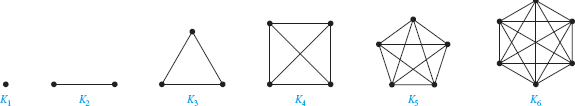
**Proof that for every simple graph with at least two vertices,**

**there must exist two vertices with the same degree**

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First, note that Section 10.1 of *Discrete Mathematics and Its Applications* (ISBN: 978-0-07-338309-5) defines a simple graph as having undirected edges, no multiple edges between the same vertices and no loops connecting a vertex to itself.  Given these limitations, the maximum number of edges a vertex can have in a simple graph is (# of vertices - 1).  We can prove that (# of vertices - 1) is the maximum degree of a vertex in a simple graph by invoking the Generalized Pigeonhole Principle, noting that k or more connections with k-1 other vertices means that at least one vertex-vertex pair has two connections (if looping is not permitted). This scenario is equivalent to the formula:

ceiling( k / (k-1) ) = 2 [when k >= 2]

The smallest number of edges incident upon a vertex is zero.  Because each vertex in a simple graph can have a maximum degree of (# of vertices - 1) and a minimum degree of zero, the set of possible degrees a vertex can have is the same size as the set of all vertices in that graph.

Now that we have established the framework for the problem, I will prove that in a simple graph with two or more vertices, there must be at least two vertices of the same degree. First, note that the negation of the statement ‘at least two vertices have the same degree’ is ‘each vertex has a unique degree value’ when the number of vertices is greater than or equal to two.

Consider a simple graph with n vertices, where n >= 2. We can represent the relationship between each vertex and its degree as an ordered pair (v,d), where ‘v’ is an arbitrary identifier representing the vertex and ‘d’ is the degree of that vertex. Recall that the set of all vertices in our simple graph is the same size as the set of all possible degrees a vertex can have. Invoking the Pigeonhole principle again, we see that each vertex must be present in exactly 1 ordered pair in this relation if all vertices have exactly one degree value. Likewise, each possible degree in the range (0,n-1) must be present in exactly 1 ordered pair in this relation if no two vertices have the same degree.

If we sort the binary relation between each vertex and its degree from smallest degree to largest, a graph where each vertex has a unique degree would have the relation:

R = { (v1,0), (v2,1), (v3,2) … (vn,n-1) }

However, this relation has a contradiction between the ordered pairs (v1,0) and (vn,n-1). The ordered pair (vn,n-1) represents a vertex in a simple graph which is connected to all vertices except itself exactly once. However, vertex v1 is not connected to any vertices at all. Because of the rules established for simple graphs in our textbook (ISBN: 978-0-07-338309-5) and the restrictions we have established above by invoking the Pigeonhole Principle, we see that the only way to resolve this contradiction is by accepting that there must be at least two vertices of identical degree in a simple graph of two or more vertices.