Mathematics

Concise Lecture Notes

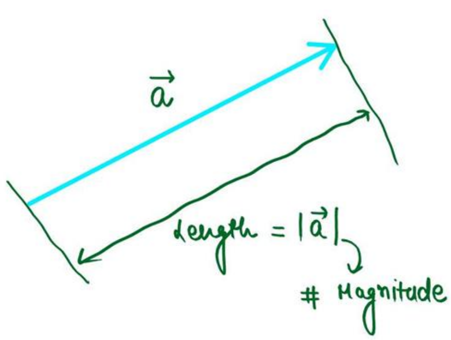
Vector Algebra

Key topics covered in the Class:

- Basics and Types of Vectors
- Vector Addition and Section Formulas
- Geometry Centers (Centroid, Incenter, etc.)
- Dot and Cross Products
- Scalar and Vector Triple Products
- Linearity and Coplanarity

BASICS OF VECTORS

- **Definition:** Physical quantities having both magnitude and direction, satisfying the law of vector addition.
- Mathematically, a **directed line segment** is called a Vector.
- Magnitude is represented by length; direction is represented by the arrow.



TYPES OF VECTORS

- **Null Vector:** Magnitude is 0; direction is not defined.
- **Unit Vector:** A vector with magnitude 1. Calculated as $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.
- **Equal Vectors:** Vectors with the same magnitude and same direction ($\vec{a} = \vec{b}$).

IMPORTANT NOTE

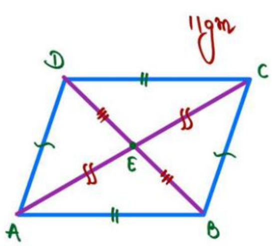
- **Multiplication by number:** Multiplying a vector by a scalar changes its magnitude (and direction if negative) but they remain collinear.
- If vectors are collinear, they can be expressed as $\vec{a} = \lambda\vec{b}$.

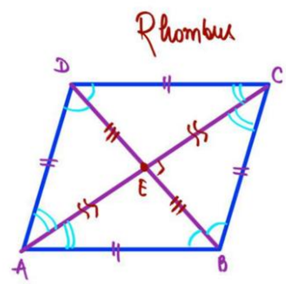
IMPORTANT NOTE

- If $\vec{a}$ and $\vec{b}$ are non-zero, non-collinear vectors, then $x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = 0$ and $y = 0$.
- This implies linear independence; no scalar combination can zero them out unless the scalars themselves are zero.

PARALLELOGRAM & RHOMBUS

- **Parallelogram:** Diagonals bisect each other.
- **Rhombus:** Diagonals bisect each other at $90°$ and bisect the vertex angles.





	Parall.	Rhomb.
① diag. bisects each other.	T	T
② diag. bisects each other at 90°	F	T
③ diag. (AC) bisects angle at A ($\measuredangle DAB$)	F	T

Triangle law of add^n

- Used to find the resultant of vectors arranged head-to-tail.
- $\overrightarrow{Resultant} = \vec{a} + \vec{b}$.

Parallelogram law of add^n

- Used when two vectors start from a common point (tail-to-tail).
- **Special Result:** The median vector $\vec{m} = \frac{\vec{a}+\vec{b}}{2}$.

VECTORS IN TERMS OF ORTHOGONAL TRIAD

- Vectors are represented using unit vectors along axes: $\hat{\imath}$ (x-axis), $\hat{\jmath}$ (y-axis), $\hat{k}$ (z-axis).
- For $\vec{a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, magnitude is $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$.

IMPORTANT NOTE

- **Vector Between Two Points:** $\overrightarrow{AB}$ = Position Vector of B − Position Vector of A.
- Formula: $\overrightarrow{AB} = \vec{b} - \vec{a}$. Note that $\overrightarrow{AB} = -\overrightarrow{BA}$.

SECTION FORMULA

- **Internal Division:** Point P dividing AB in ratio $m{:}n$ is $\vec{p} = \frac{m\vec{b}+n\vec{a}}{m+n}$.
- **Midpoint:** $\vec{p} = \frac{\vec{a}+\vec{b}}{2}$.
- **External Division:** Point P dividing AB externally is $\vec{p} = \frac{m\vec{b}-n\vec{a}}{m-n}$.

Centroid → p.o.i. of Median

- The point of intersection of medians (G).
- Divides the triangle area into three equal parts.

Orthocentre → p.o.i. of altitudes

- The point of intersection of altitudes (H).
- In a right-angled triangle, the vertex at the right angle is the orthocenter.

Incentre → p.o.i. of angle bisector

- The point equidistant from all three sides (I).
- Uses the Angle Bisector Theorem: $\frac{BD}{DC} = \frac{AB}{AC}$.

Circumcentre → p.o.i. of $\perp^r$ bisectors of sides

- The point equidistant from all vertices (O).
- In a right-angled triangle, it is the midpoint of the hypotenuse.

IMPORTANT TRICK

- **General coordinates for centers:** $\frac{\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}}{\alpha+\beta+\gamma}$.
- **Values:** Centroid $(\alpha = \beta = \gamma = 1)$, Incentre $(\alpha = a, \beta = b, \gamma = c)$, Circumcentre $(\sin 2A, \sin 2B \ldots)$.

IMPORTANT POINT TO REMEMBER

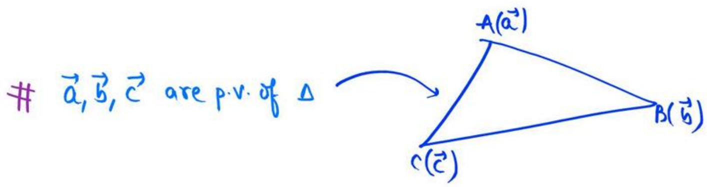
- **ONGIC Rule:** Orthocenter (H), Centroid (G), and Circumcenter (O) are collinear.
- G divides the line segment HO in the ratio **2 : 1**.

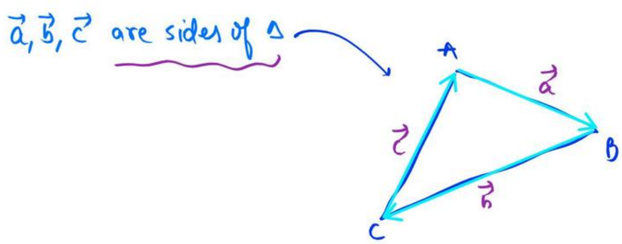
Circles

- **Inradius** (r): $r = \frac{\text{Area of } \Delta}{\text{semi-perimeter}} = \frac{\Delta}{s}$.
- **Circumradius** (R): $R = \frac{\text{product of sides}}{4\times\text{Area}} = \frac{abc}{4\Delta}$.

IMPORTANT POINT TO REMEMBER

- Distinguish between position vectors of vertices $(\vec{a}, \vec{b}, \vec{c})$ and vectors representing sides (e.g., $\vec{b} - \vec{a}$).





DOT PRODUCT (Scalar Product)

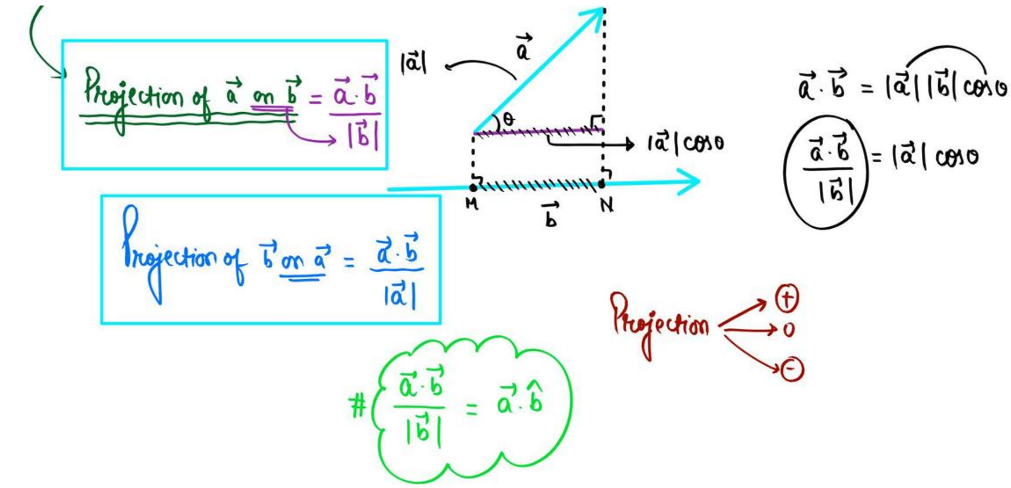
- Formula: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$. Use this to find the angle θ.
- Condition for Perpendicularity: $\vec{a} \cdot \vec{b} = 0$ (implies $\theta = 90°$).
- Self-product: $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.

Identities

- $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b})$.
- Similar to algebraic identities: $|\vec{a} + \vec{b} + \vec{c}|^2 = \sum|\vec{a}|^2 + 2\sum(\vec{a} \cdot \vec{b})$.

PROJECTION

- **Projection of $\vec{a}$ on $\vec{b}$:** Scalar component $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.
- **Vector Component:** Projection vector $= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\hat{b}$.



Cosine formula

- Used to find angles in a triangle using side lengths.
- $\cos A = \frac{b^2+c^2-a^2}{2bc}$.

Obvious Inequality

- For any three unit vectors summing to zero ($\vec{a}+\vec{b}+\vec{c}=\vec{0}$), the value is minimum.
- $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} \geq -\frac{3}{2}$.

THEOREM IN PLANE

- If $\vec{a}$ and $\vec{b}$ are non-zero, non-collinear vectors, any vector $\vec{r}$ in their plane is a linear combination.
- $\vec{r} = x\vec{a} + y\vec{b}$.

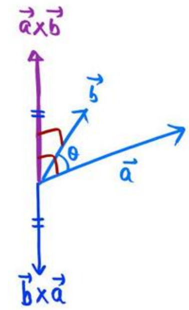
Vector Relation (Ikka)

- **Strategy for solving vector equations:**
- 1. Take dot product with any vector. 2. Take cross product. 3. Square magnitude on both sides.

CROSS (VECTOR) PRODUCT

- Formula: $\vec{a}\times\vec{b} = (|\vec{a}||\vec{b}|\sin\theta)\hat{n}$. Output is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.
- **Condition for Parallelism:** $\vec{a}\times\vec{b} = \vec{0}$ (implies $\theta = 0°$ or $180°$).

\# $\boxed{\vec{a} \times \vec{b} = \Big(\underbrace{|\vec{a}|\,|\vec{b}|\,\sin\theta}_{\text{Magnitude}}\Big)\underbrace{\hat{n}}_{\text{direc}^n}}$



\# $\vec{a}\times\vec{b}$ is $\perp^r$ to both $\vec{a}$ & $\vec{b}$ \# $\boxed{\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}}$

$$(\vec{a}\times\vec{b})\cdot\vec{a} = 0$$
$$(\vec{a}\times\vec{b})\cdot\vec{b} = 0$$

PROPERTIES

- Anti-commutative: $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.
- Matrix/Determinant method is used for calculation with components $(\hat{\imath}, \hat{\jmath}, \hat{k})$.
- Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

PROPERTIES

- **Lagrange's Identity**: $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$.
- General Use: Dot product for angles/lengths; Cross product for areas/perpendicular vectors.

AREA

- **Parallelogram**: Area $= |\vec{a} \times \vec{b}|$ (adjacent sides).
- **Quadrilateral**: Area $= \frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$ (where d_1, d_2 are diagonals).

Area of Triangle

- If sides are vectors $\vec{a}, \vec{b}$: Area $= \frac{1}{2}|\vec{a} \times \vec{b}|$.
- If vertices are position vectors $\vec{a}, \vec{b}, \vec{c}$: Area $= \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$.

SCALAR TRIPLE PRODUCT (STP)

- Represented as $[\vec{a}\vec{b}\vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$.
- Calculated using a determinant where rows are components of $\vec{a}, \vec{b}, \vec{c}$.

IMPORTANT PROPERTIES

- **Dot & Cross interchange**: $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$.
- **Cyclic Order**: $[\vec{a}\vec{b}\vec{c}] = [\vec{b}\vec{c}\vec{a}] = [\vec{c}\vec{a}\vec{b}]$. Breaking order introduces a negative sign.

MAJOR FORMULAS

- $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}\vec{b}\vec{c}]$.
- $[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}] = [\vec{a}\vec{b}\vec{c}]^2$.

Geometrical Meaning

- **Parallelopiped:** Volume = $|[\vec{a}\vec{b}\vec{c}]|$ (base area × height).
- **Tetrahedron:** Volume = $\frac{1}{6}|[\vec{a}\vec{b}\vec{c}]|$.

VECTOR TRIPLE PRODUCT

- Definition: $\vec{a}\times(\vec{b}\times\vec{c})$. A vector coplanar with $\vec{b}$ and $\vec{c}$, perpendicular to $\vec{a}$.
- **Expansion Formula:** $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ (Remember: "far · mid − close · far").

Important Properties of VTP

- Unit vector coplanar with $\vec{a}$ & $\vec{b}$ and perpendicular to $\vec{c}$ is $\pm\frac{(\vec{a}\times\vec{b})\times\vec{c}}{|(\vec{a}\times\vec{b})\times\vec{c}|}$.
- Generally, $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$.

Scalar Product of 4 vectors

- $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$.

COLLINEARITY

- **3 Points:** Form two vectors (e.g., $\overrightarrow{AB}, \overrightarrow{BC}$) and check if $\overrightarrow{AB} = \lambda\overrightarrow{BC}$.
- **Component Form:** Ratios of corresponding coefficients must be equal: $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

COPLANARITY

- **4 Points:** Form 3 vectors from the points (e.g., $\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}$).
- Condition: Their Scalar Triple Product must be zero, i.e., $[\overrightarrow{BA}\ \overrightarrow{BC}\ \overrightarrow{BD}] = 0$.

LINEARLY DEPENDENT & INDEPENDENT VECTORS

- **Linearly Dependent:** Scalars x_i exist (not all zero) such that $\sum x_i\vec{a}_i = \vec{0}$.
- **Linearly Independent:** Equation $\sum x_i\vec{a}_i = \vec{0}$ is satisfied ONLY if all scalars $x_i = 0$.

RECIPROCAL SYSTEM OF VECTORS

- Two sets of non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ where $\vec{a}\cdot\vec{a}' = 1$, etc.
- Formula: $\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$, $\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$, etc.

THEOREM IN SPACE

- Any vector in 3D space can be written as a linear combination of three non-coplanar vectors.
- $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$. Note: 4 or more vectors in 3D are always linearly dependent.