Snowball Under Heston Model

Course EN.553.649 Advanced Equity Derivatives
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Contents

1 Description

A snowball is a path-dependent structured exotic derivative that is typically linked to the performance of a specific underlying asset, such as a stock index or an individual stock. It is designed to provide investors with relatively high returns when the underlying asset price fluctuates within a specified range. Snowball is equivalent to selling a knock-in structured put option. As long as the underlying asset does not experience a significant decline, the longer the certificate is held, the more coupon income is earned, much like a snowball rolling and growing larger, provided the ground remains relatively smooth without any major potholes.

1.1 Snowball Termsheet

This is an example of a 12-month snowball linked to the SPX index table 1 9 10. According to the termsheet, the knock-out level is 110% S_0, the knock-in level is 80% S_0, and the annualized coupon rate is 20%. On predefined observation dates (usually monthly, see Key Information), if the underlying asset price reaches or exceeds the knockout level, the product terminates early, and the investor receives all margin and a pre-agreed return. If the underlying asset price is less than the knock-in level on any trading day between the start date and the end date, and no knock-out event is observed during the product's lifetime, the short put option with the same notional as the snowball knocks in. In that case, if the put option is in the money, the investor bears all losses caused by the decline in the underlying. Even if the put option is out of the money or at the money, the investor can only receive the full principle without any coupon or dividend income. If the product neither knocks in nor out, the investor receives a fixed annualized return at maturity. Different scenarios are shown in figure scenarios.

Product Type	Snowball linked to SPX Index
Maturity	12 months (Early redemption possible on knock-out observation dates)
Underlying Asset	SPX Index
Knock-Out Level	Initial Price × 105%
Knock-In Level	Initial Price × 80%
Coupon (Annualized)	20% (per annum)
Knock-Out Event	If on any knock-out observation date, the closing price of the underlying is greater than or equal to the knock-out level
Knock-In Event	If on any trading day, the closing price of the underlying is less than the knock-in level
Investment Return	**If knocked out:** Redemption amount = 20% × Nominal Principal × Accrued Days / 365 **If knocked in:** Redemption amount =min(Price at Maturity/Initial Price $-$ $1, 0$)× Nominal Principal **If non-knock:** Coupon payment = 20% × Nominal Principal × Accrued Days / 365

Table 1: Summary of Snowball Product Features

1.2 Historical Return

The historical returns of such snowball in the termsheet is shown in figure 1. The x-axis shows the start date(date) and the y-axis shows the return(index) for the snowball and the SPX separately. Results are shown in table 2. The return distribution figure 2 is left-tailed, and the returns are usually highly positive.

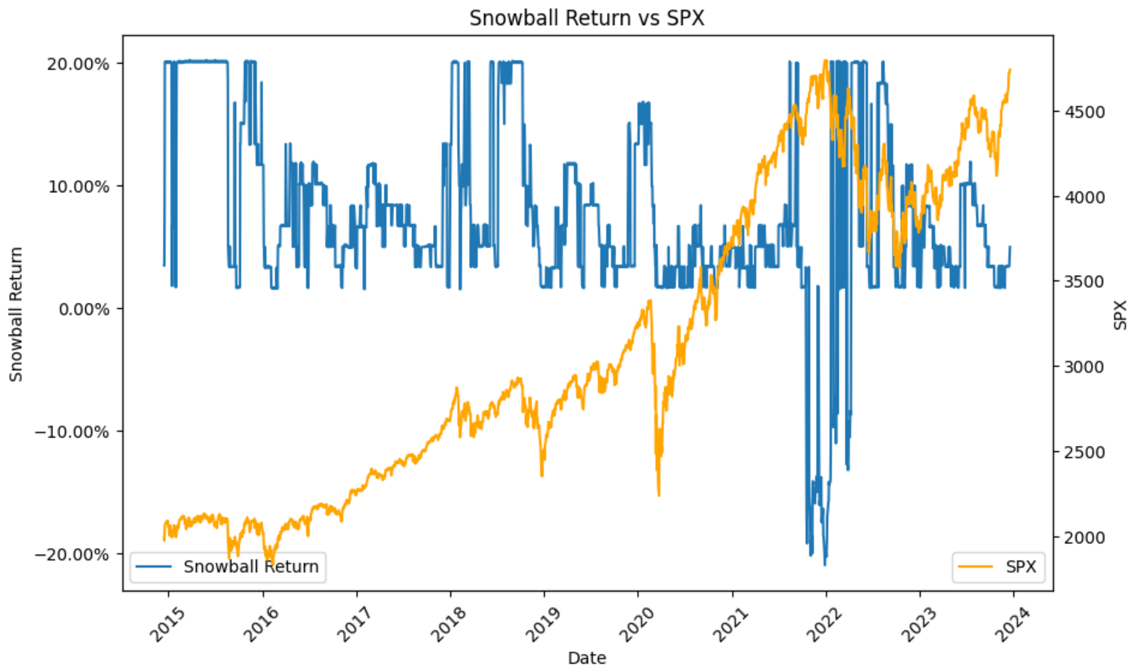


Figure 1: Snowball Historic Returns

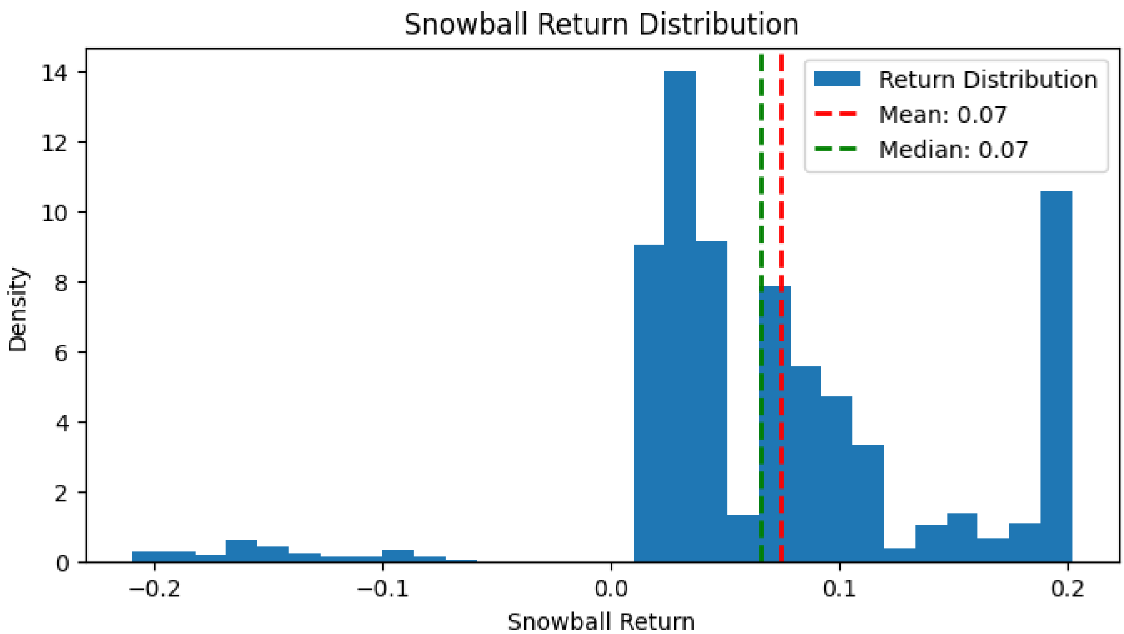


Figure 2: Snowball Historic Return Distribution

Metric	**Value**
Mean Return	7%
Median Return	7%
Maximum Return	20.21%
Minimum Return	-20.96%
Return Std	7%
Skewness	-0.36
Kurtosis	1.72
Mean Knockout Time	0.3214
Mean Knockin Loss	682.10
Prob Knockout	82.23%
Prob Knockin	3.79%
Prob Nonknock	13.98%

Table 2: Backtest Results

2 Pricing Model

The Monte Carlo simulation method is frequently used in pricing path-dependent exotic products. We first assume that the volatility is constant during the contract term and use the Monte Carlo method with antithetic variate to price the product. Convergence is checked and Greeks(Delta&Vega) are calculated. After that, the Heston model is used to model the stochastic volatility. We calibrate the Heston model with market data from the current month to maturity. The Milstein discretization is used to mitigate the effect caused by negative volatility in simulation. We compare the results between the constant volatility model and stochastic volatility models.

2.1 Constant Volatility

The snowball product that we use in this section is the same as the snowball in the section *Description*. The contract starts on December 13, 2024, and the maturity date is December 19, 2025 (see details in 9 10. The initial price is 6051.06. The risk-free rate is 4%. The dividend rate for SPX is 2%. Volatility is set to 18% due to recent volatile markets. In this Monte Carlo simulation, we use Euler discretization with 200000 path. The return distribution is shown in figure 3.The simulated results are shown in table 3.

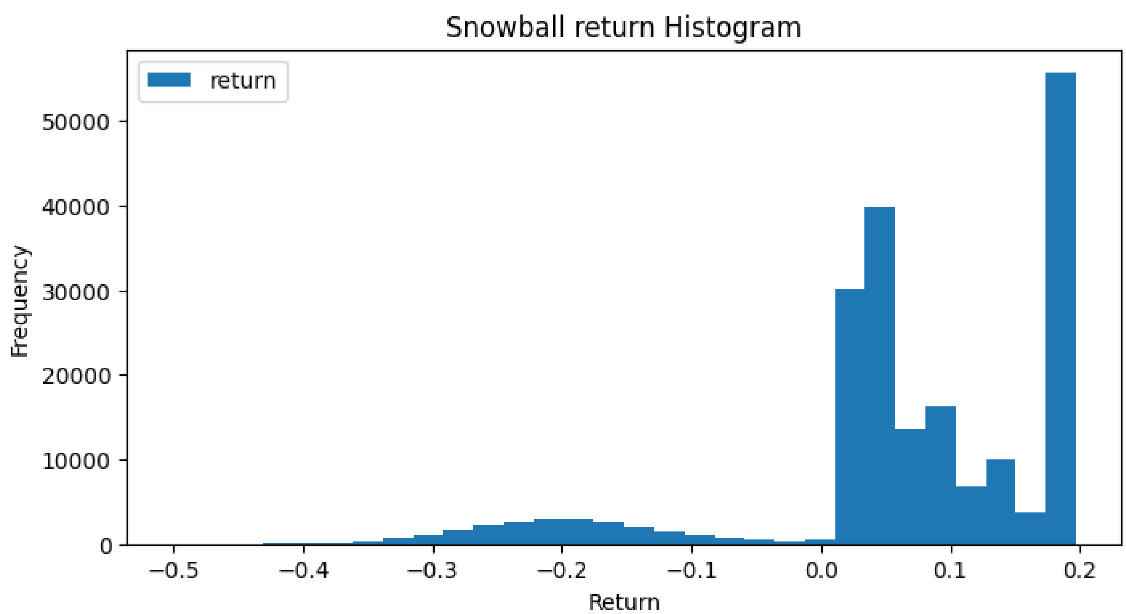


Figure 3: Constant Vol Return Distribution

We check the convergence of the simulation by figure 4.

Metric	Value
Mean Return	6.89%
95% Confidence Interval	6.8471%, 6.9525%
5%VaR	-21.27%
5%ES	-26.82%
Maximum Return	20.00%
Minimum Return	-48.97%
Return Std	12.03%
Skewness	-1.22
Kurtosis	1.54
Mean Knockout Time	0.3558
Mean Knockin Loss	19.35%
Prob Knockout	63.48%
Prob Knockin	11.77%
Prob Nonknock	24.74%

Table 3: MC Constant Vol Summary

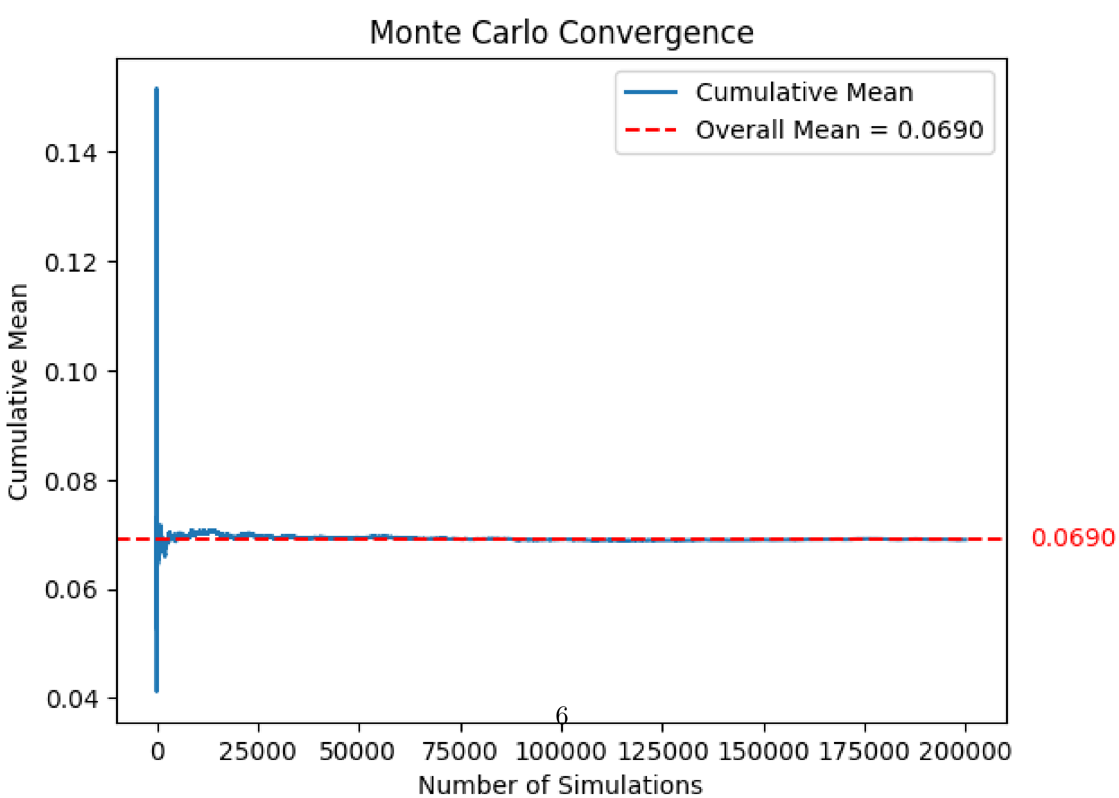
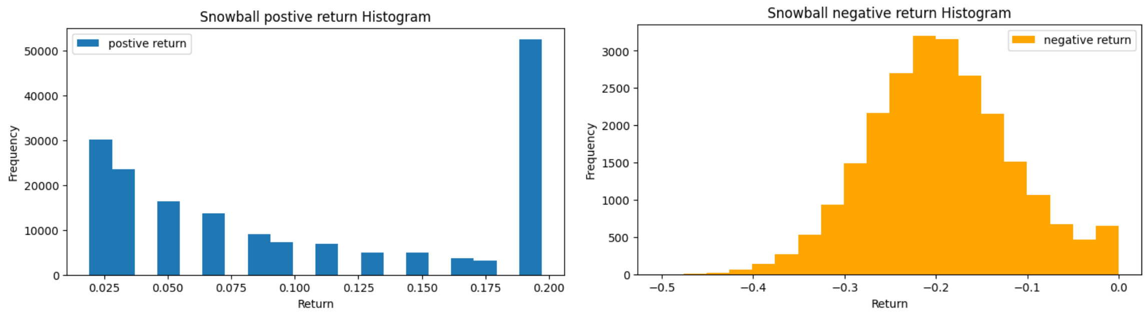


Figure 4: MC Convergence

We also demonstrate the positive and negative return distribution separately figure 5.

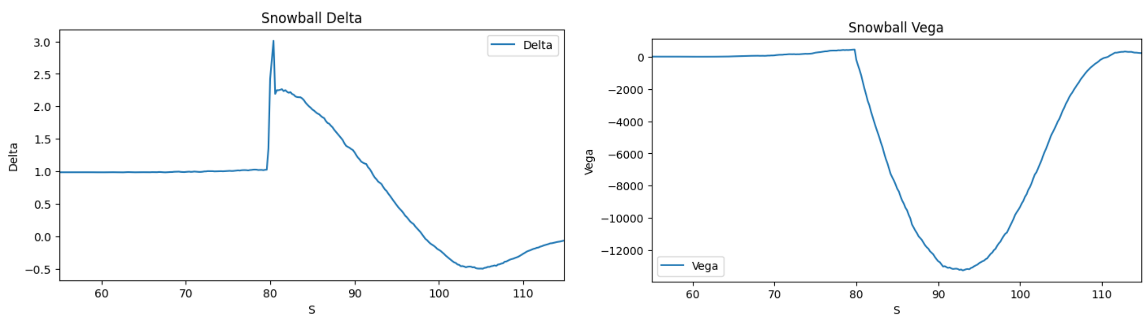


(a) Positive Return Distribution

(b) Negative Return Distribution

Figure 5: Positive and Negative Returns

The Delta and Vega over 55% to 115% of initial price is presented in figure 6



(a) Delta

(b) Vega

Figure 6: Delta & Vega

2.2 Heston Model

The Heston model extends the Black-Scholes framework by modeling volatility as a stochastic process, allowing it to vary over time. This makes the Heston model suitable for capturing volatility smiles and term structures often observed in financial markets. The asset price S_t and variance v_t evolve according to the following stochastic differential equations (SDEs):

$$dS_t = \mu S_t \, dt + \sqrt{v_t} S_t \, dW_t^1,$$

$$dv_t = \kappa(\theta - v_t) \, dt + \sigma \sqrt{v_t} \, dW_t^2,$$

where:

- S_t: Asset price at time t,

- v_t: Instantaneous variance (stochastic),
- μ: Drift (expected return),
- κ: Mean-reversion speed of the variance process,
- θ: Long-term mean variance,
- σ: Volatility of variance (vol-of-vol),
- W_t^1 and W_t^2: Two correlated Brownian motions, with correlation ρ.

2.2.1 Calibration

We calibrate the Heston model using monthly market SPX option data from December 20, 2024 to December 19, 2025. For each maturity date, call and put options' implied volatility are equally weighted to form a combined implied volatility for each strike. We have 50 strikes for each maturity date. The implied volatility surface of the market is shown in figure 7.

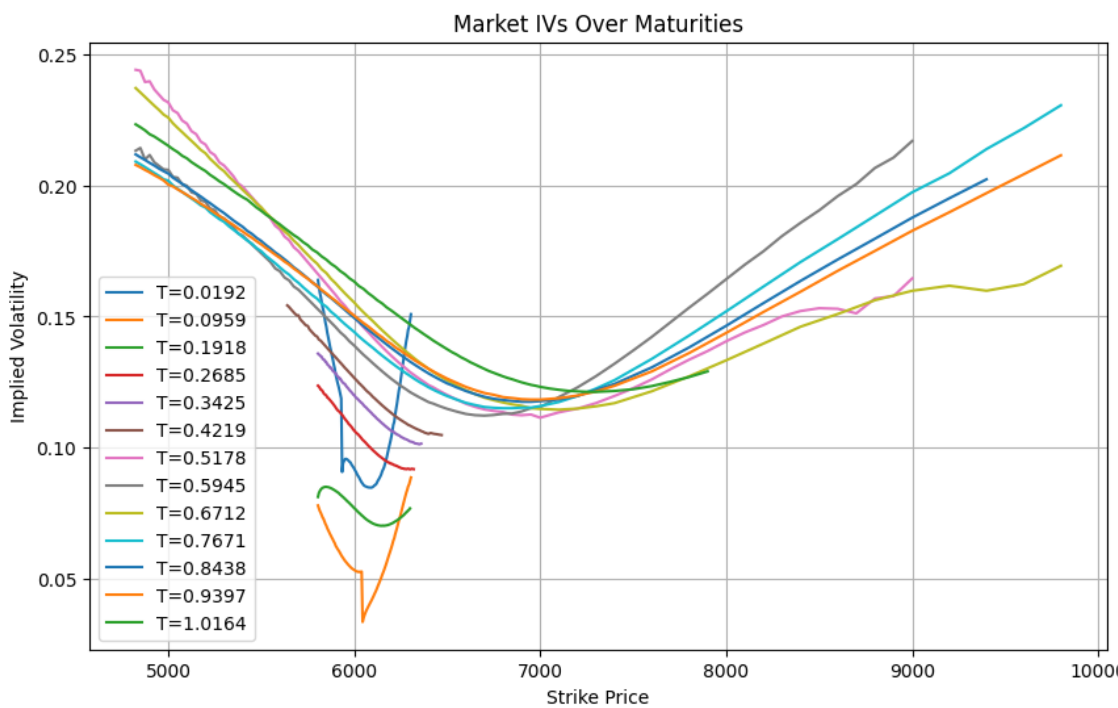


Figure 7: Market Implied Volatility

We use the **QuantLib** to calibrate with **LevenbergMarquardt** method. It minimizes the difference between the actual implied volatility and the implied volatility calculated by the Heston price.

The set of calibrated parameters are

- v_0: 0.0038
- κ: 2.9531,
- θ: 0.0487,
- σ: 1.5068,
- ρ : -0.5067

The calibration performance is illustrated by comparing the ivs by the Heston model and those from the market. Several maturities are shown in figure 8 figure 9. The market volatility surface is better fitted for intermediate and later maturities.

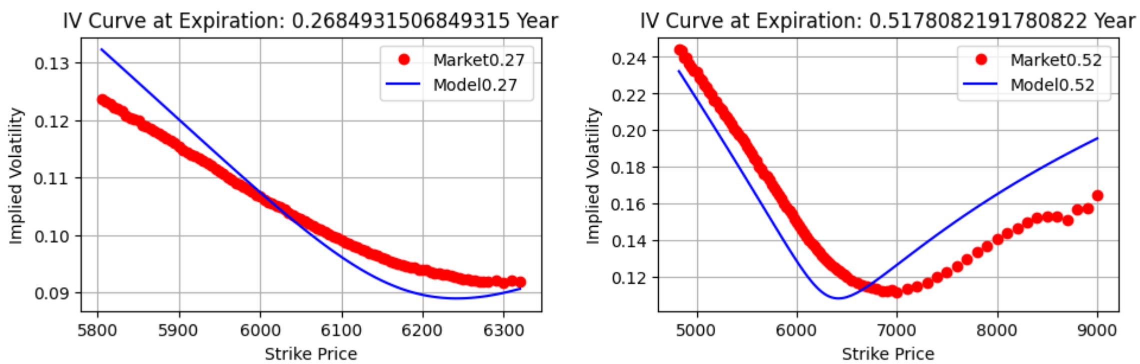


Figure 8: Calibration Performance

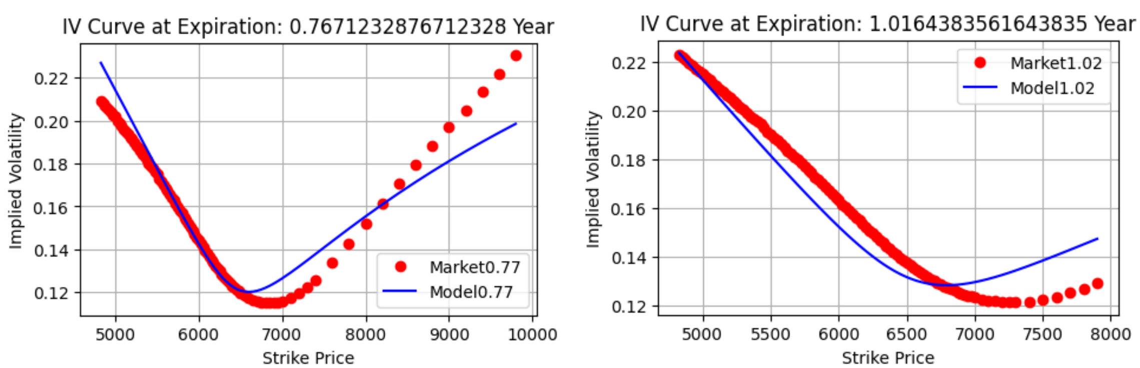


Figure 9: Calibration Performance

2.2.2 Simulation Schemes

We implemented two distinct schemes for simulating the Heston model using Milstein discretization. The first scheme follows the classic approach, where the variance path and the corresponding price path are generated simultaneously during each simulation run. The second scheme first simulates a single variance path. Based on this variance path, multiple price paths are then simulated, and the corresponding payoffs are calculated. This process is repeated multiple times, and the final payoff is determined by averaging the results across all iterations. In all daily simulations, we set risk free rate 0.04, dividend yield 0.02.

2.2.3 Results

1. For the classic methods, the number of simulations is 1000000. Results are shown below: return distribution 10, key information 4

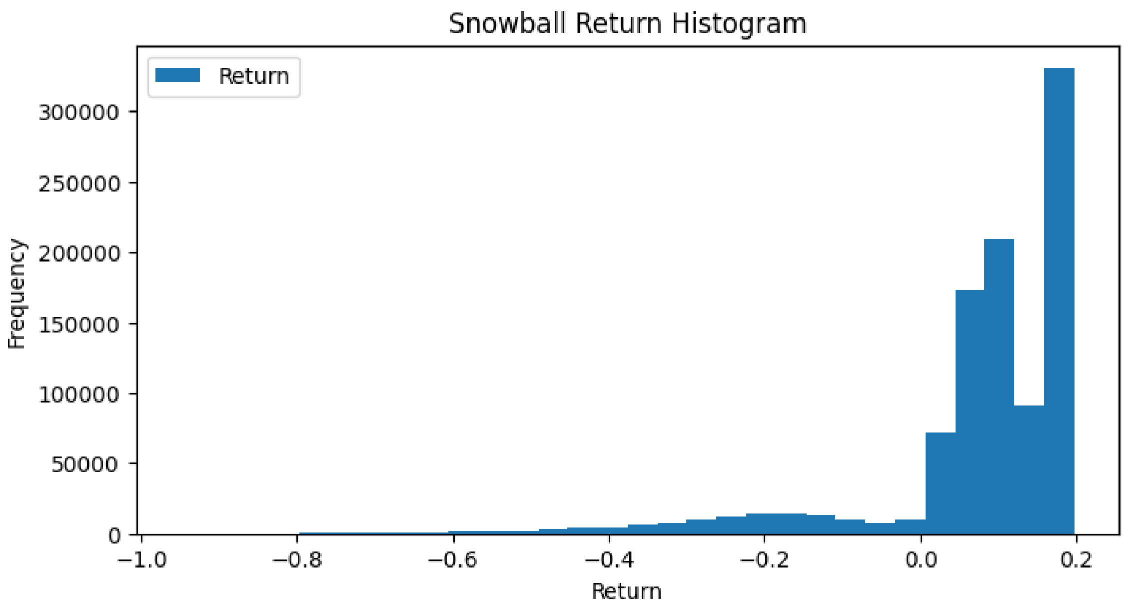


Figure 10: Heston Return Distribution

We check the convergence of the simulation by figure 11.

Metric	Value
Mean Return	8.05%
95% Confidence Interval	8.0198%, 8.0759%
5%VaR	-24.12%
5%ES	-38.90%
Maximum Return	20.00%
Minimum Return	-94.89%
Return Std	14.29%
Skewness	-2.37
Kurtosis	6.90
Mean Knockout Time	0.5128
Mean Knockin Loss	23.43%
Prob Knockout	63.14%
Prob Knockin	12.44%
Prob Nonknock	24.42%

Table 4: Heston MC Summary

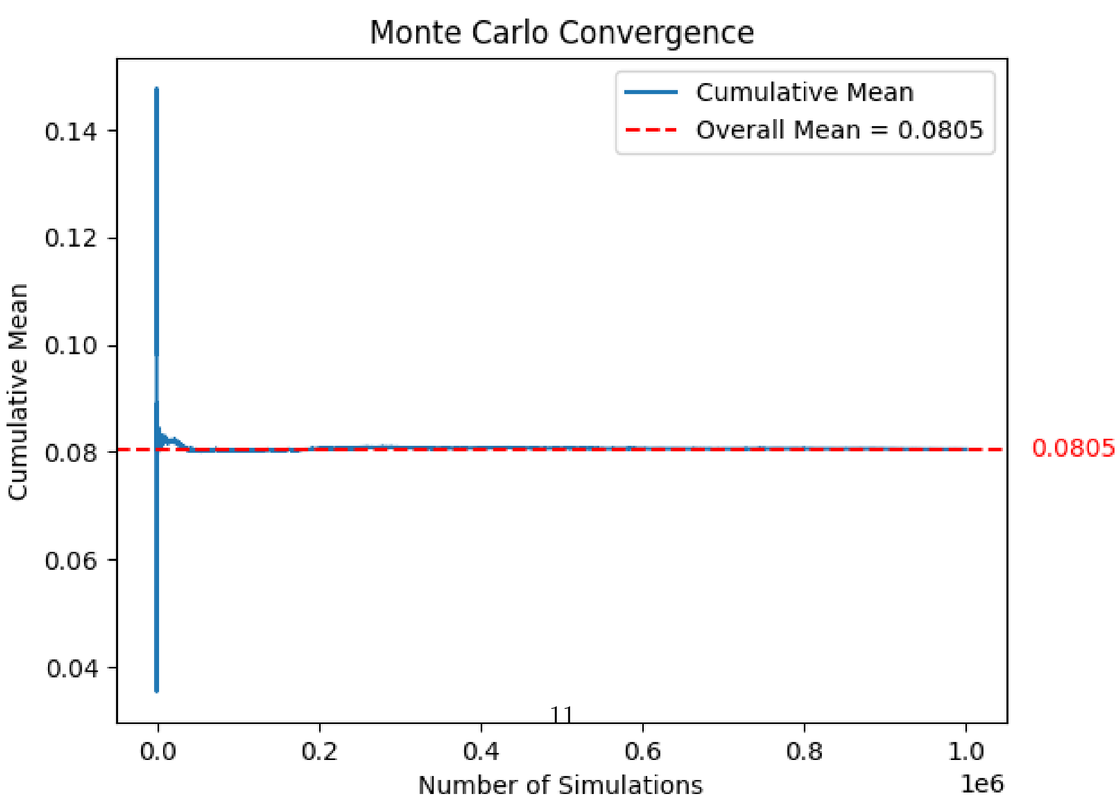
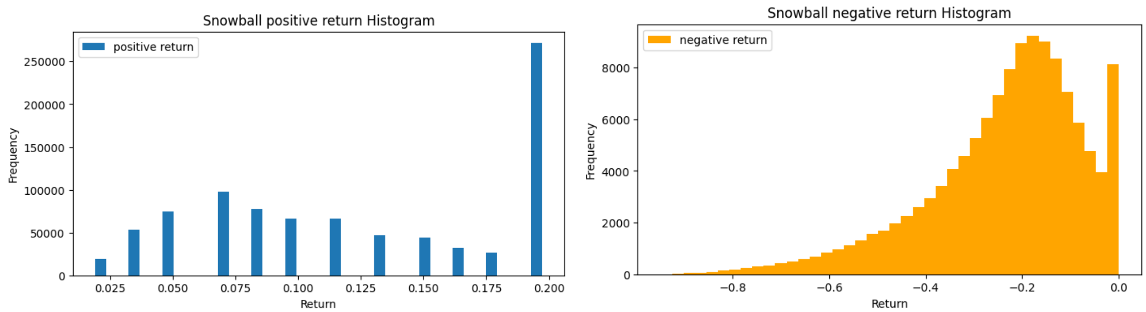


Figure 11: Heston MC Convergence

We also demonstrate the positive and negative return distribution separately figure 12.



(a) Heston Positive Return Distribution

(b) Heston Negative Return Distribution

Figure 12: Heston Positive and Negative Returns

2. For the second method, we set number of variance paths 1000 and number of price paths for each variance path 1000, which in total is the same number of simulations as in method 1. Results are shown below: return distribution 13, key information 5

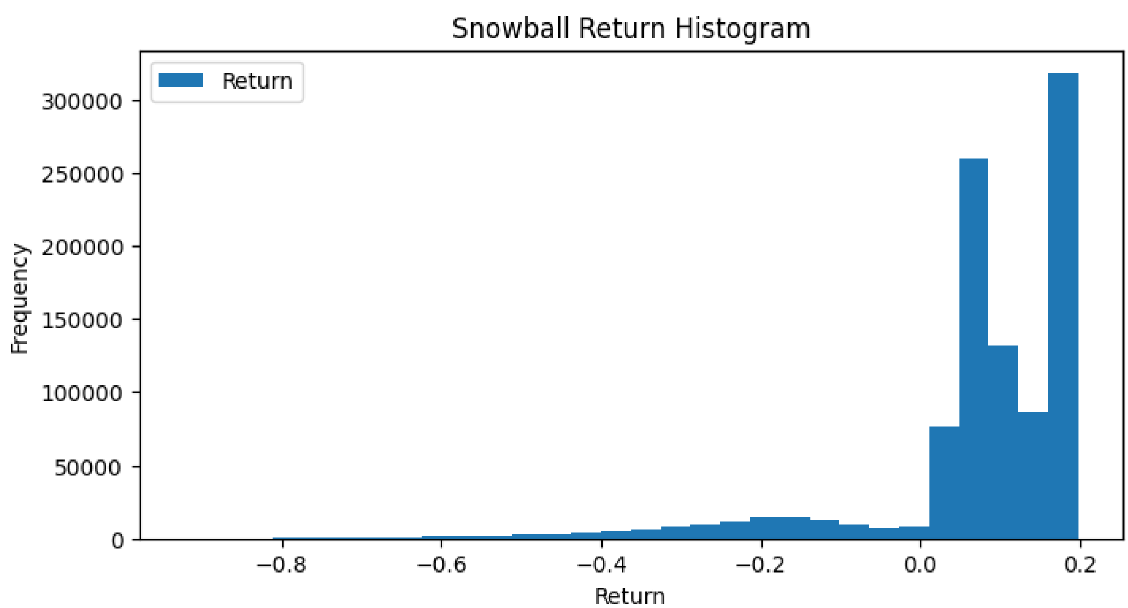


Figure 13: Heston Return Distribution

We check the convergence of the simulation by figure 14.

Metric	Value
Mean Return	7.67%
95% Confidence Interval	7.6366%, 7.6936%
Maximum Return	20.00%
Minimum Return	-92.22%
Return Std	14.54%
Skewness	-2.37
Kurtosis	6.91
Mean Knockout Time	0.49
Mean Knockin Loss	18.99%
Prob Knockout	63.79%
Prob Knockin	12.79%
Prob Nonknock	23.42%

Table 5: Heston MC Summary

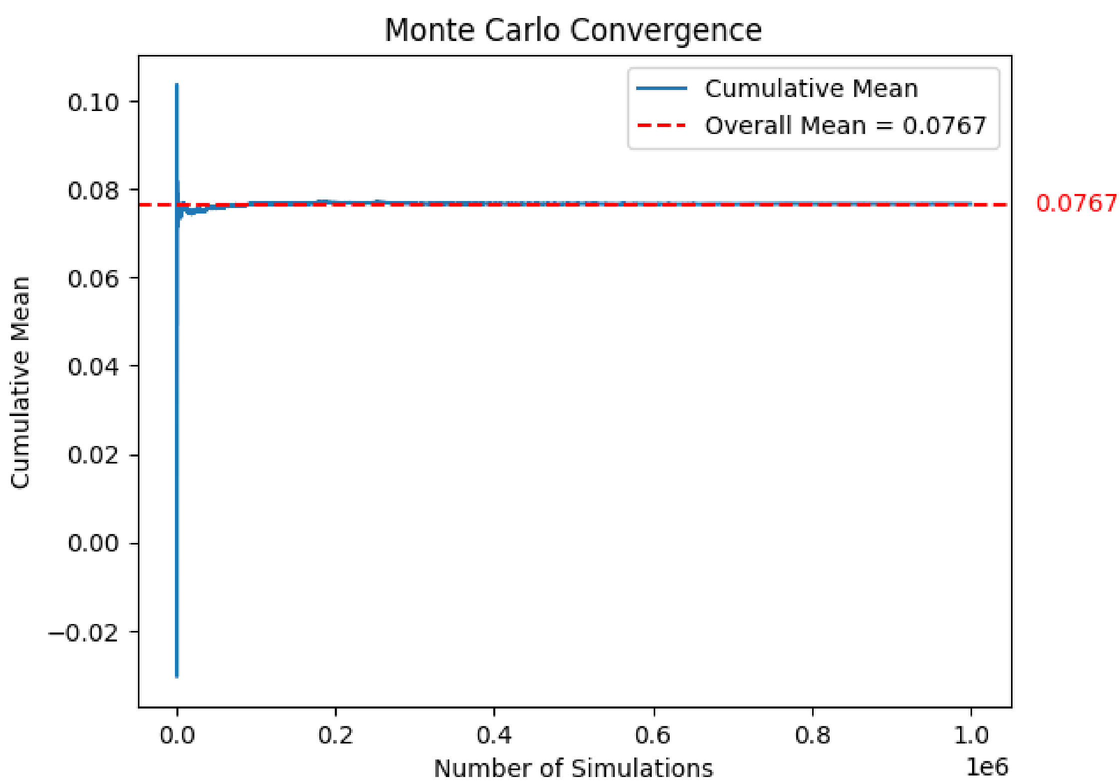
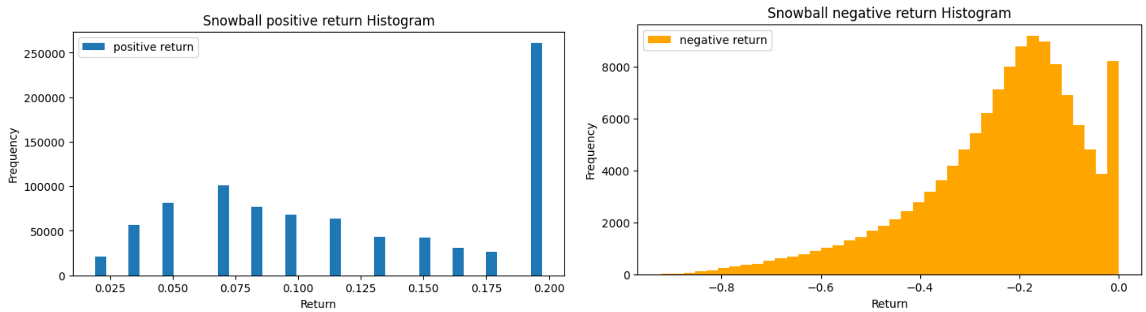


Figure 14: Heston MC Convergence

We also demonstrate the positive and negative return distribution separately figure 15.



(a) Heston Positive Return Distribution

(b) Heston Negative Return Distribution

Figure 15: Heston Positive and Negative Returns

3. **Comparison**: The return generated by Method 1 is higher than that of Method 2 due to a longer expected knockout time and a higher probability of non-knockout events. This can be attributed to more extreme cases arising in the variance path simulation. Despite this, the overall variance in the simulations remains roughly the same. When the total number of simulations is increased to 1,000,000, the primary difference observed lies in the positive return segment of the total return distribution plot. While the distribution plots for positive and negative returns appear similar when examined separately, the underlying values are not identical (as verified). Given these findings, we have opted to stick with Method 1, as it is computationally faster. Additionally, a higher calculated option premium benefits the seller, aligning with favorable pricing strategies.

2.2.4 Comparison With Constant Volatility

When comparing the returns of the Snowball under the Heston model with those under the constant volatility model, the Heston model produces higher average returns. However, it also exhibits significantly more negative minimum returns. The return distribution under the Heston model is more negatively skewed and has higher kurtosis, indicating a greater likelihood of extreme negative outcomes. Additionally, the average knockout time is longer under the Heston model, while the probabilities of knockout, knock-in, and non-knock events remain relatively similar.

A Snowball product benefits from stable or declining volatility since it is effectively short volatility. The Heston model allows for periods of lower volatility when the stochastic variance v_t reverts toward its mean θ. These periods of reduced volatility increase the likelihood of favorable conditions for the Snowball to accrue returns, resulting in higher average payoffs. The Heston model captures volatility clustering, where periods of low volatility are followed by similar periods. These low-volatility phases tend to prolong favorable conditions, such as a delayed knock-out, leading to higher average returns. The stochastic volatility in the Heston model can lead to sudden spikes in variance, which can severely impact the underlying asset's price. This increases the risk of significant negative returns due to a knock-in event or larger mark-to-market losses. The Heston model often assumes a negative correlation $\rho < 0$ between asset price and volatility. During volatility spikes, the asset price tends to drop sharply, increasing the likelihood of hitting the knock-in barrier and leading to more extreme

negative outcomes.
The negative skewness arises because the Snowball is short volatility, making it vulnerable to sudden increases in volatility. This creates asymmetric downside risk, as extreme negative returns are more likely than extreme positive returns. The higher kurtosis in the Heston model reflects a greater probability of extreme returns, both positive and negative. The stochastic nature of volatility introduces tail risks that are not captured in the constant volatility model.
Under the Heston model, the underlying asset's price path is influenced by fluctuating volatility. Periods of higher volatility can slow down the approach to the knock-out barrier, extending the time it takes for the product to knock out. While the probabilities of different scenarios are relatively unchanged, the stochastic nature of the Heston model redistributes the outcomes across the return spectrum, leading to a longer average knock-out time and more pronounced tail risks.

2.3 Heston Greeks

In Heston model, there are 2 types of vega. **Initial Volatility Vega (ν_{v_0})**: Sensitivity of the option price to changes in the initial variance (v_0).

$$\nu_{v_0} = \frac{\partial C}{\partial v_0},$$

where C is the option price.
Volatility of Volatility Vega (ν_σ): Sensitivity of the option price to changes in the volatility of variance (σ).

$$\nu_\sigma = \frac{\partial C}{\partial \sigma}.$$

The Delta and Vega(s) under Heston Model over 70% to 110% of initial price is presented in figure 16 17 18. The delta in Heston model for snowball is larger than the delta in constant volatility model. It shows more pronounced fluctuations, especially near barriers like knock-in or knock-out levels for Snowball options. This is due to the stochastic variance introducing randomness in the sensitivity of the option value to price changes. The price sensitivity with respect to initial volatility level ν_{v_0} stays negative. It reaches its highest near the strike, indicating the price is more less sensitive to initial volatility level in the Heston model. An increase in the initial volatility level will reduce the price of the snowball. The price sensitivity with respect to the vol of vol ν_σ has a huge upper jump when the price is near knockin level and knockout level, and remain negative for prices in between. It shows that when the underlying is neither knockout nor knockin, the increase in vol of vol will reduce the price of the snowball. It aligns with intuition when in this case the correlation ρ is negative. When price decreases, the volatility tends to increase and an increase in vol of vol will accelerate the increasing process of volatility, leading a potential knockin. When price increases, the volatility tends to decrease and an increase in vol of vol will accelerate the process. However, the product may knockout due to price momentum, which reduces the value of this product in another way.

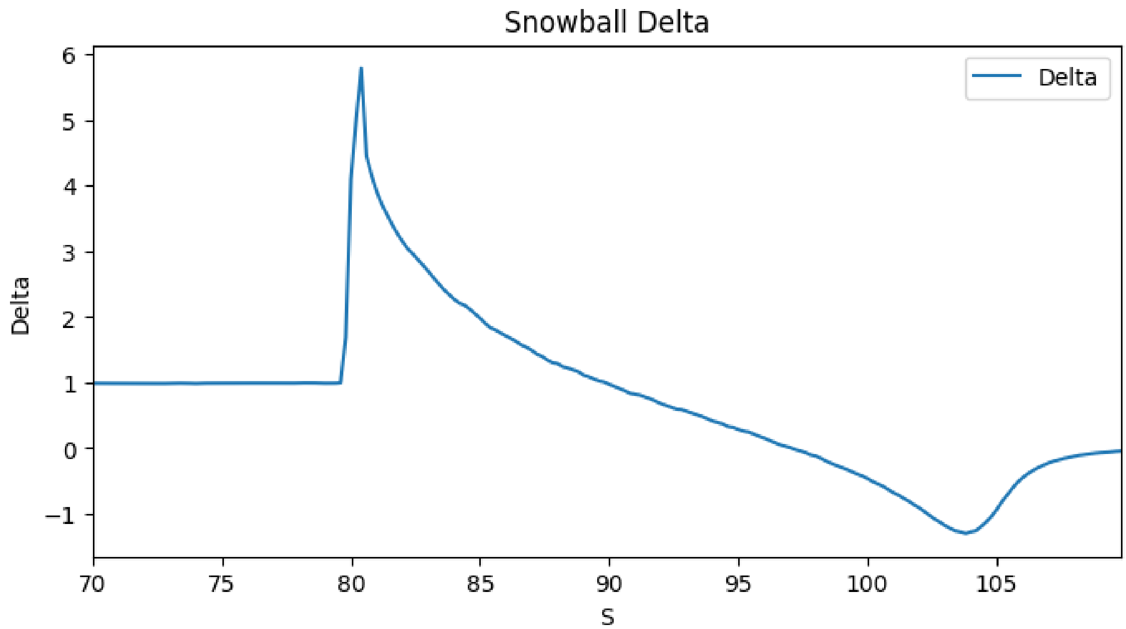


Figure 16: Delta

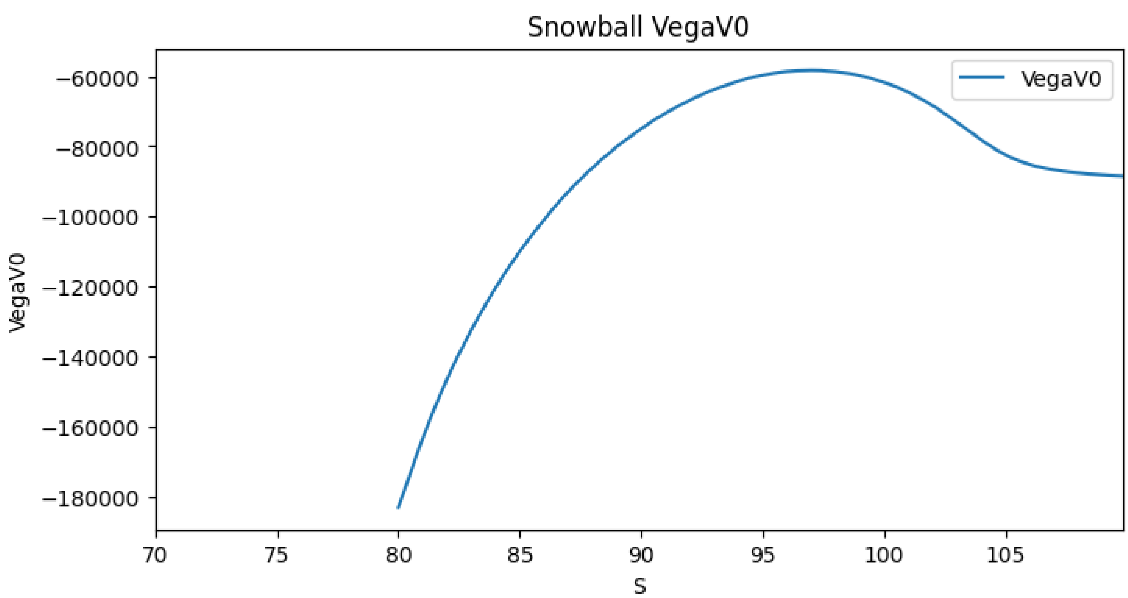


Figure 17: ν_{v_0}

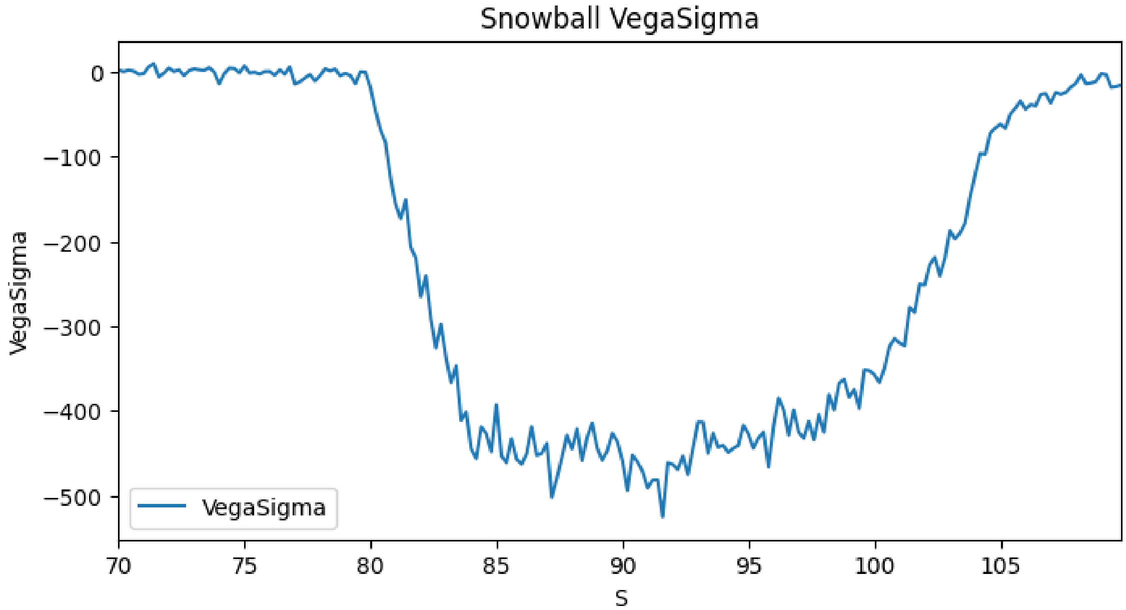


Figure 18: ν_σ

3 Extreme Case: ρ

We want to investigate the relationship between snowball return and the correlation ρ between volatility and underlying price. In this section, we show some of the price paths to better illustrate the scenarios.

1. **Case 1: ρ =-1** This is the case where underlying price and volatility are perfectly negative correlated. Price paths are shown in figure 19. From the selected price paths, this case is similar to real market dynamics, with a stable up trend in prices and huge volatility when the market crashes. The leverage effect is strong.

   The snowball return distribution is shown in figure 20 6. Higher volatility during price drops leads to increased likelihood of hitting the knock-in barrier for Snowball options. Snowball options are more likely to incur losses due to higher knock-in probabilities, resulting in lower overall pricing or increased risk premium. Additionally, the tail risk is very high, with 5% VaR -25.89% and 5% ES -44.49%. Meanwhie, the knockout and knockin probability are high compared to the uncorrelated case.

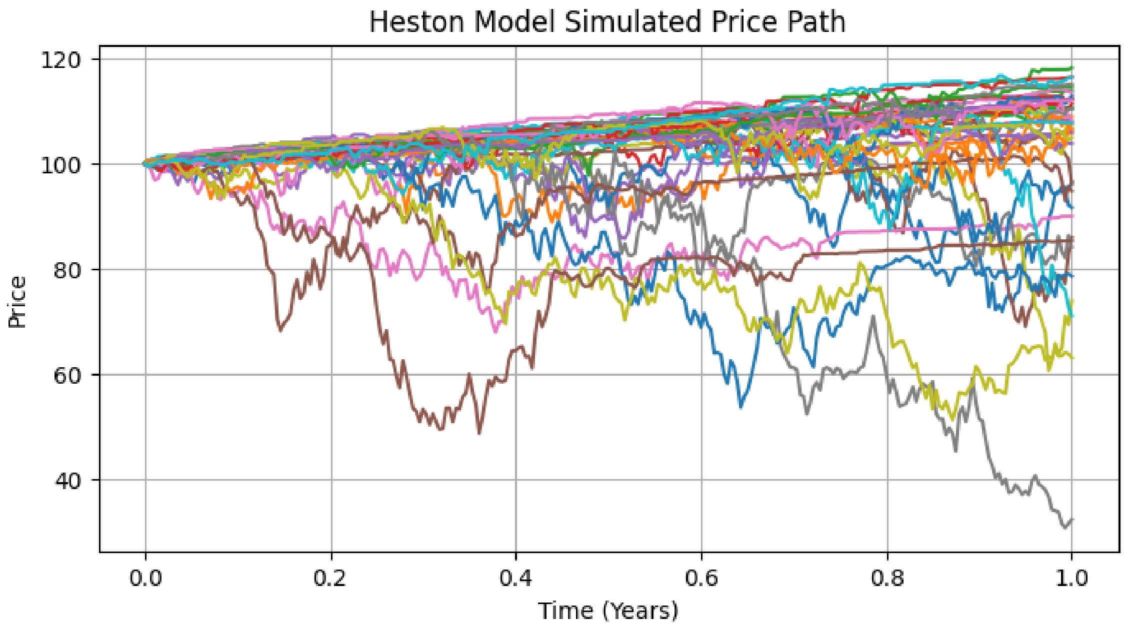


Figure 19: $\rho = -1$ Price Paths

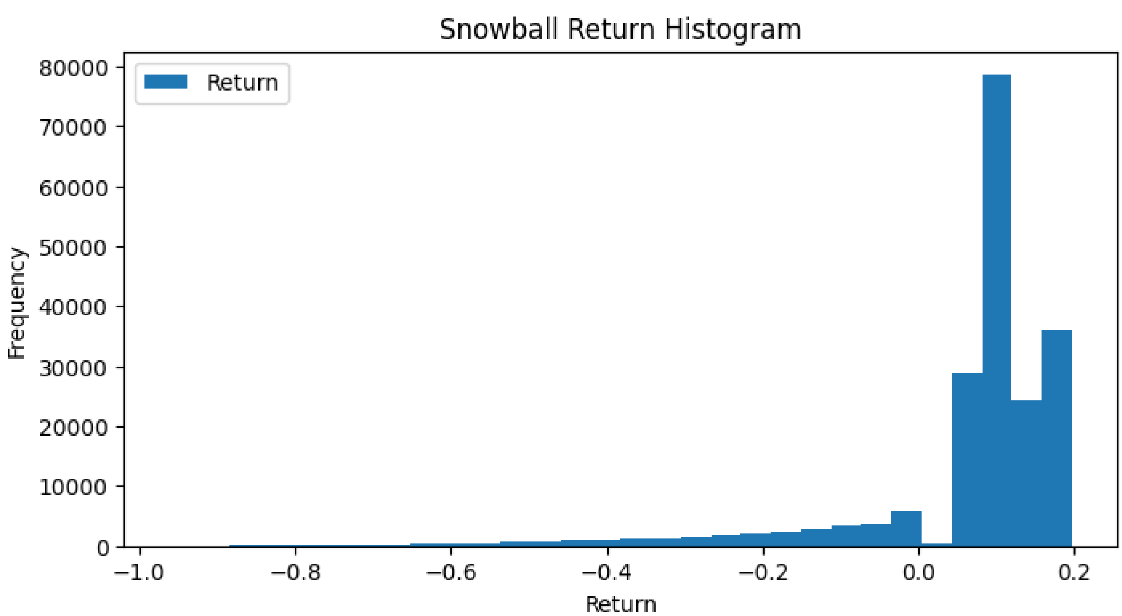
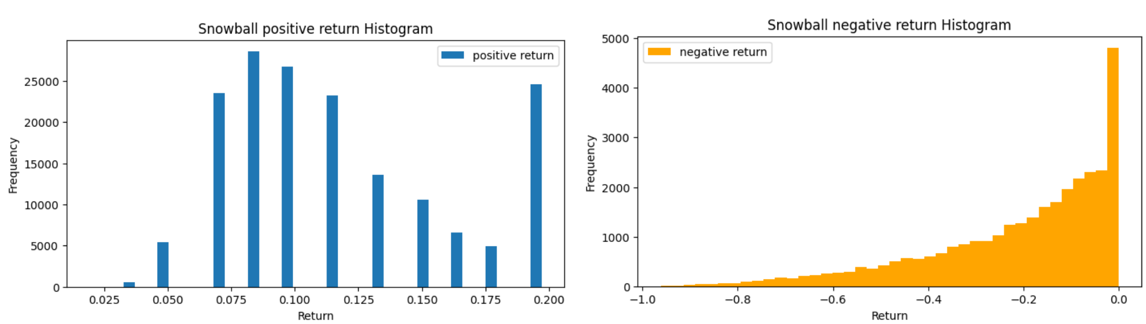


Figure 20: $\rho = -1$ Heston Return Distribution

Metric	Value
Mean Return	6.64%
95% Confidence Interval	6.5778%, 6.7064%
5%VaR	-25.89%
5%ES	-44.49%
Maximum Return	20.00%
Minimum Return	-96.18%
Return Std	14.68%
Skewness	-2.81
Kurtosis	9.35
Mean Knockout Time	0.5636
Mean Knockin Loss	20.77%
Prob Knockout	74.20%
Prob Knockin	15.77%
Prob Nonknock	10.03%

Table 6: $\rho = -1$ Summary

We also demonstrate the positive and negative return distribution separately figure 21.



(a) $\rho = -1$ Heston Positive Return Distribution (b) $\rho = -1$ Heston Negative Return Distribution

Figure 21: $\rho = -1$ Heston Positive and Negative Returns

2. **Case 2:** ρ **=0** This is the case where underlying price and volatility are uncorrelated. Price paths are shown in figure 22. This case describes the most stable market dynamics, similar to some commodity or currency market situation. Volatility evolves independently, leading to a

more symmetric impact on price paths.

The snowball return distribution is shown in figure 23 7. Results are intermediate compared to positive and negative ρ , with less skewness in the return distribution compared with the negative correlation case. The tail risk is reduced compared with extreme negative correlation. Although the mean knockout time is reduced, the probabilities of knockout and nonknock are dramatically increased.

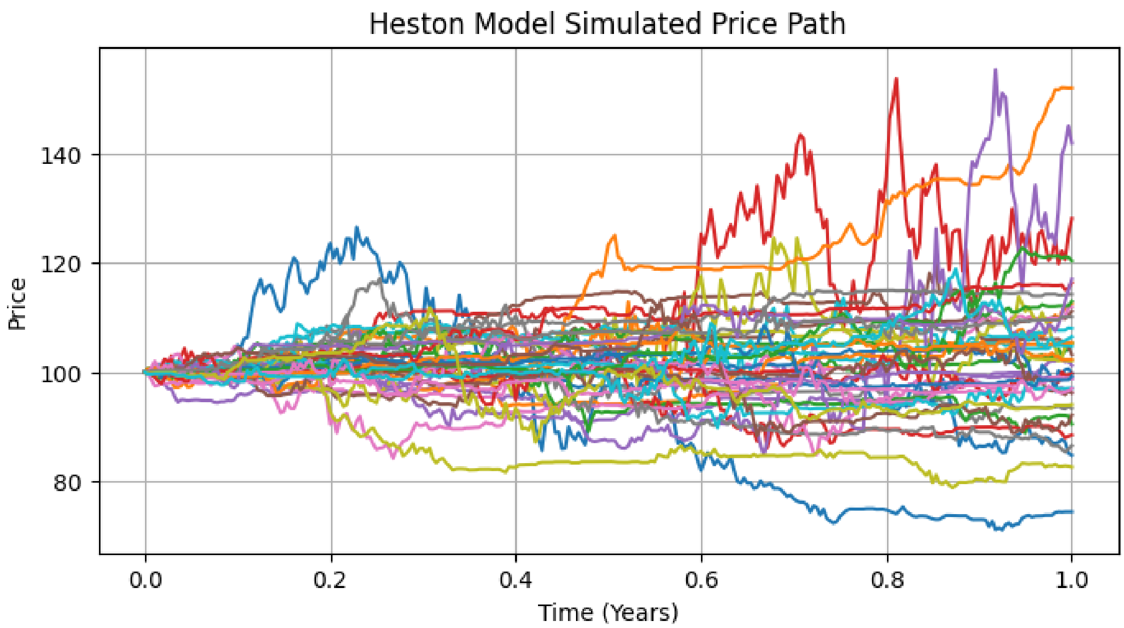


Figure 22: $\rho = 0$ Price Paths

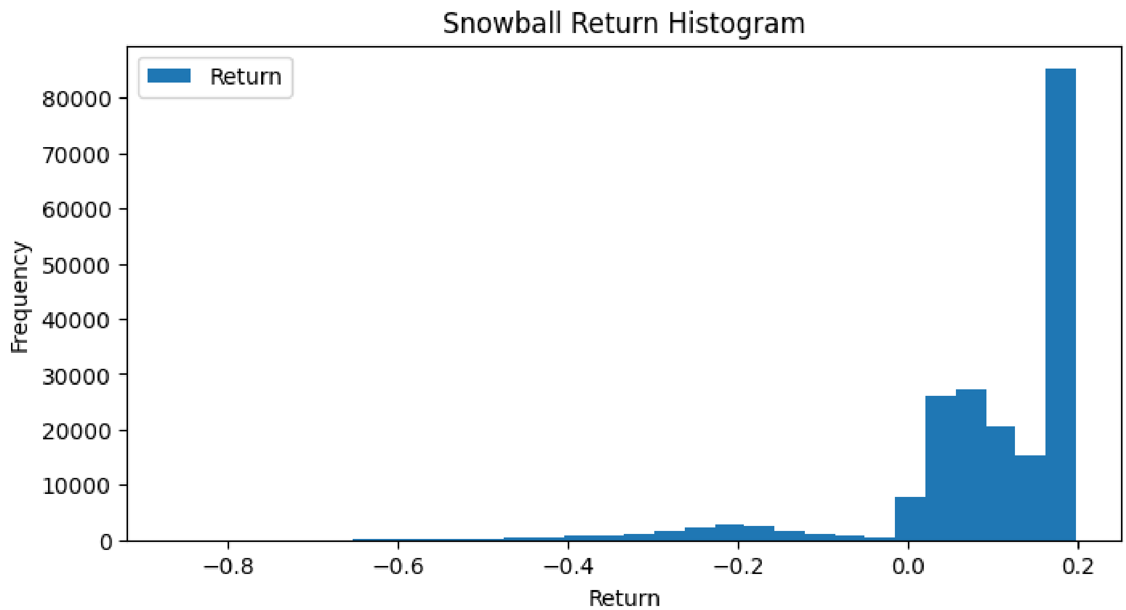
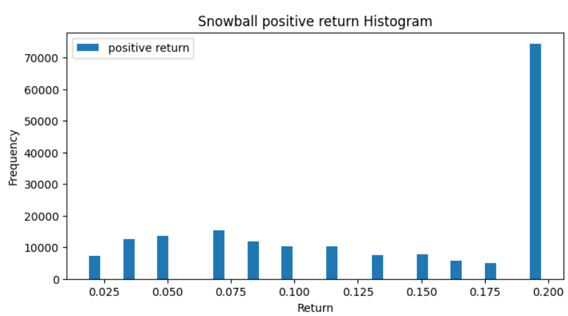
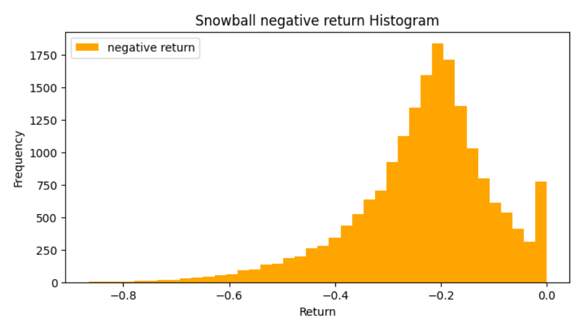


Figure 23: $\rho = 0$ Heston Return Distribution

We also demonstrate the positive and negative return distribution separately figure 24.



(a) $\rho = 0$ Heston Positive Return Distribution



(b) $\rho = 0$ Heston Negative Return Distribution

Figure 24: $\rho = 0$ Heston Positive and Negative Returns

3. **Case 3:** ρ **=1** This is the case where underlying price and volatility are perfectly positive correlated. Price paths are shown in figure 25. Bull markets or momentum-driven markets can be best described in this case, where rapid price increases lead to heightened uncertainty about sustainability.

   The snowball return distribution is shown in figure 26 8. In this case, the current calibrated Heston model will not generate negative return paths. The chance of hitting the knock-in

Metric	**Value**
Mean Return	9.90%
95% Confidence Interval	9.8484%, 9.9610%
5%VaR	-20.73%
5%ES	-31.91%
Maximum Return	20.00%
Minimum Return	-86.57%
Return Std	12.85%
Skewness	-2.22
Kurtosis	6.07
Mean Knockout Time	0.4931
Mean Knockin Loss	23.21%
Prob Knockout	56.09%
Prob Knockin	9.33%
Prob Nonknock	34.59%

Table 7: $\rho = 0$ Summary

barrier is significantly reduced. The probabilities for knockout and nonknock increases.Snowball options are priced higher, as knock-in events become less likely, and the product accrues more coupon payments over a longer time horizon.

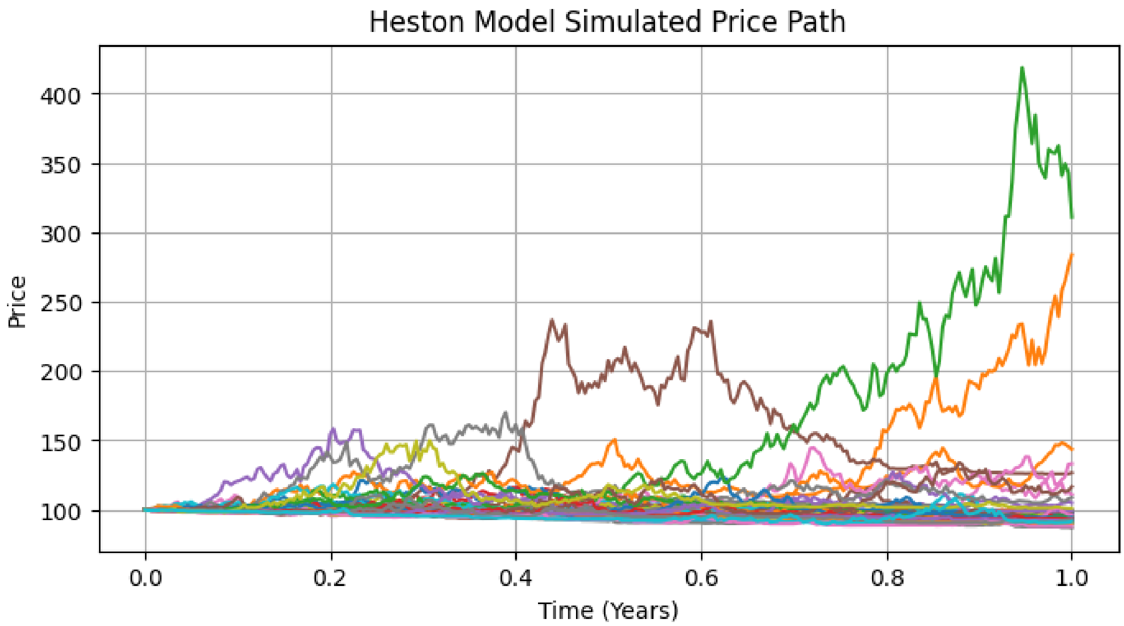


Figure 25: $\rho = 1$ Price Paths

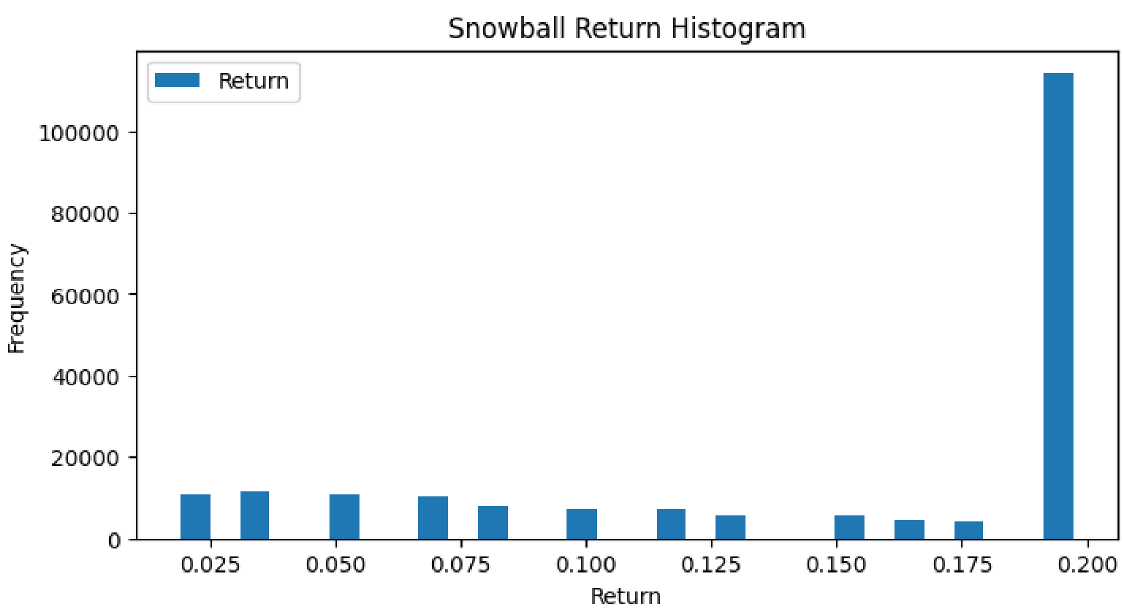


Figure 26: $\rho = 1$ Heston Return Distribution

Metric	Value
Mean Return	14.83%
95% Confidence Interval	14.7982%, 14.8550%
5%VaR	0.02%
5%ES	0.02%
Maximum Return	20.00%
Minimum Return	0.02%
Return Std	6.48%
Skewness	-0.84
Kurtosis	-0.94
Mean Knockout Time	0.4675
Mean Knockin Loss	0%
Prob Knockout	45.10%
Prob Knockin	0%
Prob Nonknock	54.89%

Table 8: $\rho = 1$ Summary

3.1 Sensitivity to ρ

We plot the snowball return over correlation ρ in Heston model from -1 to 1 in figure 27. The monotone behavior shows that under current parametrization, the snowball return increases as the correlation between the underlying price and its volatility becomes more positive.

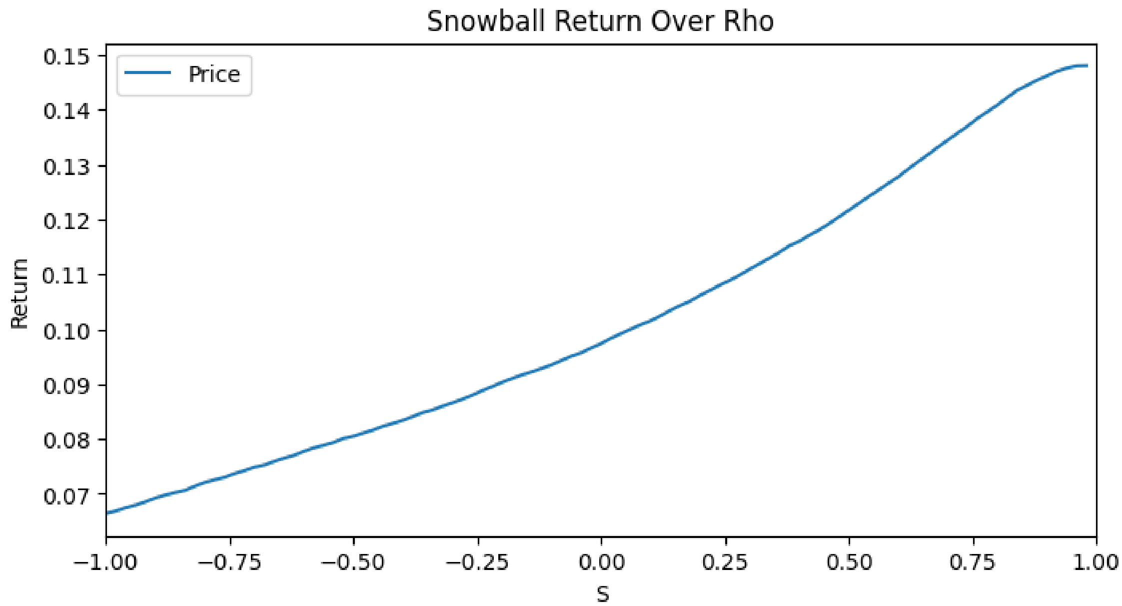


Figure 27: Snowball Return Over ρ

Define the price sensitivity with respect to ρ as $\frac{\partial V}{\partial \rho}$, often referred to as **correlation Vega**. The plot in Figure 28 illustrates how correlation Vega varies across different initial prices. For most initial prices, correlation Vega is positive, with a distinct peak near $S = 90$, indicating that correlation sensitivity is maximized at this price.

Under the calibrated Heston model, correlation Vega reaches its highest value when S is near the knock-out level and its lowest value when S is near the knock-in level. A negative correlation increases the likelihood of hitting the knock-in barrier, as price declines are typically accompanied by spikes in volatility. This results in correlation Vega becoming more negative, as even small changes in ρ significantly impact the probability of a knock-in event. The sharp dip in correlation Vega near the knock-in region highlights the critical sensitivity of the option price to ρ, where minor changes in correlation can dramatically influence knock-in probabilities.

As S approaches the knock-out level, the option enters an accrual phase. During this phase, periodic coupon payments depend on the underlying price remaining within the knock-in and knock-out boundaries, making correlation Vega highly positive and sensitive in this region.

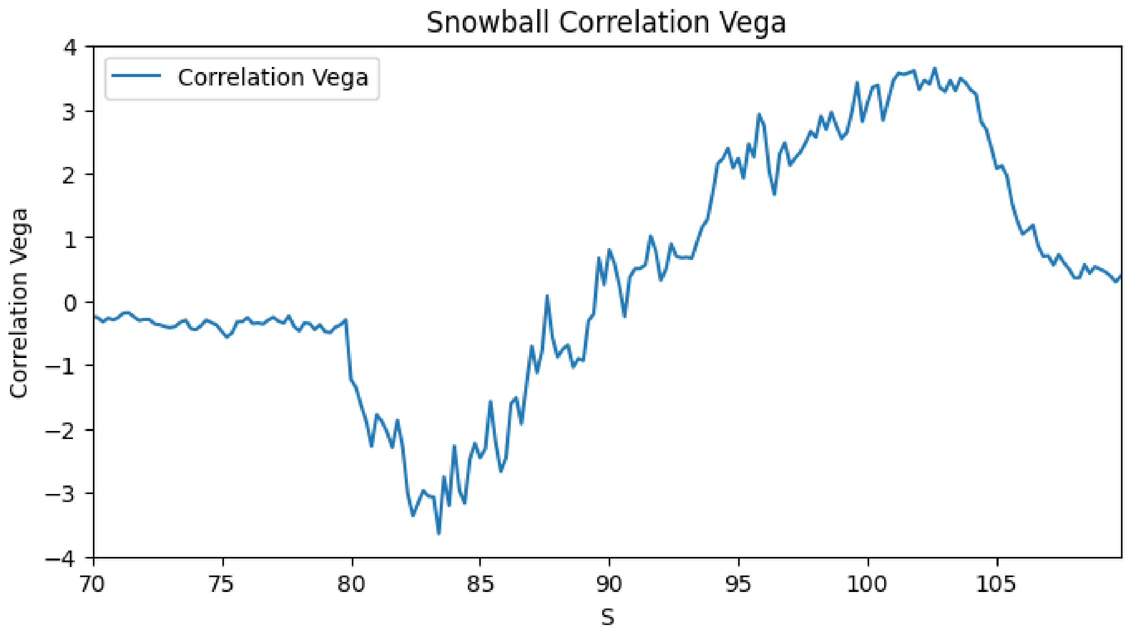


Figure 28: $\frac{\partial V}{\partial \rho}$

4 Conclusion

In this project, we analyzed the Snowball product using two different models: the constant volatility model and the Heston model. Our results demonstrate distinct differences in pricing and risk characteristics due to the stochastic nature of volatility in the Heston framework.

Under the Heston model, the Snowball option exhibited higher average returns compared to the constant volatility model. This is attributed to the stochastic variance reverting to its mean level, θ, which introduces periods of lower volatility. These periods favor prolonged accruals and delayed knock-out events, increasing average returns. However, the Heston model also demonstrated a significantly more negative minimum return and greater downside risk due to volatility clustering and sudden spikes in variance. This resulted in a negatively skewed return distribution with higher kurtosis, capturing extreme outcomes that the constant volatility model fails to account for.

The correlation parameter ρ played a critical role in shaping the Snowball's behavior under the Heston model. A negative ρ, which reflects the leverage effect commonly observed in equity markets, increased the likelihood of knock-in events due to higher volatility during price declines. This led to heightened tail risks and more pronounced downside outcomes. Conversely, a positive ρ reduced knock-in probabilities and extended knock-out times, resulting in higher returns and prolonged accrual periods. When ρ was neutral ($\rho = 0$), the model exhibited more symmetric behavior, balancing the probabilities of knock-in, knock-out, and non-knock scenarios.

Overall, the Heston model provides a more realistic and comprehensive framework for pricing Snowball products compared to the constant volatility model. By incorporating stochastic volatility and correlation dynamics, it captures key market phenomena, such as volatility clustering, leverage effects, and tail risks. However, this increased realism comes with greater computational complexity and sensitivity to parameter calibration, especially the correlation ρ.

This study highlights the importance of accurately modeling volatility and correlation dynamics in pricing path-dependent derivatives. The findings can guide practitioners in structuring Snowball products and managing their associated risks effectively.

A Key Facts Sheet

Underlying	S&P500 Index (SPX)
Notional Principle	10000$
Start Date	December 13, 2024
Maturity Date	December 19, 2025 (Early redemption possible on knock-out observation dates)
Knock-Out Level	Initial Price $\times$ 105%
Knock-In Level	Initial Price $\times$ 80%
Coupon (Annualized)	20% (per annum)
Initial Margin	20%
Prepayment Date	December 13, 2024
Maximum Loss	100%
Option Premium	9% (Absolute)
Knock-Out Observation	See Knockout Observation Date
Knock-Out Event	If on any knock-out observation date, the closing price of the underlying is greater than or equal to the knock-out level
Knock-Out Observation	Each trading day
Knock-In Event	If on any knock-in observation date, the closing price of the underlying is less than the knock-in level
Investment Return	**If knocked out:** Redemption amount = 20% $\times$ Nominal Principal $\times$ Accrued Days / 365 **If knocked in:** Redemption amount =min(Price at Maturity/Initial Price $-$ $1, 0$)$\times$ Nominal Principal **If non-knock:** Coupon payment = 20% $\times$ Nominal Principal $\times$ Accrued Days / 365

Table 9: Summary of Snowball Product Features

A.1 Knockout Observation Date

i	Knockout Observation Date T'_i	Knockout Price Ratio P_i (%)	Coupon Yield (%)
1	Jan 17, 2025	105	20
2	Feb 14, 2025	105	20
3	Mar 14, 2025	105	20
4	April 18, 2025	105	20
5	May 16, 2025	105	20
6	June 13, 2025	105	20
7	July 18, 2025	105	20
8	August 15, 2025	105	20
9	September 19, 2025	105	20
10	October 17, 2025	105	20
11	November 14, 2025	105	20
12	December 19, 2025	105	20

Table 10: Knockout Observation Dates and Price Conditions