Ideal MHD

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Chapter

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3

General properties of ideal MHD

3.1 Introduction

This chapter presents a discussion of the basic properties of the ideal MHD model. These properties include the general conservation laws satisfied by the model as well several types of boundary conditions that are of interest to fusion plasmas. The discussion demonstrates the physical foundations of ideal MHD while providing insight into its reliability in predicting experimental behavior.

The material is organized as follows. First, a short description is given of the three most common types of boundary conditions that couple the plasma behavior to the externally applied magnetic fields: (1) plasma surrounded by a perfectly conducting wall; (2) plasma isolated from a perfectly conducting wall by an insulating vacuum region; and (3) plasma surrounded by a vacuum with embedded external coils. The most complex of these provides a quite accurate description of realistic experimental conditions.

Second, it is shown that despite the significant number of approximations made in the derivation of the model, ideal MHD still conserves mass, momentum, and energy, both locally and globally. This is one basic reason for the reliability of the model.

Finally, a short calculation shows that as a consequence of the perfect conductivity assumption, the plasma and magnetic field lines are constrained to move together; that is, the field lines are "frozen" into the plasma. This leads to important topological constraints on the allowable dynamical motions of the plasma. In fact the property of "frozen-in" field line topology can be taken as the basic definition of "ideal" MHD.

3.2 Boundary conditions

In order to properly formulate an MHD problem a set of appropriate boundary conditions must be specified that couples the plasma behavior to the externally applied magnetic fields. For problems involving MHD equilibrium and stability,

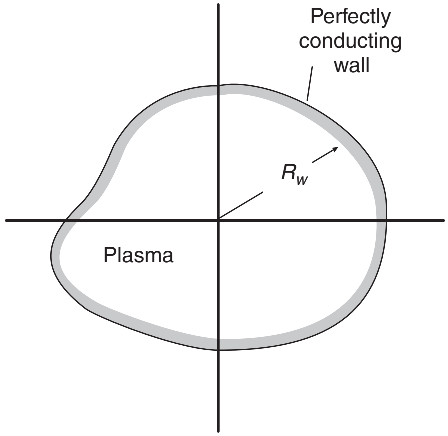

Figure 3.1 A plasma surrounded by a perfectly conducting wall whose surface is defined by $S_w(r, \theta, z) \equiv r - R_w(\theta, z) = 0$.

there are three types of boundary conditions, of varying complexity, that are often used. A discussion of each now follows.

3.2.1 Perfectly conducting wall

The first and simplest boundary condition assumes that the plasma extends out to a stationary, perfectly conducting wall whose shape is defined by $S_w(r, \theta, z) = 0$, as shown in Fig. 3.1. In this case the electromagnetic boundary conditions require that the tangential electric field and normal magnetic field vanish on the conducting wall:

$$\begin{aligned} \mathbf{n} \times \mathbf{E}|_{S_w} &= 0 \\ \mathbf{n} \cdot \mathbf{B}|_{S_w} &= 0 \end{aligned} \tag{3.1}$$

Here, $\mathbf{n}$ is the outward-pointing normal vector. In time-varying systems the magnetic condition is redundant, a consequence of Faraday's law.

Next, from the ideal Ohm's law one sees that $\mathbf{n} \times \mathbf{E} + (\mathbf{n} \cdot \mathbf{B})\mathbf{v} - (\mathbf{n} \cdot \mathbf{v})\mathbf{B} = 0$; that is, the normal component of velocity also automatically vanishes on the wall:

$$\mathbf{n} \cdot \mathbf{v}|_{S_w} = 0 \tag{3.2}$$

Thus, once appropriate initial data and the shape of the wall are specified, the conditions in Eq. (3.1) plus the condition given by Eq. (3.2) completely specify the problem. Note that there are no constraints imposed on the tangential velocity at the wall.

3.2.2 Insulating vacuum region

A more realistic set of boundary conditions assumes that the plasma is isolated from the conducting wall by a vacuum region as shown in Fig. 3.2. In most cases this model is more appropriate than the previous one for describing "confined" plasmas.

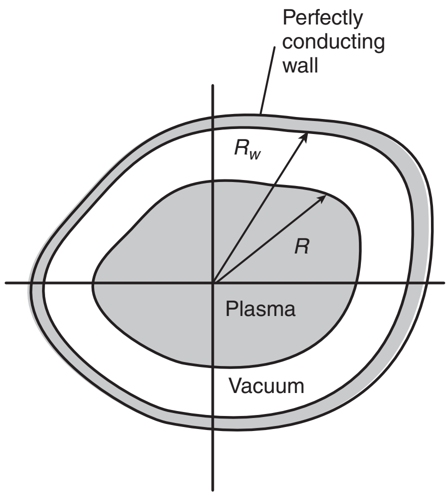

Figure 3.2 A plasma isolated from a perfectly conducting wall by a vacuum region. The plasma surface is defined by $S_p(r, \theta, z) \equiv r - R(\theta, z) = 0$ while the wall surface is defined by $S_w(r, \theta, z) \equiv r - R_w(\theta, z) = 0$.

In principle, one solves the combined plasma–vacuum system as follows. In the plasma region the ideal MHD equations apply, while in the vacuum region, where the fluid variables are not defined, the relevant equations are those determining the magnetic field:

$$\begin{aligned} \nabla \times \hat{\mathbf{B}} &= 0 \\ \nabla \cdot \hat{\mathbf{B}} &= 0 \end{aligned} \tag{3.3}$$

Here, quantities with a ^ denote vacuum variables. Equation (3.3) implies that $\hat{\mathbf{B}} = \nabla \hat{V}$ with $\nabla^2 \hat{V} = 0$. Assume now that the relevant equations can be solved in each region and consider the boundary conditions. On the perfectly conducting wall $S_w(r, \theta, z) = 0$, the normal component of magnetic field must vanish:

$$\mathbf{n} \cdot \hat{\mathbf{B}}|_{S_w} = 0 \tag{3.4}$$

Unlike the previous case, however, the plasma surface defined by $S_p(r, \theta, z, t) = 0$ is now free to move since the plasma is surrounded by vacuum. Hence, $\mathbf{n} \cdot \mathbf{v}|_{S_p}$ is arbitrary. There are, however, three non-trivial jump conditions that must be satisfied to connect the fields across the surface. These arise from the divergence $\mathbf{B}$ equation, Faraday's law, and the momentum equation. There are other jump conditions that can be derived but these in general are subsidiary relations not required to solve MHD problems.

Consider now the jump conditions. A convenient way to obtain the desired relations is to assume that the plasma surface is moving with a normal velocity

$v_n\mathbf{n} = (\mathbf{n}\cdot\mathbf{v})\mathbf{n}$. The jump conditions are straightforward to derive in a reference frame moving with the plasma surface. Once these conditions are obtained, all that is then required is to convert back to the laboratory frame using the corresponding Galilean transformation.

In the moving frame where the surface appears stationary, integrating the divergence $\mathbf{B}$ equation over a small closed volume through which the surface passes leads to the jump condition

$$[\![\mathbf{n}\cdot\mathbf{B}']\!]_{S_p} = 0 \tag{3.5}$$

Here and below, primed quantities correspond to the moving frame and $[\![Q]\!] \equiv \hat{Q} - Q$. Similarly, by integrating Faraday's law over a small open surface area whose normal vector is tangent to the surface one obtains the jump condition

$$[\![\mathbf{n}\times\mathbf{E}']\!]_{S_p} = 0 \tag{3.6}$$

The last jump condition is obtained by rewriting the momentum equation as follows:

$$\rho'\left(\frac{\partial\mathbf{v}'}{\partial t'} + \mathbf{v}'\cdot\nabla'\mathbf{v}'\right) = \frac{1}{\mu_0}\mathbf{B}'\cdot\nabla'\mathbf{B}' - \nabla'\left(p' + \frac{B'^2}{2\mu_0}\right) \tag{3.7}$$

Only the last term can exhibit a delta function behavior leading to a non-trivial jump condition. For finite accelerations the inertial terms at most have a step discontinuity across the surface. Similarly, since the operator $\mathbf{B}'\cdot\nabla'$ contains only surface derivatives the term $\mathbf{B}'\cdot\nabla'\mathbf{B}'$ also has at most a step discontinuity across the surface. Thus, integrating across a small distance perpendicular to the surface leads to the jump condition

$$[\![p' + B'^2/2\mu_0]\!]_{S_p} = 0 \tag{3.8}$$

The desired form of the jump conditions is obtained by moving back to the laboratory reference frame by means of the Galilean transformation defined by $p' = p$, $\mathbf{B}' = \mathbf{B}$, $\mathbf{n}\times\mathbf{E}' = \mathbf{n}\times(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \mathbf{n}\times\mathbf{E} - (\mathbf{n}\cdot\mathbf{v})\mathbf{B}$. Here, the unprimed coordinates correspond to the laboratory reference frame. With these substitutions, the relevant jump conditions for ideal MHD can be written as

$$\begin{aligned}
[\![\mathbf{n}\cdot\mathbf{B}]\!]_{S_p} &= 0 \\
[\![\mathbf{n}\times\mathbf{E} - (\mathbf{n}\cdot\mathbf{v})\mathbf{B}]\!]_{S_p} &= 0 \\
[\![p + B^2/2\mu_0]\!]_{S_p} &= 0
\end{aligned} \tag{3.9}$$

A final simplification occurs by making use of the fact that the plasma is a perfect conductor: $\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0$. This implies that in the plasma $[\mathbf{n}\cdot\mathbf{B}]_{S_p}$ and $[\mathbf{n}\times\mathbf{E} - (\mathbf{n}\cdot\mathbf{v})B]_{S_p}$ are both automatically zero. Therefore, Eq. (3.9) reduces to

$$\begin{aligned}\mathbf{n}\cdot\hat{\mathbf{B}}|_{S_p} &= 0\\ \mathbf{n}\times\hat{\mathbf{E}}-(\mathbf{n}\cdot\mathbf{v})\hat{\mathbf{B}}|_{S_p} &= 0\\ [\![p+B^2/2\mu_0]\!]_{S_p} &= 0\end{aligned} \tag{3.10}$$

In the interesting situation where there are no surface currents (i.e., $[\![\mathbf{n}\times\mathbf{B}]\!]_{S_p}=0$) and the pressure falls smoothly to zero at the plasma edge (i.e., $p|_{S_p}=0$), the jump conditions reduce to

$$\begin{aligned}\mathbf{n}\cdot\hat{\mathbf{B}}|_{S_p} &= 0\\ \mathbf{n}\times\hat{\mathbf{E}}|_{S_p} &= 0\\ [\![B^2/2\mu_0]\!]_{S_p} &= 0\end{aligned} \tag{3.11}$$

Although Eqs. (3.4) and (3.9) completely specify the boundary conditions, the plasma–vacuum problem is in practice difficult to solve. The reason is that a straightforward counting of boundary conditions suggests that the problem is over determined. To understand this point recall that for a vacuum field defined by $\hat{\mathbf{B}}=\nabla\hat{V}$ with $\nabla^2\hat{V}=0$, the two boundary conditions on $\mathbf{n}\cdot\hat{\mathbf{B}}$ on the wall and plasma surfaces, uniquely determine $\hat{V}$, implying that $\mathbf{n}\times\hat{\mathbf{B}}$ and hence $\hat{B}^2$ are also known quantities. However, there remains the pressure balance jump condition, which places an additional constraint on $\hat{B}^2$ apparently leading to an over-determined problem.

The problem is resolved as follows. One must treat the shape of the plasma $S_p(r,\theta,z,t)$ as an additional unknown to be self-consistently determined by the analysis. Herein lies the extra degree of freedom to make the problem well posed although difficult to solve. A problem for which $S_p(r,\theta,z,t)$ must be determined is known as a "free boundary" problem. In contrast, it is often far simpler, though less relevant, to specify the shape of the plasma surface $S_p(r,\theta,z,t)$ and then determine a self-consistent shape for the outer perfect conductor. When $S_p(r,\theta,z,t)$ is specified the problem is known as a "fixed boundary" problem.

This completes the discussion of the boundary and jump conditions for a system consisting of a plasma isolated from a conducting wall by a vacuum region.

3.2.3 Plasma surrounded by external coils

The most difficult but realistic set of boundary conditions corresponds to the situation where the plasma is confined by the magnetic fields created by a fixed set of external current-carrying conductors embedded in the vacuum region as shown in Fig. 3.3. This problem is more difficult than the previous one because the current-carrying conductors must by definition have spaces between them. Consequently, the problem must be solved in an infinite rather than finite domain.

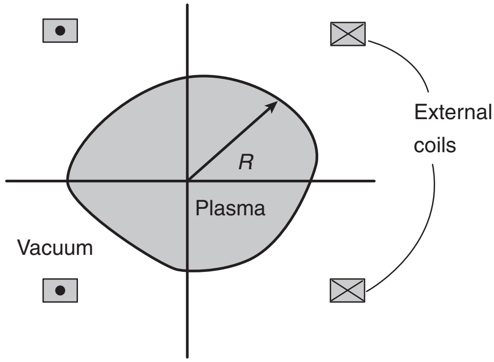

Figure 3.3 A plasma surrounded by a vacuum region into which are embedded external current-carrying coils. The plasma surface is defined by $S_p(r, \theta, z) \equiv r - R(\theta, z) = 0$.

In this regard note that the jump conditions at the plasma–vacuum interface remain unchanged: i.e., Eq. (3.9) still applies. However, with external coils the wall condition, Eq. (3.4), is no longer required. Instead, the vacuum field is written as $\hat{\mathbf{B}}(\mathbf{r}, t) = \mathbf{B}_a(\mathbf{r}) + \tilde{\mathbf{B}}(\mathbf{r}, t)$, where $\mathbf{B}_a(\mathbf{r})$ is the known steady state (or perhaps slowly varying) applied field due to the external coils and $\tilde{\mathbf{B}}(\mathbf{r}, t)$ is the induced field due to the plasma. The applied field is obtained by writing $\mathbf{B}_a = \nabla \times \mathbf{A}$ and then calculating $\mathbf{A}$ from the Biot–Savart law

$$\mathbf{A} = \frac{\mu_0}{4\pi} \sum_i \int \frac{\mathbf{J}_i}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{3.12}$$

where the sum is over the external conductors. The induced field satisfies $\nabla \times \tilde{\mathbf{B}} = \nabla \cdot \tilde{\mathbf{B}} = 0$ again implying that $\tilde{\mathbf{B}} = \nabla \tilde{V}$ with $\nabla^2 \tilde{V} = 0$. The potential $\tilde{V}$ must be regular throughout the entire vacuum region thus leading to the following boundary conditions:

$$\begin{aligned} \tilde{\mathbf{B}}|_\infty &= 0 \\ \tilde{\mathbf{B}}|_{S_p} &= -\mathbf{B}_a|_{S_p} \end{aligned} \tag{3.13}$$

The shape of the plasma surface is also an unknown for this class of "free boundary" problems and must be determined self-consistently from the analysis.

Solutions to the MHD equations satisfying the external coil boundary conditions provide an accurate description of plasma behavior in realistic experimental situations. Because of the complexity involved it is perhaps not surprising that most such applications require substantial numerical computations. The MHD equations plus any set of the boundary conditions just discussed constitute a well-posed formulation to investigate the macroscopic equilibrium and stability of fusion plasmas.

3.3 Local conservation relations

The original kinetic-Maxwell equations from which the MHD model has been derived conserve mass, momentum, and energy, not only macroscopically, but microscopically as well. Since a considerable number of assumptions have been made in the derivation of the MHD equations it is important to investigate whether the resulting model still satisfies these basic conservation laws. In this section the question of local conservation is treated by showing that the ideal MHD equations can be written in canonical conservation form. The implications regarding global conservation are discussed in the next section.

The canonical conservation form is given by

$$\frac{\partial}{\partial t}(\;) + \nabla \cdot (\;) = 0 \tag{3.14}$$

Once the mass, momentum, and energy equations can be written in this form, it is then straightforward to derive the global conservation relations.

3.3.1 Conservation of mass

To begin observe that the MHD mass equation is already in the desired conservation form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \tag{3.15}$$

The first term represents the gain in mass within a given volume element. This gain is due to a net inward flux of particles through the boundaries of the volume element as described by the second term.

3.3.2 Conservation of momentum

Consider now the momentum equation. If one makes use of the tensor identity $\nabla \cdot (\mathbf{AC}) = (\nabla \cdot \mathbf{A})\mathbf{C} + (\mathbf{A} \cdot \nabla)\mathbf{C}$ then a short calculation allows the momentum equation to be written in conservation form as follows:

$$\begin{aligned} &\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \mathsf{T} = 0 \\ &\mathsf{T} = \rho \mathbf{vv} + \left(p + \frac{B^2}{2\mu_0}\right)\mathsf{I} - \frac{1}{\mu_0}\mathbf{BB} \end{aligned} \tag{3.16}$$

The $\partial(\rho\mathbf{v})/\partial t$ term represents the increase in momentum within a volume element. This increase is generated by a net inward flux of momentum $\nabla \cdot \mathsf{T}$ through the boundaries of the volume element.

The contributions to T have the following physical interpretation. The $\rho\mathbf{vv}$ term represents the Reynolds stress and is important in systems with large fluid flows. Often it is not too important in studies of plasma stability where the equilibrium flows are assumed to be small or zero. The remaining contributions to T include the effects of the pressure and magnetic field. In a locally orthogonal coordinate system in which one coordinate is aligned along $\mathbf{B}$ these contributions can be conveniently rewritten as

$$\mathsf{T}_B = \begin{vmatrix} p_\perp & & \\ & p_\perp & \\ & & p_\parallel \end{vmatrix} \tag{3.17}$$

where

$$\begin{aligned} p_\perp &= p + \frac{B^2}{2\mu_0} \\ p_\parallel &= p - \frac{B^2}{2\mu_0} \end{aligned} \tag{3.18}$$

The quantities $p_\perp$ and $p_\parallel$ represent the total pressures perpendicular and parallel to the magnetic field respectively. Equation (3.18) implies, as expected, that the particle pressure p acts isotropically perpendicular and parallel to the field. In contrast, the magnetic pressure $B^2/2\mu_0$ adds to the total pressure perpendicular to the field, while subtracting when parallel to the field. The "negative parallel magnetic pressure" actually corresponds to a tension along the magnetic field lines. This anisotropic behavior of the magnetic field (i.e., producing pressure perpendicular to $\mathbf{B}$ and tension parallel to $\mathbf{B}$) is fundamental to the understanding of the equilibrium and stability properties of the magnetic geometries of fusion interest.

3.3.3 Conservation of energy

The last conservation equation of interest corresponds to energy. Several steps are required to obtain the desired form. To begin, form the dot product of the momentum equation with $\mathbf{v}$:

$$\rho\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \mathbf{v}\cdot(\mathbf{J}\times\mathbf{B}) - \mathbf{v}\cdot\nabla p \tag{3.19}$$

Next, by making use of the conservation of mass, one can rewrite the inertial terms as

$$\rho\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \nabla\cdot\left(\frac{1}{2}\rho v^2\mathbf{v}\right) \tag{3.20}$$

Now, by using the ideal MHD Ohm's law and Faraday's law the electromagnetic terms can be expressed as

$$\begin{aligned}\mathbf{v}\cdot\mathbf{J}\times\mathbf{B} &= \frac{1}{\mu_0}\mathbf{E}\cdot\nabla\times\mathbf{B}\\ &= -\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})+\frac{1}{\mu_0}\mathbf{B}\cdot\nabla\times\mathbf{E}\\ &= -\nabla\cdot\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)-\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right)\end{aligned}\tag{3.21}$$

Lastly, the term $\mathbf{v}\cdot\nabla p$ can be rewritten by (1) combining the adiabatic energy equation with the conservation of mass yielding $\partial p/\partial t+\mathbf{v}\cdot\nabla p+\gamma p\nabla\cdot\mathbf{v}=0$ and (2) using the definition $\nabla\cdot(p\mathbf{v})=\mathbf{v}\cdot\nabla p+p\nabla\cdot\mathbf{v}$. Eliminating $\nabla\cdot\mathbf{v}$ from these two equations yields

$$\mathbf{v}\cdot\nabla p=\frac{1}{\gamma-1}\frac{\partial p}{\partial t}+\frac{\gamma}{\gamma-1}\nabla\cdot p\mathbf{v}\tag{3.22}$$

Substituting Eqs. (3.20)–(3.22) leads to the energy equation in conservation form:

$$\begin{aligned}&\frac{\partial w}{\partial t}+\nabla\cdot\mathbf{s}=0\\ &w=\frac{1}{2}\rho v^2+\frac{p}{\gamma-1}+\frac{B^2}{2\mu_0}\\ &\mathbf{s}=\left(\frac{1}{2}\rho v^2+\frac{p}{\gamma-1}\right)\mathbf{v}+p\mathbf{v}+\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\end{aligned}\tag{3.23}$$

In this equation w represents the total energy of the system, which consists of the kinetic, internal (i.e., $3p/2$ for $\gamma = 5/3$), and magnetic energies. The quantity $\mathbf{s}$ is comprised of the net flux of kinetic plus internal energy, the mechanical work done on the plasma through compression, and the flux of electromagnetic energy as given by the Poynting vector.

Summarizing, it has just been demonstrated that the ideal MHD equations can be written in local conservation form in which each of the terms has a simple physical interpretation.

3.4 Global conservation laws

By integrating the local conservation relations over an appropriate volume it is possible to obtain a set of global conservation laws for ideal MHD. These laws are exact and are valid for general, non-linear, multidimensional, time-dependent situations. The specific forms of the global laws, as well as the choice of the appropriate integration volume, depend on the boundary conditions to be applied. Consequently, a separate derivation is required for each type of boundary condition as presented below.

3.4.1 Perfectly conducting wall

For this case the global conservation laws are obtained by integrating the local conservation relations out to the perfectly conducting wall. Making use of the boundary conditions given by Eqs. (3.1) and (3.2) then leads to the following global conservation laws:

$$\begin{aligned}
&\frac{dM}{dt} = 0 \\
&\frac{d\mathbf{P}}{dt} = -\int_{S_w} \left(p + \frac{B^2}{2\mu_0}\right)\mathbf{n}dS = 0 \\
&\frac{dW}{dt} = 0
\end{aligned} \tag{3.24}$$

where M is the total mass of the plasma

$$M = \int_{V_w} \rho \, d\mathbf{r} \tag{3.25}$$

$\mathbf{P}$ is the mechanical momentum of the plasma

$$\mathbf{P} = \int_{V_w} \rho\mathbf{v} \, d\mathbf{r} \tag{3.26}$$

and W is the sum of the kinetic, internal, and magnetic energies of the plasma

$$W = \int_{V_w} \left(\frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}\right) d\mathbf{r} \tag{3.27}$$

The first contribution to Eq. (3.24) demonstrates that the total mass of plasma is conserved. The second equation represents the conservation of momentum. The boundary term represents the total force exerted by the walls on the plasma. If, as one might reasonably expect, the wall remains in place during the duration of experimental operation, then this force must exactly vanish: $d\mathbf{P}/dt = 0$; that is, mechanical momentum is conserved. The final contribution to Eq. (3.24) shows that the total energy of the system is conserved. From the definition of W given by

Eq. (3.27) one observes that during a given dynamical motion of the plasma (an instability, for instance) energy can in general be transferred between the magnetic field, the plasma internal energy, and the plasma kinetic energy, although the sum must be conserved.

3.4.2 Insulating vacuum region

When the plasma is surrounded by a vacuum region it is of particular interest to focus attention on the conservation of energy. This relation is slightly more complicated than for the case of the perfectly conducting wall since the plasma boundary is now allowed to move. The result is that it is not the individual, but the combined plasma–vacuum energy that is conserved.

To show this, first consider a global quantity G, defined by

$$G(t) = \int_V g(\mathbf{r}, t)\, d\mathbf{r} \tag{3.28}$$

The total time derivative of G in a closed volume $V(t)$ whose surface $S(t)$ is moving with a normal velocity $v_n(S)\mathbf{n} = (\mathbf{n} \cdot \mathbf{v})\mathbf{n}$ is given by

$$\frac{dG(t)}{dt} = \int_V \frac{\partial g}{\partial t}\, d\mathbf{r} + \int_S g\, \mathbf{n} \cdot \mathbf{v}\, dS \tag{3.29}$$

where $\mathbf{n}(S)$ is the outward unit normal to the surface. For readers unfamiliar with this result a derivation is given in Appendix C. Equation (3.29) is applied to the total plasma energy by setting $G(t) = W(t)$, $g(\mathbf{r}, t) = w(\mathbf{r}, t)$, plus choosing $V = V_p$ to be the plasma volume and $S = S_p$ the plasma surface. The normal boundary velocity is $\mathbf{n} \cdot \mathbf{v}(S_p) = \mathbf{n} \cdot \mathbf{v}(\mathbf{r}, t)|_{S_p}$, corresponding to the motion of the plasma surface. Making use of the ideal Ohm's law, and the fact that on the plasma boundary $\mathbf{n} \cdot \mathbf{B}|_{S_p} = 0$, it follows that

$$\begin{aligned}
\frac{dW}{dt} &= \int_{V_p} \frac{\partial w}{\partial t}\, d\mathbf{r} + \int_{S_p} w\, \mathbf{n} \cdot \mathbf{v} dS \\
&= -\int_{V_p} \nabla \cdot \mathbf{s}\, d\mathbf{r} + \int_{S_p} w\, \mathbf{n} \cdot \mathbf{v} dS \\
&= -\int_{S_p} (\mathbf{s} - w\, \mathbf{v}) \cdot \mathbf{n}\, dS \\
&= -\int_{S_p} \left(p + \frac{B^2}{2\mu_0}\right) \mathbf{n} \cdot \mathbf{v} dS
\end{aligned} \tag{3.30}$$

Note that if the plasma surface is moving (i.e., $\mathbf{n}\cdot\mathbf{v}\neq 0$), then the boundary term is in general non-zero.

Consider now the vacuum region. Here the total energy is given by

$$\hat{W} = \int_{\hat{V}} \frac{\hat{B}^2}{2\mu_0}\, d\mathbf{r} \tag{3.31}$$

where $\hat{V}$ is the volume of the vacuum region between the plasma and the wall. From Eq. (3.29) it follows that

$$\frac{d\hat{W}}{dt} = \int_{\hat{V}} \frac{1}{\mu_0}\left(\hat{\mathbf{B}}\cdot\frac{\partial\hat{\mathbf{B}}}{\partial t}\right) d\mathbf{r} - \int_{S_p} \frac{\hat{B}^2}{2\mu_0}\,\mathbf{n}\cdot\mathbf{v}dS + \int_{S_w} \frac{\hat{B}^2}{2\mu_0}\,\mathbf{n}\cdot\mathbf{v}dS \tag{3.32}$$

The surface integral consists of two contributions, one from the wall surface and one from the plasma surface. The wall surface contribution vanishes because the wall is not moving: $\mathbf{n}\cdot\mathbf{v}|_{S_w}=0$. The plasma surface contribution has a minus sign (here and below) because $\mathbf{n}$ represents an inward normal vector to the vacuum region. The first term can be simplified by substituting Faraday's law and converting the volume integral to a surface integral by the divergence theorem. One also makes use of the facts that (1) $\nabla\times\hat{\mathbf{B}}=0$ in vacuum and (2) $\mathbf{n}\cdot\hat{\mathbf{B}}|_{S_p}=0$ and $\mathbf{n}\times\hat{\mathbf{E}}-(\mathbf{n}\cdot\mathbf{v})\hat{\mathbf{B}}|_{S_p}=0$ from Eq. (3.10). This yields

$$\begin{aligned}
\frac{d\hat{W}}{dt} &= -\int_{\hat{V}} \frac{1}{\mu_0}\left(\hat{\mathbf{B}}\cdot\nabla\times\hat{\mathbf{E}}\right) d\mathbf{r} - \int_{S_p} \frac{\hat{B}^2}{2\mu_0}\,\mathbf{n}\cdot\mathbf{v}dS \\
&= -\int_{S_p}\left(\frac{1}{\mu_0}\hat{\mathbf{E}}\times\hat{\mathbf{B}} + \frac{\hat{B}^2}{2\mu_0}\mathbf{v}\right)\cdot\mathbf{n}\, dS \\
&= \int_{S_p} \frac{\hat{B}^2}{2\mu_0}\mathbf{n}\cdot\mathbf{v}\, dS
\end{aligned} \tag{3.33}$$

The final energy relation is obtained by adding Eqs. (3.30) and (3.33):

$$\frac{d}{dt}\left(W+\hat{W}\right) = \int_{S_p}\left(\frac{\hat{B}^2}{2\mu_0} - p - \frac{B^2}{2\mu_0}\right)\mathbf{v}\cdot\mathbf{n}\, dS = 0 \tag{3.34}$$

Here, the boundary term vanishes by virtue of the pressure balance jump condition given by Eq. (3.10).

Equation (3.34) implies that when an ideal MHD plasma is isolated from a perfectly conducting wall by a vacuum region, the combined energy of the plasma–vacuum system is conserved. The fact that only the sum is conserved indicates that, in general, energy can and will flow from one region to the other as the plasma moves.

3.4.3 Plasma surrounded by external coils

If the conducting wall in the vacuum is replaced by a series of current-carrying conductors, the energy of the system is no longer conserved. The reason is that with external sources present, energy can be supplied to, or extracted from, the system. Even so, it is still possible to derive a relatively simple energy balance relation for the system.

The procedure is almost identical to that given for the vacuum region bounded by a perfectly conducting wall. The main differences are that (1) the conducting wall is now moved to infinity and (2) $\nabla \times \hat{\mathbf{B}} \neq 0$ due to the presence of external conductors. For this situation there is an additional volume contribution to the energy in the region outside the plasma due to these conductors. The energy balance relation becomes

$$\frac{d}{dt}\left(W + \hat{W}\right) = P_{\text{ext}} \tag{3.35}$$

where

$$\begin{aligned} P_{\text{ext}} &= -\int_{\hat{V}} \frac{1}{\mu_0} \hat{\mathbf{E}} \cdot \nabla \times \hat{\mathbf{B}}\, d\mathbf{r} \\ &= -\sum_i \int_{V_i} \hat{\mathbf{J}}_i \cdot \hat{\mathbf{E}}_i\, d\mathbf{r} \end{aligned} \tag{3.36}$$

represents the power delivered to the system by the external circuits. The integrals in P_{ext} are carried out over the volume of each external conductor. Equation (3.35) has the simple physical interpretation that the rate of increase of the combined plasma–vacuum energy is equal to the power delivered by the external circuits.

3.5 Conservation of flux: the "frozen-in field line" concept

The final conservation law to be considered concerns the magnetic flux. The basic result, which is a consequence of the perfect conductivity Ohm's law, is that the magnetic flux contained within an arbitrary open surface area moving with the plasma does not change; that is, the flux is "frozen" into the plasma.

To show this, one starts with the definition of the magnetic flux ψ passing through an open area S_p in the plasma as shown in Fig. 3.4:

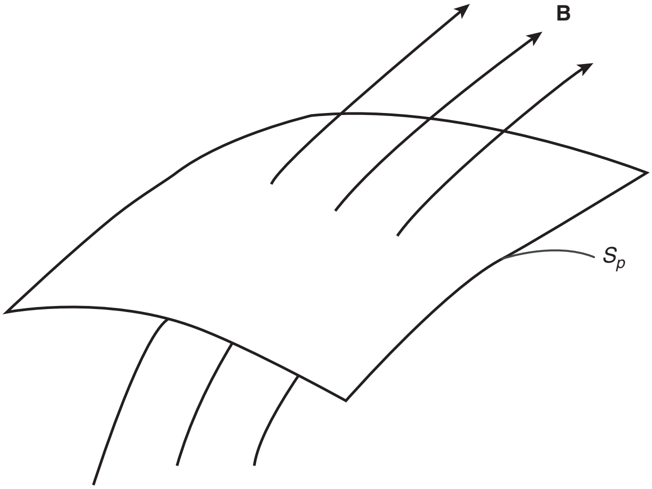

Figure 3.4 Magnetic flux passing through an open surface S_p whose surface normal vector is $\mathbf{n}$.

$$\psi = \int_{S_p} \mathbf{B} \cdot \mathbf{n}\, dS \tag{3.37}$$

Assume now that the plasma contained within S_p is moving with a velocity $\mathbf{v}$. As the surface moves the change in the flux passing through the area is given by

$$\frac{d\psi}{dt} = \int_{S_p} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS - \oint \mathbf{v} \times \mathbf{B} \cdot d\,\mathbf{l} \tag{3.38}$$

where $d\,\mathbf{l}$ is the arc length along the perimeter of the surface. For readers unfamiliar with this relation, a derivation is presented in Appendix C.

Substituting for $\partial \mathbf{B}/\partial t$ from Faraday's law and then converting the surface integral into a line integral by applying Stokes' theorem yields

$$\frac{d\psi}{dt} = -\oint (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\,\mathbf{l} \tag{3.39}$$

Clearly, if the plasma obeys the ideal MHD Ohm's law then

$$\frac{d\psi}{dt} = 0 \tag{3.40}$$

Since the derivation of Eq. (3.40) applies to any arbitrary surface area, it immediately follows that by setting S_p equal to the entire cross section of the plasma, the total flux contained within an ideal MHD plasma is conserved. An equally interesting case is to allow $S_p(l, t) = S_p(l_0, 0)$ to initially coincide with the

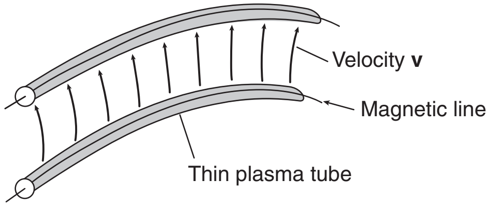

Figure 3.5 Thin tube of plasma containing a magnetic field line. As the plasma moves with a velocity **v**, the field line is "frozen" into the plasma motion.

cross section of a long thin flux tube (see Fig. 3.5). Assume now that the plasma within $S_p(l_0, 0)$ moves with a velocity $\mathbf{v}(\mathbf{r}, t)$ for a time Δt. After this period the area $S_p(l_0, \Delta t)$, which still contains the original plasma, will in general have a different shape and size (e.g., due to a non-uniform expansion of the plasma). However, Eq. (3.40) implies that the flux contained within the new cross section is the same as in the original cross section. Now, apply this result to a continuous sequence of cross sections along the original flux tube, $S_p(l, 0)$ with $l_0 \leq l \leq l_1$. Then, in the limit where $S_p \to 0$, one arrives at the well-known intuitive picture that in ideal MHD, magnetic lines move with the plasma; they are "frozen" into the fluid.

The conservation of flux relation has very important implications about the structure of the magnetic field. This follows because any allowable physical velocity **v** of a dissipationless ideal MHD plasma requires that neighboring fluid elements remain adjacent to one another; fluid elements are not allowed to tear or break into separate pieces. Since the magnetic lines move with the plasma, the field line topology must thus be preserved during any physically allowable MHD motion. This is a very strict requirement on the structure of the magnetic fields.

There are many configurations in plasma physics in which intuition suggests that it would be energetically favorable for field lines to break and reconnect, forming new configurations with lower potential energy. Such transitions are not allowed in ideal MHD because of the constraint on the topology. It is for this reason that the introduction of even a small resistivity can have a dramatic effect on plasma stability, much larger than indicated by simple dimensional arguments. A small dissipation breaks the topological constraint by allowing field lines to diffuse through the plasma. This permits a much wider class of motions to take place, although admittedly on a slower and presumably less dangerous time scale. These new motions allow the plasma to access the lower energy states that are prohibited in the ideal model. As stated earlier, the effects of resistivity lie beyond the scope of the present textbook.

To conclude, the essential distinguishing feature that serves as the definition of an "ideal" MHD plasma is that the magnetic field lines are frozen into the plasma during all allowable dynamical motions.

3.6 Summary

The results of this chapter show that the ideal MHD model conserves mass, momentum, energy, and magnetic flux. These conservation laws apply to general non-linear, time-dependent, multidimensional systems. The existence of such laws is a non-trivial consequence in view of the many assumptions made in the derivation of the model. However, having shown the existence of the conservation laws, one can proceed with some confidence to the problems of equilibrium and stability of magnetic fusion configurations knowing that the model should prove reliable and provide valuable insight because of its inherently sound foundation.

Further reading

Much of the work presented in this chapter has been derived from "classic" papers written during the early days of the fusion program. Some of these early papers are cited below as well as more recent contributions.

Boundary conditions and conservation laws

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Problems

3.1 An MHD plasma is surrounded by a rigid wall. For this problem the MHD model is generalized to include the dissipative effects of resistivity and thermal conduction. Specifically the Ohm's law and energy equation are replaced by

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$$

$$\frac{1}{\gamma - 1}\left(\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p\right) + \frac{\gamma}{\gamma - 1} p \nabla \cdot \mathbf{v} = \eta J^2 - \nabla \cdot \mathbf{q}$$

Here η is the resistivity and $\mathbf{q} = \mathbf{q}_i + \mathbf{q}_e$ is the total heat flux vector. Show that the global conservation of energy relation can be written as

$$\frac{dW}{dt} = -Q = -\int \mathbf{q} \cdot \mathbf{n}\, dS$$

where the energy W is given by Eq. (3.27) and Q represents the heat loss through the surface of the wall due to thermal conduction. What happened to the Ohmic heating term?

3.2 Consider an ideal MHD plasma surrounded by a perfectly conducting wall. The global angular momentum of the plasma is defined as

$$\mathbf{L} = \int \rho\, \mathbf{r} \times \mathbf{v}\, d\mathbf{r}$$

Prove that

$$\frac{d\mathbf{L}}{dt} = 0$$

This result demonstrates that the global angular momentum is a conserved quantity.

3.3 In this problem the global conservation of energy relation is generalized to include multiple ion species (e.g., D, T, plus impurities). The basic multiple species ideal MHD model is now written as

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot n_\alpha \mathbf{u}_\alpha = 0$$

$$m_\alpha n_\alpha \left(\frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla\right) \mathbf{u}_\alpha = Z_\alpha e(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) - \nabla p_\alpha$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla\right) \frac{p_\alpha}{n_\alpha^\gamma} = 0$$

Here α denotes species (electrons and all ions), Z_α is the charge number, and $m_e = 0$.

For a plasma surrounded by a perfectly conducting wall show that the global conservation of energy relation has the form

$$\frac{dW}{dt} = 0$$

where

$$W = \int \left[\frac{B^2}{2\mu_0} + \sum_\alpha \left(\frac{1}{2} m_\alpha n_\alpha u_\alpha^2 + \frac{p_\alpha}{\gamma - 1} \right) \right] d\mathbf{r}$$

3.4 A plasma surrounded by a perfectly conducting wall satisfies the MHD equations with the exception of Ohm's law, which now includes resistivity: $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$. Assume the current density vanishes at the wall. Show that the total magnetic flux contained within the wall is a constant. However, show that for any open surface area interior to the wall the flux is no longer a conserved quantity, implying that the magnetic field is no longer frozen into the plasma.

3.5 This problem provides an alternate demonstration that in an ideal MHD plasma the field lines are frozen into the plasma. It can be easily shown that any divergence-free magnetic field can be written as

$$\mathbf{B} = \nabla \alpha \times \nabla \beta$$

where $\alpha = \alpha(\mathbf{r}, t)$, $\beta = \beta(\mathbf{r}, t)$ represent field line coordinates; that is, the intersection of the surfaces $\alpha = \alpha_0$ and $\beta = \beta_0$ defines a line that is everywhere tangent to $\mathbf{B}$, thus defining the field line. Compute $\partial \mathbf{B}/\partial t$ and substitute the result into Faraday's law using the ideal MHD Ohm's law $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$. Show that the resulting relation is satisfied when the field lines move with the plasma:

$$\frac{d\alpha}{dt} = \frac{d\beta}{dt} = 0$$

with $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$.

3.6 The requirement $d\alpha/dt = d\beta/dt = 0$ in Problem 3.5 does not lead to a unique solution, as it is possible to add various homogeneous solutions to α and β. To illustrate this point consider a magnetic field $\mathbf{B} = B_\theta(r)\, \mathbf{e}_\theta + B_z(r)\, \mathbf{e}_z$. Write $\mathbf{B} = \nabla \alpha \times \nabla \beta$ with $\alpha = r^2/2$ and $\beta = \theta f_1(r) + (z/r) f_2(r)$.

(a) Express $f_1(r)$ and $f_2(r)$ in terms of $B_\theta(r)$ and $B_z(r)$.
(b) On any surface $\alpha = \alpha_0$ show that replacing $\beta(r, \theta, z)$ with $\beta(r, \theta, z) + g(r, t)$ leaves the magnetic field unchanged but corresponds to either a rotation in θ or a translation in z of the field line. Note that such rotations and translations have no physical significance because of the cylindrical symmetry.
(c) Assume the plasma moves with a velocity $\mathbf{v} = v_\theta(r)\, \mathbf{e}_\theta + v_z(r)\, \mathbf{e}_z$ and $v_r = 0$.

Is this velocity field consistent with Faraday's law? Could the given magnetic field be consistent with any $v_r \neq 0$? Explain.

The conclusion is that the concept of "frozen-in" field lines, while intuitively appealing, is not unique because of the ambiguity in identifying a field line from one instant of time to the next. Even so, the "frozen-in" concept represents an extremely useful interpretation of the motion of plasma and field lines.