553.628, STOCHASTIC PROCESSES AND APPLICATIONS TO FINANCE FINAL PROJECT: SNOWBALL OPTION PRICING

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Contents

1. Abstarct

The goal of this project is to set up a robust evaluation method to price the Snowball option with the underlying asset CSI 500 index, calculate Greeks, perform model tests, and perform a backtest on the delta hedging and vega hedging.

2. Introduction

The Snowball option has gained increasing attention from the recent intensive knock-in events in the Chinese stock market — A shares. For an investor longing a Snowball option, she is essentially shorting an exotic put option, and the institution from which the investor buys is longing an exotic put option. From the delta-hedge principle, the institution establishes a long position of the underlying index futures. However, when the knock-in event is approaching, the institution has to close a large portion of his hedging long position in a short amount of time due to a steep increase in the negative delta of the longing Snowball option, which leads to a drastic decrease in the index futures and leads to further more knock-in events. This phenomenon is described as the 'death spiral'.

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In this project, we derived the theoretical pricing for the Snowball option (partial cases) and simulated numerically it via the Monte Carlo method.

2.1. Assumptions.

Throughout this report, we adhere to the following assumptions: Firstly, The underlying asset price follows the Geometric Brownian Motion dynamics, i.e. under the risk-neutral measure $\tilde{\mathbb{P}}$:

$$d\log(S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma d\tilde{W}_t, \quad t \in [0, \infty)$$

where

- $\tilde{W}$ is a standard Brownian motion under measure $\tilde{\mathbb{P}}$.
- σ is implied (ATM) volatility derived from option quotes on the underlying.
- r risk-free rate.

The Snowball option is a structural product with duration T that is comprised of the following components:

- the underlying asset S_t.
- knock-in event: the underlying asset price falls below some pre-set lower barrier $L < S_0$. For theoretical purposes, we consider knock-in event can happen every single moment in $(0, T]$.
- knock-out event: the underlying assets price exceeds some pre-set upper barrier $U > S_0$. For theoretical purposes, we consider knock-out events can happen every single moment in $(0, T]$.
- coupon rate y.
- notional principal M, the amount of money the investor invests.

It has the following pay rules:

(1) Once the underlying asset price exceeds U at time t^*, i.e. the knock-out event happens, the transaction ends immediately and the investor receives an immediate annualized coupon from the institution with the amount

$$Me^{yt^*}$$

(2) If the underlying price does not touch the knock-in L or knock-out boundary U during the duration $(0, T]$, the contract will automatically end and the customer will still receive annualized coupons with the amount

$$Me^{yT}$$

(3) If at any moment during the duration $(0, T]$, the target has fallen below the knock-in threshold L and no knock-out event has occurred before expiration, the investor receives at expiry

$$M \min(1, \frac{S_T}{S_0})$$

We stress that the knock-out events dominate knock-in events, in the sense that, once the knock-out event happens, the payoff for the investor is unaffected, regardless of the previous knock-in event or not.

3. Particular cases

In this section, we derive the theoretical pricing of the Snowball option in several extreme cases.

(1) Very low volatility. The probability of knock-in or knock-out is very low. The contract is equivalent to zero coupon bond (simple compounding).
(2) Very low knock-in barrier L. Then the structure is a knock-out American-type contract with the complication that time enters the payoff.
$$Me^{yt^*}.$$
We will price it using the distribution of stopping time t^*, and do the numerical integration for comparison with Monte Carlo simulation methods.
(3) Very high knock-out barrier U. Then the structure is equivalent to a short knock-in put option with strike S_0 and longs knock-out digital call option with notional M. Can be valued analytically.

3.1. **Very Low Volatility.**

When volatility is extremely low like 0, the price of the underlying remains constant. Neither knock-out nor knock-in would happen during the duration. Thus, the client get paid Me^{yT} at expiry. The return of the snowball for the client is $M(e^{(y-r)T}-1)$ at $t_0=0$. The value of the snowball for the institution is $M(1-e^{(y-r)T})$ at $t_0=0$.

3.2. **Very Low Knock-In Barrier L.**

In this section, we consider a boundary case where the knock-in barrier L is extremely low, and theoretically, we consider $L=0$, i.e. there is no knock-in barrier. We set the notional principal $M=1$ for simplicity.

In this case, we have the profit of the investor at time 0 as

$$\begin{aligned} V(0) &= \mathbb{E}_{\tilde{\mathbb{P}}}[e^{(y-r)\cdot(\tau_U\wedge T)}-1] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}}[G(\tau_U\wedge T)]-1 \end{aligned} \tag{1}$$

where

$$\tau_U := \min\{t>0 : S_t = U\}$$

is the first passage time of S_t exceeds U and $G(x):=e^{(y-r)x}$. Denote $U_X := \frac{1}{\sigma}\log\frac{U}{S_0}$ and

$$X_t := \frac{1}{\sigma}\log\frac{S_t}{S_0} = \frac{r-\frac{1}{2}\sigma^2}{\sigma}t+\tilde{W}_t.$$

Observing that

$$\tau_U\wedge T = \min\{t>0 : \frac{1}{\sigma}\log\frac{S_t}{S_0}=U_X\}\wedge T = \begin{cases} \min\{0<t\le T : X_t = U_X\} =: \tau^X_{U_X} & \text{if } \tau_U<T \\ T & \text{otherwise,} \end{cases}$$

where the last equality holds at least in almost everywhere sense, and hence we may invoke arguments in the appendix A that the distribution of $\tau_U\wedge T$ has density on $(0,T)$

$$f(t) = \frac{U_X}{t\sqrt{2\pi t}}\exp\{-\frac{(\theta t-U_X)^2}{2t}\}$$

and mass $1-\int_0^T f(t)\mathrm{d}t$ on point T, where $\theta = \frac{r-\frac{1}{2}\sigma^2}{\sigma}$.

That is,

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{P}}}[G(\tau_U \wedge T)] &= \int_0^T G(t)f(t)\mathrm{d}t + G(T)\tilde{\mathbb{P}}(\tau_U \wedge T = T) \\ &= \int_0^T G(t)f(t)\mathrm{d}t + G(T)\Big[1 - \int_0^T f(t)\mathrm{d}t\Big],\end{aligned}$$

and hence

$$V(0) = \mathbb{E}_{\tilde{\mathbb{P}}}[G(\tau_U \wedge T)] - 1 = \Big\{\int_0^T e^{(y-r)t} f(t)\mathrm{d}t + e^{(y-r)T}\Big[1 - \int_0^T f(t)\mathrm{d}t\Big]\Big\} - 1. \tag{2}$$

3.3. **Very High Knock-Out Barrier U.**

In this section, we consider a boundary case where the knock-out barrier U is extremely high, and theoretically, we consider $U = \infty$, i.e. there is no knock-out barrier. Still, we assume $M = 1$ for simplicity.

In this case, we can divide the payoff into two parts, corresponding to two conditions. The first one corresponds to that knock-in event happens and under this condition, the return of the investor (valued at time 0) is

$$e^{-rT}\min(1, \frac{S_T}{S_0}) - 1 = -(\frac{1}{S_0}e^{-rT}\max(0, S_0 - S_T) + 1 - e^{rT}),$$

where $e^{-rT}\max(0, S_0 - S_T)$ via taking expectation is the premium of vanilla European put option with strike S_0 and expiry T. Secondly, if the knock-in event does not happen, the investor receives payment e^{yT} and hence the return is (valued at time 0) is

$$e^{(y-r)T} - 1$$

In math words, we have

$$V(0) = -\{\frac{1}{S_0}V_P(0) + 1 - e^{-rT}\}\tilde{\mathbb{P}}(\tau_L \wedge T < T) + (e^{(y-r)T} - 1)\tilde{\mathbb{P}}(\tau_L \wedge T = T)$$

where $\tau_L \coloneqq \min\{t > 0 : S_t = L\}$ is the first passage time of S_t falls below L, and V_P is the premium of the vanilla European put option, with strike S_0 and expiry T.

Similar to the previous section, $\tau_L \wedge T$ has density on $(0, T)$

$$f(t) = \frac{|L_X|}{t\sqrt{2\pi t}}\exp\{-\frac{(\theta t - L_X)^2}{2t}\}$$

and mass $1 - \int_0^T f(t)\mathrm{d}t$ on point T, where $\theta = \frac{r - 1/2\sigma^2}{\sigma}$, and $L_X = \frac{1}{\sigma}\log\frac{L}{S_0}$. Combining all information, we have

$$\begin{aligned}V(0) &= -\{\frac{1}{S_0}V_P(0) + 1 - e^{-rT}\}\tilde{\mathbb{P}}(\tau_L \wedge T < T) + (e^{(y-r)T} - 1)\tilde{\mathbb{P}}(\tau_L \wedge T = T) \\ &= -\{\frac{1}{S_0}V_P(0) + 1 - e^{-rT}\}\int_0^T f(t)\mathrm{d}t + (e^{(y-r)T} - 1)[1 - \int_0^T f(t)\mathrm{d}t]\end{aligned}$$

where V_P is the premium of vanilla European put option, with strike S_0 and expiry T.

3.4. **PDE approach: Without Upper Barrier.**

In this case, the institution

- longs $\frac{M}{S_0}$ down-and-in put barrier option with $K = S_0$ and barrier L.
- shorts yTM or $M(e^{rT} - 1)$ down-and-out digital call with $K = 0$ and barrier L. Choosing yTM or $M(e^{rT} - 1)$ depends on the way of compounding. We use $M(e^{rT} - 1)$ consistent with the way of compounding in the simulation.

The PDE approach can price both barrier options.

Notice that the institution receives the cash M from time t_0. The institution can deposit the cash into the bank to get risk-free profit $M(e^{rT} - 1)$, which has value $M(1 - e^{-rT})$ at time t_0.

For European vanilla put option:

$$\begin{cases} \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rs\frac{\partial P}{\partial S} - rP = 0 \\ P(T, S) = (K - S(T))^+ \\ P(t, S) \to Ke^{-r(T-t)}, S \to 0 \\ P(t, S) \to 0, S \to \infty \end{cases}$$

Change of variable: $x = \ln \frac{S}{K}, \tau = \frac{1}{2}\sigma^2(T - t), k_1 = \frac{r}{\frac{1}{2}\sigma^2}, P(t, S) = Kv(\tau, x)$

Then the PDE:

$$\begin{cases} \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k_1 - 1)\frac{\partial v}{\partial x} - k_1 v \\ v(0, x) = (1 - e^x)^+ \\ v(\tau, x) \to e^{-k_1 \tau}, x \to -\infty \\ v(\tau, x) \to 0, x \to \infty \end{cases}$$

Let $v = e^{\alpha x + \beta\tau}u(\tau, x)$. By letting $\alpha = -\frac{1}{2}(k_1-1), \beta = -\frac{1}{4}(k_1+1)^2, v = e^{-\frac{1}{2}(k_1-1)x - \frac{1}{4}(k_1+1)^2\tau}u(\tau, x)$, the PDE becomes:

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \\ u(0, x) = (e^{\frac{1}{2}(k_1-1)x} - e^{\frac{1}{2}(k_1+1)x})^+ e^{-\beta\tau} \\ u(\tau, x) \to e^{\frac{1}{2}(k_1-1)x + \frac{1}{4}(k_1-1)^2\tau}, x \to -\infty \\ u(\tau, x) \to 0, x \to \infty \end{cases} \tag{3}$$

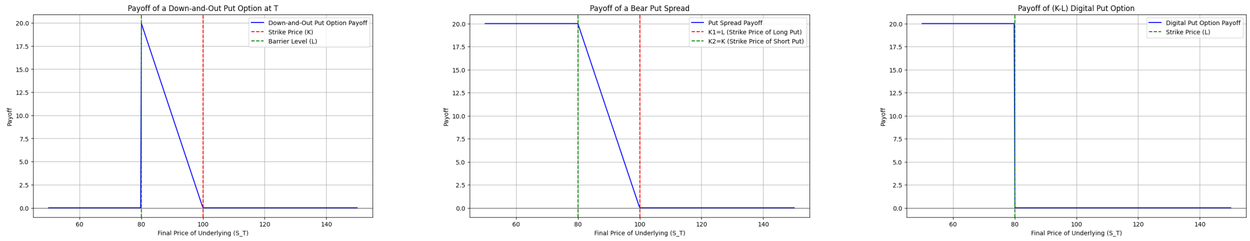
Connect the option price P with the solution u: $P(t, S) = Kv(\tau, x) = Ke^{\alpha x + \beta\tau}u(\tau, x)$

3.4.1. *Down-and-In Put.*

We first price the down-and-out put with barrier $L < K$. The payoff of down-and-out put at time T can be decomposed as a put spread minus $K - L$ digital put with strike L (see Fig 1).

Now consider the barrier put option. Let $x_0 = \ln \frac{L}{K} < 0$ since $L < K$. The PDE 3 becomes: (the third condition is imposed by barrier)

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \\ u(0, x) = (e^{\frac{1}{2}(k_1-1)x} - e^{\frac{1}{2}(k_1+1)x})^+ e^{-\beta\tau} \\ u(\tau, x_0) = 0 \\ u(\tau, x) \to 0, x \to \infty \end{cases}$$



(A) Down-and-Out Put (B) Put Spread (C) Digital Put

Figure 1. Decomposition of Down-and-Out Put

By invoking the Method of Image:

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, x \in \mathbb{R}, 0 < \tau \\ u(0,x) = (e^{\frac{1}{2}(k_1-1)x} - e^{\frac{1}{2}(k_1+1)x})^+ e^{-\beta\tau}, x > x_0 \\ u(0,x) = -(e^{(k_1-1)(x_0-\frac{1}{2}x} - e^{(k_1+1)(x_0-\frac{1}{2}x)})^+ e^{-\beta\tau}, x < x_0 \end{cases}$$

The solution has such form $V(t,S) = Ke^{\alpha x + \beta\tau}(u_1(\tau,x) + u_2(\tau,x))$, where $u_2(\tau,x)$ is the solution with anti-symmetric initial data.

Then

$$u_1(\tau,x) = P(t,S)e^{-\alpha x - \beta\tau}/K,$$

$$u_2(\tau,x) = -u_1(\tau, 2x_0 - x) = -e^{-\alpha(2x_0-x)-\beta\tau}P(t, \frac{B^2}{S})/K$$

By decomposition,

$$\begin{aligned} V_{down-and-out}(t,S) =& P(t,S,K) - P(t,S,L) - P_d(t,S,L)(K-L) \\ & - (\frac{L}{S})^{(k_1-1)}[P(t, \frac{L^2}{S}, K) - P(t, \frac{L^2}{S}, L) - P_d(t, \frac{L^2}{S}, L)(K-L)] \end{aligned}$$

By in-and-out parity, we now have the price of down-and-in:

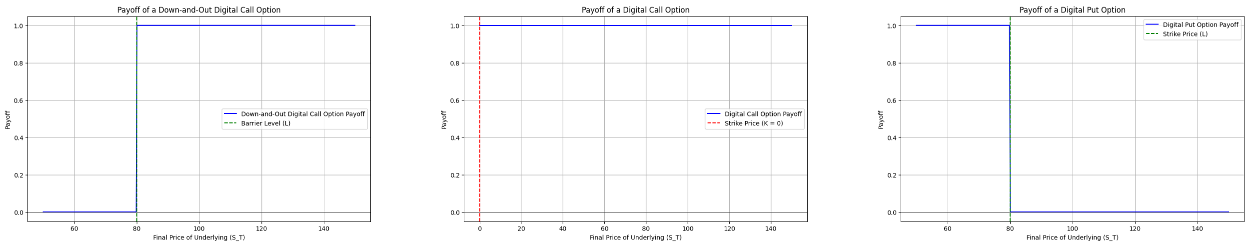
$$\begin{aligned} V_{down-and-in}(t,S) =& P(t,S,L) + P_d(t,S,L)(K-L) \\ & + (\frac{L}{S})^{(k_1-1)}[P(t, \frac{L^2}{S}, K) - P(t, \frac{L^2}{S}, L) - P_d(t, \frac{L^2}{S}, L)(K-L)] \end{aligned}$$

3.4.2. *Down-and-Out Digital Call.*

We price the down-and-out digital call by a similar argument. Notice the payoff of the down-and-out digital call can be decomposed as a digital call with strike 0 and a digital put with strike L (see Fig 2). By a similar argument, we have the price of down-and-out digital call

$$C_d = C_d(t,S,0) - P_d(t,S,L) - (\frac{L}{S})^{(k_1-1)}[C_d(t, \frac{L^2}{S}, 0) - P_d(t, \frac{L^2}{S}, L)]$$

We can calculate the value of product at time t_0 in this extreme case from the institution perspective: $V_0 = \frac{M}{S_0}P_{down-and-in}(0) - M(e^{rT}-1) \times DigitalCall_{down-and-out}(0) + M(1-e^{-rT})$. The value of the product to the client at t_0 is simply $-V_0$



(A) Down-and-Out C_d (B) Digital Call (C) Digital Put

FIGURE 2. Decomposition of Down-and-Out Digital Call

4. Evaluation, bounds, Greeks, hedging

(1) Evaluation is likely to be done by performing a Monte Carlo, use Advanced Monte Carlo techniques (like variance reduction, moment matching, stratification and other), references will be provided. Pay attention to the choice of discretization step in time. Check Monte Carlo vs analytical prices for extreme cases (barriers, level of volatility).
(2) Work out the bounds of the option (using particular cases) and behavior of the option with changing barriers L and U.
(3) Calculate sensitivities w.r.t
  - Delta: The sensitivity w.r.t the underlying price. This can be used for hedging with a position in the underlying (and possibly backtesting).
  - Vega: Volatility is a very important factor, and Vega must be calculated. Come up with a hedge buying/selling options on the underlying.

4.1. **Monte Carlo Simulation.**

Throughout this Monte Carlo simulation, we are pricing in the view of the investors. We use $S_0 = 50, \sigma = 0.55, T = 1, r = 0.04, y = 0.18, U = 1.05S_0, L = 0.7S_0, M = 1$ for our simulation. In Feb 2024, there was a huge drawback in the CSI500 and CSI1000 index, which are 2 common underlying of Snowball. Therefore many Snowball products konck-in at that time. The volatility of CSI500 was once over 50% during Feb 2024 (See Fig 3). Thus, a high volatility is assumed to test this situation. The upper and lower barrier coefficients are the usual combination institutions give.

If the observation is daily, the number of steps n in each simulated path should be $T \times 252$. If the observation is continuous, n should be as large as possible. We demonstrate the simulation results by choosing the number of steps $n : 500 : 250 : 1250$ and the number of paths $N : 1000 : 2000 : 10000$. Fig 4 is the simulation result.

When the number of steps increases, the probability of observing knock-out increases more than the probability of knock-in since we set the U closer to S_0 than L. The value to the client thus increases.

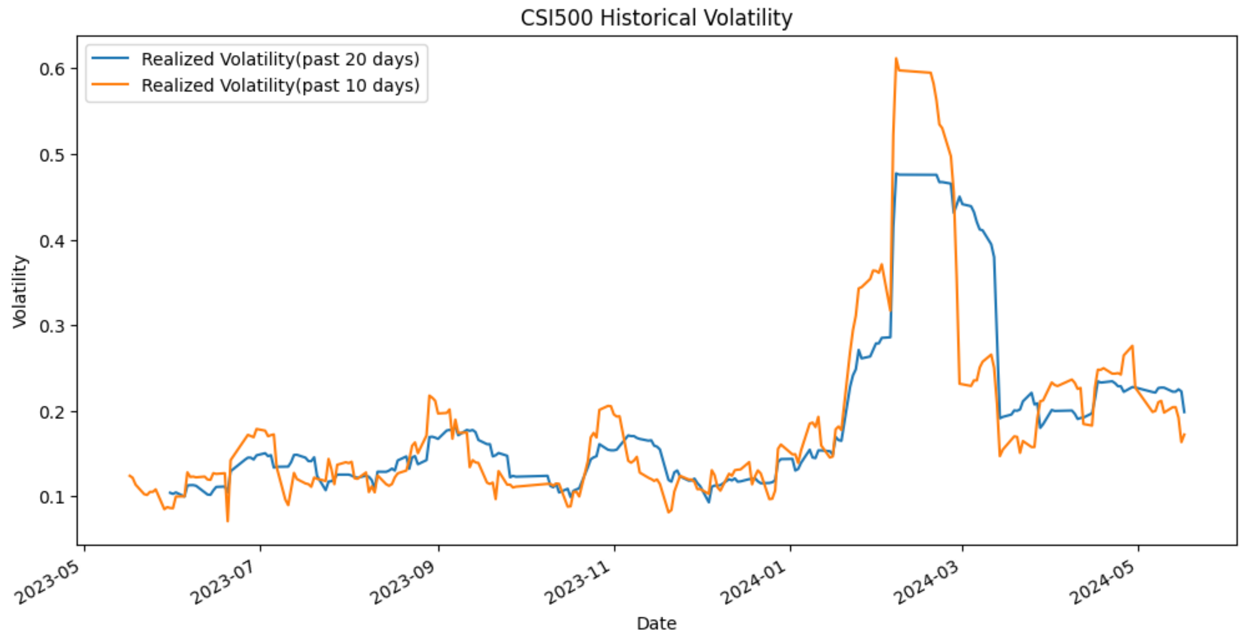


FIGURE 3. CSI500 Realized Volatility

	250	500	750	1000	1250
1000	-0.049288	-0.035757	-0.031619	-0.032447	-0.038424
3000	-0.046641	-0.039121	-0.042268	-0.038804	-0.040797
5000	-0.046929	-0.043696	-0.042848	-0.043824	-0.034846
7000	-0.045400	-0.042438	-0.039750	-0.039352	-0.041732
9000	-0.043871	-0.042678	-0.042497	-0.044208	-0.041147
10000	-0.047771	-0.039323	-0.042720	-0.040431	-0.039129

FIGURE 4. Simulation with Different n and N

4.2. **Extreme Cases.**

In this section, we still use $S_0 = 50, \sigma = 0.55, T = 1, r = 0.04, y = 0.18, U = 1.05S_0, L = 0.7S_0, M = 1$. The number of steps $n = 252$ to model daily observation and the number of paths $N = 5000$.

4.2.1. *Low Volatility.*

When setting the volatility $\sigma$to 0.00001 and all other parameters the same. the simulation and the theoretical results $M(e^{(y-r)T}-1)=0.1503$ coincide. See code for details.

4.2.2. *Low Lower Barrier.*

In this section, we set $L=0$. We can visualize the theoretical return of the investor by the probabilistic approach and the Monte Carlo counterpart in Fig 5. When we vary one parameter, we keep all the other parameters the same as the original settings. For example, when we varying σ from 0 to 2, we set $S_0=50, T=1, r=0.04, y=0.18, U=1.05S_0, L=0.7S_0, M=1$.

In fact, in this case with no knock-in barrier, when the high barrier gets higher, the return of the investor converges to the coupon rate, discounted by the risk-free rate, which coincides with $0.18-0.04=0.14$ in the picture. When volatility gets higher, the investor has a high probability of very early termination resulting in low to zero return, which coincides with our simulated and theoretical return in the probabilistic approach.

In particular, we apply the `scipy.integrate.quad` to numerically calculate the integral in 2. We denote the numerical integral of $\int_0^T e^{(y-r)t} f(t)\mathrm{d}t$ as I_1 and that of $\int_0^T f(t)\mathrm{d}t$ as I_2, and we know there are numerical errors $\epsilon_1 := |\int_0^T e^{(y-r)t} f(t)\mathrm{d}t - I_1|$, $\epsilon_2 := |\int_0^T f(t)\mathrm{d}t - I_2|$. Our final numerical result for the theoretical value is given by $I_1 + e^{(y-r)T}(1-I_2)-1$, with the following error bound

$$
\begin{aligned}
&|I_1 + e^{(y-r)T}(1-I_2) - 1 - \Big\{ \int_0^T e^{(y-r)t} f(t)\mathrm{d}t + e^{(y-r)T}\Big[1-\int_0^T f(t)\mathrm{d}t\Big] - 1\Big\}| \\
&\leq |I_1 - \int_0^T e^{(y-r)t} f(t)\mathrm{d}t| + e^{(y-r)T}|I_2 - \int_0^T f(t)\mathrm{d}t| \\
&= \epsilon_1 + e^{(y-r)T}\epsilon_2
\end{aligned} \tag{4}
$$

4.2.3. *High Upper Barrier.*

When setting the upper barrier to ∞, the simulation results are very close to the theoretical results given by the PDE approach. The plots of the theoretical value against the simulation results are shown. See Fig 6 for value change with parameters. When we vary one parameter, we keep all the other parameters the same as the original settings.

4.2.4. *Snowball with Upper and Lower Barrier.*

Lastly, we refer readers to Fig 7 for a complete setting of simulation of Snowball option pricing. When we vary one parameter, we keep all the other parameters the same as the original settings.

Generally, the higher volatility, lower barrier, and upper barrier will decrease the product value to the investor. At the same time, a longer time to maturity and yield rate will increase the product's value to the investor.

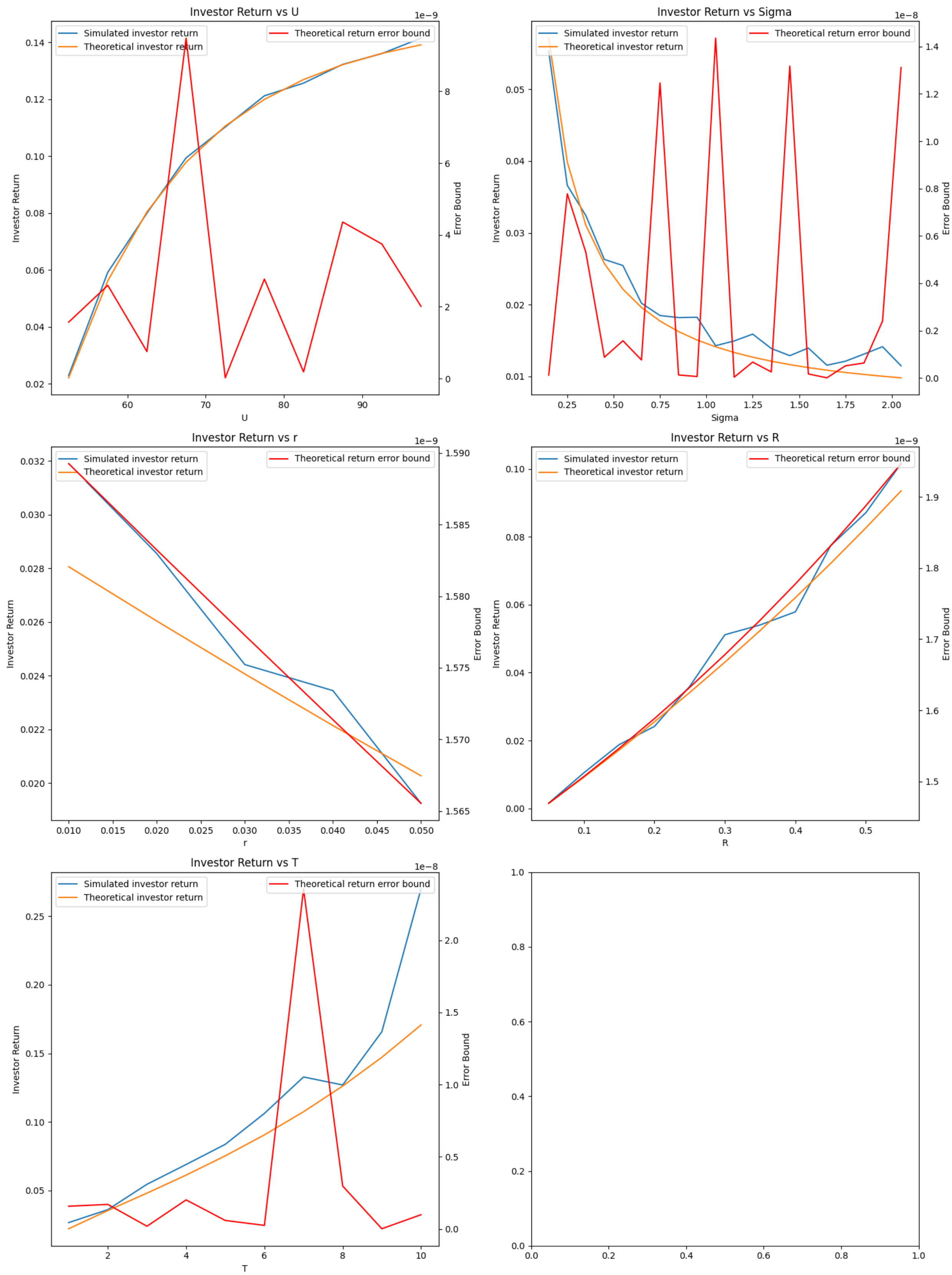
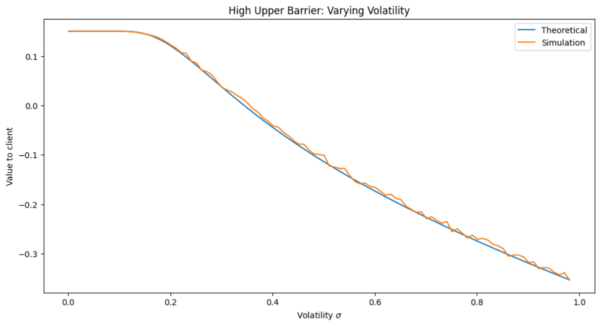
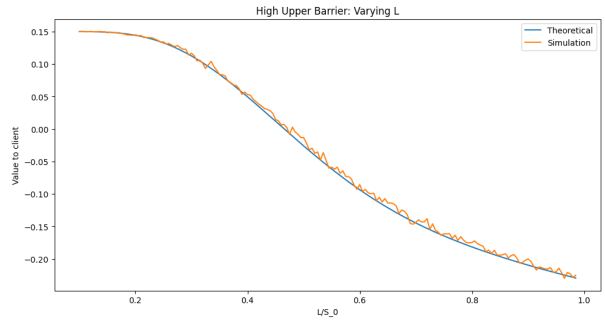


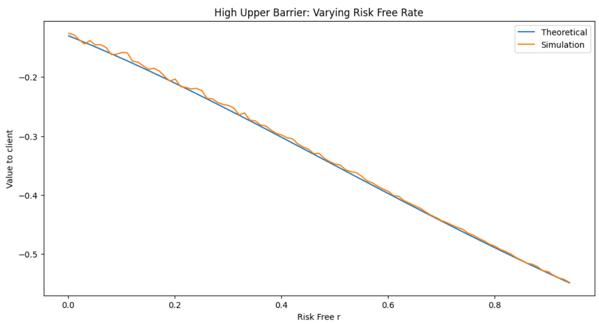
FIGURE 5. Low Lower Barrier



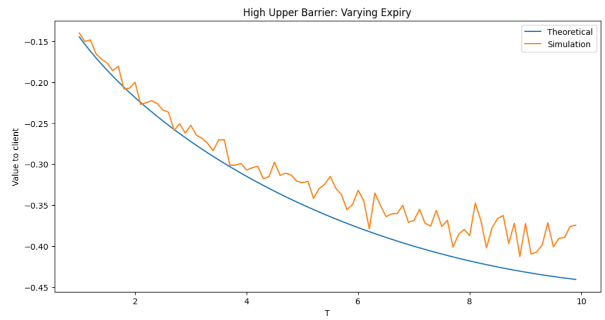
(A) Varying σ



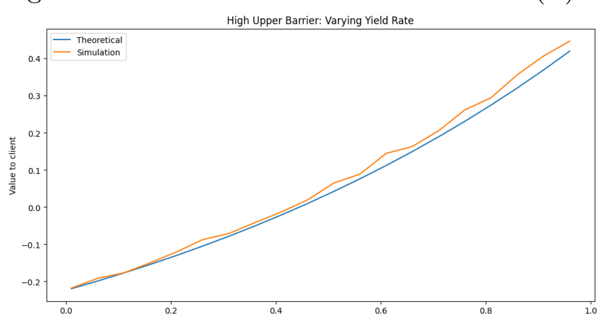
(B) Varying L



(C) Varying r

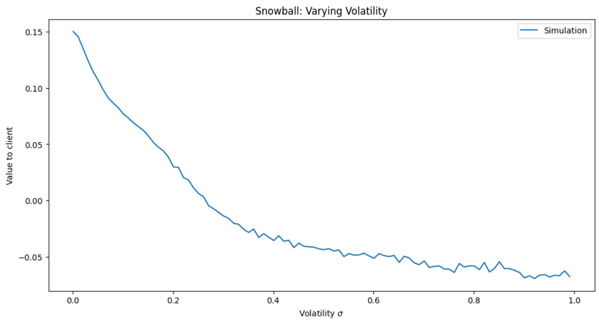


(D) Varying T

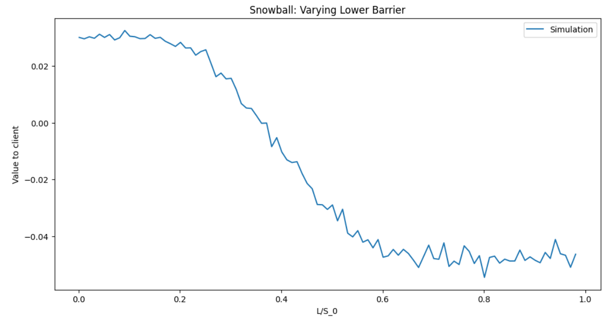


(E) Varying y

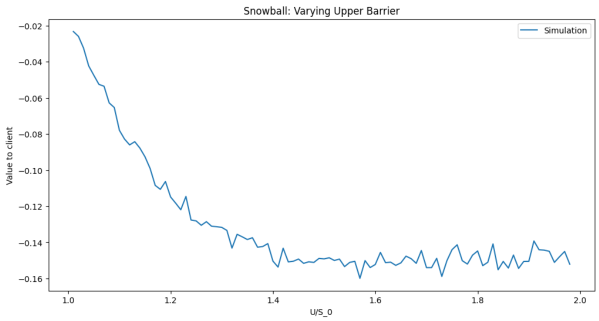
Figure 6. High Upper Barrier



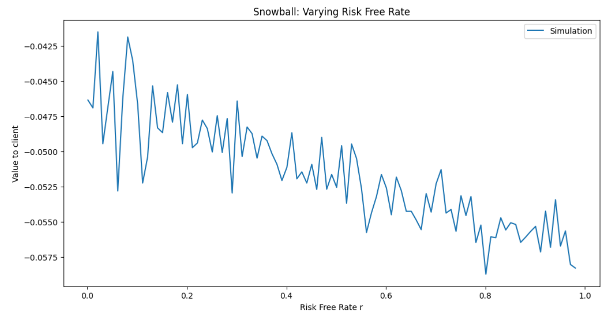
(A) Varying σ



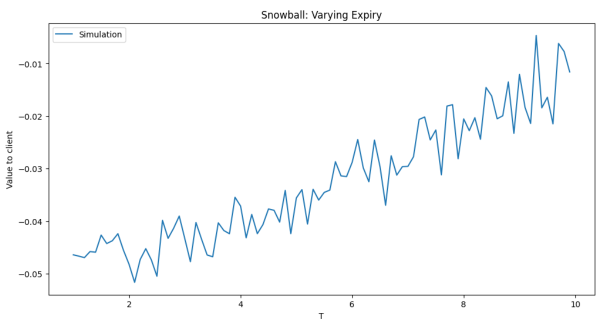
(B) Varying L



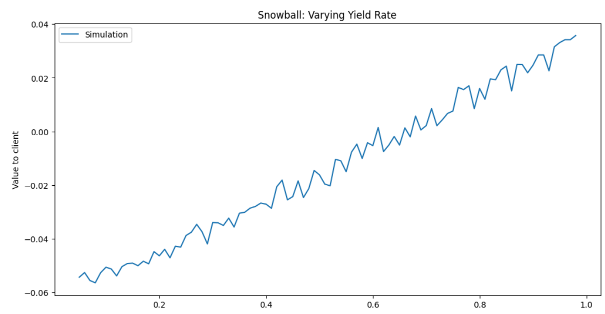
(C) Varying U



(D) Varying r



(E) Varying T



(F) Varying y

FIGURE 7. Snowball With Upper and Lower Barrier

4.3. **Volatility vs Observation Times.**

We plot the value of Snowball to the investor(client) changing the σ from 1 to 100 within the same setting, where $S_0 = 50, T = 1, r = 0.04, y = 0.18, U = 1.05S_0, L = 0.7S_0, M = 1$. Meanwhile, we also compare the plot under different numbers of steps in the simulation 8.

With all other things equal, the snowball would knock-out with high probability when σ is very small. As σ increases, the increment of the knock-in possibility is larger than that of the knock-out possibility since we set U closer to S_0. Thus, the possibility of knock-in but no knock-out before expiry increases, decreasing the product value. In contrast, when σ continues to grow, the possibility of knock-out before expiry increases, increasing the product value. Finally, when σ becomes large enough, the snowball will knock-out immediately almost surely. There is no profit or loss to the investor.

n can also be interpreted as the observation times. For fixed σ, if we observe more frequently, the possibility that paths knock out is larger, thus increasing the value to the investor

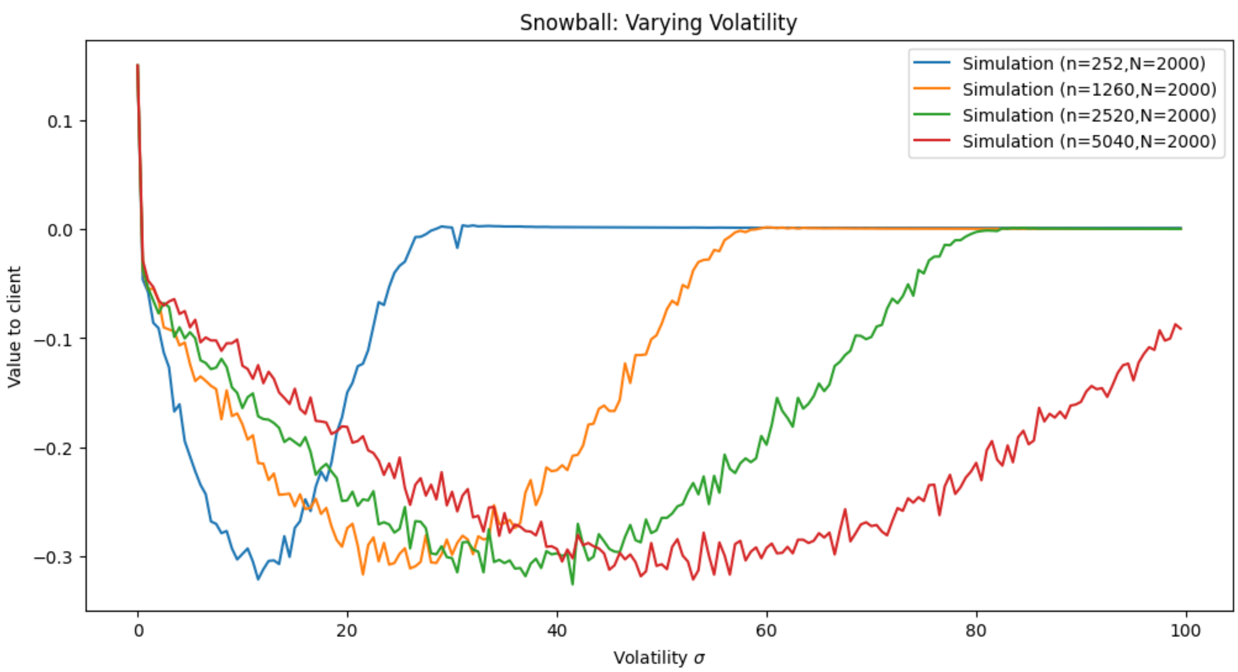
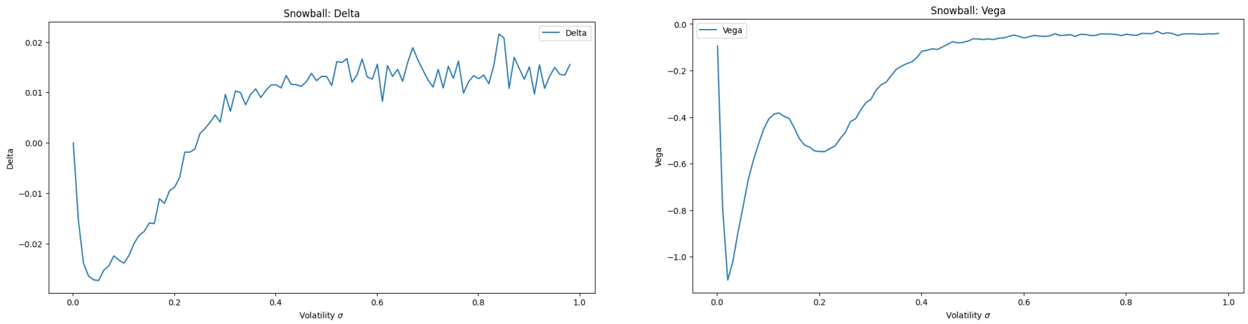


FIGURE 8. Vary σ using different n

4.4. **Greeks.**

Delta and Vega are estimated with the simulation method. For the same simulated normal random variables, increase and decrease the S_0 by a small amount ΔS and run the left procedures in simulation. We get 2 prices *price_right* and *price_left*. The delta is obtained by $\frac{price\_right - price\_left}{2\Delta S}$. Vega can be computed by a similar modification to σ.

4.4.1. *Delta.*



(A) Δ against σ (B) ν against σ

FIGURE 9. Greeks: varying σ

We use the setting $S_0 = 50, T = 1, r = 0.04, y = 0.18, U = 1.05S_0, L = 0.7S_0, M = 1$. By changing σ from 0.001 to 1, we plot the Δ against the σ. We do 100 times the above approximation and take the average as the final delta. See Fig 9.

4.4.2. *Vega.*
Similarly $S_0 = 50, T = 1, r = 0.04, y = 0.18, U = 1.05S_0, L = 0.7S_0, M = 1$, we plot the ν against the σ. See Fig 9.

4.5. **Hedging.**
Since the product's underlying is the stock index, the hedging instruments can be stock index futures and index options. We use the CSI500 index as an example. The data of CSI500 index in the period 2023-05-01 to 2024-05-17 is from *Investing.com*[2] accessed on May 19, 2024. The data for futures prices are from *Barchart*[1] accessed on May 19, 2024.

Delta. For Delta hedge, we know the futures price $F(t) = S(t)e^{(r-q)(T-t)}$. The delta of the futures contract is $\Delta_F = \frac{\partial F(t)}{\partial S(t)} = e^{(r-q)(T-t)}$. $\Delta_F(0) = e^{(r-q)T}$. Thus, after we calculate the $\Delta_{Snowball}$ of the Snowball, we can long $\frac{\Delta_{Snowball}}{\Delta_F}$ contract of futures to hedge the risk. Notice the futures contract has 0 Vega.

Vega. Ideally, we need the options on the stock index to hedge for Vega hedge. Suppose the long date call option has $\nu_C = S_0\sqrt{T}\phi(d_1)$, where $d_1 = \frac{\ln\frac{S_0}{K}+(r+0.5\sigma^2)T}{\sigma T}$, we need to long $\frac{\nu_{Snowball}}{\nu_C}$ to hedge the Vega.

To hedge Delta and Vega simultaneously, we consider the system of equations:

$$\begin{cases} \Delta_{Snowball} = -x\Delta_F - y\Delta_C \\ \nu_{Snowball} = -y\nu_C \end{cases}$$

where x, y are the numbers of futures contracts and call options to long.

References

[1] Barchart.com. Title of the webpage, 2024.
[2] investing.com. Title of the webpage, 2024.
[3] Steven E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer Finance. Springer, New York, 2004.

Appendix A. First passage time of Brownian Motion with Drift

Here we consider a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ and a stochastic process on time horizon $[0,\infty)$

$$\mathrm{d}X_t = \theta \mathrm{d}t + \mathrm{d}W_t, X_0 = 0$$

where W is a standard Brownian motion under $\mathbb{P}$. We could apply Girsanov theorem on time $[0,T]$ that X (within $[0,T]$) is a standard Brownian motion under measure $\mathbb{Q}$, where

- the Radon-Nikodym derivative process $Z_t \coloneqq \exp\Big\{-\theta W_t - \frac{1}{2}\theta^2 t\Big\}$
- $\mathrm{d}\mathbb{Q} \coloneqq Z \mathrm{d}\mathbb{P}$,
- $Z \coloneqq Z_T = \exp\Big\{-\theta\sqrt{T}N - \frac{1}{2}\theta^2 T\Big\}$, N is an independent standard normal random variable, the equality sign in the sense of distribution.

Denote the first passage time of X within finite time horizon $(0,T]$ by $\tau_m^X \coloneqq \{t \in (0,T] : X_t = m\}$, $m > 0$. We follow the convention that the minimum of the empty set is ∞. Then $\mathbb{P}(t - \mathrm{d}t < \tau_m^X \leq t)$, the probability that τ_m^X lies in some infinitesimal interval $(t-\mathrm{d}t, t]$ before t $(t < T)$, has the following:

$$\begin{aligned}
\mathbb{P}(t - \mathrm{d}t < \tau_m^X \leq t) &= \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{t-\mathrm{d}t<\tau_m^X\leq t}] \\
&= \mathbb{E}_{\mathbb{Q}}[Z_t^{-1}\mathbb{1}_{t-\mathrm{d}t<\tau_m^X\leq t}] \\
&= \mathbb{E}_{\mathbb{Q}}[\exp\{\theta W_t + \frac{1}{2}\theta^2 t\}\mathbb{1}_{t-\mathrm{d}t<\tau_m^X\leq t}] \\
&= \mathbb{E}_{\mathbb{Q}}[\exp\{\theta X_t - \frac{1}{2}\theta^2 t\}\mathbb{1}_{t-\mathrm{d}t<\tau_m^X\leq t}] \\
&= \exp\{\theta m - \frac{1}{2}\theta^2 t\}\mathbb{Q}(t - \mathrm{d}t < \tau_m^X \leq t),
\end{aligned}$$

where the last line we use the trick that in the infinitesimal interval, $X_t = m$. Note that X on $[0,T]$ is a standard Brownian motion under measure $\mathbb{Q}$, and hence we can invoke the density of the first passage time of standard Brownian motion as in [3] theorem 3.7.1,

$$f^{\mathbb{Q}}_{\tau_m^X}(s) = \frac{m}{s\sqrt{2\pi s}}\exp\{-\frac{m^2}{2s}\}, \quad s \in (0,T),$$

and hence

$$\begin{aligned}
\mathbb{P}(t - \mathrm{d}t < \tau_m^X \leq t) &= \exp\{\theta m - \frac{1}{2}\theta^2 t\}\mathbb{Q}(t - \mathrm{d}t < \tau_m^X \leq t) \\
&= \exp\{\theta m - \frac{1}{2}\theta^2 t\}\frac{m}{t\sqrt{2\pi t}}\exp\{-\frac{m^2}{2t}\}\mathrm{d}t \\
&= \frac{m}{t\sqrt{2\pi t}}\exp\{-\frac{(\theta t - m)^2}{2t}\}\mathrm{d}t
\end{aligned}$$

That is, the distribution of τ_m^X on $(0,T)$ has density

$$f_{\tau_m^X}(t) = \frac{m}{t\sqrt{2\pi t}} \exp\{-\frac{(\theta t - m)^2}{2t}\}.$$

Note this distribution has mass $1 - \int_0^T f_{\tau_m^X}(t)\mathrm{d}t$ on point ∞.