SYNTHESIS OF STABILIZING PID CONTROLLERS FOR BIOMECHANICAL MODELS

Anindo Roy[1], Kamran Iqbal[2]

[1]Department of Applied Science, University of Arkansas at Little Rock
[2]Department of Systems Engineering, University of Arkansas at Little Rock

Abstract: A framework for synthesis of stabilizing PID controllers for linear time-invariant systems using Hermite-Biehler Theorem is presented. The approach is based on the analytical characterization of the roots of the characteristic polynomial. Generalized Hermite-Biehler Theorem from functional analysis is used to derive stability results, leading to necessary and sufficient conditions for the existence of stabilizing PID controllers. An algorithm for the selection of stabilizing feedback gains using root locus techniques and Linear Matrix Inequalities (LMI) is presented.

Keywords: Biomedical systems, PID control, Stability analysis, Feedback systems, Hurwitz polynomial.

1. INTRODUCTION

Feedback control aims to design fixed-order controllers that would, taking into account the system dynamics, stabilize the closed-loop system and meet certain performance criteria. Among many of the issues that arise during the life cycle of system design, one of the first, and probably the most critical one, is stability of the feedback loop. The problem of stabilizing a given process by suitable low fixed-order compensator, and attempt to design optimal and robust controllers with constrained order, is a challenging one. Based on the methodology used to design the stabilizing controller and the manner in which the system is modeled, numerous approaches are available for stability analysis and system design; however, most lack complete analytical characterization.

Algebraic approaches to stability analysis and controller synthesis draw motivation from the Hermite-Biehler, Kharitonov, Edge, and Lipatov theorems (Bhattacharyya, *et al.*, 1995; Kharitonov, 1978; Bartlett *et al.*, 1988; Lipatov and Sokolov, 1978). Such methodologies use polynomials (or polynomial matrices) to provide conditions for the stability of linear feedback processes, or a family of processes (interval plants). Intelligent use of these theorems allows the simultaneous design of the characteristic polynomial of the closed-loop and synthesis of the controller. Furthermore, they offer the advantage of analytical characterization and fixed-order controller synthesis, which most classical control methods fail to address. For example, classical control techniques in frequency domain such as Evans root locus and Nyquist stability criteria are graphical in nature and fail to provide any analytical characterization of the stabilizing compensator parameters. The Routh-Hurwitz criterion on the other hand, does provide an analytical solution to the stability problem; however, the set of stabilizing compensators can only be determined by solving a set of nonlinear inequalities, a task that may become cumbersome for high-order processes. Similarly, other methods such as YJBK parameterization (Youla *at al.*, 1976) can be used to parameterize all proper feedback controllers that stabilize a given process, and minimize a specific ∞-norm, but the disadvantage of such an approach is that the controller order must be constrained. Nevertheless, the Hermite-Biehler framework can be used to design optimal constant gain controllers that minimize the ∞-norm to a value that is a reasonably good approximation to the unattainable infimum. Furthermore, when used in conjunction with the Kharitonov and Edge theorems, the set of all stabilizing controllers for a family of processes can be found, an issue that is central to the robust control theoretic framework.

In this paper, we present an analysis-synthesis framework for linear time-invariant systems based on the generalized Hermite-Biehler Theorem (Ho *et al.*, 1999; Ho *et al.*, 2000). The generalized Hermite-Biehler Theorem is used in functional analysis to study the stability properties of real polynomials defined over complex fields, and provides conditions for the Hurwitz stability of real polynomials. Since characteristic polynomials in linear systems with feedback are real polynomials, the Hermite-Biehler Theorem, in addition to providing information about stability also provides an elegant, easy, and analytical way of characterizing the set of all stabilizing controllers for a given process. The analytical framework developed in the paper is generic, and has been modified to apply to low order plants with feedback delays (Roy and Iqbal, 2003a). We may, however, point out that the analysis presented in this paper only concerns the stability and not the performance aspects of the system.

The stability of the closed-loop system is analyzed using generalized Hermite-Biehler Theorem in a manner that enables characterization of the stabilizing set of controller gains. It is shown how the generalized Hermite-Biehler Theorem can be used not only to derive conditions for the existence of the set of stabilizing compensators but also as a convenient and elegant analytical method to design compensators for linear control systems (Datta *et al.*, 1999). The stability problem is solved in (Datta *et al.*, 1999) for PID controllers for processes without time-delay, while in (Roy and Iqbal, 2003b) a modified and extended solution is presented for unstable processes with transport lags in the feedback path. The PID stabilization problem for first-order-plus-dead-time (FOPDT) and a fourth-order process are solved in (Roy and Iqbal, 2003a) and (Iqbal and Roy, 2002), respectively. The solution to the PI stabilizing problem can be found in (Datta *et al.*, 1999).

The organization of this paper is as follows: In Section 2, we present the stability analysis of the model in the Hermite-Biehler framework and develop an algorithm to synthesize stabilizing PID controllers. Applicability of the results is illustrated in Section 3 with a design example. Finally, conclusions are given in Section 4.

2. PROBLEM FORMULATION

In this section we develop a framework for stability analysis of a linear time-invariant system in unity gain feedback configuration based on the application of the generalized Hermite-Biehler Theorem. The system is shown in Fig. 1. Let $G_p(s) = W(s)/Q(s)$, where $W(s)$ and $Q(s)$ are relatively prime, and $G_c(s) = n_c(s)/d_c(s)$, where for simplicity we assume a PID controller given by $n_c(s) = K_d s^2 + K_p s + K_i$, $d_c(s) = s$; then, the closed-loop characteristic polynomial $\psi(s)$ is given as

$$\psi(s) = sQ(s) + (K_i + s^2 K_d)W(s) + sK_p W(s). \quad (1)$$

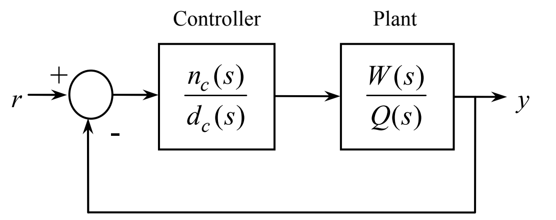


Fig. 1. Block diagram of a typical single-channel, unity feedback control system with a PID controller.

In the following, we first state the generalized Hermite-Biehler Theorem and then develop a framework for controller synthesis. For proof of the Theorem the reader is referred to (Ho *et al.*, 1999; Ho *et al.*, 2000).

2.1 The Generalized Hermite-Biehler Theorem

Generalized Hermite-Biehler Theorem (Ho *et al.*, 1999; Ho *et al.*, 2000): Let $\delta(s) = \sum_{i=0}^{n} \delta_i s^i$, $\delta_i \in \mathbb{R}\, \forall i$ with a root at the origin of multiplicity k. Writing $\delta(s) = \delta_e(s^2) + s\delta_o(s^2)$, where $\delta_{e,o}(s^2)$ are the components $\delta(s)$ made up of even and odd powers of s, respectively. For every $\omega \in \mathbb{R}$, denote $\delta(\omega) = p(\omega) + jq(\omega)$ where $p(\omega) = \delta_e(-\omega^2)$, $q(\omega) = \omega\delta_o(-\omega^2)$, and let ω_{ej} denote the real, nonnegative, and distinct zeros of $\delta_e(-\omega^2)$ and let ω_{ok} denote the real, nonnegative, and distinct zeros of $\delta_o(-\omega^2)$ $\forall j,k$, both arranged on ascending order of magnitude. Let $0 < \omega_{o1} < \omega_{o2} < \cdots < \omega_{om-1}$ be the zeros of $q(\omega)$ that are real, distinct, and nonnegative. Also, define $\omega_0 = 0$, $\omega_{om} = \infty$, $p^{(k)}(\omega_0) = \left(\frac{d^k}{d\omega^k} p(\omega) \right)\Big|_{\omega=\omega_0}$. Then $\forall m \in \mathbb{Z}_+$

$$\begin{aligned}\sigma[\delta(s)] = (-1)^{m-1}\{&\operatorname{sgn}[p^{(k)}(\omega_0)] \\ &+2\sum_{i=1}^{m-1}(-1)^i \operatorname{sgn}[p(\omega_{oi})] \\ &+(-1)^m \operatorname{sgn}[p(\omega_{om})]\}\cdot \operatorname{sgn}[q(\infty)],\ n = 2m,\end{aligned}$$

$$\begin{aligned}\sigma[\delta(s)] = (-1)^{m-1}\{&\operatorname{sgn}[p^{(k)}(\omega_0)] \\ &+2\sum_{i=1}^{m-1}(-1)^i \operatorname{sgn}[p(\omega_{oi})]\}\cdot \operatorname{sgn}[q(\infty)],\ n = 2m+1,\end{aligned}$$

where $\sigma[\delta(s)]$ denotes the signature of the polynomial defined as: $\sigma[\delta(s)] \triangleq n_\delta^{(L)} - n_\delta^{(R)}$, where $n_\delta^{(L)}$ and $n_\delta^{(R)}$ are the number of open left-half plane (LHP) and right-half plane (RHP) roots.

2.2 Stability Analysis

In order to develop a stability framework based on the Hermite-Biehler Theorem, we proceed as follows: define the signature of $\psi(s)$ as $\sigma[\psi(s)] \triangleq n_\psi^{(L)} - n_\psi^{(R)}$ and the order of $\psi(s)$ as $n_\psi^{(L)} + n_\psi^{(R)} \triangleq \Theta[\psi(s)]$; then, from a stability perspective, if $\psi(s)$ is Hurwitz, then $n_\psi^{(R)} = 0$ (no RHP poles), or equivalently, $\sigma[\psi(s)] = \Theta[\psi(s)] = m_\psi$ (i.e., if the system is Hurwitz stable, then the signature equals the order of the characteristic polynomial). In order to apply the generalized Hermite-Biehler Theorem, we decompose $W(s)$ and $Q(s)$ into polynomials with even and odd powers of s. To this effect we let $W(s) = W_e(s^2) + sW_o(s^2)$, $Q(s) = Q_e(s^2) + sQ_o(s^2)$. Also, define $W^*(s) \triangleq W(-s) = W_e(s^2) - sW_o(s^2)$ and let

$$\begin{aligned}\delta(s) &\triangleq \psi(s)W^*(s) = sQ(s)W^*(s) \\ &+ (K_i + s^2 K_d)W(s)W^*(s) + sK_p W(s)W^*(s)\end{aligned} \tag{3}$$

then, it can be verified that $\sigma[\delta(s)] = \sigma[\psi(s)W^*(s)] = \sigma[\psi(s)] - \sigma[W(s)]$.

Further, if $\psi(s)$ is Hurwitz stable, then $\sigma[\psi(s)W^*(s)] = m_\psi - \sigma[W(s)]$. Now, substituting $s = j\omega$ in $\delta(s)$ we obtain

$$\delta(\omega) = \psi(\omega)W^*(\omega) = p(\omega) + jq(\omega) \tag{4}$$

where

$$p(\omega) \triangleq p_1(\omega) + (K_i - K_d\omega^2)p_2(\omega) \tag{5}$$

$$q(\omega) \triangleq q_1(\omega) + K_p q_2(\omega) \tag{6}$$

and the polynomials $p_1(\omega), p_2(\omega), q_1(\omega)$, and $q_2(\omega)$ are given as:

$$p_1(\omega) \triangleq \omega^2[Q_e W_o - Q_o W_e],\ p_2(\omega) \triangleq [W_e^2 + \omega^2 W_o^2],$$

$$q_1(\omega) \triangleq \omega[W_e Q_e + \omega^2 W_o Q_o], q_2(\omega) \triangleq \omega\, p_2(\omega) \tag{6}$$

We note that Eqs. (5) and (6) provide a decoupling of the position gain, K_p, from the velocity and integral gains, K_d and K_i. This structure will be exploited to develop a synthesis procedure for PID controllers later in the section. For now, in order to develop stability characterization using the generalized Hermite-Biehler Theorem, let $m_q \triangleq \Theta[q(\omega)]$ and $\tilde{m}_q \le m_q$ be the number of real, nonnegative, and distinct zeros of $q(\omega)$ with odd multiplicities that satisfy the following condition for $K_p \in (-\infty, \infty)$:

$$0 = \omega_0 < \omega_{o1} < \omega_{o2} < \cdots < \omega_{o\tilde{m}_q - 1} < \omega_{o\tilde{m}_q} = \infty .$$

Furthermore, define $\rho \triangleq m_\psi - \sigma[W(s)]$, $\alpha \triangleq \frac{1}{2}[m_\psi + m_W]$, and $\gamma \triangleq (-1)^{\tilde{m}_q - 1}\operatorname{sgn}[q(\infty)]$, then application of the generalized Hermite-Biehler Theorem to Eq. (4) leads to the following:

$$\begin{aligned}\rho = \{&\operatorname{sgn}[p(0)] + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \operatorname{sgn}[p(\omega_{oi})] \\ &+ (-1)^{\tilde{m}_q}\operatorname{sgn}[p(\omega_{o\tilde{m}_q})]\}\cdot\gamma,\ \alpha \in \mathbb{Z}_+,\end{aligned} \tag{7}$$

$$\rho = \{\operatorname{sgn}[p(0)] + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \operatorname{sgn}[p(\omega_{oi})]\}\cdot\gamma,\quad \alpha \notin \mathbb{Z}_+, \tag{8}$$

where $\mathbb{Z}_+$ defines the set of positive integers. Substituting Eq. (5) in Eqs. (7) and (8) we obtain

$$\begin{aligned}\rho = \{&\operatorname{sgn}[p_1(0) + K_p p_2(0)] \\ &+2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \operatorname{sgn}[p_1(\omega_{oi}) + (K_i - K_d\omega_{oi}^2)p_2(\omega_{oi})] \\ &+(-1)^{\tilde{m}_q}\operatorname{sgn}[p_1(\omega_{o\tilde{m}_q}) \\ &+(K_i - K_d\omega_{o\tilde{m}_q}^2)p_2(\omega_{o\tilde{m}_q})]\}\cdot\gamma,\ \alpha \in \mathbb{Z}_+,\end{aligned} \tag{9}$$

$$\begin{aligned}\rho = \{&\operatorname{sgn}[p_1(0) + K_p p_2(0)] \\ &+2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \operatorname{sgn}[p_1(\omega) + (K_i - K_d\omega^2)]\}\cdot\gamma, \\ &\alpha \notin \mathbb{Z}_+\end{aligned} \tag{10}$$

We now define the following integer variables

$$\tilde{I}_0 \triangleq \operatorname{sgn}[p_1(0) + K_p p_2(0)], \tag{11a}$$

$$\tilde{I}_i \triangleq \operatorname{sgn}[p_1(\omega_{oi}) + (K_i - K_d\omega_{oi}^2)p_2(\omega_{oi})], \tag{11b}$$

where $\tilde{I}_0 \in \{-1, 0, 1\}$ and $\tilde{I}_i \in \{-1, 1\} \forall i \in [1, \tilde{m}_q]$; then the necessary and sufficient conditions for the existence of stabilizing PID controllers are given by the following theorem:

Theorem: The characteristic polynomial $\psi(s)$ is Hurwitz stable if and only if there exist a feasible non-empty solution $\{\tilde{I}_i\} \ne \varnothing$ to either of the following equations:

$$\rho = \left[\tilde{I}_0 + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \tilde{I}_i + \tilde{I}_{\tilde{m}_q}\right]\gamma,\ \alpha \in \mathbb{Z}_+ \tag{12}$$

$$\rho = \left[\tilde{I}_0 + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \tilde{I}_i\right]\gamma,\ \alpha \notin \mathbb{Z}_+, \tag{13}$$

where $\tilde{I}_i, i = 0, 1, \ldots, \tilde{m}_q$ denote the unknown integer variables as defined in Eqs. (11a) and (11b).

Proof: (*Necessity*) If Eq. (12) or (13) has a feasible solution, then by using Eqs. (11a) and (11b) in Eqs. (9) and (10), and applying the generalized Hermite-Biehler Theorem, we can ensure that $\sigma[\delta(s)] = m_\psi - \sigma[W(s)]$, so that $\sigma[\psi(s)] = \Theta[\psi(s)]$ and $\psi(s)$ is Hurwitz stable.

(*Sufficiency*) If $\psi(s)$ is Hurwitz stable, then $\sigma[\psi(s)] = \Theta[\psi(s)]$ s.t. that $\sigma[\delta(s)] = m_\psi - \sigma[W(s)]$, and from the generalized Hermite-Biehler Theorem either Eq. (7) or (8) must be satisfied. The only way to satisfy it is if Eq. (12) or (13) has a feasible solution. ♣

Characterization of the stabilizing PID gains for the system is now provided by the following results:

Corollary 1: The range of K_p for which the root distribution of $q(\omega)$ is such that Eq. (12) or (13) is satisfied, can be identified from the root locus plot of $\{1+K_p\, q_2(\omega)/q_1(\omega)\}$. This range is denoted as $S(K_p)=(\underline{K}_p,\overline{K}_p)\subseteq(-\infty,\infty)$, where the under bar and the over bar represent the lower and upper limit, respectively.

Corollary 2: Assume that a non-empty solution to Eq. (12) or (13) has been found, i.e., let $\tilde{g}\triangleq\{\tilde{I}_i\}\neq\varnothing$, then the ranges of the stabilizing gains K_i and K_d, i.e., $(\underline{K}_i,\overline{K}_i)$ and $(\underline{K}_d,\overline{K}_d)$, can be solved from the following linear matrix inequalities (LMI):

$$\operatorname{sgn}\{[P_1]+[P_2][\tilde{\kappa}]\}=(\tilde{g})^T, \tag{14}$$

where the matrices $[P_1]\in\mathbb{R}^{\tilde{m}_q,1}$, $[P_2]\in\mathbb{R}^{\tilde{m}_q,2}$, and $[\tilde{\kappa}]\in\mathbb{R}^{2,1}$ are defined as:

$$[P_1]\triangleq[p_1(0)\quad p_1(\omega_{o1})\quad p_1(\omega_{o2})\quad\cdots\quad p_1(\omega_{o\tilde{m}_q-1})]^T \tag{15}$$

$$[P_2]\triangleq\begin{bmatrix} p_2(\omega_0) & 0 \\ p_2(\omega_{o1}) & -\omega_{o1}^2 p_2(\omega_{o1}) \\ p_2(\omega_{o2}) & -\omega_{o2}^2 p_2(\omega_{o2}) \\ \vdots & \vdots \\ p_2(\omega_{o\tilde{m}_q-1}) & -\omega_{o\tilde{m}_q-1}^2 p_2(\omega_{o\tilde{m}_q-1}) \end{bmatrix} \tag{16}$$

$$[\tilde{\kappa}]\triangleq[K_i\quad K_d]^T \tag{17}$$

thus, for a given $K_p\in S(K_p)$, the stabilizing PID controller set is: $S_{\tilde{\kappa}}\triangleq\{K_p,(\underline{K}_i,\overline{K}_i),(\underline{K}_d,\overline{K}_d)\}$.

We note that the solution of the LMI given by Eq. (14) is either a convex polygon or a half-plane in the K_i-K_d space. We also note that the procedure leading to Eq. (14) and its solution can be easily coded on the computer. We further note that by repeating this process for different values of $K_p\in S(K_p)$, one can obtain a picture of stability in the 3-D space. The stability characterization in the controller parameter space is shown in Fig. 2. Finally, the above analysis can be modified to restrict $K_p, K_i, K_d\in\mathbb{R}_+$ in order to force non-negative solution to the problem, if so desired.

Based on the above discussion, we provide the following algorithm for PID controller synthesis for the single-channel unity feedback system (Fig. 1).

2.3 Algorithm for Controller Synthesis

Step 1: Given $W(s)$ and $Q(s)$, use Eq. (6b) to obtain the polynomials $q_1(\omega)$, $q_2(\omega)$; and plot the root locus for $\{1+K_p\, q_1(\omega)/q_1(\omega)\}$. Also, compute ρ.

Step 2: Select some $S(K_p)\subseteq(-\infty,\infty)$ from the root locus such that $\forall K_p\in S(K_p)$, $\tilde{m}_q$ has the potential to satisfy Eq. (12) or (13), i.e., $2\tilde{m}_q-1\geq\rho$ on the root locus plot.

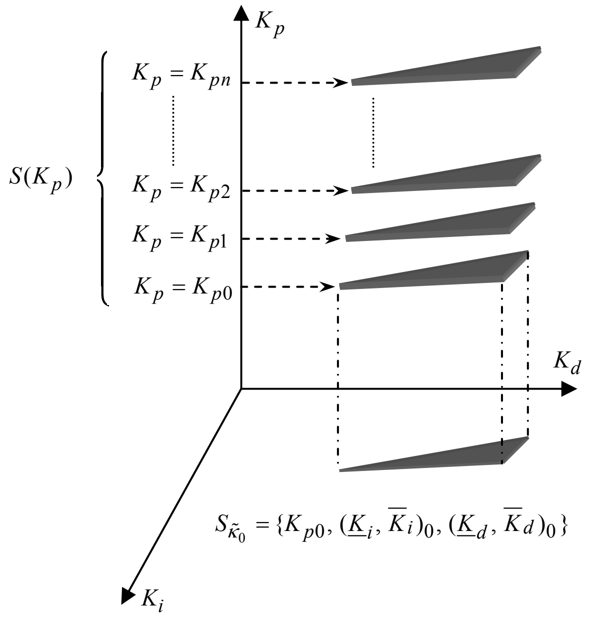


Fig. 2. The design methodology illustrated in the $K_p-K_i-K_d$ space.

Step 3: Choose a trial value $K_{p0}\in S(K_p)$; accordingly, calculate the values of γ and α.

Step 4: For the selected $K_{p0}\in S(K_p)$, use Eqs. (12) or (13) to obtain $\tilde{g}$. If $\tilde{g}\neq\varnothing$, proceed to Step 5; otherwise repeat Steps 2-3 with different ranges of $S(K_p)$ until $\tilde{g}\neq\varnothing$ is obtained.

Step 5: Compute $[P_1]$ and $[P_2]$ for $K_{p0}\in S(K_p)$. Use the set of LMI given by Eq. (14) to obtain the non-empty stabilizing set $S_{\tilde{\kappa}}$. If the solution to Eq. (14) is empty, then repeat Steps 3-5 with a different $K_{p0}\in S(K_p)$.

The algorithm and the stability characterization is illustrated in the form of a Venn diagram in Fig. 3.

3. DESIGN EXAMPLE

As an example of the stability analysis and controller synthesis, we apply the synthesis algorithm to a single-link biomechanical system with position, velocity, and force feedback, and with physiological latencies in he feedback loops (Iqbal and Roy, 2004).

Using [1/1] Padé approximation to represent the delays, the closed-loop transfer function is given as

$$\frac{y(s)}{r(s)}=\frac{n_G(s)}{sQ(s)+n_c(s)W(s)},$$

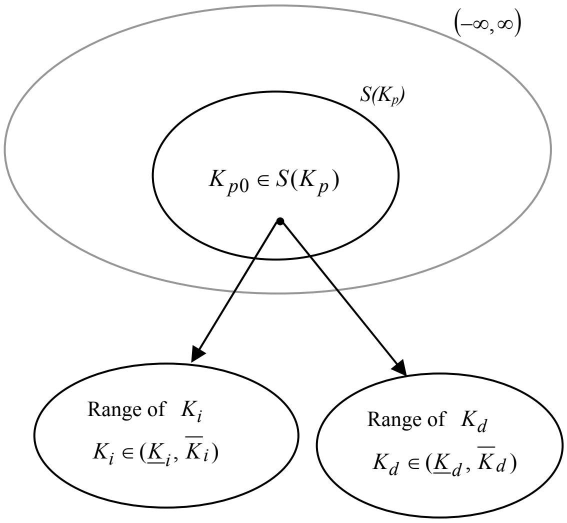


Fig. 3. Venn diagram illustrating the controller synthesis algorithm based on the Hermite-Biehler Theorem.

where

$$n_G(s) = 0.112s^3 + 13.03s^2 + 305.08s + 596.17,$$

$$W(s) = -0.2s^6 - 28.78s^5 - 383.69s^4 + 9.92\times10^4 s^3 + 4.89\times10^6 s^2 + 8.28\times10^7 s + 4.23\times10^8,$$

$$Q(s) = -4\times10^{-4}s^8 + 0.14s^7 + 22.17s^6 + 1681.63s^5 + 6.78\times10^4 s^4 + 1.36\times10^6 s^3 + 1.05\times10^7 s^2 - 6.23\times10^6 s - 5.47\times10^7,$$

The even and odd parts of $W(s)$ and $Q(s)$ are calculated as:

$$W_e(s) = -0.2s^6 - 383.69s^4 + 4.89\times10^6 s^2 + 4.23\times10^8,$$

$$W_o(s) = -28.78s^4 + 9.92\times10^4 s^2 + 8.28\times10^7,$$

$$Q_e(s) = -4\times10^{-4}s^8 + 22.17s^6 + 6.78\times10^4 s^4 + 1.05\times10^7 s^2 - 5.47\times10^7,$$

$$Q_o(s) = 0.14s^6 + 1681.63s^4 + 1.36\times10^6 s^2 - 6.23\times10^6.$$

Then, using Steps 1-4 of the controller synthesis algorithm we obtain: $\rho = 5$, $\alpha = 7.5$, and the stabilizing range of K_p is given as: $S(K_p) = (0.129,\ 13.3)$. For illustration we choose $K_{p0} = 5$; then $\mathrm{sgn}[q(\infty)] = 1$ and the real, nonnegative, and distinct zeros of $q(\omega)$ with odd multiplicities are: $\omega_0 = 0$, $\omega_{o1} = 23.35$, and $\omega_{o2} = 161.19$ rad/sec. Therefore, $\tilde{m}_q = 3$, $\gamma = 1$, and from Eq. (12) we obtain $\{\tilde{I}_0 - 2\tilde{I}_1 + 2\tilde{I}_2\} = 5$, which is solved as $\tilde{g} = \{\tilde{I}_0, \tilde{I}_1, \tilde{I}_2\} = \{1, -1, 1\}$. Use of Eqs. (13)-(16) then results in the following linear inequalities:

$$\begin{cases} K_i < 0 \\ K_i + 545.31K_d + 40.95 < 0 \\ K_i + 13502.1K_d - 4193.79 > 0 \end{cases} \qquad (17)$$

The shaded area in Fig. 4 shows the bounded feasible region in the $K_i - K_d$ space for the active constraints given by Eq. (17). Therefore, the set of all stabilizing PID controllers for $K_p = 5$ is: $S_{\tilde{K}} = \{5,\ (0,\ 220),\ (-0.075,\ 0.325)\}$. For illustration, we select $\{K_i = 10,\ K_d = 0.16\}$; then, the closed-loop poles of $G(s)$ are given as: $s_{1,2} = -34.83 \pm j80.27$, $s_3 = -85.16$, $s_4 = -66.67$, $s_5 = -28.9$, $s_6 = -20.64$, $s_{7,8} = -9.45 \pm j10$, and $s_9 = -2.15$. Since $\mathrm{Re}[s_i] < 0$, the closed-loop system is Hurwitz stable. The closed-loop step response for $K_p = 5,\ K_i = 10,\ K_d = 0.16$ is shown in Fig. 5.

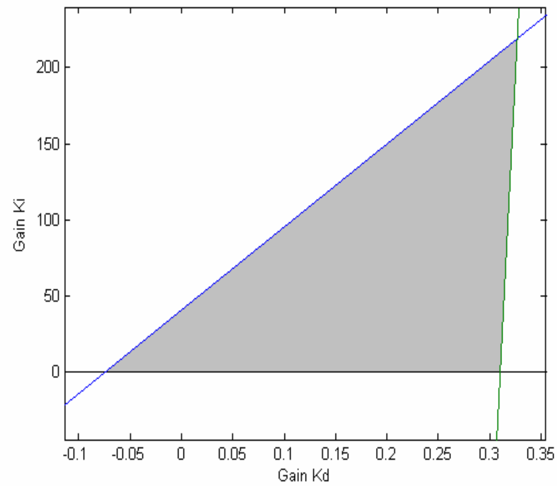


Fig. 4. Feasible $K_d - K_i$ region obtained from active constraints. The feasible region is either a convex or a hyper plane.

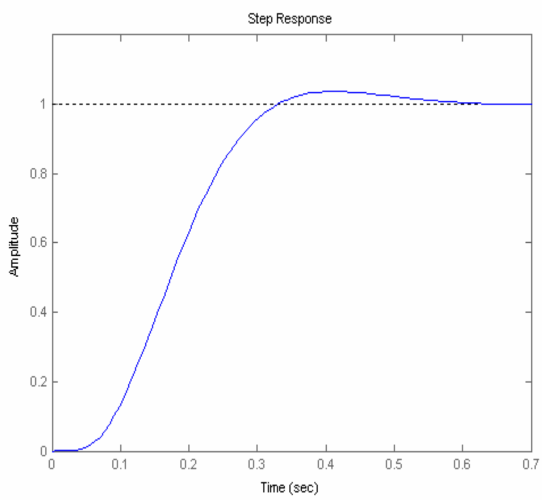


Fig. 5. Stable closed-loop response to a step input with $K_p = 5,\ K_d = 0.16$, and $K_i = 10$.

4. CONCLUSIONS

In this paper rigorous mathematical stability of linear time-invariant systems of arbitrary order is established using Hermite-Biehler framework from functional analysis. Stabilizing PID gains are chosen through a combination of root-locus techniques and linear matrix inequalities that result from application of the generalized Hermite-Biehler Theorem. The resultant controller synthesis algorithm is, in general, programmable[1] for linear systems of arbitrary order.

REFERENCES

Bartlett, A.C., C.V. Hollot and H. Lin (1988). Root Location of an Entire Polytope of Polynomials: It Suffices to Check the Edges, *Mathematics of Controls, Signals and Systems*, **Vol. 1**, pp. 61-71.

Bhattacharyya, S.P., H. Chapellat and L. Keel, (1995). *Robust Control: The Parametric Approach,* Chapter 1, pp. 38-53, Prentice Hall, New Jersey.

Datta, A., M. Ho and S.P. Bhattacharyya (1999) *Synthesis and Design of PID Controllers*, Chapter 4, pp. 51-88, Springer-Verlag, New York.

Ho, M., A. Datta and S.P. Bhattacharyya (1999). Generalizations of the Hermite-Biehler Theorem, *Linear Algebra and Its Applications*, **Vol. 302**, pp. 135-153.

Ho, M., A. Datta and S.P. Bhattacharyya (2000). Generalizations of the Hermite-Biehler Theorem: the complex case, *Linear Algebra and Its Applications*, **Vol. 320**, pp. 23-26.

Iqbal, K. and A. Roy (2002). PID Controller Design for the Human-Arm Robot Manipulator Coordination Problem, *Proc. IEEE Int. Symp. on Intelligent Control*, **Vol. 1**, pp. 121-124.

Iqbal, K. and A. Roy (2004). Stabilizing PID Controllers for a Single-Link Biomechanical Model with Position, Velocity, and Force Feedback, *ASME Trans. on Biomechanical Engineering*, **Vol. 126** (To appear).

Kharitonov, V.L. (1978). On a generalization of a stability criterion, *Izv. Akad. Nauk. Kazakh. SSR Ser. Fiz. Mat.*, **Vol. 1**, pp. 53-57.

Lipatov, A.V. and N.I. Sokolov (1978). Some sufficient conditions for stability and instability of continuous linear stationary systems, *Automation and Remote Control*, **Vol. 39**, pp. 1285–1291.

[1] See Roy (2004) for details on the programmable aspects of the controller synthesis algorithm.

Roy, A., and K. Iqbal (2003a). PID Controller Design for First-Order-Plus-Dead-Time Model via Hermite-Biehler Theorem, *Proc. American Control Conference*, Denver, **Vol. 3**, pp. 5286-5291.

Roy, A. and K. Iqbal (2003b). PID Controller Stabilization of a Single-Link Biomechanical Model with Multiple Delayed Feedbacks, *Proc. IEEE Conf. on Systems, Man, and Cyb.*, Washington, D.C., **Vol. 1**, pp. 642-647.

Roy, A. (2004). Robust Stabilization of Multi-Body Biomechanical Systems: A Control Theoretic Approach, PhD Thesis, Univ. Arkansas at Little Rock, Little Rock, USA, Appendix I, pp. 275-296.

Stroeve, S.H., (1998). Impedance Characteristics of a neuromusculoskeletal model of the human arm: I Posture Control. In: *Neuromuscular control of arm movements*, PhD Thesis, Delft Univ. of Technology, Delft, The Netherlands, Chapter 5, pp. 86-90.

Van der Helm, F. and L. Rozendaal (2000). Musculoskeletal Systems with Intrinsic and Proprioceptive Feedback. In: *Biomechanics and Neural Control of Movement and Posture* (J.M. Winters and P.E. Crago. (2nd Ed.)), pp. 164-174, Springer-Verlag, New York.

Youla, D. C., H. A. Jabr and J. J. Bongiorno (1976). Modern Wiener-Hopf Design of Optimal Controllers - Part II: The Multivariable Case, *IEEE Trans. on Automatic Control*, **Vol. 21**, 319-338.

ENPM 667: Control of Robotic Systems

Synthesis of Stabilizing PID Controllers for Biomechanical Models

Project I

Nupur Nimbekar (UID: 116894876)
Patan Sanaulla Khan (UID: 116950985)

27 November 2019

Contents

Abstract

The paper presents an analysis-synthesis framework for linear time invariant systems based on the generalized Hermite-Biehler Theorem. To understand this further, let us consider a system with PID controller which has a closed-loop characteristic polynomial. We know from Hurwitz stability analysis that 0 right-half plane roots indicate stability, to test this let us consider the odd and even powers of 's' and verify whether the characteristic polynomial is Hurwitz stable. When the Polynomial is not stable we go for the generalized Hermite-Biehler Theorem's implementation. for that, we will now substitute the values of '$j\omega$' in place of 's' gives the K_p and from the velocity and integral gains we get K_d and K_i. Now if we apply the Hermite-Biehler Theorem on the system such that we derive the signature of the polynomial, thereby achieving the conditions for the existence of stabilizing PID controller by the following:

- The Range of K_p for which the root distribution is satisfied can be identified by root locus plot over the upper limit and lower limit of K_p
- The non-empty solution can be found over the ranges of the stabilizing gains K_i and K_d which is given by linear matrix inequalities

Hence, solution is given by a convex polygon or a half-plane in the K_i - K_d space for different values possible for K_p. Therefore, the mathematical stability of linear time-variant systems of arbitrary order is established using Hermite-Biehler framework. The resultant controller synthesis algorithm is programmable for linear systems of arbitrary order.

Chapter 1

Introduction

Among countless number of the issues that rise during the presence cycle of system structure, one of the first, and apparently the most fundamental one, is stability of the feedback loop. For contribution to be an effective tool, it must be controlled as an uncontrolled structure will either influence or disregard to work. The issue of offsetting a given method by suitable low fixed-order compensator, and try to design perfect and generous controllers with constrained solicitation, is a troublesome one.

Keeping complete analytical characterization in mind is one of the key factors of approach while designing the stabilizing controller. This arithmetical methodology is seen in different sorts of hypotheses and they use polynomials to provide conditions for the stability of linear feedback processes. Clever utilization of these hypotheses allows the simultaneous design of the characteristic polynomial of the closed-loop and synthesis of the controller. Most of the classic control techniques give us either analytical characterization or fixed-order controller synthesis. Here, we will be focusing on an analysis-synthesis framework for linear time-invariant systems based on the generalized Hermite-Biehler Theorem.[1]

Stability factors is what we will be focusing on. The Hermite Biehler theorem here, will be used for multiple purposes. The uses of generalized Hermite Biehler Theorem in our study are listed below: -

- It will help us in the functional analysis which will in return help us study the stability properties of real polynomials defined over complex fields.

- It will provide conditions for the Hurwitz stability of real polynomials.
- It will also provide us with an elegant, easy, and analytical way of characterizing the set of all stabilizing controllers for a given processes.
- With the help of stability analysis, we will be studying the development of algorithm to synthesize stabilizing PID controllers.[1]

Before going to the stabilizing of PID controllers, let us first understand what a PID controller is. PID (proportional integral derivative) controllers use a control loop feedback mechanism to control process variables and are the most precise and stable controller.

The problem of determining conditions under which all of the roots of a given real polynomial lie in the open left-half complex plane is one of fundamental importance in the study of stability of a dynamic system. Hurwitz Stability of real polynomials provides us with a criteria which is a part of mathematical test which provides us with necessary and sufficient criterion of stability of a linear time invariant (LTI) control system.

Chapter 2

Literature Review

Doing research without a proper literature review is ignoring the importance of previous studies which may result in duplicate works and waste of energy! Here we will be discussing different components of the paper in detail to get a proper understanding of terminologies used throughout the paper.

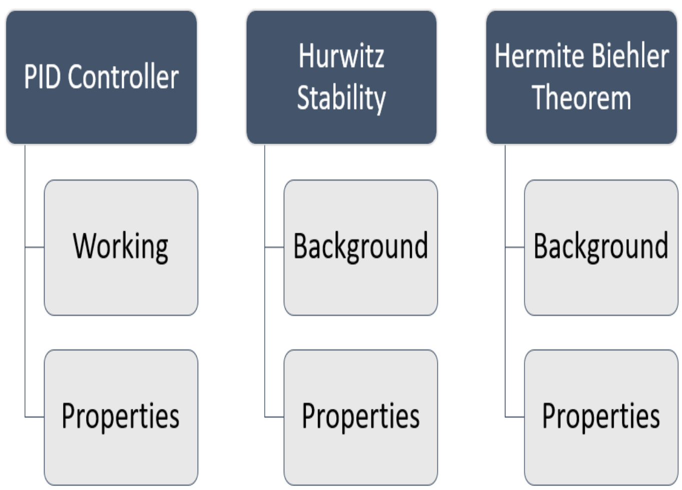


Figure 2.1: Categories of Literature Survey

2.1 PID Controllers

PID control is an entrenched method for driving a framework towards an objective position or level. As its name suggests, a PID controller combines proportional control with additional integral and derivative adjustments which help the unit automatically compensate for changes in the system.[2]

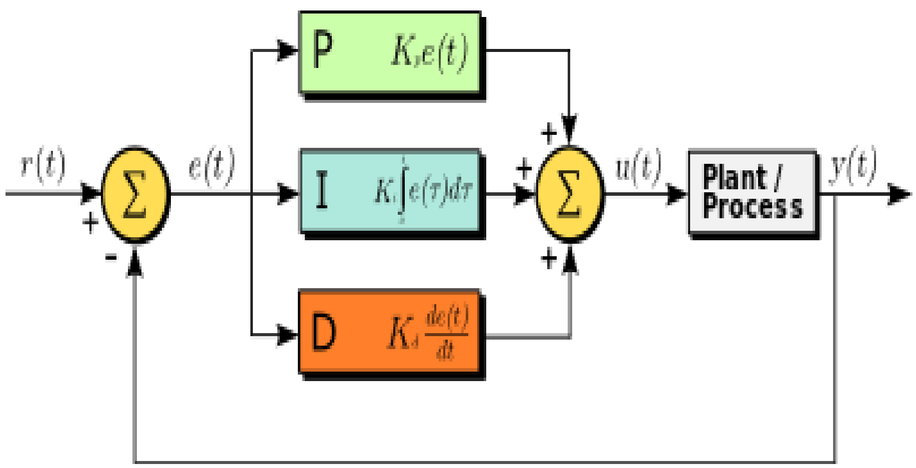


Figure 2.2: Block Diagram of PID Controller

PID stands for proportional-integral-derivative. Not every controller uses all three of these mathematical functions. Many processes can be handled to an acceptable level with just the proportional-integral terms. However, fine control, and especially overshoot avoidance, requires the addition of derivative control.

The overall control function $u(t) = K_{\mathrm{p}} e(t) + K_{\mathrm{i}} \int_0^t e(t')\, dt' + K_{\mathrm{d}} \frac{de(t)}{dt}$

As its name suggests, a PID controller combines proportional control with additional integral and derivative adjustments which help the unit automatically compensate for changes in the system.

2.1.1 Working

The working principle behind a PID controller is that the proportional, integral and derivative terms must be individually adjusted or "tuned."Based on the distinction between these qualities a correctness factor is determined and applied to the input information.[2] Here are the three stages:

- Proportional tuning involves correcting a target proportional to the difference. Thus, the target value is never achieved because as the difference approaches zero, so does the applied correction.
- Integral tuning attempts to resolve this by effectively culminating the error result from the "P" action to increase the correction factor. But this makes the output to overshoot.
- Derivative tuning attempts to minimize this overshoot by slowing the correction factor applied as the target is approached.

2.1.2 Properties

PID controller has all the necessary dynamics: fast reaction on change of the controller input (D mode), increase in control signal to lead error towards zero (I mode) and suitable action inside control error area to eliminate oscillations (P mode). Derivative mode improves stability of the system and enables increase in gain K and decrease in integral time constant Ti, which increases speed of the controller response. PID controllers are the most often used controllers in the process industry. The majority of control systems in the world are operated PID controllers. PID controller combines the advantage of proportional, derivative and integral control action.[5]

2.2 Hurwitz Stability

Hurwitz Stability of real polynomials provides us with a criteria which is a part of Routh-Hurwitz mathematical test which provides us with necessary and sufficient criterion of stability of a linear time invariant (LTI) control system.

2.2.1 Background

German mathematician Adolf Hurwitz independently proposed in 1895 to arrange the coefficients of the polynomial into a square matrix, called the Hurwitz matrix, and showed that the polynomial is stable if and only if the sequence of determinants of its principal sub-matrices are all positive.

2.2.2 Properties

For a polynomial to be Hurwitz, it is vital but not adequate that all of its coefficients be positive. A fundamental and adequate condition that a polynomial is Hurwitz is that it passes the Routh–Hurwitz stability criterion. A given polynomial can be productively tried to be Hurwitz or not by utilizing the Routh proceeded division extension procedure.[6]

The properties of Hurwitz polynomials are:

- All the poles and zeros are in the left half plane or on its boundary, the imaginary axis.
- Any poles and zeros on the imaginary axis are simple (have a multiplicity of one).
- Any poles on the imaginary axis have real strictly positive residues, and similarly at any zeros on the imaginary axis, the function has a real strictly positive derivative.
- Over the right half plane, the minimum value of the real part of a PR function occurs on the imaginary axis (because the real part of an analytic function constitutes a harmonic function over the plane, and therefore satisfies the maximum principle).
- The polynomial should not have missing powers of s.

2.3 Hermite Biehler

Hermite Biehler theorem gives necessary and enough conditions for the Hurwitz stability of a polynomial in terms of certain interlacing conditions. The result can be used to solve stability issues in control theory, and it can be proved by both academically and practically. Hermite Biehler theorem states that a given real polynomial is Hurwitz if and only if it satisfies a certain interlacing property.[3] [HBP]

Theorem (Hermite Biehler theorem).[3]

Let δ(s) = δ $_0$ + δ $_1$(s ^{2})+...+δ $_n$(s^{2n}) be a given real polynomial of degree n. Write where are the components of made up of even and odd powers of s, respectively. Let ω_{e1} , ω_{e2} ; . . . denote the distinct non-negative real zeros of deÿx2 and let ω_{o1} , ω_{o2} ; . . . denote the distinct non-negative real zeros of doÿx2, both arranged in ascending order of magnitude. Then ds is Hurwitz stable if and only if all the zeros of deÿx2, doÿx2 are real and distinct, dn and dnÿ1 are of the same sign, and the non-negative real zeros satisfy the following interlacing property

$\omega_0 < \omega_{e1} < \omega_{o1} < \omega_{e2} < \omega_{o2}$

In this paper, our objective is to obtain generalizations of the above theorem for real polynomials that are not necessarily Hurwitz. To clearly understand what it is that we are trying to generalize, we provide below some alternative characterizations and interpretations of the HermiteBiehler theorem. To do so, we introduce the standard signum function,

$$R \to \{-1, 0, 1\}$$

defined by,

$$sgn[x] = \begin{cases} -1 & if x < 0 \\ 0 & if x = 0 \\ 1 & if x > 0 \end{cases}$$

2.3.1 Properties

The conditions of Hermite-Biehler theorem from the Lemma 2.1 can be given be:

- δ(s) is Hurwitz Stable
- δ_{n} and $\delta_{\text{n-1}}$ are of the same sign and

$$n = \begin{cases} sgn\left[\delta_0\right] \cdot \{sgn[p(0)] - 2sgn\left[p\left(\omega_n\right)\right] + 2sgn\left[p\left(\omega_n\right)\right] + \cdots + (-1)^{m-1} \\ \times 2sgn\left[p\left(\omega_{o_m}\right)\right] + (-1)^m \cdot sgn[p(\infty)]\} \quad for n = 2m \\ sgn\left[\delta_0\right] \cdot \{sgn[p(0)] - 2sgn\left[p\left(\omega_{o_1}\right)\right] + 2sgn\left[p\left(\omega_{o_2}\right)\right] + \cdots + (-1)^{m-1} \\ \times 2sgn\left[p\left(\omega_{o_m}\right)\right] + (-1)^m \cdot 2sgn\left[p\left(\omega_{o_n}\right)\right]\} \quad for n = 2m+1 \end{cases}$$

- δ_{n} and $\delta_{\text{n-1}}$ are of the same sign and

$$n = \begin{cases} sgn\left[\delta_0\right] \cdot \{2sgn\left[q\left(\omega_{e_1}\right)\right] - 2sgn\left[q\left(\omega_{e_2}\right)\right] + 2sgn\left[q\left(\omega_{e_3}\right)\right] + \cdots + (-1)^{m-2} \\ \times 2sgn\left[q\left(\omega_{e_{m-1}}\right)\right] + (-1)^{m-1} \cdot 2sgn\left[q\left(\omega_{e_m}\right)\right]\} \quad for n = 2m \\ sgn\left[\delta_0\right] \cdot \{2sgn\left[q\left(\omega_{e_1}\right)\right] - 2sgn\left[q\left(\omega_{e_2}\right)\right] + 2sgn\left[q\left(\omega_{e_3}\right)\right] + \cdots + (-1)^{m-1} \\ \times 2sgn\left[q\left(\omega_{e_m}\right)\right] + (-1)^m \cdot sgn[q(\infty)]\} \quad for n = 2m+1 \end{cases}$$

The proof of this theorem and the Lemma are explained in details in [3]

Chapter 3

Problem Formulation and Designing

Here we will study the stability analysis of Hermite-Biehler Framework and also understand how it's algorithm works. Later we will analyse whether the framework actually works as expected.

3.1 Problem Formulation

A framework for stability analysis of a linear time-invariant system in unity gain feedback configuration based on the application of the generalized Hermite-Biehler Theorem can be found out by following method. First we state the generalised Hermite-Biehler Theorem and then design the framework. For let us make some assumptions to get the equations for a typical single-channelled, unity feedback control system with a PID controller.

Let, $G_p(s) = W(s)/Q(s)$, where $W(s)$ and $Q(s)$ are relatively prime

and $G_c(s) = n_c(s)/\ d_c(s)$ and the PID controller will be given by

$n_c(s) = K_d(s)^2 + K_p(s) + K_i$; $d_c(s) = s$ and let the closed loop characteristic polynomial be represented by $\psi(s)$ and is given by

$$\psi(s) = sQ(s) + (\ K_i + (s)^2 K_d)W(s) + sK_p W(s)$$

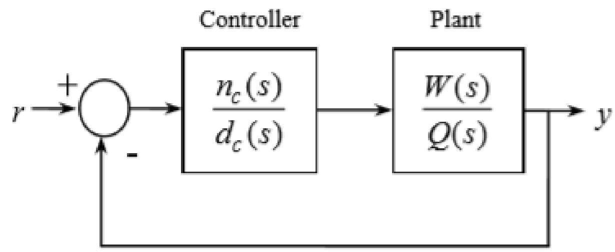


Figure 3.1: Block diagram of a typical single-channel, unity feedback control system with a PID controller.

By the Generalised theorem we can say that let δ(s) =

$$\sum_{i=0}^{n} \delta_{\mathrm{i}} s^{\mathrm{i}}$$

, $\delta_{\mathrm{i}} \in \Re \forall i.$

with a root at the origin of multiplicity k .

$\delta = \delta_{\mathrm{e}}(\mathrm{s}^2)$ + s $\delta_{\mathrm{o}}(\mathrm{s}^2)$, where $\delta_{\mathrm{e,o}}(\mathrm{s}^2)$ are the components of δ(s) made up of even and odd powers of s.

From the paper we can say that the signature of the polynomial can be represented by,

$\sigma[\delta(s)] \triangleq {n_\delta}^{(\mathrm{L})} - {n_\delta}^{(\mathrm{R})}$

Now for stability framework based on the Hermite-Biehler Theorem, signature of polynomial will be defined as,

$\sigma[\psi(s)] \triangleq {n_\psi}^{(\mathrm{L})} - {n_\psi}^{(\mathrm{R})}$
the order of ψ(s) as${\mathrm{n}_\psi}^{(\mathrm{L})}$ + ${\mathrm{n}_\psi}^{(\mathrm{R})} \triangleq \Theta$ [ψ(s)] ; then, from a stability perspective, if ψ(s) is Hurwitz, then ${\mathrm{n}_\psi}^{(\mathrm{R})} = 0$ (no RHP poles), or equivalently, $\sigma[\psi(s)] \triangleq \Theta$ ψ(s) = m_ψ (i.e., if the system is Hurwitz stable, then the signature equals the order of the characteristic polynomial).

Now after decomposing W(s) and Q(s) into polynomials with even and odd powers we get,

$$W(s) = W_e\left(s^2\right) + sW_o\left(s^2\right), Q(s) = Q_e\left(s^2\right) + sQ_o\left(s^2\right)$$

let,

$$\delta(s) \triangleq \psi(s)W^*(s) = sQ(s)W^*(s) + \left(K_i + s^2K_d\right)W(s)W^*(s) + sK_pW(s)W^*(s)$$

then, it can be verified that

$$\sigma[\delta(s)] = \sigma\left[\psi(s)W^*(s)\right] = \sigma[\psi(s)] - \sigma[W(s)]$$

if the characteristic polynomial is Hurwitz stable, then

$$\sigma\left[\psi(s)W^*(s)\right] = m_\psi - \sigma[W(s)]$$

Now, substituting s = jω in δ(s) we obtain eq(4),

$$\delta(\omega) = \psi(\omega)W^*(\omega) = p(\omega) + jq(\omega)$$

where,

$$\begin{aligned} p(\omega) &\triangleq p_1(\omega) + \left(K_i - K_d\omega^2\right)p_2(\omega) \\ q(\omega) &\triangleq q_1(\omega) + K_pq_2(\omega) \end{aligned}$$

and the polynomials, $p_1(\omega)$, $p_2(\omega)$, $q_1(\omega)$ and $q_2(\omega)$for the equation are given as:

$$p_1(\omega) \triangleq \omega^2\left[Q_eW_o - Q_oW_e\right], p_2(\omega) \triangleq \left[W_e^2 + \omega^2W_o^2\right]$$

$$q_1(\omega) \triangleq \omega\left[W_eQ_e + \omega^2W_oQ_o\right], q_2(\omega) \triangleq \omega p_2(\omega)$$

Above equations give us the decoupling for position gain K_p, from the velocirty and then integral gains, K_d and K_i.

This structure will contribute in the development of synthesis procedure for PID controllers later.

For now, in order to develop stability characterization using the generalized Hermite-Biehler Theorem, let

$$m_q \triangleq \Theta[q(\omega)]$$

and

$$\tilde{m}_q \leq m_q$$

be the number of real, non-negative, and distinct zeros of q(ω) with the old multiplicities that satisfies the following condition for $\mathrm{K_p} \in (-\infty, \infty)$:

$$0 = \omega_0 < \omega_{o1} < \omega_{o2} < \cdots < \omega_{o\tilde{m}_q - 1} < \omega_{o\tilde{m}_q} = \infty$$

if we define,

$$\rho \triangleq m_\psi - \sigma[W(s)]$$

$$\alpha \triangleq \frac{1}{2}\left[m_\psi + m_W\right]$$

$$\gamma \triangleq (-1)^{\tilde{m}_q - 1}$$

Then, application of the generalized Hermite-Biehler Theorem to eq(4), we will obtain eq(7) and eq(8),

$$\rho = \{sgn[p(0)] + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i sgn\left[p\left(\omega_{oi}\right)\right] + (-1)^{\tilde{m}_q} sgn\left[p\left(\omega_{o\tilde{m}_q}\right)\right]\} \cdot \gamma, \alpha \in \mathbb{Z}_+$$

$$\begin{array}{l} \rho = \left\{sgn[p(0)] + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i sgn\left[p\left(\omega_{oi}\right)\right]\right\} \cdot \gamma \\ \alpha \notin \mathbb{Z}_+ \end{array}$$

where + Z defines the set of positive integers. Substituting Eq. (5) in Eqs. (7) and (8) we obtain

$$\begin{array}{l} \rho = \{sgn\left[p_1(0) + K_p p_2(0)\right] \\ +2\sum_{i=1}^{m_n - 1}(-1)^i sgn\left[p_1\left(\omega_{oi}\right) + \left(K_i - K_d\omega_{oi}^2\right) p_2\left(\omega_{oi}\right)\right] \\ +(-1)^{i\pi_0} sgn\left[p_1\left(\omega_{o\bar{k}_v}\right)\right. \\ \left. + \left(K_i - K_d\omega_{om_i}^2\right) p_2\left(\omega_{om_i}\right)\right]\} \cdot \gamma_. \alpha \in \mathbb{Z}_+ \end{array}$$

$$\rho = \{sgn\,[p_1(0) + K_p p_2(0)]$$
$$+2\sum_{i=1}^{n_n-1}(-1)^i sgn\,[p_1(\omega) + (K_i - K_d\omega^2)]\} \cdot \gamma$$
$$\alpha \notin \mathbb{Z}_+$$

From these equations, we can define integer variables as,

$$\tilde{I}_0 \triangleq sgn\,[p_1(0) + K_p p_2(0)]$$

$$\tilde{I}_i \triangleq sgn\left[p_1(\omega_{oi}) + \left(K_i - K_d\omega_{oi}^2\right)p_2(\omega_{oi})\right]$$

where

$$\tilde{I}_0 \in \{-1, 0, 1\}$$

and

$$\tilde{I}_i \in \{-1, 1\} \forall i \in [1, \tilde{m}_q]$$

then the necessary and sufficient conditions for the existence of stabilizing PID controllers are given by the theorem and corollaries in the Glossary.

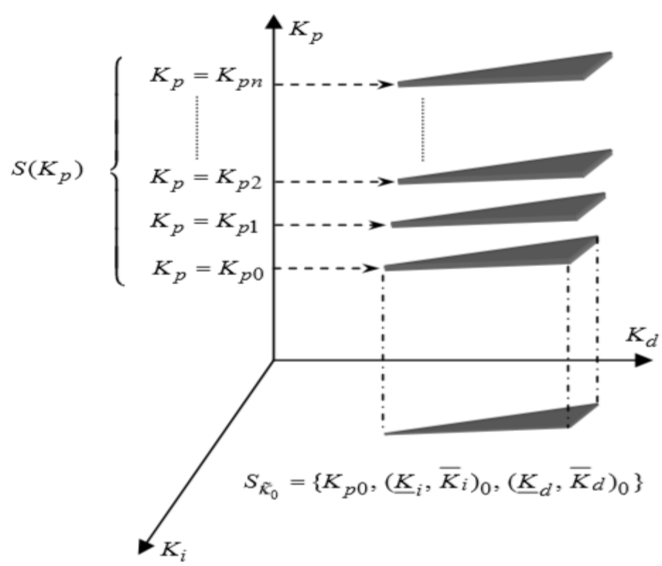


Figure 3.2: Illustration of Design Methodology

3.1.1 Observation

Similary, these Theorems and corollaries can also helped us determine following aspects:

- The solution of LMI will be given by either a convex polygon or a half plane in the $\mathrm{K_i}$-$\mathrm{K_d}$ space.
- By repeating the process for different values of $\mathrm{K_p} \in \mathrm{S(K_d)}$, we can get an idea about the stability in 3-D space.
- This analysis can be modified to restrict

  $$K_p, K_i, K_d \in \mathbb{R}_+$$

  in order to force non-negative solution to the problem.

3.2 Algorithm for Controller Synthesis

Step 1:

- Use the values of given W(s) and Q(s) to obtain the polynomials $q_1(\omega)$ and $q_2(\omega)$ in equation:

$$q_1(\omega) \triangleq \omega \left[W_e Q_e + \omega^2 W_o Q_o \right], q_2(\omega) \triangleq \omega p_2(\omega)$$

- Then plot the root locus for

$$\{1 + K_p q_1(\omega)/q_1(\omega)\}$$

- Then Compute ρ

Step 2:

Select some $S(K_p) \subseteq (-\infty, \infty)$ from the root locus such that

$$\forall K_p \in S\left(K_p\right)$$

$\tilde{m}_q$ has the potential to satisfy Eq.

$$\rho = \left[\tilde{I}_0 + 2 \sum_{i=1}^{\tilde{m}_q - 1} (-1)^i \tilde{I}_i + \tilde{I}_{\tilde{m}_q} \right] \gamma, \alpha \in \mathbb{Z}_+$$

or

$$\rho = \left[\tilde{I}_0 + 2 \sum_{i=1}^{\tilde{m}_q - 1} (-1)^i \tilde{I}_i \right] \gamma, \alpha \notin \mathbb{Z}_+$$

i.e.

$$2\tilde{m}_q - 1 \geq \rho$$

on the root locus plot.

Step 3:

Choose a trial value for

$$K_{p0} \in S\left(K_p\right)$$

such that we can calculate γ and α

Step 4:

- For the selected

  $$K_{p0} \in S\left(K_p\right)$$

  , use equations,

  $$\rho = \left[\tilde{I}_0 + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \tilde{I}_i + \tilde{I}_{\tilde{m}_q}\right]\gamma, \alpha \in \mathbb{Z}_+$$

  or

  $$\rho = \left[\tilde{I}_0 + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \tilde{I}_i\right]\gamma, \alpha \notin \mathbb{Z}_+$$

  to obtain $\tilde{g}$. If $\tilde{g} \neq \varnothing, proceed to Step 5$

- Otherwise repeat Steps 2-3 with different ranges of $S(K_p)$ until $\tilde{g} \neq \varnothing$ is obtained.

Step 5:

Compute $[P_1]$ and $[P_2]$ for $K_{p0} \in S(K_p)$. Use the set of LMI given by Equation

$$sgn\left\{[P_1] + [P_2]\left[\tilde{\kappa}\right]\right\} = \left(\tilde{g}\right)^T$$

to obtain the non-empty stabilizing set $S_{\tilde{K}}$. If the solution to above equation is empty, then repeat Steps 3-5 with a different $K_{p0} \in S(K_p)$.

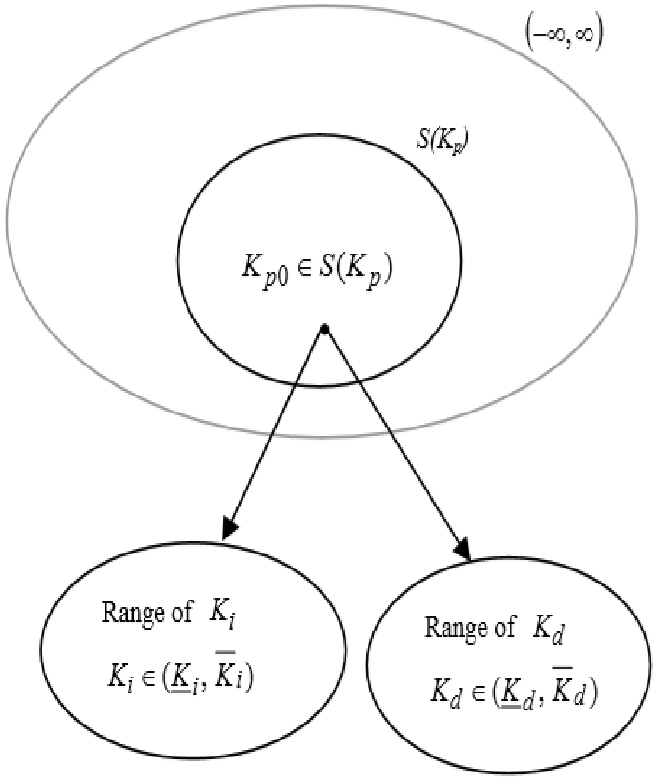


Figure 3.3: Venn Diagram illustrating the controller synthesis algorithm based on the Hermite-Biehler Theorem.

3.3 Simulation to verify the Implementation

The Simulation will experiment along the design example explained in the paper. That will help us understand the implementation of Hermite-Biehler Theorem's contribution in stabilizing the PID controller.

```
>> synthesisForStabilizingPID
The value of roe from the algorithm =
     5

The value of alpha from the algorithm  =
    7.5000
```

Figure 3.4: Computed value of ρ and α

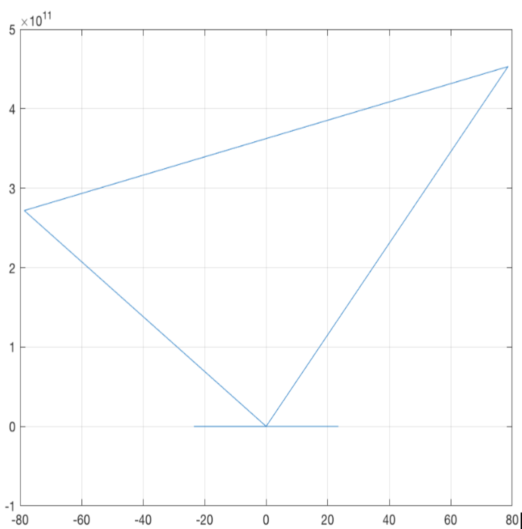
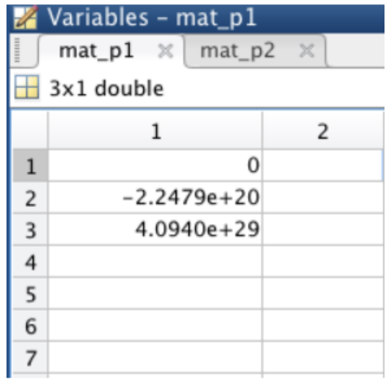


Figure 3.5: Simulation of K_p-K_i-K_d in 2-D space

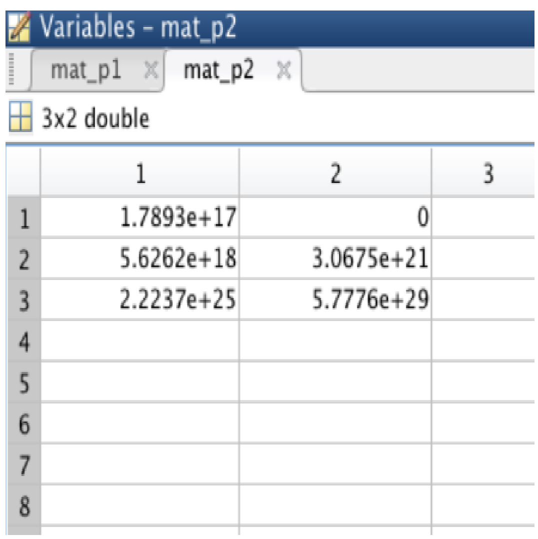

Variables - mat_p1

mat_p1 | mat_p2

3x1 double

	1	2
1	0	
2	-2.2479e+20	
3	4.0940e+29	
4		
5		
6		
7		

Figure 3.6: Computed value of the P_1 Matrix


Variables - mat_p2

mat_p1 | mat_p2

3x2 double

	1	2	3
1	1.7893e+17	0	
2	5.6262e+18	3.0675e+21	
3	2.2237e+25	5.7776e+29	
4			
5			
6			
7			
8			

Figure 3.7: Computed value of the P_2 Matrix

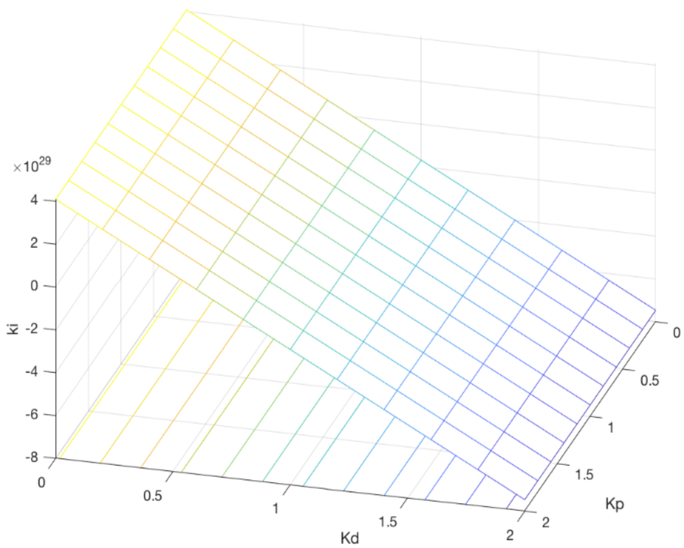


Figure 3.8: Simulation of K_p-K_i-K_d in 3-D space

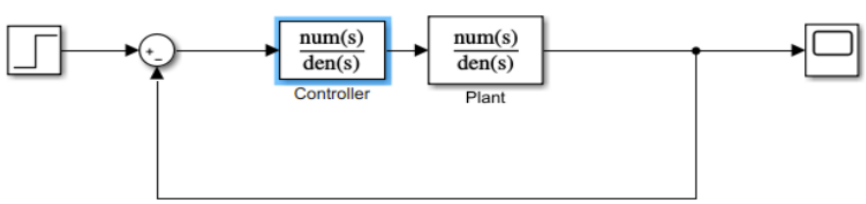


Figure 3.9: Block diagram of Closed-loop PID Controller system

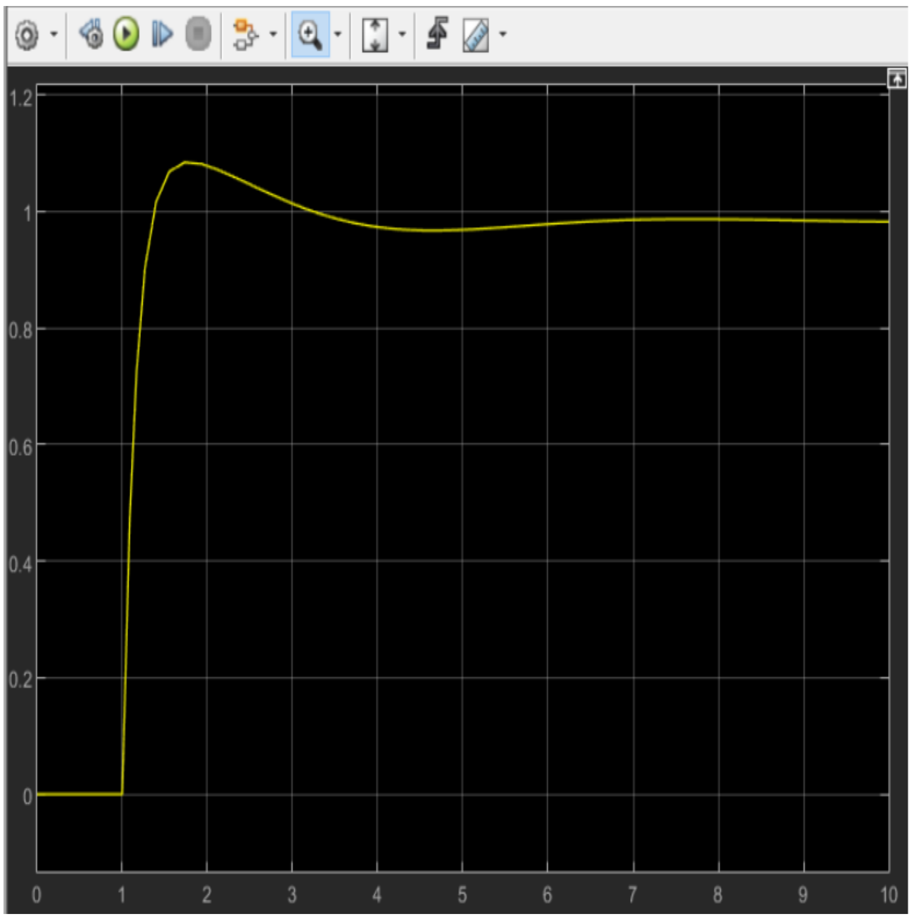


Figure 3.10: Graph representation

Chapter 4

Conclusion

The report studies the components of the paper such as PID Controller, Hurwitz Stability and Hermite Biehler so that we can understand the critical mathematics in the paper.

In the reference paper, the stability of linear time-invariant systems of arbitrary order is established mathematically using Hermite-Biehler framework from functional analysis. Stabilizing PID gains are chosen through a combination of root-locus techniques and linear matrix inequalities that result from application of the generalized Hermite-Biehler Theorem. The resultant controller synthesis algorithm is, in general, programmable for linear systems of arbitrary order.

With Simulink, we can show that the framework works and that the Hermit-Biehler Theorem can be used to stabilize the PID Controller.

Bibliography

[1] Iqbal, K. and Roy, A. (2004). Stabilizing PID Controllers for a Single-Link Biomechanical Model with Position, Velocity, and Force Feedback. Journal of Biomechanical Engineering, 126(6), pp.838-843.

[2] https://www.omega.com/en-us/. (2019). How Does a PID Controller Work?. [online] Available at: https://www.omega.com/en-us/resources/how-does-a-pid-controller-work [Accessed 27 Nov. 2019].

[3] Ho, M., A. Datta and S.P. Bhattacharyya (1999). Generalizations of the Hermite-Biehler Theorem, Linear Algebra and Its Applications, Vol. 302, pp. 135-153

[4] Bartlett, A.C., C.V. Hollot and H. Lin (1988). Root Location of an Entire Polytope of Polynomials: It Suffices to Check the Edges, Mathematics of Controls, Signals and Systems, Vol. 1, pp. 6171

[5] Mhaisgawali, M. (2013). Speed Control of Induction Motor using PI and PID Controller. IOSR Journal of Engineering, 03(05), pp.25-30.

[6] En.wikipedia.org. (2019). Routh–Hurwitz stability criterion. [online] Available at: https://en.wikipedia.org/wiki/Routh-Hurwitz-stability-criterion [Accessed 27 Nov. 2019].

Appendix

1. Theorem

2. Corollaries

3. Code

Theorem

Theorem: The characteristic polynomial $\psi(s)$ is Hurwitz stable if and only if there exist a feasible non-empty solution $\{\tilde{I}_i\} \neq \varnothing$ to either of the following equations:

$$\rho = \left[\tilde{I}_0 + 2\sum_{i=1}^{\tilde{m}_q - 1} (-1)^i \tilde{I}_i + \tilde{I}_{\tilde{m}_q} \right] \gamma, \ \alpha \in \mathbb{Z}_+ \qquad (12)$$

$$\rho = \left[\tilde{I}_0 + 2\sum_{i=1}^{\tilde{m}_q - 1} (-1)^i \tilde{I}_i \right] \gamma, \ \alpha \notin \mathbb{Z}_+, \qquad (13)$$

where $\tilde{I}_i,\ i = 0,1,\ldots,\tilde{m}_q$ denote the unknown integer variables as defined in Eqs. (11a) and (11b).

Proof: (*Necessity*) If Eq. (12) or (13) has a feasible solution, then by using Eqs. (11a) and (11b) in Eqs. (9) and (10), and applying the generalized Hermite-Biehler Theorem, we can ensure that $\sigma[\delta(s)] = m_\psi - \sigma[W(s)]$, so that $\sigma[\psi(s)] = \Theta[\psi(s)]$ and $\psi(s)$ is Hurwitz stable.

(*Sufficiency*) If $\psi(s)$ is Hurwitz stable, then $\sigma[\psi(s)] = \Theta[\psi(s)]$ s.t. that $\sigma[\delta(s)] = m_\psi - \sigma[W(s)]$, and from the generalized Hermite-Biehler Theorem either Eq. (7) or (8) must be satisfied. The only way to satisfy it is if Eq. (12) or (13) has a feasible solution. ♣

Corollaries

Corollary 1: The range of K_p for which the root distribution of $q(\omega)$ is such that Eq. (12) or (13) is satisfied, can be identified from the root locus plot of $\{1+K_p\, q_2(\omega)/q_1(\omega)\}$. This range is denoted as $S(K_p)=(\underline{K}_p, \overline{K}_p)\subseteq(-\infty,\infty)$, where the under bar and the over bar represent the lower and upper limit, respectively.

Corollary 2: Assume that a non-empty solution to Eq. (12) or (13) has been found, i.e., let $\tilde{g}\triangleq\{\tilde{I}_i\}\neq\varnothing$, then the ranges of the stabilizing gains K_i and K_d, i.e., $(\underline{K}_i, \overline{K}_i)$ and $(\underline{K}_d, \overline{K}_d)$, can be solved from the following linear matrix inequalities (LMI):

$$\operatorname{sgn}\{[P_1]+[P_2][\tilde{\kappa}]\}=(\tilde{g})^T, \tag{14}$$

where the matrices $[P_1]\in\mathbb{R}^{\tilde{m}_q,1}$, $[P_2]\in\mathbb{R}^{\tilde{m}_q,2}$, and $[\tilde{\kappa}]\in\mathbb{R}^{2,1}$ are defined as:

$$[P_1]\triangleq[p_1(0)\quad p_1(\omega_{o1})\quad p_1(\omega_{o2})\quad\cdots\quad p_1(\omega_{o\tilde{m}_q-1})]^T \tag{15}$$

$$[P_2]\triangleq\begin{bmatrix} p_2(\omega_0) & 0 \\ p_2(\omega_{o1}) & -\omega_{o1}^2 p_2(\omega_{o1}) \\ p_2(\omega_{o2}) & -\omega_{o2}^2 p_2(\omega_{o2}) \\ \vdots & \vdots \\ p_2(\omega_{o\tilde{m}_q-1}) & -\omega_{o\tilde{m}_q-1}^2 p_2(\omega_{o\tilde{m}_q-1}) \end{bmatrix} \tag{16}$$

$$[\tilde{\kappa}]\triangleq[K_i\quad K_d]^T \tag{17}$$

thus, for a given $K_p\in S(K_p)$, the stabilizing PID controller set is: $S_{\tilde{K}}\triangleq\{K_p, (\underline{K}_i, \overline{K}_i), (\underline{K}_d, \overline{K}_d)\}$.

MATLAB Code

```
%question = 'What is the value of Ng(s) equation? ';
ngs = input(question); %[0.112 13.03 305.08 596.17]   (input given in the paper)
%question2 = 'What is the value of Q(s) equation? ';
q_s = input(question2); %[ -0.0004 0.14 22.17 1681.63 67800 1360000 10500000 -6230000 -
54700000]; (input from the paper)
%question3 = 'What is the value of W(s) equation? ';
w_s = input(question3); %[-0.2 -28.78 -383.69 99200 4890000 82800000 423000000] (input
form the paper)

%Now to convolution of the polynomials to get the value of psy(s)
psy = conv([1 0],q_s) + conv(ngs,w_s);
psy_roots = roots(psy);
m_psy = length(psy_roots); %number of roots of the equation.

w_s_roots = roots(w_s);
m_w = length(w_s_roots); %number of roots of the equation.

alpha = 1/2*(m_psy+m_w);
roe = m_psy - sigma_r(w_s_roots);
printStmt = ['The value of roe from the algorithm = ',roe];
disp(printStmt);
printStmt2 = ['The value of alpha from the algorithm  = ',alpha];
disp(printStmt2);

%Generating the sigma value for the psy function
sig_psy = sigma_r(psy_roots);

%Generating the odd and even powered values in FREQUENCY DOMAIN
w_e_s = evenPowers(w_s);
q_e_s = evenPowers(q_s);
w_o_s = oddPowers(w_s);
q_o_s = oddPowers(q_s);

w_e_w = sToOmega(w_e_s);
q_e_w = sToOmega(q_e_s);
w_o_w = sToOmega(w_o_s);
q_o_w = sToOmega(q_o_s);

%Generating the values of P1, P2, Q1, and Q2
p1_w = conv([1 0 0],(conv(q_e_w,w_o_w)-conv(q_o_w,w_e_w)));
p2_w = conv([0 0 1],conv(w_e_w,w_e_w))+conv([1 0 0],conv(w_o_w,w_o_w));
q1_w = conv([0 1 0],(conv([0 0 1],conv(w_e_w,q_e_w))+conv([1 0 0],conv(w_o_w,q_o_w))));
q2_w = conv([0 1 0],p2_w);
```

```
%Estimated the Value of Kp to be equal to 5.
%Kp=5 (as mentioned in the paper)
q_w = q1_w + conv([0 0 5],q2_w);
q_roots = roots(q_w);
%Graph of the transfer function for the given values of Kp
plot(q_roots,polyval(q_w,q_roots))
grid on

%To generate the value of the Signum function
g_mat = [1,-1,1];
val = hbTheorem(p1_w,p2_w,g_mat);

function sig = sigma_r(roots)
   totalRoots = length(roots);
   rhpRoots = 0;
   for x=1:totalRoots
      if imag(roots(x))==0 && real(roots(x))>0
         rhpRoots=rhpRoots+1;
      end
   end
   if rhpRoots==0
      'Given set of equations are Hurwitz stable';
   end
   sig = totalRoots - 2*rhpRoots;
end

function odd = oddPowers(eqn) %
   eqSize = length(eqn);
   odd = zeros(1,eqSize);
   if mod(eqSize,2)==0
      %odd powered
      for x=1:eqSize-1
         if mod(x,2)~=0
           odd(x+1)=eqn(x);
         end
      end
   else
      %even powered
      for x=1:eqSize-1
         if mod(x,2)==0
           odd(x+1)=eqn(x);
         end
      end
   end
end

function even = evenPowers(eqn)
   eqSize = length(eqn);
   even = zeros(1,eqSize);
   if mod(eqSize,2)==0
```

```
        %odd powered
        for x=1:eqSize
            if mod(x,2)==0
                even(x)=eqn(x);
            end
        end
    else
        %even powered
        for x=1:eqSize
            if mod(x,2)~=0
                even(x)=eqn(x);
            end
        end
    end
end

function omegaEqn = sToOmega(eqn)
    omegaEqn = eqn;
    eqSize = length(eqn);
    for x=1:eqSize
        if mod(eqSize-x,4)~=0 && mod(eqSize-x,2)==0
            omegaEqn(x)=eqn(x)*-1;
        end
    end
end

%function to calculate the Hermite Biehler signum function to get the values of Gains
function mat = hbTheorem(p1_w,p2_w,g_mat)
    roots = [23.35 -161.19 0]; %Given value of roots for the equation in the paper.
    roots_len = length(roots);
    mat_p1 = zeros(roots_len,1);
    mat_p2 = zeros(roots_len,2);

    %default condition for the matrix
    mat_p1(1) = polyval(p1_w,0);
    mat_p2(1,1) = polyval(p2_w,0);
    mat_p2(1,2) = 0;

    %Calculating the values for the P1, P2 Matrices
    for x=1:roots_len-1
        mat_p1(x+1) = polyval(p1_w,roots(x));
        mat_p2(x+1,1) = polyval(p2_w,roots(x));
        mat_p2(x+1,2) = (-1)*power(roots(x),2)*mat_p2(x+1,1);
    end

    %Printing the matrices for the Equation
    printStmt = [‘The values of the Matrices P1 =’,mat_p1,’ and Matrices P2 =’, mat_p2,
    ‘for the values of the g Matrix =’,g_mat];
    disp(printStmt);

    syms x y
```

```
    eqn1 = sign(mat_p2(1,1)*x + mat_p2(1,2)*y + mat_p1(1)) == g_mat(1);
    eqn2 = sign(mat_p2(2,1)*x + mat_p2(2,2)*y + mat_p1(2)) == g_mat(2);
    eqn3 = sign(mat_p2(3,1)*x + mat_p2(3,2)*y + mat_p1(3)) == g_mat(3);

   %Equating the equation values to generate the 3-D Graph of the Values of Ki, Kd
    sol = solve([eqn1, eqn2], [x, y]);
    xSol = sol.x
    ySol = sol.y

    fcn0 = @(x,y) (mat_p2(1,1)*x + mat_p2(1,2)*y + mat_p1(1));
    fcn1 = @(x,y) (mat_p2(2,1)*x + mat_p2(2,2)*y + mat_p1(2));
    fcn2 = @(x,y) (mat_p2(3,1)*x + mat_p2(3,2)*y + mat_p1(3));

    [X,Y] = meshgrid(0:0.2:2);
    meshc(X, Y, fcn0(X,Y))
    meshc(X, Y, fcn1(X,Y))
    meshc(X, Y, fcn2(X,Y))
    %grid on
    xlabel('Kp')
    ylabel('Kd')
    zlabel('Ki')
    mat = mat_p2;
End
```