alpha} = \sum_\alpha \int \frac{m_\alpha w_\perp^2}{2}\bar{f}_{1\alpha}\,d\mathbf{w} = -\boldsymbol{\xi}_\perp\cdot\nabla p - p\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} - 2p\nabla\cdot\boldsymbol{\xi}_\perp \tag{10.41}$$

where $\tilde{p}_{\perp\alpha}, \tilde{p}_{\parallel\alpha}$ are the perturbed pressures for each species and $p = p_i + p_e$. Note that the contributions from $\xi_\parallel$ and $A_{\parallel 1}$ vanish because they appear in terms with odd symmetry in $w_\parallel$. The most interesting observation is that the pressures appearing in Eq. (10.41) are identical to those calculated in the modified double adiabatic model as given by Eqs. (10.12) and (10.14).

The Z-pinch Energy Relation

Since the pressures in both models are identical, so then are the stability formulations. The conclusion is that for the $m = 0$ mode in a Z-pinch the Energy Relation for kinetic MHD is actually an Energy Principle given by

$$\omega^2 = \frac{\delta W_{KIN}}{K_{KIN}} = \frac{\delta W_\perp + \delta Q_{KIN}}{K_\perp} \tag{10.42}$$

where

$$\delta Q_{KIN} = \delta Q_{CGL} = \frac{1}{2}\int \left[\frac{5}{3}p|\nabla \cdot \boldsymbol{\xi}_\perp|^2 + \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_\perp + 3\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}|^2\right] d\mathbf{r} \tag{10.43}$$

Since δW_{KIN} is self-adjoint it follows that the stability transition occurs through $\omega^2 = 0$. Detailed comparisons with other models and modes are made in Section 10.5 once the other Energy Relations have been derived.

10.4.3 Stability of an ergodic cylindrical system

The second configuration of interest is a cylindrical system in which all the magnetic lines are ergodic, covering the entire flux surface on which they lie. Isolated rational surfaces are allowed but do not change the final results. The configuration to be studied thus corresponds to a general screw pinch. The analysis also applies to a closed line Z-pinch subject to perturbations that break the closed line symmetry. In other words, modes with $m \geq 1$.

The derivation of the screw pinch Energy Relation is similar to that of the $m = 0$ mode in a Z-pinch in that a purely analytic closed form solution is obtained for the perturbed distribution function. However, the resulting parallel and perpendicular pressures are far more complicated because of the effects of resonant particles, a distinctly kinetic effect. It is the presence of resonant particles that ultimately leads to a close relation between kinetic MHD and ideal MHD.

The analysis presented here is greatly simplified by focusing, at the appropriate point, on the behavior near marginal stability which occurs when $\omega \to 0$. In kinetic MHD, like ideal MHD, the real part of the frequency at marginal stability satisfies $\omega_r = 0$. It is only when $\omega_r = 0$ that the Landau damping due to resonant particles vanishes, which thereby provides an automatic solution to the imaginary part of the dispersion relation.

As for the $m = 0$ mode in a Z-pinch the analysis for the screw pinch consists of (1) calculating the perturbed distribution function, (2) calculating the perturbed pressures, and (3) deriving the corresponding kinetic MHD Energy Relation.

Solution for the perturbed distribution function in a general screw pinch

The perturbed distribution function satisfies the general linearized kinetic MHD equation given by Eq. (10.39). Also, the modes under consideration are described by a displacement vector of the form $\boldsymbol{\xi} = \boldsymbol{\xi}(r)\exp(im\theta + ikz)$. For such modes the operator $\mathbf{b}\cdot\nabla$ acting on any scalar S is given by

$$\mathbf{b}\cdot\nabla S = ik_{\parallel}S \tag{10.44}$$

where

$$k_{\parallel}(r) = \frac{1}{B}\left(\frac{m}{r}B_{\theta} + kB_{z}\right) \tag{10.45}$$

In general, $k_{\parallel}(r) \neq 0$ except perhaps on isolated surfaces.

Substituting Eq. (10.44) into Eq. (10.39) leads to an analytic expression for the perturbed distribution function,

$$\begin{aligned}
\overline{f}_1 &= -\boldsymbol{\xi}_{\perp}\cdot\nabla\overline{f} - i\omega\xi_{\parallel}\frac{\partial\overline{f}}{\partial w_{\parallel}} + \frac{\omega}{\omega - k_{\parallel}w_{\parallel}}\left(\hat{C}_{\perp 1}\frac{\partial\overline{f}}{\partial w_{\perp}} + \hat{C}_{\parallel 1}\frac{\partial\overline{f}}{\partial w_{\parallel}}\right)\\
\hat{C}_{\perp 1} &= \frac{w_{\perp}}{2}(\nabla\cdot\boldsymbol{\xi}_{\perp} + \boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\kappa})\\
\hat{C}_{\parallel 1} &= \frac{qA_{\parallel 1}}{m} - w_{\parallel}\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\kappa}
\end{aligned} \tag{10.46}$$

where again the species subscript α has been suppressed. Note the presence of a resonant denominator $\omega - k_{\parallel}w_{\parallel}$ in the last term in $\overline{f}_1$. Specifically, those particles with a parallel velocity $w_{\parallel} = \omega_r/k_{\parallel}$ produce Landau damping thereby generating a complex dispersion relation.

It is at this point that focusing on marginal stability, $\omega \to 0$, greatly simplifies the analysis. The reason is as follows. As stated, the resonant particle effects are contained in the terms multiplied by $\omega/(\omega - k_{\parallel}w_{\parallel})$ in Eq. (10.46). For the $m = 0$ mode in a Z-pinch, $k_{\parallel} = 0$ and $\omega/(\omega - k_{\parallel}w_{\parallel}) \to 1$. The resonance is actually not a resonance, but instead leads to fluid-like contributions. In contrast, for the screw pinch $k_{\parallel}(r) \neq 0$. In this case $\omega/(\omega - k_{\parallel}w_{\parallel}) \to 0$ as $\omega \to 0$. As stated, kinetic effects, Landau damping specifically, vanish when $\omega \to 0$. This automatically guarantees that the imaginary part of the complex dispersion relation is satisfied at marginal stability. The conclusion is that for each species, $\overline{f}_{1\alpha}$ at marginal stability only consists of a simple fluid-like term

$$\overline{f}_{1\alpha} = -\boldsymbol{\xi}_{\perp}\cdot\nabla\overline{f}_{\alpha} \qquad \omega \to 0 \tag{10.47}$$

The perturbed pressures

At marginal stability it is straightforward to calculate the perturbed pressures by taking appropriate moments of $\overline{f}_{1\alpha}$. After summing over species the results are given by

$$\begin{aligned} p_{\|1} &= \sum_\alpha \int m_\alpha w_\|^2 \overline{f}_{1\alpha}\, d\mathbf{w} = -\boldsymbol{\xi}_\perp \cdot \nabla p \\ p_{\perp 1} &= \sum_\alpha \int \frac{m_\alpha w_\perp^2}{2} \overline{f}_{1\alpha}\, d\mathbf{w} = -\boldsymbol{\xi}_\perp \cdot \nabla p \end{aligned} \tag{10.48}$$

where again $p = p_i + p_e$. The perturbed pressures are quite simple. They are isotropic and describe simple convection with the fluid.

For comparison the exact pressures, valid for arbitrary ω in the limit $m_e \to 0$ are, after a lengthy calculation, given by

$$\begin{aligned} p_{\|1} &= -\boldsymbol{\xi}_\perp \cdot \nabla p - p_i(\nabla \cdot \boldsymbol{\xi}_\perp)(\zeta_i^2 Z_i)' + 2p_i(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\zeta_i^3(\zeta_i Z_i)' - (enV_{Ti}A_{\|1})(\zeta_i Z'_i) \\ p_{\perp 1} &= -\boldsymbol{\xi}_\perp \cdot \nabla p + 2p_i(\nabla \cdot \boldsymbol{\xi}_\perp)(\zeta_i Z_i) + p_i(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})(\zeta_i^2 Z_i)' + (enV_{Ti}A_{\|1})(\zeta_i^2 Z_i) \\ enV_{Ti}A_{\|1} &= \frac{2p_e}{2T_i - T_e Z'_i}\left[(\nabla \cdot \boldsymbol{\xi}_\perp)Z_i + (\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})(\zeta_i Z_i)'\right] \end{aligned} \tag{10.49}$$

Here, $\zeta_i = \omega/k_\| V_{Ti}$ and prime denotes $d/d\zeta_i$. Equation (10.49) assumes that the equilibrium distribution functions are Maxwellian,

$$\overline{f}_\alpha = \frac{n}{\pi^{3/2} V_{T\alpha}^3} \exp\left(-\frac{w_\perp^2 + w_\|^2}{V_{T\alpha}^2}\right) \qquad V_{T\alpha}^2 = 2T_\alpha/m_\alpha \tag{10.50}$$

which allows one to express the results in terms of the well-known plasma dispersion function $Z = Z(\zeta)$ (Stix, 1992)

$$Z(\zeta) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{dz}{z - \zeta} \tag{10.51}$$

Also Z' is related to Z by $Z' = -2(1 + \zeta Z)$.

The kinetic terms are not derived here since their details are not explicitly needed for the stability comparison theorems. Still there are several points to be made. First and foremost one sees how complicated kinetic MHD is compared to ideal MHD and double adiabatic MHD, even for the simple cylindrical geometry where an analytic solution is known for $\overline{f}_{1\alpha}$. These complications are a primary motivation for attempting to bracket kinetic MHD stability results with simpler fluid predictions. Second, in spite of their complexity one can easily show that the

real and imaginary parts of all kinetic terms in the pressures, those that contain Z functions, vanish in the limit $\omega \propto \zeta \to 0$, and the simpler results given by Eq. (10.48) are all that remain.

The cylindrical screw pinch Energy Relation

The kinetic MHD potential energy can essentially be determined by inspection because of the simple relations for $p_{\parallel 1}$ and $p_{\perp 1}$. Recall that the quantity needed to close the system of equations is

$$\nabla p_{\perp 1} + (p_{\parallel 1} - p_{\perp 1})\boldsymbol{\kappa} + \mathbf{bB} \cdot \nabla\left(\frac{p_{\parallel 1} - p_{\perp 1}}{B}\right) \tag{10.52}$$

In the limit of marginal stability $p_{\parallel 1} = p_{\perp 1}$ and $p_{\perp 1} = -\boldsymbol{\xi}_\perp \cdot \nabla p$. Therefore, Eq. (10.52) reduces to

$$\nabla p_{\perp 1} + (p_{\parallel 1} - p_{\perp 1})\boldsymbol{\kappa} + \mathbf{bB} \cdot \nabla\left(\frac{p_{\parallel 1} - p_{\perp 1}}{B}\right) = -\nabla(\boldsymbol{\xi}_\perp \cdot \nabla p) \tag{10.53}$$

This expression should be compared to the corresponding term in ideal MHD

$$\nabla p_1 = -\nabla(\boldsymbol{\xi}_\perp \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{\xi}) \tag{10.54}$$

Now, for ergodic systems the $\xi_\parallel$ that minimizes δW for ideal MHD is the one that sets $\nabla \cdot \boldsymbol{\xi} = 0$. Equation (10.54) thus simplifies to

$$\nabla p_1 = -\nabla(\boldsymbol{\xi}_\perp \cdot \nabla p) \tag{10.55}$$

which is identical to the kinetic MHD result given by Eq. (10.53).

The conclusion is that marginal stability for a screw pinch is identical in the ideal MHD and kinetic MHD models:

$$\begin{aligned} \delta W_{MHD} &= \delta W_\perp \\ \delta W_{KIN} &= \delta W_\perp \end{aligned} \tag{10.56}$$

implying that

$$\delta W_{KIN} = \delta W_{MHD} \tag{10.57}$$

This is the desired relation.

10.4.4 Stability of a general toroidal configuration

The insight obtained from the cylindrical analysis sets the stage to investigate stability in general toroidal configurations. The analysis is rather involved, requiring multiple steps and several clever mathematical manipulations. The end result is an Energy Relation valid for arbitrary geometry and arbitrary frequency. From this

relation it is then possible to develop stability comparisons between kinetic MHD, ideal MHD, and double adiabatic MHD.

The analysis separates into two parts. The first part is straightforward but lengthy and is presented below. It leads to an Energy Relation that consists of an ideal MHD contribution plus a positive definite contribution resulting from kinetic effects. The kinetic effects result from resonant particles, trapped particles, and particles which do not uniformly sample the entire flux surface because of toroidicity.

The second part of the analysis involves the calculation of upper and lower bounds on the kinetic contribution which are required in order to be able to make stability comparisons with the other models. Here, the details are quite complicated and are presented in Appendix H.

The analysis is carried out by reverting back to the kinetic MHD equations written in terms of ε, μ rather than $w_\parallel$, $w_\perp$. Although the general Energy Relation is valid for arbitrary ω, it is necessary to take the marginal stability limit $\omega \to 0$ in order to obtain the upper and lower bounds on the kinetic contribution.

The derivation of the Energy Relation presented in this subsection requires two steps. First, an expression is derived for the perturbed distribution function. The expression is formal in nature since it involves an integral along the unperturbed guiding center orbits which in general cannot be analytically evaluated for a 3-D geometry. Fortunately the analytic solutions are not actually required. Second, a quadratic relation is formed from the perpendicular components of the momentum equation. Setting the real and imaginary parts of this relation to zero yields the kinetic MHD Energy Relation.

The analysis proceeds as follows.

Derivation of the perturbed distribution function

The starting point for the derivation is the non-linear kinetic MHD equation given by Eq. (9.23), expressed in ε, μ coordinates. The equation is repeated here for convenience for a species with mass m and charge q:

$$\frac{\partial \overline{f}}{\partial t} + \left(\mathbf{v} + w_\parallel \mathbf{b}\right) \cdot \nabla \overline{f} + \dot{\overline{\varepsilon}} \frac{\partial \overline{f}}{\partial \varepsilon} = 0$$

$$\dot{\overline{\varepsilon}} = q w_\parallel E_\parallel - m w_\parallel \mathbf{b} \cdot \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} - \frac{m w_\perp^2}{2} \nabla \cdot \mathbf{v} - m\left(w_\parallel^2 - \frac{w_\perp^2}{2}\right)\mathbf{b} \cdot (\mathbf{b} \cdot \nabla)\mathbf{v} \tag{10.58}$$

Readers should keep in mind that in Eq. (10.58), $w_\perp$, $w_\parallel$ must be expressed in terms of ε, μ since these are the independent variables. Specifically, one must set $w_\perp^2 = (2B/m)\mu$ and $w_\parallel^2 = (2/m)(\varepsilon - \mu B)$.

The analysis begins by linearizing Eq. (10.58) using the relations $\mathbf{v}_1 = -i\omega\boldsymbol{\xi}$ and $\mathbf{b}_1 \cdot \nabla \overline{f}_0 = \mathbf{b}_0 \cdot \nabla(\boldsymbol{\xi}_\perp \cdot \nabla \overline{f}_0)$. Also, all the $\xi_\parallel$ terms are collected together. A short calculation yields a complicated differential equation for the perturbed distribution function $\overline{f}_1$,

$$\begin{aligned}
&\frac{D}{Dt}\left(\overline{f}_1 + \boldsymbol{\xi}_\perp \cdot \nabla \overline{f}\right) + \overline{\dot{\varepsilon}}_1 \frac{\partial \overline{f}}{\partial \varepsilon} = 0 \\
&\overline{\dot{\varepsilon}}_1 = i\omega\left[q w_\parallel A_{\parallel 1} + \frac{m w_\perp^2}{2}\nabla \cdot \boldsymbol{\xi}_\perp + m\left(\frac{w_\perp^2}{2} - w_\parallel^2\right)(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\right] \\
&\qquad + i\omega m\left[-i\omega w_\parallel \boldsymbol{\xi}_\parallel + \frac{w_\perp^2}{2}\mathbf{B} \cdot \nabla\left(\frac{\boldsymbol{\xi}_\parallel}{B}\right) - \left(\frac{w_\perp^2}{2} - w_\parallel^2\right)(\mathbf{b} \cdot \nabla \xi_\parallel)\right]
\end{aligned} \tag{10.59}$$

Here, all zero subscripts have been suppressed from equilibrium quantities and

$$\frac{DQ_1}{Dt} = \left(-i\omega + w_\parallel \mathbf{b} \cdot \nabla\right) Q_1 \tag{10.60}$$

is the derivative moving with the unperturbed parallel velocity of the particle. This operator should be distinguished from

$$\left(\frac{dQ}{dt}\right)_1 = \left(\frac{\partial Q}{\partial t} + \mathbf{v} \cdot \nabla Q\right)_1 = -i\omega(Q_1 + \boldsymbol{\xi}_\perp \cdot \nabla Q_0) \tag{10.61}$$

which is the linearized derivative moving with the fluid velocity. The $\xi_\parallel$ terms can be simplified as follows:

$$\begin{aligned}
Q_\parallel &= i\omega m\left[-i\omega w_\parallel \xi_\parallel + \frac{w_\perp^2}{2}\mathbf{B} \cdot \nabla\left(\frac{\xi_\parallel}{B}\right) - \left(\frac{w_\perp^2}{2} - w_\parallel^2\right)(\mathbf{b} \cdot \nabla \xi_\parallel)\right] \\
&= i\omega\left[m w_\parallel \frac{D\xi_\parallel}{Dt} - \mu \xi_\parallel \mathbf{b} \cdot \nabla B\right] \\
&= i\omega\left[\frac{D}{Dt}\left(m w_\parallel \xi_\parallel\right) - m\xi_\parallel \frac{D w_\parallel}{Dt} - \mu \xi_\parallel \mathbf{b} \cdot \nabla B\right] \\
&= i\omega \frac{D}{Dt}\left(m w_\parallel \xi_\parallel\right)
\end{aligned} \tag{10.62}$$

Substituting Eq. (10.62) into Eq. (10.58) leads to a simplified equation for the perturbed distribution function given by

$$\begin{aligned}
&\frac{D}{Dt}\left(\overline{f}_1 + \boldsymbol{\xi}_\perp \cdot \nabla \overline{f} + i\omega m w_\parallel \boldsymbol{\xi}_\parallel \frac{\partial \overline{f}}{\partial \varepsilon}\right) + \overline{\dot{\varepsilon}}_{\perp 1} \frac{\partial \overline{f}}{\partial \varepsilon} = 0 \\
&\overline{\dot{\varepsilon}}_{\perp 1} = i\omega\left[q w_\parallel A_{\parallel 1} + \frac{m w_\perp^2}{2}\nabla \cdot \boldsymbol{\xi}_\perp + m\left(\frac{w_\perp^2}{2} - w_\parallel^2\right)(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\right]
\end{aligned} \tag{10.63}$$

The formal solution for $\overline{f}_1$ is obtained by integrating Eq. (10.63) along the characteristics (i.e., the unperturbed orbits)

$$\overline{f}_1 = -\boldsymbol{\xi}_\perp \cdot \nabla \overline{f} - i\omega m w_\parallel \xi_\parallel \frac{\partial \overline{f}}{\partial \varepsilon} - i\omega \frac{\partial \overline{f}}{\partial \varepsilon} s \tag{10.64}$$

where the first terms are the fluid-like contributions and

$$s = \int_{-\infty}^{t} \left[q w_\parallel A_{\parallel 1} + (m w_\perp^2/2) \nabla \cdot \boldsymbol{\xi}_\perp + m\left(w_\perp^2/2 - w_\parallel^2 \right) (\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}) \right] dt' \tag{10.65}$$

is the kinetic contribution.

Two points are worth discussing in connection with the evaluation of $\overline{f}_1$. First, the mathematics is simplified by assuming that the spatial variation of the solution is expressed in terms of flux coordinates ψ, χ, l as described in Chapter 7. Recall that in flux coordinates $\mathbf{B} = \nabla\psi \times \nabla\chi$ and $\mathbf{b} \cdot \nabla l = 1$. Then, the integrand in Eq. (10.65) is of the form

$$I' = e^{-i\omega t'} I[\varepsilon, \mu, \psi, \chi, l(t')] \tag{10.66}$$

Here, $l(t') \equiv l'$ represents the parallel motion of a particle along its unperturbed orbit and is found, in principle, by solving

$$\begin{aligned} \frac{dl'}{dt'} &= w_\parallel' = \pm (2/m)^{1/2} [\varepsilon - \mu B(\psi, \chi, l')]^{1/2} \\ l'(t' = t) &= l \end{aligned} \tag{10.67}$$

Note that the "initial" condition corresponds to setting l' equal to its present value l when $t' = t$. The advantage of flux coordinates is that ψ, χ, ε, μ are all constants along the unperturbed orbit which in turn implies that they are constants with respect to the t' integration in Eq. (10.65). The equation for the unperturbed orbits given by Eq. (10.67) is a first order non-linear ordinary differential equation that must be solved for each particle for the given $B(\psi, \chi, l')$. Fortunately, only Eq. (10.67), but not its solution, is required for the analysis.

The second point of interest concerns the lower limit of the time integration in s which has been set to $t' = -\infty$. Observe that there are no contributions to $\overline{f}_1$ or $\boldsymbol{\xi}_\perp$ at $t' = -\infty$ in Eq. (10.64). This requires that $\omega_i = \mathrm{Im}(\omega) > 0$ and corresponds to taking the Laplace transform. Obviously $e^{-i\omega t'} \to 0$ as $t' = -\infty$ when $\omega_i > 0$. An equivalent form for $s = s(\varepsilon, \mu, \psi, \chi, l, t)$ in terms of the usual Laplace transform representation is obtained by letting $t' = t - \tau$ in Eqs. (10.65) and (10.67) where τ is the new integration variable. One finds

$$\begin{aligned} s &= e^{-i\omega t} \int_0^\infty e^{i\omega\tau} I[l'(\tau)]\, d\tau \\ &= e^{-i\omega t} \int_0^\infty e^{i\omega\tau} \left[q w_\parallel A_{\parallel 1} + (m w_\perp^2/2) \nabla \cdot \boldsymbol{\xi}_\perp + m\left(w_\perp^2/2 - w_\parallel^2 \right) (\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}) \right] d\tau \end{aligned} \tag{10.68}$$

and

$$\frac{dl'}{d\tau} = -w'_{\|} = \mp(2/m)^{1/2}[\varepsilon - \mu B(\psi,\chi,l')]^{1/2}$$
$$l'(0) = l \tag{10.69}$$

It is important to keep in mind that $\omega_i > 0$ is assumed in the derivation of the Energy Relation as this will play a major role in interpreting the kinetic MHD stability results.

Derivation of the general Energy Relation

Having obtained a formal expression for the perturbed distribution function one can now proceed to derive the general Energy Relation for kinetic MHD. The starting point is again the perpendicular components of the linearized momentum equation given by

$$-\omega^2\rho\boldsymbol{\xi}_\perp = \mathbf{J}_1 \times \mathbf{B} + \mathbf{J} \times \mathbf{Q} - \nabla_\perp p_{\perp 1} - (p_{\|1} - p_{\perp 1})\boldsymbol{\kappa} \tag{10.70}$$

Each of the perturbed pressures consists of two contributions, one from the fluid-like contributions in $\overline{f}_1$ and the other from the kinetic contributions. The fluid-like contributions can be easily evaluated by a straightforward integration. For each species this leads to

$$(p_{\perp 1})_\alpha \equiv \tilde{p}_{\perp\alpha} = -\boldsymbol{\xi}_\perp \cdot \nabla p_\alpha + \hat{p}_{\perp\alpha}$$
$$(p_{\|1})_\alpha \equiv \tilde{p}_{\|\alpha} = -\boldsymbol{\xi}_\perp \cdot \nabla p_\alpha + \hat{p}_{\|\alpha} \tag{10.71}$$

where the $\boldsymbol{\xi}_\perp \cdot \nabla p_\alpha$ terms are the fluid contributions and

$$\hat{p}_{\perp\alpha} = -i\omega \int \frac{m_\alpha w_\perp^2}{2} \frac{\partial \overline{f}_\alpha}{\partial \varepsilon} s_\alpha \, d\mathbf{w}$$
$$\hat{p}_{\|\alpha} = -i\omega \int m_\alpha w_\|^2 \frac{\partial \overline{f}_\alpha}{\partial \varepsilon} s_\alpha \, d\mathbf{w} \tag{10.72}$$

are the kinetic contributions. Observe that the contribution from $\xi_\|$ again vanishes because of odd symmetry in $w_\|$.

Next, form the dot product of Eq. (10.70) with $\boldsymbol{\xi}^*_\perp$, integrate over the plasma volume, and sum over species. Some standard integrations by parts yield

$$\omega^2 K_\perp = \delta W_\perp + \delta W_{CK}$$
$$\delta W_\perp = \delta W_S + \delta W_V + \frac{1}{2\mu_0}\int \left[|\mathbf{Q}|^2 - \mu_0 \boldsymbol{\xi}^*_\perp \cdot \mathbf{J} \times \mathbf{Q} + \mu_0(\boldsymbol{\xi}_\perp \cdot \nabla p)\nabla \cdot \boldsymbol{\xi}^*_\perp\right] d\mathbf{r}$$
$$\delta W_{CK} = -\frac{1}{2}\sum_\alpha \int \left[\hat{p}_{\perp\alpha}\nabla \cdot \boldsymbol{\xi}^*_\perp + (\hat{p}_{\perp\alpha} - \hat{p}_{\|\alpha})(\boldsymbol{\xi}^*_\perp \cdot \boldsymbol{\kappa})\right] d\mathbf{r} \tag{10.73}$$

Here, $\delta W_\perp(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp)$ is identical to the term in the ideal MHD $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})$ that depends only on $\boldsymbol{\xi}_\perp$. See Eq. (10.4). The quantity $\delta W_{CK}(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp)$ represents the "compressibility" contribution to the potential energy due to kinetic effects. It ultimately must be compared to the corresponding $\delta W_C(\boldsymbol{\xi}^*, \boldsymbol{\xi})$ term in the ideal MHD $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})$.

The analysis continues by simplifying the expression for $\delta W_{CK}(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp)$. Several steps are required. To begin substitute the expressions for $\hat{p}_{\perp\alpha}, \hat{p}_{\|\alpha}$,

$$\delta W_{CK} = \frac{i\omega}{2}\sum_\alpha \int \frac{\partial \overline{f}_\alpha}{\partial \varepsilon} s_\alpha \left[(m_\alpha w_\perp^2/2)(\nabla \cdot \boldsymbol{\xi}_\perp^*) + m_\alpha (w_\perp^2/2 - w_\|^2)(\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa}) \right] d\mathbf{w}\, d\mathbf{r} \tag{10.74}$$

Now, note that from the definition of s_α one can write

$$\left(\frac{Ds_\alpha}{Dt}\right)^* = q_\alpha w_\| A_{\|1}^* + (m_\alpha w_\perp^2/2)(\nabla \cdot \boldsymbol{\xi}_\perp^*) + m_\alpha (w_\perp^2/2 - w_\|^2)(\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa}) \tag{10.75}$$

which allows Eq. (10.74) to be rewritten as

$$\delta W_{CK} = \frac{i\omega}{2}\sum_\alpha \int \frac{\partial \overline{f}_\alpha}{\partial \varepsilon} s_\alpha \left[\left(\frac{Ds_\alpha}{Dt}\right)^* - q_\alpha w_\| A_{\|1}^* \right] d\mathbf{w}\, d\mathbf{r} \tag{10.76}$$

The $A_{\|1}^*$ term integrates to zero. This can be seen by evaluating the $w_\|$ moment of $\overline{f}_1$ which must be zero because $w_\|$ is the random velocity. For each species one finds

$$0 = q_\alpha \int w_\| \overline{f}_{1\alpha}\, d\mathbf{w} = i\omega q_\alpha n_\alpha \xi_\| - i\omega q_\alpha \int \frac{\partial \overline{f}_\alpha}{\partial \varepsilon} w_\| s_\alpha\, d\mathbf{w} \tag{10.77}$$

Now sum over species and make use of the charge neutrality condition

$$\begin{aligned} 0 &= i\omega \xi_\| \sum_\alpha q_\alpha n_\alpha \\ &= i\omega \sum_\alpha q_\alpha \int \frac{\partial \overline{f}_\alpha}{\partial \varepsilon} w_\| s_\alpha d\mathbf{w} \end{aligned} \tag{10.78}$$

The velocity integral in the second line of Eq. (10.78) is identical to the one multiplying $A_{\|1}^*$ in Eq. (10.76). Therefore, as stated, the $A_{\|1}^*$ contribution vanishes and Eq. (10.76) reduces to

$$\delta W_{CK} = \frac{i\omega}{2}\sum_\alpha \int \frac{\partial \overline{f}_\alpha}{\partial \varepsilon} s_\alpha \left(\frac{Ds_\alpha}{Dt}\right)^* d\mathbf{w}\, d\mathbf{r} \tag{10.79}$$

The next step is to note that

$$\left(\frac{Ds_\alpha}{Dt}\right)^* = \left(-i\omega s_\alpha + w_\| \mathbf{b} \cdot \nabla s_\alpha\right)^* = i\omega^* s_\alpha^* + w_\| \mathbf{b} \cdot \nabla s_\alpha^* \tag{10.80}$$

Substituting into Eq. (10.79) leads to

$$\begin{aligned}\delta W_{CK} &= \delta Q_{KIN} + \frac{i\omega}{2}\sum_{\alpha}\int \frac{\partial \bar{f}_{\alpha}}{\partial \varepsilon} s_{\alpha}\left(w_{\|}\mathbf{b}\cdot\nabla s_{\alpha}^{*}\right) d\,w d\mathbf{r} \\ \delta Q_{KIN} &= -\frac{|\omega|^{2}}{2}\sum_{\alpha}\int \frac{\partial \bar{f}_{\alpha}}{\partial \varepsilon}|s_{\alpha}|^{2} d\,\mathbf{w}\,d\mathbf{r}\end{aligned} \tag{10.81}$$

The integral in the top relation in Eq. (10.81) can be simplified by writing $s = s_r + is_i$ for each species. The terms containing s can then be rewritten as

$$s\left(\mathbf{b}\cdot\nabla s^{*}\right) = \frac{1}{2}\mathbf{b}\cdot\nabla|s|^{2} + i(s_i\mathbf{b}\cdot\nabla s_r - s_r\mathbf{b}\cdot\nabla s_i) \tag{10.82}$$

The first term on the right-hand side integrates to zero. This can be seen by switching from ε, μ to $w_{\|}$, $w_{\perp}$ coordinates. A short calculation shows that

$$\begin{aligned} w_{\|}\frac{\partial \bar{f}}{\partial \varepsilon}\mathbf{b}\cdot\nabla|s|^{2}\Big|_{\varepsilon,\mu} &= w_{\|}\frac{\partial \bar{f}}{\partial \varepsilon}\mathbf{b}\cdot\nabla|s|^{2}\Big|_{w_{\perp},w_{\|}} + \frac{w_{\|}w_{\perp}^{2}}{B}(\mathbf{b}\cdot\nabla B)\left(\frac{\partial}{\partial w_{\perp}^{2}} - \frac{\partial}{\partial w_{\|}^{2}}\right)\left(|s|^{2}\frac{\partial \bar{f}}{\partial \varepsilon}\right) \\ &= w_{\|}\frac{\partial \bar{f}}{\partial \varepsilon}\mathbf{b}\cdot\nabla|s|^{2}\Big|_{w_{\perp},w_{\|}} - w_{\|}\frac{(\mathbf{b}\cdot\nabla B)}{B}\left(|s|^{2}\frac{\partial \bar{f}}{\partial \varepsilon}\right) \\ &= w_{\|}\mathbf{B}\cdot\nabla\left(|s|^{2}\frac{1}{B}\frac{\partial \bar{f}}{\partial \varepsilon}\right)\Bigg|_{w_{\perp},w_{\|}} \end{aligned} \tag{10.83}$$

where the terms with a velocity derivative in the top line have been integrated by parts. Therefore, by Gauss' theorem (over physical space) it follows that for each species

$$\begin{aligned} I &= \frac{i\omega}{4}\int \frac{\partial \bar{f}}{\partial \varepsilon} w_{\|}\mathbf{b}\cdot\nabla|s|^{2}\big|_{\varepsilon,\mu} d\,\mathbf{w}\,d\mathbf{r} \\ &= \frac{i\omega}{4}\int w_{\|}\mathbf{B}\cdot\nabla\left(|s|^{2}\frac{1}{B}\frac{\partial \bar{f}}{\partial \varepsilon}\right)\Bigg|_{w_{\perp},w_{\|}} d\,\mathbf{w}\,d\mathbf{r} \\ &= \frac{i\omega}{4}\int w_{\|}\nabla\cdot\left[\mathbf{B}\left(|s|^{2}\frac{1}{B}\frac{\partial \bar{f}}{\partial \varepsilon}\right)\right]\Bigg|_{w_{\perp},w_{\|}} d\,\mathbf{w}\,d\mathbf{r} \\ &= 0 \end{aligned} \tag{10.84}$$

The Energy Relation is now obtained by substituting these results into Eq. (10.73),

$$\begin{aligned}\omega^{2}K_{\perp} &= \delta W_{\perp} + \delta Q_{KIN} - \omega R \\ R &= \frac{1}{2}\sum_{\alpha}\int \frac{\partial \bar{f}_{\alpha}}{\partial \varepsilon} w_{\|}(s_{\alpha i}\mathbf{b}\cdot\nabla s_{\alpha r} - s_{\alpha r}\mathbf{b}\cdot\nabla s_{\alpha i})\,d\,\mathbf{w}\,d\mathbf{r}\end{aligned} \tag{10.85}$$

where R as well $K_\perp$, $\delta W_\perp$, δQ_{KIN} are all real quantities while ω is complex with $\omega_i > 0$. One now sets the imaginary part of Eq. (10.85) to zero yielding $\omega_i R = -2\omega_i \omega_r K_\perp$ or

$$R = -2\omega_r K_\perp \tag{10.86}$$

Substituting this into the real part of Eq. (10.85) finally leads to the desired form of the kinetic MHD Energy Relation,

$$\begin{aligned} |\omega|^2 &= -\frac{\delta W_\perp + \delta Q_{KIN}}{K_\perp} \\ \delta Q_{KIN} &= -\frac{|\omega|^2}{2} \sum_\alpha \int \frac{\partial \bar{f}_\alpha}{\partial \varepsilon} |s_\alpha|^2 d\mathbf{w}\, d\mathbf{r} \end{aligned} \tag{10.87}$$

This relation is valid for general 3-D geometries with arbitrary complex ω, the only constraint being $\omega_i > 0$

Discussion

After an admittedly lengthy calculation one can now ask how an Energy Relation of the form given by Eq. (10.87) can be used to determine stability. To answer this question several points should be noted. First, observe that $|\omega|^2$ rather than ω^2 appears in the relation. Second, from Eq. (10.81) one sees that $\delta Q_{KIN} \geq 0$ when $\partial \bar{f}_\alpha / \partial \varepsilon < 0$, the usual situation. Third, although δQ_{KIN} would appear to vanish when $|\omega|^2 \to 0$ this is not the case for toroidal configurations. Here, trapped particles and the non-uniform sampling of flux surfaces by passing particles each produce a contribution to $|s_\alpha|^2 \propto 1/|\omega|^2$. The net result is that δQ_{KIN} makes a finite but positive contribution to the energy as $|\omega|^2 \to 0$.

Keeping these points in mind one can deduce information about kinetic MHD stability as follows. If

$$\delta W_{KIN} \equiv \delta W_\perp + \delta Q_{KIN}(\omega = 0) > 0 \tag{10.88}$$

then there is an obvious contradiction because of the negative sign in Eq. (10.87). The only way to resolve this contradiction is to conclude that the assumption $\omega_i > 0$ is violated implying that $\omega_i < 0$. In other words, Eq. (10.88) is a sufficient condition for kinetic MHD stability.

A second stability result can be deduced when

$$\delta W_\perp + \delta Q_{KIN}(\omega = 0) = 0 \tag{10.89}$$

This can only occur if $|\omega|^2 = 0$ (i.e. $\omega_r = \omega_i = 0$) and corresponds to a marginal stability point.

The third stability condition corresponds to

$$\delta W_\perp + \delta Q_{KIN}(\omega = 0) < 0 \tag{10.90}$$

In this case there is no contradiction in the general Energy Relation. One expects that when Eq. (10.90) is satisfied, the plasma will be unstable since the assumption $\omega_i > 0$ has not been violated.

Physically, the fact that $\omega_r = 0$ when $\omega_i \to 0$ is a consequence of the fact in a kinetic MHD plasma the dispersion relation is in general complex because of the parallel Landau damping generated by resonant particles. The damping vanishes when $\partial \overline{f}/\partial w_\parallel = 0$ for the resonant particles, thereby corresponding to the condition for marginal stability. The requirement $\partial \overline{f}/\partial w_\parallel = 0$ is satisfied when $w_\parallel = \omega_r/k_\parallel = 0$ for a distribution function of the form $\overline{f} = \overline{f}(w_\perp^2 + w_\parallel^2, \psi)$.

The discussion above describes how the Energy Relation can be used to obtain kinetic MHD stability information about a plasma. In order to make stability comparisons with other models such as ideal MHD and double adiabatic MHD it is necessary to obtain upper and lower bounds on the size of δQ_{KIN}. These bounds, which require considerable analysis, are derived in Appendix H. Their values are given as the discussion progresses in the next section.

10.5 Stability comparison theorems

All the necessary derivations for the various contributions to the potential energy have been completed. The task now is to examine these contributions in order to deduce the desired stability comparison theorems (Kruskal and Oberman, 1958; Rosenbluth and Rostoker, 1959). The results are separated into four parts representing cylindrical or toroidal and ergodic or closed line geometries. For all geometries and all MHD models the basic quantity $\delta W_\perp$ defined by

$$\delta W_\perp(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp) = \frac{1}{2\mu_0}\int \left[|\mathbf{Q}|^2 - \mu_0 \boldsymbol{\xi}_\perp^* \cdot \mathbf{J} \times \mathbf{Q} + \mu_0(\boldsymbol{\xi}_\perp \cdot \nabla p)\nabla \cdot \boldsymbol{\xi}_\perp^*\right] d\mathbf{r} + \delta W_S + \delta W_V \tag{10.91}$$

appears as one contribution to the potential energy and serves as a point of reference. The four stability comparison theorems are determined as follows.

10.5.1 Closed line cylindrical geometry

The first theorem applies to a closed line cylindrical system in which the perturbations maintain the closed line symmetry. This corresponds to the $m = 0$ mode in a

Z-pinch. The relevant potential energies can be extracted directly from the text. Specifically, for ideal MHD the potential energy is obtained from Eq. (10.4) after noting that $\nabla \cdot \boldsymbol{\xi} = \nabla \cdot \boldsymbol{\xi}_\perp$ for the $m = 0$ mode in a Z-pinch. The corresponding result for double adiabatic MHD is given in Eqs. (10.27) and (10.28). Similarly the kinetic MHD result follows from Eqs. (10.42) and (10.43). These results are summarized below:

$$\begin{aligned}
\delta W_{MHD} &= \delta W_\perp + \frac{1}{2}\int \frac{5}{3} p|\nabla \cdot \boldsymbol{\xi}_\perp|^2 d\mathbf{r} \\
\delta W_{CGL} &= \delta W_\perp + \frac{1}{2}\int \left[\frac{5}{3} p|\nabla \cdot \boldsymbol{\xi}_\perp|^2 + \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_\perp + 3\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}|^2 \right] d\mathbf{r} \\
\delta W_{KIN} &= \delta W_\perp + \frac{1}{2}\int \left[\frac{5}{3} p|\nabla \cdot \boldsymbol{\xi}_\perp|^2 + \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_\perp + 3\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}|^2 \right] d\mathbf{r}
\end{aligned} \tag{10.92}$$

It is clear from Eq. (10.92) that the stability comparison theorem for a closed line cylindrical system satisfies

$$\delta W_{MHD} < \delta W_{KIN} = \delta W_{CGL} \tag{10.93}$$

Kinetic MHD is bracketed by ideal and double adiabatic MHD and is in fact identical to double adiabatic MHD.

10.5.2 Ergodic cylindrical geometry

A similar stability comparison theorem can be derived for the cylindrical screw pinch. For this geometry the ideal MHD potential energy is again obtained from Eq. (10.4) although in this case $\xi_\parallel$ is chosen to make $\nabla \cdot \boldsymbol{\xi} = 0$. The double adiabatic potential energy follows from Eqs. (10.27) and (10.28) while the kinetic MHD potential energy is given by Eq. (10.57). A summary of the potential energies can be written as

$$\begin{aligned}
\delta W_{MHD} &= \delta W_\perp \\
\delta W_{CGL} &= \delta W_\perp + \frac{1}{2}\int \left[\frac{5}{3} p|\nabla \cdot \boldsymbol{\xi}_\perp|^2 + \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_\perp + 3\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}|^2 \right] d\mathbf{r} \\
\delta W_{KIN} &= \delta W_\perp
\end{aligned} \tag{10.94}$$

An examination of Eq. (10.94) shows that the stability comparison theorem for a cylindrical screw pinch satisfies

$$\delta W_{MHD} = \delta W_{KIN} < \delta W_{CGL} \tag{10.95}$$

Kinetic MHD is again bracketed by ideal and double adiabatic MHD although in this case it coincides with ideal MHD.

10.5.3 Closed line toroidal geometry

The next configuration of interest is the closed line toroidal geometry where the perturbations maintain the closed line symmetry. Examples of such systems include the toroidal Z-pinch, the levitated dipole, and the field reversed configuration. The potential energy for each model is summarized below.

For ideal MHD the potential energy follows from Eq. (10.4) although one must eliminate $\xi_{\|}$ by the general minimizing condition discussed in Section 8.9 and given by Eq. (8.117). The result is

$$\delta W_{MHD} = \delta W_{\perp} + \frac{1}{2}\int \frac{5}{3} p \left| \left\langle \nabla \cdot \boldsymbol{\xi}_{\perp} \right\rangle \right|^2 d\mathbf{r}$$
$$\left\langle \nabla \cdot \boldsymbol{\xi}_{\perp} \right\rangle = \frac{\oint \frac{dl}{B} \nabla \cdot \boldsymbol{\xi}_{\perp}}{\oint \frac{dl}{B}} \tag{10.96}$$

The double adiabatic result is again given by Eqs. (10.27) and (10.28),

$$\delta W_{CGL} = \delta W_{\perp} + \frac{1}{2}\int \left[\frac{5}{3} p \left| \nabla \cdot \boldsymbol{\xi}_{\perp} \right|^2 + \frac{p}{3} \left| \nabla \cdot \boldsymbol{\xi}_{\perp} + 3\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa} \right|^2 \right] d\mathbf{r} \tag{10.97}$$

Observe that ideal MHD and double adiabatic MHD contain compressibility terms proportional to $|\langle \nabla \cdot \boldsymbol{\xi}_{\perp} \rangle|^2$ and $|\nabla \cdot \boldsymbol{\xi}_{\perp}|^2$ respectively. One must determine the relative size of these two terms in order to make an accurate comparison between the two models. This comparison can be made by means of Schwartz's inequality after switching to ψ, χ, l flux coordinates and recalling that $d\mathbf{r} = d\psi d\chi dl/B$. Focusing on the l integration one notes that Schwartz's inequality can be written as

$$\int g^2 dl \int |h|^2 dl \geq \left| \int g h dl \right|^2 \tag{10.98}$$

Choosing $g = 1/B^{1/2}$ and $h = (\nabla \cdot \boldsymbol{\xi}_{\perp})/B^{1/2}$ yields

$$\int \frac{dl}{B} \int \frac{dl}{B} |\nabla \cdot \boldsymbol{\xi}_{\perp}|^2 \geq \left| \int \frac{dl}{B} \nabla \cdot \boldsymbol{\xi}_{\perp} \right|^2 \tag{10.99}$$

which in turn implies that

$$\frac{1}{2}\int \frac{5}{3} p |\nabla \cdot \boldsymbol{\xi}_{\perp}|^2 d\mathbf{r} \geq \frac{1}{2}\int \frac{5}{3} p |\langle \nabla \cdot \boldsymbol{\xi}_{\perp} \rangle|^2 d\mathbf{r} \tag{10.100}$$

Using this relation in Eq. (10.97) leads to the conclusion

$$\delta W_{CGL} \geq \delta W_{\perp} + \frac{1}{2}\int \left[\frac{5}{3}p|\langle \nabla \cdot \boldsymbol{\xi}_{\perp}\rangle|^2 + \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_{\perp} + 3\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa}|^2 \right] d\mathbf{r} > \delta W_{MHD} \tag{10.101}$$

The last result of interest involves the potential energy from the kinetic MHD model, which is written as

$$\delta W_{KIN} = \delta W_{\perp} + \delta Q_{KIN} \tag{10.102}$$

Upper and lower bounds on δQ_{KIN} have been derived in Appendix H. The relevant bounds are obtained from Eqs. (H.23) and (H.29) and are given by

$$\begin{aligned} &\delta Q_{\min} < \delta Q_{KIN} < \delta Q_{\max} \\ &\delta Q_{\min} = \frac{1}{2}\int \frac{5}{3}p|\langle \nabla \cdot \boldsymbol{\xi}_{\perp}\rangle|^2 d\mathbf{r} \\ &\delta Q_{\max} = \frac{1}{2}\int \left[\frac{5}{3}p|\nabla \cdot \boldsymbol{\xi}_{\perp}|^2 + \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_{\perp} + 3\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa}|^2 \right] d\mathbf{r} \end{aligned} \tag{10.103}$$

The desired comparison theorem is now easily obtained by examining the Energy Relations for each model:

$$\delta W_{MHD} < \delta W_{KIN} < \delta W_{CGL} \tag{10.104}$$

Kinetic MHD is bracketed by ideal MHD and double adiabatic MHD.

10.5.4 Ergodic toroidal geometry

The final configuration of interest is a general toroidal geometry with ergodic field lines or a closed line system in which the perturbations break the closed line symmetry. The ideal MHD potential energy is given by Eq. (10.4) with $\xi_{\|}$ chosen to make $\nabla \cdot \boldsymbol{\xi} = 0$. In this case

$$\delta W_{MHD} = \delta W_{\perp} \tag{10.105}$$

The double adiabatic potential energy remains unchanged and is obtained from Eq. (10.27) and (10.28):

$$\delta W_{CGL} = \delta W_{\perp} + \frac{1}{2}\int \left[\frac{5}{3}p|\nabla \cdot \boldsymbol{\xi}_{\perp}|^2 + \frac{p}{3}|\nabla \cdot \boldsymbol{\xi}_{\perp} + 3\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa}|^2 \right] d\mathbf{r} \tag{10.106}$$

Lastly, the kinetic MHD potential energy also follows from the bounds on δQ_{KIN} derived in Appendix H in Eqs. (H.23) and (H.29). The results are

$$\delta W_{KIN} = \delta W_{\perp} + \delta Q_{KIN} \tag{10.107}$$

where

$$\begin{aligned}
&\delta Q_{\min} < \delta Q_{KIN} < \delta Q_{\max} \\
&\delta Q_{\min} = \frac{1}{2}\int \frac{5}{3} p |\langle \nabla \cdot \boldsymbol{\xi}_{\perp} \rangle|^2 \, d\mathbf{r} \\
&\delta Q_{\max} = \frac{1}{2}\int \left[\frac{5}{3} p |\nabla \cdot \boldsymbol{\xi}_{\perp}|^2 + \frac{p}{3} |\nabla \cdot \boldsymbol{\xi}_{\perp} + 3\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa}|^2 \right] d\mathbf{r}
\end{aligned} \tag{10.108}$$

Combining these results again yields

$$\delta W_{MHD} < \delta W_{KIN} < \delta W_{CGL} \tag{10.109}$$

Here too, kinetic MHD is bracketed by ideal MHD and double adiabatic MHD.

10.6 Summary

Marginal stability Energy Relations have been derived for the three models of interest: ideal MHD, double adiabatic MHD, and kinetic MHD. The corresponding potential energy expressions vary somewhat depending on whether the geometry is cylindrical or toroidal and whether the magnetic surfaces are ergodic or closed line. The main result of the analysis is that regardless of which of these situations prevails, there is a hierarchy of stability predictions that is always satisfied,

$$\delta W_{MHD} \leq \delta W_{KIN} \leq \delta W_{CGL} \tag{10.110}$$

The ideal MHD model is the most unstable. Since it is a fluid model it is relatively easy to analyze but the invalidity of the collision dominated assumption used in its derivation causes one to be concerned about the reliability of its predictions. The double adiabatic MHD model is the most stable. It too is a fluid model and is thus relatively simple to analyze. Although it is a collisionless model, which makes it relevant for fusion plasmas, the closure assumptions used in its derivation cannot be mathematically justified. This is a cause of concern about the reliability of its stability predictions.

The kinetic MHD stability predictions are bracketed by those of ideal MHD and double adiabatic MHD. Kinetic MHD is a self-consistent collisionless model which, as its name implies, includes kinetic behavior along the field lines. It is the most reliable MHD model in terms of the physics. Its biggest drawback is that it is quite complicated to analyze because of the kinetic effects associated with resonant particles and trapping.

Overall, if the relatively simple to obtain stability prediction gaps between ideal MHD and double adiabatic MHD are found to be small, two main benefits follow: (1) the desired, but complicated to obtain, kinetic MHD stability boundaries are

narrowly bracketed; and (2) an explanation is provided of why ideal MHD works as well as it does when comparing to experiments.

In the remainder of the text several examples are analyzed to explicitly determine how close the predictions are between ideal MHD and double adiabatic MHD.

References

Bernstein, I.B., Frieman, E.A., Kruskal, M.D., and Kulsrud, R.M. (1958). *Proc. Royal Society (London)* **A244**, 16.
Kruskal, M.D. and Oberman, C.R. (1958). *Phys. Fluids* **1**, 275.
Rosenbluth, M.N. and Rostoker, N. (1959). *Phys. Fluids* **2**, 23.
Stix, T.H. (1992). *Waves in Plasmas*. New York: American Institute of Physics.

Further reading

Antonsen, T.M. Jr. and Lee, Y.C. (1982). *Phys. Fluids* **25**, 132.
Cerfon, A.J. and Freidberg, J.P. (2011). *Phys. Plasmas* **18**, 012505.
Connor, J.W. and Hastie, R.J. (1974). *Phys. Rev. Lett.* **33**, 202.
Grad, H. (1966). *Phys. Fluids* **9**, 225.
Kulsrud, R.M. (1962). *Phys. Fluids* **5**, 275.

Problems

10.1 Consider the original double adiabatic model. Derive the marginal stability differential equation for an equilibrium with cylindrical symmetry. Assume the equilibrium (but not the perturbed) pressures are isotropic.

10.2 Repeat Problem 10.1 using the modified double adiabatic model.

10.3 Repeat Problem 10.1 assuming the equilibrium pressures are anisotropic.

10.4 Repeat Problem 10.2 assuming the equilibrium pressures are anisotropic.

10.5 Derive an equivalent form of Suydam's criterion for the anisotropic modified double adiabatic model.

10.6 Derive Eq. (10.49).