Simulation of N body Problem using Numerical Method

1. Background

N-body Problem

N-body problem is the problem of finding the analytical solution to the N number of body interacting with each other gravitationally but, for case N>2, there exist no analytical solution for the problem hence numerical method is used to probe the dynamical behaviour of the N-body. [1] We will be considering the stability of the dynamical system resulting from the numerical solution of the N-body problem. Specifically, we are interested in the stability of the N-body dynamical system under different initial state for which the initial state can be parameterised by parameters such as the initial configuration of the N-body and their initial velocity. The stability of the dynamical system is determined by how long (in terms of timestep) the numerical solution predict the true orbital motion of the dynamical system and this can be characterised by the conservation laws.

2. Introduction

Our main objective is to simulate the dynamical system with different numerical iterative method and see how different method affect the stability of the system. We will begin with the investigation of two bodies moving under the influence of gravity where there are two cases for which both of the two bodies have either equal masses or have different masses. We will then extend our investigation to systems of several bodies. For extension, we will introduce a small perturbation to the initial state of the system and see how it will affect the stability of the system.

3. Method

In the model of N-body problem, we have the following assumptions: (a) each body is a point mass with zero volume (b) gravitational constant is 1 (G=1) (c) space and time are discrete variables. We begin our simulation with an initial configuration of the system and then propagate the system forward in time using various types of integrators which includes Euler's method, Velocity-Verlet method and Runge-Kutta method. We test the stability of the system (propagated with a specific integrator) by calculating the total energy and total angular momentum of the system after each step forward in time and observe how well and for how long the system obeys the conservation laws.

Numerical Methods:

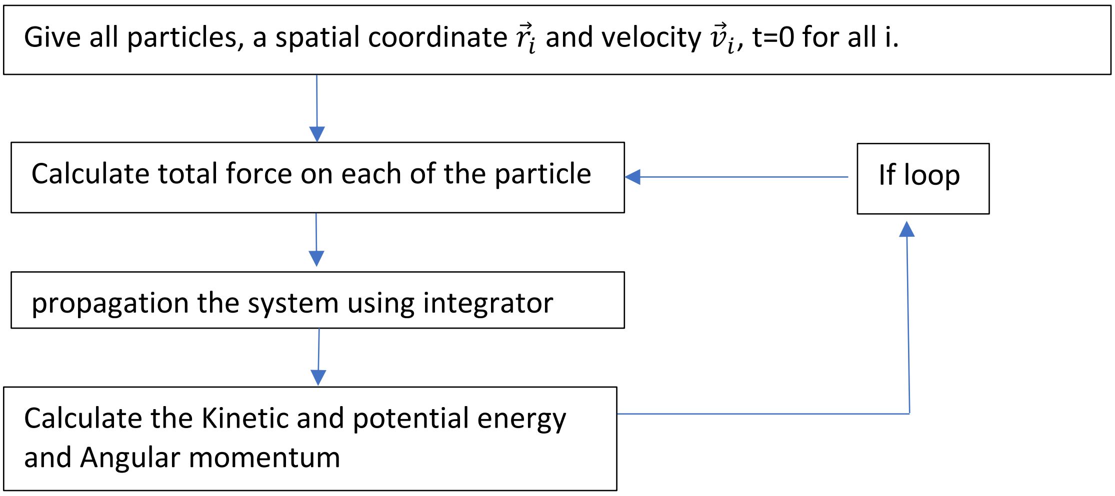
Numerical methods involve discretising the continuous equation of motion of the system and solving the resulting set of equations using a computer algorithm. We will be using three types of integrator which are Euler's method, Velocity-Verlet method and Runge-Kutta method.

The Euler method is a numerical method that approximates the solution at each time step using the slope of the tangent to the current numerical solution. This method is the fastest and is computationally the least expensive but is inevitably unstable for large time steps and the error of the solution is directly proportional to the time interval Δt.

The Velocity-Verlet method is a numerical method that is commonly used to calculate the trajectory of particle in molecular dynamics simulations involving Newtonian 's equation of motion. It approximates the solution at each time step much better than Euler method as it uses a higher order approximation and average out the slope of the tangent to the solution curve at the current and the next point. This method is energy conserving and time reversible which is important as how well the solution obeys the conservation laws determine the stability of the solution.

The Runge-Kutta (RK) method is a numerical method that involves solving a set of differential equations using a series of weighted approximations and hence averages the slope of the tangent used to approximate the next numerical solution. This method is accurate and stable but is computationally most expensive for large N as there are additional weighted coefficients to calculate.

Computer Simulation Algorithm



4. Simulation

Two-body problem
Different Mass

We consider system, like a moon orbiting a planet, having a large mass ratio and treat the problem as a central force problem where the large mass does not move. We know from classical mechanics that in order for a stable orbit to occur, we need the centripetal force on the lighter mass body to be equal to the gravitational force hence we can find the initial condition. We set the initial position of the lighter mass at (x,y)=(0,1) and found that for a circular path the velocity of the lighter mass is (v_x,v_y)=(1,0) and fix the heavier mass at the origin. We will propagate the system using Euler method and RK4 method then compare the stability of the solution.

$$\frac{\mathrm{m_2 v^2}}{\mathrm{r_2}} = \frac{\mathrm{Gm_1 m_2}}{\mathrm{r_{12}}} \rightarrow \mathrm{v_2} = \sqrt{\frac{\mathrm{Gm_1 r_2}}{\mathrm{r_{12}}}} = 1$$

$$\mathrm{KE} = \frac{1}{2}\mathrm{m_2 v_2^2}$$

$$\mathrm{PE} = \frac{\mathrm{Gm_1 m_2}}{\mathrm{r_2}}$$

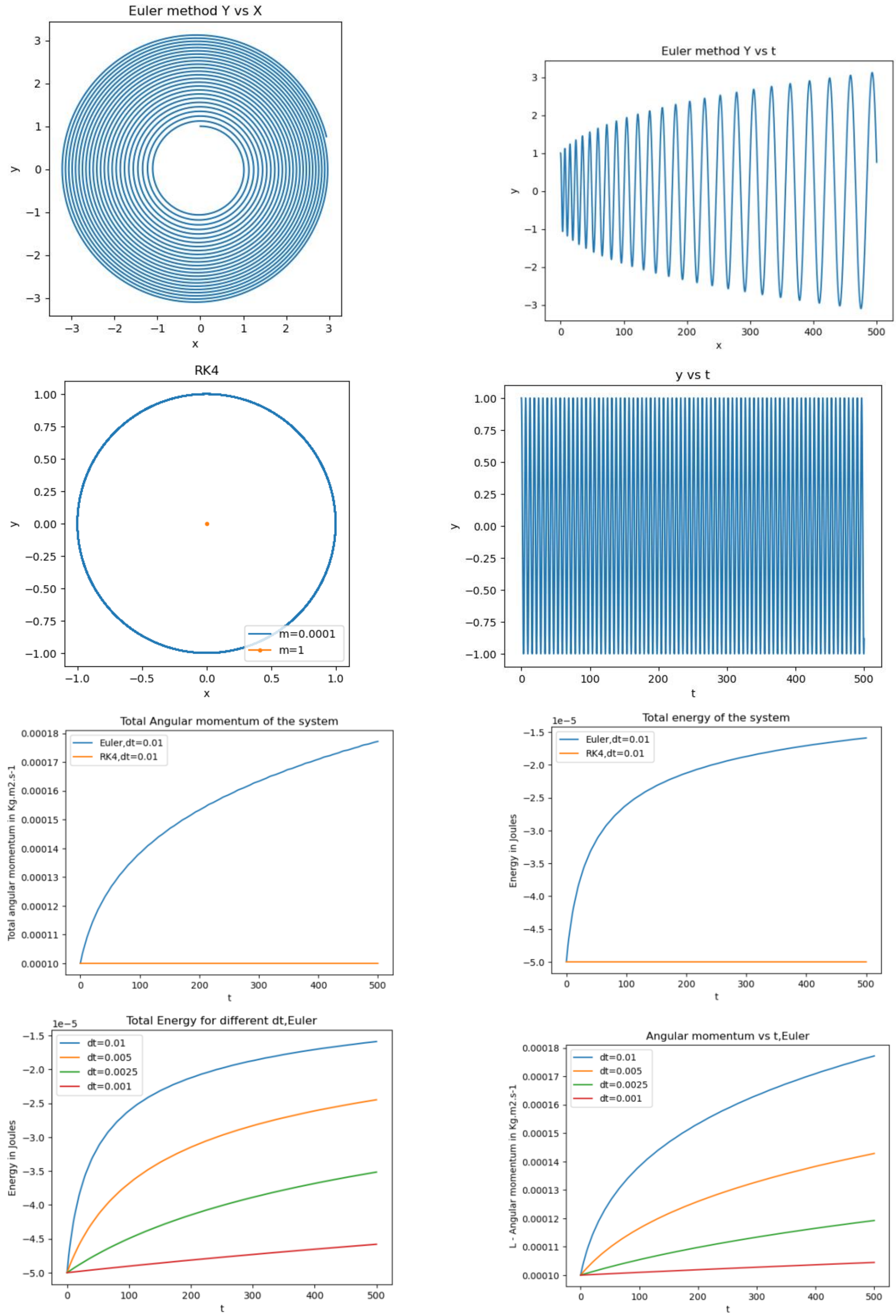


Fig1.
Column 1 shows the result of Euler method for the two-mass problem. The left graph shows the trajectory of light mass around the heavy mass while the right graph shows the time evolution of the Y-component of the light mass.
Column 2 shows the result of Runge-Kutta (order 4) method for the two-mass problem. The left graph shows the trajectory of light mass around the heavy mass while the right graph shows the time evolution of the Y-component of the light mass.
Column 3 shows how well each method obeys the conservation laws throughout the evolution of the system.
Column 4 shows the effect of reducing dt (from 0.01 to 0.001) on the stability of the system.

Similar Mass

We will consider two bodies with similar masses moving about their centre of mass. We will be using the Velocity-Verlet method to propagate the initial setup which is set such that both of the masses follow a circular orbit. To simplify the problem, we place the centre of mass of the system at the origin and set the distance between the bodies as unity and place the two bodies initially on the x-axis.

$$\mathrm{x_{com}} = \frac{\mathrm{m_1x_1 + m_2x_2}}{\mathrm{m_1+m_2}} \rightarrow \mathrm{m_1x_1 + m_2x_2 = 0}$$

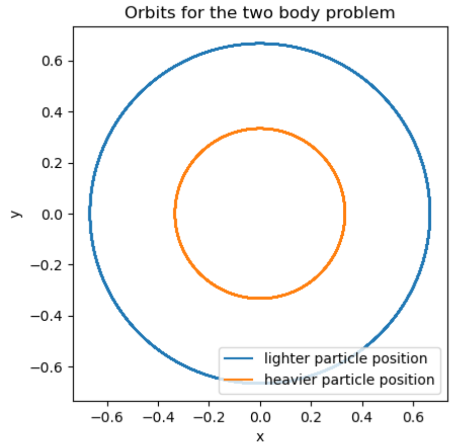
$$\frac{\mathrm{m_1v_1}}{\mathrm{r_1}} = \frac{\mathrm{Gm_1m_2}}{\mathrm{r_{12}}} \text{ and } \frac{\mathrm{m_2v_2}}{\mathrm{r_2}} = \frac{\mathrm{Gm_1m_2}}{\mathrm{r_{12}}}$$

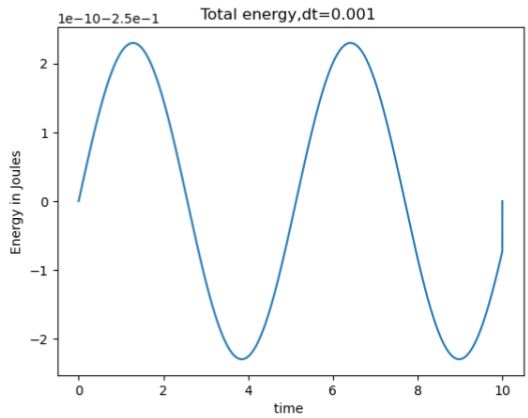
Solving the simultaneous equation above, we find that the initial positions and velocities is given as:

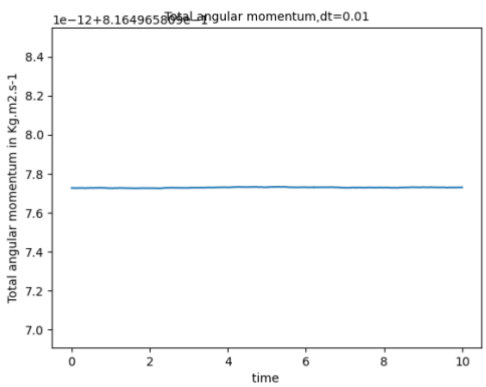
$$\mathrm{x_1} = \frac{\mathrm{m_2}}{\mathrm{m_2 + m_1}}; \mathrm{x_2} = \frac{\mathrm{m_1}}{\mathrm{m_1 + m_2}}$$

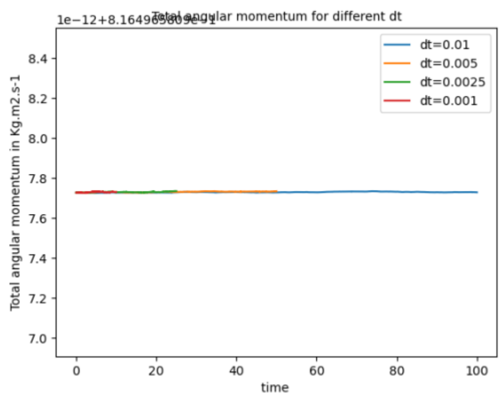
$$\mathrm{v_1} = \sqrt{\frac{(\mathrm{m_2})^2}{\mathrm{m_2 + m_1}}}; \mathrm{v_2} = \sqrt{\frac{(\mathrm{m_1})^2}{\mathrm{m_1 + m_2}}}$$

The velocities of the bodies have been set as $(v_{1,x}, v_{1,y})$=$(0, -\mathrm{v_1})$ and $(v_{1,x}, v_{1,y})$ = $(0, \mathrm{v_2})$









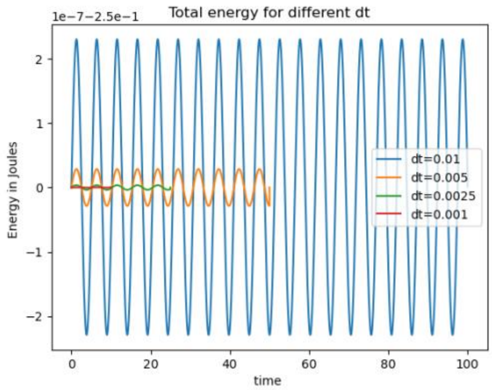


fig2.
The upper-left graph is the trajectory of the two bodies throughout the time evolution of 100 second.
The upper-right graph is the total energy of the system throughout the time evolution of 100 second.
The bottom-left graph is the total energy of the system throughout the time evolution of 100 second.

Discussion on Two-body problem

As time progresses, the plots of total angular momentum and energy against time, which is shown in fig1, have shown that the Euler method does not obey conservation laws which means it is not stable for long duration of simulation. The size of dt also play an important role in the accuracy of the propagated path. In particular, the smaller the size of dt is, the less fluctuation in total energy and total angular momentum (shown in fig1) which becomes increasing realistic as a physical system as the conservation laws are obeyed. But the cost of decreasing dt is a heavier computational power because more steps is required to propagate the same distance when comparing to a higher dt. The Euler method is in sharp contrast with the RK4 method as the total angular momentum and energy (shown in fig1) remains, in effect, constant throughout the simulation of the two-body problem because it has an error in the order of -14 compares to Euler method having an error in order of -5. Hence the RK4 method obeys conservation laws much better and is a better numerical solution to the two-body problem compares to Euler method.

In two similar mass case, we have used the Velocity-Verlet method and importantly the total energy has a stable oscillatory behaviour which indicates energy conservation even for long simulation. Surprising, the size of dt does not affect the total angular momentum as shown in Fig2. This proves to be a better method than the RK4 method because in effect the error does not accumulate as time goes on but fluctuates in a small range.

Three body problem

We will be considering the circular restricted three-body problem where we have a massive object (assumed to remain stationary) at the origin (barycentre) and the second object orbiting the stationary massive object while a third lightest object orbiting around the second object. This is similar to the Sun, Earth and moon system. To simplify the problem, we consider the following assumption: (a) All three object is located along the positive axis initially (b) ignore the effect of the third mass on the second mass (c) ignore the effect of the first mass on the third mass (d) the initial distance between the first and second mass is unity (e) the initial distance between the second and third mass is 0.0025. Using Classical mechanics, we are able to find the initial state of the system for a stable orbit.

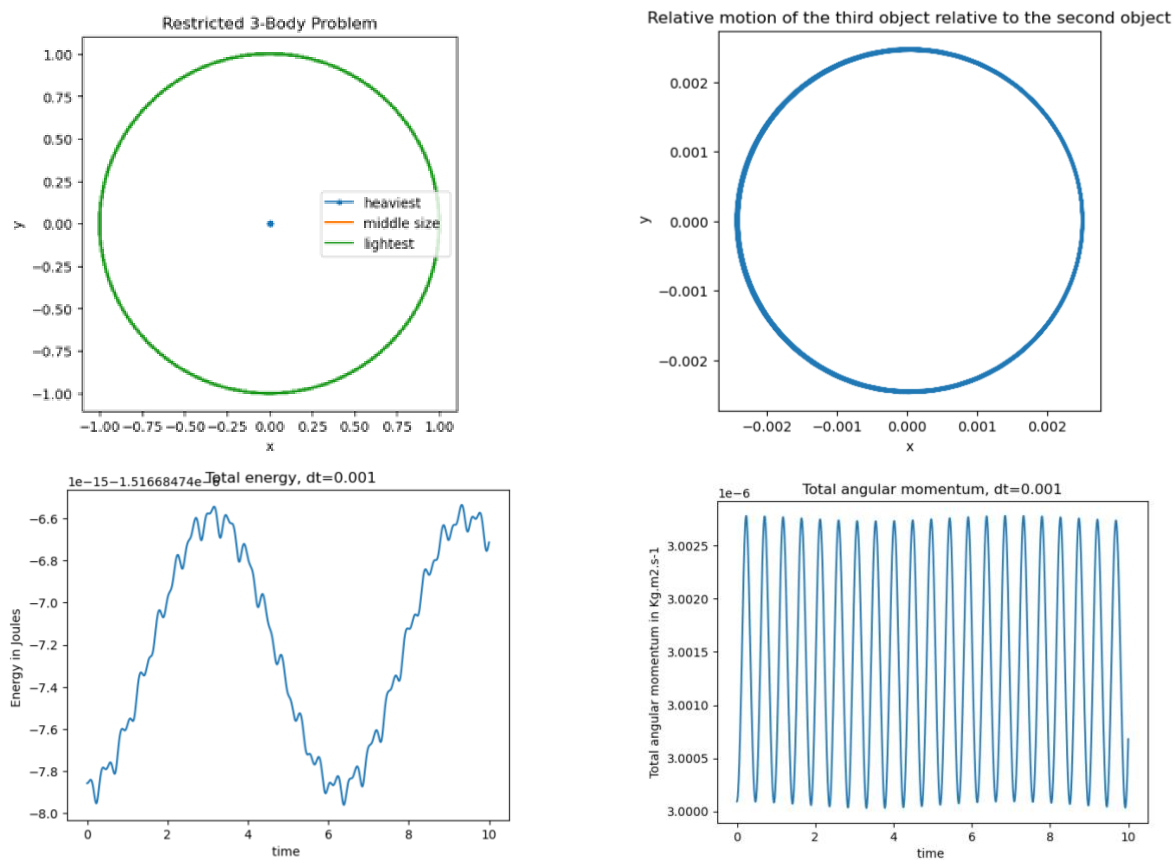


fig3. Result of restricted 3-Body Problem

Discussion on simplified Three-Body Problem

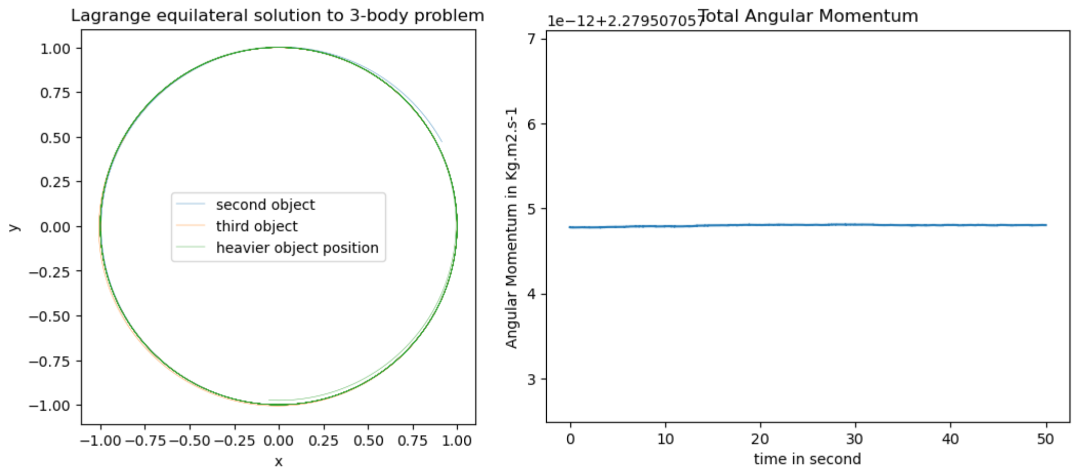
Since the distance between the second and third mass is so small, the relative motion graph (in fig3) indicates that the third mass is actually orbiting circularly around the second mass. Using the Velocity Verlet integrator, we are able to show that the simulated physical system consistently obeys conservation laws to orders of -11 and -13 for total angular momentum and total energy of the system respectively. This indicates the simulation is stable under the 100 second. To improve our simulation of the three-body problem we could consider grouping the heaviest mass with the second mass into a 2-body system then consider them orbiting around their centre of mass and then consider the lightest mass orbiting around the second mass.

Choreographies

N-body choreographies are periodic solutions to the N-body problem in which N equal masses move along the same trajectory around a fixed closed curve, equally spaced in phases along the curve. Examples of N-body choreographies includes Lagrange's equilateral solutions, figure-eight solution of the three-body problem [2] and Braids solution of the N-body problem (shapes which extend the figure-eight solution to more bodies) [3]. For the rest of our paper, we will demonstrate how each of these solutions are simulated using Velocity Verlet method and test for stability of the orbits.

Lagrange's equilateral solutions

The simplest solution is the 3-body (with equal mass) motion along a common unit circle. The mechanism of this motion can be thought of pictorially as an equilateral triangle rotating with a uniform angular velocity inside a unit circle and each of the mass is always at the vertices of the equilateral triangle. The notion of equilateral triangle is used to ensure the masses are equally spaced along the circle. I have used trigonometry to place three masses equally space along a unit circle and calculated the initial velocity of each mass using classical mechanics and evolve the system using Velocity Verlet integrator. The result of the simulation is shown in Fig4 which includes a plot of the trajectory of the masses and total energy and angular momentum. After this, we apply a small perturbation to the initial state of the system which is done by increasing the initial magnitude of the velocity for every mass by 20% compared to the magnitude of velocity before. The result of the simulation is shown in Fig5.



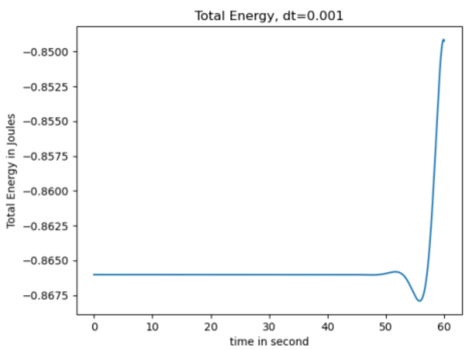
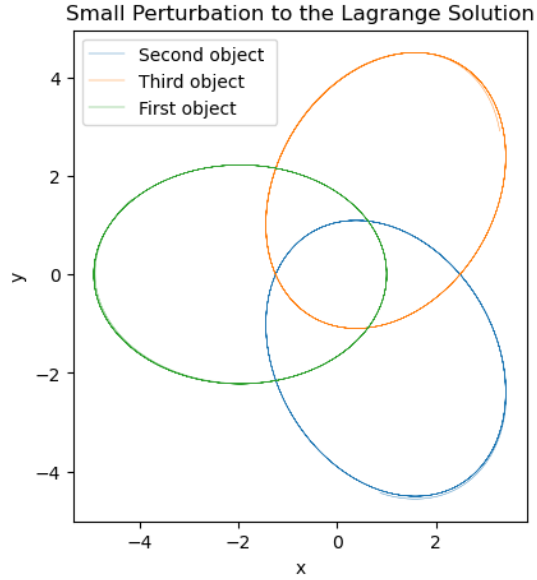
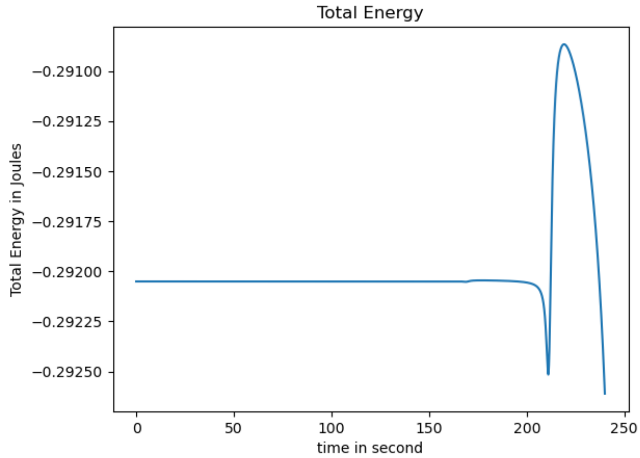


Fig4.
Results for the 3-Body Lagrange's equilateral solutions





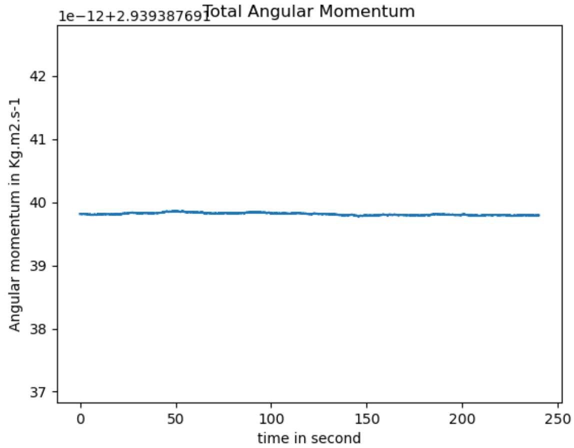


Fig5.
Result of the Perturbed Lagrange's Equilateral Solution

Discussion about the Lagrange's Equilateral Solution

For the lagrange's equilateral solution, the orbit remains stable until roughly 50,000 time step (around 50 second) with time interval dt=0.001 and total energy and angular momentum fluctuation in order of -7 and -12 respectively. After 50,000 time step, error suddenly fluctuate with great magnitude leading to the break down of the choreography. The reason might be due to the fact that system is a chaotic system which is highly sensitive to any change within the system. Surprising, by introducing a small perturbation to the system's initial condition, we can observe a longer stable orbit but however lose the choreography phenomenon. The additional perturbation causes the equilateral triangle (that have been mention before) to change in size and rotate with a constant angular momentum (as shown in Fig5.). The trajectory of each mass can be tracked by locating each vertices of the triangle. Fig.6 below illustrate the analogy used to describe the trajectory of each mass.

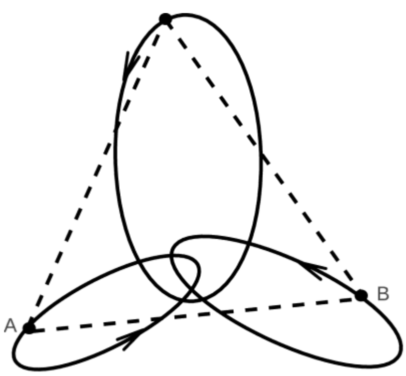
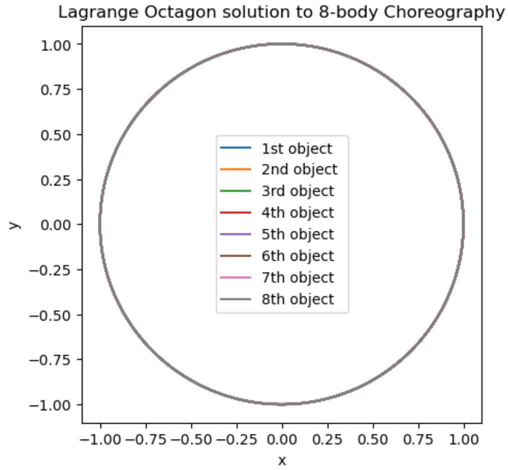


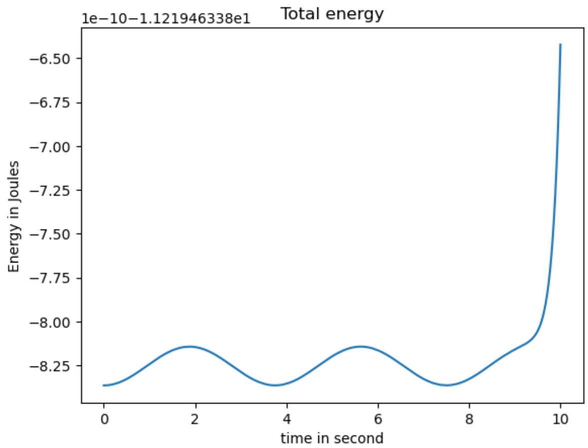
Fig6.
Triangle Analogy [4]

Extended Lagrange's solution

In theory, Lagrange's equilateral solution can be extended to N-body choreography simulation by using a regular N-gon instead of an equilateral triangle. The analogy still applies as before but in this case, it is the regular N-gon that is rotating with a constant angular velocity within a unit circle. We will simulate the 8-body Choreography moving along a common unit circle. Initially, we place 8 masses (with same mass) equally spaced along a unit circle and propagate the initial system using Velocity Verlet. The total potential energy of the system is given below and contains 28 terms.

$$PE = \sum_{i \neq j}^{N} \frac{1}{r_{ij}}$$





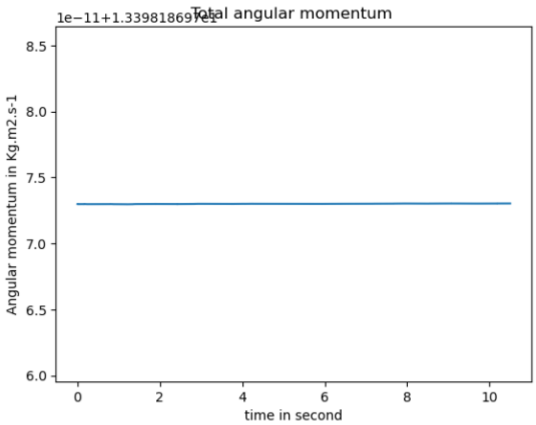


Fig7

Result of Lagrange Octagon solution to 8-body choreography

Discussion on extended Lagrange's solution

The Stability of the 8-body choreography break down after 10500 timestep (roughly 10.5 seconds) as shown in the total energy and total angular momentum plot (in Fig7). From the total energy graph, we can observe a time period of roughly 3800 timestep corresponding to 3.8 seconds. The stability, in terms of timescale, is nearly a fourth of the Lagrange's equilateral solution which indicates that the N-body Choreography system decreases in stability as the number of body increases due to the system becoming sensitive to change within the system.

8-Figure Choreography

In this section, we will try to simulate the orbit of the 8-figure 3 body choreography [2] system by using the parameter E(x0) given by the research paper [3] which are the initial condition needs to be satisfied in order to obtain the figure-eight system. The figure-eight system have properties such as (a) the sum of the positional vector of the bodies is zero (b) similar the sum of the velocity of the bodies is zero. Using these properties, we can set the initial condition according. $E(v,u)$ gives the initial condition:

$$E(v,u) = (1,0,-1,0,0,0,v,u,v,u,-2v,-2u) \quad \text{with } v = 0.347116768716, u = 0.532724944657$$

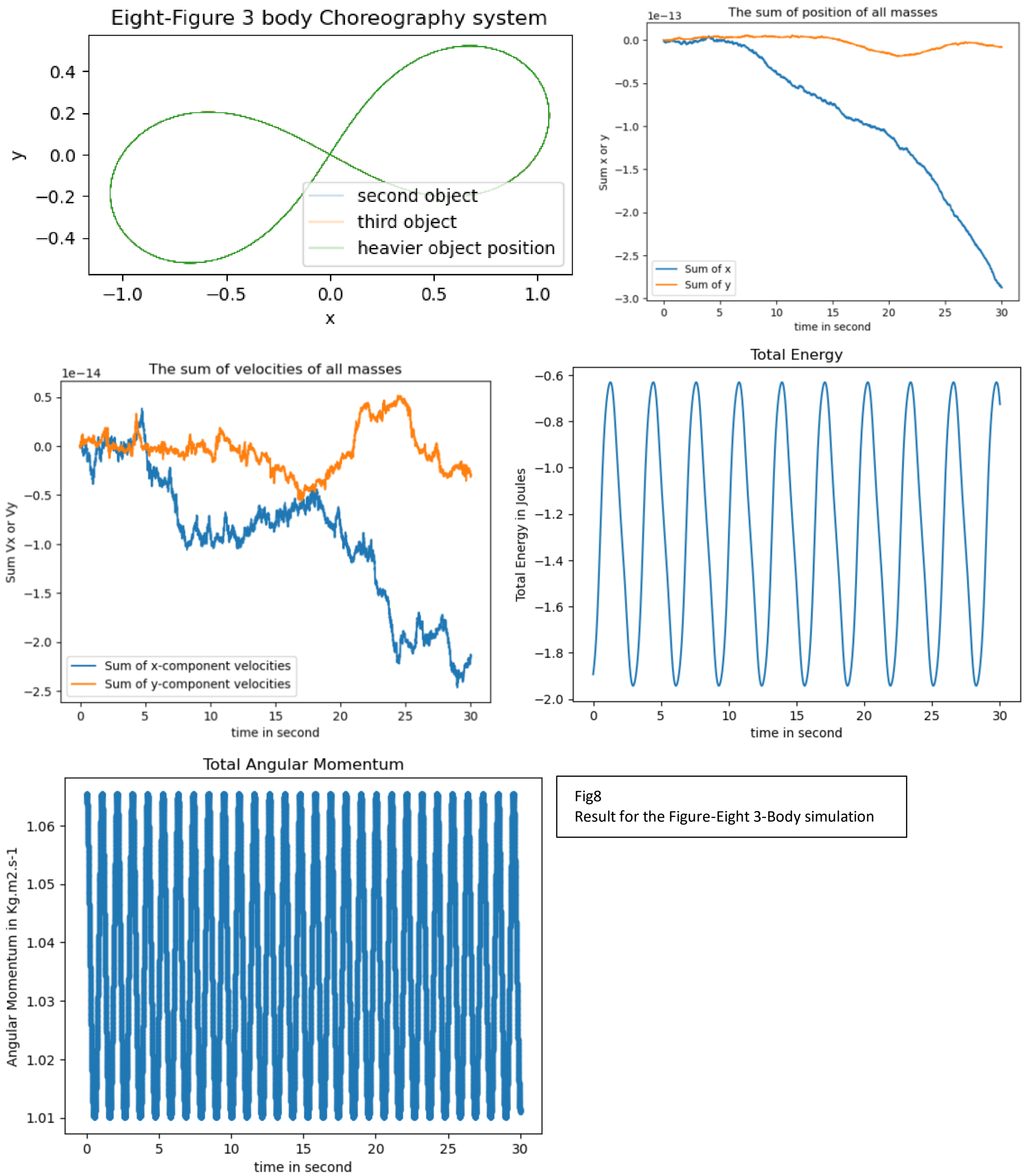


Fig8
Result for the Figure-Eight 3-Body simulation

Discussion on Figure-Eight 3-Body Choreography

Surprisingly, the system is stable even up to 500,000 timestep with dt=0.001 (around 500 second). Total energy and angular momentym is conserving although the total angular momentum is fluctuating larger than the circular 3-body problem which is expected as the bodies' velocity and positional vector are not always perpendicular especially when it is approaching the origin. Through the plot of the sum of velocities and positional vector of the bodies, we can observe a symmetry in the system which is a requirement for a stable Figure-Eight Choreography [3]. Afterall, this simulation is a very stable numerical solution.

Braids Choreography

We will now stimulate the 4-body and 6 body Braids Choreography using Velocity Verlet. The initial condition is taken from a research paper [3]. The parameters and values , for 4-body Braids, are given as:

$$E(x_1, \dot{x}_0, \dot{y}_1) = (0, a, \dot{x}_0, 0, x_1, 0, 0, \dot{y}_1, 0, -a, -\dot{x}_0, 0, -x_1, 0, 0, -\dot{y}_1)$$

$$x1 = 1.382857 \; ; \; x0\_v = 1.87193510824$$

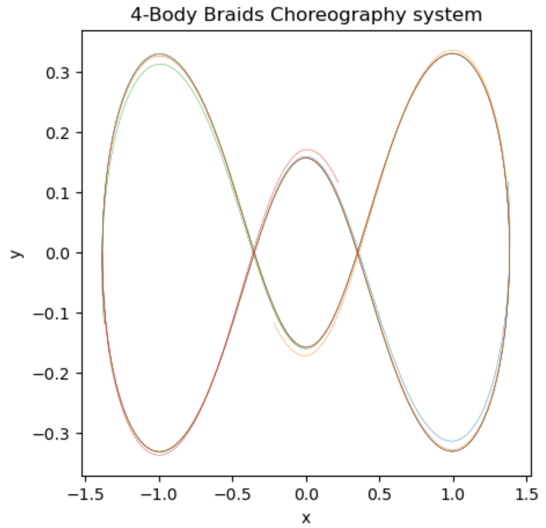
$$y1\_v = 0.584872579881 \; ; a = 0.157029944461$$

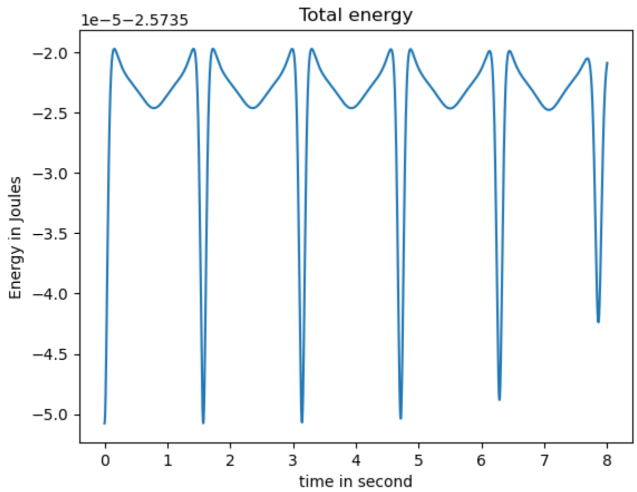
The parameters and values, for 6-body Braids, are given as and importantly need to use symmetry to assign the right initial condition for each mass:

$$E(\dot{x}_0, x_1, y_1, \dot{x}_1, \dot{y}_1) = (0, a, \dot{x}_0, 0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_1, -y_1, -\dot{x}_1, \dot{y}_1)$$

$$x0\_v = -0.635277524319 \; ; \; x1 = 0.140342838651 \; ; \; y1 = 0.797833002006$$

$$x1\_v = 0.100637737317 \; ; \; y1\_v = -2.0315222786 \; ; \; a = 1.887041548253914$$





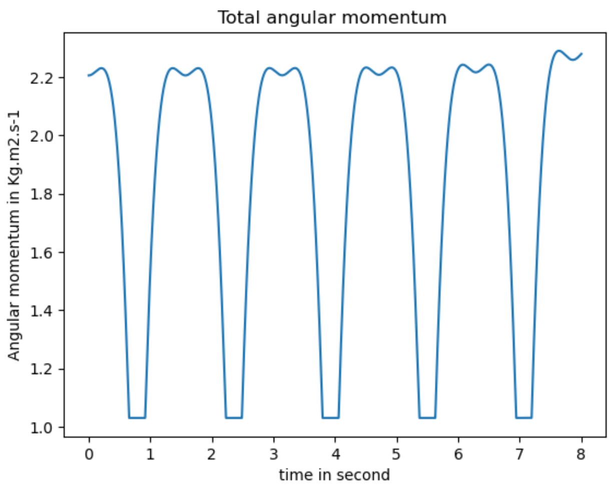
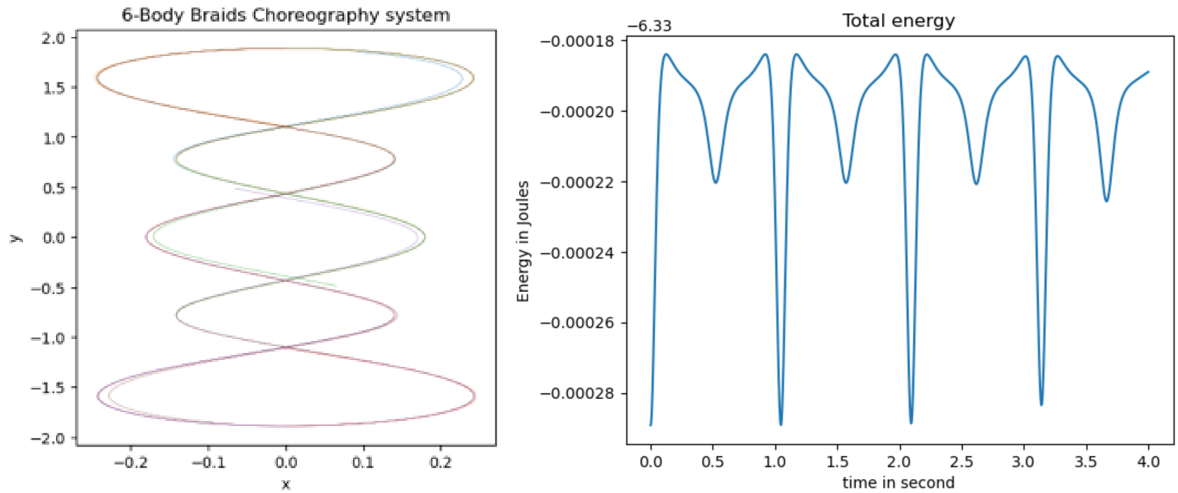


Fig9.
Simulation of the 4-Body Braids Choreography



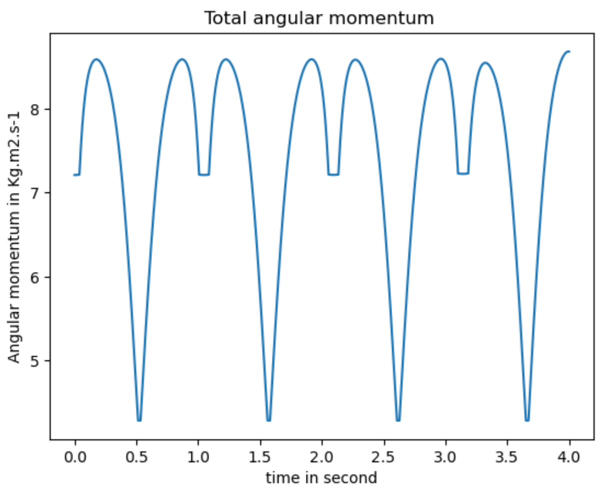


Fig10.
Simulation of the 6-Body Braids Choreography

Discussion about the Braids Choreography System

The 4-Body Braids system is stable only up to 8,000 timestep (8 seconds with dt=0.001) with time period of around1.6 second while the 6-Body Braids system is stable up to 4,000 timestep (4 seconds) with time period of around 1 second. The fluctuation in both total angular momentum and energy is bigger for the 6-Body Braids system therefore the 6-Body Braids system is less stable.

The Effect of small perturbation on Figure-Eight and Braids Choreography

By applying a small perturbation via increasing the initial velocity by a certain percentage of its original, we will test how sensitive each system is to the small change. We observe that the Figure-Eight 3-body choreography remains stable up to 2% increases of its initial velocity, 0.05% for 4-body Braid Choreography and 0.01% for 6-body Braid Choreography. This indicates that the number of object within a system increases the sensitivity of the system to any change of initial condition hence a more chaotic system.

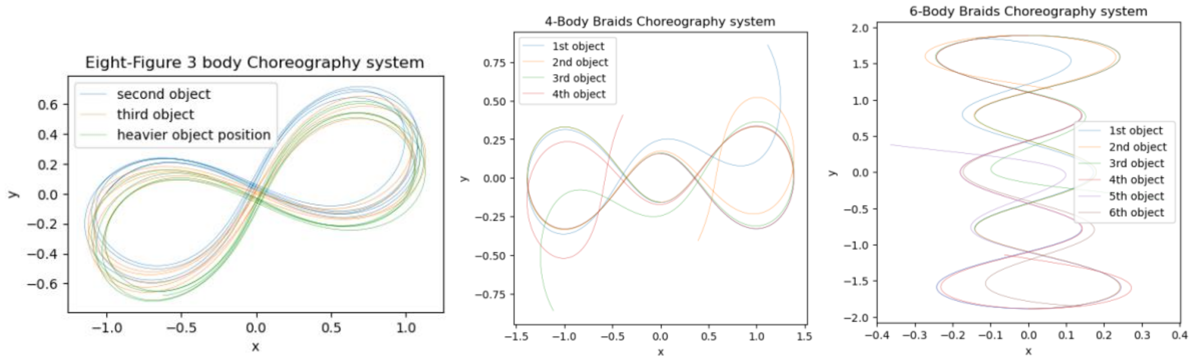
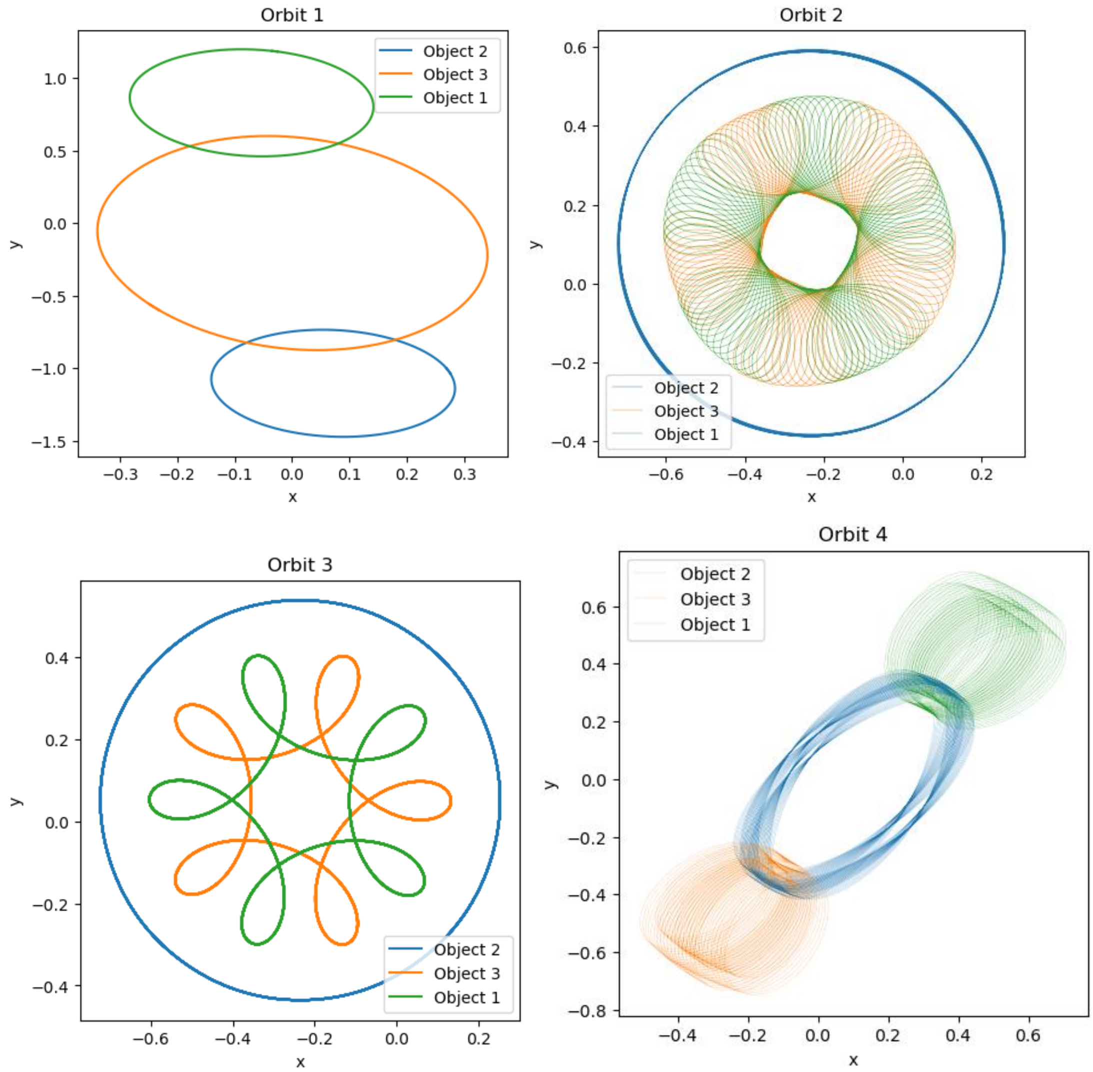


Fig11.
Result of each Choreography after subject to a small perturbation but remain relatively stable

Other Three-Body Solution

In Fig12, there are simulation of other three-body solution[4][5] which differs from the simple choreography and is highly sensitive to initial condition and have a relatively low stability in terms of timescale.



5. Future Exploration

We have only cover the stable choreography in a planar configuration. There are choreography which extends into a non-planar 3D configuration which requires an additional coordinates.

Conclusion:

Our simulation generally reflects the behaviour of many body interacting garvitationally. The general trend for stability is that the more body the system contains the lower the stability of the simulated orbit. The timescale for stability is entirely down to the type of integrator we use and the time interval for which we propagate the system forward. We have also show that the conservation laws is a key factor which can characterise the stability of a simulated orbit.

For applying a small perturbation to some of the simulated orbit, we observe that orbits having less stability is more prone to change sensitively to any small perturbation which is indicated in the choreography system.

The choreographies are interesting and impressing specially due to the fact that Poincare proof the theorem that N-body in Newtonian gravity is a chaotic system [r] but we can still observe symmetry, regularity and stability in a limited timescale.

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