Duality

1 Basics

Consider the standard form of linear programming:

$$\begin{aligned}
&\underset{\boldsymbol{x}}{\text{minimize}} \ \ \boldsymbol{c}^\top \boldsymbol{x} && (1a)\\
&\text{subject to} \ \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, && (1b)\\
&\qquad\qquad\quad \boldsymbol{x} \geq \boldsymbol{0}, && (1c)
\end{aligned}$$

where $\boldsymbol{c}^\top$ is an n dimensional row vector, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$ is an n dimensional column vector, $\boldsymbol{A}$ is an $m \times n$ matrix, and $\boldsymbol{b}$ is an n dimensional column vector. By using an m dimensional row vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)^\top$, we define a linear programming problem:

$$\begin{aligned}
&\underset{\boldsymbol{y}}{\text{maximize}} \ \ \boldsymbol{y}^\top \boldsymbol{b} && (2a)\\
&\text{subject to} \ \ \boldsymbol{A}^\top \boldsymbol{y} \leq \boldsymbol{c}. && (2b)
\end{aligned}$$

Due to the equality $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, the dual variable $\boldsymbol{y}$ is free (no constraint).

If the primal problem (1) has inequalities as:

$$\begin{aligned}
&\underset{\boldsymbol{x}}{\text{minimize}} \ \ \boldsymbol{c}^\top \boldsymbol{x} && (3a)\\
&\text{subject to} \ \ \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, && (3b)\\
&\qquad\qquad\quad \boldsymbol{x} \geq \boldsymbol{0}, && (3c)
\end{aligned}$$

we obtain the dual problem displayed below:

$$\begin{aligned}
&\underset{\boldsymbol{y}}{\text{maximize}} \ \ \boldsymbol{y}^\top \boldsymbol{b} && (4a)\\
&\text{subject to} \ \ \boldsymbol{A}^\top \boldsymbol{y} \leq \boldsymbol{c}. && (4b)\\
&\qquad\qquad\quad \boldsymbol{y} \geq \boldsymbol{0}. && (4c)
\end{aligned}$$

Due to the inequality $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$, the dual variable $\boldsymbol{y}$ is non-negative.

If the primal problem (1) has equalities and inequalities as:

$$\begin{aligned}
&\underset{\boldsymbol{x}_1, \boldsymbol{x}_2}{\text{minimize}} \ \ \boldsymbol{c}_1^\top \boldsymbol{x}_1 + \boldsymbol{c}_2^\top \boldsymbol{x}_2 && (5a)\\
&\text{subject to} \ \ \boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{x}_2 \geq \boldsymbol{b}_1, \ \ (:\boldsymbol{y}_1) && (5b)\\
&\qquad\qquad\quad \boldsymbol{A}_{21}\boldsymbol{x}_1 + \boldsymbol{A}_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2, \ \ (:\boldsymbol{y}_2) && (5c)\\
&\qquad\qquad\quad \boldsymbol{x}_1 \geq \boldsymbol{0}, \ \ (:\boldsymbol{\lambda}_1), \ \ \boldsymbol{x}_2 \geq \boldsymbol{0}, \ \ (:\boldsymbol{\lambda}_2), && (5d)
\end{aligned}$$

we obtain the dual problem displayed below:

$$\begin{aligned}
&\underset{\boldsymbol{y}_1, \boldsymbol{y}_2}{\text{maximize}} \ \ \boldsymbol{y}_1^\top \boldsymbol{b}_1 + \boldsymbol{y}_2^\top \boldsymbol{b}_2 && (6a)\\
&\text{subject to} \ \ \boldsymbol{A}_{11}^\top \boldsymbol{y}_1 + \boldsymbol{A}_{21}^\top \boldsymbol{y}_2 \leq \boldsymbol{c}_1, && (6b)\\
&\qquad\qquad\quad \boldsymbol{A}_{12}^\top \boldsymbol{y}_1 + \boldsymbol{A}_{22}^\top \boldsymbol{y}_2 \leq \boldsymbol{c}_2, && (6c)\\
&\qquad\qquad\quad \boldsymbol{y}_1 \geq \boldsymbol{0}. && (6d)
\end{aligned}$$

Due to the inequality $\boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{x}_2 \geq \boldsymbol{b}_1$, the dual variable $\boldsymbol{y}_1$ is non-negative. Due to the equality $\boldsymbol{A}_{21}\boldsymbol{x}_1 + \boldsymbol{A}_{22}\boldsymbol{x}_2 = \boldsymbol{b}_2$, the dual variable $\boldsymbol{y}_2$ is free (no constraint).

The Lagrangian function of (5) is

$$L = \boldsymbol{c}_1^\top \boldsymbol{x}_1 + \boldsymbol{c}_2^\top \boldsymbol{x}_2 + \boldsymbol{y}_1^\top (\boldsymbol{b}_1 - \boldsymbol{A}_{11}\boldsymbol{x}_1 - \boldsymbol{A}_{12}\boldsymbol{x}_2) + \boldsymbol{y}_2^\top (\boldsymbol{b}_2 - \boldsymbol{A}_{21}\boldsymbol{x}_1 - \boldsymbol{A}_{22}\boldsymbol{x}_2) - \boldsymbol{\lambda}_1 \boldsymbol{x}_1 - \boldsymbol{\lambda}_2 \boldsymbol{x}_2 \tag{7}$$

The KKT condition is:

$$\begin{aligned}
&\boldsymbol{c}_1 - \boldsymbol{A}_{11}^\top \boldsymbol{y}_1 - \boldsymbol{A}_{21}^\top \boldsymbol{y}_2 - \boldsymbol{\lambda}_1 = \boldsymbol{0}, && (8a)\\
&\boldsymbol{c}_2 - \boldsymbol{A}_{12}^\top \boldsymbol{y}_1 - \boldsymbol{A}_{22}^\top \boldsymbol{y}_2 - \boldsymbol{\lambda}_2 = \boldsymbol{0}, && (8b)\\
&\boldsymbol{0} \leq \boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{x}_2 - \boldsymbol{b}_1 \perp \boldsymbol{y}_1 \geq \boldsymbol{0}, && (8c)\\
&\boldsymbol{0} \leq \boldsymbol{x}_1 \perp \boldsymbol{\lambda}_1 \geq 0, && (8d)\\
&\boldsymbol{0} \leq \boldsymbol{x}_2 \perp \boldsymbol{\lambda}_2 \geq 0, && (8e)
\end{aligned}$$

If both the primal and dual problems are feasible, and the KKT condition (8) is satisfied, the strong duality theorem shows:

$$\boldsymbol{c}_1^\top \boldsymbol{x}_1 + \boldsymbol{c}_2^\top \boldsymbol{x}_2 = \boldsymbol{y}_1^\top \boldsymbol{b}_1 + \boldsymbol{y}_2^\top \boldsymbol{b}_2 \tag{9}$$

The dual variables $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ of equation (6), namely, the shadow prices $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are the Lagrange multipliers.

参考文献

[1] https://people.orie.cornell.edu/dpw/orie6300/Lectures/lec08.pdf